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Theme

MULTI-GRID METHODS FOR NON-LINEAR ELLIPTIC VARIATIONAL INEQUALITIES

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Abstract

This work concerns the study of a class of variational inequalities, in the sense that the obstacle does not depend on the solution, by applying the multi-grid methods V-cycle and W-cycle.

Multi-grid methods consist in successively using grids (or meshes) of different sizes, so as to obtain a detailed solution in the high frequencies, while ensuring a rapid relaxation of the low frequencies.

Multi-grid methods have been studied for linear elliptical problems. For our part, we are interested in finite difference approximation, by introducing multi-grid algorithms, for non-linear variational inequalities, insofar as the non-coercive and linear operator, the second member depends on the solution the mixture of the last two cases and the last case where the operator and the second member are non-linear.

Key words: Nonlinear elliptic variational inequalities, Multi-grid method, Finite elements approximations.

Résumé

Ce travail concerne l'étude d'une classe des inéquations variationnelles, dans le sens où l'obstacle ne dépend pas de la solution, en appliquant les méthodes multigrilles V-cycle et W-cycle.

Les méthodes multigrilles consistent à utiliser successivement des grilles (ou maillages) de différentes tailles, de manière à obtenir une solution détaillée dans les hautes fréquences, tout en assurant une relaxation rapide des basses fréquences.

Les méthode multigrilles ont été étudiées pour les problèmes elliptiques linéaires. Pour notre part, on s'intéresse à l'approximation par différences finies, en introduisant les algorithmes aux multigrilles, pour des inéquations variationnelles non linéaires, dans la mesure où l'opérateur non coercive et linéaire, le seconde membre dépend de la solution le mélange des deux derniers cas et le dernier cas où l'opérateur et le second member son non linéaires.

Mots clés: Inéquation variationnelle elliptique non linéaire, Méthodes Multigrilles, Approximations par éléments finies.

ملخص

يتعلق هذا العمل بدراسة فئة من التفاوتات المتباينة، بمعنى أن العقبة لا تعتمد على الحل، من خلال تطبيق دورة الطرق متعددة الأطراف دورة V ودورة W.

تتكون طرق متعدد الشبكات على التوالي باستخدام شبكات بأحجام مختلفة، وذلك للحصول على حل مفصل في الترددات العالية، مع ضمان الاسترخاء السريع للترددات المنخفضة.

تمت دراسة طرق متعدد الشبكات للمشاكل الإهليلجية الخطية. من جانبنا، نحن مهتمون بتقريب الاختلاف المحدود، من خلال إدخال خوارزميات متعددة العناصر، للتفاوتات المتباينة غير الخطية، طبقنا هذه الطريقة على مؤثر غير القسري والخطي، العضو الثاني يعتمد على الحل، مزيج الحالتين الأخيرتين وفي الحالة الأخيرة حيث يكون المشغل والعضو الثاني غير خطيين.

الكلمات المفتاحية: المتباينة التباينية الإهليلجية غير الخطية، طرق الشبكات المتعددة، تقريبات بواسطة العناصر المنتهية.

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A friendly greeting to all comrades of the promotion (2019 – 2020).
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Dedication

*To my **parents**, I express my deepest gratitude and thank them for having been with me in my academic journey and for their patience with me. Thank you for always being by my side and sharing my success.*
*(To my dear **father** who rests in peace that God grants him the Entry into Paradise!).*
*To my sisters **Hind** and **Hadia Chaimaa** and brothers **Hamza** and **Ilyes**, thank you for your continued support, encouragement and moral support.*
*My all **friends** who supported and trusted me. Thank you for every moment you believed in my path and success. Thank you for all the memories you left me and the strength and knowledge you gave me.*
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NESBA Nour El Houda

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List of Publications

The following results were published or submitted:

1. N. E. H. Nesba, M. Beggas, M. E. Belouafi, I. Ahmad, H. Ahmad, S. Askar. Multigrid Methods for The Solution of Nonlinear Variational Inequalities, European journal of pure and applied mathematics, Published by New York Business Global, Vol. 16, No. 3, 1956-1969, (2023).

2. N. E. H. Nesba, M. Beggas. Multigrid Methods for non Coercive Variational Inequalities.

3. N. E. H. Nesba, M. Beggas. Multigrid Methods for non Coercive Operator of Nonlinear Variational Inequalities.

4. N. E. H. Nesba, M. Beggas. Multigrid Methods for Nonlinear Elliptic Variational Inequalities.

Introduction

In the last sixty years, variational inequalities have become a tool relevant in the study of non-linear problems in physics and mechanics. The theory of variational inequalities was made from the results concerning the problems unilateral obtained by A. Signorini [5] and G. Fichera [10]. The mathematical theory has been obtained by G. Stampacchia [11], J. L. Lions and G. Stampacchia [18] and then developed by H. Brézis [13], G. Stampacchia [12], J. L. Lions [17], U. Mosco [46], D. Kinderlehrer and G. Stampacchia [7]. For the approximation of variational inequalities we recall, the contributions of U. Mosco [45], R. Glowinsky [36], J. L. Lions, R. Trémolières and R. Glowinsky [38].

The theory of variational inequalities has been used in several fields such as mechanics, physics, optimization, optimal control, linear programming, financial mathematics, etc.; Today it is considered an indispensable tool in several areas of applied mathematics.

For a long time researchers in their study of ordinary differential equations, partial differential equations, variational equations in general and in particular variational inequalities, were interested in different approximation techniques, namely the methods of finite differences, finite elements [9, 19, 39], finite volumes and methods spectral and consisted of the following problem of the obstacle

$$\begin{cases} Au \geq f & \text{in } \Omega, \\ u \leq \psi; & \psi \geq 0, \\ u = 0; & \text{on } \partial\Omega, \end{cases}$$

where

- A is an operator linear coercive.

- $f \in L^{\infty}(\Omega)$.

- Ω an open set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$.

- $\psi \in W^{2,p}(\Omega)$ an obstacle such that $\psi \geq 0$ on $\partial\Omega$.

Our work is concerned to four types as same as the previous problem with some modifications. We mention its briefly:

- The non coercive problem can solved by adding a variable to the both of members.
- The non linear second member can solved by the linearisation of the second member.
- The non coercive problem and non linear second member can solved by using the previous resolutions in the same time.
- The non linear (operator and second member) problem can solved by the linearisation of the operator and the second member.

This work is divided into four chapters:

Chapter I is devoted to presenting generalities on the elliptic variational inequality [6]. In the first section, we show some basic notations of spaces and norms. The second is for existence and uniqueness of the solution continuous with switching to the Hamilton-Jacobi-Bellman equation [21, 23] and third section of this chapter a theorem of existence and uniqueness of the solution discrete due to Stampacchia has been demonstrated. In section four, we will see the non-coercive operator problems [24, 25], the fifth is for the non-linear second member problems and the sixth for the mixing of the two previous. The last section is for the non-linear operator and second member problems [8, 37].

Chapter II is a representation of the multi-grid method [15, 35, 43, 44, 47, 48]. We start with the iterative Gauss Seidel method for pre-smoothing and post-smoothing, then the transfer operators (extension and restriction). Then, the two-grid method and its generality and finally the convergence of this method in the L^∞-norm.

For the chapter III, it is contained by three sections. Firstly, We introduce an algorithm which gives a formulation of the V.I in stationary Hamilton Jacobi Bellman equations inspired by the Hoppe multigrid method [4, 40]. This algorithm will subsequently be called M.G.H.J.B and we give the iteration matrices associated with the algorithm. We give in the third paragraph, the first original results for the approximation and smoothing properties in norm L^∞ and we show the uniform convergence of the M.G.H.J.B algorithm. Finally, we have an application numerical where the operator is linear and the second member is non linear and depend the solution and consisted of the problem

$$\begin{cases} Au + \lambda u \geq f + \lambda u & \text{in } \Omega, \\ u \leq \psi; & \psi \geq 0, \\ u = 0; & \text{on } \partial\Omega, \end{cases}$$

where

- A is an operator linear non-coercive.
- $f \in L^\infty(\Omega)$.
- Ω an open set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$.
- $\psi \in W^{2,p}(\Omega)$ an obstacle such that $\psi \geq 0$ on $\partial\Omega$.
- $\lambda > 0$ large enough.

We applied the Gauss Seidel method and the multi-grid methods V and W-cycle.

Secondly, We introduce an algorithm which gives a formulation of the V.I in stationary Hamilton Jacobi Bellman equations inspired by the Hoppe multigrid method [4]. This algorithm will subsequently be called M.G.H.J.B and we give the iteration matrices associated with the algorithm. We give in the third paragraph, the first original results for the approximation and smoothing properties in norm L^∞ and we show the uniform convergence of the M.G.H.J.B algorithm. Finally, we have an numerical application where the operator is linear and non coercive, and the second member is linear, we consisted of the problem

$$\begin{cases} Au \geq f(u) & \text{in } \Omega, \\ u \leq \psi; & \psi \geq 0, \\ u = 0; & \text{on } \partial\Omega, \end{cases}$$

where

- A is an operator linear coercive.
- $f(u) \in L^\infty(\Omega)$.
- Ω an open set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$.
- $\psi \in W^{2,p}(\Omega)$ an obstacle such that $\psi \geq 0$ on $\partial\Omega$.

We applied the Gauss Seidel method and the multi-grid methods V and W-cycle.

Lastly but not least, We introduce an algorithm which gives a formulation of the VI. in stationary Hamilton Jacobi Bellman equations inspired by the Hoppe multigrid method [4, 40]. This algorithm will subsequently be called M.G.H.J.B and we give the iteration matrices associated with the algorithm. We give in the third paragraph, the first original results for the approximation and smoothing properties in norm L^∞ and we show the uniform convergence of the M.G.H.J.B algorithm. Finally, we have an numerical application where the operator is non coercive, linear and the second member is linear, where we considred of the problem

$$\begin{cases} Au + \lambda u \geq f(u) + \lambda u & \text{in } \Omega, \\ u \leq \psi; & \psi \geq 0, \\ u = 0; & \text{on } \partial\Omega, \end{cases}$$

where

- A is an operator linear non-coercive.
- $f(u) \in L^\infty(\Omega)$.
- Ω an open set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$.
- $\psi \in W^{2,p}(\Omega)$ an obstacle such that $\psi \geq 0$ on $\partial\Omega$.
- $\lambda > 0$ large enough.

We applied the Gauss Seidel method and the multi-grid methods V and W-cycle.

In the last chapter, We introduce the method which gives a formulation of the VI in stationary Hamilton-Jacobi-Bellman equations inspired by the Hoppe multigrid method [22]. We give the iteration matrices associated with this method. In the third paragraph, we see the first original results for the approximation and smoothing properties in norm L^∞ and we show the uniform convergence of the multigrid methods [20]. Finally, we have an application numerical where the operator and the second element are non-linear and consisted of the following problem

$$\begin{cases} Au \geq f(u) + \lambda u & \text{in } \Omega, \\ u \leq \psi; & \psi \geq 0, \\ u = 0; & \text{on } \partial\Omega, \end{cases}$$

where

- A is an operator non-linear.
- $f(u) \in L^\infty(\Omega)$.
- Ω an open set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$.
- $\psi \in W^{2,p}(\Omega)$ an obstacle such that $\psi \geq 0$ on $\partial\Omega$.

We used the Gauss-Seidel method and the V-cycle and W-cycle multi-grid methods for solving the linear system obtained.

Chapter 1

Generality of the Elliptic Variational Inequalities (E.V.I) and their approximations

The objective of this chapter is to recognize the variational inequalities and their approximations and to have a brief understanding of existence and uniqueness theorems, finite element approximation, convergence, error estimation and familiarization with this kind of problems, in particular the problem of the obstacle.

1.1 Analysis tools

The object of this section is the introduction of the basic notions, necessary for the good comprehension of our thesis. On the open set Ω of $\mathbb{R}^n$, we introduce the following spaces: $L^p(\Omega)$ the space of measurable functions on Ω, provided with norms

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty, \tag{1.1.1}$$

$$\|f\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} ess|f| \text{ if } p = +\infty, \tag{1.1.2}$$

is a Banach space.
$L^2(\Omega)$ is a Hilbert space, the scalar product corresponding to the norm (1.1.1) with $p = 2$ is given by

$$(f, g) = \int_\Omega f(x)g(x)dx. \tag{1.1.3}$$

For $p \geq 1, m \in \mathbb{N}$, we call Sobolev space of order m on $L^p(\Omega)$ the space

$$W^{m,p}(\Omega) = \{v/ \quad v \in L^p(\Omega), \quad D^\alpha v \in L^p(\Omega), \quad |\alpha| \leq m\},$$

equipped with the standard

$$\|v\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \tag{1.1.4}$$

is a Banach space.
For $p = 2$, the space

$$W^{m,2}(\Omega) = H^m(\Omega) = \left\{v/ \quad v \in L^2(\Omega), \quad D^\alpha v \in L^2(\Omega), \quad |\alpha| \leq m\right\},$$

equips with the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} (D^\alpha u, D^\alpha v), \tag{1.1.5}$$

is a Hilbert space.
Space

$$H^1_0(\Omega) = \left\{v/ \quad v \in H^1(\Omega), \quad v|_\Gamma = 0\right\},$$

is a closed subspace of $H^1(\Omega)$.
So $H^1_0(\Omega)$ is a Hilbert space for the structure induced by that of $H^1(\Omega)$.

1.2 Continuous elliptic variational inequality (E.V.I)

We begin by giving notations and assumptions useful in the study of continuous variational inequalities.

1.2.1 Notations and assumptions

Let Ω be an open set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$. For $u, v \in V$, the space $H^1_0(\Omega)$ or $H^1(\Omega)$, we set the bilinear form

$$a(u, v) = \int_\Omega \left(\sum_{1\leq i,j\leq n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{1\leq i\leq n} a_i(x) \frac{\partial u}{\partial x_i} v + a_0(x) uv \right) dx, \tag{1.2.1}$$

with the coefficients $a_{ij}(x), a_i(x)$ and $a_0(x)$ are sufficiently regular and

$$a_0(x) \geq \beta > 0, \quad \forall x \in \Omega. \tag{1.2.2}$$

We assume that the bilinear form is continuous and strongly coercive

$$\exists \alpha > 0 \quad \text{such as} \quad |a(v,v)| \geq \alpha \|u\|_V^2, \quad \forall v \in V. \tag{1.2.3}$$

Moreover, we consider a second member f such that

$$f \in L^\infty(\Omega), \tag{1.2.4}$$

and an obstacle

$$\Psi \in W^{2,\infty}(\Omega) \text{ such that } \Psi > 0 \text{ on } \partial\Omega. \tag{1.2.5}$$

1.2.2 Existence and uniqueness of a continuous E.V.I solution

We are looking for u the continuous solution of the following variational inequality (abbreviated I.V):
Find $u \in V$ such that

$$\begin{cases} a(u, v-u) \geq (f, v-u), & \forall v \in V, \\ u \leq \Psi \quad \text{and} \quad v \leq \Psi. \end{cases} \tag{1.2.6}$$

The existence and uniqueness of a continuous solution of the variational inequality (1.2.6) is given by:

Theorem 1. *(cf. [7]) Under the previous assumptions and notations the problem* (1.2.6) *admits a unique solution.*

1.2.3 Regularity of continuous E.V.I. solution

The regularity of the continuous solution of the E.V.I relies on the following theorem.

Theorem 2. *(cf. [1, 14]) Under the previous assumptions, we have*

$$u \in W^{2,p}(\Omega), \quad 2 \leq p < \infty. \tag{1.2.7}$$

1.2.4 Switching from a system to the continuous H.J.B

Consider the Hamilton-Jacobi-Bellman equation (This equation is approximated by the system (1.2.6))

$$\max_{1\leq i\leq M}\left(A_i u-f_i, u_i-\Psi_i\right)=0, \tag{1.2.8}$$

where the A_i are the M second-order uniform elliptic operators and the f_i are regular functions.

Theorem 3. *[1] The equation (1.2.8) is a system of variational inequalities coercive admitting a unique solution.*

1.3 Discrete elliptic variational inequality (E.V.I)

One introduces the discrete problem and one carries out a study similar to that undertaken previously for the continuous problem. To insist on the symmetry of the study, we will follow the same approach as in the previous paragraph.

We will consider a space of conformal finite elements constructed from polynomials of degree $P1$. The introduction of polynomials of higher degree has not been considered insofar as the regularity properties encountered do not seem to make it possible to take advantage of them.

We establish on Ω a quasi-uniform regular triangulation and we introduce $\mathbb{V}_h$ the following conformal finite element space

$$\mathbb{V}_h=\left\{v_h\in C(\Omega)\cap V \text{ such as } v_h/\tau\in P_1\right\}. \tag{1.3.1}$$

Let $M_s, s=1,2,\ldots,m(h)$ be the vertices of the triangulation which do not belong to $\partial\Omega$. We denote by $\varphi_s, s=1,2,\ldots,m(h)$ the usual basis functions ($\varphi_s\left(M_l\right)=\delta_{sl}$ symbol of Kronecker). We also introduce the restriction operator r_h, for $v\in C(\bar{\Omega})\cap H_0^1(\Omega)$

$$r_h v=\sum_{s,l=1}^{m(h)} v\left(M_l\right)\varphi_s(x,y). \tag{1.3.2}$$

The order on $\mathbb{V}_h$ will be that induced by $\mathbb{R}^{m(h)}$. We naturally introduce the discretization matrix A with generic coefficients $a\left(\varphi_l,\varphi_s\right)$.

1.3.1 Existence and uniqueness of a discrete E.V.I solution

Consider the discrete problem associated with problem (1.2.6):
Find $u_h \in \mathbb{V}_h$ such that

$$\begin{cases} a\left(u_h, v_h - u_h\right) \geq \left(f, v_h - u_h\right), & \forall v_h \in \mathbb{V}_h, \\ u_h \leq r_h \Psi \quad \text{and} \quad v_h \leq r_h \Psi. \end{cases} \tag{1.3.3}$$

Theorem 4. *(cf. [30]) Under the previous assumptions, the problem* (1.3.3) *admits a unique solution.*

1.3.2 Regularity of the discrete solution of E.V.I

As in the continuous case, the regularity of the discrete solution of the E.V.I (1.3.3) relies on the following theorem.

Theorem 5. *(cf. [33]) There exists a constant C independent of h such that*

$$\left|a\left(u_h, \varphi_i\right)\right| \leq C \left\|\varphi_i\right\|_{L^1(\Omega)}, \quad \forall \varphi_i, \quad i = 1, 2, \ldots, m(h). \tag{1.3.4}$$

The assumption of the discrete maximum principle (DMP): We assume that the matrix $\mathcal{A}$ is an M-matrix. ($\mathcal{A}^{-1}$) exists and is non-negative, with moreover $\mathcal{A}_{ss} > 0$ and $\mathcal{A}_{ls} \leq 0$ for $l \neq s$).

1.3.3 H.J.B Discrete equation

The finite element analysis of the H.J.B equation (1.2.8) with an easy transfer to the discrete problem and satisfying the maximum principle [26, 31].
In effect, the discrete version of the H.J.B is to search for $u_h \in V_h$ such that

$$\max_{1 \leq i \leq M} \left(\mathcal{A}_{h,i} u_h - f_{h,i}, u_{h,i} - \Psi_{h,i}\right) = 0. \tag{1.3.5}$$

By analogy, in [31], Cortey-Dumont prove that the solution of (1.3.5) is a limit in $C(\overline{\Omega})$ of the solution of the system of discrete E.V.Is associated with the system (1.2.6).

1.3.4 Approximation in L^∞-norm

Theorem 6. *[29] According to the previous assumptions and notations, we have*

$$\left\|u - u_h\right\|_\infty \leq C h^2 \left|\log h\right|^2. \tag{1.3.6}$$

In the following section, we will see the variational inequalities with non-coercive operator in the continuous, discrete case and the associated H.J.B equation.

1.4 Non-coercive operator of E.V.I

We assume that (1.2.3) does not hold. In this case, it is well known (see [1, 2]) that the corresponding problem exists unconstrained the problem can be handled like this (see [1, 2]):
Contains $\lambda > 0$ large enough

$$a(v,v) + \lambda(v,v) \geq \delta\|v\|^2, \quad \delta > 0. \tag{1.4.1}$$

We also assume that the constants defined in (1.2.2) and (1.4.4) are l and β as follows

$$\frac{l}{\beta} < 1. \tag{1.4.2}$$

As a result of this, E.V.I (1.2.6) change into

$$b(u, v-u) \geq (f + \lambda u, v-u), \quad \forall v \in \mathbb{V}, \tag{1.4.3}$$

where the non-linearity of f is assumed to be non-decreasing and Lipschitz-continuous, i.e

$$|f(x) - f(y)| \leq l|x-y|, \quad \forall x, y \in \mathbb{R}, \tag{1.4.4}$$

where l is a normal constant, the new variational equation form $b(u,v) = a(u,v) + \lambda(u,v)$ obviously satisfies the strong coercive force assumption.
Now to approximate the continuous solution, we proceed as follows front part. In fact, we construct the corresponding fixed-point map

$$T_\lambda : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \quad w \longrightarrow T_\lambda w = \zeta_\lambda, \tag{1.4.5}$$

where ζ_λ is the solution of the following constrained variational inequality

$$b\left(\zeta_\lambda, v - \zeta_\lambda\right) \geq \left(f(w) + \lambda w, v - \zeta_\lambda\right), \quad \forall v \in \mathbb{K}.$$

Proposition 1. *The mappings T_λ is contraction in $L^\infty(\Omega)$. The contraction constant is equal to $(l+\lambda)/(\lambda+\beta)$. So they have a unique fixed point agree with the solution of E.V.I (1.4.3).*

Proof. We drafted the proof of E.V.I. Just set it up

$$\zeta_\lambda = \sigma(F,\psi), \quad \tilde{\zeta}_\lambda = \sigma(\tilde{F},\psi), \quad \Phi = \frac{1}{\lambda+\beta}\|F-\tilde{F}\|_\infty, \tag{1.4.6}$$

where $F = f(w)+\lambda w, \quad \tilde{F} = f(\tilde{w})+\lambda\tilde{w}$, it follows

$$\begin{aligned} F &\leq \tilde{F} + \|F-\tilde{F}\|_\infty \\ &\leq \tilde{F} + \frac{a_0(x)}{\lambda+\beta}\|F-\tilde{F}\|_\infty \\ &\leq \tilde{F} + a_0\Phi, \end{aligned} \tag{1.4.7}$$

(because $a_0(x) \geq \beta > 0$). Thus, using the standard comparison results in constrained variational inequalities, we get

$$\sigma(F,\psi) \leq \sigma\left(\tilde{F} + a_0(x)\cdot\Phi, \psi\right) \leq \sigma(\tilde{F}+\Phi,\psi). \tag{1.4.8}$$

So

$$\zeta_\lambda \leq \tilde{\zeta}_\lambda + \Phi. \tag{1.4.9}$$

Similarly, if we swap the roles of w and $\tilde{w}$, we get

$$\tilde{\zeta}_\lambda \leq \zeta_\lambda + \Phi. \tag{1.4.10}$$

At last,

$$\|T_\lambda w - T_\lambda\tilde{w}\|_\infty \leq \frac{1}{\lambda+\beta}\|F-\tilde{F}\|_\infty. \tag{1.4.11}$$

Thus the shrinkage properties of T result from the combination of (1.4.2) and (1.4.4). $\square$

1.4.1 The non-coercive discrete problems

The discrete non-coercive E.V.I is defined as follows

$$b\left(u_h, v-u_h\right) \geq \left(f+\lambda u_h, v-u_h\right), \quad \forall v \in \mathbb{K}_h, \tag{1.4.12}$$

her associated fixed point mappings is

$$T_{\lambda,h} : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \quad w \to T_{\lambda,h}w = \zeta_{\lambda,h}, \tag{1.4.13}$$

where $\zeta_{\lambda,h}$ is the only solution to the following mandatory E.V.I.

$$b\left(\zeta_{\lambda,h}, v - \zeta_{\lambda,h}\right) \geq \left(f + \lambda w, v - \zeta_{\lambda,h}\right), \quad \forall v \in \mathbb{V}_h. \tag{1.4.14}$$

Similar to the continuous case, under DMP it can be easily shown that $T_{\lambda,h}$ is a contraction of $L^\infty(\Omega)$ with a contraction constant of $(l+\lambda))/(\lambda+\beta)$.
The assumption of the discrete maximum principle (DMP): We assume that the matrix $\mathcal{B}$ is the M-matrix. $(\mathcal{B}^{-1})$ exists and is non-negative, with moreover $\mathcal{B}_{ss} > 0$ and $\mathcal{B}_{ls} \leq 0$ for $l \neq s$).

1.4.2 Discrete H.J.B equation of non-coercive problem

The discrete problem is satisfying the maximum principle [26, 31].
In effect, the discrete version of the H.J.B is to search for $u_h \in V_h$ such that

$$\max_{1 \leq i \leq M} \left(\mathcal{B}_{h,i} u_h - f_{h,i} - \lambda u_{h,i}, u_{h,i} - \Psi_{h,i}\right) = 0. \tag{1.4.15}$$

By analogy, in [31], Cortey-Dumont prove that the solution of (1.3.5) is a limit in $C(\overline{\Omega})$ of the solution of the system of discrete E.V.Is associated with the system:
Find $u \in V$ such that

$$\begin{cases} b(u, v-u) \geq (f - \lambda u, v-u), & \forall v \in V, \\ u \leq \Psi \quad \text{and} \quad v \leq \Psi. \end{cases}$$

1.4.3 L^∞-error estimates for the non-coercive problems

Adjusting [30] we get the following lemma:

Lemma 1. *The following inequalities apply*

$$\|T_\lambda w - T_{\lambda,h} w\|_\infty \leq C h^2 |\log h|^2 \|f + \lambda w\|_\infty. \tag{1.4.16}$$

Before proving the following theorem we need the following proposition which works by the fixed point mapping associated with the E.V.I (1.4.12):
We consider the mapping

$$T_{\lambda,h} : L^\infty(\Omega) \longrightarrow \mathbb{V}_h, \quad w \longrightarrow T_{\lambda,h} w = \zeta_{\lambda,h}, \tag{1.4.17}$$

where $\zeta_{\lambda,h} \in \mathbb{V}_h$ solves the following coercive E.V.I

$$a\left(\zeta_{\lambda,h}, v-\zeta_{\lambda,h}\right) \geq \left(f+\lambda w, v-\zeta_{\lambda,h}\right), \quad \forall v \in \mathbb{V}_h. \tag{1.4.18}$$

Proposition 2. *Under the DMP and assumptions (1.4.2) and (1.4.4), the mapping T_h is a contraction for $L^\infty(\Omega)$, namely*

$$\|T_{\lambda,h}w - T_{\lambda,h}\tilde{w}\|_\infty \leq \frac{l+\lambda}{\lambda+\beta}\|w-\tilde{w}\|_\infty. \tag{1.4.19}$$

Therefore, there exists a unique fixed point, consistent with the solution of E.V.I (1.4.12).

