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Spectral Radius Inequalities and applications

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الإهـــــــــــــداء

إلى النور الذي ينير لي درب النجــاح ... أبي .

ويا من علمتنـــي الصمود مهما تبدلت الظروف...أمي .

والى من كانوا يضيئون لي الطريق ويساندوني ويتنازلون عن حقوقهم لإرضـــائي إخوتـــــي.

إلى زملائـــي وزميلاتـــــي .

اهـــدي هدا العمـــل ...

فطـحيزة علي آمـــال

الإهـــــــــــداء

إلى مثال التفاني والإخـــلاص..... أبى الحبيـــــــب.

إلى من قدَّمت سعادتي وراحتي على سعادتها... أمي الفـاضــــلة.

إلى من لم يبخل بمساعدتي يوم ما..... زوجــي العـــزيــز.

إلى الإخوة والأخوات والى كل العائلة.

إلى جميع الأصدقاء.

إلى كل من دعا لي بالخير.

أهديكم هذا العمل المتواضع...

خلايفة ثورية

Contents

Notations

$\mathbb{N}$: The set of natural numbers $\{1, 2, 3...\}$.

$\mathbb{R}$: The set of real numbers.

$\mathbb{R}_+$: The set of positive real numbers.

$\mathbb{C}$: The set of complex numbers.

$\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$.

$\mathcal{H}$: Complex Hilbert space.

$\langle ., . \rangle$: The inner product of $\mathcal{H}$.

$\bigoplus$: The sign of direct sum.

$\mathcal{L}(\mathcal{H})$: Banach algebra of all bounded linear operators on Hilbert space $\mathcal{H}$.

$\mathcal{I}(\mathcal{H})$: The set of invertible operators in $\mathcal{L}(\mathcal{H})$.

A : A bounded linear operator defined on $\mathcal{H}(A \in \mathcal{L}(\mathcal{H}))$.

$\|A\|$: The norm of A.

A^{-1}	:	The inverse operator of A.
A^*	:	The adjoint operator of A.
$\Re(A)$	:	The real part of A.
$\Im(A)$	:	The imaginary part of A.
$\mid A \mid$	:	The absolute value of A.
$R(A)$	:	The range of A.
$N(A)$	:	The kernel of A.
$\sigma(A)$	:	The spectrum of A.
$\sigma_r(A)$	:	The residual spectrum of A.
$\sigma_p(A)$	:	The point spectrum of A.
$\sigma_c(A)$	:	The continuous spectrum of A.
$\sigma_{ap}(A)$	:	The approximate point spectrum of A.
$\rho(A)$	:	The resolvent of A.
$r(A)$	:	Spectral radius of A.
$W(A)$	:	The numerical range of A.
$w(A)$	:	The numerical radius of A.

Introduction

From the study of eigenvalues and matrices, the spectral theory of operators developed into a comprehensive and profound theory that includes many kinds of operators in functional analysis. It affects our knowledge of stability, dynamics, and the fundamental ideas of quantum mechanics, among other areas of mathematics and science.

Let A be a linear bounded operator on a Hilbert Space. The spectral radius $r(A)$ is the supremum of the absolute values of the elements in the spectrum of A. The spectral radius of an operator is a key concept in functional analysis and operator theory, crucial for understanding linear transformations.

The spectral radius inequalities play a very important role in operator theory, and several articles have been made in this area as in [13, 4], and others.

We concern this work to study the spectral radius definitions and properties, also some inequalities of spectral radius and their relation with the numerical radius and norm. Finally, we see some applications, some example of calculate the spectral radius of certain operators and Matlab program to calculate the spectral radius of certain matrix.

This work is devided on three chapter:
Chapter One, which consists of four sections, highlights the definitions and basic properties of operators and matrices in Hilbert space, which are useful throughout this thesis.

The second chapter, which consists of three sections, deals with spectral radius of linear bounded. Where, we introduces definition and properties, and deals with relation between spectral radius and numerical radius.

The third chapter discusses some applications with exercises and their solutions.

Chapter 1

Preliminaries notions

1.1 Inner product and Hilbert spaces

Definition 1.1.1 *Let X be a vector space over $\mathbb{K}$. X is said to be a normed space if there exists a map $\parallel . \parallel$ that is defined form X to $\mathbb{R}_+$ satisfies the following*

1. $\parallel x \parallel = 0 \Leftrightarrow x = 0$.

2. $\parallel \lambda x \parallel = \mid \lambda \mid \parallel x \parallel \quad \forall x \in X$ *and* $\lambda \in \mathbb{K}$.

3. $\parallel x + y \parallel \leqslant \parallel x \parallel + \parallel y \parallel \quad x, y \in \mathbb{R}$.

   And we denote $(X, \parallel . \parallel)$ *is a normed space.*

Definition 1.1.2 *Let $(X, \parallel . \parallel)$ be a normal space. Then $(X, \parallel . \parallel)$ is said to be a Banach space if every Cauchy sequence has limit in X i.e.*
If $\parallel x_n - x_m \parallel \to 0$ as $n \to \infty$ and $m \to \infty$, there exists $x \in X$ such that $\parallel x_n - x \parallel \to 0$ as $n \to \infty$.

Definition 1.1.3 *Let X be a vector space over $\mathbb{K}$, X is said to be an inner product space, if there exists a map $\langle ., . \rangle : X \times X \to \mathbb{K}$ holds the following*

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \forall x, y, z \in X$.

2. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \forall x, y \in X$ *and* $\lambda \in \mathbb{K}$.

3. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in X$.

4. $\langle x, x\rangle > 0 \quad x \in X \setminus \{0\}$.

Remarks 1.1.1 *It comes directly from the definition of inner product that*

1. $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle \quad \forall x, y, z \in X$.

2. $\langle x, \lambda z\rangle = \overline{\lambda}\langle x, y\rangle \quad \forall x, y \in X$ *and* $\lambda \in \mathbb{K}$.

Theorem 1.1.1 *(Cauchy-Schwarz inequality)*
Let $(X, \langle ., .\rangle)$ *be an inner product space. Then we have*

$$\mid \langle x, y\rangle \mid \leqslant \mid \langle x, x\rangle \mid^{\frac{1}{2}} \mid \langle y, y\rangle \mid^{\frac{1}{2}} \quad \forall x, y \in X.$$

Moreover, if $\mid \langle x, y\rangle \mid = \mid \langle x, x\rangle \mid^{\frac{1}{2}} \mid \langle y, y\rangle \mid^{\frac{1}{2}}$, *then* x *and* y *are linearly dependent.*

Remark 1.1.1 *We can write Cauchy-Schwarz inequality by*

$$\mid \langle x, x\rangle \mid \leqslant \parallel x \parallel \parallel y \parallel \quad \forall x, y \in X.$$

Lemma 1.1.1 *Let* $(X, \langle ., .\rangle)$ *be a inner product space and suppose that* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are convergent sequences in* X, *with* $\lim\limits_{n\to\infty} x_n = x$ *and* $\lim\limits_{n\to\infty} y_n = y$.
Then $\lim\limits_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle$.

