People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Echahid Hamma Lakhdar –El Oued–

Faculty of Exact Sciences

Department of Mathematics

Academic Master

Final year project

Domain: Mathematics and informatics

Field: Mathematics

Specialization: Fundamental and applied mathematics

Title:

Positive Periodic Solutions for a General Class of Non-Autonomous Lotka-Volterra Dynamic Systems on Time Scales

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Defended on: 07/06/2026, Before a jury composed of:

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Academic Year: 2025–2026

Dedication

I dedicate this thesis:

To my dear family,
for their constant support, encouragement,
and presence by my side

To all my friends,
for their kindness, support,
and sincere companionship

To all my teachers,
for their guidance, knowledge,
and valuable efforts throughout my academic journey.

Acknowledgments

First of all, I want to thank Allah for giving me strength, courage and above all knowledge. I would like to express my deepest thanks and sincere appreciation to my supervisor, Dr. **Belkis Bordj**, for her valuable guidance and wise advice throughout the preparation of this thesis.

I also extend my heartfelt thanks and appreciation to Dr. **Haitham Ali Makhzoum** for the valuable assistance he provided me, and for standing by my side-especially during moments of uncertainty. His guidance and advice greatly helped me overcome many difficulties.

My sincere thanks and appreciation also go to Dr. **Beloul Said** for his valuable help in providing some scientific references, as well as for his support and the useful advice and guidance that contributed to enriching this work.

I would also like to express my sincere thanks and appreciation to Dr. **Mhamed Eddahbi** for his valuable assistance and insightful guidance.

I am equally grateful to Dr. **Ibtissem Daira** for her valuable help in providing important scientific references that I was in great need of; she has my deepest appreciation.

I would also like to extend my sincere thanks to the honorable members of the examination committee for kindly accepting to review and discuss this work, and for their valuable remarks which will help improve it. Finally, I hope that this work is sincerely for the sake of Allah and that it will be beneficial to everyone who reads it.

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Abstract

This thesis investigates the existence of positive periodic solutions for a non-autonomous Lotka–Volterra dynamic system with a general interaction rate on periodic time scales. The time scales framework unifies the continuous case ($\mathbb{T} = \mathbb{R}$) and the discrete case ($\mathbb{T} = \mathbb{Z}$).

The analysis uses cone theory in Banach spaces and a Krasnoselskii-type homotopy fixed point theorem for compact operators. The system is transformed into an equivalent integral system, and a suitable nonlinear integral operator is defined on a cone of periodic functions. Under appropriate assumptions, we prove the existence of at least one positive periodic solution. The proof avoids classical eigenvalue conditions by employing two boundary conditions that prevent outward eigenvalues greater than one.

Two concrete applications illustrate the theoretical results. The first is an ecological fish–shark predator–prey model with a Holling type II functional response. The second is a cybersecurity model describing the interaction between polymorphic malware propagation and cyber-defense mechanisms with periodic countermeasure updates. Both examples satisfy the assumptions of the main theorem, guaranteeing positive periodic solutions.
Keywords: time scales; Lotka–Volterra dynamic systems; non-autonomous systems; positive periodic solutions; general interaction rate; Krasnoselskii-type homotopy fixed point theorem; Banach spaces; cone; predator–prey; cybersecurity.

Résumé

Cette thèse étudie l'existence de solutions périodiques positives pour un système dynamique de Lotka–Volterra non autonome avec un taux d'interaction général sur des échelles de temps périodiques. Le cadre des échelles de temps unifie le cas continu ($\mathbb{T} = \mathbb{R}$) et le cas discret ($\mathbb{T} = \mathbb{Z}$).

L'analyse utilise la théorie des cônes dans les espaces de Banach et un théorème de point fixe de type Krasnoselskii avec homotopie pour opérateurs compacts. Le système est transformé en un système intégral équivalent, et un opérateur intégral non linéaire approprié est défini sur un cône de fonctions périodiques. Sous des hypothèses adaptées, nous prouvons l'existence d'au moins une solution périodique positive. La preuve évite les conditions classiques de valeurs propres en employant deux conditions aux limites qui empêchent les valeurs propres sortantes supérieures à un.

Deux applications concrètes illustrent les résultats théoriques. La première est un modèle écologique prédateur–proie (poisson–requin) avec une réponse fonctionnelle de type Holling II. La seconde est un modèle de cybersécurité décrivant l'interaction entre la propagation de logiciels malveillants polymorphes et les mécanismes de cyberdéfense avec mises à jour périodiques. Les deux exemples satisfont les hypothèses du théorème principal, garantissant des solutions périodiques positives.

Mots-clés : time scales; Lotka–Volterra dynamic systems; non-autonomous systems; positive periodic solutions; general interaction rate; Krasnoselskii-type homotopy fixed point theorem; Banach spaces; cone; predator–prey; cybersecurity.

ملخص

تدرس هذه الأطروحة وجود حلول دورية موجبة لنظام ديناميكي من نوع لوتكا-فولتيرا غير ذاتي (غير مستقل عن الزمن) مع معدل تفاعل عام على سلاسل زمنية دورية. إطار السلاسل الزمنية يوحد الحالة المستمرة ($\mathbb{T} = \mathbb{R}$) والحالة المتقطعة ($\mathbb{T} = \mathbb{Z}$). يعتمد التحليل على نظرية المخاريط في فضاءات باناخ ومبرهنة النقطة الثابتة من نوع كراسنوسيلسكي مع التماثل للمؤثرات التراصية. يُحول النظام إلى نظام تكاملي مكافئ، ويُعرّف مؤثر تكاملي غير خطي مناسب على مخروط من الدوال الدورية. تحت فروض مناسبة، نثبت وجود حل دوري موجب واحد على الأقل. البرهان يتجنب الشروط التقليدية للقيم الذاتية باستخدام شرطين حديين يمنعان القيم الذاتية الخارجة الأكبر من الواحد. تطبيقان ملموسان يوضحان النتائج النظرية. الأول هو نموذج بيئي لمفترس وفريسة (سمك-قرش) مع استجابة وظيفية من نوع هولينغ الثاني. الثاني هو نموذج للأمن السيبراني يصف التفاعل بين انتشار البرامج الخبيثة متعددة الأشكال وآليات الدفاع السيبراني مع تحديثات دورية. كلا المثالين يحققان فروض النظرية الرئيسية، مما يضمن وجود حلول دورية موجبة.

الكلمات المفتاحية: السلاسل الزمنية، أنظمة لوتكا-فولتيرا الديناميكية، الأنظمة غير الذاتية، الحلول الدورية الموجبة، معدل التفاعل العام، مبرهنة النقطة الثابتة من نوع كراسنوسيلسكي مع التماثل، فضاءات باناخ، المخروط، المفترس-الفريسة، الأمن السيبراني.

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Notations

$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{T}$	Time scale
$\sigma(t)$	Forward jump operator
$\rho(t)$	Backward jump operator
$\mu(t)$	Graininess function
$f^{\sigma}(t)$	Composition $f(\sigma(t))$
Δ	Delta (Hilger) derivative
C_{rd}	rd-continuous
$\mathcal{R}$	Set of regressive functions
$\oplus$	Regressive addition
$\ominus p$	Circle minus of p
$p \ominus q$	Circle minus subtraction
$e_p(t,s)$	Exponential function on time scales associated with p
$e_{\ominus a}(t,s)$	Exponential function associated with $\ominus a$
$\xi_h(z)$	Cylinder transformation
$\mathbb{C}_h, \mathbb{R}_h, \mathbb{A}_h, \mathbb{I}_h$	Hilger complex numbers, Hilger real axis, alternating axis, and imaginary circle
$\|\cdot\|$	Norm on a vector space
$\|\cdot\|_{\infty}$	Supremum norm
K	Cone in a Banach space

Introduction

Mathematical modelling is one of the most important scientific tools for understanding natural and applied phenomena and predicting their behaviour. It allows real-world problems to be expressed in terms of mathematical equations, which can then be analysed in order to understand the mechanisms governing the evolution of the considered system.

Among the most classical and influential models in this direction is the Lotka–Volterra system. It was originally introduced in the context of predator–prey interactions and later became a fundamental model in the study of interacting dynamical systems. In its standard autonomous form, it is given by

$$\frac{dx}{dt} = \alpha x - \beta xy, \qquad \frac{dy}{dt} = -\gamma y + \delta xy,$$

where $\alpha, \beta, \gamma, \delta$ are positive constants. This model describes the evolution of two interacting quantities $x(t)$ and $y(t)$, where the growth or decay of each component depends on the other.

Although the Lotka–Volterra model was first introduced in ecology, its mathematical structure can be interpreted more generally as a framework for describing the interaction between two positive quantities. This broader point of view makes it suitable for applications in several fields, such as epidemiology, economics, technological diffusion, and cybersecurity. Therefore, in this thesis, the system is mainly studied in an abstract mathematical form, while concrete interpretations are presented later as applications.

However, not all real phenomena evolve continuously with respect to time. Some processes are continuous, others are discrete, and many may combine both features. For example, some biological processes occur seasonally, some economic data are recorded at fixed time intervals, and some technological or network processes may evolve through successive updates or events. This shows the need for a mathematical framework that can treat continuous and discrete dynamics in a unified way.

The theory of time scales, introduced by Stefan Hilger, provides such a framework. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. By choosing $\mathbb{T} = \mathbb{R}$, one obtains the classical continuous case described by differential equations, while choosing $\mathbb{T} = \mathbb{Z}$ leads to discrete models described by difference equations. Other choices of $\mathbb{T}$ make it possible to deal with hybrid dynamical systems. Thus, time scales theory allows

continuous, discrete, and hybrid models to be studied within the same mathematical approach.

In this thesis, we study the following non-autonomous Lotka–Volterra dynamic system with a general interaction rate on a periodic time scale:

$$\begin{cases} x^{\Delta}(t) = a(t)x(t)g(x(t)) - f(t, x(t), y(t))x(t)y(t), \\ y^{\Delta}(t) = -b(t)y^{\sigma}(t) + c(t)f(t, x(t), y(t))x(t)y(t), \end{cases}$$

where $x(t)$ and $y(t)$ are positive interacting quantities, $a(t)$, $b(t)$, and $c(t)$ are positive periodic functions, g is a decreasing function describing self-regulation, and f is a general interaction rate. Here, Δ denotes the delta derivative on the time scale $\mathbb{T}$, and σ is the forward jump operator.

The main objective of this thesis is to prove the existence of positive periodic solutions for this system. The positivity of the solutions is important because the unknown functions represent quantities that must remain nonnegative in applications. Periodicity is also natural in many situations where the system is affected by recurrent or seasonal factors. The proof is based on tools from functional analysis and fixed point theory. More precisely, the system is transformed into an equivalent integral system, and a suitable integral operator is constructed on a cone of periodic functions. The existence result is then obtained by applying a Krasnoselskii-type homotopy fixed point theorem.

The interest of this approach is that the same theoretical result covers both the continuous and discrete cases. When $\mathbb{T} = \mathbb{R}$, the system reduces to a system of differential equations. When $\mathbb{T} = \mathbb{Z}$, it becomes a system of difference equations. Hence, the time scales framework avoids treating these two cases separately and gives a unified existence result.

To illustrate the obtained result, two applications are considered in the last chapter: an ecological predator–prey model involving fish and sharks with a Holling type II functional response, and a cybersecurity model describing the interaction between polymorphic malware propagation and cyber-defense mechanisms.

This thesis is organized as follows. Chapter 1 presents the necessary preliminaries on functional analysis, including cones and completely continuous operators, together with the main notions of time scales calculus. Chapter 2 contains the main theoretical result, namely the existence of positive periodic solutions for non-autonomous Lotka–Volterra dynamic systems on periodic time scales, proved by means of a Krasnoselskii-type homotopy fixed point theorem. Chapter 3 illustrates the applicability of the result through the two models mentioned above, showing how the abstract framework can be used in both ecological and technological contexts.

Chapter 1

Preliminaries

1.1 Elements of Functional Analysis

This section collects the fundamental concepts and results from functional analysis and fixed point theory that are needed throughout this thesis. We begin with normed spaces and Banach spaces, then introduce operators, cones, compactness, and the Arzelà–Ascoli theorem. Finally, we state the Krasnoselskii fixed point theorem and the Krasnoselskii-type homotopy fixed point theorem, which will be used later in the proof of the main existence result.

1.1.1 Normed Spaces and Banach Spaces

Definition 1.1 (Normed space)**.** [12] Let X be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A *norm* on X is a function $\|\cdot\| : X \to [0, \infty)$ satisfying:

1. $\|x\| = 0 \iff x = 0$;
2. $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in X$;
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *normed space*. Every normed space is a metric space with $d(x, y) = \|x - y\|$.

Definition 1.2 (Banach space)**.** [6] A normed space X is called a *Banach space* if it is complete with respect to the metric induced by the norm, i.e., every Cauchy sequence converges to an element of X.

Example 1.1 (Spaces of continuous functions)**.** [12] Let $[a, b] \subset \mathbb{R}$ be a closed interval. The space $C([a, b], \mathbb{R})$ of all continuous real-valued functions on $[a, b]$ equipped with the supremum norm

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|$$

is a Banach space. Convergence in this norm is uniform convergence.

Example 1.2 (ℓ^p spaces)**.** [6] For $1 \leq p < \infty$, let $\ell^p = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ with norm $\|(x_n)\|_p = (\sum |x_n|^p)^{1/p}$. These are Banach spaces. ℓ^∞ with the supremum norm is also a Banach space.

1.1.2 Operators in Banach Spaces

In this subsection, we recall some basic notions concerning operators in Banach spaces. Since the operator used later in the fixed point formulation is nonlinear, we focus on the notions of nonlinear operator, boundedness, compactness and complete continuity.

Definition 1.3 (Operator)**.** [12, 7] Let X and Y be normed spaces. A mapping

$$T : X \to Y$$

is called an operator from X into Y.

