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Green's Function Approach to Entanglement Entropy on Fuzzy Spaces

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Abstract

In this dissertation, we introduce a novel Euclidean Green function approach to compute Rényi entropy on lattices and fuzzy spaces. Rényi entropy for an arbitrary subset of coupled harmonic oscillators is written as a zero temperature partition function generated by an Euclidean action with n-fold step potential. The associated Green's function is explicitly constructed and an alternative new formula for Rényi entropy is obtained. The developed approach allows one to go beyond the Gaussian case and systematically investigate interacting theories, which represents a real advance and paves the way for the investigation of entanglement entropy on lattices and fuzzy spaces for interacting theories. This approach is further applied to several systems with a focus on 1+1 scalar field theory. The case of half space entanglement was obstructed by the necessity to invert a special class of Toeplitz matrices. An asymptotic inverse for this class of matrices is conjectured and several analytical and numerical tests are in favor of its truthfulness. We finally outline how this approach can be used to investigate entanglement entropy for free and interacting scalar field theory on fuzzy spaces.

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Introduction

Entanglement, that mysterious puzzle of quantum mechanics, it is truly one of the most fascinating phenomena in the quantum world. As a matter of fact, the existence of entanglement was questionable at the beginning; it challenges classical intuition by imposing non-local effects and it was regarded as clear evidence of the non-completeness of the quantum theory [1]. However, as far as it proved, quantum mechanics can be spooky more than ghosts themselves and ordinary intuition can take no place in it. Experimental tests were always in favor of its existence [2–4].

The need to quantify entanglement has induced the development of many entanglement measures [5]. However, entanglement entropy is the most complete and coherent one. It nicely fulfills all the requirements of a proper entanglement measure [5]. In addition, its notion has a sense of physical universality; it has many clones in several fields, like Von Neumann entropy in statistical mechanics, Shannon entropy in quantum information,... etc. It thereby finds a place in different areas of physics, like quantum computation and quantum information theory [6, 7], condensed matter [8–10], black holes physics [11, 12], high energy physics and quantum field theory [13, 14].

The entropy of entanglement is considered when a two-partite system AB is in a pure state. The entanglement entropy is defined in terms of the reduced density matrix of the subsystem A or B, depending on which subsystem is considered physically inaccessible. Intuitively, the amount of entanglement between A and B should not change whether the observer has access only to the subsystem A or only to the observables of the subsystem B; the two subsystems are equally entangled. Fortunately, the entanglement entropy fulfills this requirement; that's because the eigenvalues of the reduced density matrices ρ_A and ρ_B are equal, up to zero modes, when the overall state of the system AB is a pure state.

It is important to note here that the entanglement entropy does not scale like the volume of the controlled subsystem; regardless of the volume difference between the subsystems A and B the density matrices ρ_A and ρ_B provide the same amount of entanglement. This feature of the entanglement entropy can be intuitively explained by the proportionality of the entanglement to the boundary between the two subsystems rather than their volume. This is known by the area law; the entanglement entropy for $d+1$ theory of fields is proportional to the separating area between the two considered subsystems [15] :

$$S \sim \frac{A_{d-1}}{\epsilon^{d-1}}$$

where A_{d-1} is the $d-1$ dimensional area and ϵ is the UV cutoff. The proportionality factor

depends on the regulating scheme, except in the situation of one spatial dimension $d = 1$, where the proportionality coefficient therein is universal. This represents an interesting feature for the entanglement entropy as a measure in 1+1 field theory.

This interesting behavior of entanglement entropy has been directly related to black hole physics. Remarkably, the physics of black holes showed a close analogy with thermodynamical laws. At first, it was considered as a useful analogy [16] but it turned out to be more than that; the black hole behaves as a thermodynamical object that obeys the four laws of thermodynamics and it has its own temperature and entropy [17]. The latter turns out to be proportional to the area of the black hole horizon

$$S_{BH} = \frac{1}{4l_{pl}^2} A$$

A is the area of the black hole horizon and $l_{pl} = (\hbar G/c^3)^{1/2}$ is Planck length. The early explanations of the source of black hole entropy suggested a connection between it and the entanglement entropy [11, 12, 18, 19]. Black holes give natural rise to entangling phenomena. It is by definition an information hider where any observer outside the horizon can have no access to any event inside it. However, the source of the entropy of black holes and their statistical origin is not totally understood yet. The main obstacle which prevents defining the entanglement entropy as the entropy of black hole, or at least part of it, is that the entropy of black hole is finite whereas the entanglement entropy for $d + 1$ field theory diverges in the absence of a UV cutoff [20].

Indeed, a simple comparison between the entanglement entropy and black hole entropy strongly points towards a discrete nature of space-time at the Planck scale, and the continuum is merely an effective approximation of a yet to be uncovered underlying more fundamental geometry [21].

The UV divergence of entanglement entropy for a theory of fields in the absence of UV cutoff reminds us of the black body ultraviolet catastrophe, which was overcome by discretizing the phase space into Planck cells. This discretization has a quantum mechanical origin, the uncertainty principle or the non-commutative nature of the phase space, i.e. $\{q, p\}$ do not commute, therefore they cannot be exactly determined at the same time. This is indeed a very preliminary example of non-commutative and fuzzy spaces. The fuzziness of phase space when treated quantum mechanically beautifully lifts the divergence without any need to impose a cutoff. Motivated by the black body scenario, a natural question may arise here : Can the replacement of the ordinary continuum space-time by a non-commutative version render the entanglement entropy finite or soften the UV divergences? This proposal was taken up in [22] and it was shown that entanglement entropy is UV finite, proportional to the boundary degrees of freedom and satisfies the area law [22, 23].

Many techniques have been developed in order to evaluate the entanglement entropy depending on the area of research and the treated physical situation. These techniques can be classified into two principal categories : Real time approach and the Euclidean time approach.

The real time approach was initially developed in [11, 18] where it was addressed to investigate the source of black hole entropy. A more efficient technique within real time formalism was developed by Peschel [24]. This technique is especially useful for numerical investigations; it reformulates the problem in terms of the two-point correlators of the field variables. However, this technique becomes less attractive and practically untractable when an interaction is introduced or for non-gaussian states for which Wick's theorem is not applicable and higher order correlators are needed.

The Euclidean time approach, on the other hand, provides more powerful tools for investigating the entanglement entropy analytically. It was first developed in [12,25] in the context of black hole entropy investigation. After taking Wick's rotation to the space-time, the entanglement entropy can be written in terms of the heat kernel or Green's function.

These techniques are especially useful analytically in Quantum field theory when it comes to extracting the UV divergence and establishing the area law. Moreover, it allows one to make a connection with string theory [26] and to extract finite mass dependent but cutoff independent contribution to the area law [27]. Moreover, Euclidean Green's function offers a systematic method to handle non-gaussian states, it allows one to handle interacting theories perturbatively [28], and to investigate the dependence of entanglement entropy on the renormalized mass.

Now, despite being a powerful tool in investigating the entanglement entropy in the continuum, the Euclidean time approach seems to be limited to some special geometries. For example, if the subsystem is assumed to be half of $d+1$ space-time, the Rényi entropy is governed by simple geometry, namely that of a cone with deficit angle $2\pi(n-1)$, for which Green's function or the Heat kernel expansion is known. However, if the subsystem is a finite region or a collection of disconnected regions, the problem becomes untractable and it is generally very hard to construct the associated Green's function or to recognize the underlying n-sheeted geometry over which the Heat kernel expansion is to be made.

The main motivations for this work were the lack of a Euclidean formalism for entanglement entropy on lattice and fuzzy spaces, as well as the limitation of the real-time approach to free and gaussian states. In [29, 30], the Green's function for an arbitrary chain of coupled harmonic oscillators was constructed and an alternative formula for Rényi entropy was obtained. The entangled subsystem can be randomly chosen out of the total lattice of oscillators. Unprecedently, a perturbation series can be obtained for the entanglement entropy in interacting theories in terms of the constructed Green's function [29, 30].

This dissertation is organized as follows : The first chapter gives an overview of the basic concepts of two-partite systems entanglement. The example of qubit system was taken as a toy model to introduce the concept of entanglement quantification for the entangled states. The entanglement entropy is subsequently introduced as an entanglement measure. The different methods used to evaluate the entanglement entropy are introduced through the Real time approach and the Euclidean time approach. Different examples within each approach are discussed.

In the second chapter, Green's function is constructed for an arbitrary set of intervals. The proof is well established and the necessary conditions to be fulfilled by Green's function are all presented and proved. Rényi entropy is finally written in terms of the obtained Green's function.

In the last chapter, a number of applications of Green's function are established. We start with a quick test of the method by investigating the two coupled harmonic oscillators entanglement. We further apply the formalism to a scalar field theory on a 1+1 lattice by considering the entanglement entropy in two important particular cases : one resulting from tracing out half of the space and the second resulting from tracing out an interval of infinitesimal size, in particular an interval made of a single point. We performed an analytical study for both cases.

From the technical point of view, the case of one-point turned out to be easy to handle analytically within this approach, and we could recover explicitly the formalism developed in [24]. However, the case of half-space entanglement entropy necessitated the finding of an asymptotic inverse for a special class of Toeplitz matrix with an N-dependent symbol. This problem turned

out to be a strenuous one due, as far as we are aware, to the lack of any solution for this kind of problem in the mathematical literature. A conjecture on the inverse of this class of Toeplitz matrices is proposed, suited by a numerical and analytical discussion.

The power of Green's function in treating interacting theories is presented through a quartic perturbation. We apply the present technique to evaluate the correction to the entanglement entropy due to a quartic type interaction for a system made of a finite number of coupled oscillators. In particular, we derive explicitly the first order correction to the standard entanglement entropy between two coupled harmonic oscillators due to a perturbation of the form $\lambda \sum q_i^4$. We show that to the first order in perturbation theory, the standard gaussian free entanglement entropy formula stays valid for interacting theories with a shifted parameter due to the perturbation of the ground state. Our result systematically confirm earlier conjecture that the entanglement entropy can be written exclusively in terms of the perturbed two-points correlators to the first order in perturbation theory [31] .

In addition to the above quartic interaction, we consider another natural quartic interaction which cannot be seen as arising from a discretization of a non-local term. In contrast to the first interaction, this quartic interaction appears to have an IR divergence that cannot be normalized away.

Finally, we conclude this dissertation with an outlook on the applications of the developed formalism to free and interacting scalar field theories on fuzzy spaces.

Chapter 1

Entanglement Entropy : An Overview

This chapter gives a brief overview of the basic concepts of entanglement and techniques to compute the entanglement entropy. The focus is on the entanglement entropy for quantum fields, in particular scalar theories in connection with black holes.

1.1 Entanglement of a bipartite system

The entanglement between some systems is a manifestation of a previous interaction between them. Because of the interaction, the states of these systems can not be described separately from each other. The entanglement of a two-partite system is particularly easy to recognize. Consider a two-partite system AB in a pure state $|\psi\rangle$. The density operator that describes the composed system AB is :

$$\rho_{AB} = |\psi\rangle \langle\psi| \tag{1.1}$$

Hilbert space for the system AB is of the form $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A(\mathcal{H}_B)$ is Hilbert space for the subsystem $A(B)$. Suppose that an observer can have no access to the observables of the subsystem B. From his point of view, the subsystem A is described by the reduced density operator

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_i {}_B\langle i|\rho_{AB}|i\rangle_B \tag{1.2}$$

$\{|i\rangle_B\}$ is an orthonormal basis for the subspace $\mathcal{H}_B$.

The subsystems A and B are said to be non-entangled if the tensorial product of ρ_A and ρ_B reproduces the total density operator ρ_{AB}, such that

$$\rho_{AB} = \rho_A \otimes \rho_B \tag{1.3}$$

Equivalently, the state $|\psi\rangle$ is said to be separable if the variables of the subsystem A can be separated from the variables of the subsystem B by a simple tensor product $|\psi\rangle = |\phi\rangle_A \otimes |\phi'\rangle_B$. The generalization to a multipartite system is straightforward.

The entanglement of a pure two-partite state can be defined systematically using Schmidt decomposition of the state vector $|\psi\rangle$, which can be uniquely decomposed as

$$|\psi\rangle = \sum_{j=1}^{n} \sqrt{\lambda_j} \, |j\rangle_A \otimes |j\rangle_B \tag{1.4}$$

Where $\{|j\rangle_{A(B)}\}$ is the set of eigenvectors of the reduced density matrix $\rho_{A(B)}$ which corresponds to a non-zero eigenvalues $\{\lambda_j\}$. The state $|\psi\rangle$ is entangled if Schmidt number n is greater than one, otherwise it is separable.
The variables intersection caused by entanglement makes the state of each subsystem dependent on the other, even if there is no current interaction between the two. Stranger still, the state dependence exists even if the two entangled systems are a million light years apart since manipulating one state, by some measurement, will lead to the collapse of the overall state, and the state of the other system will instantly change the moment the mentioned operation is executed, even if no physical action is applied locally on it.

This is the secret magic of entanglement and the seeds of the EPR paradox [1]. The latter attacks this weird non-local effect of entanglement via a gedanken experiment proposed by Einstein, Podolsky and Rosen in 1935. However, in 1964, J. S. Bell suggested an inequality called after his name, 'Bell inequality' [32], which could be tested experimentally to check the validity of entanglement and the general philosophical reasoning of quantum mechanics as proposed by the Copenhagen interpretation. Experimental results were always hand in hand with entanglement (see [2], [3] and [4]) and the question is no longer about its existence or its validity.

The identification of entanglement is important but not enough. There was a natural need for the concept of quantifying entanglement in order to give rise to the applications it sets up. For this purpose, many measures have been proposed and developed [5].

1.1.1 Entanglement quantification : Qubit state

In order to quantify entanglement for a certain state, it is necessary to introduce an entanglement measure. We start by discussing a simple example that intuitively illustrates how entanglement can be quantified.

The intended system is the quantum bit or Qubit. It is the simplest non-trivial quantum system with a Hilbert space of two dimensions only. Using Schmidt decomposition, the general state of two entangled Qubits AB can be written as :

$$|\psi\rangle = \cos(\theta) \, |0\rangle_A \, |0\rangle_B + \sin(\theta) \, |1\rangle_A \, |1\rangle_B \tag{1.5}$$

Where $\theta \in [0, \pi[$.

It is obvious that the state $|\psi\rangle$ is separale for $\theta = 0$ or $\theta = \pi/2$, i.e., it contains zero entanglement. The subsystems A and B are then in a pure state and there is no dependence between the two systems on each other. The state $|\psi\rangle$ becomes entangled for other values of θ and the underlying systems can no longer be described by separate state vectors.
One can depict entanglement as a band that connects the entangled systems. The strength of the band illustrates the amount of entanglement between the two systems. It might be evaluated by how close the state $|\psi\rangle$ is to a separable state or how close the reduced density matrix is to a

projector. This can be better explained as follows; the reduced density matrix for the subsytem A is :

$$\rho_A = \begin{bmatrix} \cos^2(\theta) & 0 \\ 0 & \sin^2(\theta) \end{bmatrix} \tag{1.6}$$

If $\cos^2(\theta) = 1$ or $\sin^2(\theta) = 1$, the system A is not entangled with B, and the reduced density matrix ρ_A becomes a simple projector. Other values of θ offer entanglement. However, the different values of θ give density matrices that are differently close to the non-entangled state, for example:

$$\rho_A = \begin{bmatrix} 0.99 & 0 \\ 0 & 0.01 \end{bmatrix} \tag{1.7}$$

this density matrix is clearly less entangled than

$$\rho_A = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix} \tag{1.8}$$

It is straightforward now to determine the most entangled state. Obviously, it corresponds to the density matrix ρ_A with equal eigenvalues. It is indeed the farthest density matrix, loosely speaking, from any projector on the subspace $\mathcal{H}_A$. In this situation, the entanglement makes the observer of the subsystem A loses any prediction on which state vector he will obtain when he applies a measurement, he has equal probabilities of getting either one of the two state vectors $\{|0\rangle_A, |1\rangle_A\}$. For less entangled states, he always has a bigger probability to get one state vector from the other (see figure 1.1).

However, this is just a loose and non-precise description of the situation. One needs to define a trustworthy measurement, built on firm physical and mathematical ground, which would give a precise numerical value to quantify the amount of entanglement for every given state.

1.1.2 Entanglement entropy

Entanglement entropy is an entanglement measure. It evaluates the amount of entanglement between a two-partite system in a pure overall state. It is indeed one of the widest used measures. Its importance partly comes from the manifestation of this notion, or a closed form of it, in different fields of physics for different purposes, especially : Shannon entropy in quantum information and thermodynamical entropy in statistical mechanics.

Consider a bipartite system AB in a pure state $|\psi\rangle$. The subsystem A is described by the reduced density matrix $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. The amount of entanglement between the subsystems A and B as given by the entanglement entropy is :

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \ln(\rho_A)) \tag{1.9}$$

One can easily show using Schmidt decomposition (1.85) of the state $|\psi\rangle$ that

$$S(\rho_A) = S(\rho_B) \tag{1.10}$$

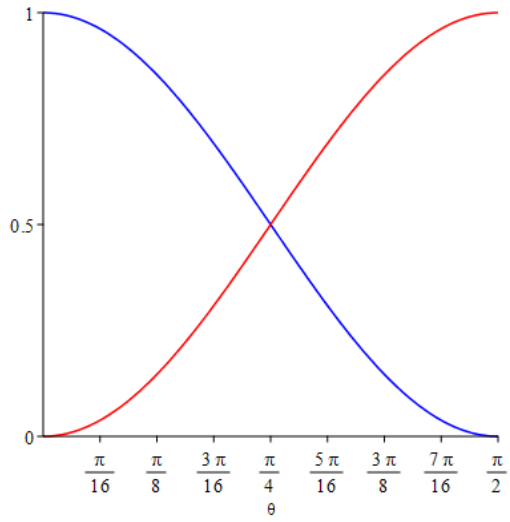

Figure 1.1: the figure represents $(\rho_A(\theta))_{11}$ in red and $(\rho_A(\theta))_{22}$ in blue for $\theta \in [0, \pi/2]$. The argument $\theta = \pi/4$ corresponds to the maximally entangled state.

This is a very important property for this measure; it states that each party is equally entangled to the other party, as it should be, regardless of the way in which the overall system AB is devided into two subsystems A and B, in other words, regardless of the volume difference between A and B.

The definition of entanglement entropy (1.9) is formal, one of the possible ways to evaluate it is by using the spectral decomposition of the reduced density matrix, it is then straightforward to show that

$$S(\rho_A) = -\sum_i \lambda_i \ln(\lambda_i) \tag{1.11}$$

Where $\{\lambda_i\}$ are the eigenvalues of ρ_A and ρ_B.
Let's go back to the two qubits example, it is easy to see that the entanglement entropy vanishes for separable states, as it should be, where $\{\lambda_i = 0, 1\}$. The amount of entanglement of other states can be easily obtained when using the expression (1.6) for ρ_A :

$$S(\rho_A) = -\cos^2(\theta)\ln\big(\cos^2(\theta)\big) - \sin^2(\theta)\ln\big(\sin^2(\theta)\big) \tag{1.12}$$

The entanglement entropy (1.12) is illustrated in figure 1.2.

Historically, within the context of black holes, the entanglement entropy was also called geometric entropy. The latter has been used as a promising candidate to explain the source of black hole entropy [11,12,18]. It has been shown, within a theory of fields, that the entanglement entropy is proportional to the separating area between the two subsystems [28]. This result was related to the entropy of black holes, since this latter has the weird property of being proportional to the area of the black hole horizon, not to its volume. The geometric entropy was named after the action of taking some geometrical region from configuration space and tracing over the field degrees of freedom inside it in order to define a black hole [12].

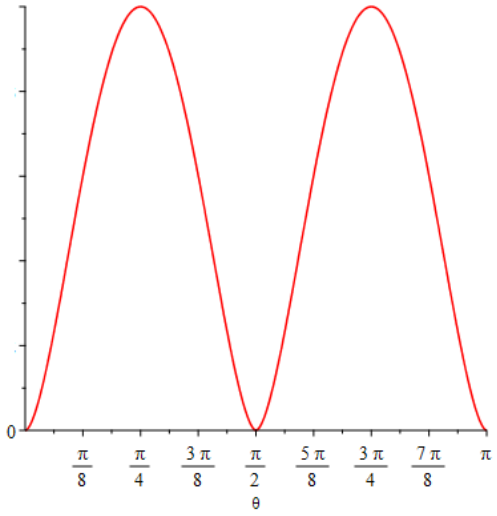

Figure 1.2: the figure represents the entanglement entropy (1.12) as a function of θ for two entangled qubits. The entanglement is zero for separable states corresponding to $\theta = 0, \pi/2$ and maximal for maximally entangled states corresponding to $\theta = \pi/4, 3\pi/4$.

1.2 Real time approach

The real time approach is a formalism to compute the entanglement entropy. It's based on evaluating the reduced density matrix for the vacuum state directly. One generally starts with a discretized version of field theory. Here the approach will be exposed for a number of models and the results will later be compared with the findings of the new approach developed in this dissertation.

1.2.1 Two coupled harmonic oscillator

It turned out that the problem of evaluating the entanglement entropy resulting from tracing out a subset of N-coupled harmonic oscillators can be reduced to the problem of two coupled harmonic oscillators. Therefore, we shall start by working out this case explicitly.

Consider the following Hamiltonian for a system of 2 coupled one-dimensional harmonic oscillators ith equal unit mass :

$$H = \frac{1}{2}\sum_{i,j=1}^{2}(\delta_{ij}\ p_i p_j + V_{ij}\ q_i q_j) \tag{1.13}$$

Where q_i is the coordinate of the i-th oscillator and p_i is its corresponding momentum. The potential matrix V is a real symmetrical positive definite matrix. To simplify the notation, define the vectors $Q = (q_1, q_2)^T$ and $P = (p_1, p_2)^T$. The Hamiltonian then takes the form :

$$H = \frac{1}{2}(P^T P + Q^T V Q) \tag{1.14}$$

The two oscillators are taken to be in their ground state. In q-basis it is given by [33] :

$$\psi_0(q_1,q_2) = (\det\left(\frac{W}{\pi}\right)\ )^{1/4} e^{-\frac{1}{2}Q^t W Q} \tag{1.15}$$

The matrix W is the positive square root of the potential matrix V. If one assumes that the information about the second oscillator are unvailable, the first oscillator is then described by the reduced density matrix :

$$\rho_{red}(q_1,q_1') = \mathrm{Tr}_2(\rho_{tot}) = \int_{-\infty}^{\infty} dq_2\ \psi_0(q_1,q_2).\psi_0^*(q_1',q_2) \tag{1.16}$$

It is not hard to show that

$$\rho_{red}(q_1,q_1') = \left(\frac{\det(W)}{\pi\omega_2}\right)^{1/2} \exp\left(-\frac{1}{2}(\omega_1 - \frac{\omega^2}{2\omega_2})(q_1^2+q_1'^2) + \frac{\omega^2}{2\omega_2}q_1q_1'\right) \tag{1.17}$$

Where the following notation $W_{11}=\omega_1$, $W_{22}=\omega_2$ and $W_{12}=W_{21}=\omega$ is adopted. Since the matrix W is positive, its trace and determinant are also positive; such that $\omega_1.\omega_2 - \omega^2 > 0$ and $\omega_1+\omega_2>0$.

Given that ρ_{red} has a gaussian form, the eigenfunctions are easy to guess; they can only have a gaussian form. The first inspection is to use a reparametrized version of the harmonic oscillator eigenfunctions from quantum mechanics [33], take for example the form :

$$f_n = H_n(\sqrt{\alpha}x)\exp\left(-\frac{\alpha}{2}x^2\right) \tag{1.18}$$

$\{H_n\}$ are Hermite polynomials and α is to be determind. The functions $\{f_n\}$ should satisfy the following relation :

$$\int_{-\infty}^{\infty} dy\ \rho_{red}(x,y)f_n(y) = \lambda_n f_n(x) \tag{1.19}$$

Let $\gamma = \omega_1 - \omega^2/(2\omega_2)$ and $\beta = \omega^2/(2\omega_2)$, the reduced density matrix takes then the form :

$$\rho_{red}(q,q') = \sqrt{\frac{\gamma-\beta}{\pi}}e^{-\frac{1}{2}\gamma(q^2+q'^2)+\beta qq'} \tag{1.20}$$

It follows that :

$$\begin{aligned}\int_{-\infty}^{\infty} dy\ \rho_{red}(x,y)f_n(y) &= \sqrt{\frac{\gamma-\beta}{\pi}}\int_{-\infty}^{\infty} dy\ e^{-\frac{\gamma}{2}(x^2+y^2)+\beta x.y}H_n(\sqrt{\alpha}y)e^{-\frac{\alpha}{2}y^2} \\ &= \sqrt{\frac{2(\gamma-\beta)}{\pi(\gamma+\alpha)}}e^{(\frac{-\gamma}{2}+\frac{\beta^2}{2(\alpha+\gamma)})x^2}\int_{-\infty}^{\infty} dy'\ e^{-(y'-x')^2}H_n(cy')\end{aligned} \tag{1.21}$$

Where $x' = \sqrt{\frac{\beta^2}{2(\gamma+\alpha)}}x$, $y' = \sqrt{\frac{\gamma+\alpha}{2}}y$ and $c = \sqrt{\frac{2\alpha}{\alpha+\gamma}}$. The previous integral can be evaluated using the standard formula [34] :

$$\int_{-\infty}^{\infty} dy\ \exp\big(-(x-y)^2\big) H_n(cy) = \sqrt{\pi}(1-c^2)^{n/2} H_n[\frac{cx}{(1-c^2)^{1/2}}] \tag{1.22}$$

So, one gets :

$$\int_{-\infty}^{\infty} dy\ \rho_{red}(x,y) f_n(y) = \sqrt{\frac{2(\gamma-\beta)}{(\gamma+\alpha)}}(1-c^2)^{n/2} e^{(\frac{-\gamma}{2}+\frac{\beta^2}{2(\alpha+\gamma)})x^2} H_n[\frac{cx'}{(1-c^2)^{1/2}}] \tag{1.23}$$

In order to prove that the functions $\{f_n\}$ are indeed the eigenfunctions of ρ_{red}; two conditions must be fulfilled simultaneously :

$$\begin{aligned} -\alpha &= -\gamma + \frac{\beta^2}{\alpha+\gamma} \\ \sqrt{\alpha} &= \frac{c}{(1-c^2)^{1/2}} \frac{\beta}{\sqrt{2(\alpha+\gamma)}} \end{aligned}$$

It is easy to check that these two equations have the same solution with respect to α, which is the following

$$\alpha = \sqrt{\gamma^2 - \beta^2} \tag{1.24}$$

Henceforth it is possible to deduce the eigenvalues of ρ_{red} from (1.23) as follows :

$$\lambda_n = \sqrt{\frac{2(\gamma-\beta)}{\gamma+\alpha}}(1-c^2)^{n/2}, \qquad n = 0..\infty \tag{1.25}$$

With a little manipulation, the eigenvalues can be rewritten in the form

$$\lambda_n = (1-\xi)\xi^n, \qquad \xi = \beta/(\gamma+\alpha) \tag{1.26}$$

It is easy to show that $\sum \lambda_n = 1$, therefore the set of eigenfunctions (1.18) is complete. As a result, the entanglement entropy can be evaluated in straightforward manner

$$S = -\sum_{n=0}^{\infty} \lambda_n \ln(\lambda_n) = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln(\xi) \tag{1.27}$$

This is the entanglement entropy for two coupled harmonic oscillators [18].

1.2.2 N-coupled harmonic oscillators

We now turn to the problem of N-coupled harmonic oscillators. It will be shown that the solution of this problem comes from a simple generalization of the problem of two coupled harmonic oscillators treated earlier. The Hamiltonian of the system is given by

$$H = \frac{1}{2}(\mathbf{P}^T\mathbf{P} + \mathbf{Q}^T V \mathbf{Q}) \tag{1.28}$$

V is an $N \times N$ real symmetric positive definite matrix. $\mathbf{Q}$ and $\mathbf{P}$ are N-component coulomn vectors; such that $\mathbf{Q} = (q_1, q_2, .., q_N)^T$ and $\mathbf{P} = (p_1, p_2, .., p_N)^T$. The ground state of the system is given by a simple generalization of the ground state of 2-coupled harmonic oscillators :

$$\psi_0(q_1, q_2, ..., q_N) = \left(\det\left(\frac{W}{\pi}\right)\right)^{1/4} e^{-\frac{1}{2}\mathbf{Q}^T W \mathbf{Q}} \tag{1.29}$$

The matrix W is the positive square root of the matrix V. The density matrix of the total system can be easily derived as follows :

$$\rho(\mathbf{Q}, \mathbf{Q'}) = (\det(\frac{\mathrm{W}}{\pi}))^{1/2} \mathrm{e}^{-\frac{1}{2}(\mathbf{Q}^{\mathrm{T}}\mathrm{W}\mathbf{Q} + \mathbf{Q'}^{\mathrm{T}}\mathrm{W}.\mathbf{Q'})} \tag{1.30}$$

The reduced density matrix that describes the first M oscillators is found by tracing the density matrix (1.30) over the variables of the last $N - M$ oscillators $\{q_i, i = M + 1..N\}$, the information on these oscillatos are considered unavailable.

$$\rho_{red}(Q, Q') = \int_{-\infty}^{\infty} \prod_{i=M+1}^{N} dq_i \ \rho(q_1..q_M, q_{M+1}..q_N;\ q'_1..q'_M, q_{M+1}..q_N) \tag{1.31}$$

Where Q is the M-component vector $Q = (q_1, q_2, .., q_M)^T$; where $\mathbf{Q} = (Q, \bar{Q})^T$ and $\bar{Q} = (q_{M+1}, q_{M+2}, .., q_N)^T$. In order to evaluate the previous integral, one uses the following decomposition for the matrix W

$$W = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{1.32}$$

Where A, B and C are $M \times M$, $(N-M) \times M$ and $(N-M) \times (N-M)$ matrices respectively. It follows that

$$\rho_{red}(Q, Q') = (\det\left(\frac{W}{\pi}\right))^{1/2} e^{-\frac{1}{2}(Q^T A Q + Q'^T A Q')} \int \prod_{m=M+1}^{N} dq_m \ e^{-(\bar{Q}^T C \bar{Q} \ + \ \bar{Q}^T B^T (Q+Q'))} \tag{1.33}$$

The normalization of ρ_{red} is going to be ignored from now on, it will be implemented in a factor N_c. This factor is going to change from one operation to another. However, the same symbol will be kept. The reasoning here is that ρ_{red} can always be renormalized to 1 to find the proper 'N_c', therefore this factor does not affect the derivation of the eigenvalues and thus the entanglement entropy.

The matrix C is a real positive symmetric matrix, so it is possible to diagonalize it through an orthogonal transformation; such that $C = U^T C_d U$ where the matrix C_d is diagonal and $U^T U = I$. Put $J = B^T.(Q + Q')$, it follows that :

$$\begin{aligned}
\int \prod_{m=M+1}^{N} dq_m e^{-(\bar{Q}C\bar{Q}+\bar{Q}^T J)} &= \int \prod_{m=M+1}^{N} dq_m e^{-(\bar{\bar{Q}}^T C_d \bar{\bar{Q}}+\bar{\bar{Q}}^T \bar{J})} \\
&= \prod_{m=M+1}^{N} \int_{-\infty}^{\infty} dq_m e^{-(\ (C_d)_m \bar{\bar{q}}_m^2 + \bar{\bar{q}}_m \bar{J}_m\ )} \\
&= \prod_{m=M+1}^{N} e^{\frac{\bar{J}^2}{4(C_d)_m}} \sqrt{\frac{\pi}{(C_d)_m}} \\
&= e^{J^T C^{-1} J/4} \frac{\pi^{\frac{N-M}{2}}}{det(C^{1/2})}
\end{aligned} \tag{1.34}$$

Where $\bar{\bar{Q}} = UQ$ and $\bar{J} = UJ$. The reduced density matrix ρ_{red} then becomes

$$\rho_{red}(Q, Q') = N_c e^{-\frac{1}{2}(Q^T A Q + Q'^T A Q' - \frac{1}{2} J^T C^{-1} J)} \tag{1.35}$$

The aim is to write ρ_{red} as a function product, each function is of the form of the reduced density matrix of two coupled harmonic oscillators (1.20). This can be done by repetitively changing the variables in order to exploit the diagonal form of the contributing matrices, so one can finally achieve the required factorization.

Let $J^T C^{-1} J = (Q^T + Q'^T) M_1 (Q + Q')$ where $M_1 = B C^{-1} B^T$, it follows that :

$$\rho_{red}(Q, Q') = N_c e^{-\frac{1}{2}(\ Q^T (A - \frac{M_1}{2}) Q + Q'^T (A - \frac{M_1}{2}) Q' - \frac{1}{2}(Q^T M_1 Q' + Q'^T M_1 Q)\ )} \tag{1.36}$$

Let $M_2 = A - M_1/2$. M_2 is a real symmetric matrix that can be diagonalized using the following orthogonal transformation :

$$M_2 = U_2^T (M_2)_d U_2 \qquad\qquad U_2^T U_2 = I$$

Let $M_3 = U_2 M_1 U_2^T$ and $\tilde{Q} = U_2 Q$, it is straightforward to show that :

$$\begin{aligned}
\rho_{red}(Q, Q') &= N_c e^{-\frac{1}{2}(\ \tilde{Q}^T (M_2)_d \tilde{Q} + \tilde{Q}'^T (M_2)_d . \tilde{Q}' - \frac{1}{2}(\tilde{Q}^T M_3 \tilde{Q}' + \tilde{Q}'^T M_3 \tilde{Q})\ )} \\
&= N_c e^{-\frac{1}{2}(\ \hat{Q}^T \hat{Q} + \hat{Q}' \hat{Q} - \frac{1}{2}(\hat{Q}' M_4 \hat{Q} + \hat{Q} M_4 \hat{Q}')\ )}
\end{aligned} \tag{1.37}$$

Where $M_4 = (M_2)_d^{-1/2} M_3 (M_2)_d^{-1/2}$ and $\hat{Q} = (M_2)_d^{1/2} \tilde{Q}$ are defined. It was assumed here that M_2 is invertible, which will be proved later.

Finally, one gets :

$$\begin{aligned}\rho_{red}(Q,Q') &= N_c.e^{-\frac{1}{2}(\breve{Q}^T\breve{Q}+\breve{Q}'^T\breve{Q}-\frac{1}{2}(\breve{Q}'^T(M_4)_d\breve{Q}+\breve{Q}^T(M_4)_d\breve{Q}'))} \\ &= N_c\prod_{m=1}^{M} e^{-\frac{1}{2}(\breve{q}_m^2+\breve{q}_m'^2-(M_{4d})_m.\breve{q}_m\breve{q}_m')} \end{aligned} \tag{1.38}$$

where $M_4 = U_4^T(M_4)_dU_4$ and $\breve{Q} = U_4\hat{Q}$. This is the desirable result! The density matrix $\rho_{red}(Q,Q')$ is factorized into a functions product, each of which takes the form of the reduced density matrix (1.20) of two coupled harmonic oscillators. The entanglement entropy can be formally obtained from (1.27).

However, it is required here to determine the eigenvalues of the matrix M_4 in order to be able to evaluate the entanglement entropy by means of (1.27). This turns out to be untractable task since the obtained form of the matrix M_4 is really complicated; it can be shown that M_4 shares the same eigenvalues with the matrix $M_1.M_2^{-1}$ which has the perplexing form $M_1.M_2^{-1} = BC^{-1}B^T(A - BC^{-1}B^T/2)^{-1}$.

To partially overcome this problem, one could rewrite the reduced density matrix of two coupled oscillators (1.20) with a little twist in order to change the matrix that should be diagonalized. The density matrix (1.20) can be rewritten as :

$$\rho_{red}(q,q') = \sqrt{\frac{\gamma-\beta}{\pi}}e^{-\frac{1}{2}((\gamma-\beta)(q^2+q'^2)+\beta(q-q')^2)} \tag{1.39}$$

Using the same tricks and tools used before, the reduced density matrix (1.36) is factorized into a function product of the same form of (1.39). The details are resumed as follows

$$\begin{aligned}\rho_{red}(Q,Q') &= N_ce^{-\frac{1}{2}(Q^T(A-\frac{M_1}{2})Q+Q'^T(A-\frac{M_1}{2})Q'-\frac{1}{2}(Q^TM_1Q'+Q'^TM_1Q))} \\ &= N_ce^{-\frac{1}{2}(Q^TM_2'Q+Q'^TM_2'Q'+\frac{1}{2}(Q-Q')^TM_1(Q-Q'))} \\ &= N_ce^{-\frac{1}{2}(\tilde{\mathcal{Q}}^T\tilde{\mathcal{Q}}+\tilde{\mathcal{Q}}'^T\tilde{\mathcal{Q}}'+\frac{1}{2}(\tilde{\mathcal{Q}}-\hat{\mathcal{Q}}')M_3'(\tilde{\mathcal{Q}}-\tilde{\mathcal{Q}}'))} \\ &= N_ce^{-\frac{1}{2}(\hat{\mathcal{Q}}^T\hat{\mathcal{Q}}+\hat{\mathcal{Q}}'^T\hat{\mathcal{Q}}'+\frac{1}{2}(\hat{\mathcal{Q}}-\hat{\mathcal{Q}}')(M_3')_d(\hat{\mathcal{Q}}-\hat{\mathcal{Q}}'))} \\ &= N_c\prod_{m=1}^{M} e^{-\frac{1}{2}(\hat{\mathfrak{q}}_m^2+\hat{\mathfrak{q}}_m'^2+\frac{1}{2}(M_{3d}')_m(\hat{\mathfrak{q}}_m-\hat{\mathfrak{q}}_m')^2)} \end{aligned} \tag{1.40}$$

The unknown matrices and vectors are listed below :

$$\begin{aligned} M_1 &= B.C^{-1}B^T \\ M_2' &= A - M_1 = A - BC^{-1}B^T = U_2'^T(M_2')_dU_2' \\ M_3' &= (M_2')_d^{-1/2}U_2'M_1U_2'^T(M_2')_d^{-1/2} = U_3'^T(M_3)_dU_3' \\ \tilde{\mathcal{Q}} &= (M_2')_dU_2'Q \\ \hat{\mathcal{Q}} &= U_3'\tilde{\mathcal{Q}} \end{aligned}$$

The obtained factorization (1.40) is useful if the eigenvalues of the matrix M_3' can be determined. It can be easily shown that M_3' has common eigenvalues with the matrix $M_1 M_2'^{-1}$ as follows

$$\begin{aligned}
\det(M_3' - \lambda I) &= \det\Big((M_2')_d^{-1/2} U_2' M_1 U_2'^T (M_2')_d^{-1/2} - \lambda I\Big) \\
&= \det\Big((M_2')_d^{1/2}[(M_2')_d^{-1/2} U_2' M_1 U_2'^T (M_2')_d^{-1/2} - \lambda I](M_2')_d^{-1/2}\Big) \\
&= \det\big(U_2' M_1 U_2'^T (M_2')_d^{-1} - \lambda I\big) \\
&= \det\big(M_1 M_2'^{-1} - \lambda I\big) \qquad (1.41)
\end{aligned}$$

The matrix $M_1 M_2'^{-1}$ can be rewritten in a more intuitive way. Actually, if the matrix W^{-1} is decomposed into a block sub-matrices in a similar manner to the matrix W :

$$W^{-1} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B}^t & \bar{C} \end{bmatrix} \qquad (1.42)$$

Where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are $M \times M$, $M \times (N-M)$ and $(N-M) \times (N-M)$ matrices respectively. Then it is not diffucult to show that

$$M_1 M_2'^{-1} = -B\bar{B}^T \qquad (1.43)$$

The key idea is to use the identity $W^{-1}.W = I$, from which two relations are needed

$$\bar{A}A + \bar{B}B^T = I_M \qquad (1.44)$$

$$\bar{A}B + \bar{B}C = 0 \qquad (1.45)$$

Solving these two equations with respect to $\bar{A}$, one finds that

$$\bar{A}(A - BC^{-1}B^T) = I_M \qquad (1.46)$$

Therefore

$$M_2' = \bar{A}^{-1} = A - BC^{-1}B^T \qquad (1.47)$$

This is an implemented argument that M_2' is a positive definite and thus invertible matrix[1]. It follows from (1.45) that $\bar{B}^T = -C^{-1}B^T\bar{A}$, hence one gets to the intended form

$$M_1 M_2'^{-1} = BC^{-1}B^T\bar{A} = -B\bar{B}^T \qquad (1.48)$$

The matrix $-B\bar{B}^T$ is mathematically a lot simpler and physically better and more intuitive. The form of the matrix depends directly on the set of the traced oscillators. Also, it has some form of discrete convolution :

$$(M_1 M_2'^{-1})_{ij} = \sum_{a=M+1}^{N} -W_{ia}(W^{-1})_{ja} \qquad i,j = (1,M) \qquad (1.49)$$

[1]Since M_2' is positive definite, M_2 is a priori positive definite and therefore invertible matrix, as it is the sum of two positive definite matrices (M_2' and $BC^{-1}B^T/2$).