Proof. Same steps as **Proposition 1**. □

Then there is the error estimate.

Theorem 7. *The following inequalities apply*

$$\|u-u_h\|_\infty \leq \frac{Ch^2|\log h|^2}{1-(l+\lambda)/(\lambda+\beta)}\|f+\lambda u\|_\infty. \tag{1.4.20}$$

Proof. It's easy to see

$$\|u-u_h\|_\infty \leq \|u-T_{\lambda,h}u\| + \|T_{\lambda,h}u{-}_h\|_\infty\,. \tag{1.4.21}$$

Further, according to Theorems 1 and 2, u and u_h are the fixed points of T_λ and $T_{\lambda,h}$ respectively, namely

$$u = T_\lambda u, \quad u_h = T_{\lambda,h}u. \tag{1.4.22}$$

If we apply the Lemma 1 with $w=u$, then

$$\begin{aligned}\|u-u_h\|_\infty &\leq \|T_\lambda u - T_{\lambda,h}u\|_\infty + \|T_{\lambda,h}u - T_{\lambda,h}u_h\|_\infty\,,\\ &\leq Ch^2|\log h|^2\|f+\lambda u\|_\infty + \frac{l+\lambda}{\lambda+\beta}\,\|u-u_h\|_\infty\,,\end{aligned} \tag{1.4.23}$$

thus,

$$\|u-u_h\|_\infty \leq \frac{Ch^2|\log h|^2\|f+\lambda u\|_\infty}{1-(l+\lambda)/(\lambda+\beta)}, \tag{1.4.24}$$

which is the desired error estimate. □

In the following section, we will see the variational inequalities with non-linear second member in the continuous, discrete case and the associated H.J.B equation.

1.5 Non-linear second member of E.V.I

We assume that (1.2.3) does not hold. In this case it is well known that the corresponding problem exists. The non-linear second term problem can be handled as follows (see [1, 2])

$$a(v,v) \geq \delta\|v\|^2, \quad \delta > 0. \tag{1.5.1}$$

We also assume that the constants l and β defined in (1.2.2) and (1.4.4) respectively are also in (1.4.2).
As a result, E.V.I (1.2.6) translates to

$$a(u,v-u) \geq (f(u),v-u), \quad \forall v \in \mathbb{V}, \tag{1.5.2}$$

where the non-linearity of f is assumed to be non-decreasing and Lipschitz continuous satisfing (1.4.4), where l is a positive constant and clearly the variational form $a(u,v)$ satisfies the strong coercivity assumption.

Now, in order to approximate the continuous solutions, we shall proceed as in the previous sections. Indeed, we construct the respective fixed point mappings

$$T : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \quad w \longrightarrow Tw = \zeta, \tag{1.5.3}$$

where ζ is the solution of the coercive variational inequality below

$$a\left(\zeta, v-\zeta\right) \geq \left(f(w), v-\zeta\right), \quad \forall v \in \mathbb{K}. \tag{1.5.4}$$

Proposition 3. *The mappings T is contraction in $L^\infty(\Omega)$. The contraction constant is equal to $(l)/(\beta)$. So they have a unique fixed point agree with the solution of E.V.I (1.5.2).*

Proof. We sketch the proof for the E.V.I. It suffices to set

$$\zeta = \sigma(F,\psi), \quad \tilde{\zeta} = \sigma(\tilde{F},\psi), \quad \Phi = \frac{1}{\beta}\|F-\tilde{F}\|_\infty, \tag{1.5.5}$$

where $F = f(w)$ and $\tilde{F} = f(\tilde{w})$, it follows

$$\begin{aligned} F &\leq \tilde{F} + \|F-\tilde{F}\|_\infty \\ &\leq \tilde{F} + \frac{a_0(x)}{\beta}\|F-\tilde{F}\|_\infty \\ &\leq \tilde{F} + a_0\Phi, \end{aligned} \tag{1.5.6}$$

(because $a_0(x) \geq \beta > 0$). Thus, using the standard comparison results in coercive variational inequalities, we get

$$\sigma(F,\psi) \leq \sigma\left(\tilde{F} + a_0(x)\cdot\Phi,\psi\right) \leq \sigma(\tilde{F}+\Phi,\psi). \tag{1.5.7}$$

So

$$\zeta \leq \tilde{\zeta} + \Phi. \tag{1.5.8}$$

Similarly, if we swap the roles of w and $\tilde{w}$, we get

$$\tilde{\zeta} \leq \zeta + \Phi, \tag{1.5.9}$$

at last,

$$\|Tw - T\tilde{w}\|_\infty \leq \frac{1}{\beta}\|F-\tilde{F}\|_\infty. \tag{1.5.10}$$

Thus the shrinkage properties of T result from the combination of (1.4.2) and (1.4.4). $\square$

1.5.1 The discrete non-linear second member problems

The discrete non-linear second member E.V.I is defined as follows

$$a\left(u_h, v-u_h\right) \geq \left(f(u_h), v-u_h\right), \quad \forall v \in \mathbb{K}_h, \tag{1.5.11}$$

her associated fixed point mappings is

$$T_h : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \quad w \to T_h w = \zeta_h, \tag{1.5.12}$$

where ζ_h is the unique solution of the following mandatory E.V.I

$$a\left(\zeta_h, v-\zeta_h\right) \geq \left(f(w), v-\zeta_h\right), \quad \forall v \in \mathbb{V}_h. \tag{1.5.13}$$

Similar to the continuous case, under the DMP it can be easily shown that T_h is a contraction of $L^\infty(\Omega)$ with a contraction constant of l/β.

The assumption of the discrete maximum principle (DMP): We assume that the matrix $\mathcal{B}$ is an M-matrix. ($\mathcal{B}^{-1}$) exists and is non-negative, with moreover $\mathcal{B}_{ss} > 0$ and $\mathcal{B}_{ls} \leq 0$ for $l \neq s$).

1.5.2 Discrete H.J.B equation of non-linear second member problem

The discrete problem is satisfying the maximum principle [26, 31].
In effect, the discrete version of the H.J.B is to search for $u_h \in V_h$ such that

$$\max_{1\leq i\leq M} (\mathcal{A}_{h,i}u_h - f_{h,i}(u_{h,i}), u_{h,i} - \Psi_{h,i}) = 0. \tag{1.5.14}$$

By analogy, in [31], Cortey-Dumont prove that the solution of (1.3.5) is a limit in $C(\overline{\Omega})$ of the solution of the system of discrete E.V.Is associated with the system
Find $u \in V$ such that

$$\begin{cases} a(u, v-u) \geq (f(u), v-u), & \forall v \in V, \\ u \leq \Psi \quad \text{and} \quad v \leq \Psi. \end{cases}$$

1.5.3 L^∞-error estimates for the non-linear second member problems

Adjusting [30], we get the following lemma:

Lemma 2. *The following inequality apply*

$$\|Tw - T_h w\|_\infty \leq Ch^2 |\log h|^2 \|f(w)\|_\infty. \tag{1.5.15}$$

Before proving the following theorem we need the following proposition which works by the fixed point mapping associated with the E.V.I (1.5.11).
We consider the mapping

$$T_h : L^\infty(\Omega) \longrightarrow \mathbb{V}_h, \quad w \longrightarrow T_h w = \zeta_h, \tag{1.5.16}$$

where $\zeta_h \in \mathbb{V}_h$ solves the following E.V.I

$$a(\zeta_h, v - \zeta_h) \geq (f(w), v - \zeta_h), \quad \forall v \in \mathbb{V}_h. \tag{1.5.17}$$

Proposition 4. *Under the DMP and assumptions (1.4.2) and (1.4.4), the mapping T_h is a contraction in $L^\infty(\Omega)$, that is*

$$\|T_h w - T_h \tilde{w}\|_\infty \leq \frac{l}{\beta}\|w - \tilde{w}\|_\infty. \tag{1.5.18}$$

Therefore, there exists a unique fixed point, consistent with the solution of E.V.I (1.5.11).

Proof. The same steps of **Proposition** 1. □

Then, there is the error estimate:

Theorem 8. *The following inequality apply*

$$\|u-u_h\|_\infty \leq \frac{Ch^2|\log h|^2}{1-(l/\beta)}\|f(u)\|_\infty. \tag{1.5.19}$$

Proof. It's easy to see that

$$\|u-u_h\|_\infty \leq \|u-T_h u\| + \|T_h u - u_h\|_\infty. \tag{1.5.20}$$

Also, by Propositions 3 and 5, u and u_h are the fixed points of T and T_h, respectively, namely

$$u = Tu, \quad u_h = T_h u. \tag{1.5.21}$$

If we apply the Lemma 2 with $w = u$, then

$$\begin{aligned} \|u-u_h\|_\infty &\leq \|Tu - T_h u\|_\infty + \|T_h u - T_h u_h\|_\infty \\ &\leq Ch^2|\log h|^2\|f(u)\|_\infty + \frac{l}{\beta}\|u-u_h\|_\infty, \end{aligned} \tag{1.5.22}$$

thus,

$$\|u-u_h\|_\infty \leq \frac{Ch^2|\log h|^2\|f(u)\|_\infty}{1-(l/\beta)}. \tag{1.5.23}$$

This is the desired error estimate. Before proving this theorem we need the following proposition which works by the fixed point mapping associated with the E.V.I (1.5.11). We consider the mapping

$$T_h : L^\infty(\Omega) \longrightarrow \mathbb{V}_h, \quad w \longrightarrow T_h w = \zeta_h, \tag{1.5.24}$$

where $\zeta_h \in \mathbb{V}_h$ solves the following E.V.I

$$a(\zeta_h, v-\zeta_h) \geq (f(w), v-\zeta_h), \quad \forall v \in \mathbb{V}_h. \tag{1.5.25}$$

Proposition 5. *Under the DMP and assumptions* (1.4.2) *and* (1.4.4)*, the mapping T_h is a contraction in $L^\infty(\Omega)$, that is*

$$\|T_h w - T_h \tilde{w}\|_\infty \leq \frac{l}{\beta}\|w-\tilde{w}\|_\infty. \tag{1.5.26}$$

Therefore, there exists a unique fixed point, consistent with the solution of E.V.I (1.5.11).

Proof. The same steps of **Proposition** 1. □

In the following section, we will see the variational inequalities with non-coercive operator and non-linear second member in the continuous, discrete case and the associated H.J.B equation:

1.6 Non-linear second member and non-coercive operator of E.V.I

We assume that (1.2.3) does not hold. In this situation It is well known (cf. [1, 2]) that the question of existence for the corresponding problem exists unconstrained problems can be handled as follows (cf. [1, 2]):
Exists $\lambda > 0$ large enough

$$a(v,v) + \lambda(v,v) \geq \delta\|v\|^2, \quad \delta > 0. \tag{1.6.1}$$

Let us also assume that the constants l and β defined in (1.2.2) and (1.4.4) are in (1.4.2). As a result of this, E.V.I (1.2.6) change into

$$b(u, v-u) \geq (f(u) + \lambda u, v-u), \quad \forall v \in \mathbb{V}, \tag{1.6.2}$$

where the non-linearity of f is assumed to be non-decreasing and Lipschitz continuous satisfing (1.4.4), where l is a positive constant, the new variational form $b(u,v) = a(u,v) + \lambda(u,v)$ obviously satisfies the strong coercivity assumption.
Now to approximate the continuous solutions, we proceed as follows front part. In fact, we construct the corresponding fixed-point maps

$$T_\lambda : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \quad w \longrightarrow T_\lambda w = \zeta_\lambda, \tag{1.6.3}$$

where ζ_λ is the solution of the coercive variational inequality below

$$b\left(\zeta_\lambda, v-\zeta_\lambda\right) \geq \left(f(w) + \lambda w, v-\zeta_\lambda\right), \quad \forall v \in \mathbb{K}. \tag{1.6.4}$$

Proposition 6. *The mappings T_λ is contraction in $L^\infty(\Omega)$. The contraction constant is equal to $(l+\lambda)/(\lambda+\beta)$. So they have unique fixed-point agree with the solution of the E.V.I (1.6.4).*

Proof. We drafted the proof for the E.V.I. Just set it up:

$$\zeta_\lambda = \sigma(F,\psi), \quad \tilde{\zeta}_\lambda = \sigma(\tilde{F},\psi), \quad \Phi = \frac{1}{\lambda+\beta}\|F-\tilde{F}\|_\infty, \tag{1.6.5}$$

where $F = f(w) + \lambda w, \quad \tilde{F} = f(\tilde{w}) + \lambda \tilde{w}$, it follows

$$\begin{aligned} F &\leq \tilde{F} + \|F-\tilde{F}\|_\infty \\ &\leq \tilde{F} + \frac{a_0(x)}{\lambda+\beta}\|F-\tilde{F}\|_\infty \\ &\leq \tilde{F} + a_0\Phi, \end{aligned} \tag{1.6.6}$$

(because $a_0(x) \geq \beta > 0$). Thus using standard comparison results in constrained variational inequalities, we get

$$\sigma(F,\psi) \leq \sigma\left(\tilde{F} + a_0(x)\cdot\Phi, \psi\right) \leq \sigma(\tilde{F}+\Phi,\psi). \tag{1.6.7}$$

So

$$\zeta_\lambda \leq \tilde{\zeta}_\lambda + \Phi. \tag{1.6.8}$$

Similarly, if we swap the roles of w and $\tilde{w}$, we get

$$\tilde{\zeta}_\lambda \leq \zeta_\lambda + \Phi. \tag{1.6.9}$$

At last,

$$\|T_\lambda w - T_\lambda \tilde{w}\|_\infty \leq \frac{1}{\lambda+\beta}\|F-\tilde{F}\|_\infty. \tag{1.6.10}$$

Thus the shrinkage properties of T result from the combination of (1.4.2) and (1.4.4). □

1.6.1 The non-linear second member and non-coercive operator discrete problems

The discrete non-coercive E.V.I is defined as follows

$$b\left(u_h, v-u_h\right) \geq \left(f(u_h) + \lambda u_h, v-u_h\right), \forall v \in \mathbb{K}_h, \tag{1.6.11}$$

her associated fixed point mappings is

$$T_{\lambda,h} : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \quad w \to T_{\lambda,h}w = \zeta_{\lambda,h}, \tag{1.6.12}$$

where $\zeta_{\lambda,h}$ is the only solution of the following mandatory E.V.I

$$b\left(\zeta_{\lambda,h}, v-\zeta_{\lambda,h}\right) \geq \left(f(w)+\lambda w, v-\zeta_{\lambda,h}\right), \quad \forall v \in \mathbb{V}_h. \tag{1.6.13}$$

Similar to the continuous case, under the DMP it can be easily shown that $T_{\lambda,h}$ is a contraction of $L^\infty(\Omega)$ with a contraction constant of: $(l+\lambda)/(\lambda+\beta)$.
The assumption of the discrete maximum principle (DMP): We assume that the matrix $\mathcal{B}$ is an M-matrix. ($\mathcal{B}^{-1}$) exists and is non-negative, with moreover $\mathcal{B}_{ss}>0$ and $\mathcal{B}_{ls} \leq 0$ for $l \neq s$).

1.6.2 Discrete H.J.B equation of non-linear second member and non-coercive operator problem

The discrete problem is satisfying the maximum principle [26, 31].
In effect, the discrete version of the H.J.B is to search for $u_h \in V_h$ such that:

$$\max_{1 \leq i \leq M} \left(\mathcal{B}_{h,i} u_h - f_{h,i}(u_{h,i}) - \lambda u_{h,i}, u_{h,i} - \Psi_{h,i}\right) = 0. \tag{1.6.14}$$

By analogy, in [31], Cortey-Dumont prove that the solution of (1.6.14) is a limit in $C(\overline{\Omega})$ of the solution of the system of discrete E.V.Is associated with the system
Find $u \in V$ such that

$$\begin{cases} b(u, v-u) \geq (f(u)-\lambda u, v-u), & \forall v \in V, \\ u \leq \Psi \quad \text{and} \quad v \leq \Psi. \end{cases}$$

1.6.3 L^∞-error estimates for the non-coercive problems

Adjusting [30] we get the following lemma:

Lemma 3. *The following inequality apply*

$$\|T_\lambda w - T_{\lambda,h} w\|_\infty \leq C h^2 |\log h|^2 \|f(w)+\lambda w\|_\infty. \tag{1.6.15}$$

Before proving the following theorem we need the following proposition which works by the fixed-point mapping associated with the E.V.I (1.6.11):
We consider the map

$$T_{\lambda,h} : L^\infty(\Omega) \longrightarrow \mathbb{V}_h, \quad w \longrightarrow T_{\lambda,h} w = \zeta_{\lambda,h}, \tag{1.6.16}$$

where $\zeta_{\lambda,h} \in \mathbb{V}_h$ solves the following E.V.I

$$a\left(\zeta_{\lambda,h}, v-\zeta_{\lambda,h}\right) \geq \left(f(w)+\lambda w, v-\zeta_{\lambda,h}\right), \quad \forall v \in \mathbb{V}_h. \tag{1.6.17}$$

Proposition 7. *Under the DMP and assumptions (1.4.2) and (1.4.4), the mapping T_h is a contraction for $L^\infty(\Omega)$ namely*

$$\|T_{\lambda,h}w - T_{\lambda,h}\tilde{w}\|_\infty \leq \frac{l+\lambda}{\lambda+\beta}\|w-\tilde{w}\|_\infty. \tag{1.6.18}$$

Therefore, there exists a unique fixed-point, consistent with the solution of E.V.I (1.6.11).

Proof. Same steps of **Proposition 1**. □

Then, there is the error estimate.

Theorem 9. *The following inequality apply*

$$\|u-u_h\|_\infty \leq \frac{Ch^2|\log h|^2}{1-(l+\lambda)/(\lambda+\beta)}\|f(u)+\lambda u\|_\infty. \tag{1.6.19}$$

Proof. It's easy to see

$$\|u-u_h\|_\infty \leq \|u-T_{\lambda,h}u\| + \|T_{\lambda,h}u-u_h\|_\infty. \tag{1.6.20}$$

Further, according to Propositions 1 and 2, u and u_h are the fixed-points of T_λ and $T_{\lambda,h}$ respectively, namely

$$u = T_\lambda u, \quad u_h = T_{\lambda,h}u. \tag{1.6.21}$$

If we apply the Lemma 1 with $w=u$, then

$$\begin{aligned} \|u-u_h\|_\infty &\leq \|T_\lambda u - T_{\lambda,h}u\|_\infty + \|T_{\lambda,h}u - T_{\lambda,h}u_h\|_\infty, \\ &\leq Ch^2|\log h|^2\|f(u)+\lambda u\|_\infty + \frac{l+\lambda}{\lambda+\beta}\|u-u_h\|_\infty, \end{aligned} \tag{1.6.22}$$

thus,

$$\|u-u_h\|_\infty \leq \frac{Ch^2|\log h|^2\|f(u)+\lambda u\|_\infty}{1-(l+\lambda)/(\lambda+\beta)}, \tag{1.6.23}$$

which is the desired error estimate. □

Corollary 1. *If the right-hand side is independent of u, the problem (1.4.3) reduces to the well-known stochastically governed linear unconstrained variational inequality [32]. In this*

situation the approximation convergence orders (1.4.16) *transforms into:*
For the E.V.I of stochastic control (cf. [32])

$$\|u-u_h\|_\infty \leq \frac{Ch^2|\log h|^2}{1-\lambda/(\lambda+\beta)}\|u\|_\infty. \tag{1.6.24}$$

In the following section, we will see the variational inequalities with non-linear operator and second member in the continuous, discrete case and the associated H.J.B equation.

1.7 Non-linear E.V.I (operator and second member)

Let Ω be an open set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$. For $u, v \in V$, the space $H_0^1(\Omega)$ or $H^1(\Omega)$, we set the non-linear operator

$$\mathscr{A}u = -\sum_{1\leq i,j\leq n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+F(u), \tag{1.7.1}$$

with the coefficient $a_{ij}(x)$ are sufficiently regular and

$$F(.) \in \mathcal{C}^2(\mathbb{R}). \tag{1.7.2}$$

In the following, C will be a positive constant involved in the calculations and independent of the discrete parameter h.
Let $\mathscr{A}: V \to V'$ be the non-linear operator verify the following proprieties:

1. The operator $\mathscr{A}$ is strongly monotone and more precisely there exists a function strictly monotone i.e

$$\chi : [0,\infty[\to \mathbb{R} \text{ such that} \chi(0)=0 \quad \text{and} \quad \lim_{t\to\infty}\chi(t)=+\infty. \tag{1.7.3}$$

   Such that for every $u, v \in V$

$$\langle \mathscr{A}u-\mathscr{A}v, u-v\rangle \geq \chi(\|u-v\|)\|u-v\|. \tag{1.7.4}$$

2. $\mathscr{A}$ is strictly T-monotone that is for every $u, v \in V (u-v)^+ \neq 0$

$$\left\langle \mathscr{A}u-\mathscr{A}v, (u-v)^+\right\rangle \geq 0.$$

3. $\mathscr{A}$ is Lipschitz continuous for a bounded argument, for any ball

$$B(0,r) = \{v \in V \quad \text{such that} \quad \|v\| \leq r\},$$

   we have for every $u, v \in B(0,r)$

$$\|\mathscr{A}u - \mathscr{A}v\|_{V'} \leq \gamma(r)\|u - v\|.$$

4. If $a(u,v)$ is a variational form associated with $\mathscr{A}$, for every positive constant δ, we have

$$\forall u, v \in V, v \geq 0 \quad a(u,v) \leq a(u+\delta, v).$$

5. Let $f(u) \in L^\infty(\Omega)$ is l-Lipschitzian such that

$$f \geq 0 \quad \text{and} \quad \frac{l}{\beta} < 1.$$

   which will be the right hand side.

6. Let Ψ be an obstacle. In the following, we will take Ψ in $W^{1,\infty}$ such that for every $x_0 \in \Omega$ there exists a function ρ such that $\Psi(x_0) = \rho(x_0)$, $\|\rho\|_{W^{2,\infty}} \leq C$ and for every x in $B(x_0, Ch)$ $\Psi(x) \leq \rho$. For example if $\Psi \in W^{2,\infty}$ we can take $\rho = \Psi_{|B(x_0,Ch)}$.

1.7.1 The Continuous Problem

Let us now consider an "obstacle problem" for the non-linear operator i.e. the following variational problem.
Find u the solution of

$$\begin{cases} a(u, v-u) \geq (f(u), v-u), & \forall v \in K, \\ u \in K, \end{cases} \tag{1.7.5}$$

where $K = \{v \in V, v \leq \Psi \text{ in } \Omega\}$ is a closed convex subset of V.
The regularity of u is not simple. For the sake of generality we will introduce the hypothesis of regularity in an indirect manner (see [7]).
The discretisation of this problem is the same of the linear part (section 1.3).
We will associate with $a(.,.)$ an approximation of the variational form noted $a_h(.,.)$. In this way, we can consider techniques of numerical integration.

1.7.2 The Discrete Problem

Let us now consider the discrete problem associated with (1.7.5). Find $u_h \in V_h$ the solution of

$$\begin{cases} a_h\left(u_h, v_h - u_h\right) \geq \left(f(u_h), v_h - u_h\right), \quad \forall v_h \in K_h, \\ u_h \in K_h, \end{cases} \tag{1.7.6}$$

where $K_h = \{v_h \in V_h \text{ such that } v_h \leq r_h \Psi\}$.
With such a definition, the constraint for the discrete function is imposed only on the nodes of the regular triangulation.
Let $\mathscr{A}$ be the application of $\mathbb{R}^{m(h)}$ belongs to $\mathbb{R}^{m(h)}$ such that

$$\mathscr{A}_k(u) = \left(\mathscr{A}_1(u), \cdots, \mathscr{A}_{m(h)}(u)\right),$$

where

$$\mathscr{A}_k(u) = a_h\left(u_h, \varphi_k\right) \quad \forall k \in (1, \cdots, m(h)).$$

The **DMP** is needed the following assumption:

1. $\mathscr{A}_k$ is a continuous surjective M-function.

2. $\mathscr{A}_k(u) \leq \mathscr{A}_k(u+\gamma)$ where $\gamma^T = (\delta, \cdots, \delta)$ is a constant vector of $\mathbb{R}^{m(h)}$ with $\delta > 0$ arbitrary.