Definition 1.4 (Linear operator)**.** [12] Let X and Y be normed spaces. An operator $T : X \to Y$ is called linear if, for all $x, y \in X$ and all scalars α, β, we have

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

Definition 1.5 (Nonlinear operator)**.** [7, 16] Let X and Y be normed spaces. An operator $T : X \to Y$ is called nonlinear if it is not linear; equivalently, there exist $x, y \in X$ and scalars α, β such that

$$T(\alpha x + \beta y) \neq \alpha Tx + \beta Ty.$$

Definition 1.6 (Continuous operator)**.** [12, 7] Let X and Y be normed spaces and let $D \subset X$. An operator $T : D \to Y$ is said to be continuous at $x_0 \in D$ if, for every sequence $(x_n) \subset D$ such that $x_n \to x_0$ in X, we have

$$Tx_n \to Tx_0 \quad \text{in } Y.$$

The operator T is said to be continuous on D if it is continuous at every point of D.

Definition 1.7 (Bounded operator)**.** [7, 16] Let X and Y be normed spaces and let $D \subset X$. An operator $T : D \to Y$ is said to be bounded if it maps bounded subsets of D into bounded subsets of Y. In other words, for every bounded set $B \subset D$, the set

$$T(B) = \{Tx : x \in B\}$$

is bounded in Y.

Definition 1.8 (Compact operator)**.** [7, 16] Let X and Y be normed spaces and let $D \subset X$. An operator $T : D \to Y$ is called compact if it maps bounded subsets of D into relatively compact subsets of Y. Equivalently, for every bounded sequence $(x_n) \subset D$, the sequence (Tx_n) has a convergent subsequence in Y.

Definition 1.9 (Completely continuous operator)**.** [7, 16] Let X and Y be normed spaces and let $D \subset X$. An operator $T : D \to Y$ is called completely continuous if it is continuous and compact, that is, if it is continuous and maps bounded subsets of D into relatively compact subsets of Y.

Definition 1.10 (Uniformly bounded set)**.** [7] Let X be a compact normed space and Ω be a subset of $C(X)$. Ω is uniformly bounded if there exists $M > 0$ such that $\|u\| < M$ for all $u \in \Omega$.

Definition 1.11 (Equicontinuous set)**.** [7] Let X be a compact normed space and Ω be a subset of $C(X)$. Ω is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in X$ and $d(t_1, t_2) < \delta$ imply

$$|u(t_1) - u(t_2)| < \varepsilon \text{ for all } u \in \Omega.$$

1.1.3 Cones in Banach Spaces

Cones allow us to impose order and to study positive solutions.

Definition 1.12 (Cone)**.** [7] Let X be a real Banach space. A nonempty closed convex set $K \subset X$ is called a *cone* if:

1. $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$;
2. $u \in K$ and $-u \in K$ imply $u = 0$.

Example 1.3 (Natural cone of nonnegative functions)**.** [7] In $X = C([a, b], \mathbb{R})$, the set $K = \{f \in X : f(t) \geq 0 \ \forall t\}$ is a cone.

Example 1.4 (Cone with interior)**.** [7] Let $X = \mathbb{R}^n$ with the Euclidean norm. The set $K = \{x = (x_1, \ldots, x_n) : x_i \geq 0, \ i = 1, \ldots, n\}$ is a cone (the positive orthant). It has nonempty interior if $n \geq 1$.

Remark. Cones are useful in the study of positive solutions because they provide an ordered structure in Banach spaces. In this thesis, a suitable cone of positive periodic functions will be used in Chapter 2 to obtain positive periodic solutions.

1.1.4 Arzelà–Ascoli Theorem

The Arzelà–Ascoli theorem is a cornerstone for establishing compactness in function spaces.

Theorem 1.1 (Arzelà–Ascoli)**.** *[15] Let $X = C([a, b], \mathbb{R})$ be equipped with the supremum norm. A subset $\mathcal{F} \subset X$ is relatively compact if and only if it is:*

1. Uniformly bounded*: $\exists M > 0$ such that $|f(t)| \leq M$ for all $f \in \mathcal{F}$ and all $t \in [a, b]$;*
2. Equicontinuous*: $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall f \in \mathcal{F}$, $\forall t, s \in [a, b]$ with $|t - s| < \delta$ we have $|f(t) - f(s)| < \varepsilon$.*

Definition 1.13 (Lipschitz continuity)**.** [15] Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is said to be Lipschitz continuous if there exists a constant $L > 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y), \qquad \text{for all } x, y \in X.$$

The constant L is called a Lipschitz constant of f.

Example 1.5. The family $\mathcal{F} = \{f_n(x) = \sin(nx) : n \in \mathbb{N}\}$ on $[0, 2\pi]$ is uniformly bounded but not equicontinuous. Hence it is not relatively compact in $C([0, 2\pi])$.

1.1.5 Fixed Point Theory in Cone

Krasnoselskii's Cone Expansion/Compression Theorem

Theorem 1.2 (Krasnoselskii)**.** *[7]*

Let X be a Banach space and $K \subset X$ a cone. Let Ω_1, Ω_2 be open bounded subsets of X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous and satisfies either:

1. *(Expansion) $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$; or*
2. *(Compression) $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.*

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Example 1.6. Let $X = \mathbb{R}$, $K = [0, \infty)$. Define $Tu = \frac{1}{2}u + 10$. Choose $\Omega_1 = (0, 10)$, $\Omega_2 = (0, 30)$. For $u = 10$ (boundary of Ω_1), $Tu = 15 \geq 10$. For $u = 30$, $Tu = 25 \leq 30$. Hence the compression condition holds, and there is a fixed point in $[10, 30]$ (indeed $u = 20$).

Krasnoselskii Type Homotopy Fixed Point Theorem

We now recall a more flexible version, which is often used to avoid eigenvalue problems on the boundary.

Definition 1.14 (Homotopy)**.** [7] A **homotopy** is a continuous deformation between two continuous functions. Formally, a homotopy between $f, g : X \to Y$ is a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

Theorem 1.3 (Krasnoselskii type homotopy)**.** *[13] Let $(X, \|.\|)$ be a Banach space, $K \subset X$ a cone, $0 < r < R$ and $T : K_R \to K$ a continuous and compact operator, where $K_R = \{u \in K : \|u\| \leq R\}$. Assume that*
(C_1) $T(u) \neq \lambda u$, for every $u \in K$, $\|u\| = r$ and all $\lambda > 1$,
(C_2) there exists $v \in K \setminus \{0\}$ such that $u - T(u) \neq \lambda v$, for every $u \in K$, $\|u\| = R$ and all $\lambda > 1$.
Then, T has a fixed point u in K with $r \leq \|u\| \leq R$.

Remark. Condition (C1) prevents the operator from having an outward eigenvalue greater than 1 on the inner boundary, while (C2) prevents the shift $x - T(x)$ from being parallel to a fixed direction on the outer boundary. This is particularly useful for systems like predator–prey models.

1.2 Time Scales Calculus

Time scales theory was introduced by Hilger [10] as a unified approach to continuous and discrete calculus. The fundamental concepts and tools used in this section are mainly based on the book of Bohner and Peterson [3], which provides a systematic introduction to dynamic equations on time scales and their applications.

This section presents the basic concepts of time scales theory that will be used in the study of dynamic equations. We begin by introducing the notion of a time scale, together with the forward and backward jump operators and the graininess function. Then, we recall the main tools of calculus on time scales, including the delta derivative, the delta integral, regressive functions, and the exponential function. Finally, we introduce dynamic equations and periodic functions on time scales, which are essential for the study of positive periodic solutions in Chapter 2.

1.2.1 General Definition

Definition 1.15 (Time Scale)**.** [3] A time scale is an arbitrary nonempty closed subset of the real numbers.
The real numbers, the integers, the natural numbers, and the nonnegative integers are

examples of time scales, as are $[0,1] \cup [2,3]$, $[0,1] \cup \mathbb{N}$ and the Cantor set, while $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{C}$, $(0,1)$ are not time scales.

Definition 1.16 (Forward and Backward Jump Operator)**.** [3] For $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition we put $\inf \varnothing = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \varnothing = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Finally, if $f :\to \mathbb{R}$ is a function, then we define the function $f^\sigma : T \to \mathbb{R}$ by

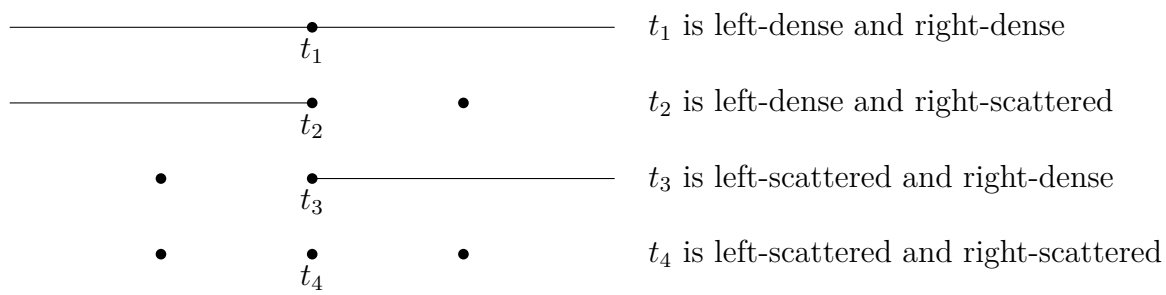
$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all } t \in \mathbb{T},$$

i.e., $f^\sigma = f \circ \sigma$.

Table 1.1: Classification of Points

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Figure 1.1: Classifications of Points



(t_1 is dense and t_4 is isolated)

Figure 1.2: Some Time Scales

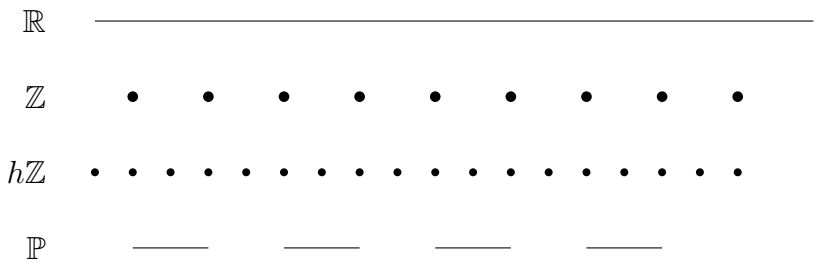


Table 1.2: Examples of Time Scales

$\mathbb{T}$	$\mu(t)$	$\sigma(t)$	$\rho(t)$
$\mathbb{R}$	0	t	t
$\mathbb{Z}$	1	$t+1$	$t-1$
$h\mathbb{Z}$	h	$t+h$	$t-h$
$q^{\mathbb{N}}$	$(q-1)t$	qt	$\frac{t}{q}$
$2^{\mathbb{N}}$	t	$2t$	$\frac{t}{2}$
$\mathbb{N}_0^2$	$2\sqrt{t}+1$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$

1.2.2 Differentiation

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called delta (or Hilger) derivative of f at a point $t \in \mathbb{T}^\kappa$.

Definition 1.17 (The Delta (or Hilger) Derivative)**.** [3] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the *delta (or Hilger) derivative* of f at t.
Moreover, we say that f is *delta (or Hilger) differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the *delta derivative* of f on $\mathbb{T}^\kappa$.

Theorem 1.4. *[3] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Example 1.7. Again we consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

if and only if f is differentiable (in the ordinary sense) at t. In this case we then have

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t),$$

(ii) If $\mathbb{T} = \mathbb{Z}$, $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t),$$

where Δ is the usual forward difference operator defined by

$$\Delta f(t) := f(t+1) - f(t).$$

Theorem 1.5. *[3] Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant α, the function $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 1.8. The delta derivative of t^2 is given by

$$(t^2)^{\Delta} = t + \sigma(t).$$

The delta derivative of $\frac{1}{t}$ is given by

$$\left(\frac{1}{t}\right)^{\Delta} = -\frac{1}{t\,\sigma(t)}.$$

1.2.3 Integration

In order to describe classes of functions that are "integrable", we introduce the following two concepts.

Definition 1.18 (Regulated)**.** [3] A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 1.19 (Rd-Continuous)**.** [3] A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{\mathrm{rd}} = C^1_{\mathrm{rd}}(\mathbb{T}) = C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

Theorem 1.6. *[3] Assume $f : \mathbb{T} \to \mathbb{R}$.*

(i) If f is continuous, then f is rd-continuous.

(ii) If f is rd-continuous, then f is regulated.

(iii) The jump operator σ is rd-continuous.

(iv) If f is regulated or rd-continuous, then so is f^σ.

(v) Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.

Definition 1.20 (Pre-Differentiable)**.** [3] A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Theorem 1.7. *Every regulated function on a compact interval is bounded.*

Theorem 1.8 (Mean Value Theorem)**.** *[3] Let f and g be real-valued functions defined on $\mathbb{T}$, both pre-differentiable with D. Then*

$$|f^\Delta(t)| \le g^\Delta(t) \quad \textit{for all } t \in D,$$

implies

$$|f(s) - f(r)| \le g(s) - g(r) \quad \textit{for all } r, s \in \mathbb{T},\ r \le s.$$

Proposition 1.1 (Uniform Boundedness of Delta Derivatives Implies Equicontinuity)**.** *Let $\mathbb{T}$ be a time scale and let $[a,b]_{\mathbb{T}}$ be a compact interval. Let $\mathcal{F} \subset C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$. Assume that there exists a constant $M > 0$ such that*

$$|u^{\Delta}(t)| \leq M, \qquad \text{for all } u \in \mathcal{F} \text{ and for all } t \in [a,b]^{\kappa}_{\mathbb{T}}.$$

Then $\mathcal{F}$ is equicontinuous on $[a,b]_{\mathbb{T}}$.