Theorem 1.1.2 *(Buzano's inequality)*
Let $(X, \langle ., .\rangle)$ *be an inner product space and* $x, y, e \in X$ *with* $\parallel e \parallel = 1$. *Then we have*

$$\mid \langle x, e\rangle\langle e, y\rangle \mid \leqslant \frac{1}{2}(\parallel x \parallel \parallel y \parallel + \mid \langle x, y\rangle \mid).$$

Proof. Let x, y, e be as in the theorem

$$\begin{aligned} 2 \mid \langle x, e\rangle\langle e, y\rangle \mid - \mid \langle x, y\rangle \mid &= \mid \langle x, 2\overline{\langle e, y\rangle} e\rangle \mid - \mid \langle x, y\rangle \mid \\ &\leqslant \mid \langle x, 2\overline{\langle e, y\rangle} e\rangle - \langle x, y\rangle \mid \\ &= \mid \langle x, 2\overline{\langle e, y\rangle} e - y\rangle \mid \\ &\leqslant \parallel x \parallel \parallel 2\overline{\langle e, y\rangle} e - y \parallel . \end{aligned}$$

So, we have

$$2 \mid \langle x, e\rangle\langle e, y\rangle \mid - \mid \langle x, y\rangle \mid \leqslant \parallel x \parallel \parallel 2\overline{\langle e, y\rangle} e - y \parallel .$$

Let us calculate $\| 2\overline{\langle e,y\rangle}e - y \|$

$$\begin{aligned}
\| 2\overline{\langle e,y\rangle}e - y \|^2 &= \langle 2\overline{\langle e,y\rangle}e - y, 2\overline{\langle e,y\rangle}e - y\rangle \\
&= 4 \mid \langle e,y\rangle \mid^2 \| e \|^2 + \| y \|^2 - 2\langle e,y\rangle\overline{\langle e,y\rangle} - 2\langle e,y\rangle\langle y,e\rangle \\
&= 4 \mid \langle e,y\rangle \mid^2 + \| y \|^2 - 4 \mid \langle e,y\rangle \mid^2 \\
&= \| y \|^2 .
\end{aligned}$$

So,

$$\| 2\overline{\langle e,y\rangle}e - y \| = \| y \|$$

and

$$2 \mid \langle x,e\rangle\langle e,y\rangle \mid - \mid \langle x,y\rangle \mid \leqslant \| x \| \| y \|$$

.

Hence

$$\mid \langle x,e\rangle\langle e,y\rangle \mid \leqslant \frac{1}{2}(\| x \| \| y \| + \mid \langle x,y\rangle \mid).$$

$\square$

Remark 1.1.2 *1. Let $x, y, z \in X$. From Buzano's inequality we obtain*

$$\mid \langle x,y\rangle\langle y,z\rangle \mid \leqslant \frac{1}{2}(\| x \| \| z \| + \mid \langle x,z\rangle \mid) \| y \|^2 . \tag{1.1.1}$$

2. Buzano's inequality is a generalization of Cauchy-Schwarz inequality , just put $z = x$ in equation (1.1.1) to obtain Cauchy-Schwarz inequality.

Proposition 1.1.1 *Let $(x, \langle.,.\rangle)$ be an inner product space. Then we have*

1. $\forall x, y \in X : \| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)$ *(parallelogram law).*
2. *If X is a real vector space, then* $\forall x, y \in X : \langle x,y\rangle = \frac{1}{4}\{\| x + y \|^2 - \| x - y \|^2\}$.
3. *If X is a complex vector space, then*
   $\forall x, y \in X : \langle x,y\rangle = \frac{1}{4}\{\| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2\}$
   (polarization identity).
   Where $\|x\| = \mid \langle x,x\rangle \mid^{\frac{1}{2}}$ *and i is the imaginary unit of complex numbers* $(i^2 = -1)$.

Definition 1.1.4 *Let $\mathcal{H}$ be vector space over $\mathbb{C}$. $\mathcal{H}$ is said to be a Hilbert space if it is an inner product space and H with associated norm is a Banach space, and we denote $(x, \langle ., . \rangle)$ is a complex Hilbert space.*

1.2 Definitions and basic properties in $\mathcal{L}(\mathcal{H})$

Definition 1.2.1 *A mapping A from $\mathcal{H}$ to $\mathcal{H}$ is said to be a linear operator if it satisfies the following*

1. *Additive : $A(x+y) = Ax + Ay \quad x, y \in \mathcal{H}$.*
2. *Homogeneous : $A(\lambda x) = \lambda Ax \quad \forall x \in \mathcal{H}$ and $\forall \lambda \in \mathbb{C}$.*

Definition 1.2.2 *Linear operator A on $\mathcal{H}$ is said to be bounded if it satisfies*

$$\exists c > 0 \quad \forall x \in \mathcal{H} : \| Ax \| \leqslant c \| x \| .$$

Definition 1.2.3 *The set of all bounded linear operators on $\mathcal{H}$ is an unitary algebra over $\mathbb{C}$ denoted by $\mathcal{L}(\mathcal{H})$, where the addition, the external product and the product are defined as follows Let $A, B \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$*

1. *$\forall x \in \mathcal{H} \quad (A+B)x = Ax + Bx$.*
2. *$\forall x \in \mathcal{H} \quad (\lambda T)x = \lambda Ax$.*
3. *$\forall x \in \mathcal{H} \quad (AB)x = A(Bx)$.*
   And the unitary element is I the identity($Ix = x, \forall x \in \mathcal{H}$), moreover $\mathcal{L}(\mathcal{H})$ is a Banach algebra when we equip it with the following norm
   Let $A \in \mathcal{L}(\mathcal{H})$, then $\| A \| = \inf\{c > 0 : \| Ax \| \leqslant c \| x \| \ \ \forall x \in \mathcal{H}\}$.

Theorem 1.2.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

$$\| A \| = \sup\{\| Ax \| : \| x \| = 1\} = \sup\{\frac{\| Ax \|}{\| x \|} : \ x \in \mathcal{H} \backslash \{0\}\} = \sup\{| \langle Ax, y \rangle | : \| x \| = \| y \| = 1\}.$$

Corollary 1.2.1 *Let $A, B \in \mathcal{L}(\mathcal{H})$, and $n \in \mathbb{N}$. Then*

1. *$\forall x \in \mathcal{H} \quad \| Ax \| \leqslant \| A \| \| x \|$.*

2. $\| AB \| \leqslant \| A \| \| B \|$.

3. $\| A^n \| \leqslant \| A \|^n$.

Definition 1.2.4 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *The range of A is the set*
$$R(A) = \{Ax : x \in \mathcal{H}\}.$$

2. *The kernel of A is the set*
$$N(A) = \{x \in \mathcal{H} : Ax = 0\}.$$

Definition 1.2.5 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *$R(A)$ is a linear subspace of $\mathcal{H}$.*

2. *$N(A)$ is a closed linear subspace of $\mathcal{H}$.*

Theorem 1.2.2 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*
$$\forall x \in \mathcal{H} \quad \langle Ax, x\rangle = 0 \Rightarrow A = 0.$$

Proof. Let $x, y \in \mathcal{H}$ suppose that $\langle Ax, x\rangle = 0$ for any $x \in \mathcal{H}$, then
Since
$$\langle Ax, y\rangle = \frac{1}{4}\left\{\langle A(x+y),(x+y)\rangle - \langle A(x-y),(x-y)\rangle + i\langle A(x+iy),(x+iy)\rangle - i\langle A(x-iy),(x-iy)\rangle\right\}.$$
Thus
$$\langle Ax, x\rangle = 0 \quad \forall x, y \in \mathcal{H}.$$
So, $\quad Ax = 0 \quad \forall x \in \mathcal{H}$.
Therefore $A = 0$. □

Corollary 1.2.2 *Let $x, y \in \mathcal{L}(\mathcal{H})$. If $\langle Ax, x\rangle = \langle Bx, x\rangle \quad \forall x \in \mathcal{H}$, then $A = B$.*

Definition 1.2.6 *Let $A \in \mathcal{L}(\mathcal{H})$ and $M \subset \mathcal{H}$ a linear subspace of $\mathcal{H}$. Then $A_{|M}$ is the restriction of A on M such that $A_{|M} : M \to \mathcal{H}$ by $A_{|M}x = Ax \quad \forall x \in \mathcal{H}$.*

Definition 1.2.7 *Let $(A_n)_n \in \mathbb{N} \subset \mathcal{L}(\mathcal{H})$ be a sequence of operators and $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *$(A_n)_{n\in\mathbb{N}}$ is said to be strongly convergent to A, if*

$$\forall x \in \mathcal{H} \quad \lim_{n\to\infty} \| A_n x - Ax \| = 0.$$

And we denote $A_n \overset{s}{\to} A$.