Which could be a very simplifying factor when taking the continuous limit.
Even after these simplifications, there is no guarantee that we will find the eigenvalues analytically. This depends directly on the form of the potential matrix V. As a matter of fact, the eigenvalues are found numerically for non-trivial physical models. Once they are determined, the entanglement entropy can be calculated directly by a straightforward generalization of the entanglement entropy of two coupled oscillators (1.27).
The density matrix (1.40) can be reparametrized in the form :

$$\rho(q_1..q_M) = \prod_{m=1}^{M} \rho_m(\mathfrak{q}_m) \tag{1.50}$$

Such that $\mathrm{Tr}(\rho_m) = 1$. It follows that

$$\begin{aligned} S &= -\mathrm{Tr}(\rho.\ln(\rho)\ ) = -\mathrm{Tr}\left(\prod_{l=1}^{M}\rho_l(\mathfrak{q}_l).\sum_{m=1}^{M}\ln(\rho_m(\mathfrak{q}_m)\ )\right) \\ &= -\sum_{m=1}^{M}\mathrm{Tr}(\ \rho_m(\mathfrak{q}_m).\ln(\rho_m(\mathfrak{q}_m))\ ) \\ &= \sum_{m=1}^{M} S_m \end{aligned} \tag{1.51}$$

The entanglement entropy is therefore the sum of the entanglement entropies of two coupled oscillators (1.27), where $\xi_m = \lambda_m(1+\sqrt{1+\lambda_m})^{-2}$. It then follows that

$$S = \sum_{m=1}^{M} -\ln\left(\frac{2}{1+\sqrt{1+\lambda_m}}\right) - \frac{\lambda_m}{2(1+\sqrt{1+\lambda_m})}.\ln\left(\frac{\lambda_m}{(1+\sqrt{1+\lambda_m})^2}\right) \tag{1.52}$$

Where $\{\lambda_m\}$ are the eigenvalues of the matrix $M_1 {M'_2}^{-1} = -B\bar{B}^T$.

1.2.3 Entanglement entropy of N-coupled harmonic oscillators through correlators

For a lattice of harmonic oscillators, there is another method to determine the entanglement entropy in real time approach. This method depends on a physically more intuitive tool, namely the two point function or the correlator.

The Hamiltonian of the lattice is defined by (1.38). As before, the accessed system is formed of the first M oscillators. For the ground state, one can define the functions :

$$X_{ij} = \langle q_i q_j\rangle, \quad P_{ij} = \langle p_i p_j\rangle \qquad\qquad i,j = 1..M \tag{1.53}$$

$\{q_i\}$ are the canonical coordinates and $\{p_i\}$ are their corresponding momenta. The $M \times M$ matrices X and P are obviously real and symmetric. Higher order correlators can be found using Wick's theorem [15, 24]:

$$\langle \mathcal{O} f_1.f_2...f_{2k}\rangle = \frac{1}{2^k k!}\sum_{\{\sigma\}}\langle \mathcal{O} f_{\sigma(1)}.f_{\sigma(2)}\rangle.\langle \mathcal{O} f_{\sigma(3)}.f_{\sigma(4)}\rangle...\langle \mathcal{O} f_{\sigma(2k-1)}.f_{\sigma(2k)}\rangle \tag{1.54}$$

Where $\mathcal{O}$ is an ordering operator, 'f' is either the momentum or the coordinate operator and $\{\sigma\}$ are all possible ordering permutations. It is useful to keep in mind that $\{q_i, p_j\}$ can be written using the creation and annihilation operators $\{c_i, c_i^\dagger\}$

$$q_i \propto c_i + c_i^\dagger, \quad p_i \propto c_i - c_i^\dagger \tag{1.55}$$

The correlators X_{ij} and P_{ij} are defined for the reduced system by means of the 'overall' pure ground state :

$$X_{ij} = \langle \Psi_0 \mid q_i q_j \mid \Psi_0 \rangle; \;\; P_{ij} = \langle \Psi_0 \mid p_i p_j \mid \Psi_0 \rangle \tag{1.56}$$

Where $\psi_0(q_1, q_2, ..., q_N) = \langle q_1, q_2, ..., q_N \mid \Psi_0 \rangle$ is the ground state defined by (1.29). However, it is completely valid to write these correlators using the reduced density matrix ρ_{red} of the reduced system :

$$X_{ij} = Tr(\rho_{red}\; q_i q_j); \;\; P_{ij} = Tr(\rho_{red}\; p_i p_j) \tag{1.57}$$

Therefore, Wick's theorem can be exclusively written using the reduced density operator ρ_{red}. Since not any form of functions admits such a property, one suspects that ρ_{red} is a function of the creation and annihilation operators $\{c_i, c_i^\dagger\}$, more precisely, a Gaussian form of them [24] :

$$\rho_{red} = \mathcal{K} \exp\left(- \sum_{ij} H_{ij} c_i^\dagger c_j \right) \tag{1.58}$$

where $\mathcal{K}$ is the normalization constant and H_{ij} is the matrix element of the matrix H, and it is to be determined. Since ρ_{red} is Hermitian (the subscript of ρ_{red} will be omitted from now on), this means that H is also Hermitian. The matrix H is then diagonalizable and its eigenvalues $\{\varepsilon_i\}$ are real :

$$\rho = \mathcal{K} \exp\left(- \sum_{i=1}^{M} \varepsilon_i a_i^\dagger a_i \right) \tag{1.59}$$

where

$$c_i = \sum_k \eta_k(i) a_k \tag{1.60}$$

$\{\eta(i)\}$ are the eigenvectors of the matrix H. It is not hard to show that $\{a_i, a_i^\dagger\}$ is a set of creation and annihilation operators. Thanks to the orthonormality property of the eigenvectors $\{\eta(i)\}$, the operators $\{a_i, a_i^\dagger\}$ fulfill the property $[a_i, a_j^\dagger] = \delta_{ij}$. All that is left now is to determine the eigenvalues $\{\varepsilon_i\}$ in terms of known quantities, in order to identify ρ and evaluate the entropy.

First, it is preferable to write $\{q_i\}$ and $\{p_i\}$ in terms of $\{a_i, a_i^\dagger\}$ directly. From (1.55) and (1.60) one can write :

$$\begin{aligned} q_i &= \sum_j \alpha_{ij} (a_j + a_j^\dagger) \\ p_i &= \sum_j i\beta_{ij} (a_j - a_j^\dagger) \end{aligned} \tag{1.61}$$

where α and β are the specific transformation matrices to the specific set of creation and annihilation operators $\{a_i, a_i^\dagger\}$. The general condition on α and β is derived from the commutation relation $[q_i, p_j] = i\delta_{ij}$ which is

$$\alpha.\beta^T = -1/2$$

The normalization constant '$\mathcal{K}$' can be easily computed as follows

$$\begin{aligned}
\mathcal{K}^{-1} &= \mathrm{Tr}\left(\exp\left(-\sum_{i=1}^{M}\varepsilon_i a_i^\dagger a_i\right)\right) = \prod_{i=1}^{M}\mathrm{Tr}\Big(\exp\left(-\varepsilon_i a_i^\dagger a_i\right)\Big) \\
&= \prod_{i=1}^{M}\sum_{m=0}^{\infty}\frac{(-\varepsilon_i)^m}{m!}\sum_{u_i=0}^{\infty}\langle u_i \mid (a_i^\dagger a_i)^m \mid u_i\rangle = \prod_{i=1}^{M}\sum_{m,u_i=0}^{\infty}\frac{(-\varepsilon_i)^m}{m!}u_i^m \\
&= \prod_{i=1}^{M}\sum_{u_i=0}^{\infty}\exp(-\varepsilon_i.u_i) = \prod_{i=1}^{M}\frac{1}{1-e^{-\varepsilon_i}} \qquad (1.62)
\end{aligned}$$

It was taken that $\{\varepsilon_i\}$ is positive, the necessary condition to have a proper density matrix. By the same previous way, one can compute the following

$$n_{ij} \equiv \langle a_i^\dagger.a_j\rangle = tr(\rho a_i^\dagger.a_j) = \frac{e^{-\varepsilon_i}}{1-e^{-\varepsilon_i}}.\delta_{ij} \qquad (1.63)$$

$$\langle a_i.a_j^\dagger\rangle = 1 + \langle a_i^\dagger.a_j\rangle = \frac{1}{1-e^{-\varepsilon_i}}.\delta_{ij} \qquad (1.64)$$

Using these two equations and the property $\alpha.\beta^T = -1/2$, one can easily find that

$$X = \alpha(2n+1)\alpha^T \qquad (1.65)$$

$$P = \beta(2n+1)\beta^T \qquad (1.66)$$

$$XP = \frac{1}{4}\alpha(2n+1)^2\alpha^{-1} \qquad (1.67)$$

Where the matrix 'n' is defined by (1.63).

Finally, by means of (1.67), it is straightforward to write $\{\varepsilon_i\}$ in terms of the eigenvalues of the matrix $C \equiv \sqrt{XP}$, call them $\{\nu_i\}$, such that

$$\nu_i = \frac{1}{2}\coth(\varepsilon_i/2) \qquad (1.68)$$

It is time now to evaluate the entanglement entropy, Using the form (1.59) for the density matrix, it is straightforward to show that

$$\begin{aligned}
S &= -\mathrm{Tr}(\rho.\ln(\rho)) = -\mathrm{Tr}\left(\rho.\left(\ln\mathcal{K} - \sum_{i=1}^{M}\varepsilon_i.a_i^\dagger.a_i\right)\right) = -\ln\mathcal{K} + \sum_{i=1}^{M}\varepsilon_i.n_{ii} \\
&= \sum_{i=1}^{M}\left[-\ln\big(1-e^{-\varepsilon_i}\big) + \frac{\varepsilon_i.e^{-\varepsilon_i}}{1-e^{-\varepsilon_i}}\right] \qquad (1.69)
\end{aligned}$$

Which could be written in a more compact form using matrix notation as follows :

$$S = \mathrm{Tr}\left((C+1/2)\ln(C+1/2) - (C-1/2)\ln(C-1/2)\right) \tag{1.70}$$

Where the identity $\exp(\varepsilon_i) = (\nu_i + 1/2)/(\nu_i - 1/2)$, derived from (1.68), has been used.

For a lattice of oscillators, the two point functions evaluated with respect to the ground state (1.29) are the following [35] :

$$\begin{aligned} X_{ij} &= \frac{1}{2}(V^{-1/2})_{ij} \equiv \frac{1}{2}(W^{-1})_{ij} \\ P_{ij} &= \frac{1}{2}(V^{1/2})_{ij} \equiv \frac{1}{2}W_{ij} \end{aligned} \tag{1.71}$$

where the indices $\{i, j\}$ refer only to the oscillators of the reduced system, such that $(i, j = 1 : M)$. As a result, using the previous section notation, the determination of the entanglement entropy requires the determination of the eigenvalues of the matrix $C = \left(\bar{A}.A\right)^{1/2}/2$. The equivalence between the current entropy and the one obtained by the previous method (1.52) is all clear and obvious when knowing that

$$\bar{A}.A = I_M - \bar{B}B^T \tag{1.72}$$

Which is deduced from (1.44). The matrix $\bar{A}.A$ has common eigenvalues with the matrix $A\bar{A} = I_M - B\bar{B}$. Finally, the form (1.52) of the entanglement entropy can be found in a straightforward manner.

1.2.4 Free massive scalar field

When adopting a theory of fields and trying to quantify the amount of entanglement between two regions of space, if the total state is not separable, then the entanglement entropy is indeed infinity! One of the ways to explain this is the following (more discussion can be found in [11]); if one assumes that each point of the field incorporates into the entanglement entropy by some amount, then the total entropy will be infinity since there are an infinity of points.

A possible way to lift this infinity is to impose a distance cutoff, i.e., to lift the continuity of the space and to take the fields to be at discrete permitted positions in space. These points form a lattice. The spacing ϵ between two successive vertices of the lattice is called the ultraviolet cutoff. This cutoff is taken to be very small, of the order of Planck's distance. This discretization of space to lift the infinity of the entanglement entropy is similar to the divergence lifting of the entropy in statistical mechanics, where the phase space was taken to be formed of cells based on the Heisenberg uncertainty principle, in order to have a finite entropy for the system of gas.

To avoid the infinite sum over the lattice vertices, it is also required to put another cutoff on the macroscopic scale, i.e., to put the lattice in a box of length L. This latter is called the infrared cutoff.

It is worth to mention here that this is not the only existing way to avoid the divergence of the entropy (again, see [11]). Another regularization can be imposed on the energy of the field modes, limiting the possible energy range by two cutoffs : ultraviolet cutoff ϵ^{-1} and infrared cutoff L^{-1}.

Let's take an example. The aim here is to illustrate the field regularization into a spatial lattice. It will show up that the bosonic field problem will be reduced to the problem of N-coupled harmonic oscillators discussed earlier. Hence, it will be possible to use the result of previous section directly.

A real massive scalar field has the lagrangian density :

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - \mid \vec{\nabla}\phi \mid^2 - \mu^2\phi^2) \tag{1.73}$$

The entanglement is going to be evaluated between a sphere and the rest of space. To exploit the spherical symmetry of the problem, the field will be expanded on the spherical harmonics basis :

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm}(r,t).Y_l^m(\theta,\varphi) \tag{1.74}$$

The field ϕ itself is real but its components $\{\phi_{lm}\}$ are not, this is due to the non reality of the spherical harmonics. Actually, the property $(Y_l^m)^* = (-1)^m Y_l^{-m}$ of spherical harmonics means that

$$(\phi_l^m)^* = (-1)^m \phi_l^{-m} \tag{1.75}$$

In order to evaluate the field Lagrangian, the orthogonality property of spherical harmonics will be used, where

$$\int d\Omega \; Y_l^m Y_{l'}^{m'} = \delta_{ll'}\delta_{mm'}$$

It follows that

$$L = \frac{1}{2}\sum_{l,m} \int r^2 dr \; \left(\dot{\phi}_{lm}.\dot{\phi}_{lm}^* + \frac{d\phi_{lm}}{dr}.\frac{d\phi_{lm}^*}{dr} + l(l+1)\frac{\phi_{lm}}{r}.\frac{\phi_{lm}^*}{r} - \mu^2\phi_{lm}.\phi_{lm}^* \right) \tag{1.76}$$

Now define the field components $\{\Phi_{lm}\}$ such that

$$\begin{cases} \Phi_{lm} &= r.\sqrt{2}\,\mathrm{Re}(\phi_{lm}) & m > 0 \\ \Phi_{lm} &= r.\phi_{lm} & m = 0 \\ \Phi_{lm} &= r.\sqrt{2}\,\mathrm{Im}(\phi_{lm}) & m < 0 \end{cases} \tag{1.77}$$

$\{\Phi_{lm}\}$ are real, they were chosen in this manner to avoid dealing with the complex components $\{\phi_{lm}\}$. By means of the property (1.75), it is not hard to show that this choice fulfills

$$\sum_{l,m} \phi_{lm}\phi_{lm}^* = \sum_{l,m} (\mathrm{Re}(\phi_{lm}))^2 + (\mathrm{Im}(\phi_{lm}))^2 = \sum_{lm} \frac{1}{r^2}\Phi_{lm}^2 \tag{1.78}$$

$\sum \partial_r\phi_{lm}\partial_r\phi_{lm}^*$ and $\sum \dot{\phi_{lm}}\dot{\phi}_{lm}^*$ can be treated in the same way, so one finally gets the following

$$L = \frac{1}{2}\int dr \sum_{lm} \left(\dot{\Phi}_{lm}^2 + r^2 \left(\frac{d}{dr}\frac{\Phi_{lm}}{r} \right)^2 + l(l+1)\frac{\Phi_{lm}^2}{r^2} - \mu^2\Phi_{lm}^2 \right) \tag{1.79}$$

It is time now to project the field into a spatial lattice. The discretization will be performed over the radial coordinate r because the Lagrangian is solely dependent on it. The radial spacing is ϵ which also represent the UV cutoff. The permitted sites in space are those which have radial coordinate of the form $\{r_j = j\epsilon\}$ where j is a positive integer. The system is assumed to be in a spherical box of radius L, where $\mathrm{L} = N\epsilon$ represents the infrared cutoff. It follows that

$$\begin{aligned} L &\equiv \frac{1}{2}\sum_{l,m}\sum_{j=1}^{N}\epsilon\left(\dot{\Phi}^2_{lmj} + (\frac{r_j + r_{j+1}}{2})^2\frac{1}{\epsilon^2}(\frac{\Phi_{lm,j+1}}{r_{j+1}} - \frac{\Phi_{lm,j}}{r_j})^2 + l(l+1)\frac{\Phi^2_{lm,j}}{r_j^2} - \mu^2\Phi^2_{lmj}\right) \\ &= \frac{1}{2}\sum_{lm}\sum_{j=1}^{N}\epsilon\left(\dot{\Phi}^2_{lmj} + \frac{1}{\epsilon^2}(j+\frac{1}{2})^2(\frac{\Phi_{lm,j+1}}{j+1} - \frac{\Phi_{lm,j}}{j})^2 + l(l+1)\frac{1}{\epsilon^2}\frac{\Phi^2_{lm,j}}{j^2} - \mu^2\Phi^2_{lmj}\right) \end{aligned} \quad (1.80)$$

Where $\Phi_{lmj} \equiv \Phi_{lm}(r_j)$. Obviously, the previous equation is the Lagrangian of N-coupled harmonic oscillators ! it just needs a simple rescaling to absorb the factor ϵ. Thus, it is justified to use the results of previous section to calculate the entanglement entropy of real scalar field. The potential matrix is block diagonal, its sub-matrices are defined as follows :

$$V^{(k)}_{ij} = \begin{cases} (2j^2 + 1/2 + l(l+1))/(j^2\epsilon^2) - \mu^2 & i = j \neq 1 \\ 9/(4\epsilon^2) + l(l+1)/\epsilon^2 - \mu^2 & i = j = 1 \\ -(j+1/2)^2/\left(\epsilon^2 j(j+1)\right) & i = j+1 \text{ or } i = j-1 \\ 0 & \text{else} \end{cases} \quad (1.81)$$

Where k is the coordinate of the block sub-matrix $V^{(k)}$ in the potential matrix V. It corresponds to a particular value of the doublet $\{l, m\}$. As a matter of fact, it is not possible to treat the matrix V directly. It is an infinite size matrix ($l \to \infty$)! Fortunately, it is possible to prove that a bounded form of V with $l < l_b$ is equivalent to the infinite size potential matrix (see [36], [18] and [22]). Finally, it is possible to get a formal expression of the entanglement entropy using (1.52). A number of works estimate the entropy numerically by this way within the context of black holes, since it is impossible to evaluate the eigenvalues $\{\lambda_k\}$ of the matrix $-B\bar{B}^T$ analytically (see [18], [22], [37] and [48]).

1.3 Euclidean time approach

In real time approach, it is nearly impossible to extract any analytic results for non trivial physical systems. However, the techniques developed therein are very suitable for numerical computation once the underlying space is discretized into a lattice. Within the context of black holes, new techniques have been developed based on the path integral formulation. The developed formalism, the Euclidean time approach, proved to be more powerful in performing analytical computation in quantum field theory and extracting the UV divergence. The present section will outline the existing techniques in this approach.

1.3.1 State construction in path integral formulation

The main tool in path integral formulation is the kernel of the transition amplitude. The latter gives the probability amplitude for the system to evolve from a given point in space to another point within a given interval of time, written as a functional integral over all possible paths in configuration space :

$$\langle \vec{x_2}, t_2 \mid \vec{x_1}, t_1 \rangle = \int_{\vec{x}(t_1)=\vec{x_1}}^{\vec{x}(t_2)=\vec{x_2}} \mathcal{D}(\vec{x}(t)) e^{i \int_{t_1}^{t_2} dt \ L(x,\dot{x})} \qquad t_2 < t_1 \tag{1.82}$$

Where $L = (dx/dt)^2/2 - V(x)$ is the classical Lagrangian of the studied system and $\mathcal{D}(\vec{x})$ is the functional measure over the space of functions. In Heisenberg formulation, the development of the state is a consequence of the action of the evolution operator on it (only one dimension of space is considered from now on, this causes no loss of generality) :

$$\begin{aligned} \mid x_2, t_2 \rangle &= U(t_2, t_1) \mid x_1, t_1 \rangle \\ &= e^{-iH(t_2 - t_1)} \mid x_1, t_1 \rangle \end{aligned} \tag{1.83}$$

It is assumed here that the Hamiltonian H does not depend explicitly on time. So the transition amplitude can now be written as :

$$\begin{aligned} \langle x_2, t_2 \mid x_1, t_1 \rangle &= \langle x_2 \mid e^{-iH(t_2 - t_1)} \mid x_1 \rangle \\ &= \sum_n \langle x_2 \mid e^{-iE_n(t_2 - t_1)} \mid E_n \rangle \langle E_n \mid x_1 \rangle \\ &= \sum_n \psi_n(x_2).\psi_n^*(x_1) e^{-iE_n(t_2 - t_1)} \end{aligned} \tag{1.84}$$

Where $\{\mid E_n\rangle\}$ is the eigenbasis of the Hamiltonian and $\{\psi_n(x)\}$ are its corresponding eigenfunctions. It is really inevitable here to notice the correspondence between the previous expression and the partition function Z of the canonical ensemble in statistical mechanics :

$$Z = Tr(e^{-\beta H}) = \sum_n e^{-\beta E_n} \tag{1.85}$$

$$= \int dx_1 \langle x_1 \mid e^{-\beta H} \mid x_1 \rangle = \int dx_1 \sum_n \psi_n(x_1).\psi_n^*(x_1) e^{-\beta E_n} \tag{1.86}$$

Actually, if one analytically continued the time in (1.84) from the real time to imaginary time, the equivalence between the partition function and the transition amplitude is all clear and obvious; the idea is to assume that the functional integral definition of the transition amplitude is valid even when the time is imaginary, this can be proved using the same derivation steps for (1.82), see for example [38,39]. This movement is known as Wick rotation, since it rotates the axis in the complex time plane from real to complex axis [40].

Therefore, the time t can be replaced by $-i\beta$, where β is real. Consider that $t_1 = t$ and $t_2 = 0$, so that the transition amplitude (1.84) becomes :

$$\langle x_2, 0 \mid x_1, -i\beta \rangle = \sum_n \psi_n(x_2).\psi_n^*(x_1) \ e^{\beta E_n} \tag{1.87}$$

At low temperature $T \to 0$ thus $\beta \to \infty$, the system tends to be in its ground state. This can be seen via the partition function Z where the dominant term in the sum (1.85) becomes $e^{-\beta.E_0}$. The same thing happens for the transition amplitude (1.87) when $\beta \to -\infty$, such that :

$$\begin{aligned} \lim_{\beta\to-\infty} \langle x_2, 0 \mid x_1, -i\beta\rangle &= \lim_{\beta\to-\infty} e^{\beta E_0}.(\ \psi_0.\psi_0^* + \psi_1.\psi_1^*.e^{\beta\Delta E_1} + \psi_2.\psi_2^*.e^{\beta\Delta E_2} + \dots\ ) \\ &\approx \lim_{\beta\to-\infty} e^{\beta E_0}.\psi_0(x_2).\psi_0^*(x_1) \end{aligned} \tag{1.88}$$

Where $\Delta E_n = E_n - E_0 > 0$ for $n > 0$.

At this point, the ground state $\psi_0(x_2)$ can be written as a functional integral using the formula (1.88). First, one needs to fix the value of x_1, the common choice is to put $x_1 = 0$, but any other value will do the job as long as one keeps the same value when deriving $\psi_0(x_2)$. It is then straightforward to see that

$$\psi_0(x_2) \propto \int_{x(-\infty)=0}^{x(0)=x_2} \mathcal{D}(x(\tau)) e^{-\int_{-\infty}^{0} d\tau\ L_E(x,\dot{x})} \tag{1.89}$$

Where $L_E = (dx/d\tau)^2/2 + V$, the subscript E stands for 'Euclidean' since Wick's rotation changes the Lorentzian metric in Minkowski space into an euclidean one.

The state ψ_0 is then fixed by the formula (1.89) up to some constant. This constant can be determined using the state normalization and it can be absorbed in the measure, so the ground state ψ_0 can be written as

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(0)=x} \mathcal{D}(x(\tau)) e^{-\int_{-\infty}^{0} d\tau\ L_E(x,\dot{x})} \tag{1.90}$$

A similar treatement can be done to get $\psi_0^*(x)$ in the form of a functional integral : let $t_2 = -i\beta$ and $t_1 = 0$, take $\beta \to \infty$, to get :

$$\psi_0^*(x) = \int_{x(0)=x}^{x(\infty)=0} \mathcal{D}(x(\tau)) e^{-\int_{0}^{\infty} d\tau\ L_E(x,\dot{x})} \tag{1.91}$$

Now, it is straightforward to write the ground state density matrix as a functional integral by means of (1.90) and (1.91), such that :

$$\begin{aligned} \rho(x,y) &= \psi_0(x).\psi_0^*(y) = \int_{x(-\infty)=0}^{x(0)=x} \mathcal{D}(x(\tau)).e^{-\int_{-\infty}^{0} d\tau\ L_E(x,\dot{x})} \int_{x(0)=y}^{x(\infty)=0} \mathcal{D}(x(\tau)).e^{-\int_{0}^{\infty} d\tau\ L_E(x,\dot{x})} \\ &\equiv \int_{x(-\infty)=0}^{x(\infty)=0} \mathcal{D}(x(\tau)).e^{-\int_{-\infty}^{0^-(x(0^-)=x)} d\tau L_E(x,\dot{x}) - \int_{0^+(x(0^+)=y)}^{\infty} d\tau L_E(x,\dot{x})} \end{aligned} \tag{1.92}$$

The generalization of (1.92) to any dimension is straightforward.

1.3.2 Rényi entropy

Rényi entropy for a system with density matrix ρ is defined as follows [41] :

$$S_\alpha = \frac{1}{1-\alpha}\ln(\mathrm{Tr}(\rho^\alpha)), \qquad \alpha > 0 \tag{1.93}$$

Apart from its own importance and the information afforded by Rényi entropy itself, it plays the role of an intermediate stage to get the entanglement entropy. In fact, it is not difficult to show that the entanglement entropy S for normalized states is obtained from Rényi entropy by taking $\alpha \to 1$:

$$\begin{aligned}
\lim_{\alpha\to 1} S_\alpha &\equiv -\frac{d}{d\alpha}\ln(\mathrm{Tr}(\rho^\alpha))\mid_{\alpha=1} = -\frac{1}{\mathrm{Tr}(\rho^\alpha)}\frac{d}{d\alpha}\mathrm{Tr}(\rho^\alpha)\mid_{\alpha=1} \\
&= -\frac{1}{\mathrm{Tr}(\rho^\alpha)}.\,\mathrm{Tr}\left(\frac{d}{d\alpha}\rho^\alpha\right)\mid_{\alpha=1} = -\frac{1}{\mathrm{Tr}(\rho^\alpha)}.\,\mathrm{Tr}(\ln(\rho).\rho^\alpha)\mid_{\alpha=1} \\
&= -\mathrm{Tr}(\rho.\ln(\rho))
\end{aligned} \tag{1.94}$$

Where $\ln(\mathrm{Tr}(\rho)) = 0$ was used in the first equality. The final expression is easily identified as the entanglement entropy.

Rényi entropy exhibits some interesting features; it is straightforward to show that it is a positive function. It vanishes when the density matrix is a pure state, and it is maximal when the state ρ is maximally entangled, which can be easily shown in finite dimensional spaces, where

$$\begin{aligned}
S_\alpha &= \frac{1}{1-\alpha}\ln(\mathrm{Tr}(\rho^\alpha)) = \frac{1}{1-\alpha}\ln\left(\sum_{i=1}^{N}(\frac{1}{N})^\alpha\right) \\
&= \frac{1}{1-\alpha}\ln\left(N^{1-\alpha}\right) = \ln(N)
\end{aligned} \tag{1.95}$$

with N is the dimension of the space. In the two limiting states, maximally and non-entangled states, the value of Rényi entropy coincides with the value of the entanglement entropy.

Furthermore, Rényi entropy is Schur concave for all α ; if the spectrum of ρ_1 is majorized by the spectrum of ρ_2, such that

$$\sum_{i=1}^{m}\lambda_i \geq \sum_{i=1}^{m}\sigma_i \tag{1.96}$$

where $1 \leq m \leq N$, N is the dimension of the space, $\{\lambda_i\}$ ($\{\sigma_i\}$) are the eigenvalues of $\rho_1(\rho_2)$ ordered in a decreasing way, such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ and $\sigma_1 \geq \sigma_2 \geq .. \geq \sigma_N$. Rényi entropy is Schur concave since it fulfills

$$S_\alpha(\rho_1) \leq S_\alpha(\rho_2) \tag{1.97}$$

On the other hand, Rényi entropy is concave only in the domain $0 < \alpha < 1$, such that

$$\sum_i p_i S_\alpha(\rho_i) \leq S_\alpha(\rho) \tag{1.98}$$

where $\sum_i p_i \rho_i = \rho$.

Besides, Rényi entropy is additive, i.e.

$$S_\alpha(\rho_A \otimes \rho_B) = S_\alpha(\rho_A) + S_\alpha(\rho_B) \tag{1.99}$$

Some particular values of α are especially useful in certain domains. For example, $\alpha = 2$ has many applications in information theory, it can be used to reduce the computational effort needed to get some entropic estimations [42], or to establish an uncertainty relation stronger than Heisenberg's one [43, 44]. Rényi entropy with $\alpha = 2$ is called the collision entropy.

There are other special values for α; when $\alpha = 0$ Rényi entropy is named Max entropy or Hartley entropy, where

$$\lim_{\alpha \to 0} S_\alpha = \ln(\text{rank}(\rho)) \tag{1.100}$$

Max entropy forms an upper bound to Rényi entropy and therefore to the entanglement entropy. The other lower bound is obtained when $\alpha = \infty$, where

$$\lim_{\alpha \to \infty} S_\alpha = -\ln\Big(\max_i \lambda_i\Big) \tag{1.101}$$

$\{\lambda_i\}$ are the eigenvalues of ρ.

Theoretically, if one could get the formula of Rényi entropy as a function of α, it would be possible to evaluate the entanglement entropy of the system. However, finding the explicit expression of Rényi entropy is not a tractable task. In the Heisenberg XY spin chain, for example, the explicit form of Rényi entropy was obtained, because Rényi entropy is an automorphic function with respect to a certain subgroup of the modular group [45]. Generally, it is not a priori obvious how to take some real power of a matrix without struggling with the eigenvalues problem directly. Fortunately, the replica trick offers a method to overcome this difficulty.

1.3.3 Replica trick

Consider a two-partite system AB with only two degrees of freedom. This model will serve as a toy model that would be easy to generalize it to higher dimensions and even toward the quantum field theory continuous limit. The overall non-normalized ground state density matrix that describes the system AB can be expressed, by means of (1.92), as follows :

$$\rho(q_A, q_B; q'_A, q'_B) = \int \mathcal{D}_Q \ \exp\Big(-\int_{-\infty}^{0^-(Q=(q_A,q_B))} d\tau L_E \ - \int_{0^+(Q=(q'_A,q'_B))}^{\infty} d\tau L_E \Big) \tag{1.102}$$

If one have access to the system A only, then the corresponding density matrix ρ_A is

$$\begin{aligned} \rho_A(q_A, q'_A) &= \int dq_B \ \rho(q_A, q_B; q'_A, q_B) \\ &= \int \mathcal{D}_Q \ \exp\Big(-\int_{-\infty}^{0^-(Q=(q_A,q_B))} d\tau L_E \ - \int_{0^+(Q=(q'_A,q_B))}^{\infty} d\tau L_E \Big) \end{aligned} \tag{1.103}$$

The functional integral includes the integration over the variable q_B in the last equality. The domain of integration is a bit tricky now! The functional integral is free to take any path in Minkowski space until it reaches $t = 0$, where it faces a jump in the variable of the system A. At this point, all possible paths of integration are not free to take any value but only a specific one for A's variable : q_A just before reaching $t = 0$ and q'_A just after passing $t = 0$. After that, the integral is allowed to proceed freely.

Now, if one wants to move towards Rényi entropy, the natural following step is to take some power n of the density matrix, where n is a positive integer. It will show up that ρ^n is nicely represented in the path integral language, such that :

$$\begin{aligned}\rho_A^n(q_A, q'_A) &= \int dq_A^1..dq_A^{n-1}\rho_A(q_A; q_A^1).\rho_A(q_A^1; q_A^2).\rho_A(q_A^2; q_A^3)...\rho_A(q_A^{n-1}; q'_A) \\ &= \int dq_B^1..dq_B^{n-1}dq_A^1..dq_A^{n-1}\ \rho(q_A, q_B^1; q_A^1, q_B^1)...\rho(q_A^{n-1}, q_B^n; q'_A, q_B^n) \qquad (1.104)\end{aligned}$$

it follows that

$$\rho_A^n(q_A; q'_A) = \int D_{\mathbb{Q}} \exp\left(- \int_{-\infty}^{0^-(\mathbb{Q}=\mathbb{Q}(0^-))} d\tau \mathbb{L}_{\mathbb{E}} \ - \int_{0^+(\mathbb{Q}=\mathbb{Q}(0^+))}^{\infty} d\tau \mathbb{L}_{\mathbb{E}} \right) \qquad (1.105)$$

Where

$$\mathbb{Q}(0^-) = \begin{pmatrix} q_A \\ q_B^1 \\ q_A^1 \\ q_B^2 \\ q_A^2 \\ . \\ . \\ q_A^{n-1} \\ q_B^n \end{pmatrix}, \quad \mathbb{Q}(0^+) = \begin{pmatrix} q_A^1 \\ q_B^1 \\ q_A^2 \\ q_B^2 \\ q_A^3 \\ . \\ . \\ q'_A \\ q_B^n \end{pmatrix} \qquad (1.106)$$

and

$$\mathbb{L}_{\mathbb{E}} = L_E \otimes I_n \qquad (1.107)$$

It is possible at this point to generalize these findings. First, for multidimensional subsystems A and B, all one needs is to consider q_A (q_B) as a vector whose dimension is equivalent to the dimension of the subsystem A (B) and its coordinates are the degrees of freedom of the subsystem $A(B)$. For example, if A (B) is an m (m') dimensional system, then q_A (q_B) is an m (m') dimensional vector, where

$$q_A = \begin{pmatrix} (q_A)_1 \\ (q_A)_2 \\ . \\ . \\ (q_A)_m \end{pmatrix}, q_B = \begin{pmatrix} (q_B)_1 \\ (q_B)_2 \\ . \\ . \\ (q_B)_m \end{pmatrix} \qquad (1.108)$$

Furthermore, in order to reformulate the result for quantum field theory, it is enough to take the continuous limit in the multidimensional case, where the vector $q_A(q_B)$ is replaced by a smooth function defined on some geometrical region in configuration space. This region of space represents the zone of the system $A(B)$, and the function represents the field $\phi_A(t, \overrightarrow{x})$ $(\phi_B(t, \overrightarrow{x}))$ on that region.

It is beneficial here to look over the space of integration in the field situation. The n power of the density matrix is represented by :

$$\rho_A^n(\phi_A; \phi'_A) = \int \mathcal{D}_\Phi \exp\left(- \int_{-\infty}^{0^-(\Phi=\Phi(0^-))} d\tau \mathbb{L}_\mathbb{E}(\Phi, \dot{\Phi}) - \int_{0^+(\Phi=\Phi(0^+))}^{\infty} d\tau \mathbb{L}_\mathbb{E}(\Phi, \dot{\Phi}) \right) \tag{1.109}$$

Where Φ is the n- fold field function. When taking the trace of ρ_A^n, the boundary conditions can be expressed as follows

$$\Phi(0^-) = \begin{pmatrix} \phi_A^1 \\ \phi_B^1 \\ \phi_A^2 \\ \phi_B^2 \\ \phi_A^3 \\ . \\ . \\ \phi_A^n \\ \phi_B^n \end{pmatrix}, \quad \Phi(0^+) = \begin{pmatrix} \phi_A^2 \\ \phi_B^1 \\ \phi_A^3 \\ \phi_B^2 \\ \phi_A^4 \\ . \\ . \\ \phi_A^1 \\ \phi_B^n \end{pmatrix} \tag{1.110}$$

The space of integration for $n = 1$ can be roughly represented by figure 1.3. The different boundary conditions at the level of system A at $t = 0$ represent a cut of discontinuity in the space of integration. For $n > 1$, the space of integration is composed of n copy sheets of the original one. Each one has a cut along the system A domain at $t = 0$. The n-fold manifold is represented by figure 1.4. The integration over the fields at the level of the cut $t = 0$ in the functional integral has the effect of stitching each sheet (label it l) along the two sides of the cut with the one just before it; where the field at this sheet $(\phi_A)^{(l)}$ at $t \sim 0^-$ equals the field $(\phi_A)^{(l-1)}$ in the $l-1$ sheet at $t \sim 0^+$, and with the copy just after it; where the field at this sheet $(\phi_A)^{(l)}$ at $t \sim 0^+$ equals the field in the next sheet $(\phi_A)^{(l+1)}$ at $t \sim 0^-$.

The expression (1.109) represents a formal statement for ρ_A^n. If the trace of ρ^n can be obtained by evaluating the functional integral of (1.109), with one way or another, one will eventually obtain some function of n. The replica trick's key movement is to accept that the obtained function is valid for all $\mathrm{Re}(n) > 1$, despite the fact that it was originally evaluated for positive integers n only. In other words, the resulting function has a unique analytic continuation for all $\mathrm{Re}(n) > 1$ (for proofs see [15], [20] and [46]). This step is essential to get Rényi entropy in a formal way and eventually the entanglement entropy.

Consider now the normalized density matrix $\rho_A \equiv \rho_A / \operatorname{Tr}(\rho_A)$, the trace of ρ_A^n can then be re-expressed as follows :

$$\operatorname{Tr}(\rho_A^n) = \int d\phi_A \ \rho_A^n(\phi_A, \phi_A) = \frac{\mathbb{Z}_n}{Z^n} \tag{1.111}$$

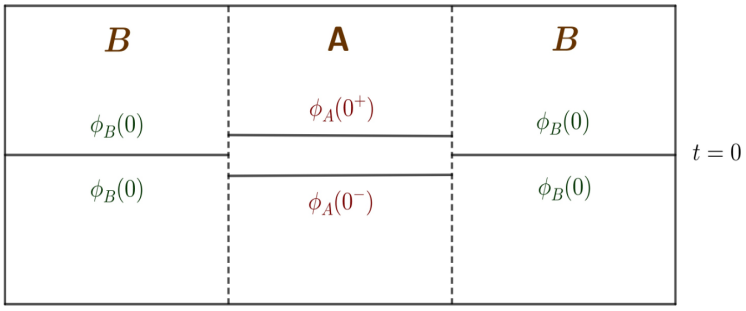

Figure 1.3: The Euclidean space of integration for $n = 1$. The path integral has a discontinuity at the level of region A at $t = 0$.