Proof. The proof follows from the mean value inequality on time scales; see [3]. Let $u \in \mathcal{F}$ and let $s, t \in [a,b]_{\mathbb{T}}$. Without loss of generality, assume that $s < t$. By the mean value inequality on time scales, we have

$$|u(t) - u(s)| \leq \left(\sup_{\tau \in [s,t]^{\kappa}_{\mathbb{T}}} |u^{\Delta}(\tau)| \right) |t - s|.$$

Since $|u^{\Delta}(\tau)| \leq M$ for all $u \in \mathcal{F}$, it follows that

$$|u(t) - u(s)| \leq M|t - s|.$$

Let $\varepsilon > 0$. Choose

$$\delta = \frac{\varepsilon}{M}.$$

Then, whenever $|t - s| < \delta$, we get

$$|u(t) - u(s)| \leq M|t - s| < M\delta = \varepsilon.$$

The number δ depends only on ε and M, and not on the particular function $u \in \mathcal{F}$. Hence $\mathcal{F}$ is equicontinuous on $[a,b]_{\mathbb{T}}$. □

Remark. Equicontinuity is not defined by the boundedness of the derivatives. However, a uniform bound on the delta derivatives is a sufficient condition for equicontinuity, because it gives a uniform Lipschitz estimate.

Theorem 1.9 (Existence of Pre-Antiderivatives)**.** *[3] Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \quad \text{for all } t \in D.$$

Definition 1.21 (Pre-Antiderivative)**.** Assume $f : T \to \mathbb{R}$ is a regulated function. Any function F is called a *pre-antiderivative* of f. We define the *indefinite integral* of a regulated function f by

$$\int f(t)\,\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the *Cauchy integral* by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \quad \text{for all } t \in \mathbb{T}^{\kappa}.$$

Theorem 1.10 (Existence of Antiderivatives)**.** *[3] Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then*

$$F(t) := \int_{t_0}^t f(\tau)\,\Delta\tau \quad \textit{for } t \in \mathbb{T},$$

is an antiderivative of f.

Theorem 1.11. *[3]*

If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{\mathrm{rd}}$, then

(i) $\displaystyle\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t;$

(ii) $\displaystyle\int_a^b (\alpha f)(t)\,\Delta t = \alpha \int_a^b f(t)\,\Delta t;$

(iii) $\displaystyle\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t;$

(iv) $\displaystyle\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t;$

(v) $\displaystyle\int_a^b f(t) g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(t)\,\Delta t;$

(vi) $\displaystyle\int_a^b f^{\sigma}(t) g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(\sigma(t))\,\Delta t;$

(vii) $\displaystyle\int_a^a f(t)\,\Delta t = 0;$

(viii) if $|f(t)| \le g(t)$ on $[a, b)$, then $\displaystyle\int_a^b f(t)\,\Delta t \le \int_a^b g(t)\,\Delta t;$

(ix) if $f(t) \ge 0$ for all $a \le t < b$, then $\displaystyle\int_a^b f(t)\,\Delta t \ge 0.$

Theorem 1.12. *[3] Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \displaystyle\sum_{t\in[a,b)} \mu(t)f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ \displaystyle -\sum_{t\in[b,a)} \mu(t)f(t), & \text{if } a > b. \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \displaystyle\sum_{k=a/h}^{b/h-1} f(kh)h, & \text{if } a < b, \\ 0, & \text{if } a = b, \\ \displaystyle -\sum_{k=b/h}^{a/h-1} f(kh)h, & \text{if } a > b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \displaystyle\sum_{t=a}^{b-1} f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ \displaystyle -\sum_{t=b}^{a-1} f(t), & \text{if } a > b. \end{cases}$$

Definition 1.22. [3] If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)$, then we define the *improper integral* by

$$\int_a^\infty f(t)\,\Delta t := \lim_{b\to\infty} \int_a^b f(t)\,\Delta t,$$

provided this limit exists, and we say that the improper integral *converges* in this case. If this limit does not exist, then we say that the improper integral *diverges*.

Theorem 1.13 (Leibniz Integral Rule on Time Scales)**.** *[8] we mean the delta derivative of $F(t, s)$ with respect to t. If F, F^Δ is continuous, and $v, h : \mathbb{T} \to \mathbb{T}$ are delta differentiable*

functions, $\forall t \in \mathbb{T}^\kappa$*:*

$$\left[\int_{v(t)}^{h(t)} F(t,s)\,\Delta s\right]^\Delta = \int_{v(t)}^{h(t)} F^{\Delta_t}(t,s)\,\Delta s + h^{\Delta_t}(t)F(\sigma(t),h(t)) - v^{\Delta_t}(t)F(\sigma(t),v(t)).$$

1.2.4 The Regressive Group

Definition 1.23. [3] We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^*,$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T},\mathbb{R}).$$

We define the set $\mathcal{R}^+$ of all *positively regressive* elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T},\mathbb{R}) = \{p \in \mathcal{R} \ : \ 1 + \mu(t)\,p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Definition 1.24. [3] If $p, q \in \mathcal{R}$, then we define the function $p \oplus q$ by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{for all } t \in \mathbb{T}^\kappa,$$

and the function $\ominus p$ defined by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Lemma 1.1. *[3] The functions* $p \oplus q$ *and* $\ominus p$ *are also elements of* $\mathcal{R}$*.*

Definition 1.25. [3] We define the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) = (p \oplus (\ominus q))(t) \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Lemma 1.2. *[3] Suppose* $p, q \in \mathcal{R}$*. Show directly from the definition that*

1. $p \ominus p = 0$,
2. $\ominus(\ominus p) = p$,
3. $p \ominus p \in \mathcal{R}$,
4. $p \ominus q = \dfrac{p - q}{1 + \mu(t)p(t)}$,
5. $\ominus(p \oplus q) = q \ominus p$,

6. $\ominus(p \oplus q) = (\ominus p) \oplus (\ominus q)$.

Lemma 1.3. *[3]*

1. $(\mathcal{R}, \oplus)$ *is an Abelian group. This group is called the regressive group.*

2. $(\mathcal{R}^+, \oplus)$ *is a subgroup of the regressive group.*

1.2.5 The Exponential Function

Definition 1.26. [3] For $h > 0$ we define the *Hilger complex numbers*, the *Hilger real axis*, the *Hilger alternating axis*, and the *Hilger imaginary circle* as

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\},$$

$$\mathbb{R}_h := \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h} \right\},$$

$$\mathbb{A}_h := \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h} \right\},$$

$$\mathbb{I}_h := \left\{ z \in \mathbb{C}_h : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\}.$$

respectively. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C},\ \mathbb{R}_0 := \mathbb{R},\ \mathbb{I}_0 := i\mathbb{R},\ \mathbb{A}_0 := \varnothing$.

Definition 1.27. [3] If $p \in \mathcal{R}$, then we define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\, \Delta\tau \right),$$

for $s, t \in \mathbb{T}$, where the cylinder transformation $\xi_h(z)$

Lemma 1.4. *[3] If* $p \in \mathcal{R}$*, then the semigroup property*

$$e_p(t, r)\, e_p(r, s) = e_p(t, s) \quad \textit{for all} \quad r, s, t \in \mathbb{T},$$

is satisfied.

Theorem 1.14. *[3] If* $p, q \in \mathcal{R}$*, then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$*;*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

(iii) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

(viii) $\left(\dfrac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Theorem 1.15. *[3] If $p,q \in \mathcal{R}$, then*

$$e_{p\ominus q}^{\Delta}(\cdot,t_0) = (p-q)\,\frac{e_p(\cdot,t_0)}{e_q^{\sigma}(\cdot,t_0)}.$$

Theorem 1.16. *[3] If $p \in \mathcal{R}$ and $a,b,c \in \mathbb{T}$, then*

$$\big[e_p(c,\cdot)\big]^{\Delta} = -p\,\big[e_p(c,\cdot)\big]^{\sigma},$$

and

$$\int_a^b p(t)e_p(c,\sigma(t))\,\Delta t = e_p(c,a) - e_p(c,b).$$

Proof. We use many of the properties from Theorem 1.14 to find

$$\begin{aligned}
p(t)e_p(c,\sigma(t)) &= p(t)e_{\ominus p}(\sigma(t),c)\\
&= p(t)\big[1+\mu(t)(\ominus p)(t)\big]e_{\ominus p}(t,c)\\
&= p(t)\left[1-\frac{\mu(t)p(t)}{1+\mu(t)p(t)}\right]e_{\ominus p}(t,c)\\
&= p(t)\frac{1}{1+\mu(t)p(t)}e_{\ominus p}(t,c)\\
&= -(\ominus p)(t)e_{\ominus p}(t,c)\\
&= -e_{\ominus p}^{\Delta}(t,c)\\
&= -[e_p(c,t)]^{\Delta}.
\end{aligned}$$

□

Lemma 1.5. *[3] If $p \in \mathbb{R}^+$, then*

$$0 < e_p(t,s) \le \exp\left(\int_s^t p(r)\Delta r\right) \text{ for all } t \in [s,\infty).$$

Corollary 1.1. *[3] If $p \in \mathbb{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(r)\Delta r\right) < 1 \text{ for all } t \in [s,\infty).$$

1.2.6 Dynamic Equation

Definition 1.28. [3] Suppose $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation

$$y^\Delta = f(t, y, y^\sigma), \tag{1.1}$$

is called a *first order dynamic equation*, sometimes also a *differential equation*. If

$$f(t,y,y^\sigma) = f_1(t)y + f_2(t) \quad \text{or} \quad f(t,y,y^\sigma) = f_1(t)y^\sigma + f_2(t),$$

for functions f_1 and f_2, then (1.1) is called a *linear equation*. A function $y : \mathbb{T} \to \mathbb{R}$ is called a solution of (1.1) if

$$y^\Delta(t) = f\big(t, y(t), y(\sigma(t))\big) \quad \text{is satisfied for all} \quad t \in \mathbb{T}^\kappa$$

The general solution of (1.1) is defined to be the set of all solutions of (1.1). Given $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, the problem

$$y^\Delta = f(t, y, y^\sigma), \qquad y(t_0) = y_0,$$

is called an *initial value problem* (abbreviated by IVP), and a solution y of (1.1) with $y(t_0) = y_0$ is called a solution of this IVP.

Definition 1.29. [3] If $p \in \mathcal{R}$, then the first order linear dynamic equation

$$y^\Delta = p(t)y, \tag{1.2}$$

is called *regressive*.

Theorem 1.17. *[3] Suppose* (1.2) *is regressive and fix $t_0 \in \mathbb{T}$. Then $e_p(\cdot, t_0)$ is a solution of the initial value problem*

$$y^\Delta = p(t)y, \qquad y(t_0) = 1, \tag{1.3}$$

on $\mathbb{T}$.

1.2.7 Periodic Functions on Time Scales

Definition 1.30. [11] We say that a time scale $\mathbb{T}$ is periodic if there exists $T > 0$ such that if t $\in \mathbb{T}$ then $t \pm T \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive T is called the period of the time scale.

Example 1.9. The following time scales are periodic:

1. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [\, 2(i-1)h,\, 2ih \,], h > 0$, with period $T = 2h$.
2. $\mathbb{T} = h\mathbb{Z}$, with period $T = h$.
3. $\mathbb{T} = \mathbb{R}$.

Remark. All periodic time scales are unbounded above and below.

Definition 1.31. [11] [14] Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period P. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *periodic* if there exists a natural number n such that $T = nP$ and $f(t+T) = f(t) \quad$ for all $t \in \mathbb{T}$, where T is the smallest number satisfying this property. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t+T) = f(t) \quad$ for all $t \in \mathbb{T}$.

Remark. If $\mathbb{T}$ is a periodic time scale with period T, then $\sigma(t+nT) = \sigma(t) + nT$. Consequently, the graininess function μ satisfies $\mu(t+nT) = \sigma(t+nT) - (t+nT) = \sigma(t) - t = \mu(t)$, and hence μ is a periodic function with period T.

Chapter 2

Existence of Positive Periodic Solutions

The mathematical framework of this chapter follows the work of Bordj and Ardjouni [4] on positive periodic solutions of non-autonomous Lotka–Volterra dynamic systems with a general attack rate on time scales. Their study extends the continuous results of Lois-Prados and Precup [13] to the setting of time scales, where both the continuous case $\mathbb{T} = \mathbb{R}$ and the discrete case $\mathbb{T} = \mathbb{Z}$ can be treated in a unified way.

This chapter is devoted to transforming system (2.1) into an equivalent integral system and proving the existence of positive periodic solutions.

2.1 System Description and Problem Formulation

Following this framework, we consider the following non-autonomous Lotka–Volterra dynamic system with a general interaction rate on a periodic time scale $\mathbb{T}$ with $0 \in \mathbb{T}$:

$$\begin{cases} x^{\Delta}(t) = a(t)x(t)g(x(t)) - f(t, x(t), y(t))x(t)y(t), \\ y^{\Delta}(t) = -b(t)y^{\sigma}(t) + c(t)f(t, x(t), y(t))x(t)y(t), \end{cases} \tag{2.1}$$

where $x(t), y(t)$ represent two interacting positive quantities, a, b, c are positive T-periodic functions, g is decreasing, and f is a general interaction rate.
Let $\mathbb{T}$ be an P-periodic time scale such that $0 \in \mathbb{T}$. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} = \mathbb{R}$, $T = nP$ for some $n \in \mathbb{N}$.

Throughout this chapter, we assume that $a, b, c \in C_{rd}(\mathbb{T}, (0, \infty))$ are T-periodic functions, $f \in C_{rd}(\mathbb{T} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ is such that $f(\cdot, x, y)$ is T-periodic for every $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, and $g \in C(\mathbb{R}_+, \mathbb{R})$ is decreasing with $g(0) \leq 1$.