2. *$(A_n)_{n\in\mathbb{N}}$ is said to be uniformly convergent to A if*

$$\lim_{n\to\infty} \| A_n - A \| = 0.$$

And we denote $A_n \to A$.

Proposition 1.2.1 *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ be a sequence of operators and $A \in \mathcal{L}(\mathcal{H})$. Then*

$(A_n)_{n\in\mathbb{N}}$ is uniformly convergent to A then $(A_n)_{n\in\mathbb{N}}$ is strongly convergent toA Briefly

$$A_n \to A \Rightarrow A_n \overset{s}{\to} A.$$

Definition 1.2.8 *Let $A \in \mathcal{L}(\mathcal{H})$. Then A is said to be invertible if there exists $B \in \mathcal{L}(\mathcal{H})$ such that $AB = BA = I$, in which case B is the inverse of A and it is denoted by A^{-1}, and $\mathcal{I}(\mathcal{H})$ denotes to the set of all invertible operators in $\mathcal{L}(\mathcal{H})$.*

Lemma 1.2.1 *Let $A, B \in \mathcal{I}(\mathcal{H})$, and $n \in \mathbb{N}$. Then*

1. *The inverse of A^{-1} is A.*
2. *The inverse of BA is $A^{-1}B^{-1}$.*
3. *The inverse of A^n is $A^{-n}(A^{-n} = (A^{-1})^n)$.*

Corollary 1.2.3 *(Banach's Isomorphism Theorem)*
Let $A \in \mathcal{L}(\mathcal{H})$. If A is bijective, then A invertible.

Remark 1.2.1 *The result of Banach's Isomorphism Theorem is that in $\mathcal{L}(\mathcal{H})$, when A^{-1} exists, then it is bounded.*

Theorem 1.2.3 *(Neumann series)*
Let $A \in \mathcal{L}(\mathcal{H})$ such that $\| A \| < 1$. Then $(I - A)$ is invertible and the inverse given by

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n \quad \text{and} \quad \| (I-A)^{-1} \| \leqslant \frac{1}{1 - \| A \|}.$$

Lemma 1.2.2 *Let $\mathcal{I}(\mathcal{H})$ be the set of all invertible operators in $\mathcal{L}(\mathcal{H})$. Then $\mathcal{I}(\mathcal{H})$ is an open set in $\mathcal{L}(\mathcal{H})$.*

Theorem 1.2.4 *$\mathcal{F} : \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{H})$ defined by $\mathcal{F}(A) = A^{-1}$ is continuous.*

Lemma 1.2.3 *Let $A \in \mathcal{L}(\mathcal{H})$ be an invertible operator. Then*

$$\forall x \in \mathcal{H} \quad \| Ax \| \geqslant \| A^{-1} \|^{-1} \| x \| .$$

Lemma 1.2.4 *Let $A \in \mathcal{L}(\mathcal{H})$. Assume that there exists $\alpha > 0$ such that*

$$\| Ax \| \geqslant \alpha \| x \| \quad \forall x \in \mathcal{H}, \text{ then } R(A) \text{ is closed.}$$

Theorem 1.2.5 *Let $A \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent*

1. *A is invertible.*

2. *$R(A)$ is dense in $\mathcal{H}$ and there exists $\alpha > 0$ such that $\| Ax \| \geqslant \alpha \| x \| \quad \forall x \in \mathcal{H}$.*

Corollary 1.2.4 *Let $A \in \mathcal{L}(\mathcal{H})$. Then A is not invertible if and only if $R(A)$ is not dense in H or there exists $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $\| x_n \| = 1 \quad \forall n \in \mathbb{N}$ such that $\lim_{n\to\infty} Ax_n = 0$.*

1.3 The adjoint of a linear operator

Theorem 1.3.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then there exists a unique $A^* \in \mathcal{L}(\mathcal{H})$ such that*

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \quad \forall x, y \in \mathcal{H}.$$

Proof. Consider $f(x) = \langle Ax, y\rangle, \quad \forall x \in \mathcal{H}$. First of all Let's prove $f \in \mathcal{H}^{'}$ and every arbitrary $y \in \mathcal{H}$, f is linear and bounded.

Let $x, z \in \mathcal{H}$ and $\lambda \in \mathbb{C}$

$$\begin{aligned} f(\lambda x + z) &= \langle A(\lambda x + z), y\rangle \\ &= \langle A\lambda x + Az, y\rangle \\ &= \lambda\langle Ax, y\rangle + \langle Az, y\rangle \\ &= \lambda f(x) + f(z) \end{aligned}$$

f is linear for every arbitrary $y \in \mathcal{H}$.

Let $x \in \mathcal{H}$, uning the boundedness of A and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \mid f(x) \mid &= \mid \langle Ax, y\rangle \mid \\ &\leqslant \parallel Ax \parallel \parallel y \parallel \\ &\leqslant \parallel A \parallel \parallel y \parallel \parallel x \parallel . \end{aligned}$$

Therefore

$$\exists C = \parallel A \parallel \parallel y \parallel \geqslant 0 \quad \forall x \in \mathcal{H} \quad \mid f(x) \mid \leqslant C \parallel x \parallel .$$

Then f is bounded.

Hence $f \in \mathcal{H}^{'}$ for every arbitrary $y \in \mathcal{H}$.

Applying Riesz's representation theorem of f, we get.

For every arbitrary $y \in \mathcal{H}$, there exists a unique $\in \mathcal{H}$ such that $f(x) = \langle Ax, y\rangle = \langle x, z\rangle$ we set for every $y \in \mathcal{H} \quad A^*y = z$.

Hence

$$\langle Ax, y\rangle = \langle x, A^*y\rangle.$$

Now we need to prove that $A^* \in \mathcal{L}(\mathcal{H})$.

Let $\lambda \in \mathbb{C}$ and $y_1, y_2 \in \mathcal{H}$, then $\langle Ax, y_1\rangle = \langle x, A^*y_1\rangle$ and $\langle Ax, y_2\rangle = \langle x, A^*y_2\rangle$ for every $x \in \mathcal{H}$

$\langle Ax, \lambda y_1 + y_2\rangle = \langle x, A^*(\lambda y_1 + y_2)\rangle \quad \forall x \in \mathcal{H}$

and

$$\begin{aligned} \langle Ax, \lambda y_1 + y_2\rangle &= \overline{\lambda}\langle Ax, y_1\rangle + \langle Ax, y_2\rangle \\ &= \overline{\lambda}\langle x, A^*y_1\rangle + \langle x, A^*y_2\rangle \\ &= \langle x, \lambda A^*y_1 + A^*y_2\rangle. \end{aligned}$$

Then,

$$\forall x \in \mathcal{H}: \quad \langle x, \lambda A^* y_1 + A^* y_2 \rangle = \langle x, A^*(\lambda y_1 + y_2) \rangle$$

which gives us

$$\forall x \in \mathcal{H} \quad \langle x, \lambda A^* y_1 + A^* y_2 - A^*(\lambda y_1 + y_2) \rangle = 0.$$

Hence

$$A^*(\lambda y_1 + y_2) = A^* y_1 + A^* y_2$$

then A^* is linear.

Let's prove that A^* is bounded, uning the boundedness of A^* and Cauchy-Schwarz inequality we get

$$\begin{aligned} \forall y \in \mathcal{H} \quad \| A^* y \|^2 = \langle A^* y, A^* y \rangle &= \langle A A^* y, y \rangle \\ &\leqslant \| A A^* y \| \| y \| \\ &\leqslant \| A \| \| A^* y \| \| y \| . \end{aligned}$$

Thus $\forall x \in \mathcal{H} \quad \| A^* y \| \leqslant \| A \| \| y \|$

therefore A^* is bounded.

Then $A^* \in \mathcal{L}(\mathcal{H})$. we just still need to prove the uniqueness of A^*.