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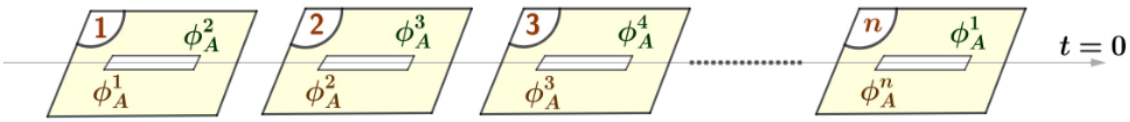

Figure 1.4: A representation of the n fold manifold, each two successive sheets are glued together at $t = 0$ when the fields at the two sides of the cut coincide, the first and the last sheet are glued by ϕ_A^1.

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$\mathbb{Z}_n$ is just the trace of non-normalized ρ_A^n and $Z \equiv \mathbb{Z}_1$. $\rho_A^n(\phi_A, \phi_A)$ is defined by (1.109). The selection of the symbol $\mathbb{Z}$ was not by coincidence, but to indicate the already mentioned parallelism between the ground state density matrix and the partition function Z at low temperatures. In fact, it is possible here to write

$$\mathbb{Z}_n = Tr(\rho_A^n) = \int_{\mathcal{M}_n} \mathcal{D}_\Phi \exp\left(-\int_{-\infty}^{\infty} d\tau \mathbb{L}_{\mathbb{E}}(\Phi, \dot{\Phi})\right) \tag{1.112}$$

Where $\mathcal{M}_n$ is the n-sheeted cover in which the functional integral is defined over. Indeed, one cannot miss the formal analogy between (1.112) and the definition of the statistical partition function in path integral formulation

$$Z = \int \mathcal{D}_x \exp\left(-\int_0^\beta d\tau H(x, \dot{x})\right)$$

Finally, by means of (1.94), the entanglement entropy can be expressed as

$$S = -(\frac{d}{dn} - 1) \ln(\mathbb{Z}_n) \mid_{n=1} \tag{1.113}$$

The good thing about this formula is that the normalization of the partition function $\mathbb{Z}_n$ is no longer required to be calculated explicitly; it is implicitly embedded in (1.113).

1.3.4 The Heat Kernel expansion

The evaluation of Rényi entropy through the formula (1.112) is not always a possible task. The formal structure of (1.112) cannot be manupilated directly. As a result, resorting to approximation methods is required. One of these ways is the heat kernel expansion.

Historically, the heat kernel $K(x, y, \tau)$ is the solution of the heat diffusion equation

$$(\partial_\tau + H(x))K(x, y, \tau) = 0 \tag{1.114}$$

with the initial condition

$$K(x, y, \tau = 0) = \delta(x, y) \tag{1.115}$$

The kernel $K(x, y, \tau)$ has the formal solution :

$$K(x, y, \tau) = \langle x \mid \exp(-\tau H) \mid y \rangle \tag{1.116}$$

Now, how could this heat kernel be useful to our problem? Well, take a free massive scalar field, it is straightforward to show that Euler- Lagrange equation for this field ϕ is formally identical to the Heat equation (1.114) :

$$(\partial_\tau - \nabla^2 + m^2)\phi = 0 \tag{1.117}$$

Where $H(x) = -\nabla^2 + m^2$. From path integral, the partition function (1.112) for this case has the formal solution [40]

$$\ln(\mathbb{Z}_n) = -\frac{1}{2}\ln\big(\det\big(-\nabla^2 + m^2\big)\big) = -\frac{1}{2}\operatorname{Tr}\big(\ln\big(-\nabla^2 + m^2\big)\big) \tag{1.118}$$

The last equality can be proved using the eigenvalues spectral decomposition of the operator $(-\nabla^2 + m^2)$. Since one tries to avoid dealing with the logarithm of an operator, it is convenient at this point to reformulate the last equation by means of an integration over an auxiallary parameter called 'proper time'

$$\ln(\mathbb{Z}_n) = \frac{1}{2}\int_0^\infty ds \frac{1}{s}\operatorname{Tr}(K_m(s)) \tag{1.119}$$

K_m is the heat kernel defined by

$$K_m(x, y, s) = \langle x \mid \exp\big(s(\nabla^2 - m^2)\big) \mid y \rangle = K_0(x, y, s)\exp\big(-sm^2\big) \tag{1.120}$$

Where $K_0 \equiv K_{m=0}$. A hint to show where (1.119) came from could be presented as follows : for a positive number $\lambda > 0$ the following identity holds

$$\ln(\lambda) = -\int_0^\infty ds \frac{1}{s}\exp(-s\lambda) \tag{1.121}$$

This identity is correct up to some constant that does not depend on λ and can be renormalized away. Taking the eigenbasis of the operator $\ln(-\nabla^2)$ with eigenvalues $\{\lambda_i\}$ it follows that :

$$\begin{aligned}\mathrm{Tr}\big(\ln\big(-\nabla^2\big)\big) &= \sum_i \ln(\lambda_i) = \int_0^\infty ds\frac{1}{s}\,\mathrm{Tr}\big(\exp\big(s\nabla^2\big)\big)\\ &= \int_0^\infty ds\frac{1}{s}\,\mathrm{Tr}(K_0(s))\end{aligned} \tag{1.122}$$

Finally, it is straightforward to rediscover the expression (1.119). A more rigorous way to find this expression can be found in [47]. It is worth to mention here that the integral of (1.122) diverges for small s due to the factor s^{-1} (the already mentioned constant which does not depend on λ). This is an ultraviolet divergence. Therefore, it is necessary here to take a UV cutoff ϵ, such that

$$\ln(\mathbb{Z}_n) = Tr(\ln\big(-\nabla^2 + m^2\big)) = \int_\epsilon^\infty ds\frac{1}{s}\exp\big(-sm^2\big)Tr(K_0(s)) \tag{1.123}$$

This expression is evaluated on a manifold with canonical singularity. The goal here is to determine the kernel K_0 in this space. Generally, one can not treat this problem directly. The suggested method is to expand $\mathrm{Tr}(K_0(s))$ around small s and to evaluate the coefficients of expansion order by order, where $\mathrm{Tr}(K_0(s))$ expansion around small s is given by [20, 35] :

$$\mathrm{Tr}(K_0(s)) = \frac{1}{(4\pi s)^{d/2}}\sum_{n=0}^{\infty} a_n s^n \tag{1.124}$$

d is the dimension of Minkowski spacetime. $\{a_n\}$ are called the heat kernel coefficients. The determination of these coefficients depends on the manifold which are taking over : on its geometry, its boundaries and the singularities it has (in a more general case). The coefficients have been determined, up to some order, for some special cases that are related to the existing problem, i.e. for a manifold with a canonical singularity. They can be mainly found in [49–52]. One can see [47] and [53] for reviews on this topic.

1.3.5 Green's Function

The heat kernel is not the only way to handle the partition function problem $\mathbb{Z}_n$ in the Euclidean framework. An alternative method is the Green function. As before, the starting point is the relation (1.118). Where the first equality that will be used :

$$\ln \mathbb{Z}_n = -\frac{1}{2}\ln\big(\det\big(-\nabla^2 + m^2\big)\big) \tag{1.125}$$

The clever trick is to note that

$$\begin{aligned}\frac{\partial}{\partial m^2}\ln \mathbb{Z}_n &= -\frac{1}{2}\frac{\partial}{\partial m^2}\sum_\lambda \ln\big(-\lambda^2 + m^2\big) = -\frac{1}{2}\sum_\lambda \frac{1}{-\lambda^2 + m^2}\\ &= -\frac{1}{2}\,\mathrm{Tr}\big(\,(-\Delta + m^2)^{-1}\big)\end{aligned} \tag{1.126}$$

Where $\{\lambda\}$ are the eigenvalues of the operator ∇. Technically, the operator $(-\Delta + m^2)^{-1}$ is Green's function for the operator $(-\Delta + m^2)$:

$$(-\Delta + m^2)G_n(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{1.127}$$

Therefore, finding the partition function $\mathbb{Z}_n$ can be accomplished by determining Green's function for the operator $-\Delta + m^2$ as a first step, then integrating the result with respect to m^2. It is worth to mention here that this technique is not limited to bosonic fields with the operator $-\Delta + m^2$; it is straightforward to generalize it to any operator O. The twist in this situation is that one should deal with the shifted operator $O + m^2 I$. At the end of calculations, it is enough to put $m^2 = 0$ to find the required partition function for the operator O.

Finally, the partition function can be written in terms of Green's function as follows

$$\frac{\partial}{\partial m^2} \ln(\mathbb{Z}_n) = -\frac{1}{2} \int d\mathbf{r}\ \mathrm{G_n}(\mathbf{r}, \mathbf{r}) \tag{1.128}$$

The normalized version of the partition function is

$$\frac{\partial}{\partial m^2} \ln(\mathrm{Tr}(\rho_A^n)) = \frac{\partial}{\partial m^2} \ln\left(\frac{\mathbb{Z}_n}{Z^n}\right) = -\frac{1}{2} \int d\mathbf{r}(\ \mathrm{G_n}(\mathbf{r}, \mathbf{r}) - \mathrm{nG_1}(\mathbf{r}, \mathbf{r})\ ) \tag{1.129}$$

As it was shown, the heat kernel and Green's function were extracted from the two parts of the last equality of (1.118). The heat kernel from the left side and Green's function from the right side. One may wonder here about the relation between the two notions. To emphasize this relation, the spectral decomposition of the two functions will be used. Take first the heat kernel, from (1.116) it is not hard to show that :

$$K(x, y, s) = \sum_l e^{-\lambda_l s} \Phi_l(x) \Phi_l^\dagger(y) \tag{1.130}$$

where

$$(-\nabla^2 + m^2)\Phi_l(x) = \lambda_l \Phi_l(x) \tag{1.131}$$

$$\int_{-\infty}^{\infty} \Phi_l(x) \Phi_{l'}^\dagger(x) = \delta_{ll'} \quad , \quad \sum_l \Phi_l(x) \Phi_l^\dagger(y) = \delta(x - y) \tag{1.132}$$

Laplace transform for the heat kernel $K(x, y, s)$ is defined by

$$\begin{aligned} \mathbf{K}(x, y, E) &= \int_0^\infty ds\ e^{-Es} K(x, y, s) = \sum_l \int_0^\infty ds\ e^{-Es - \lambda_l s} \Phi_l(x) \Phi_l^\dagger(y) \\ &= \sum_l \frac{1}{\lambda_l + E} \Phi_l(x) \Phi_l^\dagger(y) = \langle x \mid \frac{1}{-\nabla^2 + m^2 + E} \mid y \rangle \end{aligned} \tag{1.133}$$

The last term of (1.133) is just the formal solution of Green's function, which satisfies

$$(-\nabla^2 + m^2 + E)G(x, y) = \delta(x - y) \tag{1.134}$$

Finally, the relation between the heat kernel and Green's function is all clear and obvious, Green's function is just the Laplace transform of the heat kernel for the system associated with the shifted operator $-\nabla^2 + m^2 + E$. The generalization to other systems with different potentials is straightforward.

Free massive scalar field

The evaluation of Green's function depends on the geometry of space that it's taking over. In the current situation, it is evaluated over an n-fold manifold with canonical singularity. This last depends on the way in which the sheets are glued together. In other words, it depends on the choice of the extracted subsystem from the global one.

To see this in action, an example of a free bosonic field will be studied. The importance of this example comes from the fact that it is one of the very few problems treated in the literature. Moreover, it will give an essence to this method and enlight some important results which have not been possible to get analytically in the real time approach.

The total system is taken to be the overall configuration space. The subsystem A occupies the half of the space. The axes of configuration space are oriented so that the subsystem A is defined by $\{x > 0, \mathbb{R}^{d-2}\}$, Where d is the dimension of Minkowski space. The cut of discontinuity is defined by $\{\tau = 0, x > 0\}$. The subspace which contains this cut is $\mathcal{C}_1 = \{\tau, x\}$. It is convenient here to use polar coordinates $\{r \in [0, \infty[, \theta \in [0, 2\pi[\ \}$ for this subspace. The reason will be clear in a moment.

The cut of discontinuity is the half line defined by $\{r \geq 0,\ \theta = 0\}$. Now, taking an n-copy of the global space, gluing the edges of the cut as was previously explained, and the intended n-fold space is consequently formed. As a result, each subspace $\mathcal{C}_1$ of each sheet is connected to its neighbors, just before and just after, at the level of the cut. The union of these subspaces can be simply presented by extending the angle θ, and it's defined by $\mathcal{C}_n = \{r \geq 0,\ \theta \in [0, 2\pi n[\}$. The obtained geometry of $\mathcal{C}_n$ is the geometry of a cone with a deficit angle $2\pi(1 - n)$.

The tip of the cone corresponding to $r = 0$ is a singular point; it is the point where the discontinuity emerges. In the case of free bosonic fields, the operator $\mathcal{L}_E = -\nabla^2 + m^2$ is compatible with the translational invariance. However, it is the singular point that breaks this symmetry. Hence, one should not assume that Green's function G_n is a function of $(\mathbf{r} - \mathbf{r}')$.

In the current examination, the steps of [46] will be followed. The dimension of Minkowski space is taken to be two, i.e, the system is a line of fields.

By means of the identity :

$$G_n(\mathbf{r}, \mathbf{r}') = \sum_{\lambda} \frac{1}{-\lambda^2 + \mathrm{m}^2} \Phi_\lambda(\mathbf{r}) \Phi^*_\lambda(\mathbf{r}') \tag{1.135}$$

Where $\{\Phi_\lambda((\mathbf{r}))\}$ are the normalized eigenfunctions of the operator $(-\nabla^2 + m^2)$ and $\{\lambda\}$ are the eigenvalues of the operator (∇^2), It is possible to check that Green's function which fulfills the differential equation (1.127) is the following :

$$G_n(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi \mathrm{n}} \sum_{\mathrm{k}=0}^{\infty} \mathrm{d_k} \int_0^{\infty} \lambda\ \mathrm{d}\lambda \frac{\mathrm{J_{k/n}}(\lambda \mathrm{r}).\mathrm{J_{k/n}}(\lambda \mathrm{r}')}{\lambda^2 + \mathrm{m}^2} \cos(\mathrm{k}(\theta - \theta')/\mathrm{n}) \tag{1.136}$$

Where $\mathbf{r} = (\mathrm{r}, \theta)$, $\mathbf{r}' = (\mathrm{r}', \theta')$, $J_{k/n}$ are Bessel funcions of the first kind and $\{d_0 = 1,\ d_k = 2$ for $k > 0\}$. In order to use the relation (1.129) it is required to evaluate the integral over λ and $\mathbf{r}$

when $\mathbf{r} = \mathbf{r}'$. The integral over λ is standard [34], where :

$$\begin{aligned} G_n(r,\theta;r,\theta) &= \frac{1}{2\pi n}\sum_{k=0}^{\infty} d_k \int_0^{\infty} \lambda\, d\lambda \frac{J_{k/n}(\lambda r).J_{k/n}(\lambda r)}{\lambda^2+m^2}\cos(k(\theta-\theta')/n) \\ &= \frac{1}{2\pi n}\sum_{k=0}^{\infty} d_k I_{k/n}(mr)K_{k/n}(mr)\cos(k(\theta-\theta')/n) \end{aligned} \tag{1.137}$$

($\{I_{k/n}\}$, $\{K_{k/n}\}$) are the modified Bessel functions of the first and second kind, respectively. Before integrating over r, one needs first to evaluate the sum over k. For this purpose, Euler MacLaurin formula will be used :

$$\begin{aligned} \sum_{k=0}^{\infty} d_k\ I_{k/n}K_{k/n} &= 2\int_0^{\infty} dk\ I_{k/n}K_{k/n} - 2\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}(I_{k/n}K_{k/n})^{(2j-1)}\ |_{k=0} \\ &= 2\int_0^{\infty} dk\ I_{k/n}K_{k/n} - \frac{1}{6}\partial_k(I_{k/n}K_{k/n})\ |_{k=0} + \\ &\quad -2\sum_{j=2}^{\infty}\frac{B_{2j}}{(2j)!}(I_{k/n}K_{k/n})^{(2j-1)}\ |_{k=0} \end{aligned} \tag{1.138}$$

Where B_{2j} are Bernoulli numbers. The last term of the final expression can be shown that it vanishes when integrating over r [15] :

$$\partial_k^{2j-1}\int_0^{\infty} r.dr I_{k/n}(mr).K_{k/n}(mr) = \partial_k^{2j-1}.\frac{k}{2m^2} \tag{1.139}$$

Since $j > 1$ the last expression vanishes identically. The term $\partial_k(I_{k/n}.K_{k/n})\ |_{k=0}$ gives

$$\partial_k(I_{k/n}.K_{k/n})\ |_{k=0} = \partial_k I_{k/n}\ |_{k=0}\ .K_0 = \frac{-1}{n}K_0^2 \tag{1.140}$$

where $\partial_k K_k\ |_{k=0} = 0$ and $\partial_k I_k\ |_{k=0} = -K_0$ (see [54]). Hence

$$\begin{aligned} G_n(r,r) &= \int_0^{2\pi n} d\theta\ G_n(r,\theta;r,\theta) \\ &= 2\int_0^{\infty} dk\ I_{k/n}(mr).K_{k/n}(mr) + \frac{1}{12n}K_0^2(mr) \\ &= 2n\int_0^{\infty} dk I_k(mr).K_k(mr) + \frac{1}{12n}K_0^2(mr) \end{aligned} \tag{1.141}$$

Green's function $G_n(r,r)$ is formed out of two terms. The first is linear in n, where it has been factorized out by a simple change of variables. It is important to remember here that one will use the normalized partition function (1.129). The latter will cancel any term linear on n in G_n

expression. Actually, this term is proportional to the total volume of the spacetime and has no contribution to entanglement (see [28] for more discussion).

It is time to point out an important note : G_1 does not depend on the choice of the subsystems or the subspaces one wants to evaluate entanglement between them; it has no singularity caused by the decomposition of subspaces. Hence, its resulting expression does not depend on the choice of subspaces, nor does the entanglement. As a result, any term in G_n which is proportional to nG_1 should be cancelled since it has no entanglement information.

The partition function out of expression (1.129) is then :

$$\begin{aligned}\frac{\partial}{\partial m^2}\ln\left(\frac{\mathbb{Z}_n}{Z^n}\right) &= -\frac{1}{2}\int_0^\infty r.dr(\ G_n(r,r) - n.G_1(r,r)\ ) \\ &= -(\frac{1}{24n} - \frac{n}{24}).\int_0^\infty r.drK_0^2(mr) \\ &= \frac{1}{24m^2}(n-\frac{1}{n}) \end{aligned} \tag{1.142}$$

Where $\int_0^\infty r.drK_0^2(r) = 1/2$ (see [34]). The partition function is obtained by a simple integration over m^2

$$\ln\left(\frac{\mathbb{Z}_n}{Z^n}\right) = \frac{1}{24}(n-\frac{1}{n})\ln\left(\epsilon^2.m^2\right) \tag{1.143}$$

It is important to remember that the partition function is a unit free function. The integration itself produces a constant independent of m^2, which indeed has the effect of regulating the units. Here, ϵ the UV cutoff, is the proper quantity to produce a scaleless function. Therefore, the entanglement entropy for a $1+1$ free scalar field is

$$\begin{aligned} S &= -\frac{\partial}{\partial n}Tr(\rho^n)\ |_{n=1} = -\frac{\partial}{\partial n}\frac{\mathbb{Z}_n}{Z^n}\ |_{n=1} \\ &= -\frac{\partial}{\partial n}(\epsilon^2.m^2)^{\frac{1}{24}(n-\frac{1}{n})}\ |_{n=1} = -\frac{1}{6}\ln(\epsilon m) \end{aligned} \tag{1.144}$$

It can be seen here that the entropy diverges when taking the continuum limit $\epsilon \to 0$, this divergence is logarithmic. In higher dimension scalar field theories $d+1$ $(d>1)$, the entropy is proportional to the separating area between the two entangled systems, so that [28]

$$S_{d+1} \sim \frac{A_{d-1}}{\epsilon^{d-1}} \tag{1.145}$$

In 1+1 field theory, the behavior of entropy seems different from higher dimensional ones. However, the logarithmic divergence of 1+1 scalar field theory has an intuitive justification. In lattice regularization this can be seen more clearly; each oscillator of the lattice is coupled with its first neighbors only. If one lifts the oscillators of the boundary between the two entangling systems, these two will be completely separated and there will be no entanglement left between them. Therefore, it is natural for the entanglement to be proportional to the separating area. For 3+1 theory, the

area is the superficial surface of the subsystem. For the 2+1 theory, it is the circumference. For the 1+1 theory, one would expect some constant since the separating area is a point or number of them depending on the shape of the subsystem. However, the logarithmic dependence becomes totally understood when thinking of the area law (1.145) to come from an integral proportional to r^{d-2}. For $d = 1$, the logarithm is the compatible piece for an area law theory [55].

Chapter 2

Euclidean Green's Function for Lattices

The Euclidean time approach and methods developed therein are generally used in quantum field theory. Even there, they have been used in limited situations where the problem has a special symmetry that allows one to construct Green's function or to establish a Heat kernel expansion over the resulting manifold. For example, if the reduced system is half the space. Else, the problem is generally untractable to be handled analytically. On the other hand, no such methods have been developed for lattices. They have been generally treated using a real time approach only. This chapter shows how to construct the Euclidean Green's function for a lattice of oscillators, where the reduced system is an arbitrary set of lattice oscillators.

2.1 Formulation of the problem

The treated system is a lattice of coupled harmonic oscillators. The key importance of this system comes from the fact that many physical situations can be modeled by it. In particular, the space discretization of a theory of scalar fields produces a system of coupled harmonic oscillators. This process is called lattice regularization of the fields.

The Lagrangian of N-coupled harmonic oscillator is given by :

$$L = \frac{1}{2}\left(\dot{Q}^T\dot{Q} - Q^T V Q\right) \tag{2.1}$$

Where $Q^T = (q_1, q_2, ..., q_N)$ is the oscillators N-coordinate vector and V is the symmetric positive definite potential matrix.

If one takes the lattice to be in its ground state, then the density matrix which describes this state is

$$\rho(Q', Q'') = \int_{Q(-\infty)}^{Q(\infty)} \mathcal{D}_Q \exp\left(-\int_{-\infty}^{0^-(Q(0^-)=Q')} d\tau L_E(Q,\dot{Q}) - \int_{0^+(Q(0^+)=Q'')}^{\infty} d\tau L_E(Q,\dot{Q})\right) \tag{2.2}$$

Where Q denotes the coordinates of the N-oscillators of the lattice. Take the reduced system A to be formed of the first p oscillators. Its complementary system B is then formed of the rest of

the lattice oscillators. The reduced density matrix ρ_A can be written as

$$\begin{aligned}\rho_A(Q'_A,Q''_A) &= \mathrm{Tr}_B(\rho(Q',Q'')) = \int dQ_B\ \rho(Q'_A,Q_B;Q''_A,Q_B)\\ &= \int \mathcal{D}_Q \exp\left(-\int_{-\infty}^{0^-(\ Q(0^-)=(Q'_A,Q_B)\ )} d\tau L_E(Q,\dot{Q})+\right.\\ &\quad \left.-\int_{0^+(\ Q(0^+)=(Q''_A,Q_B)\ )}^{\infty} d\tau L_E(Q,\dot{Q})\right) \end{aligned} \tag{2.3}$$

where $Q^T=(Q_A,Q_B)$, $Q_A^T=(q_1,q_2,..,q_p)$, $Q_B^T=(q_{p+1},q_{p+2},...,q_N)$ and $dQ_B=\prod_{i=p+1}^{N} dq_i$. In order to be able to use the replica trick, one needs to take the power n of the reduced density matrix ρ_A. Only the trace of ρ_A^n is required :

$$\begin{aligned}\mathrm{Tr}(\rho_A^n) &= \int \rho_A(Q_A^1,Q_A^2).\rho_A(Q_A^2,Q_A^3)...\rho_A(Q_A^n,Q_A^1)\ dQ_A^1.dQ_A^2...dQ_A^n\\ &= \int \rho(Q_A^1,Q_B^1;Q_A^2,Q_B^1)\rho(Q_A^2,Q_B^2;Q_A^3,Q_B^2)..\rho(Q_A^n,Q_B^n;Q_A^1,Q_B^n)\ dQ^1...dQ^n \end{aligned} \tag{2.4}$$

Using path integral formulation, this lengthy and unappealing form of $\mathrm{Tr}(\rho_A^n)$ can be expressed in a compact formula. The trick is to construct the n-fold vector $\mathbb{Q}$ as follows

$$\mathbb{Q}=\begin{pmatrix} Q_A^1\\ Q_B^1\\ Q_A^2\\ Q_B^2\\ .\\ .\\ .\\ Q_A^n\\ Q_B^n\end{pmatrix} \tag{2.5}$$

By means of $\mathbb{Q}$ it is straightforward to show that

$$\begin{aligned}Tr(\rho_A^n) = \int \mathcal{D}_{\mathbb{Q}} \exp\left(-\int_{-\infty}^{0^-(\ \mathbb{Q}(0^-)=(Q_A^1,Q_B^1;Q_A^2,Q_B^2;...;Q_A^n,Q_B^n)\ )} d\tau \mathbb{L}_E(\mathbb{Q},\dot{\mathbb{Q}})+\right.\\ \left.-\int_{0^+(\ \mathbb{Q}(0^+)=(Q_A^2,Q_B^1;Q_A^3,Q_B^2;...;Q_A^1,Q_B^n)\ )}^{\infty} d\tau \mathbb{L}_E(\mathbb{Q},\dot{\mathbb{Q}})\right)\end{aligned} \tag{2.6}$$

$\mathbb{L}_E$ is the n-fold Euclidean Lagrangian defined by :

$$\mathbb{L}_E=\frac{1}{2}\left(\dot{\mathbb{Q}}^T\dot{\mathbb{Q}}+\mathbb{Q}^T\mathbb{V}\mathbb{Q}\right), \qquad \mathbb{V}=V\otimes\mathbb{I}_{n\times n} \tag{2.7}$$

The formal expression (2.6) of $\mathrm{Tr}(\rho_A^n)$ cannot be handled mathematically due to the cut on the boundary condition at $\tau=0$. Unlike the continuous case, when dealing with a theory of fields on

flat space, one cannot try to recast the space into an n-fold manifold with canonical singularity in order to make $\mathrm{Tr}(\rho_A^n)$ take the form of a zero partition function. However, it is still possible to rewrite (2.6) as a zero temperature partition function but with an alternative Lagrangian.

The idea is to move the cut from the boundary condition to the n-fold Lagrangian $\mathbb{L}_E$. It can be seen from (2.6) that all the values of the variables $\{Q_A^i, Q_B^i\}$ in $\mathbb{Q}(0^-)$ do exist in $\mathbb{Q}(0^+)$ but in a disorder way. The way of disorder depends on the chosen subsystem A : set of successive oscillators, disjoint sectors of them and so on. The formula (2.6) of $\mathrm{Tr}(\rho_A^n)$ considers a special choice of the subsystem A, however, the generalization to any subsystem is straightforward. So, if one properly re-order the variables $\{Q_A^i, Q_B^i\}$ when $\tau > 0$, it will be able to get a continuity on the boundary condition at $\tau = 0$. It is then possible to write :

$$Tr(\rho_A^n) = \int \mathcal{D}_{\mathbb{Q}} \exp\left(- \int_{-\infty}^{\infty} d\tau \frac{1}{2} \left(\dot{\mathbb{Q}}^T \dot{\mathbb{Q}} + \mathbb{Q}^T \mathbb{V}(\tau) \mathbb{Q} \right) \right) \tag{2.8}$$

Where

$$\mathbb{V}(\tau) = \theta(-\tau)\mathbb{V} + \theta(\tau)\mathbb{V}_p \tag{2.9}$$

$$\mathbb{V}_p = \mathbb{P}_\pi \mathbb{V} \mathbb{P}_\pi^T \tag{2.10}$$

$$\theta(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases} \tag{2.11}$$

$\theta(\tau)$ is Heaviside step function, $\mathbb{V}_p$ is the $Nn \times Nn$ permuted potential matrix and $\mathbb{P}_\pi$ is an $Nn \times Nn$ permutation matrix, its form depends on the choice of subsystem A. Its mission is to redistribute the variables $\{Q_A^i, Q_{\bar{A}}^i\}$ in the second sector of (2.6) when $\tau > 0$, in order to get the same value for $\mathbb{Q}$ over the two sides of the cut, so that $\mathbb{Q}(0^-) = \mathbb{P}_\pi \mathbb{Q}(0^+)$. As a result, the cut disappears from the integration boundaries at $\tau = 0$.

For our special case, the permutation matrix $\mathbb{P}_\pi$ is given by [1]

$$\mathbb{P}_\pi = \begin{bmatrix} P_B & P_A & 0_N & \dots & 0_N & 0_N \\ 0_N & P_B & P_A & \dots & 0_N & 0_N \\ 0_N & 0_N & P_B & \dots & 0_N & 0_N \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_N & 0_N & 0_N & \dots & P_B & P_A \\ P_A & 0_N & 0_N & \dots & 0_N & P_B \end{bmatrix} \tag{2.12}$$

The sub-matrices P_A and P_B are the projectors on the space of A and B respectively.

$$P_A = \begin{bmatrix} I_p & 0 \\ 0 & 0_{N-p} \end{bmatrix}, \quad P_B = \begin{bmatrix} 0_p & 0 \\ 0 & I_{N-p} \end{bmatrix} \tag{2.13}$$

It is not difficult to check that $\mathbb{P}_\pi$ does map the vector $Q(0^+)^T = (Q_A^2, Q_B^1; Q_A^3, Q_B^2; ...; Q_A^1, Q_B^n)$ into the vector $\mathbb{Q}^T(0^-) = (Q_A^1, Q_B^1; Q_A^2, Q_B^2; ...; Q_A^n, Q_B^n)$ as it should do, i.e., $\mathbb{P}_\pi \mathbb{Q}(0^+) = \mathbb{Q}(0^-)$. The

[1] it is important to say here that the developed method does not depend on the specific form of the matrix $\mathbb{P}_\pi$. As a result, it does not depend on the choice of the subsystem A.

action of $\mathbb{P}_\pi$ on the n-fold potential matrix $\mathbb{V}$ can be easily shown if the matrix V is decomposed as follows

$$V = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{2.14}$$

it follows that

$$\mathbb{V} = V \otimes I_{nn} = \begin{bmatrix} A & B & 0 & 0 & 0 & \dots & 0 & 0 \\ B^T & C & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & A & B & 0 & \dots & 0 & 0 \\ 0 & 0 & B^T & C & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & A & B \\ 0 & 0 & 0 & 0 & 0 & \dots & B^T & C \end{bmatrix} \tag{2.15}$$

Therefore, the permuted potential matrix $\mathbb{V}_p$ can be written as :

$$\mathbb{V}_p = \mathbb{P}_\pi \mathbb{V} \mathbb{P}_\pi^T = \begin{bmatrix} A & 0 & 0 & 0 & 0 & \dots & 0 & B \\ 0 & C & B & 0 & 0 & \dots & 0 & 0 \\ 0 & B^T & A & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & C & B & \dots & 0 & 0 \\ 0 & 0 & 0 & B^T & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & A & 0 \\ B^T & 0 & 0 & 0 & 0 & \dots & 0 & C \end{bmatrix} \tag{2.16}$$

As a matter of fact, this form of the n-fold matrix is typical in the calculations of entropy in the euclidean approach [12], [56]. Although they show up here due to the action of $\mathbb{P}_\pi$ but this forms a good sign to the coherence of the method with existing works on this domain.

Fruitfully, the normalized $\mathrm{Tr}(\rho_A^n)$ can be written as

$$\mathrm{Tr}(\rho_A^n) = \frac{\mathbb{Z}_n}{\mathbb{Z}_1^n} = \frac{\int \mathcal{D}_{\mathbb{Q}} e^{-\mathbb{S}_E}}{(\int \mathcal{D}_Q e^{-S_E})^n} \tag{2.17}$$

Where $\mathbb{S}_E$ is the n-fold Euclidean action

$$\mathbb{S}_E = \int_{-\infty}^{\infty} d\tau \frac{1}{2} \left(\dot{\mathbb{Q}}^T \dot{\mathbb{Q}} + \mathbb{Q}^T \mathbb{V}(\tau) \mathbb{Q} \right), \qquad \mathbb{V}(\tau) = \theta(-\tau)\mathbb{V} + \theta(\tau)\mathbb{V}_p$$

Finally, Rényi entropy is given by

$$\ln(\mathrm{Tr}(\rho_A^n)) = -\frac{1}{2} \ln \left(\frac{\det\left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau) \right)}{\det\left(-\frac{d^2}{d\tau^2} + V \right)^n} \right), \qquad -\infty < \tau < \infty \tag{2.18}$$

Where the n fold potential matrix $\mathbb{V}(\tau)$ is defined by (2.9).

2.2 Green's matrix function construction

In order to evaluate Rényi entropy (2.18) for a lattice of oscillators, it is required to use one of the Euclidean methods presented in the first chapter : Green's function or the Heat kernel expansion, which have been shown to be equivalent. Precisely, the first is Laplace transform of the second. The Heat kernel matrix $\mathbb{K}_n(\tau,\tau',s)$ for this system is an $nN\times nN$ matrix which satisfies the differential equation :

$$(\partial_s-\frac{d^2}{d\tau^2}+\mathbb{V}(\tau)\ )\ \mathbb{K}_n(\tau,\tau',s)=0 \tag{2.19}$$

Rényi entropy (2.18) can be written using $\mathbb{K}_n(\tau,\tau',s)$ as follows :

$$\ln(\mathrm{Tr}(\rho_A^n))=\frac{1}{2}\int_0^\infty ds\frac{1}{s}(\ \mathrm{Tr}(\mathbb{K}_n(\tau,\tau,s))-n\,\mathrm{Tr}(K(\tau,\tau,s))\ ),\quad K=\mathbb{K}_1(\tau,\tau,s) \tag{2.20}$$

It is understood here that the integration over τ is included in the trace of the kernel.

On the other hand, it is straighforward to show that Laplace transform of the heat kernel $\mathbb{G}(\tau,\tau',E)=\int_0^\infty ds\exp(-Es)\mathbb{K}(\tau,\tau',s)$ is nothing but Green's function associated with the differential equation :

$$(-\frac{d^2}{d\tau^2}+\mathbb{V}(\tau)+E)\mathbb{G}_n(\tau,\tau',E)=\delta(\tau-\tau')\mathbb{I} \tag{2.21}$$

Although the apparent source of 'E' in the previous equation is Laplace transformation, but it plays an identical role to the parameter m^2 in a massive scalar field theory, for other systems it is the extra term which can be added to any potential where the integration of Green's function with respect to it gives $\ln(\mathrm{Tr}(Z_n))$ as indicated by (1.128).

Rényi entropy can be written by means of $\mathbb{G}_n(\tau,\tau',E)$ as follows

$$\ln(\mathrm{Tr}(\rho_A^n))=\frac{1}{2}\int_0^\infty dE(\mathrm{Tr}(\mathbb{G}_n(\tau,\tau,E))-n\,\mathrm{Tr}(G(\tau,\tau,E))\ ),\quad G\equiv\mathbb{G}_1(\tau,\tau,E) \tag{2.22}$$

The advantage which comes from the definition of Green's function as Laplace transform of the heat kernel is that the obtained formula of Rényi entropy is exactly determined as a definite integral in terms of Green function. Previously, as indicated by (1.129), it was defined up to some constant since one only has the derivative of Rényi entropy with respect to m^2.

The main difficulty in evaluating $\mathbb{G}_n$ as defined by (2.21) is the discontinuity on the potential matrix $\mathbb{V}(\tau)$. Furthermore, the two-sided matrices $\mathbb{V}$ and $\mathbb{V}_p$ of the matrix function $\mathbb{V}(\tau)$ do not generally commute, which makes finding $\mathbb{G}_n$ a non-straightforward task.

It is preferable, as a beginning, not to strike directly with this problem but rather to take a simpler form of it. First, consider the case where $\mathbb{V}(\tau)$ is a real scalar rather than a matrix. In this situation, the problem reduces to the determination of Green's function for the one-dimensional step potential $V(\tau)$, where V and V_p are just numbers. The solution of Green's function is available in the literature for this situation [57], [58] :

$$\begin{aligned}G_{\mp,\mp}(\tau,\tau') &= \frac{1}{2W_\mp}e^{-W_\mp|\tau-\tau'|}+\frac{1}{2W_\mp}.\left[\frac{W_\mp-W_\pm}{W_-+W_+}\right]e^{\pm W_\mp(\tau+\tau')}\\ G_{\pm,\mp}(\tau,\tau') &= \frac{1}{W_-+W_+}e^{\mp W_\pm\tau\pm W_\mp\tau'}\end{aligned} \tag{2.23}$$

The used notation $G_{+-}(\tau,\tau')$ stands for Green function when $\tau>0,\tau'<0$, the others $\{G_{\pm,\mp},G_{\mp,\mp}\}$ can be deduced automatically, also we have

$$W_{\mp}=\sqrt{V_{\mp}},\quad V_{-}=V+E,\quad V_{+}=V_{p}+E \tag{2.24}$$

An equivalent problem to the one just discussed above is when the matrices $\mathbb{V}$ and $\mathbb{V}_p$ commute. In this situation, the related matrices act as if they were scalars, and the order of the matrices does not really matter since they commute. Therefore, Green's matrix function when $[\mathbb{V},\mathbb{V}_p]=0$ has the same form of (2.23), where $\mathbb{W}_{\mp}$ is the positive square root of $\mathbb{V}_{\mp}$ and E is multiplied by the unit matrix $\mathbb{I}$, $\mathbb{W}_{\pm}^{-1}$ and $(W_{-}+W_{+})^{-1}$ stand for the matrix inverses of $\mathbb{W}_{\pm}$ and $(W_{-}+W_{+})$ respectively.

It is now time to address the original problem directly, namely, the case where $\mathbb{V}$ and $\mathbb{V}_p$ do not necessarily commute. The sure thing is that Green's function reduces in this situation to (2.23) when taking $[\mathbb{V},\mathbb{V}_p]=0$. The fundamental question here is : what are the terms that one could miss when moving to the scalar case? In other words, are there extra terms that vanish in the commutative case? Although one cannot obtain Green's function directly by solving the differential equation (2.21) to answer such a question, one can obtain a systematic series expansion of Green's function in powers of the potential matrix $\mathbb{V}$ and $\mathbb{V}_p$. Pleasingly, the expansion series indicates that all one needs does exist in (2.23) and no more terms are needed.

2.2.1 Green's matrix function series expansion

The idea is to expand $\mathbb{G}_n$ around the free case, where $\mathbb{V}(\tau)=0$. The generic term of the series can be written as a reciprocal relation which depends only on the lower orders of the series terms.