We consider the list of useful notations

$$\underline{a} = \min_{t \in [0,T]} a(t), \qquad \underline{b} = \min_{t \in [0,T]} b(t), \qquad \underline{c} = \min_{t \in [0,T]} c(t),$$

$$\overline{a} = \max_{t\in[0,T]} a(t), \qquad \overline{b} = \max_{t\in[0,T]} b(t), \qquad \overline{c} = \max_{t\in[0,T]} c(t).$$

$$\underline{A} = \min_{(t,s)\in[0,T]\times[t,t+T]} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_a(T,0)}, \qquad \underline{B} = \min_{(t,s)\in[0,T]\times[t,t+T]} \frac{e_b(s,t)}{e_b(T,0)-1},$$

$$\overline{A} = \max_{(t,s)\in[0,T]\times[t,t+T]} \frac{e_a(\sigma(s),t)}{1-e_{\ominus a}(T,0)}, \qquad \overline{B} = \max_{(t,s)\in[0,T]\times[t,t+T]} \frac{e_b(s,t)}{e_b(T,0)-1}.$$

$$q_1 = \frac{\underline{A}}{\overline{A}}, \qquad q_2 = \frac{\underline{B}}{\overline{B}}, \qquad M = \max\{\overline{B}\overline{c}, \overline{A}\}, \qquad m = \min\{\underline{B}\underline{c}, \underline{A}\}. \tag{2.2}$$

We introduce the following conditions on f:

(H1) There exists $\eta \in C_{rd}(\mathbb{T}\times\mathbb{R}_+\times\mathbb{R}_+, \mathbb{R}_+)$ such that

- $\eta(\cdot, x, y)$ is T-periodic for every $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$,
- $\eta(t, \cdot, y)$ and $\eta(t, x, \cdot)$ are increasing functions for every $(t, y), (t, x) \in \mathbb{T} \times \mathbb{R}_+$,
- $f(t, x, y) \le \eta(t, x, y)$ for every $(t, x, y) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{R}_+$.

(H2) There exists $\phi \in C_{rd}(\mathbb{T}\times\mathbb{R}_+, \mathbb{R}_+)$ such that

- $\phi(\cdot, z)$ is T-periodic for every $z \in \mathbb{R}_+$,
- $\phi(t, \cdot)$ is decreasing for every $t \in \mathbb{T}$,
- $f(t, x, y) \ge \phi(t, x + y)$ for every $(t, x, y) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{R}_+$.

Proposition 2.1. *Define the following space,*

$$P_T = \{(x, y) \in C_{rd} \times C_{rd} : (x, y)(t + T) = (x, y)(t)\},$$

where both x and y are real valued rd-continuous functions on $\mathbb{T}$. Then P_T endowed with the norm

$$\|(x, y)\| = \|x\|_\infty + \|y\|_\infty,$$

is a Banach space where

$$\|x\|_\infty = \sup_{t\in\mathbb{T}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Proof. We prove that P_T is a Banach space. Let

$$P_T = \{(x, y) \in C_{rd}(\mathbb{T}, \mathbb{R}) \times C_{rd}(\mathbb{T}, \mathbb{R}) : \ (x, y)(t + T) = (x, y)(t), \ t \in \mathbb{T}\},$$

endowed with the norm

$$\|(x, y)\| = \|x\|_\infty + \|y\|_\infty,$$

where

$$\|x\|_\infty = \sup_{t\in\mathbb{T}} |x(t)| = \sup_{t\in[0,T]_\mathbb{T}} |x(t)|.$$

Since the functions are T-periodic, each function is completely determined by its restriction to the compact interval $[0,T]_\mathbb{T}$. Hence the supremum over $\mathbb{T}$ is finite and is equal to the supremum over $[0,T]_\mathbb{T}$.

Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a Cauchy sequence in P_T. Then

$$\|(x_n, y_n) - (x_m, y_m)\| = \|x_n - x_m\|_\infty + \|y_n - y_m\|_\infty \to 0,$$

as $n, m \to \infty$. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $C_{rd}([0,T]_\mathbb{T}, \mathbb{R})$ endowed with the supremum norm.

Since $C_{rd}([0,T]_\mathbb{T}, \mathbb{R})$ is a Banach space, there exist $x, y \in C_{rd}([0,T]_\mathbb{T}, \mathbb{R})$ such that

$$\|x_n - x\|_\infty \to 0, \qquad \|y_n - y\|_\infty \to 0.$$

We extend x and y periodically to all of $\mathbb{T}$. It remains to show that the limit is T-periodic. For every $t \in \mathbb{T}$, since x_n is T-periodic, we have

$$x_n(t+T) = x_n(t).$$

Passing to the limit as $n \to \infty$, using the uniform convergence, we obtain

$$x(t+T) = x(t).$$

Similarly,

$$y(t+T) = y(t).$$

Thus $(x, y) \in P_T$. Moreover,

$$\|(x_n, y_n) - (x, y)\| = \|x_n - x\|_\infty + \|y_n - y\|_\infty \to 0.$$

Hence every Cauchy sequence in P_T converges to an element of P_T. Therefore P_T is a Banach space.

□

Proposition 2.2. *Define the following,*

$$K = \{(x, y) \in P_T : x(t) \geq q_1\|x\|_\infty,\ y(t) \geq q_2\|y\|_\infty \text{ for every } t \in \mathbb{T}\}.$$

Thus, K is a cone in P_T.

Proof. we prove that K is a cone in P_T. Let

$$K = \{(x, y) \in P_T : \ x(t) \geq q_1\|x\|_\infty, \ y(t) \geq q_2\|y\|_\infty, \text{ for every } t \in \mathbb{T}\},$$

where $q_1, q_2 > 0$.
First, K is nonempty because $(0, 0) \in K$.

Let $(x_1, y_1), (x_2, y_2) \in K$ and let $\alpha, \beta \geq 0$. We show that

$$\alpha(x_1, y_1) + \beta(x_2, y_2) \in K.$$

For every $t \in \mathbb{T}$, we have

$$x_1(t) \geq q_1\|x_1\|_\infty, \qquad x_2(t) \geq q_1\|x_2\|_\infty.$$

Hence

$$\alpha x_1(t) + \beta x_2(t) \geq q_1\big(\alpha\|x_1\|_\infty + \beta\|x_2\|_\infty\big).$$

On the other hand, by the triangle inequality,

$$\|\alpha x_1 + \beta x_2\|_\infty \leq \alpha\|x_1\|_\infty + \beta\|x_2\|_\infty.$$

Therefore,

$$\alpha x_1(t) + \beta x_2(t) \geq q_1\|\alpha x_1 + \beta x_2\|_\infty.$$

Similarly,

$$\alpha y_1(t) + \beta y_2(t) \geq q_2\|\alpha y_1 + \beta y_2\|_\infty.$$

Thus

$$\alpha(x_1, y_1) + \beta(x_2, y_2) \in K.$$

So K is closed under nonnegative linear combinations.
Next, we show that $K \cap (-K) = \{(0, 0)\}$. Suppose that

$$(x, y) \in K \quad \text{and} \quad -(x, y) \in K.$$

Since $(x, y) \in K$, we have

$$x(t) \geq q_1\|x\|_\infty, \qquad y(t) \geq q_2\|y\|_\infty.$$

Since $-(x, y) \in K$, we also have

$$-x(t) \geq q_1\|x\|_\infty, \qquad -y(t) \geq q_2\|y\|_\infty.$$

Thus

$$x(t) \geq q_1\|x\|_\infty \quad \text{and} \quad x(t) \leq -q_1\|x\|_\infty.$$

This is possible only if $\|x\|_\infty = 0$. Hence $x = 0$. In the same way, $\|y\|_\infty = 0$, and therefore $y = 0$. Consequently,

$$K \cap (-K) = \{(0,0)\}.$$

Finally, we prove that K is closed. Let $\{(x_n, y_n)\} \subset K$ and assume that

$$(x_n, y_n) \to (x, y) \quad \text{in } P_T.$$

Then

$$\|x_n - x\|_\infty \to 0, \qquad \|y_n - y\|_\infty \to 0.$$

Also, by the reverse triangle inequality,

$$\big|\|x_n\|_\infty - \|x\|_\infty\big| \leq \|x_n - x\|_\infty \to 0,$$

and similarly,

$$\big|\|y_n\|_\infty - \|y\|_\infty\big| \leq \|y_n - y\|_\infty \to 0.$$

Since $(x_n, y_n) \in K$, for every $t \in \mathbb{T}$,

$$x_n(t) \geq q_1\|x_n\|_\infty, \qquad y_n(t) \geq q_2\|y_n\|_\infty.$$

Passing to the limit as $n \to \infty$, we get

$$x(t) \geq q_1\|x\|_\infty, \qquad y(t) \geq q_2\|y\|_\infty.$$

Thus $(x, y) \in K$. Therefore K is closed. Consequently, K is a nonempty closed convex subset of P_T, stable under nonnegative scalar multiplication, and satisfies

$$K \cap (-K) = \{(0,0)\}.$$

Hence, based on Definition 1.12 K is a cone in P_T. □

2.2 Construction of the Integral Operator

Lemma 2.1. *If $(x, y) \in P_T$, then (x, y) is a solution of the system (2.1) if and only if*

$$x(t) = \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s), t)}{1 - e_{\ominus a}(T, 0)}[a(s)x(s)(1 - g(x(s))) + f(s, x(s), y(s))x(s)y(s)]\Delta s, \quad (2.3)$$

and

$$y(t) = \int_t^{t+T} \frac{e_b(s,t)}{e_b(T,0)-1}\, c(s)f(s,x(s),y(s))x(s)y(s)\Delta s. \tag{2.4}$$

Proof. Let $(x,y) \in P_T$ be a solution of (2.1).
We put the first equation in (2.1) as

$$x^{\Delta}(t) = a(t)x(t)g(x(t)) - f(t,x(t),y(t))x(t)y(t)$$

Subtracting $a(t)x(t)$ from both sides of the equation, we obtain:

$$x^{\Delta}(t) - a(t)x(t) = a(t)x(t)g(x(t)) - f(t,x(t),y(t))x(t)y(t) - a(t)x(t)$$

$$x^{\Delta}(t) - a(t)x(t) = a(t)x(t)[g(x(t)) - 1] - f(t,x(t),y(t))x(t)y(t)$$

$$x^{\Delta}(t) - a(t)x(t) = -\left[a(t)x(t)(1-g(x(t))) + f(t,x(t),y(t))x(t)y(t)\right]$$

Multiplying both sides with $e_{\ominus a}(\sigma(t),0)$, we obtain

$$e_{\ominus a}(\sigma(t),0)[x^{\Delta}(t)-a(t)x(t)] = -e_{\ominus a}(\sigma(t),0)[a(t)x(t)(1-g(x(t)))+f(t,x(t),y(t))x(t)y(t)]. \tag{2.5}$$

Using the product rule in Theorem 1.5, we obtain:

$$[x(t)e_{\ominus a}(t,0)]^{\Delta} = x(t)[e_{\ominus a}(t,0)]^{\Delta} + x^{\Delta}(t)e_{\ominus a}(\sigma(t),0) \tag{2.6}$$

By the properties of the exponential function on time scales in Theorem 1.14, we have:

$$[e_{\ominus a}(t,0)]^{\Delta} = [\frac{1}{e_a(t,0)}]^{\Delta} = \frac{-a(t)}{e_a(\sigma(t),0)} = -a(t)e_{\ominus a}(\sigma(t),0) \tag{2.7}$$

Substituting (2.7) into equation(2.8), we have:

$$[x(t)e_{\ominus a}(t,0)]^{\Delta} = -x(t)a(t)e_{\ominus a}(\sigma(t),0) + x^{\Delta}(t)e_{\ominus a}(\sigma(t),0)$$

$$[x(t)e_{\ominus a}(t,0)]^{\Delta} = e_{\ominus a}(\sigma(t),0)[x^{\Delta}(t) - a(t)x(t)] \tag{2.8}$$

Substituting (2.8)into equation(2.6), we have:

$$[x(t)e_{\ominus a}(t,0)]^{\Delta} = -e_{\ominus a}(\sigma(t),0)[a(t)x(t)(1-g(x(t))) + f(t,x(t),y(t))x(t)y(t)]. \tag{2.9}$$

Integrating both sides from t to t + T, we obtain:

$$\int_t^{t+T} [x(s)e_{\ominus a}(s,0)]^{\Delta}\Delta s = -\int_t^{t+T} e_{\ominus a}(\sigma(s),0)[a(s)x(s)(1-g(x(s)))+f(s,x(s),y(s))x(s)y(s)]\Delta s$$

$$[x(s)e_{\ominus a}(s,0)]_t^{t+T} = -\int_t^{t+T} e_{\ominus a}(\sigma(s),0)[a(s)x(s)(1-g(x(s)))+f(s,x(s),y(s))x(s)y(s)]\Delta s$$

$$\begin{aligned} &x(t+T)e_{\ominus a}(t+T,0) - x(t)e_{\ominus a}(t,0) \\ &\qquad = -\int_t^{t+T} e_{\ominus a}(\sigma(s),0)[a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\,\Delta s \end{aligned}$$

Since $x(t)$ periodic $x(t+T) = x(t)$ So,

$$\begin{aligned} &x(t)[e_{\ominus a}(t+T,0) - e_{\ominus a}(t,0)] \\ &\qquad = -\int_t^{t+T} e_{\ominus a}(\sigma(s),0)[a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\Delta s \end{aligned}$$

$$\begin{aligned} &x(t)[e_{\ominus a}(t,0)e_{\ominus a}(t+T,t) - e_{\ominus a}(t,0)] \\ &\qquad = -\int_t^{t+T} e_{\ominus a}(\sigma(s),0)[a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\Delta s \end{aligned}$$

$$\begin{aligned} &x(t)e_{\ominus a}(t,0)[e_{\ominus a}(t+T,t) - 1] \\ &\qquad = -\int_t^{t+T} e_{\ominus a}(\sigma(s),0)[a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\Delta s \end{aligned}$$