Assume that there exists $B \in \mathcal{L}(\mathcal{H})$ such that $\forall x, y \in \mathcal{H} \quad \langle Ax, y \rangle = \langle x, By \rangle$, then

$$\langle x, A^* y \rangle = \langle x, By \rangle \quad \forall x, y \in \mathcal{H}.$$

Then,

$$\langle x, A^* y - By \rangle = 0.$$

So,

$$\forall y \in \mathcal{H} \quad A^* y = By.$$

Hence $A^* = B$, so A^* is unique. $\square$

Proposition 1.3.1 *Let $A, B \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then*

1. $(A + B)^* = A^* + B^*$.

2. $(\lambda A)^* = \overline{\lambda} A^*$.

3. $(A^*)^* = A$.

4. $\| A^* \| = \| A \|$.

5. $(AB)^* = B^* A^*$.

6. $\| A^* A \| = \| A \|^2$.

Corollary 1.3.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *Then function $f : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by $f(A) = A^*$ is bounded (continuous)antilinear.*

2. *$A = 0$ in and only $A^* A = 0$.*

3. *Let $n \in \mathbb{N}$, then $(A^n)^* = (A^*)^n$.*

Lemma 1.3.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *$N(A) = (R(A^*))^{\perp}$.*

2. *$N(A^*) = (R(A))^{\perp}$.*

3. *$N(A^*) = \{0\}$ if, and only if, $R(A)$ dense in $\mathcal{H}$.*

Lemma 1.3.2 *Let $A \in \mathcal{L}(\mathcal{H})$ be an invertible operator, then $(A^{-1})^* = (A^*)^{-1}$.*

Corollary 1.3.2 *Let $A \in \mathcal{L}(\mathcal{H})$. Then then the following are equivalent*

1. *A is invertible.*

2. *$N(A^*) = \{0\}$ are there exists $\alpha > 0$ such that $\| Ax \| \geqslant \alpha \| x \| \;\; \forall x \in \mathcal{H}$.*

1.4 Some classes of linear operators in $\mathcal{L}(\mathcal{H})$

Definition 1.4.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then A is called*

1. *Self-adjoint operator if $A^* = A$.*
2. *Normal operator if $AA^* = A^*A$.*
3. *positive operator if $\forall x \in \mathcal{H} : \langle Ax, x\rangle \geqslant 0$. and we denote $A \geqslant 0$.*
4. *Unitary operator if $AA^* = A^*A = I$.*
5. *Isometry operator if $A^*A = I$.*
6. *Projection operator if $A^2 = A$.*
7. *Orthogonal projection operator if $A^2 = A = A^*$.*
8. *Quasinormal operator if $A(A^*A) = (A^*A)A$.*
9. *Hyponormal operator if $A^*A \geqslant AA^*$ ($A \geqslant B$ if and only if $A - B \geqslant 0$).*

Theorem 1.4.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *A is self-adjoint if, and only if, $\langle Ax, x\rangle \in \mathbb{R} \quad \forall x \in \mathcal{H}$.*
2. *A is normal if and only if $\| Ax \| = \| A^*x \| \quad \forall x \in \mathcal{H}$.*
3. *A is unitary if, and only if, $\| Ax \| = \| A^*x \| = \| x \| \quad \forall x \in \mathcal{H}$.*
4. *A is hyponormal if, and only if, $\| Ax \| \geqslant \| A^*x \| \quad \forall x \in \mathcal{H}$.*

Corollary 1.4.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then the following assertions are equivalent*

1. *A is isometry.*
2. *$\langle Ax, Ay\rangle = \langle x, y\rangle \quad \forall x, y \in \mathcal{H}$.*
3. *$\| Ax \| = \| x \| \quad \forall x \in \mathcal{H}$.*

Proposition 1.4.1 *Let $A \in \mathcal{L}(\mathcal{H})$ and $\mathcal{S}$ be the set of all self-adjoint operators in $\mathcal{L}(\mathcal{H})$. Then*

1. *Let $\alpha, \beta \in \mathbb{R}$ and $A_1, A_2 \in \mathcal{S}$, then $\alpha A_1 + \beta A_2 \in \mathcal{S}$.*
2. *If $A \in \mathcal{S}$, and p is a polynomial of real coefficients, then $p(A) \in \mathcal{S}$.*
3. *If $A \in \mathcal{S}$ and A is invertible, then $A^{-1} \in \mathcal{S}$.*
4. *$AA^*, A^*A \in \mathcal{S}$.*
5. *$\mathcal{S}$ is a closed subset of $\mathcal{L}(\mathcal{H})$.*

Theorem 1.4.2 *If $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint, then $\| A \| = \sup\limits_{\|x\|=1} | \langle Ax, x \rangle |$.*

Theorem 1.4.3 *(Cartesian form)*
Let $A \in \mathcal{L}(\mathcal{H})$. Then there exist self-adjoint operators R and B such that $A = R + iB$ where $R = \frac{1}{2}(A + A^*)$ and $B = \frac{1}{2i}(A - A^*)$. R is called the real part of A and it is denoted by $\Re(A)$, while B is called the imaginary part of A and it is denoted by $\Im(A)$.

Proposition 1.4.2 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*
A is positive $\Rightarrow$ A is self-adjoint $\Rightarrow$ A is normal $\Rightarrow$ A is quasinormal $\Rightarrow$ A is hyponormal.

Theorem 1.4.4 *Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator and $n \in \mathbb{N}$, then $\| A^n \| = \| A \|^n$.*

Proposition 1.4.3 *Let $A \in \mathcal{H}$ is a normal operator.*
If there exists $\alpha > 0$ $\| Ax \| \geqslant \alpha \| x \|$ $\forall x \in \mathcal{H}$. Then $N(A^*) = \{0\}$.

Corollary 1.4.2 *Let $A \in \mathcal{H}$ is a normal operator. Then the following are equivalent*

1. *A is invertible.*
2. *There exists $\alpha > 0$ such that $\| Ax \| \geqslant \alpha \| x \| \quad \forall x \in \mathcal{H}$.*

Proposition 1.4.4 *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then*

1. *A is unitary if and only if A is an isometry with $R(A) = \mathcal{H}$.*

2. *If A is unitary, then* $\| A \| = 1$.

3. *If A and B are unitary, then AB is also unitary.*

Theorem 1.4.5 *Let $A \in \mathcal{L}(\mathcal{H})$ be a strictly positive operator with closed range $R(A)$, then A is invertible.*

Corollary 1.4.3 *If $\mathcal{H}$ is finite-dimensional and $A \in \mathcal{L}(\mathcal{H})$ is strictly positive, then A is invertible.*

Corollary 1.4.4 *Let $A, B \in \mathcal{L}(\mathcal{H})$, and $\alpha \geqslant 0$. Then*

1. *A^*A and AA^* are positive operators.*

2. *If A is positive, then αA is positive too.*

3. *If A and B are positive, then $A + B$ is positive as well.*

Theorem 1.4.6 *Let $A, B \in \mathcal{L}(\mathcal{H})$ be two positive operators.*
Then BA is positive if, and only if, B commutes A i.e. $BA = AB$.

Theorem 1.4.7 *(Generalized Schwarz inequality)*
Let $A \in \mathcal{L}(\mathcal{H})$ be a positive operator. Then

$$\forall x, y \in \mathcal{H} \quad | \langle Ax, y \rangle |^2 \leqslant \langle Ax, x \rangle \langle Ay, y \rangle.$$

1.5 Spectrum of a linear operator

Definition 1.5.1 *Let $A \in \mathcal{L}(\mathcal{H})$. The Spectrum of A is denoted by $\sigma(A)$, and it is defined as follows*

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \textit{is not invertible } \}.$$

And the resolvent of A is denoted by $\rho(A)$, and $\rho(A) = \mathbb{C} \setminus \sigma(A)$.