For completness we will recall the definition of an M-function.
Let us note e_l the l^{th}-vector of the coordinates $\left(e_l^T = (0, \cdots, 0, 1, 0, \cdots, 0)\right)$ and Θ_{kl} the following function

$$\Theta_{kl}(t) = \mathscr{A}_k\left(v + te_l\right) \quad \text{for every } t \text{ in } \mathbb{R}.$$

Definition 1. *The application $\mathscr{A}_k$ is a Z-function if for all $k \neq l, t \to \theta_{kl}(t)$ decreases. If moreover $\mathscr{A}_k(u) \leq \mathscr{A}_k(v)$ implies $u \leq v$. $\mathscr{A}_k$ is called an M-function.*

Theorem 10. *[34] Let $\mathscr{A}_k$ is a continuous surjective M-function, the existence and uniqueness of the solution of the discrete problem (1.7.6) is well-known.*

1.7.3 Discrete H.J.B equation of non-linear E.V.I

The discrete problem is satisfying the DMP.
In effect, the discrete version of the H.J.B is to search for $u_h \in V_h$ such that

$$\max_{1 \leq i \leq M} \left(\mathscr{A}_{h,i} u_h - f_{h,i}(u_{h,i}), u_{h,i} - \Psi_{h,i}\right) = 0. \tag{1.7.7}$$

1.7.4 L^∞-error estimates for the non-linear problems

As we have said above, we will consider the regularity implicitely as follows:
Let $\bar{u}_h$ be the solution of $a_h(\bar{u}_h, v_h) = a(u, v_h)$ where u is the solution of the continuous variational inequality.
We suppose

$$\begin{cases} \|u - \bar{u}_h\|_\infty \leq Ch^2 \| \log h|^2 , \\ \|u - r_h u\|_\infty \leq Ch^2 \| \log h|^2 . \end{cases} \tag{1.7.8}$$

Assumption related to the obstacle Ψ

$$\forall x \in B(x_0, Ch) \text{ such that } u(x_0) = \Psi(x_0) \text{ then } |u(x) - \rho(x)| \leq Ch^2 |\log h|^2. \tag{1.7.9}$$

The principal result will be to show the following:

Theorem 11. *(cf.[34]) Under the proprieties (1) to (6), the assumptions of **DMP** (1 and 2) and the hypothesis* (1.7.8),(1.7.9)*, we have the following estimate*

$$\|u - u_h\|_\infty \leq Ch^2 |\log h|^2. \tag{1.7.10}$$

Chapter 2

Representation of the Multi-grid Method for E.V.I

We present in this chapter the principle of multi-grid methods based on of the method developed later in the thesis, in order to reduce the computation time of the shaping simulations and to obtain a convergence more favorable to the resolution of large linear problems.

We perform here a description of the multi-grid methods. First of all we start with the iterative Gauss-Seidel method, then the transport operators (prolongation and restriction), then the two-grid and multi-grid methods, finally the convergence of this method.

2.1 Gauss-Seidel method (iterative method)

The Gauss-Seidel method is an iterative method for solving linear systems $Ax = f$, where A is a square matrix of order N and x, f are vectors of $\mathbb{R}^N$. It consists of the following manipulation:
We write A in the form $A = D - L - U$, where D is a diagonal matrix, $-L$ is a lower triangular matrix (L for Lower), and $-U$ is an upper triangular matrix (U for Upper). We can then transform the system into

$$Ax = f \Longleftrightarrow (D-L)x - Ux = f \Longleftrightarrow x = (D-L)^{-1}Ux + (D-L)^{-1}f. \tag{2.1.1}$$

We then define a sequence of vectors (x^k) by choosing a vector $x_0 \in \mathbb{R}^N$ and by the recurrence formula

$$x^{k+1} = (D-L)^{-1}Ux^k + (D-L)^{-1}f. \tag{2.1.2}$$

We hope that the sequence $\left(x^k\right)_k$ converges to a solution of $Ax = f$. Under good assumptions about the matrix A, this is indeed the case.

Theorem 12. *If A is a positive definite symmetric matrix, or if A is a strictly diagonal dominant matrix, then for any choice of $x^0 \in \mathbb{R}^N$, the sequence $\left(x^k\right)$ converges to the unique solution of $Ax = f$.*

This method calls for several comments. The first is that we can ask ourselves why we need a method which gives an approximate solution to an equation, when we have an algorithm (the Gaussian pivot) which gives in a reasonable time (from I order of N^3 operations) an exact solution. It turns out that the Gaussian pivot is numerically unstable. The calculation errors of the computer accumulate and make that the solution which one calculates is sometimes very far from the exact solution. The convergence of the Gauss-Seidel method is based on the fixed point theorem, and this method is more numerically stable.
Then, we must check that the sequence $\left(x^k\right)$ is easily computable. But since $(D-L)$ is a lower triangular matrix, it is not at all difficult to invert, and we have the following formula to calculate x^{k+1}

$$x_i^{k+1} = \frac{1}{a_{i,i}} \left(f_i - \sum_{j=1}^{i-1} a_{i,j} x_j^{k+1} - \sum_{j=i+1}^{n} a_{i,j} x_j^k \right). \tag{2.1.3}$$

This formula shows that the Gauss-Seidel method is an improvement of the Jacobi method. In the Jacobi method, the recurrence relation is

$$x_i^{k+1} = \frac{1}{a_{i,i}} \left(f_i - \sum_{j=1}^{i-1} a_{i,j} x_j^{k} - \sum_{j=i+1}^{n} a_{i,j} x_j^k \right). \tag{2.1.4}$$

The advantage of the Gauss-Seidel method is that, to calculate x_i^{k+1}, we use the already calculated values of x_j^{k+1}, for $j < i$, at instead of x_j^k as in the Jacobi method. The Gauss-Seidel method is a special case of relaxation methods.
In the next section, we will see the principal and transfer operators of the multi-grid method. They are the operator of prolongation and operator of restriction.

2.2 Nested meshes, transfer operators

One considers two uniform discretizations of the interval of study $\mathcal{M}_h$ mesh smooth, and $\mathcal{M}_{2h}$ coarse mesh. It is assumed that these meshes are "nested", i.e. say that the coarse mesh nodes are also smooth mesh nodes. For fix the ideas, and although this is not a necessary condition, it is assumed moreover, the coarse mesh is half as dense as the fine mesh.

2.2.1 Prolongation operator

Definition 2. *We call extension or prolongation any interpolation operator on the fine (smooth) mesh of functions known by their discretizations on the coarse mesh.*

Remark 1. *By using the term interpolation operator, we implicitly assume that it satisfies a condition of consistency and even precision, that we will endeavor to verify in specific cases.*

Exercise 1.

$$\mathcal{P} = I_{2h}^{h} : \mathcal{M}_{2h} \longrightarrow \mathcal{M}_h, \tag{2.2.1}$$

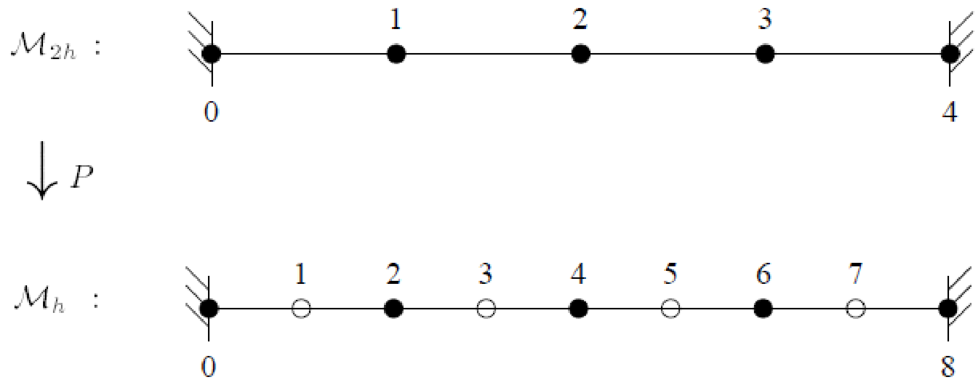

Figure 2.1: The principle of prolongation operator.

Linear interpolation

$$u_{2h} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow u_h = \begin{pmatrix} u'_1 = (u_0 + u_1)/2 \\ u'_2 = u_1 \\ u'_3 = (u_1 + u_2)/2 \\ u'_4 = u_2 \\ u'_5 = (u_2 + u_3)/2 \\ u'_6 = u_3 \\ u'_7 = (u_3 + u_4)/2 \end{pmatrix} \tag{2.2.2}$$

where the boundary values, in the example u_0 and u_4, are not degrees of freedom and do not intervene in the identification of the operator of prolongation:

$$\mathcal{P} = \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{2.2.3}$$

2.2.2 Restriction operator

Definition 3. *We call restriction any interpolation operator on the coarse mesh of functions known by their discretizations on the fine mesh.*

Remark 2. *Again, the notions of consistency and precision are implicit.*

Exercise 2.

$$\mathcal{R} = I_h^{2h} : \mathcal{M}_h \longrightarrow \mathcal{M}_{2h}, \tag{2.2.4}$$

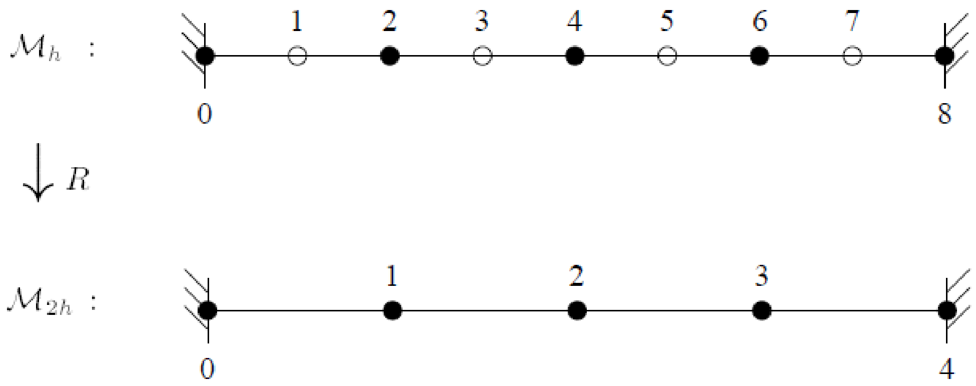

Figure 2.2: The restriction operator principle.

1. ***Injection***

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.2.5}$$

In the case of the fundamental discrete model, it is noted that by ńinjectionż eigenvectors associated with the fine discretization, one obtains the eigenvectors associated with the coarse discretization. In this sense, the ńinjection respects the proper modesż.

2. *The following restriction operator is preferable:*

$$\mathcal{R} = \frac{1}{2}\mathcal{P}^T = \begin{pmatrix} 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \end{pmatrix} \tag{2.2.6}$$

in the following section, we will show the two grid method which use only two level:

2.3 Two-grid method

The bi-grid (two-grid) method uses only two levels of calculation, i.e. it has only one only coarse grid in addition to the starting grid. We use the index h to designate the fine level and $2h$ for the coarse level.
Consider the linear system to be solved

$$\mathcal{A}_h X_h = f_h.$$

The different stages of a bi-grid cycle are as follows:

1. **Pre-smoothing**: We perform νiterations (called smoothing) of an iterative method (called smoothing) L in order to obtain a first approximation $\tilde{X}_h$ of the solution

$$\tilde{X}_h = L\left(\mathcal{A}_h, f_h, X_h, \nu\right). \tag{2.3.1}$$

2. **Calculation of the residue**:

$$\mathcal{R}_h = f_h - \mathcal{A}\tilde{X}_h. \tag{2.3.2}$$

3. **Residual restriction**: The residual is projected on the coarse grid thanks to the restriction operator R

$$\mathcal{R}_{2h} = R\mathcal{R}_h. \tag{2.3.3}$$

4. **Calculation of the correction**: The correction on the coarse grid e_2h is calculated by solving the following linear system

$$\mathcal{A}_{2h}\mathcal{E}_{2h} = \mathcal{R}_{2h}, \tag{2.3.4}$$

thanks to a resolution method Φ, where $\mathcal{A}_{2h}$ is the matrix of the linear system projected on the coarse grid

$$\mathcal{E}_{2h} = \Phi\left(\mathcal{A}_{2h}, \mathcal{R}_{2h}\right). \tag{2.3.5}$$

5. **Extension of the correction**: The correction $\mathcal{E}_{2h}$ is interpolated on the fine grid thanks to the extension operator P

$$\mathcal{E}_h = P\mathcal{E}_{2h}. \tag{2.3.6}$$

6. **Application of the correction**: The approximate solution $\tilde{X}_h$ is updated by adding the correction coming from the coarse grid

$$\tilde{X}_h \leftarrow \tilde{X}_h + \mathcal{E}_h. \tag{2.3.7}$$

7. **Post-smoothing**: We perform ν' smoothing with a smoother L' to eliminate high frequencies from the correction. Generally, $L = L'$

$$\tilde{\tilde{X}}_h = L'\left(\mathcal{A}_h, f_h, \tilde{X}_h, \nu'\right). \tag{2.3.8}$$

In the following figure, we determine the steps of the method of a one cycle (two-grid) by a diagram:

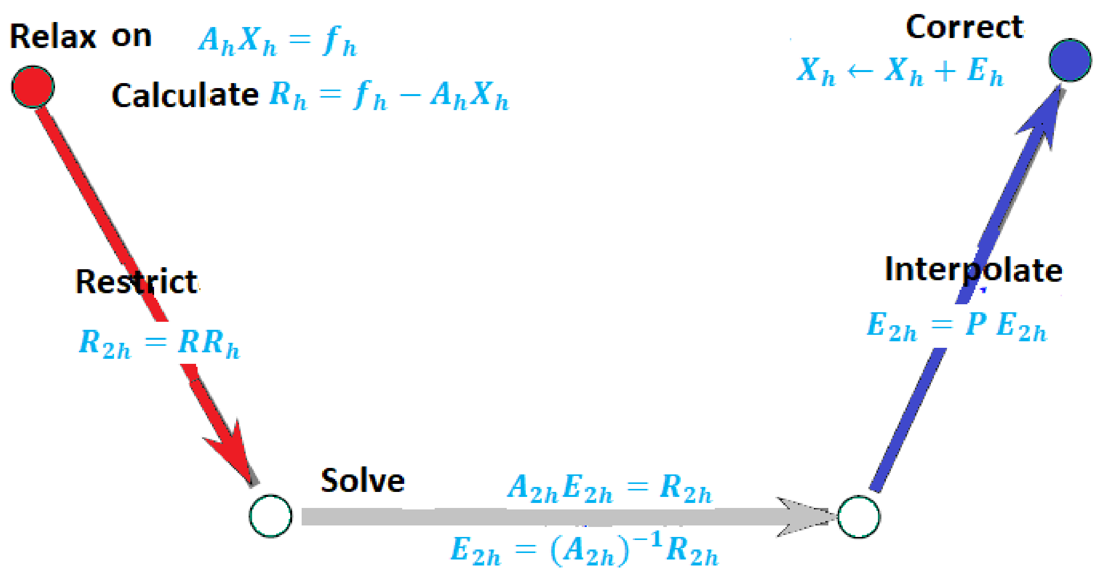

Figure 2.3: Diagram of two-grid method.

2.4 Multi-grid method

The generalization of the bi-grid method leads to multi-grid methods, in which we consider more levels of calculation. The correction is then only calculated on the coarsest level, while the pre-smoothing and post-smoothing steps are performed on the fine level and on the intermediate levels. The order in which the different grids are traversed describes the multi-grid cycle, some of which are schematized in Figure below.

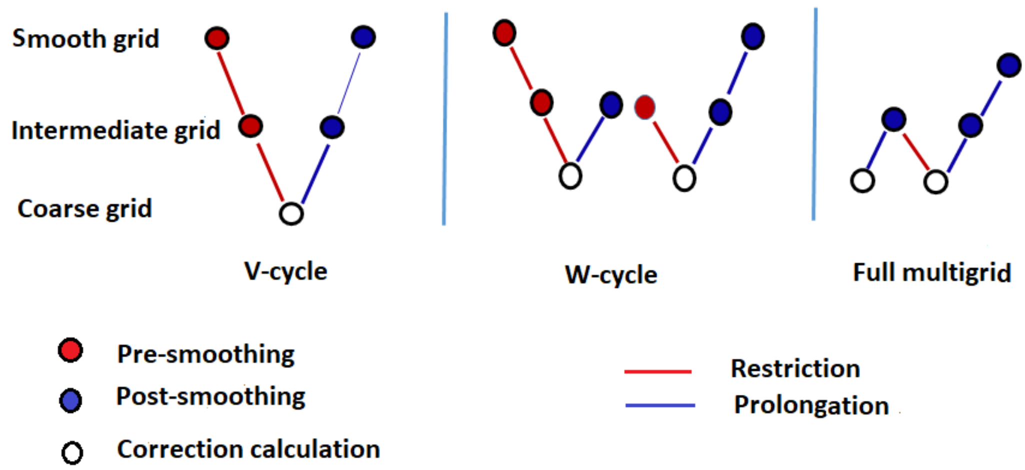

Figure 2.4: Diagrams of V-cycle, W-cycle and Full Multi-grid.

The V-cycle consists in replacing the operator by another bi-grid cycle as adding a coarser level. The transition from a bi-grid method to a V-cycle of 3 grids is schematized in Figure below. This cycle, the most commonly used in the literature.

The W-cycle consists in carrying out several operations of the big-grid method on the grids coarser before extending the correction on the fine grid. By multiplying the operations on coarse, normally inexpensive grids, a better correction is calculated and slightly improves convergence (Strang, 2007). Little used in the literature, because less natural than the V-cycle when going from 2 to more grids, the W-cycle is numerically more expensive and you have to count on a significant reduction in the number of cycles to become justified.

The Full Multi-grid method consists of initializing the multi-grid cycle on the coarsest grid and to go through the coarser levels before reaching the smoother grid.
Once the type of cycle has been chosen, the multi-grid method is defined from the parameters following resolutions:

- **The operators:** The prolongation is a linear operator, between two-dimensional

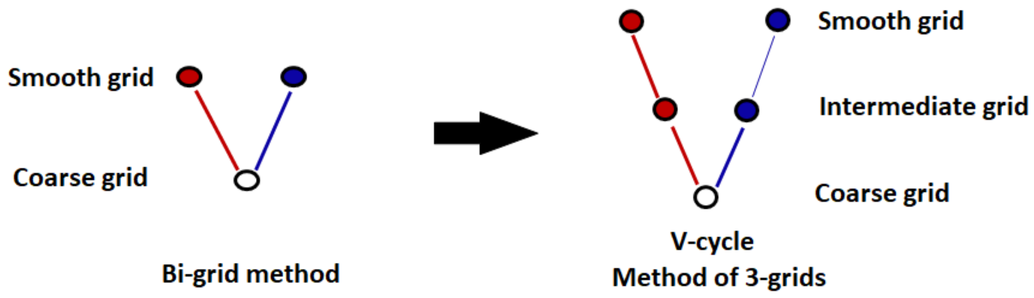

Figure 2.5: Switching from a 2-grid method to a 3-grid V-cycle.

spaces. Hence, it can be represented as a matrix. Using the standard basis of $\mathbb{R}^{N/2-1}$ and $\mathbb{R}^{N-1}$, then

$$\mathcal{P} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ 2 & & & \\ 1 & 1 & & \\ & 2 & & \\ & 1 & 1 & \\ & & 2 & \ddots & \\ & & 1 & \ddots & \\ & & & & 1 \\ & & & & 2 \\ & & & & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N/2-1)} \tag{2.4.1}$$

and if the spaces $\mathbb{R}^{N-1}$ and $\mathbb{R}^{N/2-1}$ are equipped with the standard bases, the matrix representation of the weighted restriction operator has the form

$$\mathcal{R} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{(N/2-1)\times(N-1)}. \tag{2.4.2}$$

With this representation, one can see an important connection between weighted restriction $\mathcal{R}$ and interpolation $\mathcal{P}$

$$\mathcal{R} = 2\left(\mathcal{P}\right)^T. \tag{2.4.3}$$

- The number of smoothing iterations ν.

- The resolution method (and its possible parameters) for the resolution of the system from the coarsest grid.

2.5 The Convergence of the Multi-grid Method in the L^∞ norm

2.5.1 Approximation property

Theorem 13. *The matrix* $\Upsilon_k = [J\mathcal{A}_k [u_k^*] u_k^*]^{-1} - p\left[J\mathcal{A}_{k-1} [u_k^*] r u_k^*]^{-1} r\right]$ *has the approximation property*

$$\|\Upsilon_k\|_\infty \leq Ch_k^2 \left|\ln h_k\right|^2. \tag{2.5.1}$$

Proof. According to the previous theorem (convergence of H.J.B equation), we have

$$\|u - u_k\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2 \|f_k\|_\infty \text{ and } \quad u \in W^{2,p}$$

whence

$$\|u - u_k^*\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2 \|g_k\|_\infty$$

then

$$\left\|u_k^* - u_{k-1}^*\right\|_\infty \leq Ch_k^2 \left|\log h_k\right| \|g_k\|_\infty.$$

Pose:

$$L_k = J\mathcal{A}_k [u_k^*] u_k^*$$

$$L_{k-1} = J\mathcal{A}_{k-1} [u_k^*] r u_k^*$$

then, there exists a second member g such that

$$b\left(r_k L_k^{-1} g, v\right) = \left((r_k^*)^{-1} g, v\right), \forall v \in L^2$$

and

$$b\left(r_{k-1} L_{k-1}^{-1} r g, v\right) = \left((r_k^*)^{-1} g, v\right), \forall v \in L^2$$

whence we have

$$\left\|r_k L_k^{-1} g - r_{k-1} L_{k-1}^{-1} r g\right\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2 \|g\|_\infty$$

$$\left\|\left(L_k^{-1} - r_k^{-1} r_{k-1} L_{k-1}^{-1} r\right) g\right\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2 \|g\|_\infty$$

$$\left\|L_k^{-1} - r_k^{-1} r_{k-1} L_{k-1}^{-1} r\right\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2$$

whence

$$\left\| [J\mathcal{A}_k\,[u_k^*]\,u_k^*]^{-1} - p\,[J\mathcal{A}_{k-1}\,[u_k^*]\,ru_k^*]^{-1}\,r \right\|_\infty \leq Ch_k^2\,|\log h_k|^2\,.$$

□

2.5.2 Smoothing Property

We introduce the basic lemma (cf. [42])

Lemma 4. *Let A be a matrix such that $\|A\|_\infty < 1$ then, we have*

$$\|(I-A)(I+A)^q\|_\infty \leq 2^{q+1}\sqrt{\frac{2}{\pi q}}, \quad q \geq 1. \tag{2.5.2}$$

Proof.

$$(I-A)(I+A)^q = (I-A)\sum_{k=1}^{q}\left(\begin{array}{c} q \\ k \end{array}\right)A^k = I - A^{q+1} + \sum_{k=1}^{q}\left(\left(\begin{array}{c} q \\ k \end{array}\right) - \left(\begin{array}{c} q \\ k-1 \end{array}\right)\right)A^k.$$

Whence

$$\|(I-A)(I+A)^q\|_\infty \leq 2 + \sum_{k=1}^{q}\left|\left(\begin{array}{c} q \\ k \end{array}\right) - \left(\begin{array}{c} q \\ k-1 \end{array}\right)\right|,$$

$$\left(\begin{array}{c} q \\ k \end{array}\right) \geq \left(\begin{array}{c} q \\ k-1 \end{array}\right) \Leftrightarrow k \leq \frac{1}{2}(q+1), \quad \text{and} \left(\begin{array}{c} q \\ k \end{array}\right) = \left(\begin{array}{c} q \\ q-k \end{array}\right).$$

We have

$$\sum_{k=1}^{q}\left|\left(\begin{array}{c} q \\ k \end{array}\right) - \left(\begin{array}{c} q \\ k-1 \end{array}\right)\right| = \sum_{k=1}^{\left[\frac{(q+1)}{2}\right]}\left(\left(\begin{array}{c} q \\ k \end{array}\right) - \left(\begin{array}{c} q \\ k-1 \end{array}\right)\right) + \sum_{\left[\frac{(q+1)}{2}\right]}^{q}\left(\left(\begin{array}{c} q \\ k-1 \end{array}\right) - \left(\begin{array}{c} q \\ k \end{array}\right)\right),$$

$$\begin{aligned}\sum_{k=1}^{q}\left|\left(\begin{array}{c} q \\ k \end{array}\right) - \left(\begin{array}{c} q \\ k-1 \end{array}\right)\right| &= \sum_{k=1}^{\left[\frac{q}{2}\right]}\left(\left(\begin{array}{c} q \\ k \end{array}\right) - \left(\begin{array}{c} q \\ k-1 \end{array}\right)\right) + \sum_{m=1}^{\left[\begin{array}{c} q \\ 2 \end{array}\right]}\left(\left(\begin{array}{c} q \\ k-1 \end{array}\right) - \left(\begin{array}{c} q \\ k \end{array}\right)\right), \\ &= 2\sum_{k=1}^{\left[\frac{q}{2}\right]}\left(\left(\begin{array}{c} q \\ k \end{array}\right) - \left(\begin{array}{c} q-1 \\ k-1 \end{array}\right)\right) = 2\left(\left(\begin{array}{c} q \\ \left[\frac{q}{2}\right] \end{array}\right) - \left(\begin{array}{c} q \\ 0 \end{array}\right)\right),\end{aligned}$$

as:

$$\begin{pmatrix} q \\ \left[\frac{q}{2}\right] \end{pmatrix} \leq 2^q \sqrt{\frac{2}{\pi q}} \text{ for all} q \geq 1,$$

for more detail see [42]. So

$$\|(I-A)(I+A)^q\|_\infty \leq 2 \begin{pmatrix} q \\ \left[\frac{q}{2}\right] \end{pmatrix} \leq 2^q \sqrt{\frac{2}{\pi q}} \quad \text{for all } q \geq 1.$$

□

As a smoother, we use a relaxation method with an iterative matrix S_k. For the following theorem, Arnold Reusken applied that for the equations and it work for our work because it is related uniquely by the operator.