Dividing both sides of the above equation with $e_{\ominus a}(t,0)$ and then using Theorem 1.14, we show

$$x(t)[e_{\ominus a}(t+T,t)-1] = -\int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),0)}{e_{\ominus a}(t,0)}[a(s)x(s)(1-g(x(s)))+f(s,x(s),y(s))x(s)y(s)]\Delta s$$

$$x(t)[e_{\ominus a}(t+T,t)-1] = -\int_t^{t+T} e_{\ominus a}(\sigma(s),0)e_{\ominus a}(0,t)[a(s)x(s)(1-g(x(s)))+f(s,x(s),y(s))x(s)y(s)]\Delta s$$

$$x(t)[e_{\ominus a}(t+T,t)-1] = -\int_t^{t+T} e_{\ominus a}(\sigma(s),t)[a(s)x(s)(1-g(x(s)))+f(s,x(s),y(s))x(s)y(s)]\Delta s$$

Thus,

$$x(t) = -\int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{e_{\ominus a}(t+T,t)-1}[a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\Delta s$$

Since T is the period,we have: $e_{\ominus a}(t+T,t) = e_{\ominus a}(T,0)$

$$x(t) = \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)}[a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\Delta s$$

Similarly, we put the second equation in (2.1) as

$$y^{\Delta}(t) = -b(t)y^{\sigma}(t) + c(t)f(t, x(t), y(t))x(t)y(t).$$

$$y^{\Delta}(t) + b(t)y(\sigma(t)) = c(t)f(t, x(t), y(t))x(t)y(t).$$

Multiplying both sides of the above equation with $e_b(t, 0)$ we obtain

$$e_b(t, 0)[y^{\Delta}(t) + b(t)y(\sigma(t))] = e_b(t, 0)[c(t)f(t, x(t), y(t))x(t)y(t)]. \tag{2.10}$$

Then using Theorem 1.5, we show:

$$[y(t)e_b(t, 0)]^{\Delta} = y^{\Delta}(t)e_b(t, 0) + y(\sigma(t))[e_b(t, 0)]^{\Delta}$$

$$= y^{\Delta}(t)e_b(t, 0) + y(\sigma(t))b(t)e_b(t, 0)$$

$$= e_b(t, 0)[y^{\Delta}(t) + b(t)y(\sigma(t))] \tag{2.11}$$

Substituting (2.11)into equation(2.10), we have:

$$[y(t)e_b(t, 0)]^{\Delta} = e_b(t, 0)[c(t)f(t, x(t), y(t))x(t)y(t)].$$

Integrating both sides from t to t + T, we obtain:

$$\int_t^{t+T} [y(s)e_b(s, 0)]^{\Delta}\Delta s = \int_t^{t+T} e_b(s, 0)[c(s)f(s, x(s), y(s))x(s)y(s)]\Delta s.$$

$$[y(s)e_b(s, 0)]_t^{t+T} = \int_t^{t+T} e_b(s, 0)[c(s)f(s, x(s), y(s))x(s)y(s)]\Delta s.$$

$$y(t + T)e_b(t + T, 0) - y(t)e_b(t, 0) = \int_t^{t+T} e_b(s, 0)[c(s)f(s, x(s), y(s))x(s)y(s)]\Delta s.$$

Since $y(t)$ periodic $y(t + T) = y(t)$ So,

$$y(t)e_b(t + T, 0) - y(t)e_b(t, 0) = \int_t^{t+T} e_b(s, 0)[c(s)f(s, x(s), y(s))x(s)y(s)]\Delta s.$$

$$y(t)e_b(t, 0)e_b(t + T, t) - y(t)e_b(t, 0) = \int_t^{t+T} e_b(s, 0)[c(s)f(s, x(s), y(s))x(s)y(s)]\Delta s.$$

$$y(t)e_b(t, 0)[e_b(t + T, t) - 1] = \int_t^{t+T} e_b(s, 0)[c(s)f(s, x(s), y(s))x(s)y(s)]\Delta s.$$

Dividing both sides of the above equation with $e_b(t, 0)$ and then using Theorem 1.14, we show

$$y(t)[e_b(t + T, t) - 1] = \int_t^{t+T} \frac{e_b(s, 0)}{e_b(t, 0)}[c(s)f(s, x(s), y(s))x(s)y(s)]\Delta s.$$

$$y(t)[e_b(t+T,t)-1] = \int_t^{t+T} e_b(s,0)e_b(0,t)[c(s)f(s,x(s),y(s))x(s)y(s)]\Delta s.$$

$$y(t)[e_b(t+T,t)-1] = \int_t^{t+T} e_b(s,t)[c(s)f(s,x(s),y(s))x(s)y(s)]\Delta s.$$

Thus,

$$y(t) = \int_t^{t+T} \frac{e_b(s,t)}{e_b(t+T,t)-1}[c(s)f(s,x(s),y(s))x(s)y(s)]\Delta s.$$

Since T is the period,we have: $e_b(t+T,t) = e_b(T,0)$

$$y(t) = \int_t^{t+T} \frac{e_b(s,t)}{e_b(T,0)-1}[c(s)f(s,x(s),y(s))x(s)y(s)]\Delta s.$$

□

Since each step is reversible, the converse follows easily. This completes the proof. ■

Define the mapping

$$E(x,y)(t) = \big(E_1(x,y)(t), E_2(x,y)(t)\big),$$

where

$$E_1(x,y)(t) = \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)}\big[a(s)x(s)\big(1-g(x(s))\big) + f\big(s,x(s),y(s)\big)x(s)y(s)\big]\Delta s,$$

and

$$E_2(x,y)(t) = \int_t^{t+T} \frac{e_b(s,t)}{e_b(T,0)-1}\, c(s)f\big(s,x(s),y(s)\big)x(s)y(s)\Delta s.$$

Remark. In all proofs, the denominator cannot vanish because, for all $p \in \mathcal{R}$, $e_p(t,s) \neq 1$, except when $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$ (Theorem 1.14).

Lemma 2.2. *Let $K_R = \{u \in K : \|u\| \leq R\}$ with $R > 0$. Suppose that $(H1)$ and $(H2)$ hold. Then $E : K_R \to K$ is a continuous and compact operator.*

Proof. **Step 1:** We prove that $E(x,y) \in P_T$
For any $(x,y) \in K_R$, we obtain

$$E_1(x,y)(t+T) = \int_{t+T}^{t+2T} \frac{e_{\ominus a}(\sigma(s),t+T)}{1-e_{\ominus a}(T,0)}\big[a(s)x(s)(1-g(x(s)))+f(s,x(s),y(s))x(s)y(s)\big]\Delta s.$$

We change the variable by setting $u = s-t \Rightarrow s = u+t$ and $\Delta s = \Delta u$

So, When s changes from $t+T$ to $t+2T$ u changes from t to $t+T$ We obtain,

$$= \int_t^{t+T} \frac{e_{\ominus a}(\sigma(u+T), t+T)}{1-e_{\ominus a}(T,0)} \big[a(u+T)x(u+T)(1-g(x(u+T)))$$
$$+ f(u+T, x(u+T), y(u+T))x(u+T)y(u+T)\big]\Delta u.$$

Since x, y, a, f, and g are T-periodic and $e_{\ominus a}(\sigma(u+T), t+T) = e_{\ominus a}(\sigma(u)+T, t+T) = e_{\ominus a}(\sigma(u), t)$, we have

$$= \int_t^{t+T} \frac{e_{\ominus a}(\sigma(u), t)}{1-e_{\ominus a}(T,0)} \big[a(u)x(u)(1-g(x(u))) + f(u, x(u), y(u))x(u)y(u)\big]\Delta u = E_1(x,y)(t).$$

Similarly, we get for any $(x, y) \in K_R$, we obtain

$$E_2(x,y)(t+T) = \int_{t+T}^{t+2T} \frac{e_b(s, t+T)}{e_b(T,0)-1}\, c(s) f\big(s, x(s), y(s)\big)x(s)y(s)\Delta s.$$

$$E_2(x,y)(t+T) = \int_t^{t+T} \frac{e_b(u+T, t+T)}{e_b(T,0)-1}\, c(u+T) f\big(u+T, x(u+T), y(u+T)\big)x(u+T)y(u+T)\Delta u.$$

Since x, y, c, b, f, g are T-periodic and $e_b(u+T, t+T) = e_b(u,t)$, we have

$$E_2(x,y)(t+T) = \int_t^{t+T} \frac{e_b(u,t)}{e_b(T,0)-1}\, c(u) f\big(u, x(u), y(u)\big)x(u)y(u)\Delta u = E_2(x,y)(t).$$

Thus, $E(x,y)(t+T) = E(x,y)(t)$.

Step 2: We prove that $E(K_R) \subset K$
We have for $(x, y) \in K_R$

$$\|E_1(x,y)\|_\infty = \sup_{t\in\mathbb{T}} \left| \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s), t)}{1-e_{\ominus a}(T,0)} \left[a(s)x(s)(1-g(x(s))) + f(s, x(s), y(s))x(s)y(s)\right]\Delta s \right|,$$

$$\|E_1(x,y)\|_\infty \le \int_t^{t+T} \sup_{t\in\mathbb{T}} \left| \frac{e_{\ominus a}(\sigma(s), t)}{1-e_{\ominus a}(T,0)} \right| \left[a(s)x(s)(1-g(x(s))) + f(s, x(s), y(s))x(s)y(s)\right]\Delta s,$$

$$\|E_1(x,y)\|_\infty \le \overline{A} \int_t^{t+T} \left[a(s)x(s)(1-g(x(s))) + f(s, x(s), y(s))x(s)y(s)\right]\Delta s,$$

Since all functions involved are T-periodic, the integral over $[t, t+T]_{\mathbb{T}}$ can be written over $[0, T]_{\mathbb{T}}$

$$\|E_1(x,y)\|_\infty \le \overline{A} \int_0^{T} \left[a(s)x(s)(1-g(x(s))) + f(s, x(s), y(s))x(s)y(s)\right]\Delta s, \qquad (2.12)$$

Then,

$$E_1(x,y)(t) = \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)}\big[a(s)x(s)\big(1-g(x(s))\big) + f\big(s,x(s),y(s)\big)x(s)y(s)\big]\Delta s,$$

$$E_1(x,y)(t) \geq \underline{A}\int_t^{t+T} [a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\,\Delta s.$$

Since all functions involved are T-periodic, the integral over $[t,t+T]_{\mathbb{T}}$ can be written over $[0,T]_{\mathbb{T}}$

$$E_1(x,y)(t) \geq \underline{A}\int_0^{T} [a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\,\Delta s.$$

We multiply and divide the right-hand side of the inequality $\overline{A}$ we obtain

$$E_1(x,y)(t) \geq \frac{\underline{A}}{\overline{A}}\overline{A}\int_0^{T} [a(s)x(s)(1-g(x(s))) + f(s,x(s),y(s))x(s)y(s)]\,\Delta s.$$

Since (2.12) we have:

$$E_1(x,y)(t) \geq \frac{\underline{A}}{\overline{A}}\|E_1(x,y)\|_\infty = q_1\|E_1(x,y)\|_\infty.$$

Similarly, we get for $(x,y) \in K_R$

$$\|E_2(x,y)\|_\infty = \sup_{t\in\mathbb{T}}\left|\int_t^{t+T} \frac{e_b(s,t)}{e_b(T,0)-1}c(s)f(s,x(s),y(s))x(s)y(s)\Delta s\right|,$$

$$\|E_2(x,y)\|_\infty \leq \int_t^{t+T} \sup_{t\in\mathbb{T}}\left|\frac{e_b(s,t)}{e_b(T,0)-1}\right| c(s)f(s,x(s),y(s))x(s)y(s)\Delta s,$$

$$\|E_2(x,y)\|_\infty \leq \overline{B}\int_t^{t+T} c(s)f(s,x(s),y(s))x(s)y(s)\Delta s,$$

Since the functions are T-periodic

$$\|E_2(x,y)\|_\infty \leq \overline{B}\int_0^{T} c(s)f(s,x(s),y(s))x(s)y(s)\Delta s, \tag{2.13}$$

then,

$$E_2(x,y)(t) = \int_t^{t+T} \frac{e_b(s,t)}{e_b(T,0)-1}\,c(s)f\big(s,x(s),y(s)\big)x(s)y(s)\Delta s,$$

$$E_2(x,y)(t) \geq \underline{B}\int_t^{t+T} c(s)f(s,x(s),y(s))x(s)y(s)\Delta s.$$

Since the functions are T-periodic

$$E_2(x,y)(t) \geq \underline{B}\int_0^T c(s)f(s,x(s),y(s))x(s)y(s)\Delta s.$$

We multiply and divide the right-hand side of the inequality $\overline{B}$ we obtain

$$E_2(x,y)(t) \geq \frac{\underline{B}}{\overline{B}}\overline{B}\int_0^T c(s)f(s,x(s),y(s))x(s)y(s)\Delta s.$$

Since (2.13) we have:

$$E_2(x,y)(t) \geq \frac{\underline{B}}{\overline{B}}\|E_2(x,y)\|_\infty = q_2\|E_2(x,y)\|_\infty$$

Hence, $E(K_R) \subset K$. The continuity of f and g implies the continuity of the operator E on K_R. Now, we need to show that the operator E is compact by applying Arzelà–Ascoli theorem. So we will show that $E(K_R)$ is uniformly bounded and equicontinuous subset.