First, for the free case $\mathbb{V}(\tau)=0$ the differential equation (2.21) becomes

$$(-\frac{d^2}{d\tau^2}+E)\ \mathbb{G}_0(\tau,\tau',E)=\delta(\tau-\tau')\mathbb{I} \tag{2.25}$$

The subscript n of $\mathbb{G}_n$ is abandoned in this subsection. $\mathbb{G}_0$ denotes Green's function matrix for the free case, which is well-known to be

$$\mathbb{G}_0(\tau,\tau')=\frac{1}{2\sqrt{E}}e^{-\sqrt{E}|\tau-\tau'|} \tag{2.26}$$

Now, it is possible to use $\mathbb{G}_0$ to reformulate the original differential equation of Green's function (2.21) and transform it into an integral reciprocal equation with respect to the function $\mathbb{G}$. One originally has

$$(-\frac{d^2}{d\tau^2}+\mathbb{V}(\tau)+E)\ \mathbb{G}(\tau,\tau')=\delta(\tau-\tau')\mathbb{I} \tag{2.27}$$

Multiply it by $\mathbb{G}_0(\tau'',\tau)$ and integrate it over τ to get rid of the delta function

$$\int d\tau\ \mathbb{G}_0(\tau'',\tau)(-\frac{d^2}{d\tau^2}+\mathbb{V}(\tau)+E)\mathbb{G}(\tau,\tau')=\mathbb{G}_0(\tau'',\tau') \tag{2.28}$$

integrate twice by parts the following term

$$\begin{aligned}\int d\tau \mathbb{G}_0 \frac{d^2}{d\tau^2}\mathbb{G} &= \mathbb{G}_0 \frac{d}{d\tau}\mathbb{G} \mid_{-\infty}^{\infty} - \int \frac{d}{d\tau}\mathbb{G}_0 \frac{d}{d\tau}\mathbb{G} \\ &= \mathbb{G}\frac{d}{d\tau}\mathbb{G}_0 \mid_{-\infty}^{\infty} + \int d\tau \mathbb{G}\frac{d^2}{d\tau^2}\mathbb{G}_0 \\ &= \int d\tau\ \mathbb{G}\frac{d^2}{d\tau^2}\mathbb{G}_0 \end{aligned} \tag{2.29}$$

Where

$$\lim_{\tau\to\pm\infty} \mathbb{G}_0 \frac{d}{d\tau}\mathbb{G} = \lim_{\tau\to\pm\infty} \mathbb{G}\frac{d}{d\tau}\mathbb{G}_0 = 0 \tag{2.30}$$

Which can be deduced directly from the form of $\mathbb{G}_0$ (2.26) or from the properties of Green function. As a result, (2.28) takes the following form :

$$\int d\tau\ \mathbb{G}(\tau,\tau')(-\frac{d^2}{d\tau^2} + E)\mathbb{G}_0(\tau'',\tau) + \int d\tau\ \mathbb{G}_0(\tau'',\tau)\mathbb{V}(\tau)\mathbb{G}(\tau,\tau') = \mathbb{G}_0(\tau'',\tau') \tag{2.31}$$

It is important to remember here that $\mathbb{V}$ and $\mathbb{G}$ are matrices, and their order matters. By means of the differential equation (2.25) of the function $\mathbb{G}_0$, it is straightforward to show that (2.31) takes the form

$$\mathbb{G}(\tau'',\tau') = \mathbb{G}(\tau'',\tau') - \int d\tau\ \mathbb{G}_0(\tau'',\tau)\mathbb{V}(\tau)\mathbb{G}(\tau,\tau') \tag{2.32}$$

The repetitive substitution of $\mathbb{G}$ in the right-hand side of the previous equation produces the series

$$\mathbb{G}(\tau,\tau') = \sum_{n=0}^{\infty} \mathbb{G}_n(\tau,\tau') \tag{2.33}$$

where

$$\mathbb{G}_n(\tau,\tau') = (-1)^n \int d\tau_1 ... \int d\tau_n \mathbb{G}_0(\tau,\tau_1)\mathbb{G}_0(\tau_1,\tau_2)...\mathbb{G}_0(\tau_{n-1},\tau_n)\mathbb{G}_0(\tau_n,\tau')\mathbb{V}(\tau_1)...\mathbb{V}(\tau_n) \tag{2.34}$$

$\mathbb{G}_n$ can be simply written in terms of $\mathbb{G}_{n-1}$ to give the desired reciprocal relation :

$$\mathbb{G}_n(\tau,\tau') = -\int dt\ \mathbb{G}_0(\tau,t)\mathbb{V}(t)\mathbb{G}_{n-1}(t,\tau') \tag{2.35}$$

Hence, the solution of the differential equation of Green's function (2.27) reduces to the solution of the integral equation (2.35). The solution is evaluated order-by-order. Since $\mathbb{V}(\tau)$ has a finite cut at $\tau = 0$ this will produce a parallel decomposition for $\mathbb{G}$ at $\tau = 0$ and $\tau' = 0$ as it does in the one step potential. This problem is a typical form of the problems solved by the Wiener-Hopf

method : an integral equation with different boundary conditions on the two sides of the same boundary. The method usually suggests the following decomposition for $\mathbb{G}_n$:

$$\begin{aligned} \mathbb{G}_n^+(\tau,\tau') &= \begin{cases} \mathbb{G}_n(\tau,\tau') & \tau<0 \\ 0 & \tau>0 \end{cases} \\ \mathbb{G}_n^-(\tau,\tau') &= \begin{cases} 0 & \tau<0 \\ -\mathbb{G}_n(\tau,\tau') & \tau>0 \end{cases} \end{aligned} \tag{2.36}$$

The indices $\pm$ in $\mathbb{G}_n^\pm$ should not be mixed with the signs of the arguments τ and τ' in Green's function (2.23); in expression (2.36) they only refer to the relative sign between $\mathbb{G}_n^\pm$ and $\mathbb{G}_n$.

By means of the two functions $\mathbb{G}_n^\pm$, the function $\mathbb{G}_n$ defined by (2.35) can then be rewritten as

$$\mathbb{G}_n^+(\tau,\tau') - \mathbb{G}_n^-(\tau,\tau') = -\mathbb{V}\int_{-\infty}^{0} \mathbb{G}_0(\tau,t)\mathbb{G}_{n-1}^+(t,\tau')dt + \mathbb{V}_p\int_0^{\infty} \mathbb{G}_0(\tau,t)\mathbb{G}_{n-1}^-(t,\tau')dt \tag{2.37}$$

The form of $\mathbb{G}_0$ given by (2.26) suggests taking the two-sided Laplace transform for the previous equation, it then becomes

$$\mathbb{F}_n^+(s,\tau') - \mathbb{F}_n^-(s,\tau') = -\mathbb{V}\ \mathbb{F}_0(s)\mathbb{F}_{n-1}^+(s,\tau') + \mathbb{V}_p\ \mathbb{F}_0(s)\mathbb{F}_{n-1}^-(s,\tau') \equiv D_n(s,\tau') \tag{2.38}$$

Where

$$\mathbb{F}_n^\pm(s,\tau') = \int_{-\infty}^{\infty} d\tau e^{-s\tau}\mathbb{G}_n^\pm(\tau,\tau'), \qquad \text{analytic for} \quad \mp\,\mathrm{Re}(s) > 0 \tag{2.39}$$

$$\mathbb{F}_0(s) = \frac{1}{2}\int_{-\infty}^{\infty} d\tau e^{-s\tau-\sqrt{E}|\tau|} = \frac{1}{E-s^2}\mathbb{I}, \qquad \text{analytic for} \quad |\,\mathrm{Re}(s)| < \sqrt{E} \tag{2.40}$$

One should note that the domain of the definition of 's' for Laplace transforms $(\mathbb{F}_n^\pm, \mathbb{F}_0)$ are necessary to have proper transform functions; for example, in the domain $|\,\mathrm{Re}(s)| > \sqrt{E}$ the integral of the transform $\mathbb{F}_0$ diverges!

The form of equation (2.38) is a typical form of Hilbert problem, where the contour of discontinuity is the imaginary axis $\mathrm{Re}(s) = 0$. The function $\mathbb{F}_n^\pm$ can then be determined using the contour integration if one has already determined $\mathbb{F}_{n-1}^\pm$. Therefore, the determination of $\{\mathbb{F}_n^\pm\}$, and then $\{\mathbb{G}_n^\pm\}$, is accomplished order by order.

The region $(\tau'<0, \tau<0)$ will be worked out explicitly until the second order. However, as it will be seen, it is also necessary to determine lower order functions of the region $(\tau'<0, \tau>0)$ in order to compute the functions of the desired region. The other regions can be treated in a similar way.

- **Zeroth order**

In this order, the function $\mathbb{G}_0$ is already determined by (2.26). The function $\mathbb{F}_0^\pm$ can be directly computed using Laplace transform as follows

$$\begin{aligned} \mathbb{F}_0^+(s,\tau') &= \int_{-\infty}^{0} d\tau\ e^{-s\tau}\mathbb{G}_0(\tau,\tau') \\ &= \frac{-1}{2\sqrt{E}}\left(\frac{e^{\sqrt{E}\tau'}}{s+\sqrt{E}} + \frac{e^{-s\tau'}}{s-\sqrt{E}} - \frac{e^{-s\tau'}}{s+\sqrt{E}}\right), \quad \mathrm{Re}(s)<0,\ \tau'<0 \end{aligned} \tag{2.41}$$

Whereas

$$\mathbb{F}_0^-(s,\tau') = -\int_0^\infty d\tau\ e^{-s\tau}\mathbb{G}_0(\tau,\tau') = \frac{-e^{\sqrt{E}\tau'}}{2\sqrt{E}(s+\sqrt{E})}, \quad \mathrm{Re}(s)>0,\ \tau'<0 \tag{2.42}$$

It is straightforward to note that

$$\mathbb{F}_0^+(s,\tau') = \mathbb{F}_0^-(s,\tau') + \frac{e^{-s\tau'}}{2\sqrt{E}}\mathbb{F}_0(s) \tag{2.43}$$

- **First order**

The functions $\mathbb{F}_1^\pm$ are derived from (2.38), where

$$\mathbb{F}_1^+ - \mathbb{F}_1^- = -\mathbb{V}\ \mathbb{F}_0\ \mathbb{F}_0^+ + \mathbb{V}_p\ \mathbb{F}_0\ \mathbb{F}_0^- = \delta\mathbb{V}\ \mathbb{F}_0\ \mathbb{F}_0^- - \mathbb{V}\ \mathbb{F}_0^2\ e^{s\tau'}, \qquad \delta\mathbb{V} = \mathbb{V}_p - \mathbb{V} \tag{2.44}$$

Hence

$$D_1(x) = \delta V F_0 F_0^- - V F_0^2 e^{-sy} \tag{2.45}$$

From which we can compute $F_1^\pm(s,y)$ as follows

$$\mathbb{F}_1^+(s,\tau') = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{D_1(z)}{z-s} = c_1 + c_2 \tag{2.46}$$

Where

$$\begin{aligned} D_1(s) &= \delta\mathbb{V}\ \mathbb{F}_0(s)\ \mathbb{F}_0^-(s,\tau') - \mathbb{V}\ \mathbb{F}_0^2(s)\ e^{s\tau'} && (2.47)\\ c_1 &= \frac{\delta\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\mathbb{F}_0(z)\mathbb{F}_0^-(z,\tau')}{z-s} && (2.48)\\ c_2 &= \frac{-\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\mathbb{F}_0^2(z)e^{-z\tau'}}{z-s} && (2.49) \end{aligned}$$

At this point, one needs to add an arbitrary polynomial of s to the solution of $\mathbb{F}_1^+$. Fortunately, the property of $\mathbb{F}_n$ to vanish at infinity; $\lim_{s\to-\infty}\mathbb{F}_n(s)\to 0$, guaranties that this polynomial is identically zero. Finally, it is not hard to show that (see the appendices for mathematical details) :

$$\begin{aligned} c_1 &= \frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}} \\ c_2 &= -\mathbb{V}\frac{e^{-s\tau'}}{(s^2-E)^2} - \mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+s} - \frac{1}{\sqrt{E}(\sqrt{E}+s)} - \frac{1}{(\sqrt{E}+s)^2}\right] \end{aligned} \tag{2.50}$$

Hence

$$\begin{aligned} \mathbb{F}_1^+(s,\tau') = \frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}} - \mathbb{V}\frac{e^{-s\tau'}}{(s^2-E)^2} - \mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+s} + \right. \\ \left. - \frac{1}{\sqrt{E}(\sqrt{E}+s)} - \frac{1}{(\sqrt{E}+s)^2}\right], \qquad \mathrm{Re}(s)<0 \end{aligned} \tag{2.51}$$

To obtain $\mathbb{F}_1^-(s,\tau')$, one can either use the same method used to obtain $\mathbb{F}_1^+(s,\tau')$, or simply deduce it directly from (2.44) such that

$$\mathbb{F}_1^-(s,\tau') = \frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[1-\frac{4E}{(s+\sqrt{E})^2}\right] - \mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+s}+\right.$$
$$\left.-\frac{1}{\sqrt{E}(\sqrt{E}+s)}-\frac{1}{(\sqrt{E}+s)^2}\right], \qquad \mathrm{Re}(s)>0 \tag{2.52}$$

Now, to get the first order expansion of Green's function $\mathbb{G}_1^+(\tau,\tau')$ all one needs is to use the inverse Laplace transform over $\mathbb{F}_1^+$ (see Appendix A), where

$$\mathbb{G}_1^+(\tau,\tau') = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s\tau}\mathbb{F}_1^+(s,\tau')ds \tag{2.53}$$

Therefore

$$\mathbb{G}_1(\tau,\tau') = -\frac{\delta\mathbb{V}}{(2\sqrt{E})^3}e^{\sqrt{E}(\tau+\tau')} - \frac{\mathbb{V}}{2\sqrt{E}}\mathbb{G}_0(\tau,\tau')\left(\mid\tau-\tau'\mid+\frac{1}{\sqrt{E}}\right), \quad \tau<0,\ \tau'<0 \tag{2.54}$$

- **Second order**

In order to evaluate the second order contribution to Green's function, it is required to determine $D_2(s)$. By means of (2.38) it follows that

$$D_2 = -\mathbb{V}\mathbb{F}_0\mathbb{F}_1^+ + \mathbb{V}_p\mathbb{F}_0\mathbb{F}_1^- \tag{2.55}$$

Where

$$\mathbb{F}_2^+ - \mathbb{F}_2^- = D_2(s) \tag{2.56}$$

By means of (2.44), one can rewrite $\mathbb{F}_1^+$ in terms of $\mathbb{F}_1^-$, $D_2(s)$ then becomes :

$$\begin{aligned} D_2 &= -\mathbb{V}\mathbb{F}_0(\mathbb{F}_1^- + \mathbb{V}_p\mathbb{F}_0\mathbb{F}_0^- - \mathbb{V}\mathbb{F}_0\mathbb{F}_0^+) + \mathbb{V}_p\mathbb{F}_0\mathbb{F}_1^- \\ &= \delta\mathbb{V}\mathbb{F}_0\mathbb{F}_1^- - \mathbb{V}\ \mathbb{V}_p\ \mathbb{F}_0\ \mathbb{F}_0^- + \mathbb{V}^2\ \mathbb{F}_0^2\ \mathbb{F}_0^+ \end{aligned} \tag{2.57}$$

Therefore $\mathbb{F}_2^+$ can be written as

$$\begin{aligned} \mathbb{F}_2^+(s,\tau') &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\frac{D_2(z)}{z-s} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{z-s}(\delta\mathbb{V}\mathbb{F}_0\mathbb{F}_1^- - \mathbb{V}\ \mathbb{V}_p\ \mathbb{F}_0\ \mathbb{F}_0^- + \mathbb{V}^2\ \mathbb{F}_0^2\ \mathbb{F}_0^+) \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{z-s}\left(\ \delta\mathbb{V}\mathbb{F}_0\mathbb{F}_1^- - \mathbb{V}\ \mathbb{F}_0\left[\delta\mathbb{V}\ \mathbb{F}_0\ \mathbb{F}_0^- - \mathbb{V}\ \mathbb{F}_0^2\ e^{-z\tau'}\right]\ \right) \\ &\equiv c_1 + c_2 + c_3 \end{aligned} \tag{2.58}$$

In order to avoid lengthy expressions, the term c_1 will be decomposed into two parts : $c_1 = c_1^1 + c_1^2$ where (See appendix A for mathematical details),

$$\begin{aligned} c_1^1 &= \frac{\delta\mathbb{V}}{2\pi i}\int \frac{dz}{z-s}\frac{1}{E-z^2}\left(\frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{z-\sqrt{E}}\left[1-\frac{4E}{(z+\sqrt{E})^2}\right]\right) \\ &= \frac{-1}{16\sqrt{E}}\left(\frac{\delta\mathbb{V}}{E}\right)^2\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}} \end{aligned} \tag{2.59}$$

Whereas

$$\begin{aligned} c_1^2 &= \frac{\delta\mathbb{V}}{2\pi i}\int \frac{dz}{z-s}\frac{1}{E-z^2}\left(-\mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+z}-\frac{1}{\sqrt{E}(\sqrt{E}+z)}-\frac{1}{(\sqrt{E}+z)^2}\right]\right) \\ &= \frac{1}{16}\frac{\delta\mathbb{V}\,\mathbb{V}}{E^2}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\tau'-\frac{3}{2\sqrt{E}}\right] \end{aligned} \tag{2.60}$$

The term c_2 is the following

$$\begin{aligned} c_2 &= -\frac{\mathbb{V}\,\delta\mathbb{V}}{2\pi i}\int\frac{dz}{z-s}\mathbb{F}_0^2\,\mathbb{F}_0^- = -\frac{\mathbb{V}\,\delta\mathbb{V}}{2\pi i}\int\frac{dz}{z-s}\frac{1}{(z^2-E)^2}\frac{-e^{\sqrt{E}\tau'}}{2\sqrt{E}(z+\sqrt{E})} \\ &= \frac{1}{16}\frac{\mathbb{V}\,\delta\mathbb{V}}{E^2}.\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\frac{1}{s-\sqrt{E}}-\frac{3}{2\sqrt{E}}\right] \end{aligned} \tag{2.61}$$

c_3 by its turn is given by

$$\begin{aligned} c_3 &= \frac{1}{2\pi i}\int\frac{dz}{z-s}\mathbb{V}^2\mathbb{F}_0^3 e^{-z\tau'} \\ &= -\mathbb{V}^2\left[\frac{e^{-s\tau'}}{(s^2-E)^3}+\frac{e^{\sqrt{E}\tau'}}{2(2\sqrt{E})^3(\sqrt{E}+s)}\left(\left[\tau'-\frac{3}{2\sqrt{E}}-\frac{1}{\sqrt{E}+s}\right]^2+\right.\right. \\ &\quad \left.\left.+\frac{3}{4E}+\frac{1}{(s+\sqrt{E})^2}\right)\right] \end{aligned} \tag{2.62}$$

It is possible now to write down $\mathbb{F}_2^+$ expression

$$\begin{aligned} \mathbb{F}_2^+(s,\tau') &= \frac{-1}{16\sqrt{E}}\left(\frac{\delta\mathbb{V}}{E}\right)^2\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}+\frac{1}{16}\frac{\delta\mathbb{V}\,\mathbb{V}}{E^2}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\tau'-\frac{3}{2\sqrt{E}}\right]+ \\ &\quad +\frac{1}{16}\frac{\mathbb{V}\,\delta\mathbb{V}}{E^2}.\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\frac{1}{s-\sqrt{E}}-\frac{2}{3\sqrt{E}}\right]-\mathbb{V}^2\frac{e^{-s\tau'}}{(s^2-E)^3}-\mathbb{V}^2\frac{e^{-\sqrt{E}\tau'}}{2[2\sqrt{E}(\sqrt{E}+s)]^3} \\ &\quad -\mathbb{V}^2\left[\frac{e^{\sqrt{E}\tau'}}{2(2\sqrt{E})^3(\sqrt{E}+s)}\left(\left[\tau'-\frac{3}{2\sqrt{E}}-\frac{1}{\sqrt{E}+s}\right]^2+\frac{3}{4E}\right)\right] \end{aligned} \tag{2.63}$$

Finally, one can recover $\mathbb{G}_2^+(\tau,\tau')$ by taking the inverse of Laplace transform for $\mathbb{F}_2^+(s,y)$:

$$\begin{aligned}\mathbb{G}_2(\tau,\tau') &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\; e^{s\tau}\mathbb{F}_2^+(s,\tau') \\ &= \frac{e^{-\sqrt{E}|\tau-\tau'|}}{2\sqrt{E}}\left[\frac{1}{16}\left(\frac{\mathbb{V}}{E}\right)^2\left(\sqrt{E}\mid\tau-\tau'\mid+\frac{3}{2}\right)^2+\frac{3}{4}\right]+\frac{e^{\sqrt{E}(\tau+\tau')}}{2\sqrt{E}}\left[\frac{1}{8}\left(\frac{\delta\mathbb{V}}{E}\right)^2+\right. \\ &\left.-\frac{\mathbb{V}\delta\mathbb{V}}{16E^2}(\sqrt{E}\tau-\frac{3}{2})-\frac{\delta\mathbb{V}\,\mathbb{V}}{16E^2}\left(\sqrt{E}\tau'-\frac{3}{2}\right)\right], \qquad \tau<0,\ \tau'<0 \end{aligned} \tag{2.64}$$

At this stage, it would be enough to see if the expansion of Green's function defined by (2.23) coincides with the expansion obtained by the Wiener-Hopf method. In other words, to test if the expression of Green's function (2.23) is also valid when $\mathbb{V}$ is a matrix not just a scalar. As a matter of fact, the Green function (2.23) just needs a simple rearrangement of its terms to match the expansion obtained before. To be more specific, the proper Green's function is

$$\begin{aligned}\mathbb{G}_{\mp,\mp}(\tau,\tau') &= \frac{1}{2}\mathbb{W}_{\mp}^{-1}e^{-\mathbb{W}_{\mp}|\tau-\tau'|}+\frac{1}{2}\mathbb{W}_{\mp}^{-1}e^{\pm\mathbb{W}_{\mp}\tau}\left(\mathbb{W}_{\mp}-\mathbb{W}_{\pm}\right)\left(\mathbb{W}_{-}+\mathbb{W}_{+}\right)^{-1}e^{\pm\mathbb{W}_{\mp}\tau'} \\ \mathbb{G}_{\pm,\mp}(\tau,\tau') &= e^{\mp\mathbb{W}_{\pm}\tau}\left(\mathbb{W}_{-}+\mathbb{W}_{+}\right)^{-1}e^{\pm\mathbb{W}_{\mp}\tau'}\end{aligned} \tag{2.65}$$

where $\mathbb{W}_{\pm}$ are given by

$$\mathbb{W}_{-}=\sqrt{(V+E)}\otimes\mathbb{I}_n, \qquad \text{and} \quad \mathbb{W}_{+}=\mathbb{P}_{\pi}\mathbb{W}_{-}\mathbb{P}_{\pi}^{T}$$

The expansion of $\mathbb{G}(\tau,\tau')$ for $\tau,\tau'<0$ obtained by Wiener-Hopf method is

$$\begin{aligned}\mathbb{G}(\tau,\tau') &= \mathbb{G}_0(\tau,\tau')+\mathbb{G}_1(\tau,\tau')+\mathbb{G}_2(\tau,\tau')+..., \quad \tau,\tau'<0 \\ &= \frac{1}{2\sqrt{E}}e^{-\sqrt{E}|\tau-\tau'|}\left[1-\frac{\mathbb{V}}{E}(\sqrt{E}\mid\tau-\tau'\mid+1)+\frac{1}{16}\left(\frac{V}{E}\right)^2\left(\left(\sqrt{E}\mid\tau-\tau'\mid+\frac{3}{2}\right)^2+\right.\right. \\ &\left.\left.+\frac{3}{4}\right)+...\right]+\frac{e^{\sqrt{E}(\tau+\tau')}}{2\sqrt{E}}\left[-\frac{\delta\mathbb{V}}{4E}+\frac{1}{8}\left(\frac{\delta\mathbb{V}}{E}\right)^2-\frac{\mathbb{V}\delta\mathbb{V}}{16E^2}(\sqrt{E}\tau'-\frac{3}{2})+...\right]\end{aligned} \tag{2.66}$$

As anyone can check, Green's function given by (2.65), precisely $\mathbb{G}_{--}$, does have the same expansion (2.66) when $\tau,\tau'<0$. This is a good sign for (2.65) to be the correct solution of Green's function. However, there are more requirements to be fulfilled in order to consider it as a proper solution :

1. it should satisfy Green's differential equation :

$$(-\frac{d^2}{d\tau^2}+\mathbb{V}(\tau)+E)\mathbb{G}(\tau,\tau')=\delta(\tau-\tau').\mathbb{I}$$

2. $\lim_{\tau,\tau'\to\pm\infty}\mathbb{G}(\tau,\tau')=0$

3. All continuity conditions are satisfied.

4. The Hermiticity constraint $\mathbb{G}(\tau,\tau')^\dagger = \mathbb{G}(\tau',\tau)$

The first condition is a priori satisfied. The second condition is trivial to check. The third condition stands for the continuity of all parts of Green's function and their first derivatives at $\tau,\tau' = 0$. Such a condition is extracted from the spectral decomposition of $\mathbb{G}$. Where all the eigenfunctions $\{\Phi_l(\tau)\}$ and their first derivatives should be continuous at the cut $\tau = 0$, since the jump in the potential at $\tau = 0$ is finite, so that

$$\lim_{\tau\to 0^-}\Phi_l(\tau) = \lim_{\tau\to 0^+}\Phi_l(\tau), \qquad \lim_{\tau\to 0^-}\frac{d}{d\tau}\Phi_l(\tau) = \lim_{\tau\to 0^+}\frac{d}{d\tau}\Phi_l(\tau) \tag{2.67}$$

The fourth condition is easily proved using the spectral decomposition of $\mathbb{G}$:

$$\mathbb{G}(\tau,\tau') = \sum_l \frac{1}{\lambda_l}\Phi_l(\tau)\Phi_l^\dagger(\tau') \tag{2.68}$$

From which it follows directly that $\mathbb{G}(\tau,\tau')^\dagger = \mathbb{G}(\tau',\tau)$.

The last two conditions are not directly obvious. For this reason, they will be detailed separately in the next section.

2.2.2 Hermiticity constraint and Continuity conditions

The starting point will be the hermiticity constraint, since it will allow us to reduce the number of continuity conditions by half, as it will be shown later on.

This constraint can be stated for $\mathbb{G}(\tau,\tau')$ given by (2.65) as follows :

$$\mathbb{G}_{+-}(\tau,\tau')^\dagger = \mathbb{G}_{-+}(\tau',\tau) \tag{2.69}$$
$$\mathbb{G}_{\pm\pm}(\tau,\tau')^\dagger = \mathbb{G}_{\pm\pm}(\tau',\tau) \tag{2.70}$$

The first condition (2.69) is trivial to check. The second condition is a bit more complicated. To see that it is fulfiled by $\mathbb{G}(\tau,\tau')$, only one of these two conditions of (2.70) will be proven, take for example :

$$\mathbb{G}_{--}^\dagger(\tau,\tau') = \mathbb{G}_{--}(\tau',\tau) \tag{2.71}$$

where

$$\mathbb{G}_{--}(\tau',\tau) = \frac{1}{2\mathbb{W}_-}e^{-\mathbb{W}_-|\tau-\tau'|} + \frac{1}{2\mathbb{W}_-}e^{\mathbb{W}_-\tau'}\left(\mathbb{W}_- - \mathbb{W}_+\right)\left(\mathbb{W}_- + \mathbb{W}_+\right)^{-1}e^{\mathbb{W}_-\tau} \tag{2.72}$$
$$\mathbb{G}_{--}(\tau,\tau')^\dagger = \frac{1}{2\mathbb{W}_-}e^{-\mathbb{W}_-|\tau-\tau'|} + \frac{1}{2}e^{\mathbb{W}_-\tau'}\left(\mathbb{W}_- + \mathbb{W}_+\right)^{-1}\left(\mathbb{W}_- - \mathbb{W}_+\right)\frac{1}{\mathbb{W}_-}e^{\mathbb{W}_-\tau} \tag{2.73}$$

In order to show that $\mathbb{G}_{--}^\dagger(\tau,\tau') = \mathbb{G}_{--}(\tau',\tau)$, it is required to show first that

$$\mathbb{W}_-^{-1}(\mathbb{W}_- - \mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1} = (\mathbb{W}_- + \mathbb{W}_+)^{-1}(\mathbb{W}_- - \mathbb{W}_+)\mathbb{W}_-^{-1} \tag{2.74}$$

where

$$\begin{aligned}(\mathbb{W}_- + \mathbb{W}_+)^{-1}(\mathbb{W}_- - \mathbb{W}_+)\mathbb{W}_-^{-1} &= (\mathbb{W}_- + \mathbb{W}_+)^{-1} - (\mathbb{W}_- + \mathbb{W}_+)^{-1}\mathbb{W}_+\mathbb{W}_-^{-1} \\ &= (\mathbb{W}_- + \mathbb{W}_+)^{-1} - (\mathbb{W}_-\mathbb{W}_+^{-1}(\mathbb{W}_- + \mathbb{W}_+))^{-1} \\ &= (\mathbb{W}_- + \mathbb{W}_+)^{-1} - \mathbb{W}_-^{-1}(\mathbb{W}_-\mathbb{W}_+^{-1} + 1)^{-1} \\ &= (\mathbb{W}_- + \mathbb{W}_+)^{-1} - \mathbb{W}_-^{-1}\mathbb{W}_+(\mathbb{W}_- + \mathbb{W}_+)^{-1} \\ &= (1 - \mathbb{W}_-^{-1}\mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1} \\ &= \mathbb{W}_-^{-1}(\mathbb{W}_- - \mathbb{W}_+)^{-1}(\mathbb{W}_- + \mathbb{W}_+)^{-1} \end{aligned} \tag{2.75}$$

Therefore, condition (2.71) is fulfilled.

- **Continuity conditions**

There are 6 continuity conditions to be proved, listed as follows

$$\begin{array}{rcl} \lim_{\tau'\to 0}\mathbb{G}_{-+}(\tau,\tau') &=& \lim_{\tau'\to 0}\mathbb{G}_{--}(\tau,\tau') \\ \lim_{\tau'\to 0}\mathbb{G}_{+-}(\tau,\tau') &=& \lim_{\tau'\to 0}\mathbb{G}_{++}(\tau,\tau') \\ \lim_{\tau'\to 0}\partial_\tau\mathbb{G}_{-+}(\tau,\tau') &=& \lim_{\tau'\to 0}\partial_\tau\mathbb{G}_{--}(\tau,\tau') \\ \lim_{\tau\to 0}\partial_\tau\mathbb{G}_{-+}(\tau,\tau') &=& \lim_{\tau\to 0}\partial_\tau\mathbb{G}_{++}(\tau,\tau') \\ \lim_{\tau'\to 0}\partial_\tau\mathbb{G}_{+-}(\tau,\tau') &=& \lim_{\tau'\to 0}\partial_\tau\mathbb{G}_{++}(\tau,\tau') \\ \lim_{\tau\to 0}\partial_\tau\mathbb{G}_{+-}(\tau,\tau') &=& \lim_{\tau\to 0}\partial_\tau\mathbb{G}_{--}(\tau,\tau') \end{array} \tag{2.76}$$

The other 6 continuity conditions, with respect to the other argument of $\mathbb{G}$, are satisfied identically by the hermiticity constraint.
The first condition of (2.76) can be simply proved :

$$\begin{aligned}\lim_{\tau'\to 0}\partial_\tau\mathbb{G}_{-+}(\tau,\tau') &= e^{\mathbb{W}_-\tau}\frac{1}{\mathbb{W}_- + \mathbb{W}_+} \\ \lim_{\tau'\to 0}\partial_\tau\mathbb{G}_{--}(\tau,\tau') &= \frac{1}{2\mathbb{W}_-}.e^{\mathbb{W}_-\tau} + \frac{1}{2\mathbb{W}_-}e^{\mathbb{W}_-\tau}(\mathbb{W}_- - \mathbb{W}_+)\frac{1}{\mathbb{W}_- + \mathbb{W}_+} \\ &= e^{\mathbb{W}_-\tau}\frac{1}{2\mathbb{W}_-}\left[\ \mathbb{W}_- + \mathbb{W}_+ + (\mathbb{W}_- - \mathbb{W}_+)\ \right]\frac{1}{\mathbb{W}_- + \mathbb{W}_+} \\ &= e^{\mathbb{W}_-\tau}\frac{1}{\mathbb{W}_- + \mathbb{W}_+}\end{aligned}$$

If one takes the hermitian conjugate of this condition, one will get :

$$\lim_{\tau\to 0}\mathbb{G}_{+-}(\tau,\tau') = \lim_{\tau\to 0}\mathbb{G}_{--}(\tau,\tau') \tag{2.77}$$

Which is a different condition; the condition for the other argument τ, thanks to the hermiticity constraint, it is identically fulfilled.

The second condition of (2.76) can be handled by the same previous manipulation.

The continuity of the derivatives can be handled in a similar manner, where

$$\begin{aligned}
\partial_\tau\ \mathbb{G}_{++}(\tau,\tau') &= -\frac{|\ \tau-\tau'\ |}{2(\tau-\tau')}e^{-\mathbb{W}_+|\tau-\tau'|}-\frac{1}{2}e^{-\mathbb{W}_+\tau}(\mathbb{W}_+-\mathbb{W}_-)(\mathbb{W}_-+\mathbb{W}_+)^{-1}e^{-\mathbb{W}_+\tau'}\\
\partial_\tau\ \mathbb{G}_{--}(\tau,\tau') &= -\frac{|\ \tau-\tau'\ |}{2(\tau-\tau')}e^{-\mathbb{W}_-|\tau-\tau'|}+\frac{1}{2}.e^{\mathbb{W}_-\tau}(\mathbb{W}_+-\mathbb{W}_-)(\mathbb{W}_-+\mathbb{W}_+)^{-1}e^{\mathbb{W}_-\tau'}\\
\partial_\tau\ \mathbb{G}_{+-}(\tau,\tau') &= -\mathbb{W}_+e^{-\mathbb{W}_+\tau}(\frac{1}{\mathbb{W}_-+\mathbb{W}_+})e^{\mathbb{W}_-\tau'}\\
\partial_\tau\ \mathbb{G}_{-+}(\tau,\tau') &= \mathbb{W}_-e^{\mathbb{W}_-\tau}(\frac{1}{\mathbb{W}_+-\mathbb{W}_+})e^{-\mathbb{W}_+\tau'}
\end{aligned}$$

For the third condition of (2.76), it simple to show that it is satisfied :

$$\partial_\tau\mathbb{G}_{-+}(\tau,\tau') = \mathbb{W}_-e^{\mathbb{W}_-\tau}\frac{1}{\mathbb{W}_-+\mathbb{W}_-} \tag{2.78}$$

$$\begin{aligned}
\partial_\tau\mathbb{G}_{--}(\tau,\tau') &= \frac{1}{2}e^{-\mathbb{W}_-\tau}+\frac{1}{2}e^{\mathbb{W}_-\tau}(\mathbb{W}_+-\mathbb{W}_-)(\mathbb{W}_-+\mathbb{W}_+)^{-1}\\
&= \frac{1}{2}e^{\mathbb{W}_-\tau}\left[\mathbb{W}_-+\mathbb{W}_++(\mathbb{W}_--\mathbb{W}_+)\right](\mathbb{W}_-+\mathbb{W}_+)^{-1}\\
&= \mathbb{W}_-e^{\mathbb{W}_-\tau}(\mathbb{W}_-+\mathbb{W}_+)^{-1}
\end{aligned} \tag{2.79}$$

As it was shown, no new ideas were used to prove the third condition;the same trick was used to prove the first one. This is the case for the rest of the conditions. Therefore, their proof will be skipped and all continuity conditions will be considered fulfilled.

To this end, Green's function matrix given by (2.65) satisfies all requirements. As a result, **it is the proper solution for a chain of coupled harmonic oscillator systems. The differential equation for its Green's function is defined by (2.27).**

2.3 Rényi entropy via Green's matrix function

After obtaining the formula of Green's function for a subset of harmonic oscillators arbitrarily chosen from the chain of oscillators, it is straightforward to write down Rényi entropy for the system. From the expression of Rényi entropy (2.22) and after a simple integration over τ for $\mathbb{G}_n(\tau,\tau,E)$, Rényi entropy is found to be

$$\boxed{\ln(\mathrm{Tr}(\rho^n))=\frac{1}{8}\int_0^\infty dE\ \ \mathrm{Tr}\left[\ (\mathbb{V}_-^{-1}-\mathbb{V}_+^{-1})(\mathbb{W}_--\mathbb{W}_+)(\mathbb{W}_-+\mathbb{W}_+)^{-1}\right]} \tag{2.80}$$

This is the formal expression of Rényi entropy, which can be used to find the entanglement entropy via the replica trick.

The alternative new formula (2.80) of Rényi entropy directly shows the dependence of the entropy on the chosen subsystem A, which is implemented on ($\mathbb{W}_+$, $\mathbb{V}_+^{-1}$) via the permutation matrix $\mathbb{P}_\pi$.

When the subsystem A is assumed to be the overall system so that $\rho_A = \rho_{AB}$, the permutation matrix becomes the identity matrix, and Rényi entropy vanishes, as it should because $\mathbb{V}_+ = \mathbb{V}_-$.

In general, when the permutation matrix and the potential matrix commutes $[\mathbb{V}, \mathbb{P}_\pi] = 0$, the permuted potential coincides with the potential matrix, $\mathbb{V} = \mathbb{V}_p$, there is no entanglement in this situationand Rényi entropy vanishes identically. Therefore, the quantity $\| [\mathbb{V}, \mathbb{P}_\pi] \| / \| \mathbb{V} \|$ can be used as some sort of indicator or a measure of the degree of entanglement.

A final remark about (2.80) is that it's a scaleless quantity, i.e., it is invariant under an overall rescaling of the potential. A property which should be intuitively satisfied by a proper entanglement measure.

Chapter 3

Applications and Outlook

The obtained Green's function for a lattice of oscillators gave a formal expression for Rényi entropy. The aim of this chapter is to see the possible applications of this expression and compare it with the already existing results. The possibility of using the obtained Rényi entropy to treat interacting theory problems is shown via $\lambda\phi^4$ type of interaction. The interaction was treated as a perturbation, and Rényi entropy was expanded as a power series of λ.

3.1 Entanglement of 2-coupled oscillators

The preliminary example of the entanglement of two coupled harmonic oscillators, already treated in the first chapter within the real time approach, is a perfect candidate as a first test of the validity of the obtained Green's function and the method as a whole.