Theorem 14. *Assuming the previous assumptions and notations are satisfied, there exists a constant C independent of k and α such that the following smoothness properties hold*

$$\|(\mathcal{B}_k^\nu)\, S_k^\alpha\|_\infty \leq C \frac{1}{\sqrt{\alpha}} h_k^{-2}. \tag{2.5.3}$$

Proof. We pose $L_k = J\mathcal{A}_k\,[u_k^*]\,u_k^* = M_k - N_k$ and we consider the relaxation method $\left(\varpi = \frac{1}{2}\right)$ with the iteration matrix

$$S_k = I - \frac{1}{2} M_k^{-1} L_k.$$

As the $\mathcal{A}_k$ are M-matrices we can verify that $\left\|M_k^{-1}N_k\right\|_\infty < 1$ and $\|M_k\|_\infty \leq C h_k^{-2}$. Then

$$\|[J\mathcal{A}_k\,[u_k^*]\,u_k^*]\,[JS_k\,[u_k^*]\,ru_k^*]^p\|_\infty = \|L_k \left(I - \frac{1}{2} M_k^{-1} L_k\right)^p \|_\infty.$$

Pose

$$A = M_k^{-1} N_k = I - M_k^{-1} L_k,$$

we have

$\|[J\mathcal{A}_k\,[u_k^*]\,u_k^*]\,[JS_k\,[u_k^*]\,ru_k^*]^p\|_\infty = \left\|M_k(I-A)\left(\frac{1}{2}\right)^p (I+A)^p\right\|_\infty \leq \|M_k\|_\infty \left(\frac{1}{2}\right)^p 2^{p+1} \sqrt{\frac{2}{2p}}.$
Then

$$\|[J\mathcal{A}_k\,[u_k^*]\,u_k^*]\,[JS_k\,[u_k^*]\,ru_k^*]^p\|_\infty \quad \leq C \frac{1}{\sqrt{p}} h_k^{-2}.$$

□

2.5.3 L^∞-error estimates for the multi-grid method

We obtain the result by using the norm in (2.3.7) and considering (2.5.1) and (2.5.3).

Theorem 15. *Under the previous assumptions and symbols, the iteration $u_k^\nu, \nu \geq 0$ of the two networks k and $k-1$ satisfies*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_\infty \leq \left(\frac{C}{\sqrt{\alpha}}\left|\mathrm{Logh}_k\right|^2\right)\|u_k^\nu - u_k^*\|_\infty .. \tag{3.6}$$

Proof. We have

$$\begin{aligned}
\left\|u_k^{\nu+1} - u_k^*\right\|_\infty = & \quad \left\|\left(\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{B}_k^\nu\right)S_k^{\alpha_1}\right)\left(u_k^\nu - u_k^*\right)\right\|_\infty \\
\leq & \quad \left\|\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\right\|_\infty \|\mathcal{B}_k^\nu S_k^{\alpha_1}\|_\infty \|u_k^\nu - u_k^*\|_\infty \\
\leq & \quad \left(\frac{C_2}{\sqrt{\alpha}}h_k^{-2}\right)\left(C_1 h_k^2\left|\log h_k\right|^2\right)\|(u_k^\nu - u_k^*)\|_\infty \\
\leq & \quad \left(\frac{C_1 C_2}{\sqrt{\alpha}}\right)\left|\log h_k\right|^2\|u_k^\nu - u_k^*\|_\infty
\end{aligned}$$

□

Chapter 3

Approximation of the multi-grid method for linear variational inequalities

In this chapter, we study the numerical solution of elliptic problems by a multi-grid method. We prove the uniform convergence of the multi-grid algorithm using the elementary sub-differential calculus and ideas from the convergence theory of non-linear multi-grid methods.

3.1 Approximation of the multi-grid method for non-coercive variational inequality

Let us give $u_k^{(\nu)} \in \mathbb{R}^{n_k}, \nu \geq 0$ and calculate $u_k^{(\nu+1)} \in \mathbb{R}^{n_k}$ as being the solution of equation

$$\mathcal{B}_k^{\nu} u_k^{\nu+1} - Z_k^{\nu} = 0, \tag{3.1.1}$$

such that

$$Z_k^{\nu} = F_k^{\nu} + \lambda u_k^{\nu}$$

where

$$\mathcal{B}_{k,i}^{\nu} = \begin{cases} \mathcal{B}_{k,i}\left(u_k\right) & \text{if} \quad \mathcal{B}_{k,i} u_{k,i}^{\nu} - Z_{k,i} > u_{k,i}^{\nu} - \psi_{k,i}, \\ u_{k,i} & \text{if} \quad 1 \leq i \leq N, \end{cases} \tag{3.1.2}$$

$$Z_{k,i}^{\nu} = \begin{cases} Z_{k,i} & \text{if} \quad \mathcal{B}_{k,i} u_{k,i}^{\nu} - Z_{k,i} > u_{k,i}^{\nu} - \psi_{k,i}, \\ u_{k,i} & \text{if} \quad 1 \leq i \leq N. \end{cases} \tag{3.1.3}$$

Let u_k^* be the unique solution of the discrete H.J.B equation (1.4.15)

$$\max_{1\leq i\leq N}\left(\mathcal{B}_{k,i}u_k^* - Z_{k,i}, u_{k,i}^* - \psi_{k,i}\right) = 0. \tag{3.1.4}$$

3.1.1 Multi-grid (M.G.H.J.B) algorithm for V.Is.

If we choose iterative $u_k^\nu, \nu > 0$ for the multi-grid method, we obtain $\bar{u}_k^\nu$ by applying the iterative method to the system (3.1.1) α, expressed as

$$\bar{u}_k^\nu = S_k^\alpha\left(u_k^\nu\right) \tag{3.1.5}$$

where S_k is the iteration or smoothing operator and α is the number of iterations performed.

We denote the solution of (3.1.1) as u_k^*. Error setting $e_k^\nu = \bar{u}_k^\nu - u_k^*$ and residual $d_k^{(\nu)} = Z_k^\nu - \mathcal{B}_k^\nu\bar{u}_k^\nu$, we can write the equation (3.1.1) as

$$\mathcal{B}_k^\nu\left(\bar{u}_k^\nu + e_k^\nu\right) = Z_k^\nu.$$

This leads to the residual equation

$$\mathcal{B}_k^\nu e_k^\nu = Z_k^\nu - \mathcal{B}_k^\nu\bar{u}_k^\nu = d_k^{(\nu)}.$$

On the fine grid, after the relaxation on $\mathcal{B}_k^\nu\bar{u}_k^\nu = Z_k^\nu$ the error will be smooth, while on the coarse grid this error seems to be more oscillatory, and the relaxation will be so efficient. Therefore, to fully determine e_k^ν, we must compute e_{k-1}^ν at the level of $(k-1)$ as the solution for the coarse grid system

$$\mathcal{B}_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{3.1.6}$$

We can explain $e_{k-1}^\nu\left(\text{resp}\,\mathcal{B}_{k-1}^\nu, d_{k-1}^{(\nu)}\right)$ as an approximate operator of $e_k^\nu\left(\text{resp}\,\mathcal{B}_k^\nu, d_k^{(\nu)}\right)$ on $k-1$ layer $\mathcal{R}_k$ and its inverse $\mathcal{P}_k$.

Therefore, we determine k-level improvement iterations

$$u_k^{\nu+1} = \bar{u}_k^\nu + \mathcal{P}_k\left(e_{k-1}^\nu\right). \tag{3.1.7}$$

Due to nesting, we use the well-defined identity operator

$$\begin{aligned}\pi\quad &: \mathbb{V}_{k-1} \longrightarrow \mathbb{V}_k,\\ \pi v\quad &= v.\end{aligned}$$

Define the prolongation and restriction operators, i.e.

$$\mathcal{P}_k = r_k^{-1} r_{k-1}, \quad \mathcal{R}_k = \mathcal{P}_k^t. \tag{3.1.8}$$

Remark 3. *The previous algorithm describes a loop of two grid iterations to solve (3.1.1) for two grid levels Ω_{k-1}. Clearly, the coarse mesh system (3.1.6) has the same shape as the system (3.1.1). Therefore, we can approximate the system (3.1.6) by recursively performing two-grid iterations on all grid layers $\{\Omega_k, k = 0, \ldots, m_k\}$ untie.*

3.1.2 Matrix of the M.G.H.J.B Algorithm

The iteration matrix for the two-grid method with α_1 pre-smoothing and α_2 post-smoothing iterations at the k level is given by

$$TG_k(\alpha_1, \alpha_2) = S_k^{\alpha_2}\left((\mathcal{B}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{B}_{k-1}^\nu\right)^{-1} \mathcal{R}_k\right)(\mathcal{B}_k^\nu)\, S_k^{\alpha_1}. \tag{3.1.9}$$

Theorem 16. *(see [3]) The multi-grid method is a linear iterative method whose iteration matrix MG_k is given by*

$$MG_0 = 0, \tag{3.1.10}$$

$$MG_k = S_k^{\alpha_2}\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)(\mathcal{B}_k^\nu)\, S_k^{\alpha_1}, \tag{3.1.11}$$

$$= TG_k + S_k^{\alpha_2}\mathcal{P}_k MG_{k-1}\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\left(\mathcal{B}_k^\nu\right) S_k^{\alpha_1}, \quad k = 1, 2, .. \tag{3.1.12}$$

Proof. The result in (3.1.10) is trivial. The result in (3.1.12) follows from (3.1.11) and the definition of TG_k. We now prove the result in (3.1.11) by induction. For $k = 1$ it follows from (3.1.10) and (3.1.9). Assume that the result is correct for $k-1$. Then $\mathrm{MGM}_{k-1}(\mathbf{y}_{k-1}, \mathbf{z}_{k-1})$ defines a linear iterative method and for arbitrary $\mathbf{y}_{k-1}, \mathbf{z}_{k-1} \in \mathbb{R}^{n_{k-1}}$, we have

$$\mathrm{MGM}_{k-1}(\mathbf{y}_{k-1}, \mathbf{z}_{k-1}) - (\mathcal{B}_{k-1}^\nu)^{-1}\mathbf{z}_{k-1} = MG_{k-1}\left(\mathbf{y}_{k-1} - (\mathcal{B}_{k-1}^\nu)^{-1}\mathbf{z}_{k-1}\right). \tag{3.1.13}$$

We write the algorithm as follows:

$$\begin{aligned}
\mathrm{x}^1 &:= S_k^{\alpha_1}\left(\mathrm{x}_k^{\mathrm{old}}, Z_k^\nu\right) \\
\mathrm{x}^2 &:= \mathrm{x}^1 + \mathcal{P}_k\, \mathrm{MGM}_{k-1}\left(0, \mathcal{R}_k\left(Z_k^\nu - \mathcal{B}_k^\nu \mathrm{x}^1\right)\right) \\
\mathrm{x}_k^{\mathrm{new}} &:= S_k^{\alpha_2}\left(\mathrm{x}^2, Z_k^\nu\right).
\end{aligned}$$

From this we get

$$\begin{aligned}
\mathrm{x}_k^{\text{new}} - \mathrm{x}_k^* = \mathrm{x}_k^{\text{new}} - (\mathcal{B}_k^\nu)^{-1} Z_k^\nu &= S_k^{\alpha_2}\left(\mathrm{x}^2 - \mathrm{x}_k^*\right) \\
&= S_k^{\alpha_2}\left(\mathrm{x}^1 - \mathrm{x}_k^* + \mathcal{P}_k \,\mathrm{MGM}_{k-1}\left(0, \mathcal{R}_k\left(Z_k^\nu - \mathcal{B}_k^\nu \mathrm{x}^1\right)\right)\right).
\end{aligned}$$

Now we use the result (3.1.13) with $\mathbf{y}_{k-1} = 0, \mathrm{z}_{k-1} := \mathcal{R}_k\left(Z_k^\nu - \mathcal{B}_k^\nu \mathrm{x}^1\right)$. This yields

$$\begin{aligned}
\mathbf{x}_k^{\text{new}} - \mathbf{x}_k^* &= S_k^{\alpha_2}\left(\mathbf{x}^1 - \mathbf{x}_k^* + \mathcal{P}_k\left(\mathcal{B}_{k-1}^{-1}\mathbf{z}_{k-1} - \mathrm{MGM}_{k-1}(\mathcal{B}_{k-1}^\nu)^{-1}\mathbf{z}_{k-1}\right)\right.\\
&= S_k^{\alpha_2}\left(\mathbf{I} - \mathcal{P}_k\left(\mathbf{I} - \mathrm{MGM}_{k-1}\right)(\mathcal{B}_{k-1}^\nu)^{-1}\mathcal{R}_k\mathcal{B}_k^\nu\right)\left(\mathbf{x}^1 - \mathbf{x}_k^*\right)\\
&= S_k^{\alpha_2}\left(\mathbf{I} - \mathcal{P}_k\left(\mathbf{I} - \mathrm{MGM}_{k-1}\right)(\mathcal{B}_{k-1}^\nu)^{-1}\mathcal{R}_k\mathcal{B}_k^\nu\right) S_k^{\alpha_1}\left(\mathbf{x}^{\text{old}} - \mathbf{x}_k^*\right).
\end{aligned}$$

This completes the proof. □

3.1.3 Convergence of the Multi-grid algorithm in the L^∞-norm

In this section, we present a unified convergence analysis for multi-grids. The algorithm was described in the previous section using maximum norm.

Approximation property

Theorem 17. *(see [28]) The matrix* $\Upsilon_k = \left[(\mathcal{B}_k^\nu)^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right]$ *has the approximation property*

$$\|\Upsilon_k\|_\infty \leq C h_k^2 \left|\ln h_k\right|^2. \tag{3.1.14}$$

Proof. The proof of approximation property introduced by Arnold in [4] and based on Theorem in Hoppe (see [41]). □

Property of Smoothing

To demonstrate the smoothness property, we decompose $\mathcal{B}_k^\nu = E_k - N_k$ and use the following assumptions

$$E_k \text{ is regular and } \left\|E_k^{-1}N_k\right\|_\infty \leq 1, \text{ for all } \quad k. \tag{3.1.15}$$

$$\|E_k\|_\infty \leq C h_k^{-2}, \text{for all} \quad k, \text{ with C independent of k.} \tag{3.1.16}$$

As a smoother, we use a relaxation method with an iterative matrix

$$S_k = I_k - \omega E_k^{-1} N_k, \quad \omega \in (0,1).$$

For the following theorem, Arnold Reusken applied that for the equations and it work for our work because it is related uniquely by the operator.

Theorem 18. *Assuming the previous assumptions and notations are satisfied, there exists a constant C independent of k and α such that the following smoothness properties hold*

$$\|(\mathcal{B}_k^{\nu})\, S_k^{\alpha}\|_{\infty} \leq C\frac{1}{\sqrt{\alpha}}h_k^{-2}. \tag{3.1.17}$$

Considering the norm in (3.1.7) and considering (3.1.14) and (3.1.17), we need to prove the following stability limit

$$\exists C_s : \|S_k^{\alpha}\|_{\infty} \leq C_s, \text{ for all } k \text{ and } \alpha. \tag{3.1.18}$$

The convergence analysis is based on the following two mesh divisions Iterate the matrix using $\alpha_2 = 0$

$$\begin{aligned}\|TG_k(\alpha_1, 0)\|_{\infty} &= \left\| \left((\mathcal{B}_k^{\nu})^{-1} - \mathcal{P}_k \left(\mathcal{B}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k \right) (\mathcal{B}_k^{\nu})\, S_k^{\alpha_1} \right\|_{\infty} \\ &\leq \left\| \left((\mathcal{B}_k^{\nu})^{-1} - \mathcal{P}_k \left(\mathcal{B}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k \right) \right\|_{\infty} \|(\mathcal{B}_k^{\nu})\, S_k^{\alpha_1}\|_{\infty}.\end{aligned}$$

Often, we will select a hierarchy from more than two grids. In this case, the iteration matrix (3.1.10), (3.1.11) and (3.1.12) can be defined by recursion using the iteration matrix (3.1.9) for all levels k. Therefore, if we assume that (3.1.18) holds, the result that L^{∞} converges can be easily deduced from the previous results.

Theorem 19. *([3]) Consider the multi-grid method given the iterative matrix, input* (3.1.10), (3.1.11) *and* (3.1.12). *Under the previous assumptions, then for the parameter value $\alpha_2 = 0, \alpha_1 = \alpha > 0, \tau \geq 2$. For each $\zeta \in (0, 1)$ there is aα^* such that for all $\alpha \geq \alpha^*$*

$$\|MG_k\|_{\infty} \leq \zeta, k = 0, 1, ... \tag{3.1.19}$$

hold.

Proof. If the approximation and smoothness properties are combined with (3.1.18), then we can apply the same parameters as in [[3], Theorem 7.20]. □

The following sentence represents the main result of our work.

Theorem 20. *Under the previous assumptions and symbols, the iteration $u_k^{\nu}, \nu \geq 0$ of the* *two networks k and $k-1$ satisfies*

$$\left\|u_k^{\nu+1}-u_k^*\right\|_\infty \leq \left(\frac{C}{\sqrt{\alpha}}\left|\mathrm{Logh}_k\right|^2\right)\left\|u_k^\nu-u_k^*\right\|_\infty. \tag{3.1.20}$$

Proof. We have

$$\begin{aligned}
\left\|u_k^{\nu+1}-u_k^*\right\|_\infty = & \left\|\left(\left(I_k-\mathcal{P}_k\left(I_k-MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{B}_k^\nu\right)S_k^{\alpha_1}\right)\left(u_k^\nu-u_k^*\right)\right\|_\infty \\
\leq & \left\|\left(I_k-\mathcal{P}_k\left(I_k-MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\right\|_\infty \left\|\mathcal{B}_k^\nu S_k^{\alpha_1}\right\|_\infty \left\|u_k^\nu-u_k^*\right\|_\infty \\
\leq & \left(\frac{C_2}{\sqrt{\alpha}}h_k^{-2}\right)\left(C_1 h_k^2\left|\log h_k\right|^2\right)\left\|\left(u_k^\nu-u_k^*\right)\right\|_\infty \\
\leq & \left(\frac{C_1C_2}{\sqrt{\alpha}}\right)\left|\log h_k\right|^2\left\|u_k^\nu-u_k^*\right\|_\infty.
\end{aligned}$$

□

3.1.4 Numerical Simulation

In this section we present numerical examples of non-linear variational inequalities. To apply this method to our example, we assume that the data of our problem should be smooth enough and apply Bellman's principle dynamic programming, then we solve (1.4.3) as we discussed before, with the following data:

$$\begin{cases} Au \leq f, & \text{in } \Omega=[0,1]^2, \\ \langle Au-f, u-\psi\rangle = 0, \\ u \leq \psi, \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{3.1.21}$$

Where

$$\begin{aligned}
Au &= -\Delta u - 0.02\frac{\partial^2 u}{\partial x\partial y} + 0.15\tfrac{\partial u}{\partial x} + 0.1\tfrac{\partial u}{\partial y} + (1+\lambda)u, \\
f &= \sin(\pi x)\sin(2\pi y)sin(\pi(x+y)) + \lambda u, \\
\lambda &= 2, \\
\psi &= 0.
\end{aligned}$$

We restrict ourselves to FEM discretizations with uniform triangulation and P1 nested finite element function spaces. For domain discretization, we use the PDE toolbox in MATLAB (R2017b) to generate meshes that can then be efficiently solved using multi-grid FEM as described above. The domain consists of 64 triangles and 41 nodes.

This numerical example is intended to demonstrate the high efficiency of the multi-grid method. We choose the Gauss-Seidel method for pre/post-smoothing in the multi-grid code. For recursion in the multi-grid method, we stop the recursive multi-grid algorithm when the degrees of freedom (number of internal grid points) are less than 5.

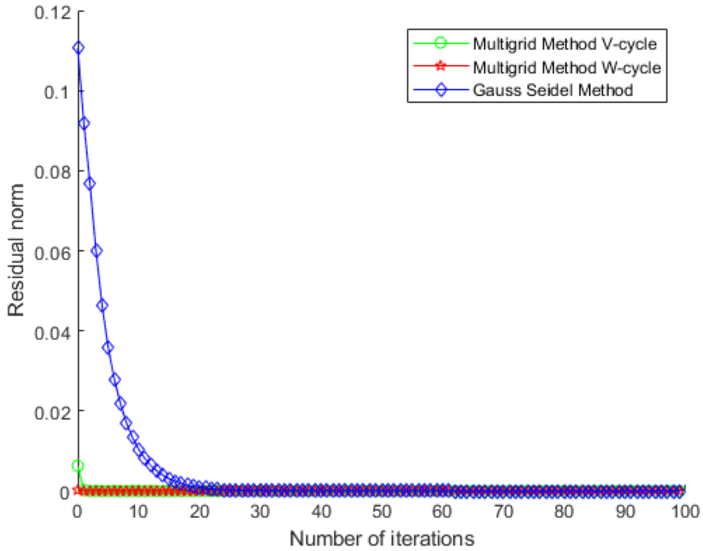

Figure 3.1: Comparison between the convergence behaviour of Multi-grid and Gauss-Seidel methods.

The figure above illustrates the convergence behaviour of the multi-grid solver (green). The red curve represents the maximum norm of the multi-grid residuals (V and W cycles) versus the number of iterations performed. For comparison, Gauss-Seidel convergence behaviour (blue curve) included.

Execute multi-grid V loop, we get the finest mesh with 41 nodes and the coarsest with 4 nodes, then we apply the Matlab backslash operator and Gauss-Seidel to this finest mesh and get the solution in the figure below.

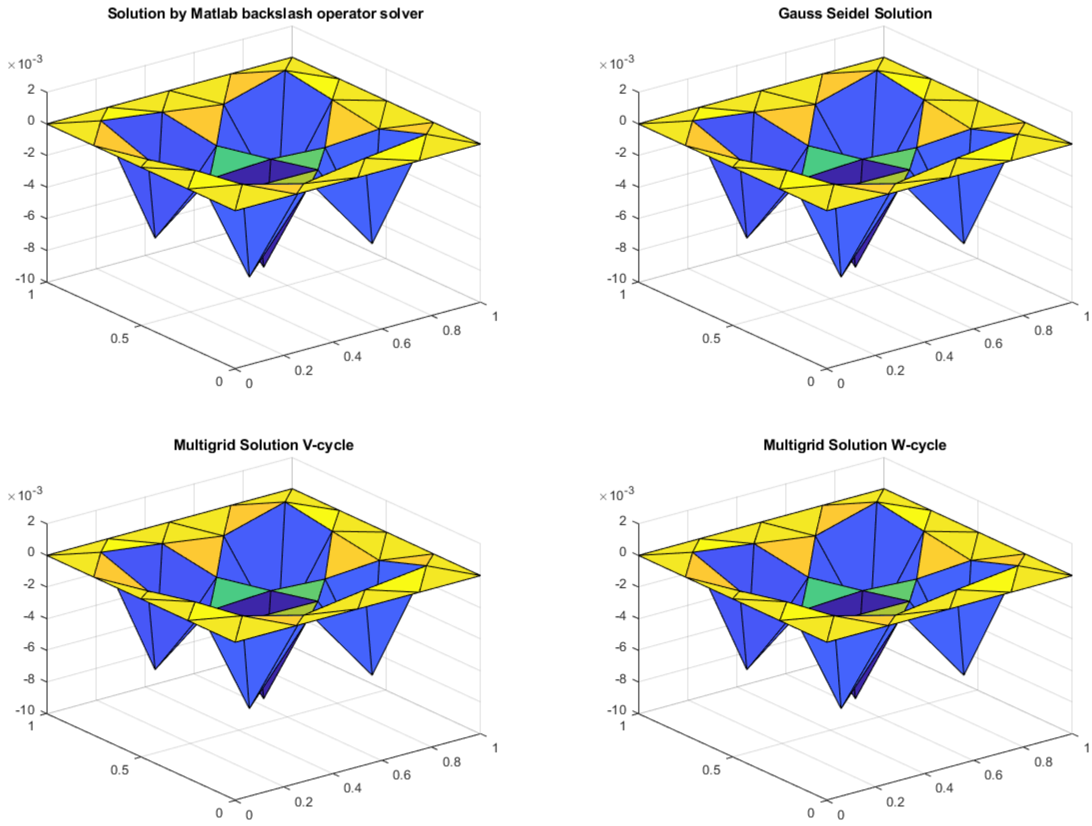

Figure 3.2: Solution of the problem (3.1.21) on fine grid with 41 DOFs using Matlab backslash operator solver, Gauss Seidel Method, Multi-grid Method V-cycle and W-cycle after 100 iterations.

Iterations number	Gauss-Seidel method	Multi-grid V-cycle method	Multi-grid W-cycle method
5	0.067462393216819	$3.476538790181394e^{-08}$	$4.884981308350689e^{-15}$
20	0.001927545543214	$2.220446049250313e^{-16}$	$2.220446049250313e^{-16}$
50	$1.549240267406660e^{-06}$	$2.220446049250313e^{-16}$	$2.220446049250313e^{-16}$
100	$1.076361222374089e^{-11}$	$2.220446049250313e^{-16}$	$2.220446049250313e^{-16}$

Table 3.1: The norm of residual with three method in the iterations.

Notting that, if we perform more than 20 iterations, the multi-grid solution is better than the Matlab backslash-operator (M.B.O) solution.