Example 1.5.1 $\sigma(\alpha I) = \{\alpha\}$.

Remark 1.5.1 *It is clear that* $\sigma(A) = \{\lambda \in \mathbb{C} : N(A - \lambda I) \neq \{0\} \, or R(A - \lambda I) \neq \mathcal{H}\}$.

Definition 1.5.2 *Let $A \in \mathcal{L}(\mathcal{H})$, then the resolvent function of A is the map $R : \rho(A) \to \mathcal{I}(\mathcal{H})$ defined by $R(\lambda) = (A - \lambda I)^{-1}$.*

Proposition 1.5.1 *Let $A \in \mathcal{L}(\mathcal{H})$, and let R be the resolvent function of A. Then*

1. $R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu) \quad \forall \lambda, \mu \in \rho(A)$.
2. $R(\lambda)R(\mu) = R(\mu)R(\lambda) \quad \forall \lambda, \mu \in \rho(A)$.
3. *R is continuous.*

Proof.

1.

$$\begin{aligned} R(\lambda) - R(\mu) &= (A - \lambda I)^{-1} - (A - \mu I)^{-1} \\ &= (A - \lambda I)^{-1}(A - \mu I)(A - \mu I)^{-1} - (A - \lambda I)^{-1}(A - \lambda I)(A - \mu I)^{-1} \\ &= (A - \lambda I)^{-1}((A - \mu I) - (A - \lambda I))(A - \mu I)^{-1} \\ &= (A - \lambda I)^{-1}(\lambda - \mu)I(A - \mu I)^{-1} \\ &= (\lambda - \mu)R(\lambda)R(\mu). \end{aligned}$$

2. If $\mu = \lambda$, then $R(\lambda)R(\mu) = R(\mu)R(\lambda)$. Assume that $\mu \neq \lambda$, we have

$$R(\lambda)R(\mu) = \frac{R(\lambda) - R(\mu)}{\mu - \lambda} = \frac{R(\mu) - R(\lambda)}{\mu - \lambda} = R(\mu)R(\lambda).$$

3. We have that $\mathcal{F} : \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{H})$ defined by $\mathcal{F}(\mathcal{H}) = A^{-1}$ is continuous.

   Let $G : \rho(A) \to \mathcal{I}(\mathcal{H})$ defined by $G(\lambda) = A - \lambda I$, it is clear that G is continuous, since

$$\| G(\lambda) - G(\mu) \| = \| \lambda I - \mu I \| = | \lambda - \mu | \, .$$

   And, since $R = \mathcal{F} \circ G$, then R is continuous.

$\square$

Theorem 1.5.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *If $| \lambda | > \| A \|$, then $\lambda \in \rho(A)$.*

2. $\sigma(A)$ is a closed set.

Proof.

1. If $\mid \lambda \mid > \parallel A \parallel$, then $\parallel \lambda^{-1} A \parallel < 1$, we have that $I - \lambda^{-1}A$ is invertible, then $A - \lambda I$ is invertible. Hence $\lambda \in \rho(A)$.

2. Define $G : \mathbb{C} \to \mathcal{L}(\mathcal{H})$ by $G(\lambda) = A - \lambda I$, and let N be the set of non-invertible operators.
   Then $\sigma(A) = \{\lambda \in \mathcal{L}(\mathcal{H}) : G(\lambda) \in N\} = G^{-1}(N)$, we have that N is a closed of $\mathcal{L}(\mathcal{H})$, and we have that G is continuous, then $\sigma(A)$ is closed of $\mathbb{C}$.

□

Corollary 1.5.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. $\sigma(A)$ is compact.

2. $\rho(A)$ is an open non-empty set.

Proposition 1.5.2 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $\sigma(A)$ is non-empty.*

Proof. Suppose that $\sigma(A) = \emptyset$, then $\rho(A) = \mathbb{C}$. Let $f \in \mathcal{L}(H)^{'}$
First, let's prove that $f \circ R : \mathbb{C} \to \mathbb{C}$ is bounded
we have that, R is continuous, then $\|R\|$ is also continuous, by Bolzano-Weierstrass theorem, we get

$$\sup_{|\lambda| \leqslant \|A\|} \|R(\lambda)\| < \infty.$$

When

$$\mid \lambda \mid < \parallel A \parallel$$

then,

$$\|(I - \lambda^{-1}A)^{-1}\| \leqslant \frac{\mid \lambda \mid}{\mid \lambda \mid - \parallel A \parallel}$$

and

$$\parallel \lambda(A - \lambda I)^{-1} \parallel \leqslant \frac{\mid \lambda \mid}{\mid \lambda \mid - \parallel A \parallel}.$$

So,

$$\| (A - \lambda I)^{-1} \| \leqslant \frac{1}{| \lambda | - \| A \|} \underset{|\lambda|\to\infty}{\longrightarrow} 0.$$

Then,

$$\lim_{|\lambda|\to\infty} \| (A - \lambda I)^{-1} \| = 0.$$

Thus,

$$\sup_{|\lambda|>\|A\|} \|R(\lambda)\| < \infty.$$

Therefore

$$\sup_{|\lambda|\in\mathbb{C}} \|R(\lambda)\| < \infty.$$

Then,

$$\sup_{|\lambda|\in\mathbb{C}} \|(f \circ R)(\lambda)\| \leqslant \| f \| \sup_{|\lambda|\in\mathbb{C}} \|R(\lambda)\| < \infty.$$

Thus $f \circ R$ is bounded.

Second, let's prove that $f \circ R : \mathbb{C} \to \mathbb{C}$ is analytic

Let $\lambda, \mu \in \mathbb{C}$ such that $\lambda \neq \mu$, we have that, we get

$$\frac{R(\lambda) - R(\mu)}{\lambda - \mu} - R(\lambda)^2 = (R(\mu) - R(\lambda))R(\lambda).$$

Set $g = f \circ R$, and let $g^{'} : \mathbb{C} \to \mathbb{C}$ defined by $g^{'}(\lambda) = f(g(\lambda)^2)$, then

$$\begin{array}{rcl} | \frac{g(\lambda)-g(\mu)}{\lambda-\mu} - g^{'}(\lambda) | & = & | f(\frac{R(\lambda)-R(\mu)}{\lambda-\mu} - R(\lambda)^2) | \\ & = & | f((R(\mu) - R(\lambda))R(\lambda)) | \end{array}.$$

Then $\quad | \frac{g(\lambda)-g(\mu)}{\lambda-\mu} - g^{'}(\lambda) | \leqslant \| f \| \| R(\lambda) \| \| R(\mu) - R(\lambda) \| \underset{\mu\to\lambda}{\to} 0$ (since R is constant).

So,

$$\lim_{\mu\to\lambda} \frac{g(\lambda) - g(\mu)}{\lambda - \mu} = g^{'}(\lambda).$$

Therefore $f \circ R$ is analytic.

Then $\forall f \in \mathcal{L}(\mathcal{H})^{'}$, $f \circ R$ is bounded and analytic over all $\mathbb{C}$, then by the Liouville Theorem $f \circ R$ is constant $\forall \in \mathcal{L}(\mathcal{H})^{'}$, but we have that

$$\| R(\lambda) \| \underset{|\lambda|\to\infty}{\to} 0$$

and

$$f(R(\lambda)) \underset{|\lambda|\to\infty}{\to} 0 \ (f \text{ is continuous}).$$

Then $f \circ R = 0 \quad \forall f \in \mathcal{L}(H)'$, so $R = 0$.

Thus $(A - \lambda I)^{-1} = 0 \quad \forall \lambda \in \mathbb{C}$, which is a contradiction because 0 is not invertible i.e. $0 \notin \mathcal{I}(\mathcal{H})$.