As a starting point, take two oscillators with the same frequency. This system is described by the Lagrangian :

$$L = \frac{1}{2}\left(\dot{Q}^T\dot{Q} - Q^T V Q\right) \tag{3.1}$$

where

$$V = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3.2}$$

q_1 (q_2) is the coordinate of the first (second) oscillator. The potential V is rescaled from the beginning, so that $V_{12} = V_{21} = 1$. To evaluate Rényi entropy, one first needs to determine the matrices $\mathbb{V}_\pm^{-1}$ and $\mathbb{W}_\pm$). The fact that V is a 2×2 circulant matrix simplifies the determination of V^{-1} and W, where

$$V^{-1} = \frac{1}{a^2-1}\begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix}, \; W = \frac{1}{2}\begin{bmatrix} \alpha_+ & \alpha_- \\ \alpha_- & \alpha_+ \end{bmatrix} \tag{3.3}$$

With $\alpha_\pm = \sqrt{a+1} \pm \sqrt{a-1}$. It is straightforward to determine the matrices $\mathbb{V}_-^{-1}$ and $\mathbb{W}_-$ since $\mathbb{W}_- = (V+E)^{1/2} \otimes I_n$ and $\mathbb{V}_-^{-1} = (V+E)^{-1} \otimes I_n$. The matrices $\mathbb{V}_+^{-1}$ and $\mathbb{W}_+$ are easily obtained

by the permutation matrix (2.12). The situation here is similar to the one taken in the previous chapter, where half of the chain is traced over, so that

$$\mathbb{V}_+^{-1} = \mathbb{P}_\pi \mathbb{V}_-^{-1} \mathbb{P}_\pi^T = \frac{1}{a^2-1} \begin{bmatrix} a & 0 & 0 & 0 & 0 & \dots & -1 \\ 0 & a & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & a & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & a & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & 0 & \dots & a \end{bmatrix} \tag{3.4}$$

$$\mathbb{W}_+ = \mathbb{P}_\pi \mathbb{V}_-^{1/2} \mathbb{P}_\pi^T = \frac{1}{2} \begin{bmatrix} \alpha_+ & 0 & 0 & 0 & 0 & \dots & \alpha_- \\ 0 & \alpha_+ & \alpha_- & 0 & 0 & \dots & 0 \\ 0 & \alpha_- & \alpha_+ & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha_+ & \alpha_- & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_- & 0 & 0 & 0 & 0 & \dots & \alpha_+ \end{bmatrix} \tag{3.5}$$

It is understood here that E is implemented in the term a, in other words, $a \equiv a + E$. It is simple to show that the corresponding matrix product $(\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+)$ is given by

$$(\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+) = -\frac{\alpha_-}{2(a^2-1)} \mathbb{C} \tag{3.6}$$

Where $\mathbb{C}$ is $2n \times 2n$ circulant matrix defined as follows

$$\mathbb{C}_{ij} = \begin{cases} 2 & i-j = 0 \\ -1 & i-j = 2,\ 2n-2 \\ 0 & \text{else} \end{cases} \tag{3.7}$$

In this case, the matrix $(\mathbb{W}_+ + \mathbb{W}_-)$ is a circulant matrix. Its inverse, call it $\mathbb{C}'$, is also a circulant matrix. It can be easily shown to be :

$$\mathbb{C}'_{ij} = \begin{cases} 2\alpha_+ & i-j = 0 \\ \alpha_- & i-j = 1,\ 2n-1 \\ 0 & \text{else} \end{cases} \tag{3.8}$$

Thus

$$\mathrm{Tr}\big(\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1}\big)(\mathbb{W}_- - \mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1} = -\frac{\alpha_-}{2(a^2-1)} \mathrm{Tr}(\mathbb{C}\mathbb{C}') \tag{3.9}$$

Both matrices $\mathbb{C}$ and $\mathbb{C}'$ are circulant, thus they can be diagonalized simultaneously. The eigenvalues of $\mathbb{C}$ and $\mathbb{C}'$ are given by

$$\lambda_j = 2\left(1 - \cos\left(\frac{2\pi j}{n}\right)\right), \quad \lambda'_j = \left(\alpha_+ + \alpha_- \cos\left(\frac{\pi j}{n}\right)\right)^{-1} \qquad j = 0..2n-1 \tag{3.10}$$

Rényi entropy can be easily obtained using the formula (2.80), such that

$$\ln(\mathrm{Tr}(\rho^n)) = \frac{1}{8}\sum_{j=0}^{2n-1}\left(1-\cos\left(\frac{2\pi j}{n}\right)\right)\int_a^\infty dx \frac{-\alpha_-(x)}{(x^2-1)(\alpha_+(x)+\alpha_-(x)\cos(\pi j/n)\ )} \tag{3.11}$$

Where $x \equiv a+E$. The integration in (3.11) can be evaluated exactly. Taking into consideration that the two terms $j = 0, n$ of the sum vanish from the beginning, the result after some rearrangements and simplifications is

$$\ln(\mathrm{Tr}(\rho^n)) = -\sum_{j=1}^{n-1}\ln\left(\frac{\cos\left(\frac{\pi j}{n}\right)+\beta}{\sqrt{\beta^2-1}}\right) \tag{3.12}$$

where $\beta = a+\sqrt{a^2-1}$. To achieve the sum, one can use the formula [34] :

$$\prod_{k=1}^{n-1}\left(\beta+\cos\left(\frac{k\pi}{n}\right)\right) = \frac{1}{(2\alpha)^{n-1}}\frac{\alpha^{2n}-1}{\alpha^2-1}, \qquad \alpha = \beta+\sqrt{\beta^2-1} \tag{3.13}$$

it follows that

$$\ln(\mathrm{Tr}(\rho^n)) = \ln\left(\frac{(\alpha^2-1)^n}{\alpha^{2n}-1}\right) \tag{3.14}$$

Finally, the entanglement entropy can be easily shown to be

$$S = -\frac{d}{dn}\ln(\mathrm{Tr}(\rho^n))\ |_{n=1} = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln(\xi) \tag{3.15}$$

This is the standard relation for the entanglement entropy of two coupled harmonic oscillators. This realtion was already derived in the first chapter within the real time approach, relation (1.27), where it turns out that $\sqrt{\xi} = 1/\alpha$. Pleasingly, the new method gives exactly the same standard result!

3.1.1 Entanglement of 2 harmonic oscillators with different frequencies

The situation when the two coupled oscillators have different frequencies is a bit more complicated than the previous one. In this situation, the rescaled potential matrix is given by

$$V = \begin{bmatrix} a & 1 \\ 1 & c \end{bmatrix} \tag{3.16}$$

whereas the matrices V^{-1} and W are given by

$$V^{-1} = \frac{1}{ac-1}\begin{bmatrix} c & -1 \\ -1 & a \end{bmatrix},\ W = \begin{bmatrix} \omega_1 & \omega \\ \omega & \omega_2 \end{bmatrix} \tag{3.17}$$

with

$$\omega_1 = \frac{1}{\lambda_+ - \lambda_-}(\sqrt{\lambda_-}(\lambda_+ - a) - \sqrt{\lambda_+}(\lambda_- - a)\ ) \tag{3.18}$$

$$\omega_2 = \frac{1}{\lambda_+ - \lambda_-}.(\ \sqrt{\lambda_+}(\lambda_+ - a) - \sqrt{\lambda_-}(\lambda_- - a)\ ) \tag{3.19}$$

$$\omega = \frac{1}{\lambda_+ - \lambda_-}(\ \sqrt{\lambda_+} - \sqrt{\lambda_-}\ ) \tag{3.20}$$

$$\lambda_\pm = \frac{1}{2}(c + a \pm \sqrt{(a-c)^2 + 4}) \tag{3.21}$$

The evaluation of the entanglement between the two oscillators using Green's function requires the determination of $\mathbb{V}_+^{-1}$ and $\mathbb{W}_+$, where

$$\mathbb{V}_+^{-1} = \frac{1}{ac-1}\begin{bmatrix} c & 0 & 0 & 0 & 0 & \dots & -1 \\ 0 & a & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & c & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & a & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & 0 & \dots & a \end{bmatrix},\ \mathbb{W}_+ = \begin{bmatrix} \omega_1 & 0 & 0 & 0 & 0 & \dots & \omega \\ 0 & \omega_2 & \omega & 0 & 0 & \dots & 0 \\ 0 & \omega & \omega_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \omega_2 & \omega & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \omega & 0 & 0 & 0 & 0 & \dots & \omega_2 \end{bmatrix} \tag{3.22}$$

Again, it is understood here that 'E' is implemented in the terms a and c, such that $a \equiv a + E$ and $c \equiv c + E$. Now, it is possible to calculate the trace of the matrix product. The starting point will be :

$$(\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+) = \omega\frac{-1}{ac-1}\mathbb{C} \tag{3.23}$$

Where $\mathbb{C}$ is the same circulant matrix defined before by (3.7). It is then diagonalizable by Fourier discrete transformation; $\mathbb{C} = F^{-1}\mathbb{C}_d F$, where F is Fourier discrete transform :

$$F = (F^{-1})^* = \frac{1}{\sqrt{2n}}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{2n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(2n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{2n-1} & w^{2(2n-1)} & \dots & w^{(2n-1)(2n-1)} \end{bmatrix}, \quad w = e^{-2\pi i/2n} \tag{3.24}$$

The eigenvalues of $\mathbb{C}$ are the following :

$$(\mathbb{C}_d)_{jj} = 2\,(1 - \cos(2\pi j/n))\,, \qquad j = 0..2n-1 \tag{3.25}$$

The trace of the matrix product can then be written as follows :

$$\begin{aligned} \mathrm{Tr}\big((\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1}\big) &= \frac{-\omega}{ac-1}\ \mathrm{Tr}\big[\mathbb{C}_d F(\mathbb{W}_+ + \mathbb{W}_-)^{-1}F^{-1}\big] \\ &= \frac{-\omega}{ac-1}\ \mathrm{Tr}\big[\mathbb{C}_d(F(\mathbb{W}_+ + \mathbb{W}_-)F^{-1})^{-1}\big] \end{aligned} \tag{3.26}$$

The matrix $\mathbb{W}_- + \mathbb{W}_+$ can be decomposed to the sum of a circulant matrix, call it $\mathcal{C}$, and an extra matrix, call it M, such that

$$\mathcal{C} = \begin{bmatrix} 2\omega_1 & \omega & 0 & \dots & \omega \\ \omega & 2\omega_1 & \omega & \dots & 0 \\ 0 & \omega & 2\omega_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega & 0 & 0 & \dots & 2\omega_1 \end{bmatrix}, \quad M = 2(\omega_2 - \omega_1) \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{3.27}$$

When a Fourier transformation is applied to the matrix $\mathcal{C}$, it is diagonalized; such that $F\mathcal{C}F^{-1} = \mathcal{C}_d$. The eigenvalues of $\mathcal{C}$ are the following :

$$(\mathcal{C}_d)_{jj} = 2\left(\omega_1 + \omega\cos\left(\frac{\pi j}{n}\right)\right), \qquad j = 0..2n-1 \tag{3.28}$$

On the other hand, the matrix FMF^{-1} is given by :

$$(FMF^{-1})_{ij} = \begin{cases} \omega_2 - \omega_1 & i = j \\ \omega_1 - \omega_2 & |i-j| = n \\ 0 & \text{otherwise} \end{cases} \tag{3.29}$$

The inverse of the matrix $F(W_- + W_+)F^{-1}$ can be obtained by the Gaussian elimination. Take a $2n \times 2n$ matrix T of the form :

$$T_{ij} = \begin{cases} a_i & i = j \\ d & \mid i-j \mid = n \\ 0 & \text{otherwise} \end{cases} \tag{3.30}$$

The inverse of the matrix T is given by :

$$T^{-1}_{ij} = \begin{cases} a_{n+i}/(a_{n+i}a_i - d^2) & i = j < n \\ a_{i-n}/(a_{i-n}a_i - d^2) & i = j \geq n \\ d/(d^2 - a_j a_i) & \mid i-j \mid = n \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

The inverse matrix $(F(\mathbb{W}_- + \mathbb{W}_+)F^{-1})^{-1}$ can be easily deduced from (3.31) since the matrix $F(\mathbb{W}_- + \mathbb{W}_+)F^{-1}$ has the same form of the matrix T. Therefore, it is straightforward to compute the matrix product $\mathrm{Tr}\big((\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1}\big)$ given by (3.26). Rényi entropy is then given by

$$\ln(Tr(\rho^n)) = \sum_{j=0}^{2n-1} \int_{x_0}^{\infty} dx \frac{-1}{8} \frac{(1 - \cos\left(\frac{2\pi j}{n}\right))}{x^2 - \delta^2} \frac{\delta^2 - (x - \sqrt{x^2 - \delta^2})\cos(\pi j/n)}{2\delta^2\sqrt{x^2 - \delta^2} + (x - \sqrt{x^2 - \delta^2})\sin^2(\pi j/n)} \tag{3.32}$$

where :

$$x = E + \frac{a+c}{2},\ x_0 = \frac{a+c}{2},\ \delta = \sqrt{\left(\frac{a-c}{2}\right)^2 + 1} \tag{3.33}$$

It is not difficult to show that the term multiplied by $\cos(\pi j/n)$ in (3.32) is summed to zero. Also, the two terms $j = 0, n$ of the sum vanish from the beginning.

Consider the following change of variable $y = (x + \sqrt{x^2 - \delta^2})^2$, Rényi entropy then becomes :

$$\begin{aligned}
\ln(Tr(\rho^n)) &= 2\int_{y_0}^{\infty} dy \sum_{j=1}^{n-1} \frac{-(1-\cos(2\pi j/n))}{8} \frac{1}{(\ y - \delta^2 + \sin^2(\pi j/n)\ )(y-\delta^2)} \\
&= 2\sum_{j=1}^{n-1} \frac{-(1-\cos(2\pi j/n))}{8} \frac{1}{\sin^2(\pi j/n)} \ln\left(\frac{y_0 - \delta^2 + \sin^2(\pi j/n)}{y_0 - \delta^2}\right) \\
&= \frac{-1}{2} \ln[\prod_{j=1}^{n-1} (\frac{y_0 - \delta^2 + \sin^2(\pi j/n)}{y_0 - \delta^2})]
\end{aligned} \tag{3.34}$$

By means of the identity (3.13), one can evaluate the following product

$$\begin{aligned}
\prod_{j=1}^{n-1} (y_0 - \delta^2 + \sin^2(\pi j/n)) &= \prod_{j=1}^{n-1} (\sqrt{y_0 - \delta^2 + 1} - \cos(\pi j/n)\ )(\sqrt{y_0 - \delta^2 + 1} + \cos(\pi j/n)\ ) \\
&= \left(\prod_{j=1}^{n-1} (\sqrt{y_0 - \delta^2 + 1} + \cos(\pi j/n))\ \right)^2 \\
&= \frac{1}{(2\alpha)^{2n-2}} \frac{(\alpha^{2n}-1)^2}{(\alpha^2-1)^2}
\end{aligned} \tag{3.35}$$

Where $\alpha = \sqrt{y_0 - \delta^2} + \sqrt{y_0 - \delta^2 + 1}$. In the second equality, the product was raised to power two because $\cos(\pi(n-j)/n) = -\cos(\pi j/n)$, and thus the two products are equal.

Hence, Rényi entropy can be written as

$$\ln(Tr(\rho^n)) = -\ln\left(\frac{\alpha^{2n}-1}{(\alpha^2-1)^n}\right) \tag{3.36}$$

Finally, the entanglement entropy for two coupled oscillators with different frequencies is

$$S = -\frac{d}{dn} \ln(\mathrm{Tr}(\rho^n))\ |_{n=1} = -\ln\big(\alpha^2 - 1\big) - \frac{\alpha^2}{1-\alpha^2} \ln\big(\alpha^2\big) \tag{3.37}$$

Where $\alpha = \sqrt{y_0 - \delta^2} + \sqrt{y_0 - \delta^2 + 1}$ with $y_0 = (\ (a+c)/2 + \sqrt{a.c-1}\ )^2$ and $\delta = [\ (a-c)^2/4 + 1\ ]$. It can be checked that $\alpha^2 = \xi^{-1}$, it then follows that (3.37) gives the same standard formula (1.27) of the entanglement of two coupled harmonic oscillators.

3.2 Entanglement of 1+1 Massive scalar field

The investigation of the case of two coupled harmonic oscillators formed a good exercise to check the validity of the method. It is time now to move into a more explored physical situation : the case of a massive Klein-Gordon field. This problem has been extensively treated in the literature, especially in the continuum.

In this section, the problem is treated discretely by applying a lattice regularization to the field. After that, it is possible to apply Green's function approach over it.

The Lagrangian of 1+1 real massive scalar field is given by

$$L = \frac{1}{2}\int dx \left(\dot{\phi}^2 - (\partial_x \phi)^2 + m^2\phi^2\right) \tag{3.38}$$

This spatial line of fields is discretized into a chain of oscillators as follows : The position axis x is discretized so that only specific locations are allowed. Precisely, the permitted positions are of the form $x_i = i\epsilon$ where $i \in \mathbb{Z}$ and ϵ is the lattice spacing. It also represents the UV cutoff. For the infrared cutoff, the line is taken to be limited to $2N$ sites only, with periodic boundary conditions. Therefore, the field function $\phi(x)$ becomes

$$\phi(x) \to q_i \equiv \phi(x_i), \qquad i = -N..N \tag{3.39}$$

The lagrangian then becomes of the form

$$L = \frac{1}{2}\sum_i \epsilon \left(\dot{q}_i^2 - \left(\frac{q_{i+1} - q_i}{\epsilon}\right)^2 + m^2 q_i^2\right) \tag{3.40}$$

By the simple rescaling $q_i^2 \to q_i^2/\epsilon$, the Lagrangian can be rewritten as

$$\begin{aligned} L &= \frac{1}{2}\sum_i \left(\dot{q}_i^2 - \frac{1}{\epsilon^2}\left(q_{i+1}^2 + q_i^2 - 2q_i q_{i+1} + \epsilon^2 m^2 q_i^2\right)^2\right) \\ &= \frac{1}{2}\left(\dot{Q}^T\dot{Q} - \frac{1}{\epsilon^2}Q^T V Q\right) \end{aligned} \tag{3.41}$$

The rescaled potential V will be considered from now on instead of V/ϵ^2. Since the boundary conditions are periodic, this means that the last coordinate and the first are identical. Therefore, Q is a $2N$ coordinate vector and V is $2N \times 2N$ matrix given by :

$$V = \begin{pmatrix} 2+\mu^2 & -1 & 0 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2+\mu^2 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2+\mu^2 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2+\mu^2 & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2+\mu^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 2+\mu^2 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & 2+\mu^2 \end{pmatrix} \tag{3.42}$$

Where $\mu = m\epsilon$.

Fortunately, V is a symmetrical circulant matrix. Its elements can be formally written in terms of its eigenvalues as follows

$$V_{lm} = \frac{1}{2N} \sum_{k=0}^{2N-1} \varphi_k e^{i\pi k(l-m)/N} \tag{3.43}$$

$\{\varphi_k\}$ are the eigenvalues of V given by

$$\varphi_k = (\mu^2 + E + 2(1 - \cos(\pi k/N)) \ ), \quad k = 0..2N-1 \tag{3.44}$$

Where the matrix $E \otimes I_{2N}$ is implemented in V upon this stage. Similarly, the circulant matrices V^{-1} and W have the following matrix elements:

$$V^{-1}_{lm} = \frac{1}{2N} \sum_{k=0}^{2N-1} \varphi_k^{-1} e^{i\pi k(l-m)/N} \tag{3.45}$$

$$W_{lm} = \frac{1}{2N} \sum_{k=0}^{2N-1} \varphi_k^{1/2} e^{i\pi k(l-m)/N} \tag{3.46}$$

The determination of $\mathbb{W}_+$ and $\mathbb{V}_+^{-1}$ necessitates the determination of the reduced system A. Two situations will be explored; the first takes half the space to be under supervision. The second situation is a more peculiar one, where the reduced system is formed of only one vertex of the lattice, namely the origin. This latter turned out to be exactly solvable, and it gave some interesting results.

3.2.1 Situation one : Entanglement of half space

In this situation, half of the oscillators in the chain are traced over, keeping only the first N-oscillators under supervision and averaging over the rest. The matrices $\mathbb{W}_+$ and $\mathbb{V}_+^{-1}$ are found using the permutation matrix (2.12). It is preferable first to decompose the matrices W and V^{-1} as follows

$$V^{-1} = \frac{1}{ac-1} \begin{bmatrix} A & B \\ B^T & A \end{bmatrix}, \ W = \begin{bmatrix} G & F \\ F^T & G \end{bmatrix} \tag{3.47}$$

$A,\ B,\ G$ and F are $N \times N$ matrices, their matrix elements are given as follows

$$A_{lm} = \frac{1}{2N} \sum_{k=0}^{2N-1} \varphi_k^{-1} e^{i\pi k(l-m)/N}, \qquad l, m = 0..N-1 \tag{3.48}$$

$$B_{lm} = \frac{1}{2N} \sum_{k=0}^{2N-1} \varphi_k^{-1} (-1)^k e^{i\pi k(l-m)/N}, \qquad l, m = 0..N-1 \tag{3.49}$$

$$G_{lm} = \frac{1}{2N}\sum_{k=0}^{2N-1}\varphi_k^{1/2}e^{i\pi k(l-m)/N}, \qquad l,m = 0..N-1 \tag{3.50}$$

$$F_{lm} = \frac{1}{2N}\sum_{k=0}^{2N-1}\varphi_k^{1/2}(-1)^k e^{i\pi k(l-m)/N}, \qquad l,m = 0..N-1 \tag{3.51}$$

The related matrix product $(\mathbb{V}_+^{-1}-\mathbb{V}_-^{-1})(\mathbb{W}_+-\mathbb{W}_-)$ is given by

$$(\mathbb{V}_-^{-1}-\mathbb{V}_+^{-1})(\mathbb{W}_--\mathbb{W}_+) = B.F\otimes\mathbb{C} \tag{3.52}$$

$\mathbb{C}$ is the same circulant matrix defined by (3.7). On the other hand

$$(\mathbb{W}_++\mathbb{W}_-) = \begin{pmatrix} 2G & F & 0 & 0 & 0 & \dots & 0 & F \\ F & 2G & F & 0 & 0 & \dots & 0 & 0 \\ 0 & F & 2G & F & 0 & \dots & 0 & 0 \\ 0 & 0 & F & 2G & F & \dots & 0 & 0 \\ 0 & 0 & 0 & F & 2G & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 2G & F \\ F & 0 & 0 & 0 & \dots & 0 & F & 2G \end{pmatrix} \tag{3.53}$$

$(\mathbb{W}_++\mathbb{W}_-)$ is a block circulant matrix. It has an explicit block inverse when applying a block-discrete Fourier transformation on it. The trace of the matrix product $(\mathbb{V}_+^{-1}-\mathbb{V}_-^{-1})(\mathbb{W}_+-\mathbb{W}_-)(\mathbb{W}_++\mathbb{W}_-)^{-1}$ can be partially computed in terms of the block submatrices as follows

$$\begin{aligned}\mathrm{Tr}\big((\mathbb{V}_+^{-1}-\mathbb{V}_-^{-1})(\mathbb{W}_+-\mathbb{W}_-)(\mathbb{W}_++\mathbb{W}_-)^{-1}\big) &= \\ \sum_{j=0}^{2n-1}(1-\cos\frac{2\pi j}{n}) \quad \mathrm{Tr} &\left(B.F\left(G+F.\cos\left(\frac{\pi j}{n}\right)\right)^{-1}\right)\end{aligned} \tag{3.54}$$

It is straightforward to show that the two coupled harmonic oscillator's problem is a special case of (3.54), where $N=1$ and $B,\ F$and G are just scalars.

The determination of (3.54) turns out to be a strenuous task. The main problem is to determine the inverse of the matrix $(G+F\cos(\pi j/n)\ )$, which is a special Toeplitz matrix with N-dependent symbol and, as far as we are aware, there is no solution for this kind of problem in the mathematical literature. In fact, finding the exact inverse of $(G+F\cos(pij/n))$ is not required; only an asymptotic inversion for large N is needed. Thus, it is necessary to overcome this problem by trying to find an asymptotic inverse to it. More details will be explored in the next section.

3.2.2 On the asymptotic inverse of a special class of Toeplitz matrices

In simple words, the task is to find the asymptotic inverse for Toeplitz matrix of the form of $(G + F\cos(\pi j/n)\ )$ in large N limit. In this limit, it turns out to be more convenient to look for the inverse of the $N \times N$ matrix $T_N(\alpha)$, where

$$\begin{aligned} T_N(\alpha)_l &\equiv \sqrt{2}(G + F\cos(\pi j/n)\ )_l\ )\ |_{N>>1} \\ &\backsimeq \frac{1}{N}\sum_{k=0}^{N-1}\sqrt{c_k}(\alpha_+ + \alpha_- e^{i\pi l/N})e^{\frac{2\pi ikl}{N}} = \alpha_+ C + \alpha_- \tilde{C} \end{aligned} \tag{3.55}$$

where $\alpha_\pm = 1 \pm \cos\left(\frac{\pi j}{n}\right)$ and $c_k = a + \cos(2\pi k/N)$ with $a = (\ (\mu^2 + E)/2 + 1)$. It is understood here that $(T_N)_{ij} \equiv (T_N)_{i-j}$. The matrices C and $\tilde{C}$ are defined as follows

$$C_l = \frac{1}{N}\sum_{k=0}^{N-1}\sqrt{c_k}e^{i2\pi kl/N}, \quad \tilde{C}_l = \frac{e^{i\pi l/N}}{N}\sum_{k=0}^{N-1}\sqrt{c_k}e^{i2\pi kl/N} \tag{3.56}$$

C is a circulant matrix, whereas $\tilde{C}$ is an anti-circulant matrix. They are both invertible matrices, such that :

$$\begin{aligned} C_l^{-1} &= \frac{1}{N}\sum_{k=0}^{N-1}\sqrt{c_k^{-1}}e^{i2\pi kl/N} \\ \tilde{C}_l^{-1} &= \frac{e^{i\pi l/N}}{N}\sum_{k=0}^{N-1}\sqrt{c_k^{-1}}e^{i2\pi kl/N} \end{aligned}$$

The matrix elements of T_N mainly live in two specific regions : $\mid l \mid << N$ and $\mid l \mid \sim N$. This is easily deduced by rearranging the expression of T_N as follows

$$T_N(\alpha)_l = \frac{1}{N}(1 + e^{i\pi l/n})\sum_k \sqrt{c_k}e^{i2\pi kl/N} + \cos(\pi j/N)\frac{1}{N}(1 - e^{i\pi l/N})\sum_k \sqrt{c_k}e^{i2\pi kl/N} \tag{3.57}$$

When $\mid l \mid << N$, the first sum survives, whereas when $\mid l \mid \sim N$, the second sum survives. The other elements are damped by the harsh oscillating sum when taking very large N. Therefore, the distribution of the elements of T_N is known, however, the extent or the width of this distribution depends upon the value of N, which gives a meaning for $\mid l \mid << N$ and $\mid l \mid \sim N$. This makes T_N an N-dependent symbol matrix. As a matter of fact, the N-independent symbol matrix admits a Wiener form, which represents an asymptotic inverse circulant matrix. Unfortunately, no explicit treatment for the N-dependent symbol matrix has been found in the mathematical literature.

The search for the inverse of the matrix T_N was led by two principal clues : The first clue is that the inverse of the Toeplitz matrix should be persymmetric. The second clue is that it should satisfy the condition $\mathrm{Tr}\left(T_N T_N^{-1}\right) = N$. Hand in hand with numerical and analytical investigations, an ansatz M_N of the inverse of T_N is suggested

$$(M_N)_{lm} = \frac{1}{N^2}\sum_{p,k=0}^{N-1}\left[\alpha_{kp}e^{-i\pi m/N}k_+(p-k) + \alpha_{pk}e^{i\pi l/N}k_-(p-k)\right]e^{i2\pi kl/N}e^{-i2\pi pm/N} \tag{3.58}$$

where α_{pk} is a function of p and k to be determined and $k_\pm$ are defined as follows

$$k_\pm(x) = \frac{1}{2}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}(x\pm 1/2)m} = \frac{1}{1-e^{2\pi i(x\pm 1/2)/N}} \tag{3.59}$$

M_N can be easily seen to be persymmetric for arbitrary α_{kp}. The next step is to recognize that the function α_{kp} given by

$$\alpha_{kp} = \frac{1}{\alpha_+\sqrt{c_k} + \alpha_-\sqrt{c_p}} \tag{3.60}$$

This α_{kp} solves the equation $\mathrm{Tr}(T_N.M_N) = N$ exactly for any N. Furthermore, this choice of α_{kp} coincides exactly with the matrix T_N^{-1} for the two invertible special cases $\alpha_+ = 0$ and $\alpha_- = 0$. Therefore, one can suggest the following conjecture.

Conjecture:

Let T_N be an $N \times N$ Toeplitz matrix defined by

$$T_N(\alpha) = \alpha_+ C + \alpha_- \tilde{C} \tag{3.61}$$

C and $\tilde{C}$ are also Toeplitz matrices given by

$$C_l = \frac{1}{N}\sum_{k=0}^{N-1}\sqrt{c_k}e^{i2\pi kl/N}, \quad \tilde{C}_l = \frac{e^{i\pi l/N}}{N}\sum_{k=0}^{N-1}\sqrt{c_k}e^{i2\pi kl/N} \tag{3.62}$$

where $\alpha_\pm = 1 \pm \cos\frac{\pi j}{n}$, and $c_k = a - \cos\frac{2\pi k}{N}$.
Then the following matrix

$$(M_N)_{lm} = \frac{1}{N^2}\sum_{p,k=0}^{N-1}\left[\alpha_{kp}e^{-i\pi m/N}k_+(p-k) + \alpha_{pk}e^{i\pi l/N}k_-(p-k)\right]e^{i2\pi kl/N}e^{-i2\pi pm/N} \tag{3.63}$$

with

$$k_\pm(x) = \frac{1}{2}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}(x\pm 1/2)m}, \qquad \alpha_{kp} = \frac{1}{\alpha_+\sqrt{c_k} + \alpha_-\sqrt{c_p}} \tag{3.64}$$

is an asymptotic inverse of $T_N(\alpha)$.

By asymptotic inverse here we mean that M_N fulfills the following condition

$$\lim_{N\to\infty}\delta_N(a) = |T_N.M_N - I| \to 0, \quad \forall a > 1 \tag{3.65}$$

Numerical investigation for δ_N shows its beautiful sliding to zero depending on the value of a, where it turns out that a and N have an anti-proportional relation. For example, for $a > 1.2$ and

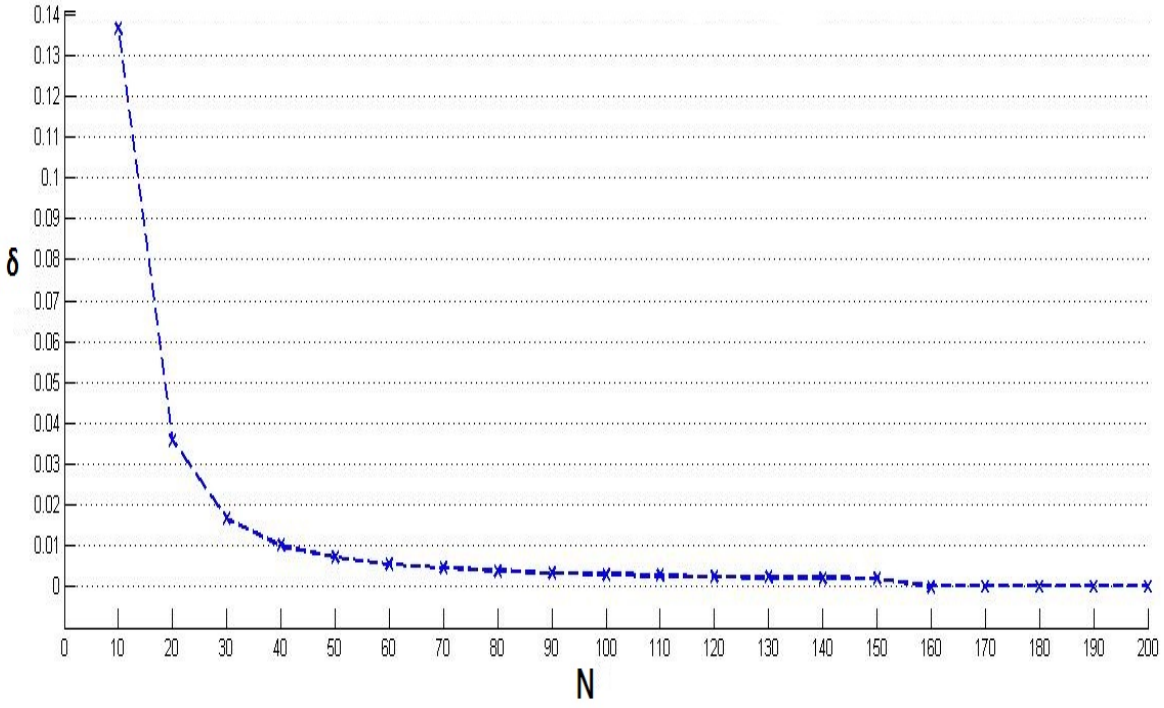

Figure 3.1: The curve represents δ as a function of N for a fixed $\alpha_{\pm}$, where $\alpha_{+} = 1.5$ and $\alpha_{-} = 0.5$.

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$N = 30$ one finds that $\delta < 10^{-3}$ (using the weak norm or Hilbert-Schmidt norm). The related interesting case is when $a \to 1$, i.e. the continuum limit $\mu = m\epsilon \to 0$. Here one needs to take larger N. Numerical calculation suggests that δ vanishes like $1/N$. For example, for $a = 1.002$ and $N = 100$ one finds that $\delta = 0.003$, for $a = 1.002$ and $N = 200$ one gets $\delta = 0.0015$. Figure 3.1 shows the way in which δ tends to zero when taking larger and larger N.

Another encouraging sign for the conjecture is that Neumann expansion of M_N for large a matches the T_N^{-1} expansion order by order, at least for the first few investigated orders.

The theory of Generalized locally Toeplitz GLT [59,60] provides a tighter test on the conjecture. This machinery estimates the trace of T_N^{-1} in the limit $N \to \infty$ by

$$\lim_{N\to\infty} \frac{\mathrm{Tr}(T_N(\alpha)^{-1})}{N} = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{\sqrt{a-\cos\theta}} = \frac{1}{\pi\sqrt{a+1}} K\left(\sqrt{\frac{2}{a+1}}\right) \tag{3.66}$$

Where K is the complete elliptic integral of the first kind. One hopes that the trace of M_N matches the value predicted by GLT machinery (3.66), as this should represent an asymptotic inverse for T_N when $N \to \infty$. Pleasingly, it turns out that the trace of M_N does have the same value as (3.66), as it will be shown later.

One cannot claim that all these clues form a proof of the conjecture. However, it does give it a good sense of credibility since it gave accurate outcomes, which are unlikely to be by coincidence.

Interestingly, numerical studies of several Toeplitz matrices similar to T_N but with different c_k and $\alpha_{\pm}$, revealed that the ansatz extends beyond this particular problem. More precisely, if T_N is

a Toeplitz matrix defined by

$$T_N(\alpha) = \alpha C_1 + \beta \tilde{C}_1$$

$$C_l = \frac{1}{N}\sum_{k=0}^{N-1} a_k e^{i2\pi kl/N}, \quad \tilde{C}_l = \frac{e^{i\pi l/N}}{N}\sum_{k=0}^{N-1} a_k e^{i2\pi kl/N}$$

a_k and b_k are two functions of k, which need to satisfy certain technical conditions, otherwise arbitrary.

Then the following matrix

$$(M_N)_{lm} = \frac{1}{N^2}\sum_{p,k=0}^{N-1}\left[\alpha_{kp}e^{-i\pi m/N}k_+(p-k) + \alpha_{pk}e^{i\pi l/N}k_-(p-k)\right]e^{i2\pi kl/N}e^{-i2\pi pm/N}$$

with

$$\alpha_{pk} = \frac{1}{\alpha a_k + \beta b_p}$$

is an asymptotic inverse for T_N with $\lim_{N\to\infty}\delta_N \to 0$.

This results, if confirmed by analytic calculation, will provide a general solution for the asymptotic inverse of a wide class of Toeplitz matrices, with N-dependent symbol, which are not covered by the known results in literature [61, 62].

In the situation of half-chain entanglement, it is possible to use the conjecture inverse to compute the entropy. However, it turns out that the obtained expression is complicated by first inspection. The aim is to find a way to manipulate it and extract the diverging term,and this will be the subject of our next research.

Estimation of the trace of the matrix M_N

In this section, the average of the trace of M_N will be evaluated in large N limit, where

$$\begin{aligned}\lim_{N\to\infty}\frac{\mathrm{Tr}M_N(\alpha)}{N} &= \lim_{N\to\infty}\frac{1}{N^3}\sum_{k,p,q,l=0}^{N-1}\alpha_{kp}e^{2\pi il(p-k+1/2)/N}.e^{2\pi iq(k-p-1/2)/N} \\ &= \lim_{N\to\infty}\frac{1}{N^2}\sum_{k,p=0}^{N-1}\sum_{l=-(N-1)}^{N-1}(1-\frac{|l|}{N})\alpha_{kp}e^{i2\pi(p-k+1/2)l/N} \qquad (3.67)\end{aligned}$$

For abitrary N, it is not hard to show that

$$\sum_{l=1}^{N-1}\alpha_{kp}e^{2\pi il(p-k+1/2)/N} + \sum_{l=1}^{N-1}\alpha_{kp}e^{2\pi i(-l).(p-k+1/2)/N} = 0 \qquad (3.68)$$

To prove it, one just needs to achieve the sum over 'l'. It follows that

$$\sum_{k,p=0}^{N-1}\sum_{l=-(N-1)}^{N-1}\alpha_{kp}e^{2\pi il(p-k+1/2)/N} = \sum_{k,p=0}^{N-1}\alpha_{kp} \qquad (3.69)$$

Hence we are left with three contributions :

$$I_0 = \frac{1}{N^2}\sum_{k,p=0}^{N-1}\alpha_{kp} \tag{3.70}$$

$$I_- = (I_+)^* = \sum_{k,p=0,l=1}^{N-1}\frac{l}{N^3}\alpha_{kp}e^{\frac{2\pi il}{N}(p-k+1/2)} \tag{3.71}$$

Let's start with I_0, one can transform the sum into an integral since we are interested in the limit $N\to\infty$. Take the sum over p as a beginning, define the complex integration for $z\equiv\exp(2\pi ip/N)$ over the unit circle C_1 as follows

$$\frac{1}{N^2}\sum_{k,p=0}^{N-1}\alpha_{kp} = \frac{1}{2\pi i.N}\sum_{k=0}^{N-1}\oint_{C_1}\frac{dz}{z}\frac{1}{\alpha_+.\sqrt{ck}+\alpha_-.\sqrt{\mathcal{R}_z}} \tag{3.72}$$

where $\mathcal{R}_z=(1/r-z)(z-r)/(2z)$. It is not hard to show that $r, 1/r$ and 0 are branch points for the integrand where $r=a-\sqrt{a^2-1}<1$. The branch point $z=1/r$ is outside the contour therefore it will not be considered in the calculations. Consider the contour $\mathcal{W}$ illustrated in figure 3.2. This contour with the branch cut over the positive real axis can be decomposed into three circles (the unit circle C_1 around $z=0$ and two infinitesimal circles around $z=0$ and $z=r$) and two lines (back and forth over the segment $x\in]0,1[$). It follows that

$$\begin{aligned}\oint_{\mathcal{W}}\frac{dz}{z}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\mathcal{R}_z}} &= \oint_{C_1}\frac{dz}{z}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\mathcal{R}_z}}+\\ &+\int_0^r\frac{dx}{x}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-i\sqrt{R_x}}+\int_r^0\frac{dx}{x}\frac{1}{\alpha_+\sqrt{ck}-\alpha_-i\sqrt{R_x}}\end{aligned}$$

where $R_x=(1/r-x)(r-x)/(2x)$.

The integration over the components of the contour $\mathcal{W}$ cancels identically, except for the ones shown in the previous equation. These integrals are listed below

$$\begin{aligned}\oint_{c_\epsilon(0)}\frac{dz}{z}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\mathcal{R}_z}} &= \lim_{\epsilon\to0}\int_0^{2\pi}i\epsilon e^{i\theta}d\theta\frac{1}{\epsilon e^{i\theta}}\frac{1}{\alpha_+\sqrt{c_k}+i\alpha_-e^{-i\theta/2}/\sqrt{2\epsilon}}=0\\ \oint_{c_\epsilon(r)}\frac{dz}{z}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\mathcal{R}_z}} &= \lim_{\epsilon\to0}\int_{2\pi}^{\pi}i\epsilon e^{i\theta}d\theta\frac{1}{r}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\epsilon}e^{i\theta/2}\sqrt{1/r-r}/(-\sqrt{r})}+\\ &+\lim_{\epsilon\to0}\int_{\pi}^{0}i\epsilon e^{i\theta}d\theta\frac{1}{r}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\epsilon}e^{i\theta/2}\sqrt{1/r-r}/(\sqrt{r})}=0\\ \int_{x(\to)}\frac{dz}{z}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\mathcal{R}_z}} &= \int_0^r\frac{dx}{x}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-i\sqrt{R_x}}+\int_r^1\frac{dx}{x}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{R_x}}\\ \int_{x(\leftarrow)}\frac{dz}{z}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-\sqrt{\mathcal{R}_z}} &= \int_r^0\frac{dx}{x}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-(-i)\sqrt{R_x}}+\int_1^r\frac{dx}{x}\frac{1}{\alpha_+\sqrt{ck}+\alpha_-(-\sqrt{R_x})}\end{aligned}$$

Therefore

$$I_0=\frac{1}{\pi N}\sum_{k=0}^{N-1}\int_0^r\frac{dx}{x}\frac{\alpha_-\sqrt{R_x}}{\alpha_+^2c_k+\alpha_-^2R_x} \tag{3.73}$$

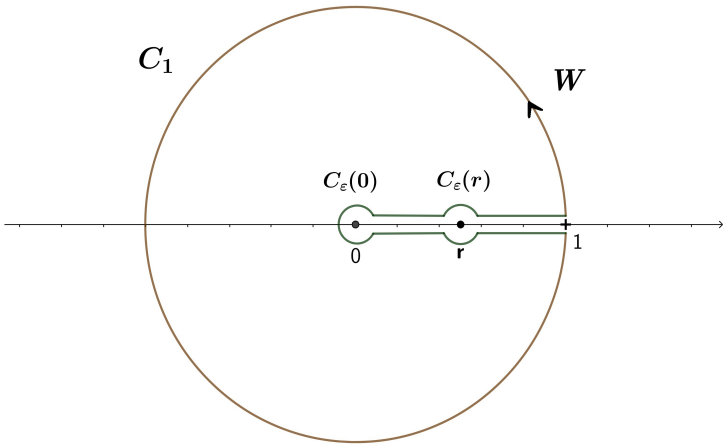

Figure 3.2: The contour of integration $\mathcal{W}$.