3.2 Approximation of the multi-grid method for variational inequality where the second member is non-linear

Let us give $u_k^{(\nu)} \in \mathbb{R}^{n_k}, \nu \geq 0$ and calculate $u_k^{(\nu+1)} \in \mathbb{R}^{n_k}$ as being the solution of equation

$$\mathcal{A}_k^{\nu} u_k^{\nu+1} - Z_k^{\nu} = 0, \tag{3.2.1}$$

such that:

$$Z_k^{\nu} = F_k^{\nu}(u_k^{\nu}),$$

where

$$\mathcal{A}_{k,i}^{\nu} = \begin{cases} \mathcal{A}_{k,i}\left(u_k\right) & \text{if} \quad \mathcal{A}_{k,i} u_{k,i}^{\nu} - Z_{k,i} > u_{k,i}^{\nu} - \psi_{k,i}. \\ u_{k,i} & \text{if} \qquad 1 \leq i \leq N \end{cases}. \tag{3.2.2}$$

$$Z_{k,i}^{\nu} = \begin{cases} Z_{k,i} & \text{if} \quad \mathcal{A}_{k,i} u_{k,i}^{\nu} - Z_{k,i} > u_{k,i}^{\nu} - \psi_{k,i} \\ u_{k,i} & \text{if} \qquad 1 \leq i \leq N \end{cases} \tag{3.2.3}$$

Let u_k^* be the unique solution of the discrete H.J.B equation (1.5.14)

$$\max_{1 \leq i \leq N} \left(\mathcal{A}_{k,i} u_k^* - Z_{k,i}, u_{k,i}^* - \psi_{k,i} \right) = 0. \tag{3.2.4}$$

3.2.1 Multi-grid (M.G.H.J.B) algorithm for V.Is

If we choose iterative $u_k^\nu, \nu > 0$ for the multi-grid method, we obtain $\bar{u}_k^\nu$ by applying the iterative method to the system (3.2.1) α, expressed as

$$\bar{u}_k^\nu = S_k^\alpha \left(u_k^\nu\right) \tag{3.2.5}$$

where S_k is the iteration or smoothing operator and α is the number of iterations performed. We denote the solution of (3.2.1) as u_k^*. Error setting $e_k^\nu = \bar{u}_k^\nu - u_k^*$ and residual $d_k^{(\nu)} = Z_k^\nu - \mathcal{A}_k^\nu \bar{u}_k^\nu$, we can write the equation (3.2.1) as

$$\mathcal{A}_k^\nu \left(\bar{u}_k^\nu + e_k^\nu\right) = Z_k^\nu.$$

This leads to the residual equation

$$\mathcal{A}_k^\nu e_k^\nu = Z_k^\nu - \mathcal{A}_k^\nu \bar{u}_k^\nu = d_k^{(\nu)}.$$

On the fine grid, after the relaxation on $\mathcal{A}_k^\nu \bar{u}_k^\nu = Z_k^\nu$ the error will be smooth, while on the coarse grid this error seems to be more oscillatory, and the relaxation will be so efficient. Therefore, to fully determine e_k^ν, we must compute e_{k-1}^ν at the level of $(k-1)$ as the solution for the coarse grid system

$$\mathcal{A}_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{3.2.6}$$

We can explain $e_{k-1}^\nu \left(\text{resp}\, \mathcal{A}_{k-1}^\nu, d_{k-1}^{(\nu)}\right)$ as approximation operator of $e_k^\nu \left(\text{resp}\, \mathcal{A}_k^\nu, d_k^{(\nu)}\right)$ on $k-1$ layer $\mathcal{R}_k$ and its inverse $\mathcal{P}_k$.
Therefore, we determine k-level improved iterations

$$u_k^{\nu+1} = \bar{u}_k^\nu + \mathcal{P}_k \left(e_{k-1}^\nu\right). \tag{3.2.7}$$

Due to nesting, we use the well-defined identity operator

$$\begin{aligned} \Pi : \mathbb{V}_{k-1} &\longrightarrow \mathbb{V}_k, \\ \Pi v &= v. \end{aligned}$$

Define the prolongation and restriction operators, i.e.,

$$\mathcal{P}_k = r_k^{-1} r_{k-1}, \quad \mathcal{R}_k = \mathcal{P}_k^t. \tag{3.2.8}$$

Remark 4. *The preceding algorithm describes a loop of two grid iterations to solve* (3.2.1) *for two grid levels* Ω_{k-1}. *Clearly, the coarse mesh system* (3.2.6) *has the same shape as the* *system* (3.2.1). *Therefore, we can approximate the system* (3.2.6) *by recursively performed two-grid iterations on all grid layers* $\{\Omega_k, k = 0, \dots, m_k\}$ *untie.*

3.2.2 Matrix of the M.G.H.J.B Algorithm

The iteration matrix for the two-grid method with α_1 pre-smoothing and α_2 post-smoothing iterations at the k level is given by

$$TG_k(\alpha_1, \alpha_2) = S_k^{\alpha_2} \left((\mathcal{A}_k^{\nu})^{-1} - \mathcal{P}_k \left(\mathcal{A}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k \right) (\mathcal{A}_k^{\nu}) S_k^{\alpha_1}. \tag{3.2.9}$$

Theorem 21. *([3]) The multi-grid method is a linear iterative method whose iteration matrix MG_k is given by*

$$MG_0 = 0, \tag{3.2.10}$$

$$MG_k = S_k^{\alpha_2} \left(I_k - \mathcal{P}_k (I_k - MG_{k-1}) \left(\mathcal{A}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k \right) (\mathcal{A}_k^{\nu}) S_k^{\alpha_1}, \tag{3.2.11}$$

$$= TG_k + S_k^{\alpha_2} \mathcal{P}_k MG_{k-1} \left(\mathcal{A}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k (\mathcal{A}_k^{\nu}) S_k^{\alpha_1}, \quad k = 1, 2, .. \tag{3.2.12}$$

Proof. The result in (3.2.10) is trivial. The result in (3.2.12) follows from (3.2.11) and the definition of TG_k. We now prove the result in (3.2.11) by induction. For $k = 1$ it follows from (3.2.10) and (3.2.9). Assume that the result is correct for $k-1$. Then $\mathrm{MGM}_{k-1}(\mathbf{y}_{k-1}, \mathbf{z}_{k-1})$ defines a linear iterative method and for arbitrary $\mathbf{y}_{k-1}, \mathbf{z}_{k-1} \in \mathbb{R}^{n_{k-1}}$, we have

$$\mathrm{MGM}_{k-1}(\mathbf{y}_{k-1}, \mathbf{z}_{k-1}) - (\mathcal{A}_{k-1}^{\nu})^{-1}\mathbf{z}_{k-1} = MG_{k-1}\left(\mathbf{y}_{k-1} - (\mathcal{A}_{k-1}^{\nu})^{-1}\mathbf{z}_{k-1}\right). \tag{3.2.13}$$

We write the algorithm as follows:

$$\begin{aligned}
\mathrm{x}^1 &:= S_k^{\alpha_1}\left(\mathrm{x}_k^{\mathrm{old}}, Z_k^{\nu}\right) \\
\mathrm{x}^2 &:= \mathrm{x}^1 + \mathcal{P}_k \, \mathrm{MGM}_{k-1}\left(0, \mathcal{R}_k \left(Z_k^{\nu} - \mathcal{A}_k^{\nu}\mathrm{x}^1\right)\right) \\
\mathrm{x}_k^{\mathrm{new}} &:= S_k^{\alpha_2}\left(\mathrm{x}^2, Z_k^{\nu}\right).
\end{aligned}$$

From this, we get

$$\begin{aligned}
\mathrm{x}_k^{\mathrm{new}} - \mathrm{x}_k^* &= \mathrm{x}_k^{\mathrm{new}} - (\mathcal{A}_k^{\nu})^{-1} Z_k^{\nu} = S_k^{\alpha_2}\left(\mathrm{x}^2 - \mathrm{x}_k^*\right) \\
&= S_k^{\alpha_2}\left(\mathrm{x}^1 - \mathrm{x}_k^* + \mathcal{P}_k \, \mathrm{MGM}_{k-1}\left(0, \mathcal{R}_k\left(Z_k^{\nu} - \mathcal{A}_k^{\nu}\mathrm{x}^1\right)\right)\right).
\end{aligned}$$

Now we use the result (3.2.13) with $\mathbf{y}_{k-1} = 0, \mathrm{z}_{k-1} := \mathcal{R}_k \left(Z_k^\nu - \mathcal{A}_k^\nu \mathrm{x}^1\right)$. This yields

$$\begin{aligned}
\mathbf{x}_k^{\text{new}} - \mathbf{x}_k^* &= S_k^{\alpha_2}\left(\mathbf{x}^1 - \mathbf{x}_k^* + \mathcal{P}_k\left(\mathcal{A}_{k-1}^{-1}\mathbf{z}_{k-1} - \mathrm{MGM}_{k-1}(\mathcal{A}_{k-1}^\nu)^{-1}\mathbf{z}_{k-1}\right)\right.\\
&= S_k^{\alpha_2}\left(\mathbf{I} - \mathcal{P}_k\left(\mathbf{I} - \mathrm{MGM}_{k-1}\right)(\mathcal{A}_{k-1}^\nu)^{-1}\mathcal{R}_k\mathcal{A}_k^\nu\right)\left(\mathbf{x}^1 - \mathbf{x}_k^*\right)\\
&= S_k^{\alpha_2}\left(\mathbf{I} - \mathcal{P}_k\left(\mathbf{I} - \mathrm{MGM}_{k-1}\right)(\mathcal{A}_{k-1}^\nu)^{-1}\mathcal{R}_k\mathcal{A}_k^\nu\right)S_k^{\alpha_1}\left(\mathbf{x}^{\text{old}} - \mathbf{x}_k^*\right).
\end{aligned}$$

This completes the proof. □

3.2.3 Convergence of the Multi-grid algorithm in the L^∞ norm

In this section, we present a unified convergence analysis of multi-grids. The algorithm was described in the previous section using maximum norm.

Property of approximation

Theorem 22. *([28]) The matrix $\Upsilon_k = \left[\left(\mathcal{A}_k^\nu\right)^{-1} - \mathcal{P}_k\left(\mathcal{A}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right]$ has the approximation property*

$$\|\Upsilon_k\|_\infty \le C h_k^2 \left|\ln h_k\right|^2 .. \tag{3.2.14}$$

Proof. The proof of approximation property introduced by Arnold in [4] and based on Theorem in Hoppe (see [41]). □

Property of smoothing

To demonstrate the smoothness property, we decompose $\mathcal{A}_k^\nu = E_k - N_k$ and use the following assumptions:

$$E_k \text{ is regular and } \left\|E_k^{-1}N_k\right\|_\infty \le 1, \text{ for all } \quad k, \tag{3.2}$$

$$\|E_k\|_\infty \le C h_k^{-2}, \text{for all} \quad k, \text{ with C independent of k.} \tag{3.3}$$

As a smoother, we use a relaxation method with an iterative matrix

$$S_k = I_k - \omega E_k^{-1} N_k, \quad \omega \in (0,1).$$

For the following theorem, Arnold Reusken applied that for the equations and it work for our work because it is related uniquely by the operator:

Theorem 23. *([4]) Assuming the previous assumptions and notations are satisfied, there exists a constant C independent of k and α such that the following smoothness properties* *hold*

$$\|(\mathcal{A}_k^\nu)\, S_k^\alpha\|_\infty \leq C \frac{1}{\sqrt{\alpha}} h_k^{-2}. \tag{3.2.15}$$

Considering the norm in (3.2.7), and considering (3.2.14) and (3.2.15), we need to prove the following stability limit

$$\exists C_s : \|S_k^\alpha\|_\infty \leq C_s, \text{ for all } k \text{ and } \alpha. \tag{3.2.16}$$

The convergence analysis is based on the following two meshes divisions iterate the matrix using $\alpha_2 = 0$

$$\begin{aligned}\|TG_k(\alpha_1, 0)\|_\infty &= \left\| \left((\mathcal{A}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{A}_{k-1}^\nu\right)^{-1} \mathcal{R}_k \right) (\mathcal{A}_k^\nu)\, S_k^{\alpha_1} \right\|_\infty \\ &\leq \left\| \left((\mathcal{A}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{A}_{k-1}^\nu\right)^{-1} \mathcal{R}_k \right) \right\|_\infty \|(\mathcal{A}_k^\nu)\, S_k^{\alpha_1}\|_\infty .\end{aligned}$$

Often, we will select a hierarchy from more than two grids. In this case, the iterative matrix (3.2.10), (3.2.11) and (3.2.12) can be defined by recursion using the iterative matrix (3.2.9) for all levels k. Therefore, if we assume (3.2.16) holds, the result that L^∞ converges can be easily deduced from previous results.

Theorem 24. *([3]) Consider a multi-grid method given the iterative matrix, input* (3.2.10), (3.2.11) *and* (3.2.12). *Under the previous assumptions, then for the parameter value* $\alpha_2 = 0, \alpha_1 = \alpha > 0, \tau \geq 2$. *For each* $\zeta \in (0,1)$ *there is a*α^* *such that for all* $\alpha \geq \alpha^*$

$$\|MG_k\|_\infty \leq \zeta, k = 0, 1, ... \tag{3.2.17}$$

hold.

Proof. If the approximation and smoothness properties are combined with (3.2.16), then we can apply the same parameters as in [[3], Theorem 7.20]. □

The following theorem represents the main result of our work.

Theorem 25. *Under the previous assumptions and notations the iterated* $u_k^\nu, \nu \geq 0$ *for two meshes* k *and* $k-1$ *satisfy*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_\infty \leq \left(\frac{C}{\sqrt{\alpha}} \left|\mathrm{Logh}_k\right|^2 \right) \|u_k^\nu - u_k^*\|_\infty .. \tag{3.2.18}$$

Proof. We have

$$\begin{aligned}
\left\|u_k^{\nu+1} - u_k^*\right\|_\infty = & \left\|\left(\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{A}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{A}_k^\nu\right)S_k^{\alpha_1}\right)\left(u_k^\nu - u_k^*\right)\right\|_\infty \\
\leq & \left\|\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{A}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\right\|_\infty \left\|\mathcal{A}_k^\nu S_k^{\alpha_1}\right\|_\infty \left\|u_k^\nu - u_k^*\right\|_\infty \\
\leq & \left(\frac{C_2}{\sqrt{\alpha}}h_k^{-2}\right)\left(C_1 h_k^2 \left|\log h_k\right|^2\right)\left\|\left(u_k^\nu - u_k^*\right)\right\|_\infty \\
\leq & \left(\frac{C_1 C_2}{\sqrt{\alpha}}\right)\left|\log h_k\right|^2 \left\|u_k^\nu - u_k^*\right\|_\infty
\end{aligned}$$

□

3.2.4 Numerical Simulation

In this section, we present numerical examples of a non-linear variational inequality. To apply this method to our example, we assume that the data of Our problem should be smooth enough and apply the Bellmans principle Dynamic programming, then we solve (1.5.2) As we discussed before, with the following data:

$$\begin{cases}
Au \leq f(u), \quad \text{in } \Omega = \left\{(x,y) \mid x^2 + y^2 \leq 1\right\}, \\
\langle Au - f(u), u - \psi\rangle = 0, \\
u \leq \psi, \\
u = 0, \quad \text{in } \partial\Omega.
\end{cases} \tag{3.2.19}$$

Where

$$\begin{aligned}
Au &= -\Delta u, \\
f(u) &= \cos u, \\
\psi &= 0.
\end{aligned}$$

We restrict ourselves to FEM discretization with a uniform triangulation and $P1$ nested finite element function spaces. For domain discretization, we use the PDE toolbox in MATLAB (R2017b) to generate meshes that can then be efficiently solved using multi-grid FEM as described above. The domain is as following:

This numerical example is intended to demonstrate the high efficiency of the multi-grid method. We choose the Gauss-Seidel method for pre/post-smoothing in the multi-grid code. For recursion in the multi-grid method, we stop the recursive multi-grid algorithm when the degrees of freedom (number of interior grid points) are less than 5.

The figure above illustrates the convergence behaviour of the multi-grid solver (green).The red curves represents the maximum norm of the multi-grid residuals (V and W cycle) versus

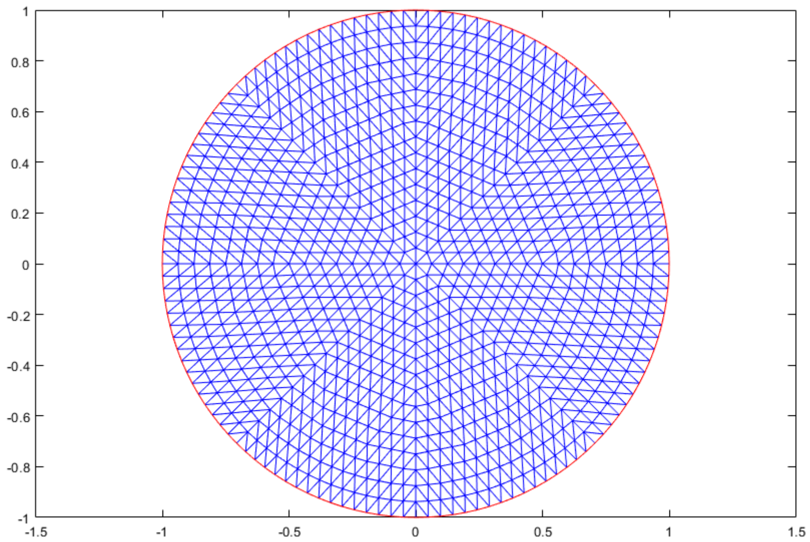

Figure 3.3: Domain of our problem with 2048 triangle and 1089 nodes.

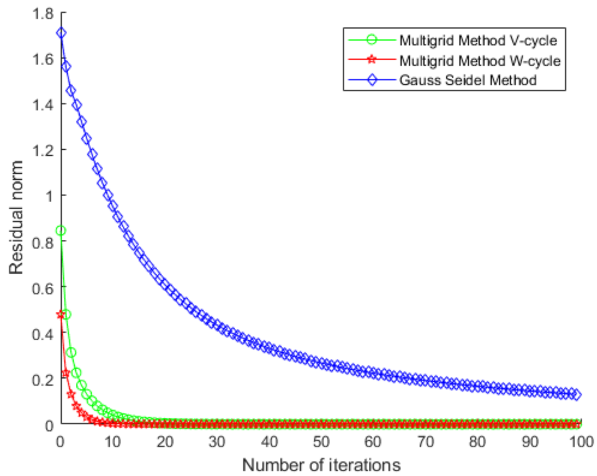

Figure 3.4: Comparison between the convergence behaviour of Multi-grid and Gauss-Seidel methods.

the number of iterations performed. For comparison, Gauss-Seidel convergence behaviour (blue curve) included.
Execute multi-grid V loop, we get the finest mesh with 1089 nodes and the coarsest with 4 nodes then we apply the Matlab backslash operator and Gauss-Seidel on this finest mesh and get the solution in the figure below.

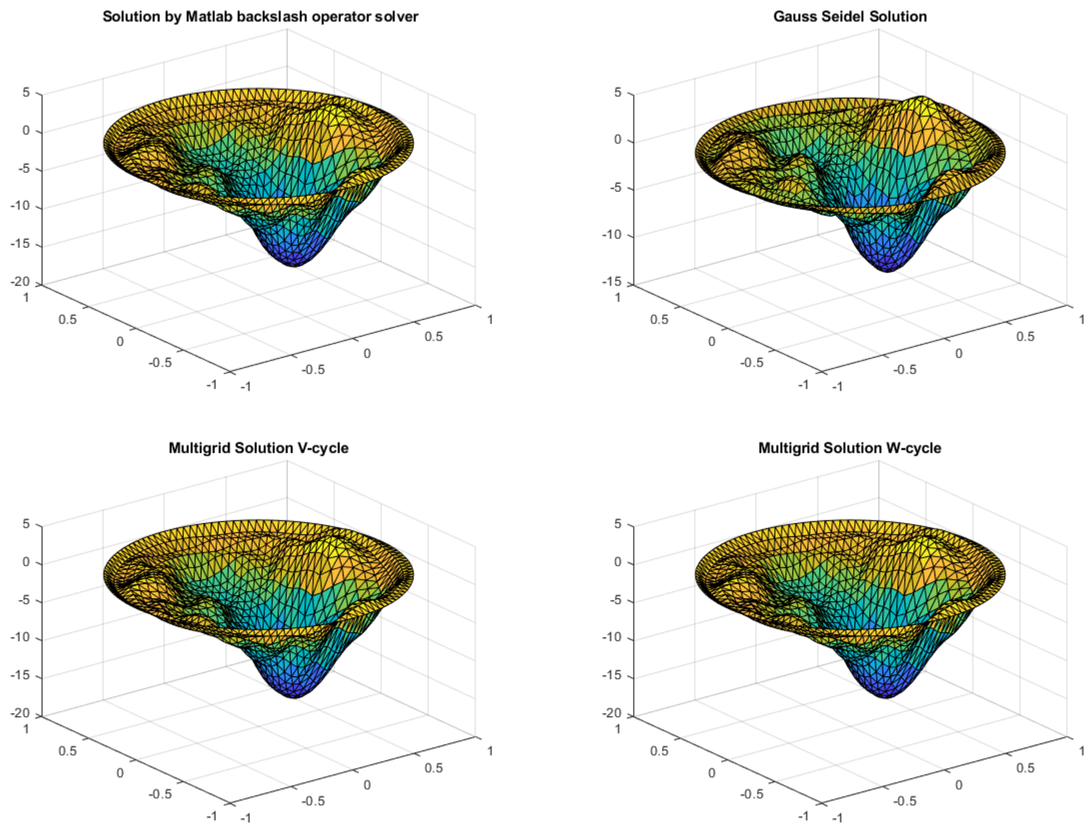

Figure 3.5: Solution of the problem (3.2.19) on fine grid with 1089 DOFs using Matlab backslash operator solver, Gauss Seidel Method, Multi-grid Method V-cycle and W-cycle after 100 iterations.

Iterations number	Gauss-Seidel method	Multi-grid V-cycle method	Multi-grid W-cycle method
5	1.3214	0.1699	0.0498
20	0.6334	0.0049	$4.5777e^{-05}$
50	0.2723	$4.4366e^{-06}$	$3.7937e^{-11}$
100	$4.058087199609872e^{-12}$	$4.440892098500626e^{-16}$	$4.440892098500626e^{-16}$

Table 3.2: The norm of residual with three method in the iterations.

Notting that, if we perform more than 20 iterations, the multi-grid solution is better than the Matlab backslash-operator (M.B.O) solution.

3.3 Approximation of the multi-grid method for non-coercive variational inequality and the non-linear second member

Let us give $u_k^{(\nu)} \in \mathbb{R}^{n_k}, \nu \geq 0$ and calculate $u_k^{(\nu+1)} \in \mathbb{R}^{n_k}$ as being the solution of the equation

$$\mathcal{B}_k^{\nu} u_k^{\nu+1} - Z_k^{\nu} = 0, \tag{3.3.1}$$

such that

$$Z_k^{\nu} = F_k^{\nu}(u_k^{\nu}) + \lambda u_k^{\nu}$$

where

$$\mathcal{B}_{k,i}^{\nu} = \begin{cases} \mathcal{B}_{k,i}\left(u_k\right) & \text{if} \quad \mathcal{B}_{k,i} u_{k,i}^{\nu} - Z_{k,i} > u_{k,i}^{\nu} - \psi_{k,i}, \\ u_{k,i} & \text{if} \quad 1 \leq i \leq N, \end{cases} \tag{3.3.2}$$

$$Z_{k,i}^{\nu} = \begin{cases} Z_{k,i} & \text{if} \quad \mathcal{B}_{k,i} u_{k,i}^{\nu} - Z_{k,i} > u_{k,i}^{\nu} - \psi_{k,i}, \\ u_{k,i} & \text{if} \quad 1 \leq i \leq N. \end{cases} \tag{3.3.3}$$

Let u_k^* be the unique solution of the discrete H.J.B equation (1.6.14)

$$\max_{1 \leq i \leq N} \left(\mathcal{B}_{k,i} u_k^* - Z_{k,i}, u_{k,i}^* - \psi_{k,i} \right) = 0. \tag{3.3.4}$$

3.3.1 Multi-grid (M.G.H.J.B) algorithm for V.Is

If we choose iteration $u_k^{\nu}, \nu > 0$ for the multi-grid method, we obtain $\bar{u}_k^{\nu}$, by applying the iterative method to e the system (3.3.1) by α, expressed as

$$\bar{u}_k^{\nu} = S_k^{\alpha}\left(u_k^{\nu}\right) \tag{3.3.5}$$

where S_k is the iteration or smoothing operator and α is the number of iterations performed.