Hence $\sigma(A) \neq \emptyset$. □

Lemma 1.5.1 *1. Let $A \in \mathcal{L}(\mathcal{H})$, then $\sigma(A^*) = \{\overline{\lambda},\ \lambda \in \sigma(A)\}$.*

2. If is invertible then $\sigma(A^{-1}) = \{\lambda^{-1}, \lambda \in \sigma(A)\}$.

Proof.

1.
$$\begin{aligned}
\lambda \in \rho(A) &\Leftrightarrow A - \lambda I \quad \textit{is invertible}\\
&\Leftrightarrow (A - \lambda I)^* \quad \textit{is invertible}\\
&\Leftrightarrow A^* - \overline{\lambda} I \quad \textit{is invertible}\\
&\Leftrightarrow \overline{\lambda} \in \rho(A^*).
\end{aligned}$$

   Therefore $\qquad \lambda \notin \rho(A) \Leftrightarrow \overline{\lambda} \notin \rho(A^*)$.

   Thus $\sigma(A^*) = \{\overline{\lambda},\ \lambda \in \sigma(A)\}$.

2. Since A is invertible, then $0 \notin \sigma(A)$, therefore λ^{-1} is defined when $\lambda \in \sigma(A)$.

   Let $\lambda \in \mathbb{C} \setminus \{0\} \quad A^{-1} - \lambda^{-1} I = -\lambda^{-1} A^{-1}(A - \lambda I)$, and $\lambda^{-1} A^{-1}$ is invertible $\forall \lambda \in \mathbb{C} \setminus \{0\}$.

   Then
$$\begin{aligned}
\lambda \in \sigma(A) &\Leftrightarrow A - \lambda I \quad \textit{is not invertible}\\
&\Leftrightarrow -\lambda^{-1} A^{-1}(A - \lambda I) \quad \textit{is not invertible}\\
&\Leftrightarrow A^{-1} - \lambda^{-1} I \quad \textit{is not invertible}\\
&\Leftrightarrow \lambda^{-1} \in \sigma(A^{-1}).
\end{aligned}$$

   Hence $\sigma(A^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(A)\}$.

□

Theorem 1.5.2 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *If, p is a polynomial then $\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda), \lambda \in \sigma(A)\}$.*

2. *$f(\sigma(A)) = \sigma(f(A))$, A self-adjoint.*

Proof.

1. Let $\mu \in \mathbb{C}$ and $q(z) = p(z) - \mu$, it is obvious that q is a polynomial, so it has factorization of the form $q(z) = c(z-\mu_1)...(z-\mu_n)$ where $c, \mu_1, ..., \mu_n \in \mathbb{C}$ with $c \neq 0$. Then
Let $\mu \in \sigma(p(A))$ then $p(A) - \mu I$ is not invertible imply that $q(A)$ is not invertible then,

$$c(A-\mu_1 I)...(A-\mu_n I) \text{ is not invertible}$$

and

$$\exists k \text{ such that } 1 \leqslant k \leqslant n \text{ and } (A-\mu_k I) \text{ is not invertible.}$$

So, $\mu_k \in \sigma(A)$
and $\mu \in p(\mu_k)$.
Thus $\mu \in p(\sigma(A))$.
Then $\sigma(p(A) \subset p(\sigma(A))$.
Let $\mu \in p(\sigma(A))$, then $\exists \lambda \in \sigma(A)$ such that $\mu = p(\lambda)$, so $p(\lambda) - \mu = 0$.
Thus $\exists k \in \{1, 2, ..., n\}$ such that $\lambda = \mu_k$.
Since $(A-\mu_k I)$ commutes with $(A-\mu_j I)$ for all $1 \leqslant j \leqslant n$, then

$$\begin{aligned} p(A) - \mu I &= c(A-\mu_1 I)...(A-\mu_k I)...(A-\mu_n I) \\ &= (A-\mu_k I)(c(A-\mu_1 I)...(A-\mu_n I)) \\ &= (c(A-\mu_1 I)...(A-\mu_n I))(A-\mu_k I). \end{aligned}$$

If $\mu \in \rho((p(A))$, then $p(A) - \mu I$ is invertible, then we have the following

$$(A-\mu_k I)(c(A-\mu_1 I)...(A-\mu_n I))(p(A)-\mu_k I)^{-1} = (p(A)-\mu I)^{-1}(c(A-\mu_1 I)...(A-\mu_n I))(A-\mu_k I) = I.$$

Which means that $A - \mu_k I$ has right and left inverses, therefore $A - \mu_k I$ is injective and surjective, so that $A - \mu_k I$ is bijective, then $A - \mu_k I$ is invertible, thus $\mu_k \in \rho(A)$, and this contradict $\mu_k \in \sigma(A)$, then $\mu \notin \rho(p(A))$.

Thus $\mu \in \sigma(A)$, hence $p(\sigma(A)) \subset \sigma(p(A))$. Overall $p(\sigma(A)) = \sigma(p(A))$.

$\square$

Corollary 1.5.2 *Let $A \in \mathcal{L}(\mathcal{H})$, $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, then*

1. $\sigma(\alpha A) = \alpha(\sigma(A)) = \{\alpha\lambda : \lambda \in \sigma(A)\}$.

2. $\sigma(A^n) = (\sigma(A))^n = \{\lambda^n : \lambda \in \sigma(A)\}$.

Theorem 1.5.3 *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.*

Proof. Let $\lambda \in \rho(AB) \setminus \{0\} \Leftrightarrow AB - \lambda I$ is invertible.
Set $R = (AB - \lambda I)^{-1}$, then

$$(AB - \lambda I)R = R(AB - \lambda I) = I \Leftrightarrow ABR - \lambda R = RAB - \lambda R = I$$

therefore $\quad ABR = RAB = I + \lambda R$

$$\begin{aligned} (BA - \lambda I)(BRA - I) &= BABRA - BA - \lambda BRA + \lambda I \\ &= B(I + \lambda R)A - BA - \lambda BRA + \lambda I \\ &= BA + \lambda BRA - BA - \lambda BRA + \lambda I \\ &= \lambda I. \end{aligned}$$

Since $\lambda \neq 0$, then $\quad (BA - \lambda I)\frac{1}{\lambda}(BRA - I) = I$

$$\begin{aligned} (BRA - \lambda I)(BA - I) &= BRABA - \lambda BRA - BA + \lambda I \\ &= B(I + \lambda R)A - \lambda BRA - BA + \lambda I \\ &= BA + \lambda ARB - \lambda BRA - BA + \lambda I \\ &= \lambda I. \end{aligned}$$

Since $\lambda \neq 0$, then

$$\frac{1}{\lambda}(ARB - I)(BA - \lambda I) = I.$$

Therefore, $BA - \lambda I$ is invertible, then $\lambda \in \rho(BA) \setminus \{0\}$. Hence

$$\rho(AB) \setminus \{0\} = \rho(BA) \setminus \{0\} \Leftrightarrow \sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

$\square$

Theorem 1.5.4 *Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$.*

Proof. Let $\lambda \in \mathbb{C}$ such that $\Im\lambda \neq 0$, and let $x \in \mathcal{H}$. Since A is self-adjoint $\langle Ax, x\rangle \in \mathbb{R}$, then

$$-\Im\lambda \parallel x \parallel^2 = \Im\langle (A-\lambda I)x, x\rangle.$$

However,

$$\begin{aligned} \mid \Im\lambda \mid \parallel x \parallel^2 &= \mid \Im\langle (A-\lambda I)x, x\rangle \mid \\ &\leqslant \mid \langle (A-\lambda I)x, x\rangle \mid \\ &\leqslant \parallel (A-\lambda I)x \parallel \parallel x \parallel . \end{aligned}$$

Then $\quad \mid \Im\lambda \mid \parallel x \parallel \leqslant \parallel (A-\lambda I)x \parallel$.

We have that (since $\mid \Im\lambda \mid \neq 0$), then $A-\lambda I$ is invertible, thus $\lambda \in \rho(A)$.