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Similarly, the sum over k can be converted into an integral and evaluated. One may easily note that there are no branch points in this situation, and it is possible to use the Residues theorem directly.

$$I_0 = \frac{\alpha_-}{\pi(2\pi i)} \int_0^r dx \frac{\sqrt{R_x}}{x} \int \frac{dz}{z} \frac{1}{\alpha_+^2 \mathcal{R}_z + \alpha_-^2 R_x} \tag{3.74}$$

The integrand has only one simple pole inside the unit circle, which is $u = e - \sqrt{e^2 - 1} \leq r$ with $e = a + \alpha_-^2 R_x / \alpha_+^2$. Thus

$$I_0 = \frac{\alpha_-}{\pi} \int_0^r \frac{\sqrt{R_x}}{\alpha_+^2 x \sqrt{e^2 - 1}} \tag{3.75}$$

The same manipulation can be done for I_-, starting with the sum over p, one finds

$$\begin{aligned} I_- &= \sum_{k,p=0,l=1}^{N-1} \frac{l}{N^3} \alpha_{kp} e^{\frac{2\pi i l}{N}(p-k+1/2)} \\ &= \frac{1}{2\pi i N^2} \sum_{k=0,l=1}^{N-1} l . e^{\frac{2\pi i l}{N}(-k+1/2)} \int_{C_1} dz \left[\frac{z^{l-1}}{\alpha_+ \sqrt{c_k} + \alpha_- \sqrt{R_z}} + \frac{z^{l-1-N}}{\alpha_+ \sqrt{c_k} + \alpha_- \sqrt{R_z}} \right] \end{aligned} \tag{3.76}$$

The extra term proportional to z^{l-N} comes from the Euler-Maclaurin formula. When turning the sum into an integral, this term is the only surviving term from the reminder.

It is not hard to show that the first term of the previous expression does not contribute when taking large N limit. One can evaluate the integral by taking the same previous contour $\mathcal{W}$. It

follows that

$$\sum_{l=1}^{N-1} e^{\frac{2\pi i l}{N}(-k+1/2)} \frac{l}{N} \frac{z^{l-1}}{\alpha_+\sqrt{c_k}+\alpha_-\sqrt{R_z}} = \sum_{l=1}^{N-1} e^{\frac{2\pi i l}{N}(-k+1/2)} \frac{l}{N} \int_0^r dx\ x^{l-1} \frac{2i\alpha_-\sqrt{R_x}}{\alpha_+^2 c_k + \alpha_-^2 R_x} \tag{3.77}$$

where

$$\lim_{N\to\infty} \sum_{l=1}^{N-1} \frac{l}{N} x^l e^{\frac{2\pi i l}{N}(-k+1/2)} \to 0 \tag{3.78}$$

To evaluate the second term of I_- it is preferable to take its complex conjugate in order to avoid some divergent difficulties at the origin. After that, the integral can be evaluated using the complex contour $\mathcal{W}$, where

$$\left(\oint_{C_1} dz \frac{z^{l-1-N}}{\alpha_+\sqrt{c_k}+\alpha_-\sqrt{R_z}}\right)^* = -\oint_{C_1} dz \frac{z^{N-1-l}}{\alpha_+\sqrt{c_k}+\alpha_-\sqrt{R_z}} = \int_0^r dx.x^{N-1-l} \frac{2i\alpha_-\sqrt{R_x}}{\alpha_+^2 c_k + \alpha_-^2 R_x} \tag{3.79}$$

It is straightforward to evaluate the sum over l, omitting any vanishing terms when taking the limit $N \to \infty$, one then has :

$$\sum_{l=1}^{N-1} \frac{l}{N} x^{N-1-l} e^{-\frac{2\pi i l}{N}(-k+1/2)} \to \frac{1}{x - e^{\frac{2\pi i}{N}k}} \tag{3.80}$$

When turning the sum over k into an integral, the integrand shows no branch points and the integral can be achieved by means of the Residue's theorem, thus

$$\begin{aligned} I_- &= \frac{\alpha_-}{\pi N} \int_0^r dx\ \sqrt{R_x} \sum_{k=0}^{N-1} \left(\frac{1}{x - e^{\frac{2\pi i}{N}k}} \frac{1}{\alpha_+^2 c_k + \alpha_-^2 R_x}\right)^* \\ &= \frac{\alpha_-}{\pi} \int_0^r dx\ \sqrt{R_x} \left(\frac{1}{2\pi i} \oint_{C_1} \frac{dz}{z} \frac{1}{(x-z)(\alpha_+^2 \mathcal{R}_z + \alpha_-^2 R_x)}\right)^* \\ &= \frac{\alpha_-}{\pi} \int_0^r dx\ \sqrt{R_x} \left(-\frac{1}{x(\alpha_-^2 - \alpha_+^2)R_x} + \frac{1}{\alpha_+^2 (x-u)\sqrt{e^2-1}}\right) \end{aligned} \tag{3.81}$$

Consequently

$$I_0 - I_- - I_+ = \frac{\alpha_-}{\alpha_+^2 \pi} \int_0^r \frac{\sqrt{R_x}}{x\sqrt{e^2-1}} + 2\frac{\alpha_-}{(\alpha_-^2 - \alpha_+^2)\pi} \int_0^r dx \frac{1}{x\sqrt{R_x}} - 2\int_0^r dx \frac{\alpha_-}{\alpha_+^2 \pi} \frac{\sqrt{R_x}}{(x-u)\sqrt{e^2-1}}$$

The last term can be manipulated as follows

$$\begin{aligned} \frac{\alpha_-\sqrt{R_x}}{\alpha_+^2 (x-u)\sqrt{e^2-1}} &= \frac{\alpha_-\sqrt{R_x}}{\sqrt{e^2-1}} \frac{x - u^{-1}}{\alpha_+^2 (x-u)(x-u^{-1})} \\ &= \frac{\alpha_-\sqrt{R_x}(x - e - \sqrt{e^2-1})}{\sqrt{e^2-1}} \frac{1}{2x(\alpha_+^2 - \alpha_-^2)R_x} \\ &= -\frac{\alpha_-(x-e)}{2(\alpha_-^2 - \alpha_+^2)x\sqrt{R_x}.\sqrt{e^2-1}} + \frac{\alpha_-}{2(\alpha_-^2 - \alpha_+^2)x\sqrt{R_x}} \end{aligned} \tag{3.82}$$

Define the term Q as follows

$$\begin{aligned} Q &= \frac{1}{\pi}\int_0^r \frac{\alpha_-\sqrt{R_x}}{\alpha_+^2 x\sqrt{e^2-1}} - \frac{1}{\pi}\int_0^r dx \frac{-\alpha_-(x-e)}{(\alpha_-^2-\alpha_+^2)x\sqrt{R_x}.\sqrt{e^2-1}} \\ &= \frac{1}{\pi}\int_0^r \frac{\alpha_-}{\alpha_+^2}.(\frac{\alpha_+^2}{\alpha_+^2-\alpha_-^2})\frac{\sqrt{R_x}}{x\sqrt{e^2-1}} + \frac{1}{\pi}\int_0^r dx \frac{\alpha_-(x-a)}{(\alpha_-^2-\alpha_+^2).x\sqrt{R_x}.\sqrt{e^2-1}} \\ &= \frac{1}{\pi}\frac{\alpha_-}{\alpha_+^2-\alpha_-^2}\int_0^r dx \frac{R_x - x + a}{x\sqrt{R_x}.\sqrt{e^2-1}} \\ &= \frac{1}{\pi}\frac{\alpha_-}{\alpha_+^2-\alpha_-^2}\int_0^r dx \frac{1}{x\sqrt{R_x}.\sqrt{e^2-1}}.\left(-x\frac{dR_x}{dx}\right) \end{aligned} \tag{3.83}$$

Therefore one can write

$$\lim_{N\to\infty}\frac{\mathrm{Tr}M_N(\alpha)}{N} = I_0 - I_+ - I_- = Q + \frac{\alpha_-}{\pi(\alpha_-^2-\alpha_+^2)}\int_0^r \frac{1}{x\sqrt{R_x}} \tag{3.84}$$

To evaluate Q, one can make the change of variable $x \to (\alpha_-^2/\alpha_+^2)R_x$ and use standard integral formulas [34] to find :

$$Q = \frac{1}{\pi}\frac{\alpha_+}{\alpha_+^2-\alpha_-^2}\frac{2}{\sqrt{a+1}}K\left(\sqrt{\frac{2}{a+1}}\right) \tag{3.85}$$

On the other hand

$$\begin{aligned} \int_0^r \frac{dx}{x\sqrt{R_x}} &= 2\sqrt{2r}K(r) \\ &= \frac{2\sqrt{2r}}{1+r}K\left(\frac{2\sqrt{r}}{1+r}\right) \end{aligned} \tag{3.86}$$

Where the integral (3.131) and the identity (8.126) of [34] have been used. One can easily show that

$$\frac{2\sqrt{r}}{1+r} = \sqrt{\frac{2}{1+a}} \tag{3.87}$$

where $r = a - \sqrt{a^2-1}$.
Finally, one finds

$$\lim_{N\to\infty}\frac{Tr(M_N)}{N} = \frac{1}{\pi}\frac{1}{\sqrt{a+1}}K\left(\sqrt{\frac{2}{a+1}}\right) \tag{3.88}$$

Which is identical to the result found by GLT machinery given by expression (3.66).

3.2.3 Situation 2 : One point entanglement

The situation presented here, namely the entanglement of the oscillator at the origin with the rest of the lattice oscillators, indeed forms a peculiar situation. However, it does come together with other physical situations and gives compatible results with them.

As a matter of fact, the entanglement of a finite interval L out of line of 1+1 massive scalar field is extensively studied in the literarture. The universal part of the entanglement entropy in 1+1 has been derived using different techniques, see [15, 46] and references therein. This leading universal divergences is given for a massive scalar theory by

$$S = \frac{1}{3}\ln\left(\frac{L}{\epsilon}\right) + ... \tag{3.89}$$

Where ϵ is the UV cut off and L is generally considered very large comparing to ϵ.

In the current situation, the subsystem consists of only one oscillator. Therefore, L is of the order of ϵ. The divergence seems to disappear here. So, by first inspection, one would expect to have a finite entropy. However, this is not the case as it will show up later.

Surprisingly, another softer divergence shows up. This divergence is related to the IR origins of massless scalar fields, which turns out to have a fundamental analogy with one-point entanglement.

In order to add some symmetry to the related matrices of the problem. The lattice is considered to be made up of $2N+1$ vertices instead of $2N$. The subsystem A is formed by the oscillator at the origin. It is straightforward to show that

$$\rho^n(q_1;q_1) = \int dQ_{\{i\}}dq_{\{j\neq 1\}}\ \rho_{tot}(\begin{bmatrix}Q_0\\ q_1\\ Q_2\end{bmatrix};\begin{bmatrix}Q_0\\ q_4\\ Q_2\end{bmatrix}).\rho_{tot}(\begin{bmatrix}Q_3\\ q_4\\ Q_5\end{bmatrix};\begin{bmatrix}Q_3\\ q_7\\ Q_5\end{bmatrix})...\rho_{tot}(\begin{bmatrix}Q_{3n-3}\\ q_{3n-2}\\ Q_{3n-1}\end{bmatrix};\begin{bmatrix}Q_{3n-3}\\ q_1\\ Q_{3n-1}\end{bmatrix})$$

Where ρ_{tot} is the density matrix which describes the total lattice and Q is a vector with N coordinates describing N successive oscillators of the lattice. Henceforth, it is possible to deduce the permutation matrix $\mathbb{P}_\pi$:

$$\mathbb{P}_\pi(\{q_0\}) = \begin{pmatrix} P_{2N} & 0_{2N} & 0_{2N} & \dots & 0_{2N} & P_0 \\ P_0 & P_{2N} & 0_{2N} & \dots & 0_{2N} & 0_{2N} \\ 0_{2N} & P_0 & P_{2N} & \dots & 0_{2N} & 0_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2N} & 0_{2N} & \dots & P_0 & P_{2N} & 0_{2N} \\ 0_{2N} & 0_{2N} & 0_{2N} & \dots & P_0 & P_{2N} \end{pmatrix} \tag{3.90}$$

Where 0_{2N} stands for $(2N+1)\times(2N+1)$ zero matrix, P_{2N} and P_0 are $(2N+1)\times(2N+1)$projector matrices given as follows

$$P_{2N} = \begin{pmatrix} I_N & 0 & 0_N \\ 0 & 0 & 0 \\ 0_N & 0 & I_N \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0_N & 0 & 0_N \\ 0 & 1 & 0 \\ 0_N & 0 & 0_N \end{pmatrix} \tag{3.91}$$

I_N represents $N \times N$ identity matrix and 0_N is an $N \times N$ zero matrix. The other zeroes above represent either zero columns or vectors, depending on their position. P_0 represent the projector into the subsystem A whereas P_{2N} projects into the complement system.

The form of the permutation matrix leads to a natural decomposition of the potential matrix V, it follows that

$$V_- = V + E = \begin{bmatrix} A & b & B \\ b^T & 2a & b'^T \\ B^T & b' & A \end{bmatrix} \tag{3.92}$$

Where A and B are $N \times N$ matrices, b and b' are N-coordinates vectors and a is the same scalar defined earlier $a = (\mu^2 + E)/2 + 1$. A similar decomposition for V_-^{-1} and W_- can be done

$$V_-^{-1} = \begin{bmatrix} C & d^T & D \\ d & c & d' \\ D^T & d'^T & C \end{bmatrix}, \quad W_- = \begin{bmatrix} E & f & F \\ f^T & e & f'^T \\ F^T & f' & E \end{bmatrix} \tag{3.93}$$

$C,\ D, E$ and F are $N \times N$ matrices, d and f are N-coordinate vectors, c and e are only scalars.

Next step is to evaluate $\mathbb{V}_+^{-1}$ and $\mathbb{W}_+$, It is not hard to show that

$$\mathbb{V}_+^{-1} = \mathbb{P}_\pi \mathbb{V}_-^{-1} \mathbb{P}_\pi^T = \begin{bmatrix} M_C & M_d^T & 0 & 0 & \dots & 0 & M_d \\ M_d & M_C & M_d^T & 0 & \dots & 0 & 0 \\ 0 & M_d & M_C & M_d^T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_d^T & 0 & 0 & 0 & \dots & M_d & M_C \end{bmatrix} \tag{3.94}$$

$$\mathbb{W}_+ = \mathbb{P}_\pi \mathbb{W}_- \mathbb{P}_\pi^T = \begin{bmatrix} M_E & M_f^T & 0 & 0 & \dots & 0 & M_f \\ M_f & M_E & M_f^T & 0 & \dots & 0 & 0 \\ 0 & M_f & M_E & M_f^T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_f^T & 0 & 0 & 0 & \dots & M_f & M_E \end{bmatrix} \tag{3.95}$$

Where

$$M_C = \begin{bmatrix} C & 0 & D \\ 0 & c & 0 \\ D^T & 0 & C \end{bmatrix}, \quad M_d = \begin{bmatrix} 0 & 0 & 0 \\ d^T & 0 & d'^T \\ 0 & 0 & 0 \end{bmatrix}, \quad M_E = \begin{bmatrix} E & 0 & F \\ 0 & e & 0 \\ F^T & 0 & E \end{bmatrix}, \quad M_f = \begin{bmatrix} 0 & 0 & 0 \\ f^T & 0 & f'^T \\ 0 & 0 & 0 \end{bmatrix}$$

It is then straightforward to evaluate the matrix product

$$(\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+) = \begin{bmatrix} 2M & -M & 0 & 0 & \dots & 0 & -M \\ -M & 2M & -M & 0 & \dots & 0 & 0 \\ 0 & -M & 2M & -M & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -M & 0 & 0 & 0 & \dots & -M & 2M \end{bmatrix} \tag{3.96}$$

Where $M = (M_d + M_d^T)(M_f + M_f^T)$. Obviously, the resulting matrix is block circulant, a very important property that will be used later. In fact, the matrix $\mathbb{W}_- + \mathbb{W}_+$ is also a block circulant matrix :

$$\mathbb{W}_- + \mathbb{W}_+ = \begin{pmatrix} W_- + M_G & M_f^T & 0 & 0 & \dots & 0 & M_f \\ M_f & W_- + M_G & M_f^T & 0 & \dots & 0 & 0 \\ 0 & M_f & W_- + M_G & M_f^T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & M_f & W_- + M_G & M_f^T & 0 \\ 0 & 0 & 0 & \dots & M_f & W_- + M_G & M_f^T \\ M_f^T & 0 & 0 & \dots & 0 & M_f & W_- + M_G \end{pmatrix}$$

This means that the matrix product $(\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1}$ can be simultaneously block diagonalized as follows

$$\mathrm{Tr}((\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1}) = 2\sum_{j=0}^{n-1}(1 - \cos\left(\frac{2\pi j}{n}\right)\mathrm{Tr}MT^{-1} \tag{3.97}$$

where

$$T = W_- + M_E + e^{-\frac{2\pi ij}{n}}M_f + e^{\frac{2\pi ij}{n}}M_f^T$$

The evaluation of the previous trace requires the determination of the inverse of the matrix T. Noting that T can be rewritten as

$$T = W_- + M_E + e^{-\frac{2\pi j}{n}}M_f + e^{\frac{2\pi ij}{n}}M_f^T = 2W_- + \xi^* M_f + \xi M_f^T \tag{3.98}$$

Where $\xi = 1 - e^{\frac{2\pi ij}{n}}$. It is then possible to calculate its inverse by a recursive use of a theorem introduced by K. Miller [63]. Since W_- is an invertible matrix and both M_f and M_f^T are rank-one matrices,the matrix T can be inverted as follows :

$$\begin{aligned} (2W_- + \xi^* M_f)_{lm}^{-1} &= \frac{(W_-^{-1})_{lm}}{2} + \frac{\xi^*}{2\Delta}(\delta_{Nj} - (W_-)_{NN}(W_-^{-1})_{Nm}) \\ T_{lm}^{-1} &= (2W_- + \xi^* M_f + \xi M_f^T)_{lm}^{-1} = (2W_- + \xi^* M_f)_{lm}^{-1} + \frac{\xi\Delta}{\Delta'}\Big(\ \delta_{lN} - (W_-)_{NN}(W_-^{-1})_{lN} + \\ &\quad + \frac{\xi^*}{\Delta}(W_-)_{NN}(W_-^{-1})_{lN}\ \Big(\ (W_-)_{NN}(W_-^{-1})_{NN} - 1\ \Big)\ \Big)(2W_- + \xi^* M_f)_{Nm}^{-1} \end{aligned} \tag{3.99}$$

Where

$$\Delta = 2 - \xi^*(1 - (W_-)_{NN}(W_-^{-1})_{NN}) \tag{3.100}$$

$$\Delta' = 2\Delta - \xi\big(1 - (W_-)_{NN}(W_-^{-1})_{NN}\big)\ \big(\Delta - \xi^*(W_-)_{NN}(W_-^{-1})_{NN}\ \big) \tag{3.101}$$

It is only a matter of lengthy calculations to find out the result :

$$\begin{aligned} \mathrm{Tr}\big((\mathbb{V}_-^{-1} - \mathbb{V}_+^{-1})(\mathbb{W}_- - \mathbb{W}_+)(\mathbb{W}_- + \mathbb{W}_+)^{-1}\big) &= \sum_{j=0}^{n-1}\frac{4}{\Delta'}\left(1 - \cos\left(\frac{2\pi j}{n}\right)\right)\Big(\Big((W_-^{-1})_{NN}\Big)^2 + \\ &\quad -(V_-^{-3/2})_{NN}(W_-)_{NN}\Big) \end{aligned} \tag{3.102}$$

It is then straightforward to obtain Rényi entropy

$$\ln \mathrm{Tr}\rho_A^n = -\frac{1}{2}\sum_{j=1}^{n-1}\left(1-\cos\left(\frac{2\pi j}{n}\right)\right)\int_0^\infty \left(\left((W_-^{-1})_{NN}\right)^2 - (V_-^{-3/2})_{NN}(W_-)_{NN} \right)\frac{dE}{\Delta'} \quad (3.103)$$

This expression is valid for any N. In order to evaluate the precedent integral, one suggests the change of variable $\omega = (W_-)_{NN}(W_-^{-1})_{NN}$, so that

$$\begin{aligned}
\ln \mathrm{Tr}\rho_A^n &= -\frac{1}{2}\sum_{j=1}^{n-1}\left(1-\cos\left(\frac{2\pi j}{n}\right)\right)\int_1^{\omega_0}\frac{d\omega}{(1+\omega)+(1-\omega)\cos\frac{2\pi j}{n}} \\
&= \frac{1}{2}\sum_{j=1}^{n-1}\ln\left((1+\omega_0)+(1-\omega_0)\cos\frac{2\pi j}{n}\right) \\
&= \ln[(C+1/2)^n-(C-1/2)^n] \qquad (3.104)
\end{aligned}$$

where $C=\sqrt{\omega_0}/2$ and $\omega_0 = (W_-^{-1})_{NN}(W_-)_{NN}|_{E=0}$.

It is fruitful to know that the correlators method presented in the first chapter gives exactly the same Rényi entropy. Here the subsystem is formed of only one oscillator, therefore the correlator matrix in this situation is $C=\sqrt{XP}=\sqrt{X_{NN}P_{NN}}$, where (see [35])

$$X_{NN} = < q_N q_N >= \frac{1}{2}W_{NN}^{-1}|_{E=0} = \frac{1}{2}(V^{-1/2})_{NN} \qquad (3.105)$$

$$P_{NN} = < \dot{q}_N \dot{q}_N >= \frac{1}{2}W_{NN}|_{E=0} = \frac{1}{2}(V^{1/2})_{NN} \qquad (3.106)$$

Thus $C=\sqrt{\omega_0}/2$ as found before. Until now, we are steel taking N arbitrary. To discuss the relevant behavior of the entropy, one needs to take the limit $N\to\infty$. It follows that

$$\begin{aligned}
W_{NN}|_{E=0} &= \frac{1}{2\pi}\int_0^{2\pi}(\mu^2+2-2\cos q)^{1/2}dq = \frac{(a_0^2-1)^{1/4}}{2^{1/2}}P_{1/2}(\frac{a_0}{\sqrt{a_0^2-1}}) \\
W_{NN}^{-1}|_{E=0} &= \frac{1}{2\pi}\int_0^{2\pi}(\mu^2+2-2\cos q)^{-1/2}dq = \frac{2^{1/2}}{(a_0^2-1)^{1/4}}P_{-1/2}(\frac{a_0}{\sqrt{a_0^2-1}})
\end{aligned}$$

where $a_0=\mu^2/2+1$, $P_{1/2}$ and $P_{-1/2}$ are the associated Legendre functions of the first kind.

In the continuum limit $\mu = m\epsilon \to 0$ and using the asymptotic form of $P_{1/2}$ and $P_{-1/2}$ [64], one finds

$$C = \frac{2}{\pi}(-\ln\mu)^{1/2} \qquad (3.107)$$

It follows that Rényi entropy is

$$\ln \mathrm{Tr}\rho_A^n = (n-1)(\frac{1}{2}\ln(-\ln m\epsilon)+\ln\frac{2}{\pi})+\ln n \qquad (3.108)$$

Finally, the entanglement entropy can be easily deduced

$$S = \frac{d}{dn} \ln \mathrm{Tr}\rho_A^n \mid_{n=1} = \frac{1}{2} \ln(-\ln m\epsilon) + \ln \frac{2}{\pi} + 1 \tag{3.109}$$

Interestingly, the entropy diverges when taking the continuum limit $\epsilon \to 0$! This divergence is related to the one of 1+1 massless scalar field. Investigations on the latter showed that the entropy difference in the limit $m \to 0$ behaves as follows [15]

$$S(L) - S(L_0) \sim \frac{1}{3} \ln(L/L_0) + \frac{1}{2} \ln(-\ln(mL)) - \frac{1}{2} \ln(-\ln(mL_0)) \tag{3.110}$$

This shows that the entropy diverges in the massless limit by the term $\ln\left(\frac{1}{2}\ln(mL)\right)$ for any set. Usually, when $L >> \epsilon$ it becomes irrelative and the dominant divergent coefficient is $\ln\left(\frac{1}{2}\ln(m)\right)$. It is inevitable here to see the analogy between the entropy (3.110) and the one point entanglement entropy (3.109).

The divergence in the massless 1+1 scalar field is due to the development of zero modes. The ground state of a massless field is therefore non-normalizable in the full Minkowski space nor in $R \times S_1$ (which is the present situation where we took periodic boundary condition).

In the absence of a normalizable ground state, one should not expect to have a finite entropy. Nevertheless, the divergence of the entropy when $m \to 0$ reflects the disturbance that happened to the theory due to the creation of zero modes. In one point entanglement, the divergence of the entropy has fundamentally the same origin.

Roughly speaking, since the entropy is a scaleless quantity, it should be a function of scaleless parameters. Here one has only two available parameters (m, ϵ), since $(L \sim \epsilon)$ it is not considered a different parameter than ϵ. Therefore, entanglement entropy should be a function of $(m\epsilon)$. In this situation, the continuum limit $\epsilon \to 0$ is technically equivalent to the massless limit. When $\mu \to 0$ there is noway to distinguish if it is $\epsilon \to 0$ or it is $m \to 0$. In both situations, zero modes will rise and produce divergence.

3.3 Perturbative evaluation for Interacting Theories, Beyond the Gaussian State

In real time approach, which is practically the only way used to treat lattice models or systems with finite degrees of freedom, one usually needs to solve an eigenvalue problem, whether it was the density matrix eigenvalues or the related correlator matrix $C = \sqrt{XP}$ eigenvalues. Not to mention the difficulty arising from tackling such a problem itself, there is no systematic method to extract a perturbative contribution to the entropy caused by some interacting potential. It is worth mentioning here that there was an attempt to generalize the entanglement entropy of two harmonic oscillators (1.27) by taking a non-Gaussian density matrix. However, the result is limited to the first order of the perturbation series only, and one cannot a priori see or choose the perturbed potential. A brief illustration of this method will be presented later.

An essential profit of developing Green's matrix approach for lattices is that it gives a systematic method to deal with interacting theories. The method provides a systematic expansion of the

partition function in powers of the perturbation. This allows one to have a more accurate entropy when dealing with more realistic systems.

To illustrate the method, ϕ^4 interaction will be examined in two versions. The first version takes a perturbative potential of the form $\sum q_i^4$ which is obtained out of the discretization of ϕ^4 field. The other version takes a little bit different perturbation, namely of the form $(Q^TQ)^2$. It is a quartic interaction, but obviously cannot be interpreted as a discretization of ϕ^4 field.

3.3.1 Local interaction

In this situation, the Euclidean Lagrangian is the following

$$L_E = \frac{1}{2}\left(\dot{Q}^T\dot{Q} + Q^TVQ\right) + \frac{\lambda}{4!}\sum_i q_i^4 \tag{3.111}$$

The interaction term $\frac{\lambda}{4!}\sum_i q_i^4$ is named local interaction, since there is no mixing between the oscillators degrees of freedom. Each oscillator is locally perturbed in a distinct way from the other oscillators.

To construct the related action $\mathbb{S}_E$, one should remember that the cut at $\tau = 0$ was surmounted with a rearrangement of the coordinate $\{q_i\}$ when $\tau > 0$, so that $\mathbb{Q} \to \mathbb{P}_\pi\mathbb{Q}$. Fortunately, this rearrangement does not change the sum $\{\sum q_i^4\}$ since it will only translate each coordinate q_l into another coordinate q_m in the set $\{q_i\}$, therefore the sum $\sum q_i^4$ remains unchanged, and it follows that

$$\mathbb{S}_E(\lambda) = \int_{-\infty}^{\infty} d\tau\ \frac{1}{2}\left(\dot{\mathbb{Q}}^T\dot{\mathbb{Q}} + \mathbb{Q}^T\mathbb{V}(\tau)\mathbb{Q}\right) + \frac{\lambda}{4!}\sum_i q_i^4 \tag{3.112}$$

The partition function takes the form :

$$\mathbb{Z}_n(\lambda) = \int D_{\mathbb{Q}} \exp\left\{-\int_{-\infty}^{\infty} d\tau \frac{1}{2}\left(\dot{\mathbb{Q}}^T\dot{\mathbb{Q}} + \mathbb{Q}^T\mathbb{V}(\tau)\mathbb{Q}\right) + \frac{\lambda}{4!}\sum_i q_i^4\right\} \tag{3.113}$$

Consider now the generating partition function $\mathbb{Z}_n(J)$ which is written in terms of an external source J as follows :

$$\begin{aligned}\mathbb{Z}_n(J) &= \int D_{\mathbb{Q}} \exp\left\{-\int_{-\infty}^{\infty} d\tau \frac{1}{2}\left(\dot{\mathbb{Q}}^T\dot{\mathbb{Q}} + \mathbb{Q}^T\mathbb{V}(\tau)\mathbb{Q}\right) - \sum_{i=1}^{nN} J_i(\tau)q_i(\tau)\right\} \\ &= \int D_{\mathbb{Q}} \exp\left\{-\int_{-\infty}^{\infty} d\tau\ \frac{1}{2}\ \mathbb{Q}^T\left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau)\right)\mathbb{Q} - J^T(\tau).\mathbb{Q}(\tau)\right\}\end{aligned} \tag{3.114}$$

Where J is a Nn-dimensional coordinate vector. The term $\int_{-\infty}^{\infty} d\tau\ \dot{\mathbb{Q}}^T\dot{\mathbb{Q}}$ has been integrated by parts and the limit $\lim_{\tau\to\pm\infty}\mathbb{Q}^T\dot{\mathbb{Q}} \to 0$ was taken.

On the other hand, define the partition function $\mathbb{Z}_n(\lambda, J)$ as follows

$$\mathbb{Z}_n(\lambda, J) = \int D_{\mathbb{Q}} \exp\left\{-\int_{-\infty}^{\infty} d\tau\ \frac{1}{2}\ \mathbb{Q}^T\left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau)\right)\mathbb{Q} + \frac{\lambda}{4!}\sum_i q_i^4 - J^T.\mathbb{Q}\right\} \tag{3.115}$$

Where $\mathbb{Z}_n(\lambda, J=0) \equiv \mathbb{Z}_n(\lambda)$. The partition function $\mathbb{Z}_n(\lambda, J)$ can be evaluated perturbatively in powers of the interaction parameter λ, but first consider Green's function $\mathbb{G}_n$ defined by (2.65) which satisfies the following differential equation

$$\left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau)\right) \mathbb{G}_n(\tau, \tau') = \delta(\tau, \tau')\mathbb{I}, \qquad E = 0 \tag{3.116}$$

This identity will be used later.

Consider now the change to a new variable

$$Q \to Q = \tilde{Q} + \int_{-\infty}^{\infty} d\tau_1 \mathbb{G}_n(\tau, \tau') J(\tau') \tag{3.117}$$

It is straightforward to show that

$$\mathbb{Q}^T \left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau)\right) \mathbb{Q} = \tilde{\mathbb{Q}}^T \left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau)\right) \tilde{\mathbb{Q}} + 2J^T \tilde{\mathbb{Q}} + \int d\tau' J^T(\tau) \mathbb{G}_n(\tau, \tau') J(\tau')$$

Where the differential equation (3.116) has been used besides the fact that

$$\int d\tau \ \mathbb{G}_{\ltimes} \partial\tau^2 \mathbb{Q} = \int d\tau \ \partial\tau^2 \mathbb{G}_n \mathbb{Q}$$

It follows that

$$\begin{aligned}
\mathbb{Z}_n(J) &= \int D_{\mathbb{Q}} \exp\left\{ - \int_{-\infty}^{\infty} d\tau \ \frac{1}{2} \ \mathbb{Q}^T \left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau)\right) \mathbb{Q} - J^T(\tau).\mathbb{Q}(\tau) \right\} \\
&\to \int D_{\tilde{\mathbb{Q}}} \exp\left\{ - \int_{-\infty}^{\infty} d\tau \ \frac{1}{2} \ \tilde{\mathbb{Q}}^T \left(-\frac{d^2}{d\tau^2} + \mathbb{V}(\tau)\right) \tilde{\mathbb{Q}} + \frac{1}{2} \iint d\tau d\tau' J^T(\tau) \mathbb{G}_n(\tau, \tau') J(\tau') \right\} \\
&= \mathbb{Z}_n(0). \exp\left\{ \frac{1}{2} \iint d\tau d\tau' J^T(\tau) \mathbb{G}_n(\tau, \tau') J(\tau') \right\}
\end{aligned} \tag{3.118}$$

The perturbative expansion of $\mathbb{Z}_n(\lambda, J)$ in powers of λ can be written as

$$\begin{aligned}
\mathbb{Z}_n(\lambda, J) &= \int D_{\mathbb{Q}} \exp\left\{ - \int_{-\infty}^{\infty} d\tau \frac{1}{2} \left(\dot{\mathbb{Q}}^T \dot{\mathbb{Q}} + \mathbb{Q}^T \mathbb{V}(\tau) \mathbb{Q} \right) + \sum_i \frac{\lambda}{4!} q_i^4 - J_i q_i \right\} \\
&= \int D_{\mathbb{Q}} \left(1 - \frac{\lambda}{4!} \sum_i \int d\tau q_i^4(\tau) + \frac{\lambda^2}{2(4!)^2} \sum_{ij} \iint d\tau d\tau' q_i^4(\tau) q_i^4(\tau') + \ldots \right). \\
&\qquad \times \exp\left\{ - \int_{-\infty}^{\infty} d\tau \frac{1}{2} \left(\dot{\mathbb{Q}}^T \dot{\mathbb{Q}} + \mathbb{Q}^T \mathbb{V}(\tau) \mathbb{Q} \right) - \sum_i J_i q_i \right\}
\end{aligned}$$

One can easily note that $q_i^4(\tau)$ in the perturbation series can be replaced by $\delta^4/(\delta J_i^4(\tau))$, this follows from the identity

$$\frac{\delta J_j(y)}{\delta J_i(x)} = \delta_{ij} \ \delta(x - y)$$

It follows that

$$\begin{aligned}
\mathbb{Z}_n(\lambda, J) &= \int D_{\mathbb{Q}} \left(1 - \frac{\lambda}{4!}\sum_i \int d\tau \frac{\delta^4}{\delta J_i^4(\tau)} + \frac{\lambda^2}{2(4!)^2}\sum_{ij}\frac{\delta^4}{\delta J_i^4(\tau)}\frac{\delta^4}{\delta J_i^4(\tau')} + ...\right) \\
&\quad \times \exp\left[- \int_{-\infty}^{\infty} d\tau \frac{1}{2}\left(\dot{\mathbb{Q}}^T\dot{\mathbb{Q}} + \mathbb{Q}^T\mathbb{V}(\tau)\mathbb{Q}\right) - \sum_i J_i q_i\right] \\
&= \left(1 - \frac{\lambda}{4!}\sum_i \int d\tau \frac{\delta^4}{\delta J_i^4(\tau)} + \frac{\lambda^2}{2(4!)^2}\sum_{ij}\frac{\delta^4}{\delta J_i^4(\tau)}\frac{\delta^4}{\delta J_j^4(\tau')} + ...\right) \\
&\quad \times \int D_{\mathbb{Q}} \exp\left[- \int_{-\infty}^{\infty} d\tau \frac{1}{2}\left(\dot{\mathbb{Q}}^T\dot{\mathbb{Q}} + \mathbb{Q}^T\mathbb{V}(\tau)\mathbb{Q}\right) - \sum_i J_i q_i\right] \\
&= \left(1 - \frac{\lambda}{4!}\sum_i \int d\tau \frac{\delta^4}{\delta J_i^4(\tau)} + \frac{\lambda^2}{2(4!)^2}\sum_{ij}\frac{\delta^4}{\delta J_i^4(\tau)}\frac{\delta^4}{\delta J_j^4(\tau')} + ...\right)\mathbb{Z}_n(J)
\end{aligned}$$

The expansion of the partition function $\mathbb{Z}_n(\lambda)$ can be easily evaluated by means of the previous equation and the formula (3.118) for $\mathbb{Z}_n(J)$. Satisfied with the first order expansion, it is not hard to show that

$$\begin{aligned}
\mathbb{Z}_n(\lambda) &\simeq \left(1 - \frac{\lambda}{4!}\sum_i \int d\tau \frac{\delta^4}{\delta J_i^4(\tau)}\right)\mathbb{Z}_n(J)\Big|_{J=0} \\
&= \mathbb{Z}_n(0)\left(1 - \frac{\lambda}{4!}\sum_i \int d\tau \frac{\delta^4}{\delta J_i^4(\tau)}\right)\exp\left\{\frac{1}{2}\iint d\tau d\tau' J^T(\tau)\mathbb{G}_n(\tau,\tau')J(\tau')\right\}\Big|_{J=0} \\
&= \mathbb{Z}_n(0)\left(1 - \frac{\lambda}{4!}.(3\sum_i \int d\tau \mathbb{G}_{ii}^2(\tau,\tau)\ )\right) \qquad (3.119)
\end{aligned}$$

It is then straightforward to obtain the first order expansion of Rényi entropy

$$\begin{aligned}
\ln \mathrm{Tr}(\rho^n(\lambda)) &= \ln \mathbb{Z}_n(\lambda) - n \ln \mathbb{Z}_1(\lambda) \\
&= \ln \mathbb{Z}_n(0) - n \ln \mathbb{Z}_1(0) - \frac{3\lambda}{4!}\int_{-\infty}^{\infty} d\tau \left[\sum_{i=1}^{nN} \mathbb{G}_{ii}(\tau,\tau)^2 + \right. \\
&\quad \left. -n\sum_{i=1}^{N} G_{ii}(\tau,\tau)^2\right] + \mathcal{O}(\lambda^2) \qquad (3.120)
\end{aligned}$$

Where $\mathbb{G}_{ii}$ is evaluated for $E = 0$.

To illustrate the applicability of (3.120), an example of two coupled harmonic oscillators will be examined. But before using (3.120), it will be useful to present the result of [31] for later comparison. As mentioned before, there was an attempt to generalize the standard formula of the entanglement entropy (1.27) of two coupled harmonic oscillators, where the state is gaussian, into non-gaussian states by means of the correlators method. In [31], they were able to prove that the entropy of a non-gaussian state has the same form as the gaussian one but with correlators

evaluated with respect to the non-gaussian state. However, this result is valid for the first order correction only.

Consider a non-Gaussian state of the form

$$\rho_\lambda(q,q') = Ne^{-\frac{A}{2}(q^2+q'^2)-\frac{C}{2}(q-q')^2-\lambda[\alpha_1(q^4+q'^4)+\alpha_2(q^3q'+qq'^3)+\alpha_3 q^2q'^2]} \tag{3.121}$$

The entanglement entropy of this state as found by [31] is

$$S = -\ln(1-\xi_{corr}) - \frac{\xi_{corr}}{1-\xi_{corr}}\ln(\xi_{corr}) \tag{3.122}$$

where

$$\xi_{corr} = \xi + \lambda\left[\frac{3\xi}{(\xi+1)(\xi-1)^3}\alpha_1 + \frac{3(\xi+1)}{2(\xi-1)^3\beta^2}\alpha_2 + \frac{1+\xi+\xi^2}{(\xi+1)(\xi-1)^3\beta^2}\alpha_3\right] \tag{3.123}$$

ξ is the parameter of the two coupled harmonic oscillator's entanglement entropy. The entanglement entropy (3.122) is valid only to the first order in λ. A brief overview of the method used by [31] can be found in the appendices, complete details can be found in [31].

Back to Green's method, a quartic perturbation on the bosonic lattice of two oscillators has the following Euclidean lagrangian

$$L_E = \frac{1}{2}[\dot{q}_1^2+\dot{q}_2^2+\omega^2q_1^2+\omega^2q_2^2+2bq_1q_2] + \frac{\lambda}{4!}(q_1^4+q_2^4) \tag{3.124}$$

The corresponding potential matrix has already been treated before, namely

$$V = b\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \qquad a = \omega^2/b \tag{3.125}$$

Back to [31], it was the state that suffered from a quartic perturbation. The ground state density matrix of the lagrangian (3.124) is unknown. However, a simple comparison between the entropy (B.8) and the one will be obtained here shows that the density matrix of (3.124) has the same form as ρ_λ and it can be determined by matching the coefficient $(\alpha_1,\alpha_2,\alpha_3)$.