We denote the solution of (3.3.1) as u_k^*. Error setting $e_k^{\nu} = \bar{u}_k^{\nu} - u_k^*$ and residual $d_k^{(\nu)} = Z_k^{\nu} - \mathcal{B}_k^{\nu} \bar{u}_k^{\nu}$, we can write the equation (3.3.1) as

$$\mathcal{B}_k^{\nu}\left(\bar{u}_k^{\nu} + e_k^{\nu}\right) = Z_k^{\nu}.$$

This leads to the residual equation

$$\mathcal{B}_k^{\nu} e_k^{\nu} = Z_k^{\nu} - \mathcal{B}_k^{\nu} \bar{u}_k^{\nu} = d_k^{(\nu)}.$$

On the fine grid, after the relaxation on $\mathcal{B}_k^{\nu} \bar{u}_k^{\nu} = Z_k^{\nu}$ the error will be smooth, while on the coarse grid this error seems to be more oscillatory, and the relaxation will be so efficient. Therefore, to fully determine e_k^ν, we must compute e_{k-1}^ν at the level of $(k-1)$ as the solution for the coarse grid system

$$\mathcal{B}_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{3.3.6}$$

We can explain e_{k-1}^ν $\left(\text{resp}\, \mathcal{B}_{k-1}^\nu, d_{k-1}^{(\nu)}\right)$ as approximation operator of e_k^ν $\left(\text{resp}\, \mathcal{B}_k^\nu, d_k^{(\nu)}\right)$ on $k-1$ layer $\mathcal{R}_k$ and its inverse $\mathcal{P}_k$.
Therefore, we determine k-level improved iterations

$$u_k^{\nu+1} = \bar{u}_k^\nu + \mathcal{P}_k\left(e_{k-1}^\nu\right). \tag{3.3.7}$$

Due to nesting, we use the well-defined identity operator

$$\begin{aligned} \pi \quad &: \mathbb{V}_{k-1} \longrightarrow \mathbb{V}_k, \\ \pi v \quad &= v. \end{aligned}$$

Define the prolongation and restriction operators, i.e.,

$$\mathcal{P}_k = r_k^{-1} r_{k-1}, \quad \mathcal{R}_k = \mathcal{P}_k^t. \tag{3.3.8}$$

Remark 5. *The preceding algorithm describes a loop of two grid iterations to solve (3.3.1) for two grid levels Ω_{k-1}. Clearly, the coarse mesh system (3.3.6) has the same shape as the system (3.3.1). Therefore, we can approximate the system (3.3.6) by recursively performed two-grid iterations on all grid layers $\{\Omega_k, k = 0, \ldots, m_k\}$ untie.*

3.3.2 Matrix of the M.G.H.J.B Algorithm

The iteration matrix for the two-grid method with α_1 pre-smoothing and α_2 post-smoothing iterations at the k level is given by

$$TG_k(\alpha_1, \alpha_2) = S_k^{\alpha_2}\left(\left(\mathcal{B}_k^\nu\right)^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{B}_k^\nu\right) S_k^{\alpha_1}. \tag{3.3.9}$$

Theorem 26. *(see [3]) The multi-grid method is a linear iterative method whose iteration matrix MG_k is given by*

$$MG_0 = 0, \tag{3.3.10}$$

$$MG_k = S_k^{\alpha_2}\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{B}_k^\nu\right) S_k^{\alpha_1}, \tag{3.3.11}$$

$$= TG_k + S_k^{\alpha_2}\mathcal{P}_k MG_{k-1}\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\left(\mathcal{B}_k^\nu\right) S_k^{\alpha_1}, \quad k = 1, 2, .. \tag{3.3.12}$$

Proof. The result in (3.3.10) is trivial. The result in (3.3.12) follows from (3.3.11) and the definition of TG_k. We now prove the result in (3.3.11) by induction. For $k=1$ it follows from (3.3.10) and (3.3.9). Assume that the result is correct for $k-1$. Then $\mathrm{MGM}_{k-1}(\mathbf{y}_{k-1}, \mathbf{z}_{k-1})$ defines a linear iterative method and for arbitrary $\mathbf{y}_{k-1}, \mathbf{z}_{k-1} \in \mathbb{R}^{n_{k-1}}$ we have

$$\mathrm{MGM}_{k-1}(\mathbf{y}_{k-1}, \mathbf{z}_{k-1}) - (\mathcal{B}_{k-1}^{\nu})^{-1}\mathbf{z}_{k-1} = MG_{k-1}\left(\mathbf{y}_{k-1} - (\mathcal{B}_{k-1}^{\nu})^{-1}\mathbf{z}_{k-1}\right). \tag{3.3.13}$$

We write the algorithm as follows:

$$\begin{aligned}
\mathrm{x}^1 &:= S_k^{\alpha_1}\left(\mathrm{x}_k^{\mathrm{old}}, Z_k^{\nu}\right)\\
\mathrm{x}^2 &:= \mathrm{x}^1 + \mathcal{P}_k\,\mathrm{MGM}_{k-1}\left(0, \mathcal{R}_k\left(Z_k^{\nu} - \mathcal{B}_k^{\nu}\mathrm{x}^1\right)\right)\\
\mathrm{x}_k^{\mathrm{new}} &:= S_k^{\alpha_2}\left(\mathrm{x}^2, Z_k^{\nu}\right).
\end{aligned}$$

From this we get

$$\begin{aligned}
\mathrm{x}_k^{\mathrm{new}} - \mathrm{x}_k^* = \mathrm{x}_k^{\mathrm{new}} - (\mathcal{B}_k^{\nu})^{-1}Z_k^{\nu} &= S_k^{\alpha_2}\left(\mathrm{x}^2 - \mathrm{x}_k^*\right)\\
&= S_k^{\alpha_2}\left(\mathrm{x}^1 - \mathrm{x}_k^* + \mathcal{P}_k\,\mathrm{MGM}_{k-1}\left(0, \mathcal{R}_k\left(Z_k^{\nu} - \mathcal{B}_k^{\nu}\mathrm{x}^1\right)\right)\right).
\end{aligned}$$

Now we use the result (3.3.13) with $\mathbf{y}_{k-1} = 0, \mathrm{z}_{k-1} := \mathcal{R}_k\left(Z_k^{\nu} - \mathcal{B}_k^{\nu}\mathrm{x}^1\right)$. This yields

$$\begin{aligned}
\mathbf{x}_k^{\mathrm{new}} - \mathbf{x}_k^* &= S_k^{\alpha_2}\left(\mathbf{x}^1 - \mathbf{x}_k^* + \mathcal{P}_k\left(\mathcal{B}_{k-1}^{-1}\mathbf{z}_{k-1} - \mathrm{MGM}_{k-1}(\mathcal{B}_{k-1}^{\nu})^{-1}\mathbf{z}_{k-1}\right)\right.\\
&= S_k^{\alpha_2}\left(\mathbf{I} - \mathcal{P}_k\left(\mathbf{I} - \mathrm{MGM}_{k-1}\right)(\mathcal{B}_{k-1}^{\nu})^{-1}\mathcal{R}_k\mathcal{B}_k^{\nu}\right)\left(\mathbf{x}^1 - \mathbf{x}_k^*\right)\\
&= S_k^{\alpha_2}\left(\mathbf{I} - \mathcal{P}_k\left(\mathbf{I} - \mathrm{MGM}_{k-1}\right)(\mathcal{B}_{k-1}^{\nu})^{-1}\mathcal{R}_k\mathcal{B}_k^{\nu}\right)S_k^{\alpha_1}\left(\mathbf{x}^{\mathrm{old}} - \mathbf{x}_k^*\right).
\end{aligned}$$

This completes the proof. □

3.3.3 Convergence of the Multi-grid Method in the L^{∞}-norm

In this section, we present a unified convergence analysis of multi-grid. The algorithm was described in the previous section using the maximum norm.

Property of approximation

Theorem 27. *(see [28]) The matrix* $\Upsilon_k = \left[(\mathcal{B}_k^{\nu})^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right]$ *has the approximation property*

$$\|\Upsilon_k\|_{\infty} \le Ch_k^2\,|\ln h_k|^2\,.. \tag{3.3.14}$$

Proof. The approximation property proof introduced by Arnold in [4] and based on Theorem in Hoppe (see [41]). □

Property of smoothing

To demonstrate the smoothness property, we decompose $\mathcal{B}_k^{\nu} = E_k - N_k$ and use the following assumptions

$$E_k \text{ is regular and } \left\|E_k^{-1}N_k\right\|_{\infty} \leq 1, \text{ for all } \quad k. \tag{3.3.15}$$

$$\|E_k\|_{\infty} \leq Ch_k^{-2}, \text{for all } \quad k, \text{ with C independent of k.} \tag{3.3.16}$$

As a smoother, we use a relaxation method with an iterative matrix

$$S_k = I_k - \omega E_k^{-1}N_k, \quad \omega \in (0,1).$$

For the following theorem, Arnold Reusken [4] applied that for the equations and it work for our work because it is related uniquely by the operator.

Theorem 28. *([4]) Assuming the previous assumptions and notations are satisfied, there exists a constant C independent of k and α such that the following smoothness properties hold*

$$\|(\mathcal{B}_k^{\nu})\, S_k^{\alpha}\|_{\infty} \leq C\frac{1}{\sqrt{\alpha}}h_k^{-2}. \tag{3.3.17}$$

Considering the norm in (3.3.7), and considering (3.3.14) and (3.3.17), we need to prove the following stability limit

$$\exists C_s : \|S_k^{\alpha}\|_{\infty} \leq C_s, \text{ for all } k \text{ and } \alpha. \tag{3.3.18}$$

The convergence analysis is based on the following two meshes divisions Iterate the matrix using $\alpha_2 = 0$

$$\begin{aligned}\|TG_k(\alpha_1, 0)\|_{\infty} &= \left\|\left((\mathcal{B}_k^{\nu})^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right)(\mathcal{B}_k^{\nu})\, S_k^{\alpha_1}\right\|_{\infty} \\ &\leq \left\|\left((\mathcal{B}_k^{\nu})^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right)\right\|_{\infty}\|(\mathcal{B}_k^{\nu})\, S_k^{\alpha_1}\|_{\infty}.\end{aligned}$$

Often we will select a hierarchy from more than two grids. In this case, the iterative matrix (3.3.10), (3.3.11) and (3.3.12) can be defined by recursion using the iterative matrix (3.3.9) for all levels k. Therefore, if we assume (3.3.18) holds, the result that L^{∞} converges can be easily deduced from previous results.

Theorem 29. *([3]) Consider a multi-grid method given the iterative matrix, input (3.3.10), (3.3.11) and (3.3.12). Under the previous assumptions, then for the parameter value $\alpha_2 =$*

$0, \alpha_1 = \alpha > 0, \tau \geq 2$. *For each* $\zeta \in (0,1)$ *there is a*α^* *such that for all* $\alpha \geq \alpha^*$

$$\|MG_k\|_\infty \leq \zeta, k = 0, 1, \dots \tag{3.3.19}$$

hold.

Proof. If the approximation and smoothness properties are combined with (3.3.18). Then, we can apply the same parameters as in [[3], Theorem 7.20]. □

The following theorem represents the main result of our work.

Theorem 30. *Under the previous assumptions and notations the iterated* $u_k^\nu, \nu \geq 0$ *for two meshes* k *and* $k-1$ *satisfy*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_\infty \leq \left(\frac{C}{\sqrt{\alpha}} \left|\mathrm{Logh}_k\right|^2\right) \|u_k^\nu - u_k^*\|_\infty .. \tag{3.3.20}$$

Proof. We have

$$\begin{aligned}
\left\|u_k^{\nu+1} - u_k^*\right\|_\infty = & \left\|\left(\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{B}_k^\nu\right)S_k^{\alpha_1}\right)\left(u_k^\nu - u_k^*\right)\right\|_\infty \\
\leq & \left\|\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)\right\|_\infty \|\mathcal{B}_k^\nu S_k^{\alpha_1}\|_\infty \|u_k^\nu - u_k^*\|_\infty \\
\leq & \left(\frac{C_2}{\sqrt{\alpha}} h_k^{-2}\right)\left(C_1 h_k^2 \left|\log h_k\right|^2\right) \|(u_k^\nu - u_k^*)\|_\infty \\
\leq & \left(\frac{C_1 C_2}{\sqrt{\alpha}}\right) \left|\log h_k\right|^2 \|u_k^\nu - u_k^*\|_\infty
\end{aligned}$$

□

3.3.4 Numerical Simulation

In this section, we present numerical examples of a non-linear variational inequality. To apply this method to our example, we assume that the data of Our problem should be smooth enough and apply the Bellman's principle of Dynamic programming, then we solve (1.6.2) as we discussed before, with the following data:

$$\begin{cases}
Bu \leq f(u) + \lambda u, & \text{in } \Omega = [0,1]^2 \\
\langle Bu - f(u) + \lambda u, u - \psi \rangle = 0, & \\
u \leq \psi, & \\
u = 0, & \text{in } \partial\Omega.
\end{cases} \tag{3.3.21}$$

Where

$$
\begin{aligned}
&Bu = -\Delta u - C_1\frac{\partial u}{\partial x} + C_1\frac{\partial u}{\partial y} + (C_2+\lambda)u,\\
&f(u) = \cos(u),\\
&C_1 = 0.5, C_2 = 0.045,\\
&\psi = 0.
\end{aligned}
$$

We restrict ourselves to FEM discretization with a uniform triangulation and $P1$ nested finite element function spaces. For domain discretization, we use the PDE toolbox in MATLAB (R2017b) to generate meshes that can then be efficiently solved using multi-grid FEM as described above. The domain is with 64 triangle and 41 nodes. This numerical example is intended to demonstrate the high efficiency of the multi-grid method. We choose the Gauss-Seidel method for pre/post-smoothing in the multi-grid code. For recursion in the multi-grid method, we stop the recursive multi-grid algorithm when the degrees of freedom (number of interior grid points) are less than 5.

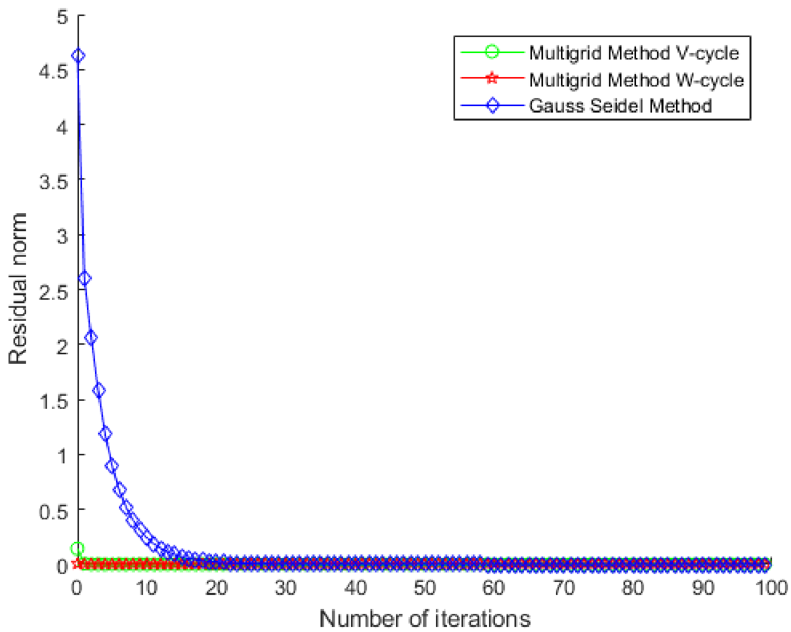

Figure 3.6: Comparison between the convergence behaviour of Multi-grid and Gauss-Seidel methods with $\lambda = 80$.

The figure above illustrates the convergence behaviour of the multi-grid solver (green). The red curves represents the maximum norm of the multi-grid residuals (V and W cycle) versus the number of iterations performed. For comparison, Gauss-Seidel convergence behaviour (blue curves) included.

Execute multi-grid V loop We get the finest mesh with 41 nodes and the coarsest with 4 nodes then we apply the Matlab backslash operator and Gauss-Seidel on this finest mesh and get the solution in the figures below.

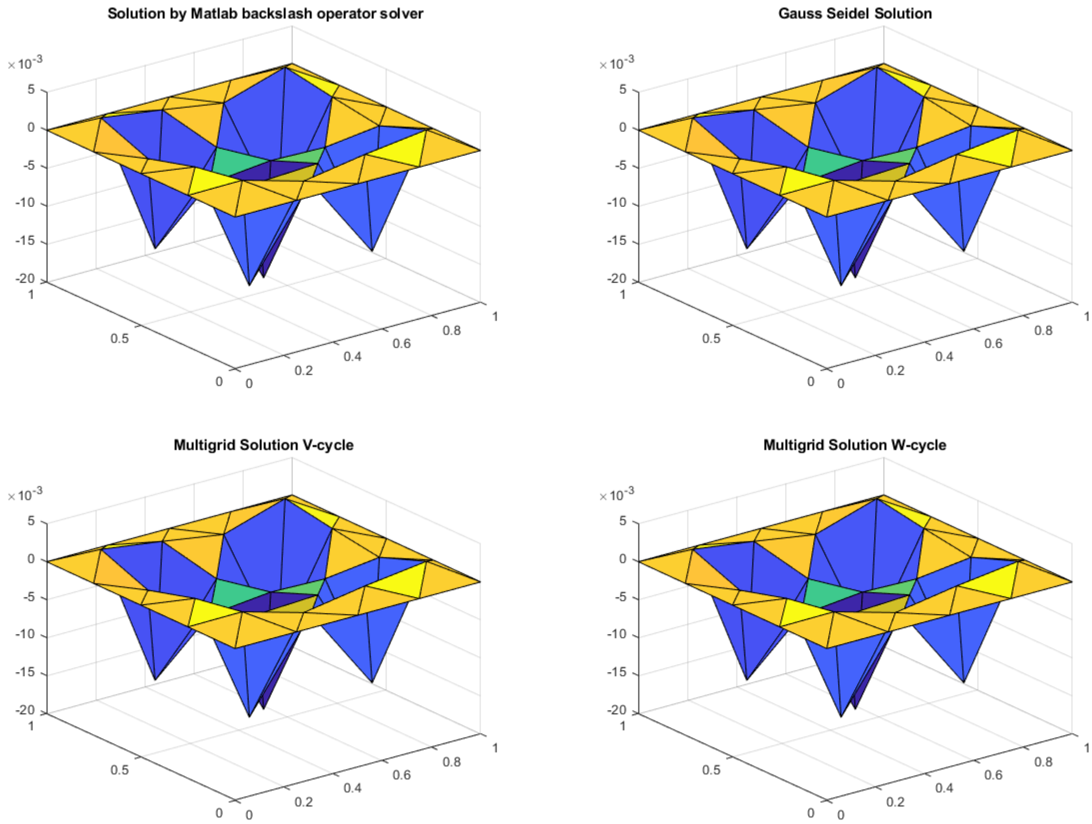

Figure 3.7: Solution of the problem (3.3.21) on fine grid with 41 DOFs using Matlab backslash operator solver, Gauss Seidel Method,Multi-grid Method V-cycle and W-cycle after 100 iterations.

Iterations number	Gauss-Seidel method	Multi-grid V-cycle method	Multi-grid W-cycle method
5	1.185927829687301	$2.215893886159392e^{-07}$	$1.421085471520200e^{-14}$
20	0.025268278660178	$2.842170943040401e^{-14}$	$2.842170943040401e^{-14}$
50	$1.470710959949884e^{-05}$	$2.842170943040401e^{-14}$	$2.842170943040401e^{-14}$
100	$5.977085493213963e^{-11}$	$2.842170943040401e^{-14}$	$2.842170943040401e^{-14}$

Table 3.3: The norm of residual with three method in the iterations.

Notting that, if we perform more than 20 iterations, the multi-grid solution is better than the Matlab backslash-operator(M.B.O) solution.

Remark 6. *In this example, the λ is important because when the $\lambda \geq 44$ the problem is converge.*

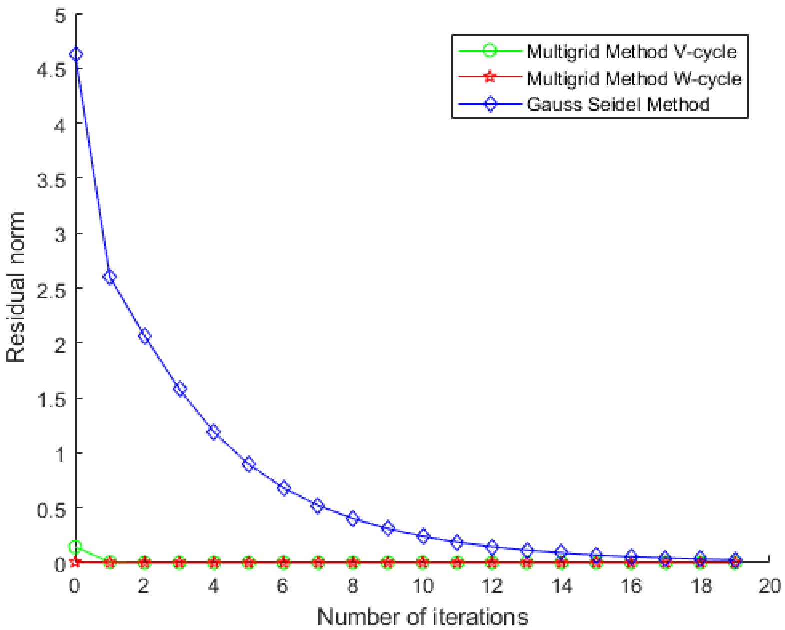

Figure 3.8: Comparison between the convergence behavior of Multigrid method and Gauss-Seidel methods with $\lambda = 80$ in 20 iterations.

Chapter 4

Approximation of the multi-grid method for non-linear variational inequality (the operator and the second member)

In this chapter, we study the multi-grid method for a non-linear elliptic variational inequality, where the operator and the second member depend the solution.

4.1 Proposition of problem

We consider the following non-linear stationary variational inequality (abbreviated VI). If V is a subspace of $H^1(\Omega)$ and $K \subset V$ a closed convex,
find $u \in K$ solution of

$$< Au, v-u > \quad \geq \quad < f(u), v-u > \quad , v \in K, \tag{4.1.1}$$

where A is a non linear application of V in its dual V' and f is l-Lipschitzian such that

$$f(u) \in L^\infty(\Omega), \quad f \geq 0 \quad \text{and} \quad \frac{l}{\beta} < 1.$$

In the following, the focus is on obstacle problems where K is given by

$$K = \{v \in V \mid v \leq \psi\}, \tag{4.1.2}$$

or

$$K = \{v \in V \mid v \geq \psi\}, \tag{4.1.3}$$

the function ψ noting an upper obstacle in (4.1.2) and a lower obstacle in (4.1.3).

Furthermore, it is assumed that the obstacle problem (4.1.1), (4.1.2), (resp. (4.1.1), (4.1.3)) is equivalent to the following complementary non-linear problem (abbreviated to CNP). Find $u \in V$ solution of

$$Au \quad \leq \quad f(u), \quad u \leq \psi \qquad (resp \quad Au \quad \geq \quad f(u), \quad u \geq \psi), \tag{4.1.4}$$

$$< Au - f(u), u - \psi >= 0, \tag{4.1.5}$$

where "$< .,. >$" refers to the canonical order in $L^2(\Omega)$.
The numerical approximation of VI (4.1.1) by finite elements leads classically to the solution of VI in finite dimension .
Find $u_h \in K_h$ solution of

$$\langle A_h u_h, v_h - u_h \rangle \quad \geq \quad \langle f_h(u_h), v_h - u_h \rangle, \quad v_h \in K_h, \tag{4.1.6}$$

where K_h is a closed convex of $\mathbb{R}^{N_h}$, $N_h \in \mathbb{N}$, $A_h : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$and $\quad f_h \in L^\infty(\Omega)$.
We give each other a suite $\{h_k\}_{k=0}^{l}$ decreasing of positive real numbers, ie $h_{k+1} < h_k$, $0 \leq k \leq l-1$,and we associate to each h_k an analogous discretization of VI (4.1.1) by finite element method.
Let $K_k \subset \mathbb{R}^{N_h}$ be the closed convexes, $A_k : \mathbb{R}^{N_k} \to \mathbb{R}^{N_k}$ be the nonlinear application and $f_k \in L^\infty(\Omega)$ the vectors generated by the chosen discretization (for simplicity the notation we write K_k instead of K_{h_k} etc).
Concerning the applications A_k, we assume that
A_k, $O \leq k \leq l$, is a Z-function continuously differentiable and

$$< A_k u_k - A_k v_k, u_k - v_k > \quad \geq \quad C \quad \|u_k - v_k\|^2, \quad 0 \leq k \leq l, \tag{4.1.7}$$

where $c > 0$ and $\|.\| =< .,. >^{1/2}$.
It is well known (see [7]) that under previous assumptions VI (4.1.6) with $h = h_k$, $O \leq k \leq l$, has a unique solution $u_k^* \in K_k$.
In addition, giving discrete obstacles $\psi_k \in \mathbb{R}^{N_k}$, $\quad 0 \leq k \leq l$, it is easy to check in the case of upper obstacle problems that the VIs (4.1.6) are equivalent to the following CNPs.
Find the solution $u_k \in \mathbb{R}^{N_k}$ of

$$A_k u_k \quad \leq \quad f_k(u_k), \quad u_k \leq \psi_k \qquad , \tag{4.1.8}$$

$$< A_k u_k - f_k(u_k), u_k - \psi_k >= 0. \tag{4.1.9}$$

Formally the CNPs (4.1.8)-(4.1.9) can be written asthe following Hamilton-Jacobi-Bellman (H.J.B) equations.
Find $u_k \in \mathbb{R}^{N_k}$ solution of

$$max_{1\leq i\leq N_k}(A_{k,i}u_{k,i} - f_{k,i}(u_{k,i}), u_{k,i} - \psi_{k,i}) = 0, \tag{4.1.10}$$

Note that when the direction of the inequality changes in (4.1.8) that is, the equation (4.1.10) replaces the "max" by " min".
In the multigrid method presented here, we will use this scheme in the smoothing processes at the level $1 \leq k \leq l$ and as an iterative solution procedure at the level $k = 0$.
In the following, we confine ourselves to the problems of the upper obstacle. The necessary modifications in case of a lower obstacle are immediate.
Then, at level k the mentioned algorithm is given as follows:
Starting from the iterate $u_k^\nu \in \mathbb{R}^{N_k}$, $\nu \geq 0$, we decompose the set $I_k = 1, 2, ..., N_k$ in $I_k = \cup_{p=1}^3 I_k^p(u_k^\nu)$ where

$$\begin{aligned} I_k^1\left(u_k^\nu\right) &= \left\{ i \in I_k \mid \left(A_k u_k^\nu - f_k\left(u_k^\nu\right)\right)_i > u_{k,i}^\nu - \psi_{k,i} \right\}, \\ I_k^2\left(u_k^\nu\right) &= \left\{ i \in I_k \mid \left(A_k u_k^\nu - f_k\left(u_k^\nu\right)\right)_i < u_{k,i}^\nu - \psi_{k,i} \right\}, \\ I_k^3\left(u_k^\nu\right) &= \left\{ i \in I_k \mid \left(A_k u_k^\nu - f_k\left(u_k^\nu\right)\right)_i = u_{k,i}^\nu - \psi_{k,i} \right\}. \end{aligned} \tag{4.1.11}$$

Then, we define an application $A_k[u_k^\nu] : \mathbb{R}^{N_k} \to \mathbb{R}^{N_k}$ and a vector $f_k(u_k)[u_k^\nu] \in \mathbb{R}^{N_k}$ by

$$(A_k[u_k^\nu]v_k)_i = \begin{cases} (A_k v_k)_i, & \text{if} \quad i \in I_k^1(u_k^\nu), \\ v_{k,i}, & \text{if} \quad i \in I_k^2(u_k^\nu) \cup I_k^3(u_k^\nu). \end{cases} \tag{4.1.12}$$

$$f_{k,i}(u_k)[u_k^\nu] = \begin{cases} f_{k,i}(u_k), & \text{if} \quad i \in I_k^1(u_k^\nu), \\ \psi_{k,i}, & \text{if} \quad i \in I_k^2(u_k^\nu) \cup I_k^3(u_k^\nu). \end{cases} \tag{4.1.13}$$

Then, we determine the iterate $u_k^{\nu+1}$ by the solution of the non linear system

$$A_k[u_k^\nu]u_k^{\nu+1} = f_k(u_k)[u_k^\nu]. \tag{4.1.14}$$

Note the difference between $I_k^2(u_k^\nu)$ and $I_k^3(u_k^\nu)$ in (4.1.11), which is not used in (4.1.12), (4.1.13) will have its effect on the convergence analysis in the next theorem.