Step 3: Now we prove that E is uniformly bounded

For $(x,y) \in K_R$, it follows that

$$|E_1(x,y)(t)| = \left|\int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)}\big[a(s)x(s)\big(1-g(x(s))\big)+f\big(s,x(s),y(s)\big)x(s)y(s)\big]\Delta s\right|,$$

$$|E_1(x,y)(t)| \leq \int_t^{t+T}\left|\frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)}\left[a(s)x(s)(1-g(x(s)))+f(s,x(s),y(s))x(s)y(s)\right]\right|\Delta s$$

$$\leq \overline{A}\overline{a}R(1-g(R))\int_t^{t+T}\Delta s + \overline{A}R^2\int_t^{t+T}\eta(s,R,R)\,\Delta s$$

$$\leq \overline{A}\overline{a}R(1-g(R))\int_t^{t+T}\Delta s + \overline{A}R^2\overline{\eta}(s,R,R)\int_t^{t+T}\Delta s$$

since:

$$\overline{\eta} = \max_{(t,x,y)\in\mathbb{T}\times\mathbb{R}_+\times\mathbb{R}_+}\eta(s,R,R)$$

$$|E_1(x,y)(t)| \leq \overline{A}\overline{a}R(1-g(R))T + \overline{A}R^2\overline{\eta}T = R_1.$$

Similarly, we get for $(x,y) \in K_R$

$$|E_2(x,y)(t)| = \left|\int_t^{t+T}\frac{e_b(s,t)}{e_b(T,0)-1}\,c(s)f\big(s,x(s),y(s)\big)x(s)y(s)\Delta s.\right|,$$

$$|E_2(x,y)(t)| \leq \int_t^{t+T} \left| \frac{e_b(s,t)}{e_b(T,0)-1}\, c(s) f\big(s,x(s),y(s)\big)x(s)y(s) \right| \Delta s$$

$$\leq \overline{B}\bar{c}R^2\overline{\eta}T = R_2.$$

such that,

$$\|E(x,y)\|_\infty = \|E_1(x,y)\|_\infty + \|E_2(x,y)\|_\infty \leq R_1 + R_2$$

therefore,based on Definition 1.10 E is uniformly bounded.

Step 4: We prove that $E(K_R)$ is equicontinuous
By Theorem 1.8 and Proposition 1.1 We use the derivative to prove it,

$$E(x,y)^\Delta(t) = E_1(x,y)^\Delta(t) + E_2(x,y)^\Delta(t)$$

We use Leibniz integral rule on time scales to compute the derivative for $E_1(x,y)(t)$ Theorem 1.13

$$E_1(x,y)(t) = \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)} \Big[a(s)\,x(s)\big(1-g(x(s))\big) + f(s,x(s),y(s))\,x(s)\,y(s)\Big]\Delta s.$$

Let us set,

$$E_1(x,y)(t) = \int_{v(t)}^{h(t)} F(s,t)\,\Delta s,$$

therefore $\quad v(t)=t, \quad h(t)=t+T, \quad v^\Delta(t)=1, \quad h^\Delta(t)=1$

$$F(s,t) = \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)} \Big[a(s)\,x(s)\big(1-g(x(s))\big) + f(s,x(s),y(s))\,x(s)\,y(s)\Big],$$

$$E_1(x,y)^\Delta(t) = h^\Delta(t)F(h(t),\sigma(t)) - v^\Delta(t)F(v(t),\sigma(t)) + \int_t^{t+T} \frac{\Delta F}{\Delta t}(s,t)\,\Delta s$$

$$E_1(x,y)^\Delta(t) = F(h(t),\sigma(t)) - F(v(t),\sigma(t)) + \int_t^{t+T} \frac{\Delta F}{\Delta t}(s,t)\,\Delta s$$

$$F(h(t),\sigma(t)) = \frac{e_{\ominus a}(\sigma(t+T),\sigma(t))}{1-e_{\ominus a}(T,0)} \Big[a(t+T)\,x(t+T)\big(1-g(x(t+T))\big)$$

$$+f(t+T,x(t+T),y(t+T))\,x(t+T)\,y(t+T)\Big],$$

since periodic and $e_{\ominus a}(\sigma(t+T),\sigma(t)) = e_{\ominus a}(\sigma(t)+T,\sigma(t)) = e_{\ominus a}(T,0)$

$$F(h(t),\sigma(t)) = \frac{e_{\ominus a}(T,0)}{1-e_{\ominus a}(T,0)} \Big[a(t)\,x(t)\big(1-g(x(t))\big) + f(t,x(t),y(t))\,x(t)\,y(t)\Big]$$

$$F(v(t),\sigma(t)) = \frac{e_{\ominus a}(\sigma(t),\sigma(t))}{1-e_{\ominus a}(T,0)}\Big[a(t)\,x(t)\big(1-g(x(t))\big)+f(t,x(t),y(t))\,x(t)\,y(t)\Big],$$

since $e_{\ominus a}(\sigma(t),\sigma(t)) = 1$

$$F(v(t),\sigma(t)) = \frac{1}{1-e_{\ominus a}(T,0)}\Big[a(t)\,x(t)\big(1-g(x(t))\big)+f(t,x(t),y(t))\,x(t)\,y(t)\Big]$$

$$\int_t^{t+T}\frac{\Delta F}{\Delta t}(s,t)\,\Delta s = \int_t^{t+T}\frac{1}{1-e_{\ominus a}(T,0)}\Big[a(s)\,x(s)\big(1-g(x(s))\big)+f(s,x(s),y(s))\,x(s)\,y(s)\Big]$$

$$[e_{\ominus a}(\sigma(s),\sigma(t))]^{\Delta}(t)\Delta s = a(t)\int_t^{t+T}\frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)}\big[a(s)x(s)\big(1-g(x(s))\big)+f\big(s,x(s),y(s)\big)x(s)y(s)\big]\Delta s$$

$$= a(t)E_1(x,y)(t).$$

We obtain,

$$E_1(x,y)^{\Delta}(t) = \frac{e_{\ominus a}(T,0)}{1-e_{\ominus a}(T,0)}\Big[a(t)\,x(t)\big(1-g(x(t))\big)+f(t,x(t),y(t))\,x(t)\,y(t)\Big]$$

$$-\frac{1}{1-e_{\ominus a}(T,0)}\Big[a(t)\,x(t)\big(1-g(x(t))\big)+f(t,x(t),y(t))\,x(t)\,y(t)\Big]+a(t)E_1(x,y)(t)$$

$$E_1(x,y)^{\Delta}(t) = -\Big[a(t)\,x(t)\big(1-g(x(t))\big)+f(t,x(t),y(t))\,x(t)\,y(t)\Big]+a(t)E_1(x,y)(t).$$

We use Leibniz integral rule on time scales to compute the derivative for $E_2(x,y)(t)$ Theorem 1.13

$$E_2(x,y)(t) = \int_t^{t+T}\frac{e_b(s,t)}{e_b(T,0)-1}\,c(s)f\big(s,x(s),y(s)\big)x(s)y(s)\Delta s.$$

Let us set,

$$E_2(x,y)(t) = \int_{v(t)}^{h(t)} F(s,t)\,\Delta s \qquad ,$$

therefore, $\quad v(t)=t, \quad h(t)=t+T, \quad v^{\Delta}(t)=1, \qquad h^{\Delta}(t)=1,$

$$F(s,t) = \frac{e_b(s,t)}{e_b(T,0)-1}\,c(s)f\big(s,x(s),y(s)\big)x(s)y(s),$$

$$E_2(x,y)^{\Delta}(t) = h^{\Delta}(t)F(h(t),\sigma(t)) - v^{\Delta}(t)F(v(t),\sigma(t)) + \int_t^{t+T}\frac{\Delta F}{\Delta t}(s,t)\,\Delta s.$$

Therefore

$$E_2(x,y)^{\Delta}(t) = F(h(t),\sigma(t)) - F(v(t),\sigma(t)) + \int_t^{t+T}\frac{\Delta F}{\Delta t}(s,t)\,\Delta s.$$

$$F(h(t),\sigma(t)) = \frac{e_b(t+T,\sigma(t))}{e_b(T,0)-1}\, c(t+T) f\big(t+T, x(t+T), y(t+T)\big) x(t+T) y(t+T),$$

since periodic and $e_b(t+T,\sigma(t)) = e_b(T,0)$

$$F(h(t),\sigma(t)) = \frac{e_b(T,0)}{e_b(T,0)-1}\, c(t) f\big(t, x(t), y(t)\big) x(t) y(t)$$

$$F(v(t),\sigma(t)) = \frac{e_b(t,\sigma(t))}{e_b(T,0)-1} c(t) f\big(t, x(t), y(t)\big) x(t) y(t),$$

since $e_b(t,\sigma(t)) = 1$

$$F(v(t),\sigma(t)) = \frac{1}{e_b(T,0)-1} c(t) f\big(t, x(t), y(t)\big) x(t) y(t)$$

$$\begin{aligned}\int_t^{t+T} \frac{\Delta F}{\Delta t}(s,t) &= \int_t^{t+T} \frac{1}{e_b(T,0)-1}\, c(s) f\big(s, x(s), y(s)\big) x(s) y(s) [e_b(s,t)]^{\Delta}(t) \Delta s \\ &= -b(t) \int_t^{t+T} \frac{e_b(s,\sigma(t))}{e_b(T,0)-1}\, c(s) f\big(s, x(s), y(s)\big) x(s) y(s) \Delta s \\ &= -b(t) E_2^{\sigma}(x,y)(t).\end{aligned}$$

We obtain,

$$E_2(x,y)^{\Delta}(t) = \frac{e_b(T,0)}{e_b(T,0)-1}\, c(t) f\big(t, x(t), y(t)\big) x(t) y(t) - \frac{1}{e_b(T,0)-1} c(t) f\big(t, x(t), y(t)\big) x(t) y(t)$$
$$-b(t) E_2^{\sigma}(x,y)(t)$$

$$E_2(x,y)^{\Delta}(t) = c(t) f\big(t, x(t), y(t)\big) x(t) y(t) - b(t) E_2^{\sigma}(x,y)(t)$$

Finally, we obtain,

$$E_1(x,y)^{\Delta}(t) = -\Big[a(t)\, x(t)\big(1 - g(x(t))\big) + f(t, x(t), y(t))\, x(t)\, y(t)\Big] + a(t) E_1(x,y)(t)$$

$$\begin{aligned}\big|E_1(x,y)^{\Delta}(t)\big| &= \big|-\big[a(t)x(t)\big(1-g(x(t))\big) + f\big(t,x(t),y(t)\big)x(t)y(t)\big] + a(t)E_1(x,y)(t)\big| \\ &\le \overline{a}R(1-g(R)) + R^2\overline{\eta} + \overline{a}R_1 = D_1\end{aligned}$$

$$E_2(x,y)^{\Delta}(t) = c(t) f\big(t, x(t), y(t)\big) x(t) y(t) - b(t) E_2(x,y)^{\sigma}(t)$$

$$\begin{aligned}\big|E_2(x,y)^{\Delta}(t)\big| &= \big|c(t) f\big(t, x(t), y(t)\big) x(t) y(t) - b(t) E_2(x,y)^{\sigma}(t)\big| \\ &\le \overline{c}R^2\overline{\eta} + \overline{b}R_2 = D_2,\end{aligned}$$

therefore

$$\big|E(x,y)^{\Delta}(t)\big| \le D_1 + D_2 = D.$$

Then based on Proposition 1.1 $E(K_R)$ is equicontinuous. Hence, by the Arzelà–Ascoli

theorem, $E(K_R)$ is relatively compact. So, E is a compact operator. Therefore,

$$E : K_R \to K$$

is continuous and compact operator. ■ □

2.3 Main Existence Results Via Krasnoselskii's type Homotopy Fixed Point Theorem

Theorem 2.1. *Assume that* (H1) *and* (H2) *hold. Let* $r, R \in \mathbb{R}$, $0 < r < R$. *If*

$$\overline{A}\overline{a}\big(1 - g(r)\big)T \leq 1, \tag{2.14}$$

$$\overline{A}\overline{a}\big(1 - g(r)\big)T + Mr \int_0^T \eta(s, r, r)\,\Delta s \leq 2, \tag{2.15}$$

and

$$mq_1q_2R \int_0^T \phi(s, R)\,\Delta s \geq 2, \tag{2.16}$$

where m *and* M *are given by* (2.2)*, then the system* (2.1) *has a* T*-periodic solution* $(x, y) \in K$ *such that*

$$r \leq \|(x, y)\| \leq R.$$

Proof. We start by proving condition (C1) of Theorem 1.3, which in our case is as follows

$$\big(E_1(x, y), E_2(x, y)\big) \neq \lambda(x, y) \quad \text{for every } (x, y) \in K,\ \|(x, y)\| = r \text{ and all } \lambda > 1. \tag{2.17}$$

To this aim we consider three cases.

a) Assume $x = 0$ and $y \neq 0$, then $E_1(x, y) = E_2(x, y) = 0$, so (2.17) trivially holds.

b) If $x \neq 0$ and $y = 0$, then $E_2(x, y) = 0$ and (2.17) becomes

$$E_1(x, 0) \neq \lambda x \quad \text{for all } (x, 0) \in K,\ \|x\|_\infty = r \text{ and all } \lambda > 1. \tag{2.18}$$

To show this, assume the contrary, namely there exists $(x, 0) \in K$, $\|x\|_\infty = r$ and $\lambda > 1$ such that

$$E_1(x, 0)(t) = \lambda x(t) \quad \text{for every } t \in [0, T].$$

Let $t_0 \in [0, T]$ be such that $\|x\|_\infty = x(t_0) = r > 0$. Then, by using the property that $-g$ is increasing, we get

$$r = x(t_0) < \lambda x(t_0) = \int_{t_0}^{t_0+T} \frac{e_{\ominus a}(\sigma(s), t_0)}{1 - e_{\ominus a}(T, 0)}\, a(s)x(s)\big(1 - g(x(s))\big)\,\Delta s \leq \overline{A}\overline{a}r\big(1 - g(r)\big)T.$$

Dividing by r yields $\overline{A}\overline{a}\big(1-g(r)\big)T>1$, which contradicts condition (2.14). So, condition (2.18) holds.

c) For $x\neq 0$ and $y\neq 0$, if (2.17) does not hold, then there exists a pair $(x,y)\in K$, $\|(x,y)\|=r$ and $\lambda>1$, with

$$E_1(x,y)(t)=\lambda x(t) \quad \text{and} \quad E_2(x,y)(t)=\lambda y(t) \quad \text{for every } t\in[0,T].$$