In order to evaluate the first order of the perturbation series, it is required to determine the diagonal matrix elements $\{\mathbb{G}_{ii}(\tau,\tau)\}$, since

$$\delta\ln\mathrm{Tr}(\rho^n) = -\frac{3\lambda}{4!}\int_{-\infty}^{\infty}d\tau\left[\sum_{i=1}^{2n}\mathbb{G}_{ii}(\tau,\tau)^2 - n\sum_{i=1}^{2}G_{ii}(\tau,\tau)^2\right] \tag{3.126}$$

To avoid lengthy mathematical details, the determination of $\{\mathbb{G}_{ii}(\tau,\tau)\}$ will be found in the appendices, where

$$\begin{aligned}\int_{-\infty}^{\infty}d\tau\sum_{i=1}^{2n}(\mathbb{G})_{ii}^2(\tau,\tau) - n\sum_{i=1}^{2}(G_{ii})^2(\tau,\tau) &= \frac{1}{b\sqrt{b}}\Bigg[n\left[-\frac{3}{16\lambda_+\lambda_-}-\frac{3}{16\lambda_+^2}-\frac{1}{4\lambda_+^2\lambda_-}-\frac{1}{4\lambda_+\lambda_-^2}+\right.\\ &\left.+\frac{1}{4\lambda_-\lambda_+(\lambda_-+\lambda_+)}\right] + \left[\frac{1}{2\lambda_+\lambda_-}+\frac{1}{4\lambda_+^2}-\frac{1}{2\lambda_-(\lambda_++\lambda_-)}\right]B_- +\\ &+\left[\frac{1}{2\lambda_+\lambda_-}+\frac{1}{4\lambda_-^2}-\frac{1}{2\lambda_+(\lambda_++\lambda_-)}\right]B_+ + \frac{1}{n}\left[\frac{1}{4\lambda_+}B_-^2+\frac{1}{4\lambda_-}B_+^2+\frac{1}{\lambda_-+\lambda_+}B_-B_+\right]\Bigg]\end{aligned} \tag{3.127}$$

where

$$\lambda_\pm = \sqrt{a \pm 1}, \quad B_\pm = \frac{1}{2}\sum_{j=1}^{n-1} \frac{2\mu \pm \nu(1+\cos\frac{2\pi j}{n})}{2\mu^2 - \nu^2(1+\cos\frac{2\pi j}{n})}, \quad \mu = \frac{\lambda_+ + \lambda_-}{2}, \quad \nu = \frac{\lambda_+ - \lambda_-}{2}$$

The sum $B_\pm$ can be achieved using standard formulas. After rewriting $\lambda_\pm$ in terms of ξ (using $\sqrt{\xi} = \beta - \sqrt{\beta^2 - 1}$ where $\beta = a + \sqrt{a^2-1}$), the first order correction to R'enyi entropy can be obtained as follows :

$$\begin{aligned}\delta \ln \mathrm{Tr}\rho_A^n(\lambda) &= -n\frac{\lambda}{32b\sqrt{b}}(x^2-1)^{3/4}\sqrt{2x}\,\big[\, x^2(1-12x^2) + 2x(4x^2-1)B_n + \\ &\qquad + (4x^2+1)B_n^2\,\big] \end{aligned} \tag{3.128}$$

where $B_n = (1+\xi^n)/(1-\xi^n)$, $x \equiv B_1 = (1+\xi)/(1-\xi)$.

Finally, the first order correction of the entanglement entropy from a local quartic perturbation obtained by Green's method is

$$\delta S_{ent} = -\frac{d}{dn}\delta \ln \mathrm{Tr}\rho_A^n|_{n=1} = \frac{2\lambda}{b\sqrt{b}}\sqrt{1+\xi}\,(\frac{1+\xi}{1-\xi})^3\xi^{7/4}\frac{\ln\xi}{(1-\xi)^2} \tag{3.129}$$

This correction was obtained systematically. Higher-order corrections are valid to compute. The first order correction obtained here can be shown to confirm the result of [31]. The entropy S_λ has the same form of the free case (B.5) until the first order, but for the perturbed ξ_λ, such that

$$S_{ent}(\lambda) = -\ln(1-\xi_\lambda) - \frac{\xi_\lambda}{1-\xi_\lambda}\ln\xi_\lambda \tag{3.130}$$

where

$$\xi_\lambda = \xi + \delta\xi, \qquad \delta\xi = \frac{2\lambda}{b\sqrt{b}}\sqrt{1+\xi}\,(\frac{1+\xi}{1-\xi})^3\xi^{7/4} \tag{3.131}$$

This property for the first order correction of the entropy is not valid in general for higher order corrections. According to [31], higher order correlators will start to contribute to the non-gaussian state when exceeding the first order. This may be a reflection of the contribution of the tunneling terms $\mathbb{G}_{\pm,\mp}$ of Green's function, since these terms did not contribute in the zeroth nor in the first order correction to the entropy.

3.3.2 Non-local interaction

The perturbation in this situation is also quartic. However, it takes a little different form from the standard one. The Euclidean Lagrangian is given by

$$L_E = \frac{1}{2}\left(\dot{Q}^T\dot{Q} + Q^T V Q\right) + \frac{\lambda}{4!}(Q^T Q)^2 \tag{3.132}$$

The perturbation term $(Q^TQ)^2$ is highly entangled. Each oscillator in the lattice is coupled to all other oscillators[1].

The Euclidean action $\mathbb{S}_E$ can be written as follows

$$\mathbb{S}_E = \int_{-\infty}^{\infty} d\tau \frac{1}{2}\left(\dot{\mathbb{Q}}^T\dot{\mathbb{Q}} + \mathbb{Q}^T\mathbb{V}(\tau)\mathbb{Q}\right) + \frac{\lambda}{4!}\sum_{i,j=1}^{nN} q_i^2 q_j^2 \tag{3.133}$$

Similar to the previous section, the leading correction of the entropy is given by means of Wick's theorem as follows

$$\begin{aligned}\delta \ln \mathrm{Tr}(\rho^n) = \frac{-\lambda}{4!}\int d\tau \left[2\left(\ \mathrm{Tr}\big(\mathbb{G}^2(\tau,\tau)\big) - n\,\mathrm{Tr}\big(G^2(\tau,\tau)\big)\ \right) + \right. \\ \left. + \left(\mathrm{Tr}(\mathbb{G}(\tau,\tau))\ \right)^2 - n\left(\mathrm{Tr}(G(\tau,\tau))\ \right)^2\right]\end{aligned} \tag{3.134}$$

There are two contributions to the perturbation correction. The first contribution is the term $[\mathrm{Tr}(\mathbb{G}^2(\tau,\tau)) - n\,\mathrm{Tr}(G^2(\tau,\tau))]$, which is represented by connected Feynman diagrams. This term gives a finite contribution, which is found to be (see the appendices)

$$(\delta \ln \mathrm{Tr}\rho^n)_c = -\frac{2\lambda}{4!}\left[-\frac{3n}{8}\mathrm{Tr}V^{-3/2} + \frac{1}{4}\mathrm{Tr}(\mathbb{V}_p^{-1}+\mathbb{V}^{-1})(\mathbb{W}+\mathbb{W}_p)^{-1} + 2F(\mathbb{W},\mathbb{W}_p)\right] \tag{3.135}$$

where the function F is defined by

$$F(\mathbb{W},\mathbb{W}_p) = \int_{-\infty}^{0} d\tau\,\mathrm{Tr}\left[(\mathbb{W}+\mathbb{W}_p)^{-1}e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}e^{2\mathbb{W}\tau}\right] \tag{3.136}$$

Whereas the second contribution is given by the term $\left[\left(\mathrm{Tr}(\mathbb{G}(\tau,\tau))\ \right)^2 - n\left(\mathrm{Tr}(G(\tau,\tau))\ \right)^2\right]$, which is represented by disconnected Feynmann diagrams. This term has unremovable infinity, such that

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau \left(\mathrm{Tr}(\mathbb{G}(\tau,\tau))\ \right)^2 - n\left(\mathrm{Tr}(G(\tau,\tau))\ \right)^2 = \int_{-\infty}^{0} d\tau \left[\frac{1}{2}\left(\mathrm{Tr}\big(\mathbb{W}^{-1}\big)\right)^2 - \frac{n}{2}\left(\mathrm{Tr}\big(W^{-1}\big)\right)^2 + \right. \\ + \frac{1}{2}\left(\mathrm{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}\right)\right)^2 + 2\left(\mathrm{Tr}\left(e^{2\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}\right)\right)^2 - 2\,\mathrm{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}\right)\mathrm{Tr}\left(e^{2\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}\right)\Bigg] + \\ \left. - \frac{1}{2}\,\mathrm{Tr}\big(\mathbb{W}^{-1}\big)\,\mathrm{Tr}\big(\mathbb{W}^{-2}\big) + \mathrm{Tr}\big(\mathbb{W}^{-1}\big)\,\mathrm{Tr}\left(\mathbb{W}^{-1}\frac{1}{\mathbb{W}+\mathbb{W}_p}\right)\right.\end{aligned} \tag{3.137}$$

See the appendices for details. The divergence comes from the term

$$\int_{-\infty}^{0} d\tau \frac{1}{2}\left(\mathrm{Tr}\big(\mathbb{W}^{-1}\big)\right)^2 - \frac{n}{2}\left(\mathrm{Tr}\big(W^{-1}\big)\right)^2 = \frac{n^2-n}{2}\,\mathrm{Tr}\big(W^{-1}\big)\int_0^{\infty} d\tau = \infty \tag{3.138}$$

This divergence has not been removed by the scaling $n\ln(\mathbb{Z}_1)$, as what happened with $(\delta \ln \mathrm{Tr}\,\rho^n)_c$ or with local interaction. As a matter of fact, non-local interaction or long range interaction in the

[1] Therefore the notion non-local interaction.

continuum does not produce such IR divergence, it just rescales the total entanglement entropy (see for example [65]). However, it seems illegal to compare the treated problem with the continuous case, since the interaction $(Q^TQ)^2$ cannot be seen as a discretization of some field interaction.

It is hard to tell whether this IR divergence is a feature of all long-range interactions or even to exclude its existence from all short-range interactions. All that can be said is that it does not appear in the first neighbor interaction; like one of the form $(q_{i+1}-q_i)^4$. In this example, even the disconnected Feynman diagram does not produce divergence, precisely the term $\mathbb{G}_{i+1,i+1}\mathbb{G}_{ii}$. A simple manipulation on the matrix G shows that $G_{i+1,i+1}G_{ii} = G_{ii}^2 = ((2W)_{ii}^{-1})^2$. Its counterpart in the supermatrix $\mathbb{G}$ behaves in the same way, and the divergence in this situation is cancelled out in the same manner to local interaction in the previous section.

3.4 Applications to fuzzy spaces and outlook

Fuzzy or non-commutative spaces are spaces in which the coordinates do not commute. It is then impossible to define a point in this space exactly and one can only get a fuzzy estimation of it, depending on the non-commutativity or fuzziness parameter. The first familiar fuzzy space is phase space in quantum mechanics, where the position coordinate and its conjugate momentum do not commute.

The investigation of the entanglement entropy in fuzzy spaces was first considered in [22]. It was possible to show in [22], that the problem of a scalar field in non-commutative geometry can be transformed into a problem of a set of coupled harmonic oscillators for the considered models. Therefore, the use of the obtained Green's function (2.65) is valid to evaluate the entanglement entropy, at least formally, in these fuzzy models. Moreover, it would be possible to handle interaction theory perturbatively.

A rigorous mathematical description of fuzzy spaces is very technical mathematically. However, a rough investigation into one of the models considered in [22] will be done in order to give a concrete example of the way that fuzzy spaces can be handled.

The example is free scalar field in fuzzy sphere. The sphere in commutative space is defined by the identity

$$x^2 + y^2 + z^2 = R^2 \tag{3.139}$$

A sphere in a non-commutative space is defined by the same identity. However, since the coordinates do not commute, they represent operators rather than scalars. A fuzzy sphere is then defined by

$$\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{R}^2 \tag{3.140}$$

$\hat{R}^2$ here is an operator, but in order to make analogy with the commutative sphere it should be represented by a scalar, more precisely a scalar times a unit operator. It is then required to determine the non-commuting operators $\{\hat{x}, \hat{y}, \hat{z}\}$ which satisfy the sphere equation.

The rotational symmetry of space suggests the use of angular momentum operators, since these last do not commute, it follows that

$$\{\hat{x}, \hat{y}, \hat{z}\} \to \{\theta\hat{L}_x, \theta\hat{L}_y, \theta\hat{L}_z\} \tag{3.141}$$

Where θ is a scalar with dimension of length, such that

$$[\hat{x}, \hat{y}] = i\theta\hat{z} \quad \text{rotational} \tag{3.142}$$

The radius of the sphere is then given by

$$\hat{R}^2 = \theta^2 \hat{L}^2 \tag{3.143}$$

By taking the basis in which z is diagonal, it follows that $\hat{R}^2$ is also diagonal since $[\hat{z}, \hat{R}^2] = 0$. The eigenvalues of $\hat{L}^2$ are of the form $l(l+1)$ with $l = 0..\infty$. Each eigenvalue is $2l+1$ degenerate. The eigenvalues of L_z are of the form m_l where $-l \leq m_l \leq l$ for each specific value l.

The identification of a fuzzy sphere reads as follows : each value l corresponds to a sphere with radius $R = \theta\sqrt{l(l+1)}$, the space dimension of the sphere is $N = 2l+1$. The larger l is, the larger R and its corresponding sphere are. The commutative sphere is obtained by taking the limit ($\theta \to 0, l \to \infty, R \to$ constant). In this limit, the coordinate commutes since $\theta \to 0$.

Roughly speaking, the commutative sphere is obtained by reducing the fuziness parameter θ and blowing up the number of degrees of freedom indicated by l (the total number of degrees of freedom is given by $N^2 = (2l+1)^2$) in order to reidentify the point on continuous surface.

Take a sphere with a given value for l. The fields defined on the sphere are $(2l+1) \times (2l+1)$ matrices. The integral over the fields in the commutative limit is substituted by the trace as follows

$$\int \phi_1\phi_2 \; d\Omega \to \frac{1}{N} \operatorname{Tr}(\phi_1\phi_2) \tag{3.144}$$

The laplacian Δ in the commutative sphere is given by

$$\Delta = \frac{L^2}{R^2} \tag{3.145}$$

One then assumes that the Laplacian on a fuzzy sphere is defined by means of L^2. Indeed, the Laplacian of a fuzzy sphere is given by

$$\Delta = \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{L}_3^2 \tag{3.146}$$

Where $\mathcal{L}_i(\phi) = [L_i, \phi]$ and $\mathcal{L}_i^2(\phi) = [L_i, [L_i, \phi]]$.

Let us now consider a real scalar free field on spacetime with a topology of $R \times S_N^2$, where S_N^2 is a fuzzy sphere of space dimension $N = 2l+1$. The Lagrangian is given by :

$$L = \frac{1}{2N} \operatorname{Tr}\Big(\dot{\phi}^2 - \phi(\mathcal{L}_i^2 + \mu^2)\phi\Big) \tag{3.147}$$

Where μ is the mass parameter.

Define the basis $\{|\ a\rangle\}$ which corresponds to the set of eigenvectors of L_3 with eigenvalues $\{-a + l + 1\}$ where $a = (1..2l+1)$. L_3 is diagonal in this basis since it is its own eigenbasis. As a matter of fact, the argument given below does not depend on this particular choice of the basis. One can choose the eigenvectors of an arbitrary component of the angular momentum $\vec{L}.\vec{n}$ without affecting the argument.

First, the Laplacian term $\mathrm{Tr}(\phi\mathcal{L}_i^2\phi)$ can be written as :

$$\mathrm{Tr}\big(\phi\mathcal{L}_i^2\phi\big) = 2\,\mathrm{Tr}(\phi_{ab}M_{ab,cd}\phi_{cd}) \tag{3.148}$$

Where

$$M_{ab,cd} = (L^2)_{bc}\delta_{da} - (L_i)_{bc}(L_i)_{da} = l(l+1)\delta_{bc}\delta_{da} - (L_i)_{bc}(L_i)_{da} \tag{3.149}$$

The aim here is to reformulate the problem into a theory of a set of real coupled harmonic oscillators. For this purpose, one decomposes ϕ into two real scalar field matrices as follows : $\phi = \phi_1 + i\phi_2$. The Hermiticity of ϕ implies that ϕ_1 is real symmetric matrix and ϕ_2 is real antisymmetric matrix. In the eigenbasis of L_3, the matrix elements of L_1, L_2 and L_3 are given by

$$\begin{aligned}(L_1)_{ab} &= \frac{1}{2}[_b\delta_{a,b+1} + B_a\delta_{a,b-1}]\\ (L_2)_{ab} &= \frac{i}{2}[_b\delta_{a,b+1} - B_a\delta_{a,b-1}]\\ (L_3)_{ab} &= A_a\delta_{ab}\end{aligned}$$

Where $B_a = \sqrt{a(N-a)}$ and $A_a = -a + \frac{N+1}{2}$ with $(a,b) = (1:N)$.

It is easy to note that $M_{ba,dc} = M_{ab,cd}$, this can be easily shown using the form of $\{L_1, L_2, L_3\}$ given above. It follows that

$$\begin{aligned}\mathrm{Tr}\big(\phi\mathcal{L}_i^2\phi\big) &= 2(\phi_1+\phi_2)_{ab}M_{ab,cd}(\phi_1+\phi_2)_{cd}\\ &= 2\Phi_{ab}M_{ab,cd}\Phi_{cd}\end{aligned} \tag{3.150}$$

Where $\Phi \equiv \phi_1 + \phi_2$. Using the explicite form of $\{L_1, L_2, L_3\}$ it is straightforward to show that

$$\begin{aligned}\mathrm{Tr}\big(\phi\mathcal{L}_i^2\phi\big) &= 2\Phi_{ab}(l(l+1) - A_aA_b)\Phi_{ab} - \Phi_{ab}(B_{a-1}B_{b-1})\Phi_{a-1,b-1} +\\ &\quad -\Phi_{ab}(B_aB_b)\Phi_{a+1,b+1}\end{aligned} \tag{3.151}$$

It is not hard to note that $\Phi_{ab}, \Phi_{a-1,b-1}$ and $\Phi_{a+1,b+1}$ lie on the same diagonal of the field matrix Φ. This suggests the introduction of the field vectors $\{Q^{(p)}\}$ which represents the set of the matrix Φ diagonals, where the positive (negative) 'p' stand for upper (lower) diagonals where $p = -(N-1) : N-1$, such that

$$\begin{aligned}Q^{(0)} &= (\Phi_{11}, \Phi_{22}, ..., \Phi_{NN})\\ Q^{(-1)} &= (\Phi_{21}, \Phi_{32}, ..., \Phi_{N,N-1})\\ Q^{(+1)} &= (\Phi_{12}, \Phi_{23}, ..., \Phi_{N-1,N})\end{aligned}$$

For general $\mid p \mid$, the vectors $Q^{(|p|)}$ and $Q^{(-|p|)}$ has $N- \mid p \mid$ degrees of freedom, such that

$$\begin{aligned}Q^{(-|p|)} &= (\Phi_{1+|p|,1}, \Phi_{2+|p|,2}, ..., \Phi_{N,N-|p|})\\ Q^{(|p|)} &= (\Phi_{1,1+|p|}, \Phi_{2,2+|p|}, ..., \Phi_{N-|p|,N})\end{aligned}$$

Since the sum over a and b in (3.151) includes all the matrix elements of the matrix Φ, the sum on (3.151) can be turned into the sum over all components of the vectors $\{Q^{(p)}\}$. It is not hard to show that

$$\begin{aligned}\mathrm{Tr}\big(\phi\mathcal{L}_i^2\phi\big) \;=\; 2\sum_{m=-(N-1)}^{N-1}\sum_{a=1}^{N-|m|} Q_a^{(m)}\big[\;(\;l(l+1)-A_aA_{a+|m|}\;)\delta_{ab}-\frac{1}{2}B_{a-1}B_{a-1+|m|}\delta_{a-1,b}+\\ -\frac{1}{2}B_aB_{a+|m|}\delta_{a+1,b}\;\big]Q_b^{(m)}\end{aligned} \tag{3.152}$$

The other part of the Lagrangian can be computed in the same way, such that

$$\mathrm{Tr}\Big(\dot\phi^2-\mu^2\phi^2\Big)=\sum_{m=-(N-1)}^{N-1}\sum_{a=1}^{N-|m|}\dot Q_a^{(m)}\dot Q_a^{(m)}+\mu^2Q_a^{(m)}Q_a^{(m)} \tag{3.153}$$

Hence the Lagrangian can be written as :

$$L=\sum_{m=-(N-1)}^{N-1}L^{(m)}=\sum_{m=-(N-1)}^{N-1}\sum_{a,b=1}^{N-|m|}\frac{1}{2}[\dot Q_a^{(m)}\dot Q_a^{(m)}-Q_a^{(m)}V_{ab}^{(m)}Q_b^{(m)}] \tag{3.154}$$

where

$$V_{ab}^{(m)}=2\bigg[\big(\;l(l+1)+\frac{\mu^2}{2}-A_aA_{a+|m|}\big)\delta_{a,b}-\frac{1}{2}B_{a-1}B_{a-1+|m|}\delta_{a-1,b}-\frac{1}{2}B_aB_{a+|m|}\delta_{a+1,b}\bigg]. \tag{3.155}$$

Therefore, the Lagrangian of free scalar field theory on a fuzzy sphere naturally splits into $2(2l)+1$ independent sectors $\{L^{(m)}\}$ with $m=-(N-1),\cdots,(N-1)$, each sector $L^{(m)}$ has $N-|m|$ degrees of freedom ($N-|m|$ coupled harmonic oscillator).

To evaluate the entanglement entropy for the fuzzy sphere, one first needs to divide the field degrees of freedom into two sets, each residing in two, or several, disjoint regions. Let's take the case of the upper and lower fuzzy hemispheres. It has been conjectured in [22] using a heuristic argument that the two hemispheres are represented by the elements of the upper left triangular part of the field matrix Φ_+ and the right lower triangular part Φ_-. For example, take the fuzzy sphere where $N=5$, the matrices Φ_- and Φ_+ are the following

$$\Phi_+=\begin{pmatrix}\Phi_{11}&\Phi_{12}&\Phi_{13}&\Phi_{14}&0\\ \Phi_{21}&\Phi_{22}&\Phi_{23}&0&0\\ \Phi_{31}&\Phi_{32}&0&0&0\\ \Phi_{41}&0&0&0&0\\ 0&0&0&0&0\end{pmatrix},\;\Phi_-=\begin{pmatrix}0&0&0&0&\Phi_{15}\\ 0&0&0&\Phi_{24}&\Phi_{25}\\ 0&0&\phi_{33}&\phi_{34}&\Phi_{35}\\ 0&\Phi_{42}&\Phi_{43}&\Phi_{44}&\Phi_{45}\\ \Phi_{51}&\Phi_{52}&\Phi_{53}&\Phi_{54}&\Phi_{55}\end{pmatrix} \tag{3.156}$$

This conjecture was subsequently proven in [66] using rigourous mathematical argument.

It is fruitful to see that this splitting of the fuzzy sphere across the spatial axis z leads to a natural splitting to the field vectors $\{Q^{(m)}\}$ into halfs. This proves that $\{Q^{(m)}\}$ are the proper coordinates to handle this problem.

For free scalar field theory in a non-commutative sphere, it has been shown that the Lagrangian splits into sectors of independent variables. Each sector has the standard form of the Lagrangian of a set of coupled harmonic oscillators. The global ground state density matrix is given by

$$\rho = \bigoplus_{m=-(N-1)}^{N-1} \rho^{(m)}. \tag{3.157}$$

Whereas the reduced density matrix is obtained as follows

$$\rho_+ = \mathrm{Tr}_-(\rho) = \bigoplus_{m=-(N-1)}^{N-1} \mathrm{Tr}_-(\rho^{(m)}) = \bigoplus_{m=-(N-1)}^{N-1} \rho_+^{(m)}. \tag{3.158}$$

It has been shown in [22] that the entanglement entropy for the system defined by the Lagrangian (3.154) is proportional to the number of degrees of freedom residing at the boundary, which can also be interpreted as proportional to the circumference of the separating area, the fuzzy circle. This was shown using numerical methods whithin the real time approach.

The developed Green's function (2.65) can be used in straightforward manner in the fuzzy sphere system, at least numerically, since the obtained Lagrangian (3.154) for this model has a quadratic positive potential. Green's function itself is splitted into sectors. Each sector of the potential corresponds to a sector of Green's function as follows

$$\mathbb{G}_n(\tau,\tau') = \bigoplus_{m=-(N-1)}^{N-1} \mathbb{G}_n^{(m)}(\tau,\tau') \tag{3.159}$$

Unlike the case of $1+1$ lattice the potentials $V^{(m)}$ are not circulant matrices, which poses an additional complication to the analytical investigation of this model. However, it was shown in [23] that the entanglement entropy is essentially captured by the near boundary approximation where the potentials are Toeplitz matrices. In addition to this it can be shown that not all sectors, different m, are equally relevant, there are sectors which are weakly entangled and hence give negligible contributions to the entanglement entropy in the large N limit [36], and the relevant sectors are governed by Toeplitz potentials in the near boundary approximation . Putting all this together, it is conceivable that in the large N limit, one can approximate the potentials for the relevant sectors with circulant matrices, which would greatly simplify the problem.

One of the motivations for developing Green's function (2.65) was to investigate the interacting theories in fuzzy spaces. Take a quartic interaction term of the form

$$V_{int} = \frac{\lambda}{4}\,\mathrm{Tr}\,\phi^4 \tag{3.160}$$

In order to apply the Green's function formalism, the interaction term should be rewritten in terms of Φ first. It is not hard to show that

$$\begin{aligned}\mathrm{Tr}\big(\phi^4\big) &= \frac{1}{2}\left[\mathrm{Tr}\big(\phi^4\big) + \mathrm{Tr}\big(\phi^4\big)^*\right] = \frac{1}{2}\left[\mathrm{Tr}\big((\phi_1^4 + i\phi_2)^4\big) + \mathrm{Tr}\big((\phi_1^4 - i\phi_2)^4\big)\right] \\ &= \mathrm{Tr}\big(\phi_1^4 + \phi_2^4 - 4\phi_1^2\phi_2^2 - 2(\phi_1\phi_2)^2\big)\end{aligned} \tag{3.161}$$

Since $\Phi = \phi_1 + \phi_2$ with ϕ_1 (ϕ_2) is a symmetric (antisymmetric) matrix, it follows that

$$2\phi_1 = \Phi + \Phi^T, \quad 2\phi_2 = \Phi - \Phi^T \tag{3.162}$$

It is straightforward to show that the interaction term can be written as

$$V_{int} = \frac{\lambda}{4} \operatorname{Tr}\left(\Phi^2 \Phi^{T2} + \frac{1}{2}(\Phi\Phi^T)^2 - \frac{1}{2}\Phi^4 \right) \tag{3.163}$$

It can be seen here that the interaction term produces new couplings between the degrees of freedom which were not coupled in the free theory. In addition, it is not invariant under the action of the permutation matrix P_π once this interaction is expressed using the n-fold variables. This makes the application of Wick's theorem a bit subtle.

The interaction term (3.163) has some form of similarity with the non-local interaction, which has been discussed in the previous section. It is natural to wonder if one would get an irremovable IR infinity, for finite N, as in the non-local interaction. Although only an explicit calculation will give a decisive answer to this question, we do not expect such irremovable IR divergence to be present in this model. Actually, working the case $N = 2$ would be enough, because as long as this particular IR problem is concerned, the size of the fuzzy sphere should play no role.

The main challenge which faces the investigation of this model, and other lattice and fuzzy models, is the lack of information about the asymptotic behavior of N-dependent index Toeplitz matrices, on which very little if any is known. Therefore, the development of new mathematical tools on this issue is required.

Conclusion

Entanglement entropy is not just a reliable tool that promises unprecedented applications in quantum computation and quantum information only; the ability to identify it and control it will enable us to understand a number of physical phenomena in different areas of physics. This indeed makes entanglement entropy an active area of research.

The determination of entanglement entropy steel represents a challenge to most physical systems. A number of methods have been developed in order to evaluate it, depending on the corresponding area of research. They are mainly decomposed into two categories : the real-time approach and the Euclidean time approach. This last offers more powerful tools in the investigation of entanglement entropy and in extracting some analytical results. These two formalisms were presented in the first chapter, in addition to a quick overview of the main concepts in entanglement.

The lack of methods and tools in the investigation of entanglement entropy on lattices within the Euclidean time approach was the motivation of this work. The importance of lattice models comes from the speculation that the physical world is essentially discrete at the Planck scale [21]. In addition, the entanglement of a theory of fields is infinity in the absence of distance or energy cutoff. Therefore, an Euclidean Green's function on discrete spaces is presented in this thesis. The discrete space could be a lattice or a fuzzy space. The main advantage that comes from the development of this method is the ability to investigate the interacting theories and non-Gaussian states systematically by means of Wick's theorem, something that was not possible before in lattice models.

The possible applications of this method are presented in the last chapter. The well-known formula for the entanglement entropy of two coupled harmonic oscillators has been rederived using the new method. The investigation of N-coupled harmonic oscillators lattice was obstructed by the lack of information about the inverse of a special class of Toeplitz matrices. A contribution to the asymptotic inverse of this class of matrices was suggested. The investigation of the entanglement entropy of the 1+1 Klein Gordon field in lattice regularization is the object of a subsequent work by the authors.

An example of a quartic interacting theory was treated. Two types of interaction have been investigated, local and non-local interaction. The local interaction comes from a discretization of ϕ^4 interaction. The non-local interaction cannot be interpreted as a discretization of some field interaction theory. The first order of the interaction has been explicitly computed in a lattice of two coupled harmonic oscillators. Non-local interaction shows a divergence of IR origins. It is the object of future work to investigate this infinity and give a criterion for its appearance.

A fuzzy space is a space in which the coordinates do not commute. It has been shown that a scalar field theory can be regularized by a lattice of harmonic oscillators in a number of fuzzy space models [22]. It is then possible to use the developed Euclidean Greens function on these systems in order to calculate the entanglement entropy. The authors are interested, as a future work, in the possibility of extracting some analytical results from these models and investigating the interaction theories.

Appendix A

Green's function expansion

The aim of this appendix is to give a detailed calculation for the determination of the first and second orders of Green's matrix function expansion. Zeroth order are straightforward integrals.

One originally has

$$\mathbb{F}_1^+(s,\tau') = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{D_1(z)}{z-s} = c_1 + c_2 \tag{A.1}$$

where

$$\begin{aligned} D_1(s) &= \delta\mathbb{V}\ \mathbb{F}_0(s)\ \mathbb{F}_0^-(s,\tau') - \mathbb{V}\ \mathbb{F}_0^2(s)\ e^{s\tau'} && \text{(A.2)}\\ c_1 &= \frac{\delta\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\mathbb{F}_0(z)\mathbb{F}_0^-(z,\tau')}{z-s} && \text{(A.3)}\\ c_2 &= \frac{-\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\mathbb{F}_0^2(z)e^{-z\tau'}}{z-s} && \text{(A.4)} \end{aligned}$$

Let's start with c_1, take a complex contour C around the left complex plane $\mathrm{Re}(z) < 0$, as it is shown in figure A.1. The inegrand in c_1 has one simple pole $z = s$ inside the contour $z = s$ since $\mathrm{Re}(s) < 0$ for $\mathbb{F}_1^+$ and a second order pole $z = -\sqrt{E}$ in this region. By means of the Residues theorem, one gets :

$$\begin{aligned} c_1 &= \frac{\delta\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\mathbb{F}_0(z)\mathbb{F}_0^-(z,\tau')}{z-s} = \frac{\delta\mathbb{V}}{2\pi i}\oint_C \frac{dz}{z-s}\frac{-1}{z^2-E}\frac{-e^{\sqrt{E}\tau'}}{2\sqrt{E}(z+\sqrt{E})}\\ &= \frac{\delta\mathbb{V}e^{\sqrt{E}\tau'}}{2\sqrt{E}}\left[\frac{1}{s^2-E}\frac{1}{s+\sqrt{E}} + \frac{d}{dz}\frac{1}{z-s}\frac{1}{z-\sqrt{E}}\ |_{z=-\sqrt{E}}\right]\\ &= \frac{\delta\mathbb{V}e^{\sqrt{E}\tau'}}{2\sqrt{E}}\left[\frac{1}{(s-\sqrt{E})(s+\sqrt{E})^2} - \frac{1}{(-\sqrt{E}-s)^2(-2\sqrt{E})} - \frac{1}{(-\sqrt{E}-s)(-2\sqrt{E})^2}\right]\\ &= \frac{\delta\mathbb{V}e^{\sqrt{E}\tau'}}{(2\sqrt{E})^3}\frac{1}{s-\sqrt{E}} \end{aligned} \tag{A.5}$$

The equivalence between the integration over imaginary line and contour C comes from the fact

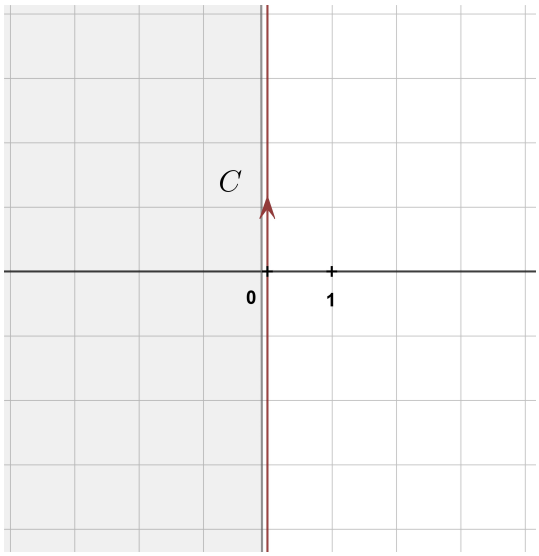

Figure A.1: The contour of integration C.

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that the integrand vanishes on all parts of the contour, where z tends to infinity in them, except for the imaginary line.

In a similar way, taking the same contour C, one can evaluate c_2 as follows

$$\begin{aligned} c_2 &= \frac{-\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\mathbb{F}_0^2(z)e^{-z\tau'}}{z-s} = \frac{-\mathbb{V}}{2\pi i}\oint_C \frac{dz}{z-s}\frac{e^{-z\tau'}}{(z^2-E)^2} \\ &= -\mathbb{V}\left[\frac{e^{-s\tau'}}{(E-s^2)^2} + \frac{e^{-z\tau'}}{(z-s)(z-\sqrt{E})^2}\,|_{z=-\sqrt{E}}\right] \\ &= -\mathbb{V}\left[\frac{e^{-s\tau'}}{(E-s^2)^2} + \frac{\tau' e^{\sqrt{E}\tau'}}{(\sqrt{E}+s)(2\sqrt{E})^2} - \frac{e^{\sqrt{E}\tau'}}{(\sqrt{E}+s)^2(2\sqrt{E})^2} + \frac{2e^{\sqrt{E}\tau'}}{(\sqrt{E}+s)(-2\sqrt{E})^3}\right] \end{aligned} \tag{A.6}$$

Finally, it follows that $\mathbb{F}_1^+$ is

$$\begin{aligned} \mathbb{F}_1^+ = \mathbb{F}_1^+(s,\tau') = \frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}} - \mathbb{V}\frac{e^{-s\tau'}}{(s^2-E)^2} - \mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+s} + \right. \\ \left. -\frac{1}{\sqrt{E}(\sqrt{E}+s)} - \frac{1}{(\sqrt{E}+s)^2}\right], \qquad \mathrm{Re}(s) < 0 \end{aligned} \tag{A.7}$$

$\mathbb{F}_1^-$ can be found from the identity

$$\mathbb{F}_1^+(s,\tau') - \mathbb{F}_1^-(s,\tau') = D_1(s) \tag{A.8}$$

Therefore

$$\begin{aligned}\mathbb{F}_1^-(s,\tau') &= \mathbb{F}_1^+(s,\tau') - D_1(s) \\ &= \frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[1-\frac{4E}{(s+\sqrt{E})^2}\right] - \mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+s}+\right. \\ &\quad \left.-\frac{1}{\sqrt{E}(\sqrt{E}+s)} - \frac{1}{(\sqrt{E}+s)^2}\right], \qquad \mathrm{Re}(s) > 0 \end{aligned} \tag{A.9}$$

$\mathbb{G}_1^+(\tau,\tau')$ is determind by performing inverse Laplace transformation over $\mathbb{F}_1^+(s,\tau')$, as follows :

$$\begin{aligned}\mathbb{G}_1^+(\tau,\tau') &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\ \mathbb{F}_1^+(s,\tau') \quad \tau,\tau' < 0 \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\left(\frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}} - \mathbb{V}\frac{e^{-s\tau'}}{(s^2-E)^2} - \mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+s}+\right.\right. \\ &\quad \left.\left.-\frac{1}{\sqrt{E}(\sqrt{E}+s)} - \frac{1}{(\sqrt{E}+s)^2}\right]\right) \end{aligned} \tag{A.10}$$

The terms of (A.10) will be treated separately. Each one with its appropriate contour of integration, which vanishes in all parts of the contour except at the imaginary line. There are two contours which are used : the old contour C and the contour over the right complex plane $\bar{C}$. After that, the Residues theorem will be used. The integrals are listed below :

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\frac{1}{s-\sqrt{E}} = -\frac{1}{2\pi i}\oint_{\bar{C}} ds\ e^{s\tau}\frac{1}{s-\sqrt{E}} = -e^{\sqrt{E}\tau} \tag{A.11}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\frac{1}{s+\sqrt{E}} = -\frac{1}{2\pi i}\oint_{\bar{C}} ds\ \frac{e^{s\tau}}{s+\sqrt{E}} = 0 \tag{A.12}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\frac{1}{(s+\sqrt{E})^2} = -\frac{1}{2\pi i}\oint_{\bar{C}} ds\ \frac{e^{s\tau}}{(s+\sqrt{E})^2} = 0 \tag{A.13}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\frac{e^{-s\tau'}}{(s^2-E)^2} = \frac{e^{-\sqrt{E}(\tau-\tau')}}{(2\sqrt{E})^2}\left[\mid \tau-\tau' \mid + \frac{1}{\sqrt{E}}\right] \tag{A.14}$$

One can evaluate the last integral (A.14) as follows

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\frac{e^{-s\tau'}}{(s^2-E)^2} = \frac{1}{2\pi i}\begin{cases} \oint_C ds\ e^{s(\tau-\tau')}(s^2-E)^{-2} & \text{if } \tau-\tau' > 0 \\ -\oint_{\bar{C}} ds\ e^{s(\tau-\tau')}(s^2-E)^{-2} & \text{if } \tau-\tau' < 0 \end{cases} \tag{A.15}$$

where

$$\frac{1}{2\pi i}\oint_C ds\ e^{s(\tau-\tau')}\frac{1}{(s^2-E)^2} = \frac{d}{ds}\frac{e^{s(\tau-\tau')}}{(s-\sqrt{E})^2}\bigg|_{s=-\sqrt{E}} = \frac{(\tau-\tau')e^{-\sqrt{E}(\tau-\tau')}}{(2\sqrt{E})^2} - 2\frac{e^{-\sqrt{E}(\tau-\tau')}}{(-2\sqrt{E})^3}$$

$$\frac{1}{2\pi i}\oint_{\bar{C}} ds\ e^{s(\tau-\tau')}\frac{1}{(s^2-E)^2} = \frac{d}{ds}\frac{e^{-s(\tau-\tau')}}{(s+\sqrt{E})^2}\bigg|_{s=\sqrt{E}} = \frac{(\tau-\tau')e^{\sqrt{E}(\tau-\tau')}}{(2\sqrt{E})^2} - 2\frac{e^{\sqrt{E}(\tau-\tau')}}{(2\sqrt{E})^3}$$

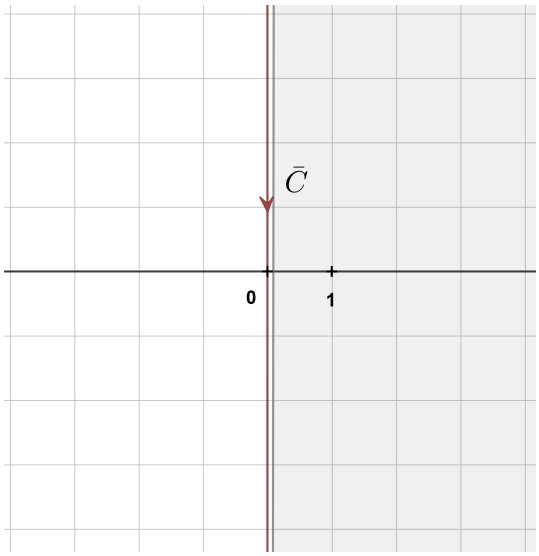

Figure A.2: The contour of integration $\bar{C}$.