Therefore $\forall \lambda \in \mathbb{C}$, such that $\Im\lambda \neq 0$ we have $\lambda \in \rho(A)$.

Thus if $\lambda \in \sigma(A)$, then $\Im\lambda = 0$, which means that $\sigma(A) \subset \mathbb{R}$. □

Corollary 1.5.3 *Let $A \in \mathcal{L}(\mathcal{H})$ be positive. Then*

1. $\sigma(A) \subset \mathbb{R}_+$.
2. $\sigma(\sqrt{A}) = \sqrt{\sigma(A)} = \{\sqrt{\lambda} : \lambda \in \sigma(A)\}$.

Proof.

1. Since A is positive, then A is self-adjoint, therefore $\sigma(A) \subset \mathbb{R}$.

   Let $\lambda \in \mathcal{H}$. Recall that $\langle Ax, x\rangle \geqslant 0$, then

   $$\begin{aligned} -\lambda \parallel x \parallel^2 &= -\lambda\langle x, x\rangle \\ &\leqslant \langle Ax, x\rangle - \lambda\langle x, x\rangle \\ &= \langle (A-\lambda I)x, x\rangle \\ &\leqslant \parallel (A-\lambda I)x \parallel \parallel x \parallel . \end{aligned}$$

   Thus

   $$-\lambda \parallel x \parallel \leqslant \parallel (A-\lambda I)x \parallel .$$

   We have that, $A-\lambda I$ is invertible (since $-\lambda > 0$), hence $\lambda \in \rho(A)$.

   Therefore $\forall \lambda \in < 0$, then $\lambda \in \rho(A) \Leftrightarrow \lambda \notin \sigma(A)$, thus if $\lambda \in \sigma(A)$, then $\lambda \geqslant 0$.

   Hence $\sigma(A) \subset \mathbb{R}$.

2. First of all, the set $\sqrt{\sigma(A)}$ is well-defined since $\sigma(A) \subset \mathbb{R}_+$, we get

$$\sigma(A) = \sigma(\sqrt{A}^2) = \sigma(\sqrt{A})^2.$$

Then $\sigma(A) = \{\lambda^2 : \lambda \in \sigma(\sqrt{A})\}$, thus $\sigma(\sqrt{A}) = \sqrt{\sigma(A)} = \{\sqrt{\lambda} : \lambda \in \sigma(A)\}$.

$\square$

Definition 1.5.3 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *The point of spectrum A is the set*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : N(A - \lambda I) \neq \{0\}\}$$

$\lambda \in \sigma_p(A)$ is called eigenvalue of A. and the $x \in \mathcal{H} \setminus \{0\}$ that verifies $Ax = \lambda x$ is called eigenvector of A.

2. *The continuous spectrum of A is the set*

$$\sigma_c(A) = \{\lambda \in \mathbb{C} : N(A - \lambda I) = \{0\} \text{ and } \overline{R(A - \lambda I)} = \mathcal{H}\}.$$

3. *The residual spectrum of A is the set*

$$\sigma_r(A) = \{\lambda \in \mathbb{C} : N(A - \lambda I) = \{0\} \text{ and } \overline{R(A - \lambda I)} \subsetneq \mathcal{H}\}.$$

4. *The approximate point spectrum of A is the set*

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : \exists (x_n) \subset \mathcal{H} : \| x_n \| = 1 \text{ and } \lim_{n \to \infty} (A - \lambda I)x_n = 0\}.$$

Remark 1.5.2 1. *$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$, where $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ are mutually disjoint,i,e.*

$$\sigma_p(A) \cap \sigma_c(A) = \sigma_p(A) \cap \sigma_r(A) = \sigma_c(A) \cap \sigma_r(A) = \emptyset.$$

2. *We have that $\sigma(A) = \sigma_{ap}(A) \cup \sigma_r(A)$, also $\sigma_p(A) \subset \sigma_{ap}(A)$.*

Corollary 1.5.4 *If $\mathcal{H}$ is a finite-dimensional Hilbert space, then $\sigma(A) = \sigma_p(A)$ $\forall A \in \mathcal{L}(\mathcal{H})$.*

Proof. Let $A \in \mathcal{L}(\mathcal{H})$, by the dimensional theorem,

we have $\dim(\mathcal{H}) = \dim(N(A)) + \dim(R(A))$.

Therefore A is inverible $\Leftrightarrow N(A) = \{0\}$.

Then if $A - \lambda I$ is not inverible $\Leftrightarrow N(A) \neq \{0\}$, hence $\sigma(A) = \sigma_p(A)$. $\square$

Proposition 1.5.3 *Let $A \in \mathcal{L}(\mathcal{H})$ be normal, if λ and μ are disjoint ($\lambda \neq \mu$) eigenvalues i.e. $\exists x, y \in \mathcal{L}(\mathcal{H}) \setminus \{0\} : Ax = \lambda x$ and $Ay = \mu y$, then $\langle x, y \rangle = 0$.*

Proof.

$$\begin{aligned} \| A^*y - \overline{\mu}y \|^2 = \langle A^*y - \overline{\mu}y, A^*y - \overline{\mu}y \rangle &= \langle A^*y, A^*y \rangle + | \mu |^2 \langle y, y \rangle - \mu \langle A^*y, y \rangle - \overline{\mu} \langle y, A^*y \rangle \\ &= \langle AA^*y, y \rangle + \| \mu y \|^2 - \langle \mu y, Ay \rangle - \langle Ay, \mu y \rangle \\ &= \langle A^*Ay, y \rangle + \| Ay \|^2 - \langle Ay, Ay \rangle - \langle Ay, Ay \rangle \\ &= \langle Ay, Ay \rangle + \langle Ay, Ay \rangle - 2\langle Ay, Ay \rangle = 0. \end{aligned}$$

Therefore $A^*y = \overline{\mu}y$. Since $\lambda \neq \mu$, then

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Ax, y \rangle = \langle x, A^*y \rangle = \langle x, \overline{\mu}y \rangle = \mu \langle x, y \rangle.$$

Then $\lambda \langle x, y \rangle = \mu \langle x, y \rangle$

and $(\lambda - \mu)\langle x, y \rangle = 0$.

Hence $\langle x, y \rangle$. $\square$

Theorem 1.5.5 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $\sigma_{ap}(A)$ is closed and $\partial\sigma(A) \subset \sigma_{ap}(A)$.*

Proof. First, let's prove that $\sigma_{ap}(A)$ is closed

Let $\lambda \notin \sigma_{ap}(A)$, then $\forall (x_n) \subset \mathcal{H}$ such that $\| x_n \| = 1 : \lim\limits_{n\to\infty} (A - \lambda I)x_n \neq 0$.

Then $\exists \alpha \in \mathcal{H}$ $\| (A - \lambda I)x \| \geqslant \alpha \| x \|$. Set $\varepsilon = \frac{\alpha}{2}$ and let $\mu \in \mathbb{C}$ such that $| \lambda - \mu | < \varepsilon$, then

$$\begin{aligned} \| (A - \lambda I)x \| = \| (A - \mu I)x + \mu x - \lambda x \| &\leqslant \| (A - \mu I)x \| + \| \lambda x - \mu x \| \\ &= \| (A - \mu I)x \| + | \lambda - \mu | \| x \| \end{aligned}$$

Thus $\| (A - \lambda I)x \| \leqslant \| (A - \mu I)x \| + | \lambda - \mu | \| x \|$. Therefore

$$\begin{aligned} \alpha \| x \| - | \lambda - \mu | \| x \| &\leqslant \| (A - \lambda I)x \| - | \lambda - \mu | \| x \| \\ &\leqslant \| (A - \mu I)x \| . \end{aligned}$$

Then $\quad (\alpha- \mid \lambda - \mu \mid) \parallel x \parallel \leqslant \parallel (A-\mu I)x \parallel$.