Let $t_0\in[0,T]$ be such that $\|x\|_\infty=x(t_0)$. Then, by using condition *(H1)*, we obtain

$$\begin{aligned}\|x\|_\infty=x(t_0)&<\lambda x(t_0)=E_1(x,y)(t_0)\\ &=\int_{t_0}^{t_0+T}\frac{e_{\ominus a}(\sigma(s),t_0)}{1-e_{\ominus a}(T,0)}\left[a(s)x(s)\big(1-g(x(s))\big)+f(s,x(s),y(s))x(s)y(s)\right]\Delta s\\ &\leq \overline{A}\overline{a}\big(1-g(\|x\|_\infty)\big)\|x\|_\infty T+\overline{A}\|x\|_\infty\|y\|_\infty\int_0^T\eta(s,\|x\|_\infty,\|y\|_\infty)\,\Delta s.\end{aligned}$$

Dividing by $\|x\|_\infty$ and using the fact that $\|x\|_\infty,\|y\|_\infty\leq\|(x,y)\|=r$, we get

$$1<\overline{A}\overline{a}\big(1-g(r)\big)T+\overline{A}\|y\|_\infty\int_0^T\eta(s,r,r)\,\Delta s. \tag{2.19}$$

Similarly, from $E_2(x,y)(t)=\lambda y(t)$, we have

$$\begin{aligned}\|y\|_\infty=y(t_0)&<\lambda y(t_0)=E_2(x,y)(t_0)\\ &=\int_{t_0}^{t_0+T}\frac{e_b(s,t_0)}{e_b(T,0)-1}\,c(s)f(s,x(s),y(s))x(s)y(s)\,\Delta s\\ &\leq \overline{B}\overline{c}\,\|x\|_\infty\|y\|_\infty\int_0^T\eta(s,r,r)\,\Delta s,\end{aligned}$$

then

$$1<\overline{B}\overline{c}\,\|x\|_\infty\int_0^T\eta(s,r,r)\,\Delta s. \tag{2.20}$$

By adding (2.19) and (2.20) and by using $\|x\|_\infty+\|y\|_\infty=\|(x,y)\|=R$, we get

$$2<\overline{A}\overline{a}\big(1-g(r)\big)T+\|y\|_\infty\overline{A}\int_0^T\eta(s,r,r)\,\Delta s+\overline{B}\overline{c}\|x\|_\infty\int_0^T\eta(s,r,r)\,\Delta s$$

$$\leq\overline{A}\overline{a}\big(1-g(r)\big)T+M\big(\|x\|_\infty+\|y\|_\infty\big)\int_0^T\eta(s,r,r)\,\Delta s.$$

Thus

$$\overline{A}\overline{a}\big(1-g(r)\big)T+Mr\int_0^T\eta(s,r,r)\,\Delta s>2,$$

which contradicts assumption (2.15). Therefore, (2.17) holds.

Now we prove condition (C2) of Theorem 1.3, which in our case is as follows

$$(x,y) - \big(E_1(x,y), E_2(x,y)\big) \neq \lambda(x_0,y_0) \quad \text{for } (x,y) \in K,\ \|(x,y)\| = R \text{ and all} \qquad \lambda > 0. \tag{2.21}$$

To this aim, we distinguish three cases.

a) If $x = 0$ and $y \neq 0$, then $E_1(x,y) = E_2(x,y) \equiv 0$, so (2.21) trivially holds.

b) If $x \neq 0$ and $y = 0$, then $E_2(x,y) = 0$ and (2.21) is satisfied since $0 < \lambda y_0$.

c) For $x \neq 0$ and $y \neq 0$, if (2.21) does not hold, then there exists a pair $(x,y) \in K$ with $\|(x,y)\| = R$ such that

$$x(t) > E_1(x,y)(t), \qquad y(t) > E_2(x,y)(t) \quad \text{for every } t \in [0,T],$$

where we use that $\lambda, x_0, y_0 > 0$. On the one hand, by using condition *(H2)*, we obtain

$$\begin{aligned}
\|x\|_\infty \geq x(t) &> E_1(x,y)(t) \\
&= \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)} \left[a(s)x(s)\big(1-g(x(s))\big) + f(s,x(s),y(s))x(s)y(s)\right] \Delta s \\
&\geq \int_t^{t+T} \frac{e_{\ominus a}(\sigma(s),t)}{1-e_{\ominus a}(T,0)} f(s,x(s),y(s))x(s)y(s)\,\Delta s \\
&\geq \underline{A} q_1 \|x\|_\infty\, q_2 \|y\|_\infty \int_0^T \phi(t, x(s)+y(s))\,\Delta s \\
&\geq \underline{A} q_1 q_2 \|x\|_\infty \|y\|_\infty \int_0^T \phi(t, \|(x,y)\|)\,\Delta s.
\end{aligned}$$

Dividing by $\|x\|_\infty$ yields

$$1 > \underline{A} q_1 q_2 \|y\|_\infty \int_0^T \phi(t,R)\,\Delta s. \tag{2.22}$$

On the other hand, in a similar way, we get

$$\begin{aligned}
\|y\|_\infty \geq y(t) &> E_2(x,y)(t) \\
&= \int_t^{t+T} \frac{e_b(s,t)}{e_b(T,0)-1} c(s) f(s,x(s),y(s))x(s)y(s)\,\Delta s \\
&\geq \underline{Bc}\, q_1 \|x\|_\infty\, q_2 \|y\|_\infty \int_0^T \phi(t, \|(x,y)\|)\,\Delta s \\
&\geq \underline{Bc}\, q_1 q_2 \|x\|_\infty \|y\|_\infty \int_0^T \phi(t,R)\,\Delta s.
\end{aligned}$$

Dividing by $\|y\|_\infty$ yields

$$1 > \underline{Bc}\, q_1 q_2 \|x\|_\infty \int_0^T \phi(t,R)\,\Delta s. \tag{2.23}$$

By adding (2.23) and (2.22) and by using $\|x\|_\infty + \|y\|_\infty = \|(x, y)\| = R$, we get

$$2 > mq_1q_2R \int_0^T \phi(t, R)\, \Delta s,$$

which contradicts assumption (2.16).

Therefore all the conditions of Theorem 1.7 are satisfied. Hence the operator E has a fixed point $(x, y) \in K$ with

$$r \leq \|(x, y)\| \leq R.$$

This fixed point (x, y) is a T-periodic solution of (2.1). □

Chapter 3

Applications in Population Dynamics and Cybersecurity

In this chapter, we illustrate the generality and applicability of the non-autonomous Lotka–Volterra dynamic system on periodic time scales studied in this thesis. While Chapter 2 established the existence of positive periodic solutions in an abstract framework, the objective here is to demonstrate how these results can be interpreted in concrete models.

We present two applications from different fields: a classical ecological predator–prey model involving fish and sharks, and a modern cybersecurity model describing the interaction between polymorphic malware propagation and malware detection mechanisms.

3.1 Lotka–Volterra as a General Framework

The Lotka–Volterra system can be viewed as a general mathematical structure describing interactions between positive quantities. In this thesis, we consider the non-autonomous Lotka–Volterra dynamic system with a general interaction rate on periodic time scales, following the framework of Bordj and Ardjouni [4]. The system studied in Chapter 2 is recalled as follows:

$$\begin{cases} x^{\Delta}(t) = a(t)x(t)g(x(t)) - f(t, x(t), y(t))x(t)y(t), \\ y^{\Delta}(t) = -b(t)y^{\sigma}(t) + c(t)f(t, x(t), y(t))x(t)y(t), \end{cases} \tag{3.1}$$

where $a(t), b(t), c(t)$ are positive periodic functions, g is decreasing, and f is a general interaction rate.

This formulation unifies continuous, discrete, and hybrid models. Indeed, when $\mathbb{T} = \mathbb{R}$, system (3.1) reduces to a differential system; when $\mathbb{T} = \mathbb{Z}$, it becomes a difference system; and for a general periodic time scale, it describes hybrid dynamics. Thus, the theoretical results obtained in Chapter 2 can be applied to a wide variety of real-world

systems. Moreover, the Lotka–Volterra framework is not restricted to classical ecological predator–prey interactions. Its generality is supported by recent theoretical work of Frezzato [9], who showed that a wide class of autonomous dynamical systems can be embedded into a Lotka–Volterra-like quadratic form. This supports the idea that Lotka–Volterra-type structures can serve as a broad mathematical framework for modelling interacting dynamical systems. In the following sections, we illustrate this flexibility through two applications. The first one is a classical ecological predator–prey model involving fish and sharks, which corresponds to the original biological motivation of the Lotka–Volterra model. The second one is a modern cybersecurity application inspired by the work of Chaikovskyi *et al.* [5], where the Lotka–Volterra model is used to describe the interaction between polymorphic malware propagation and malware detection mechanisms.

3.2 Application in Population Dynamics

3.2.1 Model Description

The ecological predator–prey interpretation is the classical motivation of the Lotka–Volterra model. In this application, we consider the interaction between a prey population and a predator population. More precisely, let:

- $x(t)$: prey population (fish),
- $y(t)$: predator population (sharks),
- $f(t)$: intrinsic growth rate of the prey population,
- $g(t)$:predation rate coefficient,
- $a(t)$:natural death rate of the predator population,
- $b(t)$:conversion rate of consumed prey into predator growth,
- $c(t)$: intra-specific competition coefficient of the predator population.

We consider logistic growth for the prey:

$$g(x) = 1 - \frac{x}{N},$$

where $N > 0$ is the carrying capacity. The coefficients $a(t), b(t), c(t)$ are positive periodic functions reflecting seasonal variations (e.g., temperature, food availability).

3.2.2 Mathematical Formulation

System (3.1) becomes:

$$\begin{cases} x^{\Delta}(t) = a(t)x(t)\left(1 - \dfrac{x(t)}{N}\right) - f(t, x(t), y(t))x(t)y(t), \\ y^{\Delta}(t) = -b(t)y^{\sigma}(t) + c(t)f(t, x(t), y(t))x(t)y(t). \end{cases} \tag{3.2}$$

A common choice for the attack rate is the saturated Holling type II form:

$$f(t, x, y) = \frac{\beta(t)}{1 + \alpha(t)x},$$

where $\alpha(t)$ and $\beta(t)$ are positive T-periodic functions.

We choose a saturated response rather than a linear response because a linear interaction rate assumes that the predator can consume prey without limitation as the prey population increases. This is not realistic in biological systems. In practice, predation is limited by several factors, such as searching time, handling time, limited energy, and the handling capacity of the predator.

For this reason, the saturated response gives a more realistic description of the predator–prey interaction. It reflects the fact that, when the prey population becomes large, the predation rate does not increase indefinitely but approaches a bounded level.

3.2.3 Verification of Hypotheses (H1) and (H2)

Verification of (H1): Since $\dfrac{1}{1 + \alpha(t)x} \leq 1$ for all $x \geq 0$, we have:

$$f(t, x, y) = \frac{\beta(t)}{1 + \alpha(t)x} \leq \beta(t).$$

Choosing $\eta(t, x, y) = \beta(t)$ (which is constant, hence increasing in x and y trivially), we see that (H1) holds.

Verification of (H2): Note that $f(t, x, y) = \dfrac{\beta(t)}{1 + \alpha(t)x}$ is decreasing in x. Since $x \leq x + y$ (because $y \geq 0$) and f is decreasing in its second argument, we obtain:

$$f(t, x, y) = \frac{\beta(t)}{1 + \alpha(t)x} \geq \frac{\beta(t)}{1 + \alpha(t)(x + y)}.$$

Define:

$$\phi(t, z) = \frac{\beta_{\min}}{1 + \alpha_{\max} z},$$

where $\beta_{\min} = \min_{t \in [0,T]} \beta(t)$ and $\alpha_{\max} = \max_{t \in [0,T]} \alpha(t)$. Then clearly $f(t, x, y) \geq \phi(t, x + y)$. Moreover, $\phi(t, z)$ is decreasing in z because the denominator increases with z. Thus

(H2) holds.

Therefore, Theorem 2.1 applies and guarantees the existence of a positive T-periodic solution for appropriate choices of r and R satisfying conditions (2.16)–(2.18) from Chapter 2.

3.2.4 Interpretation

Existence of a positive periodic solution means:

- Both fish and sharks persist over time (no extinction).
- The system exhibits recurrent oscillations: fish increase, sharks increase, fish decrease, sharks decrease, then the cycle repeats.
- Seasonal effects (temperature, migration) are naturally included through the periodic coefficients.

3.3 Application in Cybersecurity

3.3.1 Problem Description

In modern computer networks, cyber threats such as malware, viruses, ransomware, and polymorphic malicious software represent a major challenge for information security. These threats may spread dynamically through networks and continuously evolve in order to avoid detection. On the other hand, malware detection mechanisms, such as antivirus software, intrusion detection systems (IDS), firewalls, and real-time monitoring tools, are developed to limit this propagation and protect digital infrastructures.

This interaction between malware propagation and detection mechanisms can be interpreted by means of a Lotka–Volterra-type system. In this context, the predator–prey terminology is used only as a mathematical analogy: the variables do not represent biological populations, but rather technological quantities describing the evolution of malicious activity and the response of detection tools.

We denote by

- $x(t)$: the level of polymorphic malware activity, for example the density of infected devices or the intensity of malicious propagation,
- $y(t)$: the level of malware detection mechanisms, for example the effectiveness of antivirus tools, intrusion detection systems, firewalls, or network protection mechanisms.

The objective is to study whether these two quantities may coexist in a recurrent periodic regime under operational variations, such as periodic software updates, changing attack strategies, scheduled security scans, or regular improvements of detection tools.

This application is motivated by the work of Chaikovskyi *et al.* [5], who used a Lotka–Volterra predator–prey model to describe the detection process of polymorphic malware. In their interpretation, the malware level and the available detection methods play the role of two interacting quantities.