Theorem 31. *(see [20]) Under the assumption (4.1.7), it can be shown that, starting from u_k^0, the sequence of iterate u_k^ν, $\nu \geq 1$ is decreasing and converges to the unique solution u_k^**

of the CNP (4.1.8)-(4.1.9).

In the multigrid method, given an iterate $u_k^\nu, \nu \geq 0$ we first determine first determine $\overline{u}_k^\nu$ by $\kappa_1 \geq 0$ applications of a nonlinear iterative method for the the solution of the system (4.1.14). In the convergence analysis in the next paragraph, we will choose nonlinear Gauss-Seidel iteration as a representative of these methods. If we denote an application of this procedure by $S_k[u_k^\nu](., f_k(u_k)[u_k^\nu])$ we determine $\overline{u}_k^\nu = v_k^{\kappa_1}, \kappa \geq 0$ according to

$$v_k^{\iota+1} = S_k[u_k^\nu](v_k^\iota, f_k(u_k)[u_k^\nu]), \quad 0 \leq \iota \leq \kappa_1 - 1$$

where $v_k^0 = u_k^\nu$.
Than, if we set $d_k^\nu = A_k \overline{u}_k^\nu - f_k(u_k)$, it is immediate that the solution u_k^* of CNP (4.1.8)-(4.1.9) at level k satisfies the following CNP:

$$A_k u_k^* \quad \leq \quad A_k \overline{u}_k^\nu - d_k^\nu, \quad u_k^* \leq \psi_k \qquad , \tag{4.1.15}$$

$$< A_k u_k^* - (A_k \overline{u}_k^\nu - d_k^\nu), u_k^* - \psi_k >= 0. \tag{4.1.16}$$

In the multi-grid method, the CNP (4.1.15)-(4.1.16) is approximated by a CNP corresponding to the $k-1$ level.
Find $u_{k-1} \in \mathbb{R}^{N_{k-1}}$ solution of

$$A_{k-1} u_{k-1} \quad \leq \quad g_{k-1}, \quad u_{k-1} \leq r_k^{k-1} \psi_k \qquad , \tag{4.1.17}$$

$$< A_{k-1} u_{k-1} - g_{k-1}, u_{k-1} - r_k^{k-1} \psi_k >= 0, \tag{4.1.18}$$

where

$$g_{k-1} = A_{k-1} r_k^{k-1} \overline{u}_k^\nu - r_k^{k-1} d_k^\nu, \tag{4.1.19}$$

and $r_k^{k-1} : \mathbb{R}^{N_k} \to \mathbb{R}^{N_{k-1}}$ denote a suitable restriction (see the note below).
In view of (4.1.15)-(4.1.18) we can interpret $u_{k-1} - r_k^{k-1} \overline{u}_k^\nu$ as an approximation to $k-1$ level of error $u_k^* - \overline{u}_k^\nu$ and consequently, by using an appropriate extension $p_{k-1}^k : \mathbb{R}^{N_{k-1}} \to \mathbb{R}^{N_k}$ we determine an iterate improved at k level by

$$u_k^{\nu+1} = \overline{u}_k^\nu + p_{k-1}^k (u_{k-1} - r_k^{k-1} \overline{u}_k^\nu). \tag{4.1.20}$$

Remark 7. *The restriction r_k^{k-1} must be chosen carefully to ensure that the solution u_k^* of the CNP* (4.1.8)-(4.1.9) *is a fixed point of the multi-grid iteration.*

Remark 8. (*i*) *The multi-grid cycle is called V-cycle, if $\gamma = 1$, and W-cycle, if $\gamma = 2$.*

(*ii*) *Usually the CNP solution* (4.1.17)-(4.1.18) *at level* $k = 0$ *is replaced by* μ *iterations with the scheme* (4.1.12) - (4.1.14) *using* κ_3 *iterations for auxiliary problems* (4.1.14).

(*iii*) *Another modification is to apply the smoothing process again with* κ_2 *iterations at each intermediate level* $1 \leq k \leq l-1$ *after running a reduced multi-grid cycle.*

In the next paragraph we will demonstrate the local convergence of the multi-grid procedure $\mathbf{MGVI}(l, u_l, g_l)$ (multigrid method for variational inequality). Consequently, it is necessary to determine an initial iteration in a sufficiently small neighborhood of the exact solution u_l^* of the CNP (4.1.8)-(4.1.9) at the l level. In multi-grid methods this can be achieved by using a nested iteration method where, starting from an approximation of the CNP solution (4.1.8)-(4.1.9) at level $k = 0$, at each level $1 \leq k \leq l$ we determine an approximation of the CNP (4.1.8)-(4.1.9) using $\mathbf{MGVI}(k, u_k, g_k)$ with $u_k = \widehat{p}_{k-1}^k u_{k-1}$ and $g_k = f_k(u_k)$. Usually the extension $\widehat{p}_{k-1}^k$ is based on the interpolation of a higher order than in the multi-grid cycle.

4.2 Convergence results

In the following we stick to the case where the discrete VIs (4.1.6) are obtained by a finite element method applied to the VI (4.1.1). Note that, in the case of piecewise linear finite element discretization, the resulting algebraic systems are essentially the same.
We start with a stability result for the CNP (4.1.8)-(4.1.9) using from which we which we can demonstrate the convergence of the solutions u_k^* of VIs (4.1.6) to the solution u^* of VI (4.1.1).
If we denote by $JA_k\left[z_k\right](u_k)$, $z_k \in \mathbb{R}^{N_k}$, the Jacobian at the point $u_k \in \mathbb{R}^{N_k}$ of the application $A_k\left[z_k\right]$, we first show:

Lemma 5. *([16]) Under the assumptions* (4.1.7), *for all* $u_k, z_k \in \mathbb{R}^{N_k}, 0 \leq k \leq l$, *the Jacobian* $JA_k\left[z_k\right](u_k)$ *is a* M*-matrix.*

From to the previous result it follows:

Theorem 32. *Let* u_k^1 *(resp.* u_k^2*) be the solutions of CNP* (4.1.8)-(4.1.9) *to data* A_k, f_k^1, ψ_k^1 *(resp.* $A_k, f_k^2, \psi_k^2, 0 \leq k \leq l$*). Then, under the assumptions* (4.1.7) *and*

$$\left\|\left(JA_k\left(v_k\right)\right)^{-1}\right\|_{p,p+2} \leqq C, \quad p = 0, 1, \quad k \geq 0 \tag{4.2.1}$$

we have

$$\left\|u_k^1-u_k^2\right\|_2 \leq C \max \left(\left\|f_k^1(u_k)-f_k^2(u_k)\right\|_0, \left\|\psi_k^1-\psi_k^2\right\|_2\right). \tag{4.2.2}$$

Proof. According to (4.1.8)-(4.1.9), we have

$$B_k^i\left(u_k^j-u_k^i\right)=A_k\left[u_k^i\right] u_k^j-A_k\left[u_k^i\right] u_k^i \leq f_k^j(u_k)\left[u_k^i\right]-f_k^i(u_k)\left[u_k^i\right]. \tag{4.2.3}$$

where

$$B_k^i=\int_0^1 J A_k\left[u_k^i\right]\left(u_k^i+t\left(u_k^j-u_k^i\right)\right) dt, \quad i \in\{1,2\}, j \in\{1,2\} \backslash\{i\}.$$

We deduce from Lemma 5 that the B_k^i are M-matrices and consequently, we have from (4.2.3), that

$$\begin{aligned}-\left(B_k^1\right)^{-1}\left(f_k^2(u_k)\left[u_k^1\right]-f_k^1(u_k)\left[u_k^1\right]\right) &\leq u_k^1-u_k^2 \\ &\leq\left(B_k^2\right)^{-1}\left(f_k^1(u_k)\left[u_k^2\right]-f_k^2(u_k)\left[u_k^2\right]\right)\end{aligned}$$

and therefore

$$\begin{aligned}\left|u_k^1-u_k^2\right| \leq \max \Big(&\left|\left(B_k^1\right)^{-1}\left(f_k^2(u_k)\left[u_k^1\right]-f_k^1(u_k)\left[u_k^1\right]\right)\right|, \\ &\left|\left(B_k^2\right)^{-1}\left(f_k^1(u_k)\left[u_k^2\right]-f_k^2(u_k)\left[u_k^2\right]\right)\right|\Big),\end{aligned}$$

where $|v_k|=\left(\left|v_{k,1}\right|, \ldots,\left|v_{k,N_k}\right|\right)$.
We conclude, from the monotony of the norm $\|\cdot\|_2$ and to (4.2.10). □

To apply the previous theorem we denote by r_k: $V \cap C(\bar{\Omega}) \rightarrow \mathbb{R}^{N_k}$, $0 \leq k \leq l$ the application defined by the trivial restriction of $V \cap C(\bar{\Omega})$ to the space $C\left(\Omega_k\right)$ of the discrete functions defined on the set Ω_k, and we suppose that the VIs (4.1.6) and the VI (4.1.1) are consistent in the sense that we have a positive constant $\tilde{\alpha}$ such that

$$\left\|r_k\left(A u^*-f(u)\right)-\left(A_k r_k u^*-f_k(u_k)\right)\right\|_0=O\left(h_k^{\tilde{\alpha}}\right) \quad\left(h_k \rightarrow 0\right), \tag{4.2.4}$$

$$\left\|r_k \psi-\psi\right\|_2=O\left(h_k^{\tilde{\alpha}}\right) \quad\left(h_k \rightarrow 0\right). \tag{4.2.5}$$

Furthermore, it is assumed that

$$\left\|\hat{p}_{k-1}^k r_{k-1} u^*-r_k u^*\right\|_2=O\left(h_k^{\tilde{\alpha}}\right) \quad\left(h_k \rightarrow 0\right), \tag{4.2.6}$$

$$\left\|\hat{p}_{k-1}^k\right\|_{2,2} \leq C, \tag{4.2.7}$$

where $\hat{p}_{k-1}^{k}, 1 \leq k \leq l$, is the extension used for the nested iteration $\mathbf{NMGVI}(l, u_l, g_l)$ (nonlinear multigrid method for variational inequality).
Note that the consistency conditions (4.2.4)-(4.2.5) and the condition (4.2.6) presuppose a sufficient regularity of the data and of the solution of the VI (4.1.1).

Corollary 2. *Under the hypotheses* (4.1.7), (4.2.4)-(4.2.5), (4.2.6)-(4.2.7) *and*

$$h_k < h_{k-1} \leq C h_k, \quad 1 \leq k \leq l, \tag{4.2.8}$$

$$\|r_k^{k-1}\|_{p,p} \leq C, \quad 0 \leq p \leq 3, \tag{4.2.9}$$

$$\left\|(J A_k (v_k))^{-1}\right\|_{p,p+2} \leq C, \quad p = 0, 1, \quad k \geq 0, \tag{4.2.10}$$

we have

$$\|u_k^* - r_k u^*\|_2 = O\left(h_k^{\tilde{\alpha}}\right)(h_k \to 0), \quad k \geq 0, \tag{4.2.11}$$

$$\|\hat{p}_{k-1}^{k} u_{k-1}^* - u_k^*\|_2 = O\left(h_k^{\tilde{\alpha}}\right)(h_k \to 0), \quad k \geq 1. \tag{4.2.12}$$

Proof. If we set

$$u_k^2 = r_k u^*, \quad f_k^2(u_k) = f_k(u_k) - r_k\left(A u^* - f(u)\right) + \left(A_k r_k u^* - f_k(u_k)\right),$$

and

$$\psi_k^2 = \psi_k - \left(\psi_k - r_k \psi\right),$$

it is easy to see that u_k^2 is the solution of the CNP (4.1.8)-(4.1.9) of data A_k, f_k^2, ψ_k^2. Then, taking $u_k^1 = u_k^*$, $f_k^1(u_k) = f_k(u_k)$ and $\psi_k^1 = \psi_k$, the assertion (4.2.6)-(4.2.7) is an immediate consequence of (4.2.2). Finally, using (4.2.8), (4.2.6)-(4.2.7) the assertion (4.2.12) is deduced without difficulty of (4.2.6) and (4.2.11). □

Before we turn to the convergence analysis of the $\mathbf{MGVI}(l, u_l, g_l)$ it is necessary to specify the discrete free boundary of VIs (4.1.6). In this regard, we note that there is an univoc correspondence between the discrete functions u_k on the meshes $\Omega_k, 0 \leq k \leq l$, and the vectors $(u_{k,1}, \ldots, u_{k,N_k}) \in \mathbb{R}^{N_k}$. Indeed, for any $x_\eta \in \Omega_k$ there exists only one index $i(\eta) \in I_k$ such that $u_k(x_\eta) = u_{k,i(\eta)}$, and conversely, for any $i \in I_k$ there is a node $x_{\eta(i)} \in \Omega_k$ such that $u_k\left(x_{\eta(i)}\right) = u_{k,i}$. Consequently, using index sets $I_k^p\left(u_k^\nu\right), 1 \leq p \leq 3$, given by (4.1.11), we can define

$$\Omega_k^p\left(u_k^\nu\right) = \left\{x_\eta \in \Omega_k \mid i(\eta) \in I_k^p\left(u_k^\nu\right)\right\}, \quad 1 \leq p \leq 3. \tag{4.2.13}$$

In particular, a node $x_\eta \in \Omega_k$ is said to be active, if $x_\eta \in \Omega_k^2\left(u_k^\nu\right) \cup \Omega_k^3\left(u_k^\nu\right)$, and inactive, if $x_\eta \in \Omega_k^1\left(u_k^\nu\right)$. Note that $\Omega_k^2\left(u_k^*\right) \cup \Omega_k^3\left(u_k^*\right)$ corresponds to the set of coincidence of the solution u_k^* of the VI (4.1.6) with the discrete obstacle ψ_k.
In the following, it is assumed that the CNPs (4.1.8)-(4.1.9) are strictly complementary in the following sense:

$$\Omega_k^3\left(u_k^*\right) = \varnothing, \quad 0 \leq k \leq l. \tag{4.2.14}$$

Moreover, we denote by $N_k\left(x_\eta\right)$ the set containing $x_\eta \in \Omega_k$ and all the nodes $x \in \Omega_k$ in the neighbourhood of x_η . So we define

$$\tilde{\Omega}_k^p\left(u_k^*\right) = \left\{x_\eta \in \Omega_k \mid N_k\left(x_\eta\right) \subseteq \Omega_k^p\left(u_k^*\right)\right\}, \tag{4.2.15}$$

$$\partial\tilde{\Omega}_k^p\left(u_k^*\right) = \Omega_k^p\left(u_k^*\right) \backslash \tilde{\Omega}^p\left(u_k^*\right), \quad 1 \leq p \leq 2. \tag{4.2.16}$$

For the next theorem, it is an adaptation of a theorem [22] for the convergence of the finite elements and it is work for the nonlinear operator:

Theorem 33. *Under the previous assumptions and notation, we have*

$$\|u - u_k^*\|_\infty \leq C h_k^2 \left|\log h_k\right|^2 \|f(u)\|_\infty. \tag{4.2.17}$$

Let us denote by $\bar{\Omega}_k^P\left(\bar{u}_k^\nu\right), \quad 1 \leq p \leq 2$, the sets defined as in (4.2.15) with u_k^* replaced by $\bar{u}_k^\nu$, this strategy can be put into practice as follows: if $\tilde{p}_{k-1}^k, 1 \leq k \leq l$, designate the prolongation based on the bilinear interpolation, we pose

$$\left(p_{k-1}^k u_{k-1}\right)(x) = \begin{cases} \left(\tilde{p}_{k-1}^k u_{k-1}\right)(x) & \text{if} \quad x \in \bigcup_{p=1}^2 \tilde{\Omega}_k^p\left(\bar{u}_k^\nu\right) \\ 0 & \text{else.} \end{cases} \tag{4.2.18}$$

The usefulness of all the above conditions will become apparent when calculating the iteration operator of the **MGVI**. Moreover, in this point we need some elementary results of subdifferential calculus.
We consider the non linear function $F_k : \mathbb{R}^{N_k} \to \mathbb{R}^{N_k}$ given by

$$F_k u_k = \max\left(A_k u_k - f_k(u_k), u_k - \psi_k\right). \tag{4.2.19}$$

It is a max-function with two differentiable arguments, which has a generalized Jacobian $\partial F_k\left(u_k\right), u_k \in \mathbb{R}^{N_k}$, (cf. [16]), so that

$$\partial F_k\left(u_k\right) \subseteq \prod_{i=1}^{N_k} \partial F_{k,i}\left(u_k\right) \tag{4.2.20}$$

where $\partial F_{k,i}(u_k)$ denote the set of matrices whose i-th row is an element of the generalized gradient

$$\partial F_{k,i}(u_k) = \begin{cases} \{\nabla A_{k,i}(u_k)\}, & \text{if } i \in I_k^1(u_k), \\ \{e_k^i\}, & \text{if } i \in I_k^2(u_k), \\ \text{co}\{\nabla A_{k,i}(u_k), e_k^i\}, & \text{if } i \in I_k^3(u_k), \end{cases} \tag{4.2.21}$$

where $\text{co}(z_k^1, z_k^2)$ denoting the convex envelope of vectors z_k^1, z_k^2.
Moreover, we have

$$F_k u_k - F_k v_k \in \text{co}\partial F_k([u_k, v_k])(u_k - v_k) \tag{4.2.22}$$

where the right side above denotes the convex hull of all vectors in the form $DF_k(u_k - v_k)$ where $DF_k \in \partial F_k(w_k), \quad w_k \in [u_k, v_k] =$
$\{z_k | z_k = tu_k + (1-t)v_k, t \in [0,1]\}$.
In particular, it follows from (4.2.22) that there exists a matrix $DF_k[u_k, v_k] \in$
$co\partial F_k([u_k, v_k])$ not necessarily unique such that

$$F_k u_k - F_k v_k = DF_k[u_k, v_k](u_k - v_k). \tag{4.2.23}$$

Using the previous results, for the non-linear Gauss-Seidel smoothing process we have this lemma is valid for our work because it is related with the first member (matrix):

Lemma 6. *Let u_k^ν be the ν-th iteration of the multi-grid method at level k and let $\overline{u}_k^\nu$be the smoothed iterate obtained by κ applications of non linear Gauss-Seidel iteration starting from u_k^ν. Then, under the assumptions (4.1.7), (4.2.14) there exists a matrix $D^\kappa S_k[u_k^\nu, u_k^*] \in co\partial F_k[u_k, v_k]$ such that*

$$\bar{u}_k^\nu - u_k^* = D^\kappa S_k[u_k^\nu, u_k^*](u_k^\nu - u_k^*). \tag{4.2.24}$$

Moreover, for any $\varepsilon_k > 0$ there exists $\delta_k > 0$ such that

$$\|D^\kappa S_k[u_k^\nu, u_k^*] - (JS_k[u_k^*](u_k^*))^\kappa\|_{2,2} < \varepsilon_k, \tag{4.2.25}$$

provided that $\|u_k^\nu - u_k^*\|_2 < \delta_k$, where $JS_k[u_k^*](u_k^*)$ denotes the Jacobian of the application $S_k[u_k^*](., g_k[u_k^*])$ to the point u_k^*.

It is well known that the convergence of the multi-grid method in case $l > 1$ can be deduced from that in the case of two meshes (cf.[28]). For that we demonstrate:

Lemma 7. *Let $u_l^\nu, \nu \geq 1$, be the iteration obtained by applying the **MGVI**(l, u_l, g_l) for two* *meshes (two grids). Under the assumptions* (4.1.7), (4.2.8)-(4.2.9), (4.2.14) *and*

$$\begin{array}{c} \|JA_k\left[u_k^*\right](u_k^*)\left(JS_k\left[u_k^*\right](u_k^*)\right)^{\kappa}\|_{2,1} \leq C_0(\kappa)h_k^{-\alpha}, 0 \leq \kappa \leq \kappa_{\max}\left(h_1\right), \quad k \geq 1, \\ \textit{where} \quad C_0(\kappa) \to 0 \quad (\kappa \to \infty) \quad \textit{and} \quad \kappa_{\max}(h) \to \infty \quad (h \to 0)(\textit{matrix norm}). \end{array} \tag{4.2.26}$$

we have

$$u_l^{\nu+1} - u_l^* = \left(M_l^{l-1} + Z_l\right)\left(u_l^{\nu} - u_l^*\right), \tag{4.2.27}$$

where

$$\begin{aligned} M_l^{l-1} = & \left[\left(JA_l\left[u_l^*\right](u_l^*)\right)^{-1} - p_{l-1}^l\left(JA_{l-1}\left[r_l^{l-1}u_l^*\right]\left(r_l^{l-1}u_l^*\right)\right)^{-1}r_l^{l-1}\right] \times \\ & \times\left[JA_l\left[u_l^*\right](u_l^*)\left(JS_l\left[u_l^*\right](u_l^*)\right)^{\kappa}\right], \end{aligned} \tag{4.2.28}$$

and

$$\begin{array}{c} \|Z_l\|_{2,2} \leqq C(\kappa)\eta^{(\nu)}, \\ \left(\|M_k\|_{p,q} = \sup \quad \{\|M_k u_k\|_q / \|u_k\|_p \quad | \quad u_k \in \mathbb{R}^{N_k}, u_k \neq 0\}\right), \end{array} \tag{4.2.29}$$

$C(\kappa)$ be a positive constant, which depends κ, and $\eta^{(\nu)}$ denoting a function so that $\eta^{(\nu)} \to 0$, if $\|u_l^{\nu} - u_l^*\|_2 \to 0$.

Proof. We define $\tilde{F}_{l-1}\left[z_l\right], z_l \in \mathbb{R}^{N_l}$, and $\tilde{F}_l^{l-1}$ by

$$\begin{aligned} \tilde{F}_{l-1}\left[z_l\right](v_{l-1}) = \max & \left(A_{l-1}v_{l-1} - A_{l-1}r_l^{l-1}z_l + r_l^{l-1}\left(A_l z_l - f_l(z_l)\right),\right. \\ & \left. v_{l-1} - r_l^{l-1}\psi_l\right), \end{aligned} \tag{4.2.30}$$

$$\tilde{F}_l^{l-1}v_l = \max\left(r_l^{l-1}\left(A_l v_l - f_l(v_l)\right), r_l^{l-1}\left(v_l - \psi_l\right)\right). \tag{4.2.31}$$

Then, observing that $\tilde{F}_{l-1}\left[\bar{u}_l^{\nu}\right](u_{l-1}) = 0$, $\tilde{F}_{l-1}\left[\bar{u}_l^{\nu}\right]\left(r_l^{l-1}\bar{u}_l^{\nu}\right) = \tilde{F}_l^{l-1}\bar{u}_l^{\nu}$ and $\tilde{F}_l^{l-1}u_l^* = 0$, using (4.2.22) we deduce the existence of the matrices

$$\left(D\tilde{F}_{l-1}\left[\bar{u}_l^{\nu}\right]\right)\left[u_{l-1}, r_l^{l-1}\bar{u}_l^{\nu}\right] \in \operatorname{co}\left(\partial\tilde{F}_{l-1}\left[\bar{u}_l^{\nu}\right]\right)\left(\left[u_{l-1}, r_l^{l-1}\bar{u}_l^{\nu}\right]\right),$$

and

$$D\tilde{F}_l^{l-1}\left[u_l^*, \bar{u}_l^{\nu}\right] \in \operatorname{co}\partial\tilde{F}_l^{l-1}\left(\left[u_l^*, \bar{u}_l^{\nu}\right]\right),$$

so that

$$\begin{aligned} \left(D\tilde{F}_{1-1}\left[\bar{u}_l^{\nu}\right]\right)\left[u_{l-1}, r_l^{l-1}\bar{u}_l^{\nu}\right]\left(u_{l-1} - r_l^{l-1}\bar{u}_l^{\nu}\right) = & \\ = & \tilde{F}_{l-1}\left[\tilde{u}_l^{\nu}\right](u_{l-1}) - \tilde{F}_{l-1}\left[\bar{u}_l^{\nu}\right]\left(r_l^{l-1}\bar{u}_l^{\nu}\right) \\ = & \tilde{F}_l^{l-1}u_l^* - F_l^{l-1}\bar{u}_l^{\nu} = D\tilde{F}_l^{l-1}\left[u_l^*, \bar{u}_l^{\nu}\right]\left(u_l^* - \bar{u}_l^{\nu}\right), \end{aligned}$$

hence, in view of (4.1.20) and (4.2.24)

$$u_l^{\nu+1} - u_l^* = \left(\left(I_l - p_{l-1}^l \left(\left(D\tilde{F}_{l-1} \left[\bar{u}_l^\nu \right] \right) \left[u_{l-1}, r_l^{l-1} \bar{u}_l^\nu \right] \right)^{-1} \times \right.\right.$$
$$\left. \times D\tilde{F}_l' - 1 \left[u_l^*, \bar{u}_l^\nu \right] \times D^\kappa S_l \left[u_l^\nu, u_l^* \right] \right) \left(u_l^\nu - u_l^* \right).$$