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Therefore

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{s\tau}\frac{e^{-s\tau'}}{(s^2-E)^2} = \frac{e^{-\sqrt{E}|\tau-\tau'|}}{(2\sqrt{E})^2}\left[\mid \tau-\tau' \mid + \frac{1}{\sqrt{E}}\right] \tag{A.16}$$

Finally

$$\mathbb{G}_1(\tau,\tau') = -\frac{\delta\mathbb{V}}{(2\sqrt{E})^3}e^{\sqrt{E}(\tau+\tau')} - \frac{\mathbb{V}}{2\sqrt{E}}\mathbb{G}_0(\tau,\tau')\left(\mid \tau-\tau' \mid + \frac{1}{\sqrt{E}}\right), \quad \tau<0,\ \tau'<0 \tag{A.17}$$

Second order

The second order can be manipulated in the same way, where

$$\mathbb{F}_2^+ - \mathbb{F}_2^- = D_2(s) \tag{A.18}$$

$D_2(s)$ is given by

$$D_2(s) = \delta\mathbb{V}\mathbb{F}_0\mathbb{F}_1^- - \mathbb{V}\ \mathbb{F}_0\left[\delta\mathbb{V}\ \mathbb{F}_0\ \mathbb{F}_0^- - \mathbb{V}\ \mathbb{F}_0^2\ e^{-s\tau'}\right] \tag{A.19}$$

It follows that

$$\mathbb{F}_2^+(s,\tau') \ = \ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\frac{D_2(z)}{z-s} = c_1+c_2+c_3 \tag{A.20}$$

The contours of integration are C and $\bar{C}$ as before. The integrals are evaluated using the Residues theorem, the integrals c_1, c_2 and c_3 are listed below :

$$\begin{aligned}
c_1^1 &= \frac{\delta\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{z-s}\frac{1}{E-z^2}\left(\frac{\delta\mathbb{V}}{(2\sqrt{E})^3}\frac{e^{\sqrt{E}\tau'}}{z-\sqrt{E}}\left[1-\frac{4E}{(z+\sqrt{E})^2}\right]\right)\\
&= -\frac{(\delta\mathbb{V})^2}{2\pi i}\frac{e^{\sqrt{E}\tau'}}{(2\sqrt{E})^3}\left[\oint_C\frac{1}{(z-s)(z-\sqrt{E})^2(z+\sqrt{E})}+\right.\\
&\qquad\left.+4E\oint_{\bar{C}}\frac{1}{(z-s)(z-\sqrt{E})^2(z+\sqrt{E})^3}\right]\\
&= -(\delta\mathbb{V})^2\frac{e^{\sqrt{E}\tau'}}{(2\sqrt{E})^3}\left[\frac{-1}{(\sqrt{E}+s)(2\sqrt{E})^2}+\frac{1}{(s-\sqrt{E})^2(s+\sqrt{E})}+\right.\\
&\qquad\left.+4E\left(\frac{-1}{(\sqrt{E}-s)^2(2\sqrt{E})^3}-\frac{3}{(\sqrt{E}-s)(2\sqrt{E})^4}\right)\right]\\
&= \frac{-1}{16\sqrt{E}}\left(\frac{\delta\mathbb{V}}{E}\right)^2\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}} \qquad (A.21)
\end{aligned}$$

$$\begin{aligned}
c_1^2 &= \frac{\delta\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{z-s}\frac{1}{E-z^2}\left(-\mathbb{V}\frac{e^{\sqrt{E}\tau'}}{4E}\left[\frac{\tau'}{\sqrt{E}+z}-\frac{1}{\sqrt{E}(\sqrt{E}+z)}-\frac{1}{(\sqrt{E}+z)^2}\right]\right)\\
&= -\frac{\delta\mathbb{V}\mathbb{V}e^{\sqrt{E}\tau'}}{2\pi i(4E)}\oint_{\bar{C}}\left[(\tau'-\frac{1}{\sqrt{E}})\frac{1}{(z-s)(z-\sqrt{E})(z+\sqrt{E})^2}-\frac{1}{(z-s)(z-\sqrt{E})(z+\sqrt{E})^3}\right]\\
&= -\frac{\delta\mathbb{V}\mathbb{V}e^{\sqrt{E}\tau'}}{4E}\left[(\tau'-\frac{1}{\sqrt{E}})\frac{1}{(2\sqrt{E})^2(\sqrt{E}-s)}-\frac{1}{(2\sqrt{E})^3(\sqrt{E}-s)}\right]\\
&= \frac{1}{16}\frac{\delta\mathbb{V}\,\mathbb{V}}{E^2}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\tau'-\frac{3}{2\sqrt{E}}\right] \qquad (A.22)
\end{aligned}$$

Where $c_1 = c_1^1 + c_1^2$. The other integrals c_2 and c_3 are the following

$$\begin{aligned}
c_2 &= -\frac{\mathbb{V}\,\delta\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{z-s}\mathbb{F}_0^2\,\mathbb{F}_0^- = -\frac{\mathbb{V}\,\delta\mathbb{V}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{z-s}\frac{1}{(z^2-E)^2}\frac{-e^{\sqrt{E}\tau'}}{2\sqrt{E}(z+\sqrt{E})}\\
&= -\frac{\mathbb{V}\,\delta\mathbb{V}\,e^{\sqrt{E}\tau'}}{2\pi i(2\sqrt{E})}\oint_{\bar{C}}\frac{1}{(z-s)(z-\sqrt{E})^2(z+\sqrt{E})^3}\\
&= -\frac{\mathbb{V}\,\delta\mathbb{V}\,e^{\sqrt{E}\tau'}}{2\sqrt{E}}\left[\frac{-1}{(\sqrt{E}-s)^2(2\sqrt{E})^3}-\frac{3}{(\sqrt{E}-s)(2\sqrt{E})^4}\right]\\
&= \frac{1}{16}\frac{\mathbb{V}\,\delta\mathbb{V}}{E^2}\cdot\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\frac{1}{s-\sqrt{E}}-\frac{3}{2\sqrt{E}}\right] \qquad (A.23)
\end{aligned}$$

$$
\begin{aligned}
c_3 &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{z-s}\mathbb{V}^2\mathbb{F}_0^3 e^{-z\tau'} = \frac{\mathbb{V}^2}{2\pi i}\int_{-i\infty}^{i\infty}dz\frac{e^{-z\tau'}}{(z-s)(E-z^2)^3} \\
&= \frac{-\mathbb{V}^2}{2\pi i}\oint_C dz\frac{1}{(z-s)(z-\sqrt{E})^3(z+\sqrt{E})^3} \\
&= -\mathbb{V}^2\left[\frac{e^{-s\tau'}}{(s^2-E)^3}+\frac{1}{2}\frac{d^2}{dz^2}\frac{e^{-z\tau'}}{(z-s)(z-\sqrt{E})^3}\Big|_{z=-\sqrt{E}}\right] \\
&= -\mathbb{V}^2\left[\frac{e^{-s\tau'}}{(s^2-E)^3}+\frac{e^{\sqrt{E}\tau'}}{2(2\sqrt{E})^3(\sqrt{E}+s)}\left(\left[\tau'-\frac{3}{2\sqrt{E}}-\frac{1}{\sqrt{E}+s}\right]^2+\right.\right. \\
&\qquad \left.\left.+\frac{3}{4E}+\frac{1}{(s+\sqrt{E})^2}\right)\right] \qquad \text{(A.24)}
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\mathbb{F}_2^+(s,\tau') &= \frac{-1}{16\sqrt{E}}\left(\frac{\delta\mathbb{V}}{E}\right)^2\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}+\frac{1}{16}\frac{\delta\mathbb{V}\,\mathbb{V}}{E^2}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\tau'-\frac{3}{2\sqrt{E}}\right]+ \\
&+\frac{1}{16}\frac{\mathbb{V}\,\delta\mathbb{V}}{E^2}.\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\frac{1}{s-\sqrt{E}}-\frac{2}{3\sqrt{E}}\right]-\mathbb{V}^2\frac{e^{-s\tau'}}{(s^2-E)^3}-\mathbb{V}^2\frac{e^{-\sqrt{E}\tau'}}{2[2\sqrt{E}(\sqrt{E}+s)]^3}+ \\
&-\mathbb{V}^2\left[\frac{e^{\sqrt{E}\tau'}}{2(2\sqrt{E})^3(\sqrt{E}+s)}\left(\left[\tau'-\frac{3}{2\sqrt{E}}-\frac{1}{\sqrt{E}+s}\right]^2+\frac{3}{4E}\right)\right] \qquad \text{(A.25)}
\end{aligned}
$$

Green's function $\mathbb{G}_2^+(\tau,\tau')$ is obtained by taking Laplace transform for $\mathbb{F}_2^+(s,y)$:

$$
\begin{aligned}
\mathbb{G}_2(\tau,\tau') &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\; e^{s\tau}\mathbb{F}_2^+(s,\tau') \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\; e^{s\tau}\left[\frac{-1}{16\sqrt{E}}\left(\frac{\delta\mathbb{V}}{E}\right)^2\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}+\frac{1}{16}\frac{\delta\mathbb{V}\,\mathbb{V}}{E^2}\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\tau'-\frac{3}{2\sqrt{E}}\right]+\right. \\
&+\frac{1}{16}\frac{\mathbb{V}\,\delta\mathbb{V}}{E^2}.\frac{e^{\sqrt{E}\tau'}}{s-\sqrt{E}}\left[\frac{1}{s-\sqrt{E}}-\frac{2}{3\sqrt{E}}\right]-\mathbb{V}^2\frac{e^{-s\tau'}}{(s^2-E)^3}-\mathbb{V}^2\frac{e^{-\sqrt{E}\tau'}}{2[2\sqrt{E}(\sqrt{E}+s)]^3}+ \\
&\left.-\mathbb{V}^2\left[\frac{e^{\sqrt{E}\tau'}}{2(2\sqrt{E})^3(\sqrt{E}+s)}\left(\left[\tau'-\frac{3}{2\sqrt{E}}-\frac{1}{\sqrt{E}+s}\right]^2+\frac{3}{4E}\right)\right]\right] \qquad \text{(A.26)}
\end{aligned}
$$

The terms of the integral (A.26) are evaluated separately, each integral with its appropriate contour of integration, the used contours are again C and $\bar{C}$. The contributing integrals are listed below

$$
\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\;\frac{e^{s\tau}}{s-\sqrt{E}} = -\oint_{\bar{C}}ds\frac{e^{s\tau}}{s-\sqrt{E}} = -e^{\sqrt{E}\tau} \qquad \text{(A.27)}
$$

$$
\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\;\frac{e^{s\tau}}{(s-\sqrt{E})^2} = -\oint_{\bar{C}}ds\frac{e^{s\tau}}{(s-\sqrt{E})^2} = \frac{d}{ds}\,e^{s\tau}|_{s=\sqrt{E}} = -\tau e^{\sqrt{E}\tau} \qquad \text{(A.28)}
$$

$$
\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\;\frac{e^{s\tau}}{s+\sqrt{E}} = -\oint_{\bar{C}}ds\frac{e^{s\tau}}{s+\sqrt{E}} = 0 \qquad \text{(A.29)}
$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ \frac{e^{s\tau}}{(s+\sqrt{E})^2} = -\oint_{\bar{C}} ds \frac{e^{s\tau}}{(s+\sqrt{E})^2} = 0 \tag{A.30}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ \frac{e^{s\tau}}{(s+\sqrt{E})^3} = -\oint_{\bar{C}} ds \frac{e^{s\tau}}{(s+\sqrt{E})^3} = 0 \tag{A.31}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ \frac{e^{s(\tau-\tau')}}{(s^2-E)^3} = -\frac{e^{-\sqrt{E}|\tau-\tau'|}}{16E\sqrt{E}}\left[\left(|\tau-\tau'|+\frac{3}{2\sqrt{E}}\right)^2+\frac{3}{4E}\right] \tag{A.32}$$

The last integral is evaluated as follows

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ \frac{e^{s(\tau-\tau')}}{(s^2-E)^3} = \frac{1}{2\pi i}\begin{cases} \oint_C ds\ e^{s(\tau-\tau')}(s^2-E)^{-3} & \text{if } \tau-\tau'>0 \\ -\oint_{\bar{C}} ds\ e^{s(\tau-\tau')}(s^2-E)^{-3} & \text{if } \tau-\tau'<0 \end{cases} \tag{A.33}$$

where

$$\begin{aligned} \frac{1}{2\pi i}\oint_C ds\ \frac{e^{s(\tau-\tau')}}{(s^2-E)^3} &= \frac{1}{2}\frac{d^2}{ds^2}\frac{e^{s(\tau-\tau')}}{(s-\sqrt{E})^3}\bigg|_{s=-\sqrt{E}} = -\frac{1}{2}\frac{e^{-\sqrt{E}(\tau-\tau')}}{(2\sqrt{E})^3}\left[\left(\tau-\tau'+\frac{3}{2\sqrt{E}}\right)^2+\frac{3}{4E}\right] \\ \frac{1}{2\pi i}\oint_{\bar{C}} ds\ \frac{e^{s(\tau-\tau')}}{(s^2-E)^3} &= \frac{1}{2}\frac{d^2}{ds^2}\frac{e^{s(\tau-\tau')}}{(s+\sqrt{E})^2}\bigg|_{s=\sqrt{E}} = \frac{1}{2}\frac{e^{\sqrt{E}(\tau-\tau')}}{(2\sqrt{E})^3}\left[\left(\tau-\tau'-\frac{3}{2\sqrt{E}}\right)^2+\frac{3}{4E}\right] \end{aligned} \tag{A.34}$$

Therefore

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\ \frac{e^{s(\tau-\tau')}}{(s^2-E)^3} = -\frac{e^{-\sqrt{E}|\tau-\tau'|}}{16E\sqrt{E}}\left[\left(|\tau-\tau'|+\frac{3}{2\sqrt{E}}\right)^2+\frac{3}{4E}\right] \tag{A.35}$$

Finally, it is straightforward to write down $\mathbb{G}_2(\tau,\tau')$ as follows

$$\begin{aligned} \mathbb{G}_2(\tau,\tau') &= \frac{e^{-\sqrt{E}|\tau-\tau'|}}{2\sqrt{E}}\left[\frac{1}{16}\left(\frac{\mathbb{V}}{E}\right)^2\left(\sqrt{E}\mid\tau-\tau'\mid+\frac{3}{2}\right)^2+\frac{3}{4}\right]+\frac{e^{\sqrt{E}(\tau+\tau')}}{2\sqrt{E}}\left[\frac{1}{8}\left(\frac{\delta\mathbb{V}}{E}\right)^2+\right. \\ &\left.-\frac{\mathbb{V}\delta\mathbb{V}}{16E^2}(\sqrt{E}\tau-\frac{3}{2})-\frac{\delta\mathbb{V}\ \mathbb{V}}{16E^2}\left(\sqrt{E}\tau'-\frac{3}{2}\right)\right], \qquad \tau<0,\ \tau'<0 \end{aligned} \tag{A.36}$$

Appendix B

Interacting theory

B.1 The entanglement entropy of a non-Gaussian state

This section will present a brief overview of the result of [31].

The reduced density matrix of 2-coupled harmonic oscillators is Gaussian. It can be generally written as

$$\rho(q,q') = \sqrt{\frac{A}{\pi}} e^{-\frac{A}{2}(q^2+q'^2)-\frac{c}{2}(q-q')^2} \tag{B.1}$$

The corresponding correlators can be easily evaluated (see [31] for details)

$$\langle \hat{q}\hat{q} \rangle = (2A)^{-1}, \quad \langle \hat{p}\hat{p} \rangle = A/2 + c, \quad \langle \hat{q}\hat{p} \rangle = \langle \hat{p}\hat{q} \rangle^* = i/2 \tag{B.2}$$

The related ξ can be written by means of σ, where

$$\sigma = \langle \hat{q}\hat{q} \rangle \langle \hat{p}\hat{p} \rangle - (\mathrm{Re}\langle \hat{q}\hat{p} \rangle\ )^2 \tag{B.3}$$

Whereas

$$\xi = \frac{2\sigma - 1}{2\sigma + 1} \tag{B.4}$$

The entanglement entropy is evaluated by means of the correlators method. It gives the well-known formula

$$S = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln(\xi) \tag{B.5}$$

Now, take a non-gaussian state of the form

$$\rho_\lambda(q,q') = N e^{-\frac{A}{2}(q^2+q'^2)-\frac{C}{2}(q-q')^2-\lambda[\alpha_1(q^4+q'^4)+\alpha_2(q^3q'+qq'^3)+\alpha_3 q^2 q'^2]} \tag{B.6}$$

N is the normalization constant, λ is the perturbation parameter and $(A, C, \alpha_1, \alpha_2, \alpha_3)$ are all scalars.

ρ_λ is not gaussian, therefore Wick's theorem is not applicable in this situation, and ρ_λ cannot be written by means of the two point correlators only (see [24]). Therefore, higher order correlators are required. However, it has been shown that two point correlators are all one needs to determine the entanglement entropy up to the first order correction in λ.

As before, the modified ξ_{corr} is determined out of the two point correlators up to the first order in λ as follows :

$$\begin{aligned}\xi_{corr} &= \frac{2\sigma_\lambda - 1}{2\sigma_\lambda + 1} \\ &= \xi + \lambda\left[\frac{3\xi}{(\xi+1)(\xi-1)^3}\alpha_1 + \frac{3(\xi+1)}{2(\xi-1)^3\beta^2}\alpha_2 + \frac{1+\xi+\xi^2}{(\xi+1)(\xi-1)^3\beta^2}\alpha_3\right] \end{aligned} \tag{B.7}$$

where $\beta = \frac{1}{2}(A\sqrt{1+2c/A} + A + c)$.

On the other hand, the entanglement entropy can be determined by means of the replica trick applied over ρ_λ. For the first order in λ, it is given by [31]

$$\begin{aligned} S_\lambda &= -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi - \lambda\frac{3\xi\ln\xi}{(\xi+1)(\xi-1)^5\beta^2}\alpha_1 \\ &- \lambda\frac{3(\xi+1)\ln\xi}{2(\xi-1)^2\beta^2}\alpha_2 - \lambda\frac{3(\xi^2+\xi+1)\ln\xi}{(\xi+1)(\xi-1)^5\beta^2}\alpha_3 \end{aligned} \tag{B.8}$$

ξ is the old parameter for the non-pertured state. The main result of [31] was to show that the entropy S_λ has the same form as (B.5) but for ξ_{corr} :

$$S = -\ln(1-\xi_{corr}) - \frac{\xi_{corr}}{1-\xi_{corr}}\ln(\xi_{corr}) \tag{B.9}$$

A simple expansion of the entropy S around λ and one will rediscover (B.8) for the first order expansion.

B.2 Local interaction

This part of the appendix is devoted to calculating the first order correction in Rényi entropy caused by a quartic perturbation, which is given by (3.127). The main task is to compute the diagonal elements of the matrix $\mathbb{G}$, in order to be able to compute the first correction expression :

$$\int_{-\infty}^{\infty} d\tau \sum_{i=1}^{2n} (\mathbb{G})^2_{ii}(\tau,\tau) - n\sum_{i=1}^{2}(G_{ii})^2(\tau,\tau) \tag{B.10}$$

For $(\tau < 0)$, Green's function is given by :

$$\mathbb{G}(\tau,\tau) = \frac{1}{2\mathbb{W}} + e^{\mathbb{W}\tau}\frac{\mathbb{W}-\mathbb{W}_p}{\mathbb{W}+\mathbb{W}_p}e^{\mathbb{W}\tau} \tag{B.11}$$

Where $\mathbb{W} = \sqrt{\mathbb{V}}$ and $\mathbb{W}_p = \sqrt{\mathbb{V}_p}$, remmember that (B.10) is evaluated for $E = 0$. The determination of $\{\mathbb{G}_{ii}\}$ can be easily done using the following decomposition :

$$\mathbb{G}(\tau,\tau) = \frac{1}{2\mathbb{W}} - \frac{e^{2\mathbb{W}\tau}}{2\mathbb{W}} + e^{\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}e^{\mathbb{W}\tau} \tag{B.12}$$

Let's start with the matrix $\mathbb{W}^{-1}$, apparently $\mathbb{W}^{-1} = W^{-1} \otimes I_n$ where W is a 2×2 matrix such that

$$W^{-1} = TW_d^{-1}T \tag{B.13}$$

Where

$$T = \frac{1}{\sqrt{2}}.\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad W_d = \sqrt{b}\begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}, \quad \lambda_\pm = \sqrt{a \pm 1} \tag{B.14}$$

Hence, it is straightforward to show that :

$$\mathbb{W}_{ii}^{-1} = \frac{1}{2\sqrt{b}}(\lambda_-^{-1} + \lambda_+^{-1}) \tag{B.15}$$

Next, moving to the matrix $e^{2\mathbb{W}\tau}(2\mathbb{W})^{-1}$ and using the n-fold matrix $\mathbb{T} = T \otimes I_n$, one can easily show that

$$\begin{aligned}
(\frac{e^{2\mathbb{W}\tau}}{2\mathbb{W}})_{ii} &= \frac{1}{2}\langle i|\,\mathbb{T}\mathbb{T}e^{2\mathbb{W}\tau}\mathbb{T}\mathbb{T}\mathbb{W}^{-1}\mathbb{T}\mathbb{T}\,|i\rangle && \text{(B.16)}\\
&= \frac{1}{2}\langle i|\,\mathbb{T}e^{2W_d\tau}W_d^{-1}\mathbb{T}\,|i\rangle && \text{(B.17)}\\
&= \frac{1}{2}\sum_{k=0}^{\infty} T_{i1}T_{1i}\frac{(2\tau)^k}{k!}(\sqrt{b}\lambda_+)^{k-1} + T_{i2}T_{2i}\frac{(2\tau)^k}{k!}(\sqrt{b}\lambda_-)^{k-1} && \text{(B.18)}\\
&= \frac{1}{4\sqrt{b}}(\frac{e^{2\sqrt{b}\lambda_+\tau}}{\lambda_+} + \frac{e^{2\sqrt{b}\lambda_-\tau}}{\lambda_-}) && \text{(B.19)}
\end{aligned}$$

Finally, the matrix $(e^{\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}e^{\mathbb{W}\tau})$ can be handled in the same manner, but this requires dealing with the matrix $(\mathbb{W}+\mathbb{W}_p)^{-1}$ first. It is not difficult to show that $(\mathbb{T}(\mathbb{W}+\mathbb{W}_p)\mathbb{T})$ is a block circulant matrix, call it $\mathbb{C}$, its first column is given by $(c_0, c_1, 0, ..., 0, c_{n-1})^T$, where :

$$c_0 = \sqrt{b}\begin{bmatrix} 2\mu+\nu & 0 \\ 0 & 2\mu-\nu \end{bmatrix}, \quad c_1 = \frac{\sqrt{b}\ \nu}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad c_{n-1} = \frac{\sqrt{b}\ \nu}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \tag{B.20}$$

$$\mu = \frac{\lambda_+ + \lambda_-}{2}, \quad \nu = \frac{\lambda_+ - \lambda_-}{2}$$

The block eigenvalues of the matrix C are :

$$B_j = \sqrt{b}\begin{bmatrix} 2\mu + \nu(1+\cos(2\pi j/n)\ ) & i\nu\sin(2\pi j/n) \\ -i\nu\sin(2\pi j/n) & 2\mu - \nu(1+\cos(2\pi j/n)) \end{bmatrix} \qquad j = 0..n-1 \tag{B.21}$$

The inverses of $\{B_j\}$ are needed for later calculations :

$$B_j^{-1} = \frac{b^{-1/2}}{(2\mu)^2 - 2\nu^2(1+cos(2\pi j/n)\ )} \begin{bmatrix} 2\mu - \nu(1+\cos(2\pi j/n)\ ) & -i\nu\sin(2\pi j/n) \\ i\nu\sin(2\pi j/n) & 2\mu + \nu(1+\cos(2\pi j/n)) \end{bmatrix} \quad \text{(B.22)}$$

Hence

$$\begin{aligned} (e^{\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}e^{\mathbb{W}\tau})_{ii} &= \langle i|\,\mathbb{T}e^{W_d\tau}\mathbb{C}^{-1}e^{W_d\tau}\mathbb{T}\,|i\rangle & \text{(B.23)} \\ &= \langle i|\,\mathbb{T}e^{W_d\tau}\mathbb{F}^{-1}\mathbb{C}_d^{-1}\mathbb{F}e^{W_d\tau}\mathbb{T}\,|i\rangle & \text{(B.24)} \\ &= \langle i|\,\mathbb{T}\mathbb{F}^{-1}e^{W_d\tau}\mathbb{C}_d^{-1}e^{W_d\tau}\mathbb{F}\mathbb{T}\,|i\rangle & \text{(B.25)} \end{aligned}$$

Where $\mathbb{F} = I_2 \otimes F$ is the discrete, block, Fourrier transform. To move on, the vector $|i\rangle$ will be written in a block sub-vectors manner (2×1 block sub-vectors). When taking $\{|i\rangle\}$ to be the canonical basis, all sub-vectors are null except for one, which can take one of two possibilities :

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{(B.26)}$$

One can easily show that :

$$\mathbb{F}\mathbb{T}\,|i\rangle = Tu_m(l) \otimes \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ w^l \\ w^{2l} \\ . \\ . \\ . \\ w^{l(n-1)} \end{pmatrix}, \quad w = e^{2\pi i/n} \quad \text{(B.27)}$$

Where $l = 1 : n$ and $m = 1, 2$, l indicates to the coordinate of sub-vector u_m in the vector $|i\rangle$.

Let $M_j = e^{w_d\tau}B_j^{-1}e^{w_d\tau}$, by means of (B.27) it is not hard to show that :

$$\begin{aligned} (e^{\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}e^{\mathbb{W}\tau})_{ii} &= \frac{1}{n}(Tu_m)^T\sum_{j=0}^{n-1}M_jTu_m \\ &= \sum_{j=0}^{n-1}\frac{1}{2n}\big[e^{2\sqrt{b}\lambda_+\tau}(B_j^{-1})_{11} + e^{2\sqrt{b}\lambda_-\tau}(B_j^{-1})_{22} + \\ &\quad -(-1)^m e^{\sqrt{b}(\lambda_-+\lambda_+)\tau}((B_j^{-1})_{12} + (B_j^{-1})_{21})\big] \\ &= \frac{1}{2n}\big[e^{2\sqrt{b}\lambda_+\tau}B_+ + e^{2\sqrt{b}\lambda_-\tau}B_-\big] \quad \text{(B.28)} \end{aligned}$$

Where $B_+ = \sum_{j=0}^{n-1}(B_j^{-1})_{11}$ and $B_- = \sum_{j=0}^{n-1}(B_j)_{22}^{-1}$. The term $(B_j^{-1})_{21} + (B_j)_{12}^{-1}$ cancelled identically.

To this end, it is now possible to write $\mathbb{G}_{ii}$ for $(\tau < 0)$:

$$\mathbb{G}_{ii}(\tau,\tau) \;=\; \frac{1}{4\sqrt{b}}\left[\,\lambda_+^{-1}+\lambda_-^{-1}+\frac{e^{2\lambda_-\tau}}{\lambda_-}+\frac{e^{2\lambda_+\tau}}{\lambda_+}+\frac{2\sqrt{b}}{n}(e^{2\lambda_+\tau}B_+ + e^{2\lambda_-\tau}B_-)\,\right] \tag{B.29}$$

For $(\tau > 0)$ one needs not to repeat all these calculations, it is enough to note that positive and negative times have an equal contribution to the correction statement, i.e.

$$\sum_{i=1}^{2n}\int_{-\infty}^{0} d\tau\; \mathbb{G}_{ii}^2(\tau,\tau) = \sum_{i=1}^{2n}\int_{0}^{\infty} d\tau\; \mathbb{G}_{ii}^2(\tau,\tau) \tag{B.30}$$

Since $\mathbb{V}_p = \mathbb{P}_\pi \mathbb{V} \mathbb{P}_\pi^T$ and $\mathbb{P}_\pi.\mathbb{P}_\pi^T = 1$, one can easily show that for $(\tau > 0)$

$$\mathbb{G}_{ii}(\tau,\tau) = \langle i|\,\mathbb{P}_\pi.[\frac{1}{2\mathbb{W}} + e^{-\mathbb{W}\tau}\frac{\mathbb{W}-\mathbb{W}_p}{\mathbb{W}+\mathbb{W}_p}e^{-\mathbb{W}\tau}].\mathbb{P}_\pi^T\,|i\rangle \tag{B.31}$$

The fact that $\mathbb{P}_\pi$ is a permutation matrix and $\{|i\rangle\}$ is the canonical basis, means that $\{\mathbb{P}_\pi^T\,|i\rangle\} = \{|i\rangle\}$. Making the suitable change of variable $\tau \to -\tau$ completes the proof of the earlier claim (B.30). So that :

$$\int_{-\infty}^{\infty} d\tau \sum_{i=1}^{2n}(\mathbb{G})_{ii}^2(\tau,\tau) - n\sum_{i=1}^{2}(G_{ii})^2(\tau,\tau) = 2\int_{-\infty}^{0} d\tau \sum_{i=1}^{2n}(\mathbb{G})_{ii}^2(\tau,\tau) - n\sum_{i=1}^{2}(G_{ii})^2(\tau,\tau) \tag{B.32}$$

Where $G = (2W)^{-1}$, which means that $G_{ii}^2 = (\lambda_-^{-1}+\lambda_+^{-1})^2/(16b)$. Whereas $(\mathbb{G}_{ii})^2$ is given by :

$$\begin{aligned}(\mathbb{G}_{ii})^2 \;=\; \frac{1}{16b}\Big[\; & \lambda_+^{-2}+\lambda_-^{-2}+2\lambda_+^{-1}\lambda_-^{-1}+\frac{e^{4\sqrt{b}\lambda_+\tau}}{\lambda_+^2}+\frac{e^{4\sqrt{b}\lambda_-\tau}}{\lambda_-^2}+\frac{2e^{2\sqrt{b}(\lambda_-+\lambda_+)\tau}}{\lambda_+\lambda_-}+\\ & -2\big(\lambda_+^{-1}+\lambda_-^{-1}\big)\left(\frac{e^{2\sqrt{b}\lambda_+\tau}}{\lambda_+}+\frac{e^{2\sqrt{b}\lambda_-\tau}}{\lambda_-}\right)+\frac{2b}{n^2}\Big(\; e^{4\sqrt{b}\lambda_+\tau}B_+^2+e^{4\sqrt{b}\lambda_-\tau}B_-^2+\\ & +2e^{2\sqrt{b}(\lambda_-+\lambda_+)\tau}B_-B_+\Big)+\frac{2\sqrt{b}}{n}\Big(\lambda_+^{-1}+\lambda_-^{-1}-\lambda_-^{-1}e^{2\sqrt{b}\lambda_-\tau}+\\ & -\lambda_+^{-1}e^{2\sqrt{b}\lambda_+\tau}\Big)\Big(\; e^{2\sqrt{b}\lambda_+\tau}B_+ + e^{2\sqrt{b}\lambda_-\tau}B_-\Big)\;\Big]\end{aligned} \tag{B.33}$$

The integration and summation are straightforward. One can easily found that :

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau \sum_{i=1}^{2n}(\mathbb{G})_{ii}^2(\tau,\tau) - n\sum_{i=1}^{2}(G_{ii})^2(\tau,\tau) \;=\; & \frac{1}{b\sqrt{b}}\Big(\; n\left[-\frac{3}{16\lambda_+\lambda-}-\frac{3}{16\lambda_+^2}-\frac{1}{4\lambda_+^2\lambda_-}-\frac{1}{4\lambda_+\lambda_-^2}\right.\\ & \left.+\frac{1}{4\lambda_-\lambda_+(\lambda_-+\lambda_+)}\right] \;+\; \left[\frac{1}{2\lambda_+\lambda_-}+\frac{1}{4\lambda_+^2}-\frac{1}{2\lambda_-(\lambda_++\lambda_-)}\right]B_- +\\ +\left[\frac{1}{2\lambda_+\lambda_-}+\frac{1}{4\lambda_-^2}-\frac{1}{2\lambda_+(\lambda_++\lambda_-)}\right]B_+ \;+\; & \frac{1}{n}\left[\frac{1}{4\lambda_+}B_-^2+\frac{1}{4\lambda_-}B_+^2+\frac{1}{\lambda_-+\lambda_+}B_-B_+\right]\;\Big)\end{aligned} \tag{B.34}$$

B.3 Non-Local interaction

The aim of this appendix is to prove the first order perturbation of the non-local interaction.

Let's start with (??) or $(\delta \ln \operatorname{Tr} \rho^n)_c$. It is preferable to take the following decomposition for $\mathbb{G}_{--}(\tau,\tau)$

$$\mathbb{G}_{--}(\tau,\tau) = \frac{1}{2\mathbb{W}} - \frac{e^{2\mathbb{W}\tau}}{2\mathbb{W}} + e^{\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}.e^{\mathbb{W}\tau} \tag{B.35}$$

Remmember that $\mathbb{G}$ is evaluated for $E=0$. It follows that

$$\begin{aligned}\operatorname{Tr}\big(\mathbb{G}^2_{--}(\tau,\tau)\big) &= \operatorname{Tr}\frac{1}{4\mathbb{W}^2} + \operatorname{Tr}\frac{e^{4\mathbb{W}\tau}}{4\mathbb{W}^2} - \operatorname{Tr}\frac{e^{2\mathbb{W}\tau}}{2\mathbb{W}^2} + \operatorname{Tr}\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}(\mathbb{W}+\mathbb{W}_p)^{-1} + \\ &\quad - \operatorname{Tr}\frac{e^{4\mathbb{W}\tau}}{\mathbb{W}}(\mathbb{W}+\mathbb{W}_p)^{-1} + \operatorname{Tr} e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}\end{aligned} \tag{B.36}$$

Therefore

$$\int_{-\infty}^{0} d\tau \operatorname{Tr}\big(\mathbb{G}^2(\tau,\tau)\big) - n\operatorname{Tr}\big(G^2(\tau,\tau)\big) = \operatorname{Tr}\frac{-3}{16\mathbb{W}^3} + \operatorname{Tr}\frac{1}{4\mathbb{W}^2}\frac{1}{\mathbb{W}+\mathbb{W}_p} + F(\mathbb{W},\mathbb{W}_p) \tag{B.37}$$

Where

$$F(\mathbb{W},\mathbb{W}_p) = \int_{-\infty}^{0} d\tau \operatorname{Tr}\left[(\mathbb{W}+\mathbb{W}_p)^{-1}e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}e^{2\mathbb{W}\tau}\right] \tag{B.38}$$

Using the same reasoning used to prove (B.30), one may easily show that

$$\int_{-\infty}^{0} d\tau \operatorname{Tr}\big(\mathbb{G}^2\big) = \int_{0}^{\infty} d\tau \operatorname{Tr}\big(\mathbb{G}^2\big) \tag{B.39}$$

Finally, it is straightforward to find the contribution of $(\delta \ln \operatorname{Tr} \rho^n)_c$ in the first order correction

$$\begin{aligned}(\delta \ln \operatorname{Tr} \rho^n)_c &= 2\int_{-\infty}^{0} d\tau \operatorname{Tr}\big(\mathbb{G}^2(\tau,\tau)\big) - n\operatorname{Tr}\big(G^2(\tau,\tau)\big) \\ &= -\frac{2\lambda}{4!}\left[-\frac{3n}{8}\operatorname{Tr}V^{-3/2} + \frac{1}{2}\operatorname{Tr}\mathbb{V}^{-1}(\mathbb{W}+\mathbb{W}_p)^{-1} + 2F(\mathbb{W},\mathbb{W}_p)\right]\end{aligned} \tag{B.40}$$

Since $\operatorname{Tr}\mathbb{V}^{-1}(\mathbb{W}+\mathbb{W}_p)^{-1} = \operatorname{Tr}\mathbb{V}_p^{-1}(\mathbb{W}+\mathbb{W}_p)^{-1}$, one can easily rediscover (3.135).

The second part of the perturbation correction (3.137) can be proved by the same way. This contribution is given by

$$\int_{-\infty}^{\infty} d\tau \left(\operatorname{Tr}(\mathbb{G}(\tau,\tau))\right)^2 - n\left(\operatorname{Tr}(G(\tau,\tau))\right)^2 \tag{B.41}$$

Where

$$\operatorname{Tr}(\mathbb{G}(\tau,\tau)) = \operatorname{Tr}\left(\frac{1}{2\mathbb{W}}\right) - \operatorname{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{2\mathbb{W}}\right) + \operatorname{Tr}\big(e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}\big) \tag{B.42}$$

it follows that

$$\begin{aligned}[\mathrm{Tr}(\mathbb{G}(\tau,\tau))]^2 &= \left[\mathrm{Tr}\left(\frac{1}{2\mathbb{W}}\right)\right]^2+\frac{1}{4}\left[\mathrm{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}\right)\right]^2+\left[\mathrm{Tr}\big(e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}\big)\right]^2+\\ &\quad -\frac{1}{2}\,\mathrm{Tr}\left(\frac{1}{\mathbb{W}}\right)\mathrm{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}\right)-\mathrm{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}\right).\mathrm{Tr}\big(e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}\big)+\\ &\quad +\mathrm{Tr}\left(\frac{1}{\mathbb{W}}\right)\mathrm{Tr}\big(e^{2\mathbb{W}\tau}(\mathbb{W}+\mathbb{W}_p)^{-1}\big)\end{aligned}\tag{B.43}$$

By means of the identity

$$\int_{-\infty}^{0}d\tau\,\mathrm{Tr}(\mathbb{G})^2=\int_{0}^{\infty}d\tau\,\mathrm{Tr}(\mathbb{G})^2\tag{B.44}$$

it is not hard to determine the second contribution to the perturbation correction of non-local interaction as follows

$$\begin{aligned}&\int_{-\infty}^{\infty}d\tau\,(\mathrm{Tr}(\mathbb{G}(\tau,\tau))\,)^2-n\,(\mathrm{Tr}(G(\tau,\tau))\,)^2=\int_{-\infty}^{0}d\tau\left[\frac{1}{2}\left(\mathrm{Tr}\big(\mathbb{W}^{-1}\big)\right)^2-\frac{n}{2}\left(\mathrm{Tr}\big(W^{-1}\big)\right)^2+\right.\\ &+\frac{1}{2}\left(\mathrm{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}\right)\right)^2+2\left(\mathrm{Tr}\left(e^{2\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}\right)\right)^2-2\,\mathrm{Tr}\left(\frac{e^{2\mathbb{W}\tau}}{\mathbb{W}}\right)\mathrm{Tr}\left(e^{2\mathbb{W}\tau}\frac{1}{\mathbb{W}+\mathbb{W}_p}\right)\Bigg]+\\ &\qquad -\frac{1}{2}\,\mathrm{Tr}\big(\mathbb{W}^{-1}\big)\,\mathrm{Tr}\big(\mathbb{W}^{-2}\big)+\mathrm{Tr}\big(\mathbb{W}^{-1}\big)\,\mathrm{Tr}\left(\frac{1}{\mathbb{W}}.(\mathbb{W}+\mathbb{W}_p)^{-1}\right)\end{aligned}\tag{B.45}$$

This completes the proof of this appendix.

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