Additional recent works further support the relevance of the Lotka–Volterra framework in cybersecurity. In particular, Atıcı and Tuna [1, 2] developed deterministic and stochastic Lotka–Volterra models to analyze the dynamics among vulnerabilities, cyberattacks, and security hardening. In these models, vulnerabilities are interpreted as prey-like components, cyberattacks as predator-like components, and security hardening as a regulatory defensive component.

The fact that independent research groups have successfully applied Lotka–Volterra models to different cybersecurity scenarios [5, 1, 2] supports the relevance of the mathematical framework considered here. It shows that Lotka–Volterra-type models are not merely theoretical constructions, but can also provide useful tools for analyzing real-world cyber dynamics.

Following these ideas, we consider a more general non-autonomous framework on periodic time scales. This allows us to treat continuous systems ($\mathbb{T} = \mathbb{R}$), discrete systems ($\mathbb{T} = \mathbb{Z}$), and hybrid models within the same theory.

3.3.2 Mathematical Model

We assume that malware activity follows a logistic self-regulation law. This means that malware propagation cannot grow indefinitely because of limited system capacity, restricted resources, or partial user awareness. Thus, we choose

$$g(x) = 1 - \frac{x}{N}, \qquad N > 0,$$

where N represents the maximal infection capacity of the network.

The coefficient functions $a(t), b(t), c(t)$ are assumed to be positive and T-periodic, where:

- $a(t)$: intrinsic malware growth rate;
- $b(t)$: natural decay rate of the effectiveness of malware detection mechanisms;
- $c(t)$: efficiency of converting malware pressure into detection response.

To describe the interaction between malware propagation and detection mechanisms,

we introduce a nonlinear saturated interaction rate:

$$f(t,x,y)=\frac{\lambda(t)+\alpha(t)y}{1+\beta(t)\big(\lambda(t)+\alpha(t)y\big)x},$$

where:

- $\lambda(t)$: the basic malware detection rate;
- $\alpha(t)y$: the improvement of detection due to stronger detection mechanisms;
- $\beta(t)$: the saturation parameter.

We choose a saturated interaction rate rather than a linear one because a linear response assumes that detection mechanisms can react to malware activity without limitation as the malware level increases. This is not realistic in cybersecurity systems. In practice, malware detection is limited by several factors, such as computational resources, response time, processing capacity, and the complexity of polymorphic malware.

For this reason, the saturated response is more appropriate in this context. It reflects the fact that, when malware becomes very abundant, the efficiency of detection does not continue to increase indefinitely. Instead, detection mechanisms may become saturated because of limited resources, delayed response, or the adaptive nature of polymorphic malware.

Therefore, the cybersecurity model is given by

$$\begin{cases} x^{\Delta}(t)=a(t)x(t)\left(1-\dfrac{x(t)}{N}\right)-f(t,x(t),y(t))x(t)y(t), \\ \\ y^{\Delta}(t)=-b(t)y^{\sigma}(t)+c(t)f(t,x(t),y(t))x(t)y(t). \end{cases} \tag{3.3}$$

This system fits the general framework studied in Chapter 2.

3.3.3 Verification of Hypotheses (H1) and (H2)

Verification of (H1)

We first verify condition $(H1)$ of Theorem 2.1. Since

$$\frac{1}{1+\beta(t)(\lambda(t)+\alpha(t)y)x}\leq 1 \quad \text{for all } x,y\geq 0,$$

we immediately obtain

$$f(t,x,y)=\frac{\lambda(t)+\alpha(t)y}{1+\beta(t)(\lambda(t)+\alpha(t)y)x}\leq \lambda(t)+\alpha(t)y.$$

Define

$$\eta(t, x, y) = \lambda(t) + \alpha(t)y.$$

Clearly, η is increasing with respect to y, independent of x and hence nondecreasing with respect to x. Moreover, since $\lambda(t)$ and $\alpha(t)$ are positive T-periodic functions, η is also T-periodic in t. Thus, condition $(H1)$ is satisfied.

Verification of (H2)

We now verify condition $(H2)$. Recall that

$$f(t, x, y) = \frac{\lambda(t) + \alpha(t)y}{1 + \beta(t)(\lambda(t) + \alpha(t)y)x}.$$

Let

$$\lambda_{\min} = \min_{t\in[0,T]} \lambda(t), \qquad \lambda_{\max} = \max_{t\in[0,T]} \lambda(t),$$

$$\alpha_{\max} = \max_{t\in[0,T]} \alpha(t), \qquad \beta_{\max} = \max_{t\in[0,T]} \beta(t).$$

Since $\lambda(t) > 0$, we have

$$\lambda(t) + \alpha(t)y \geq \lambda_{\min}.$$

Moreover, for all $x, y \geq 0$, we have $x \leq x + y$ and $y \leq x + y$. Hence,

$$\lambda(t) + \alpha(t)y \leq \lambda_{\max} + \alpha_{\max}(x + y).$$

It follows that

$$1 + \beta(t)(\lambda(t) + \alpha(t)y)x \leq 1 + \beta_{\max}\left(\lambda_{\max} + \alpha_{\max}(x + y)\right)(x + y).$$

Therefore,

$$f(t, x, y) \geq \frac{\lambda_{\min}}{1 + \beta_{\max}\left(\lambda_{\max} + \alpha_{\max}(x + y)\right)(x + y)}.$$

Define

$$\phi(z) = \frac{\lambda_{\min}}{1 + \beta_{\max}\left(\lambda_{\max} + \alpha_{\max}z\right)z}, \qquad z \geq 0.$$

Then ϕ is positive and decreasing with respect to z. Moreover,

$$f(t, x, y) \geq \phi(x + y) \quad \text{for all } x, y \geq 0.$$

Thus, condition $(H2)$ is satisfied.

Therefore, provided that the constants $r > 0$ and $R > 0$ satisfy the conditions of Theorem 2.1, this theorem guarantees the existence of at least one positive T-periodic solution for the cybersecurity model (3.3).

3.3.4 Interpretation

The existence of a positive periodic solution has an important practical meaning within the model. It implies that:

- the malware level remains positive and does not vanish;
- malware detection mechanisms remain active over time;
- both quantities coexist in a balanced recurrent regime.

This reflects a realistic cybersecurity situation in which new malware variants may appear repeatedly, while detection systems are continuously updated and improved.

The periodic nature of the solution describes recurrent cycles such as

$$\begin{aligned} \text{Malware increase} &\to \text{Detection strengthening} \\ &\to \text{Malware decrease} \\ &\to \text{Detection relaxation} \\ &\to \text{New malware emergence.} \end{aligned}$$

Moreover, the time scales framework allows the same model to be applied to:

- continuous monitoring systems ($\mathbb{T} = \mathbb{R}$);
- discrete security reports ($\mathbb{T} = \mathbb{Z}$);
- hybrid environments combining both.

This illustrates that the Lotka–Volterra framework can also be interpreted as a rigorous mathematical tool for studying modern cybersecurity dynamics.

The time scales framework provides a significant advantage for the proposed cybersecurity model, since cyber processes may combine continuous and discrete behaviours. By choosing $\mathbb{T} = \mathbb{R}$, the system describes a continuous monitoring environment in which detection mechanisms respond continuously to malware activity. On the other hand, choosing $\mathbb{T} = \mathbb{Z}$ corresponds to discrete-time security processes, such as daily antivirus updates, weekly vulnerability scans, or periodic security reports.

More generally, a periodic time scale allows us to describe hybrid situations where continuous network monitoring coexists with defensive actions performed at specific time instants, such as signature database updates, patch installation, or firewall rule modifications. Therefore, the same mathematical model can represent continuous, discrete, and hybrid cybersecurity dynamics without requiring separate analyses for each case. This shows that the use of time scales is not only a theoretical generalization, but also a suitable tool for reflecting the recurrent and mixed nature of cybersecurity processes.

3.4 Special Cases

The general system (2.1) contains several important special cases. We begin with the simplest case: a constant (or time-dependent but population-independent) interaction rate.

Constant interaction rate

Assume that the attack rate does not depend on the populations, i.e., $f(t, x, y) = \beta(t)$, where $\beta : \mathbb{T} \to \mathbb{R}_+$ is T-periodic and rd-continuous. Define

$$\beta_{\min} = \min_{t\in[0,T]} \beta(t), \qquad \overline{\beta} = \max_{t\in[0,T]} \beta(t).$$

Then the following hold:

- Condition (H1) is satisfied with $\eta(t, x, y) = \beta(t)$, which is increasing in x, y (trivially) and T-periodic.
- Condition (H2) is satisfied with $\phi(t, z) = \beta_{\min}$, which is decreasing in z (constant) and T-periodic.

Substituting these into the general conditions (2.16)–(2.18) yields:

$$\overline{A}\overline{a}\big(1 - g(r)\big)T \;\le\; 1,$$

$$\overline{A}\overline{a}\big(1 - g(r)\big)T \;+\; Mr\int_0^T \beta(s)\,\Delta s \;\le\; 2,$$

$$m\,q_1 q_2\,R\,\beta_{\min}\,T \;\ge\; 2,$$

where the constants $A, \bar{a}, M, m, q_1, q_2$ are defined in Chapter 2 (see equations (2.4)).

These inequalities are satisfied whenever $r > 0$ is chosen sufficiently small (so that $1 - g(r)$ is small) and $R > 0$ is chosen sufficiently large. Consequently, under these conditions, system (3.3) with $f(t, x, y) = \beta(t)$ admits at least one positive T-periodic solution $(x, y) \in K$ such that

$$r \le \|(x, y)\| \le R.$$

Remark. The constant interaction rate case includes the classical autonomous Lotka–Volterra model as a special subcase when $\beta(t) \equiv \beta$ (constant) and $\mathbb{T} = \mathbb{R}$. Our result provides a rigorous existence proof for positive periodic solutions in this setting, even when the coefficients $a(t), b(t), c(t)$ remain periodic.

3.5 Other Special Cases

Beyond Corollary 1.1, other important special cases follow directly from Theorem 1.4. These include:

- **Time-independent interaction**: If f does not depend explicitly on t, the system remains non-autonomous through $a(t), b(t), c(t)$. The verification of (H1) and (H2) simplifies because the periodic dependence of f disappears, but the inequalities (2.16)–(2.18) remain essentially the same.
- **Linear interaction**: Choose $f(t, x, y) = \lambda(t) + \alpha(t)y$. Here (H1) holds with $\eta = \lambda + \alpha y$, and (H2) holds with $\phi(t, z) = \lambda_{\min}$ (a constant, hence decreasing in z). The existence conditions become explicit smallness conditions on r and largeness conditions on R.
- **Nonlinear saturated interaction**: The form $f(t, x, y) = \dfrac{\lambda + \alpha y}{1 + \beta(\lambda + \alpha y)x}$ used in the cybersecurity application is the most general example. The existence conditions (2.16)–(2.18) reflect the saturation effect: R must be large enough to overcome the denominator.

In each case, Theorem 2.1 guarantees the existence of positive periodic solutions for appropriate choices of r and R satisfying the corresponding inequalities.

Conclusion

In this thesis, we investigated a non-autonomous Lotka–Volterra dynamic system with a general attack rate on periodic time scales. By applying a Krasnoselskii-type homotopy fixed point theorem, we established the existence of at least one positive periodic solution under suitable assumptions. The framework of time scales provides a unified approach that allows the study of both continuous and discrete dynamical systems within the same mathematical setting.

The proof was carried out by transforming the considered dynamic system into an equivalent integral system and constructing an appropriate integral operator on a cone of periodic functions. The assumptions imposed in the main theorem guarantee that this operator satisfies the conditions required by the fixed point theorem, ensuring the existence of a positive periodic solution.

To demonstrate the applicability of the theoretical results, two applications were presented in Chapter 3. The first application concerns an ecological predator–prey model, whereas the second deals with a cybersecurity model describing the interaction between polymorphic malware propagation and cyber-defense mechanisms. In both cases, the models were formulated within the general framework of the studied system, and the hypotheses of the main existence theorem were verified.

From an applied perspective, the existence of a positive periodic solution indicates that the interacting quantities remain positive and evolve in a recurrent manner over time. This interpretation is consistent with the nature of the variables involved in the considered applications, where the unknown functions represent positive quantities such as population densities or levels of cyber activity.

An important feature of the obtained results is that they remain valid for both continuous and discrete time models. Indeed, since the system is studied on a T-periodic time scale, several particular cases can be derived naturally. If $\mathbb{T} = \mathbb{R}$, the studied system reduces to the differential Lotka–Volterra system

$$\begin{cases} x'(t) = a(t)x(t)g(x(t)) - f(t, x(t), y(t))x(t)y(t), \\ y'(t) = -b(t)y(t) + c(t)f(t, x(t), y(t))x(t)y(t), \end{cases} \tag{3.4}$$

and therefore this system admits a positive T-periodic solution. In this case, Theorem 2.1

extends Theorem 3.1 in [13]. On the other hand, if $\mathbb{T} = \mathbb{Z}$, the system becomes the difference system

$$\begin{cases} \Delta x(t) = a(t)x(t)g(x(t)) - f(t, x(t), y(t))x(t)y(t), \\ \Delta y(t) = -b(t)y(t+1) + c(t)f(t, x(t), y(t))x(t)y(t), \end{cases} \tag{3.5}$$

where $\Delta x(t) = x(t+1) - x(t)$. Consequently, the obtained theorem also guarantees the existence of a positive T-periodic solution for the discrete Lotka–Volterra system, which provides a new result for system (3.5).

Therefore, the theory developed in this work demonstrates the effectiveness of the time scales framework in unifying continuous and discrete models and in extending existence results to a broader class of dynamical systems.

Several directions may be explored in future research. These include developing numerical simulations for particular parameter values and extending the model to higher-dimensional systems or systems involving delays. Numerical simulations could also provide useful comparisons between the continuous and discrete cases and help illustrate the dynamical behaviour of the solutions.

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