It follows from (4.2.14) that the generalized Jacobians $\partial\tilde{F}_{l-1}\left[u_l^*\right]\left(r_l^{l-1}u_l^*\right)$ and $\partial\tilde{F}_l^{l-1}\left(u_l^*\right)$ are univalent and therefore it is justified to pose

$$X_{l-1} = \left(\partial\tilde{F}_{l-1} \left[u_l^* \right] \left(r_l^{l-1} u_l^* \right) \right)^{-1} \left(D\tilde{F}_{l-1} \left[\bar{u}_l^\nu \right] \right) \left[u_{l-1}, r_l^{l-1} \bar{u}_l^\nu \right] - I_{l-1}, \tag{4.2.32}$$

$$X_l = \left(\partial\tilde{F}_l^{l-1} \left(u_l^* \right) \right)^{-1} D\tilde{F}_l^{l-1} \left[u_l^*, \bar{u}_l^\nu \right] - I_l, \tag{4.2.33}$$

$$Y_l = D^\kappa S_l \left[u_l^\nu, u_l^* \right] - \left(JS_l \left[u_l^* \right] \left(u_l^* \right) \right)^\kappa. \tag{4.2.34}$$

Using the derivation of compound functions, we have

$$\partial\tilde{F}_l^{l-1} \left(u_l^* \right) = r_l^{l-1} \partial F_l \left(u_l^* \right),$$

and therefore, it follows that

$$u_l^{\nu+1} - u_l^* = \left[\left(\partial F_l \left(u_l^* \right) \left(I_l + X_l \right) \right)^{-1} - p_{l-1}^l \left(\partial\tilde{F}_{l-1} \left[u_l^* \right] \left(r_l^{l-1} u_l^* \right) \right] \right.$$
$$\left. \times \left(I_{l-1} + X_{l-1} \right) \right)^{-1} r_l^{l-1} \Big] \left[\partial F_l \left(u_l^* \right) \left(I_l + X_l \right) \right.$$
$$\times \left(\left(JS_l \left[u_l^* \right] \left(u_l^* \right) \right)^\kappa + Y_l \right) \Big] \left(u_l^\nu - u_l^* \right), \tag{4.2.35}$$

so we affirm that

$$\left\| X_{l-1} \right\|_{2,2} \leq C(\kappa)\eta^{(\nu)}, \quad \left\| X_l \right\|_{2,2} \leq C(\kappa)\eta^{(\nu)}, \quad \left\| Y_l \right\|_{2,2} \leq C(\kappa)\eta^{(\nu)}. \tag{4.2.36}$$

The last two assertions are an immediate consequence of (**??**) and (4.2.25). By writing

$$X_{l-1} = \left(\partial\tilde{F}_{l-1} \left[u_l^* \right] \left(r_l'^{-1} u_l^* \right) \right)^{-1} \left(D\tilde{F}_{l-1} \left[u_l^* \right] \right) \left[u_{l-1}, r_l^{l-1} \bar{u}_l^\nu \right] - I_{l-1} +$$
$$+ \left(\partial\tilde{F}_{l-1} \left[u_1^* \right] \left(r_l^{l-1} u_l^* \right) \right)^{-1} \left[\left(D\tilde{F}_{l-1} \left[\bar{u}_l^\nu \right] \right) \left[u_{l-1}, r_l^{l-1} \bar{u}_l^\nu \right] \right.$$
$$\left. - \left(D\tilde{F}_{l-1} \left[u_l^* \right] \right) \left[u_{l-1}, r_l^{l-1} \bar{u}_l^\nu \right] \right],$$

where $\left(DF_{l-1}\left[u_l^*\right]\right)\left[u_{l-1}, r_l^{l-1}\bar{u}_l^\nu\right] \in \operatorname{co}\left(\partial F_{l-1}\left[u_l^*\right]\right)\left(\left[u_{l-1}, r_l^{l-1}\bar{u}_l^\nu\right]\right)$, the first assertion is proved, if we verify

$$\left\| u_{l-1} - r_l^{l-1} u_l^* \right\|_2 \leq C(\kappa)\eta^{(\nu)}, \tag{4.2.37}$$

$$\left\| r_l^{l-1} \left(\bar{u}_l^\nu - u_l^* \right) \right\|_2 \leq C(\kappa)\eta^{(\nu)}. \tag{4.2.38}$$

In this point, observing that u_{l-1} is the solution of the CNP (4.1.8)-(4.1.9) of data A_{l-1}, g_{l-1}, $r_l^{l-1}\psi_l$ and that $r_l^{l-1}u_l^*$ satisfies the equation:

$$\max\left(A_{l-1}r_l^{l-1}u_l^* - A_{l-1}r_l^{l-1}u_l^* + r_l^{l-1}\left(A_l u_l^* - f_l(u_l^*)\right), r_l^{l-1}\left(u_l^* - \psi_l\right)\right) = 0,$$

it follows from Theorem 32 that

$$\left\|u_{l-1} - r_l^{l-1}u_l^*\right\|_2 \leqq C\left\|\left(B_{l-1}r_l^{l-1} - r_l^{l-1}B_l\right)D^\kappa S_l\left[u_l, u_l^*\right]\left(u_l^\nu - u_l^*\right)\right\|_0,$$

where

$$B_{l-1} = \int_0^1 JA_{l-1}\left(r_l^{l-1}z_l(t)\right)dt, \quad B_l = \int_0^1 JA_l\left(z_l(t)\right)dt, \quad z_l(t) = u_l^* + t\left(\bar{u}_l^\nu - u_l^*\right),$$

$$t \in [0,1].$$

We deduce from it the assertion (4.2.37) using (4.2.9) and from to Lemma 6.
By writing

$$r_l^{l-1}\left(\bar{u}_l^\nu - u_l^*\right) = r_l^{l-1}D^\kappa S_l\left[u_l^\nu, u_l^*\right]\left(u_l^\nu - u_l^*\right),$$

the assertion (4.2.38) is an immediate consequence of (4.2.9) and Lemma 6.
Finally, using (4.2.8) , (4.2.26), (4.2.36) and observing (4.2.14), (4.2.21) the assertions (4.2.27), (4.2.28) and (4.2.29) can easily be deduced from (4.2.35). □

In view of (4.2.27) and (4.2.29) the previous lemma implies that the convergence of the multi-grid method for two meshes depends essentially on the iteration matrix M_l^{l-1} given by (4.2.28). We demonstrate:

Lemma 8. *([26]) Under the assumptions of the previous lemma, we have*

$$\left\|M_l^{l-1}\right\|_{2,2} \leqq CC_0(\kappa). \tag{4.2.39}$$

If we use the estimates (4.2.29) and (4.2.39) in (4.2.27), we get the convergence of this method for our work:

Theorem 34. *In the iterates $u_l^\nu, \nu \geq 0$, of the $\boldsymbol{MGVI}(l, u_l, g_l)$ for two meshes satisfy*

$$\left\|u_l^{\nu+1} - u_l^*\right\|_2 \leqq \left[CC_0(\kappa) + C(\kappa)\eta^{(\nu)}\right]\left\|u_l^\nu - u_l^*\right\|_2. \tag{4.2.40}$$

Proof. Under hypotheses (4.1.7), (4.2.8)- (4.2.26) and (4.2.14) the iterates $u_l^\nu, \nu \geq 0$, of the

MGVI(l, u_l, g_l) for two meshes satisfy

$$\left\|u_l^{\nu+1} - u_l^*\right\|_2 \leqq \left[CC_0(\kappa) + C(\kappa)\eta^{(\nu)}\right] \|u_l^\nu - u_l^*\|_2 . \tag{4.2.41}$$

this estimate work for V and W cycle of multigrid method. □

By Theorem 4.2 there exists a neighborhood

$$\mathfrak{U}_{\varepsilon_k}(u_k^*) = \{v_k \in \mathbb{R}^{N_k} | \|v_k - u_k^*\|_2 \leq \varepsilon_k\}$$

of the solution of CNP (4.1.8)-(4.1.9) at the level $1 \leq k \leq l$ so that for any $u_k^0 \in \mathfrak{U}_{\varepsilon_k}(u_k^*)$ the iteration $u_k^\nu, \nu \geq 1$, obtained by the **MGVI**(k, u_k, g_k) satisfy

$$\|u_k^{\nu+1} - u_k^*\|_2 \leq q\|u_k^\nu - u_k^*\|_2, \nu \geq 0, q < 1.$$

Moreover, under the conditions (4.2.4)- (4.2.7), according to Corollory 2 we have the existence of a positive constant $\tilde{C}$ such that

$$\left\|\hat{p}_{k-1}^k u_{k-1}^* - u_k^*\right\|_2 \leq \tilde{C} h_k^{\tilde{\alpha}}, \quad 1 \leq k \leq l. \tag{4.2.42}$$

After these preparatory considerations we are able to deduce the following convergence result for the nested iteration **NMGVI**(l, u_l, g_l) (nonlinear multigrid method for variational inequality)(cf. [[27]; Theorem 3.9]).

Theorem 35. *We assume that the assumptions* (4.1.7), (4.2.8), (4.2.26), (4.2.4)-(4.2.7) *and* (4.2.14) *are true and further that the constants $\tilde{\alpha}$ and $\tilde{C}$ in* (4.2.42) *are such that*

$$2\tilde{C} h_k^{\tilde{\alpha}} \leq \varepsilon_k, \quad 1 \leq k \leq l. \tag{4.2.43}$$

Then, if we choose τ in the ***NMGVI***(l, u_l, g_l) *so that*

$$q^\tau \leq 1/(2\hat{C}), \quad \hat{C} = \sup_{1 \leqq k \leq I} \left\|\hat{p}_{k-1}^k\right\|_{2,2} (h_{k-1}/h_k)^{\tilde{\alpha}} , \tag{4.2.44}$$

and if we determine an approximation u_0 of u_0^* such that

$$\|u_0 - u_0^*\|_2 \leq (\tilde{C}/\hat{C}) h_0^{\tilde{\alpha}}, \tag{4.2.45}$$

the iterations $u_k, 1 \leq k \leq l$, obtained by ***NMGVI***(l, u_l, g_l) *satisfy*

$$\|u_k - u_k^*\|_2 \leq 2\tilde{C} h_k^{\tilde{\alpha}} q^\tau . \tag{4.2.46}$$

Proof. If we pose $u_k^{(0)} = \hat{p}_{k-1}^k u_{k-1}, 1 \leqq k \leq l$, we have

$$\left\| u_k^{(0)} - u_k^* \right\|_2 \leq \left\| \hat{p}_{k-1}^k u_{k-1}^* - u_k^* \right\|_2 + \left\| \hat{p}_{k-1}^k \right\|_{2,2} \left\| u_{k-1} - u_{k-1}^* \right\|_2 .$$

Then, using (4.2.42) and the conditions (4.2.43)- (4.2.45) the assertion (4.2.46) is easily deduced by induction. □

4.3 Numerical results

In this section, we present numerical examples of a non-linear variational inequality. To apply this method to our example, we assume that the data of Our problem should be smooth enough and apply the Bellman's principle Dynamic programming, then we solve (1.7.5) as we discussed before, with the following data:

1. **Linear second member**:

$$\begin{cases} Au \leq f(u), & \text{in } \Omega = (0,1)^2 \\ \langle Au - f(u), u - \psi \rangle = 0, \\ u \leq \psi, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{4.3.1}$$

where

$$\begin{aligned} &Au = -\Delta u - \frac{\partial u}{\partial x} - 0.1\frac{\partial u}{\partial y} + u^2, \\ &f(u) = \sin(2\pi x)\sin(2\pi y) + 10u, \\ &\psi = 0. \end{aligned}$$

We restrict ourselves to the FEM discretization with a uniform triangulation and $P1$ nested finite element function spaces. For domain discretization, we use the PDE toolbox in MATLAB (R2017b) to generate the meshes that can then be efficiently solved using multi-grid FEM as described above. The domain is with 64 triangle and 41 nodes. This numerical example is intended to demonstrate the high efficiency of the multi-grid method. We chose the Gauss-Seidel method for pre/post-smoothing in the multi-grid code. For recursion in the multi-grid method, we stop the recursive multi-grid algorithm when the degrees of freedom (number of interior grid points) are less than 5.

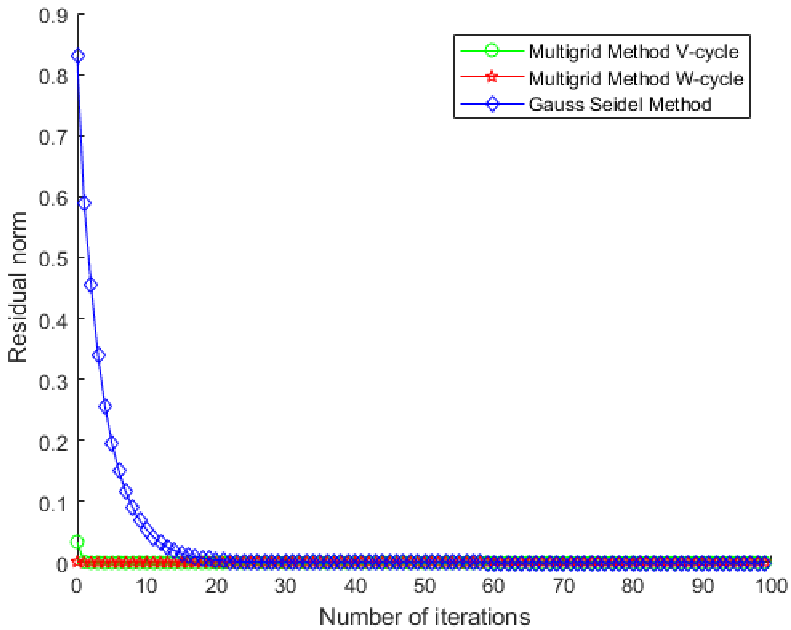

Figure 4.1: Comparison between the convergence behaviour of Multi-grid and Gauss-Seidel methods.

The figure above illustrates the convergence behaviour of the multi-grid solver (green). The red curves represents the maximum norm of the multi-grid residuals (V and W cycle) versus the number of iterations performed. For comparison, Gauss-Seidel convergence behaviour (blue curves) included.

Execute multi-grid V loop, we get the finest mesh with 41 nodes and the coarsest with 4 nodes then we apply the Matlab backslash operator and Gauss-Seidel on this finest mesh and get the solution in the figure below.

Iterations number	Gauss-Seidel method	Multi-grid V-cycle method	Multi-grid W-cycle method
5	0.256358806186284	$8.321445754688739e^{-08}$	$7.105427357601002e^{-15}$
20	0.005519442045605	$3.552713678800501e^{-15}$	$3.552713678800501e^{-15}$
50	$2.797008283650371e^{-06}$	$3.552713678800501e^{-15}$	$3.552713678800501e^{-15}$
100	$9.007905532598670e^{-12}$	$3.552713678800501e^{-15}$	$3.552713678800501e^{-15}$

Table 4.1: The norm of residual with three method in the iterations.

Notting that, if we perform more than 20 iterations, the multi-grid solution is better than the Matlab backslash operator (M.B.O) solution.

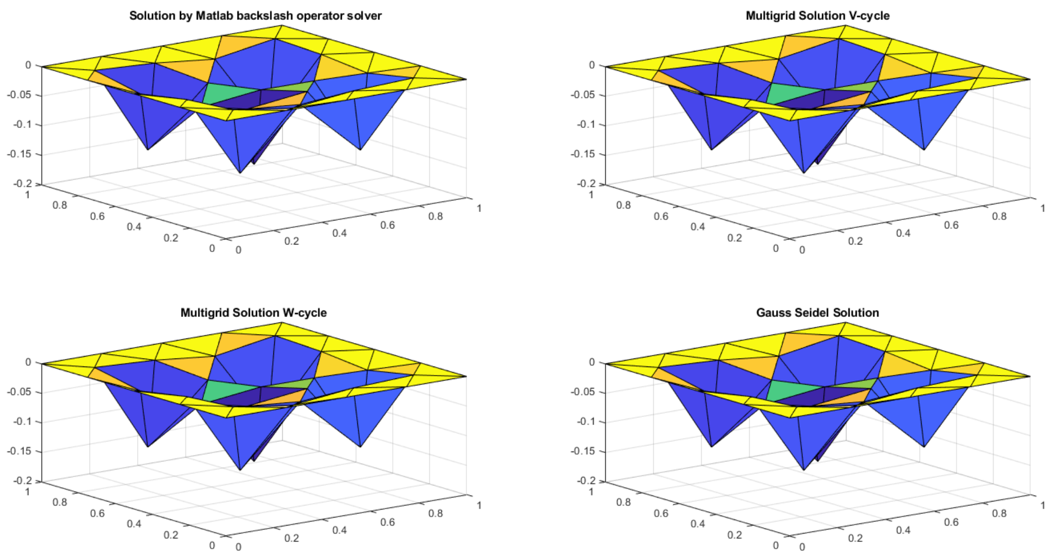

Figure 4.2: Solution of the problem (4.3.1) on fine grid with 41 DOFs using Matlab backslash operator solver, Gauss Seidel Method, Multi-grid Method V-cycle and W-cycle after 100 iterations.

2. **Non-linear second member**: We discussed with the following data

$$\begin{cases} Au \geq f(u), & \text{in } \Omega = \{(x,y) | x^2 + y^2 \leq 1\} \\ \langle Au - f(u), u - \psi \rangle = 0, \\ u \geq \psi, \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{4.3.2}$$

where

$$\begin{aligned} &Au = -(0.6)\Delta u + 0.15\frac{\partial u}{\partial x} + 0.1\frac{\partial u}{\partial y} + u^2, \\ &f(u) = \cos 2u, \\ &\psi = 0. \end{aligned}$$

We do the same steps of the linear second member. So we have

The figure above illustrates the convergence behaviour of the multi-grid solver (green). The red curves represents the maximum norm of the multi-grid residuals (V and W cycle) versus the number of iterations performed. For comparison, Gauss-Seidel convergence behaviour (blue curves) included.

Execute multi-grid V loop We get the finest mesh with 41 nodes and the coarsest with 4 nodes then we apply the Matlab backslash operator and Gauss-Seidel on this finest

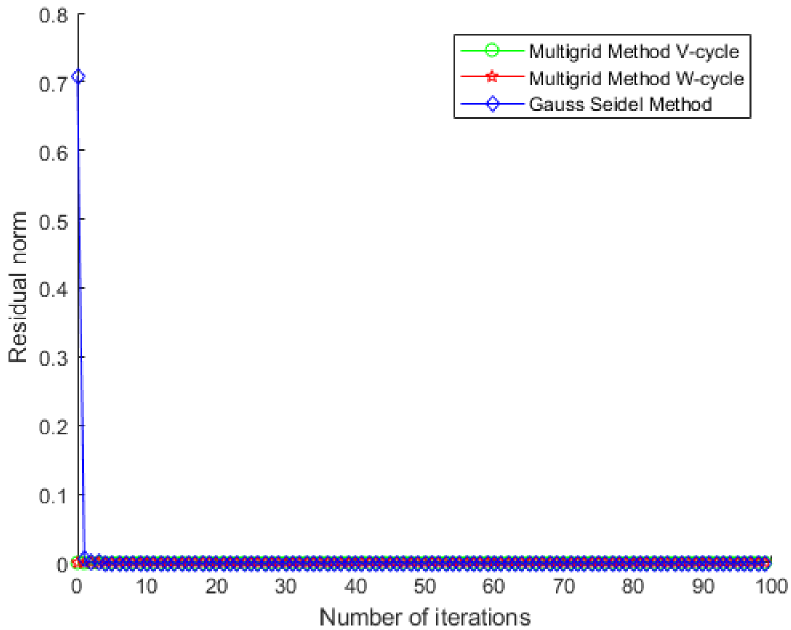

Figure 4.3: Comparison between the convergence behaviour of Multi-grid and Gauss-Seidel methods.

mesh and get the solution in the figure below.

Iterations number	Gauss-Seidel method	Multi-grid V-cycle method	Multi-grid W-cycle method
5	$6.449078715498047e^{-09}$	$4.440892098500626e^{-16}$	$4.440892098500626e^{-16}$
20	$.440892098500626e^{-16}$	$4.440892098500626e^{-16}$	$4.440892098500626e^{-16}$
50	$.440892098500626e^{-16}$	$4.440892098500626e^{-16}$	$4.440892098500626e^{-16}$
100	$4.440892098500626e^{-16}$	$4.440892098500626e^{-16}$	$4.440892098500626e^{-16}$

Table 4.2: The norm of residual with three method in the iterations.

Notting that, if we perform more than 5 iterations, the multi-grid solution is better than the Matlab backslash operator (M.B.O) solution.

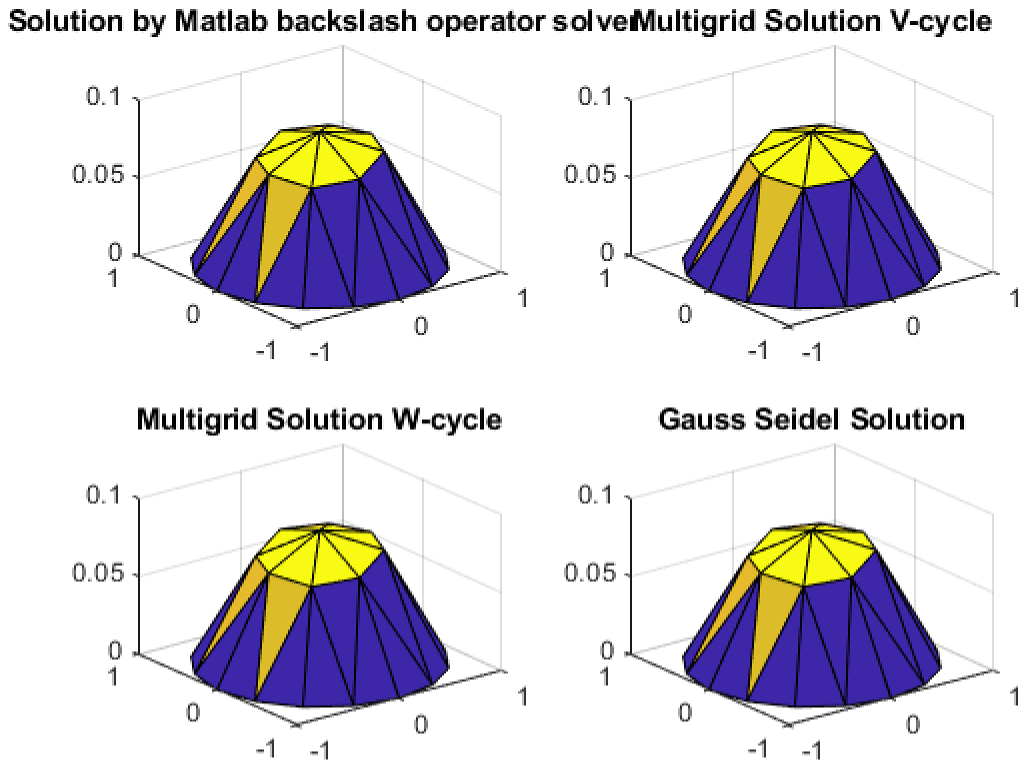

Figure 4.4: Solution of the problem (4.3.2) on fine grid with 25 DOFs using Matlab backslash operator solver, Gauss Seidel Method, Multi-grid Method V-cycle and W-cycle after 100 iterations.

Conclusion and Future works

In this work, we apply the algebraic multi-grid method Efficient iterative solutions to the discretizing elliptic variational inequalities. Adaptive finite element approximation for discretization cycle domain. After discretization, we declare a multi-grid Methods for solving discrete problems. we introduced uniforms convergence of our problem and show that the multi-grid method has a shrinkage number with respect to the maximum norm. In Figures we present experimental examples of variational inequalities. This numerical results show that Gauss-Seidel is still not good even after many iterations. Compared to multi-grid method with debugging capabilities (reducing high-frequency error through relaxation while low-frequency errors are mapped to the coarse grid and reduced there), it only takes a few iterations to converge. Many extensions of the above techniques are possible. An interesting future case is the application of parallel full multi-grid methods to solve unconstrained elliptical variations.

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Abstract

This work concerns the study of a class of variational inequalities, in the sense that the obstacle does not depend on the solution, by applying the multi-grid methods V-cycle and W-cycle.
Multi-grid methods consist in successively using grids (or meshes) of different sizes, so as to obtain a detailed solution in the high frequencies, while ensuring a rapid relaxation of the low frequencies.
Multi-grid methods have been studied for linear elliptical problems. For our part, we are interested in finite difference approximation, by introducing multi-grid algorithms, for non-linear variational inequalities, insofar as the non-coercive and linear operator, the second member depends on the solution the mixture of the last two cases and the last case where the operator and the second member are non-linear.

Résumé

Ce travail concerne l'étude d'une classe des inéquations variationnelles, dans le sens où l'obstacle ne dépend pas de la solution, en appliquant les méthodes multigrilles V-cycle et W-cycle.
Les méthodes multigrilles consistent à utiliser successivement des grilles (ou maillages) de différentes tailles, de manière à obtenir une solution détaillée dans les hautes fréquences, tout en assurant une relaxation rapide des basses fréquences.
Les méthode multigrilles ont été étudiées pour les problèmes elliptiques linéaires. Pour notre part, on s'intéresse à l'approximation par différences finies, en introduisant les algorithmes aux multigrilles, pour des inéquations variationnelles non linéaires, dans la mesure où l'opérateur non coercive et linéaire, le seconde membre dépend de la solution le mélange des deux derniers cas et le dernier cas où l'opérateur et le second member son non linéaires.

الخلاصة

يتعلق هذا العمل بدراسة فئة من التفاوتات المتباينة، بمعنى أن العقبة لا تعتمد على الحل، من خلال تطبيق دورة الطرق متعددة الأطراف دورة V ودورة W.
تتكون طرق متعدد الشبكات على التوالي باستخدام شبكات بأحجام مختلفة، وذلك للحصول على حل مفصل في الترددات العالية، مع ضمان الاسترخاء السريع للترددات المنخفضة.
تمت دراسة طرق متعدد الشبكات للمشاكل الإهليلجية الخطية. من جانبنا، نحن مهتمون بتقريب الاختلاف المحدود، من خلال إدخال خوارزميات متعددة العناصر، للتفاوتات المتباينة غير الخطية، طبقنا هذه الطريقة على مؤثر غير القسري والخطي، العضو الثاني يعتمد على الحل، مزيج الحالتين الأخيرتين وفي الحالة الأخيرة حيث يكون المشغل والعضو الثاني غير خطيين.