Since $\quad \mid \lambda - \mu \mid < \varepsilon = \frac{\alpha}{2}$, then

$$\frac{\alpha}{2} \parallel x \parallel \leqslant \parallel (A-\mu I)x \parallel .$$

So, $\quad \forall (x_n) \subset \mathcal{H}$ where $\parallel x_n \parallel = 1 : \lim\limits_{n\to\infty} (A-\lambda I)x_n \neq 0$.

Then $\quad \mu \in \sigma_{ap}(A)$, thus $\mathbb{C} \setminus \sigma_{ap}(A)$ is open, hence $\sigma_{ap}(A)$ is closed.

Second, prove that $\partial\sigma(A) \subset \sigma_{ap}(A)$:

We have that $\partial\sigma(A) = \overline{\sigma(A)} \cap \overline{\rho(A)}$, since $\sigma(A)$ is closed $\partial\sigma(A) = \sigma(A) \cap \overline{\rho(A)}$.

Let $\lambda \in \partial\sigma(A)$, then $\exists (\lambda_n) \subset \rho(A)$ such that $\lambda_n \underset{n\to\infty}{\rightarrow} \lambda$, then

$\parallel (A - \lambda_n I)^{-1} \parallel \underset{n\to\infty}{\rightarrow} \infty$, if not $\exists c > 0$ such that $\parallel (A-\lambda_n I)^{-1} \parallel \leqslant c$, for n sufficiently , we have

$$\parallel A - \lambda I - A + \lambda_n I \parallel = \mid \lambda_n - \lambda \mid \leqslant c^{-1} \leqslant \parallel (A-\lambda_n I)^{-1} \parallel^{-1} .$$

So, $\quad \parallel A - \lambda I - A + \lambda_n I \parallel \leqslant \parallel (A-\lambda_n I)^{-1} \parallel^{-1}$.

We have that, $A - \lambda I$ is invertible, and this contradicts $\lambda \in \partial\sigma(A) = \sigma(A) \cap \overline{\rho(A)}$.

Therefore $\parallel (A-\lambda_n I)^{-1} \parallel \underset{n\to\infty}{\rightarrow} \infty$.

We have that $\parallel (A-\lambda_n I)^{-1} \parallel = \sup\limits_{\parallel x \parallel = 1} \parallel (A-\lambda_n I)^{-1} x \parallel$, then

$$\forall n \in \mathbb{N}, \exists (x_n) \subset \mathcal{H} \text{ where } \parallel x_n \parallel = 1 \text{ and } \parallel (A-\lambda_n I)^{-1} x_n \parallel > \parallel (A-\lambda_n I)^{-1} \parallel - \tfrac{1}{n}.$$

Set $\alpha_n = \parallel (A-\lambda_n I)^{-1} x_n \parallel > \parallel (A-\lambda_n I)^{-1} \parallel - \frac{1}{n}$, then $\alpha_n \underset{n\to\infty}{\rightarrow} \infty$

put $y_n = \alpha_n^{-1} (A-\lambda_n I)^{-1} x_n$, using that fact that $\parallel y_n \parallel = 1$ we get

$$(A-\lambda I)y_n = (A-\lambda_n I)y_n + (\lambda_n - \lambda)y_n = \alpha_n^{-1} x_n + (\lambda_n - \lambda)y_n.$$

Then,

$$\parallel (A-\lambda I)y_n \parallel \leqslant \parallel \alpha_n^{-1} x_n \parallel + \parallel (\lambda_n - \lambda)y_n \parallel = \alpha_n^{-1} + \mid \lambda_n - \lambda \mid \underset{n\to\infty}{\rightarrow} 0.$$

Therefore, $\quad \parallel (A-\lambda I)y_n \parallel \underset{n\to\infty}{\rightarrow} 0$, thus $\lambda \in \sigma_{ap}(A)$.

Hence $\quad \partial\sigma(A) \subset \sigma_{ap}(A)$. □

Corollary 1.5.5 *Let $A \in \mathcal{L}(\mathcal{H})$, then $\sigma_{ap}(A)$ is a compact non-empty set.*

Proof. Since $\sigma_{ap}(A) \subset \sigma(A)$, then $\sigma_{ap}(A)$ is close in compact, then $\sigma_{ap}(A)$ is compact as well.

On the other hand, we have that $\partial\sigma(A) \subset \sigma_{ap}(A)$, we have that $\partial\sigma(A) \neq \emptyset$.
Because $\partial\sigma(A) = \sigma(A) \setminus \sigma(A)^\circ$, and since $\sigma(A) \neq \emptyset$ nor $\mathbb{C}$ and we know that $\mathbb{C}$ is connected, then $\sigma(A)^\circ \subsetneq \sigma(A)$.
Thus $\sigma(A) \setminus \sigma(A)^\circ \neq \emptyset$, therefore $\partial\sigma(A) \neq \emptyset$.
Hence $\sigma_{ap}(A) \neq \emptyset$. □

1.6 Numerical Range and Radius of linear bounded operator

Definition 1.6.1 *Let $A \in \mathcal{L}(\mathcal{H})$, we define the Numerical Range of A by*

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}\}.$$

And, the Numerical Radius of A by

$$w(A) = \sup_{\lambda \in W(A)} \mid \lambda \mid = \sup_{\|x\|=1} \mid \langle Ax, x\rangle \mid .$$

Proposition 1.6.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

$$\sigma(A) \subset \overline{W(A)}.$$

Proof. Since $\sigma(A) = \sigma_{ap}(A) \bigcup \sigma_r(A)$, and we have that $\sigma_{ap}(A) \subset \overline{W(A)}$ and

$$\sigma_r(A) \subset W(A).$$

Then,

$$\sigma(A) \subset \overline{W(A)}.$$

Another way, we have that $\partial\sigma(A) \subset \sigma_{ap}(A) \subset \overline{W(A)}$,
and since $\overline{W(A)}$ is convex (the closure of convex is convex).
thus,

$$\sigma(A) \subset \overline{W(A)}.$$

□

Definition 1.6.2 *Spaces ℓ^p :*

1. *For $p \in [1, +\infty[$, we consider*

$$\ell^p(\mathbb{N}) = \{x = (x_1, x_2, x_3, ...),\ \sum_{n=1}^{\infty} \mid x_n \mid^p < \infty\}.$$

2. *For $p = \infty$*

$$\ell^\infty = \{x = (x_1, x_2, x_3, ...),\ \sup_{n\in\mathbb{N}} \mid x_n \mid < \infty\}.$$

   $\| x \|_\infty = \sup_{n\in\mathbb{N}} \mid x_n \mid.$

Chapter 2

Spectral Radius of linear bounded operator

2.1 Definitions and Properties

2.1.1 Definitions

Definition 2.1.1 *Let $A \in \mathcal{L}(H)$, we define the spectral radius $r(A)$ of A. By*

$$r(A) := \{\sup \mid \lambda \mid, \, \lambda \in \sigma(A)\}.$$

As first property of the spectral radius is the relationship between the spectral radius and the norm of operator.

2.1.2 Properties

Lemma 2.1.1 *Let $A \in \mathcal{L}(H)$, we have*

$$r(A) \leq \|A\|.$$

Proof. We have

$$\mid \lambda \mid > \parallel A \parallel \Rightarrow \lambda \in \rho(A).$$

Then,

$$\forall \lambda \in \sigma(A) : \mid \lambda \mid \leqslant \parallel A \parallel$$

and

$$\sup_{\lambda \in \sigma(A)} \mid \lambda \mid \leqslant \parallel A \parallel .$$

Thus,

$$r(A) \leqslant \parallel A \parallel .$$

$\square$

Proposition 2.1.1 *Let $A, B \in \mathcal{L}(\mathcal{H})$, $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, then*

1. $r(AB) = r(BA)$.
2. $r(\alpha A) = \mid \alpha \mid r(A)$.
3. $r(A^n) = r^n(A)$.
4. $r(A^*) = r(A)$.
5. $r(\sqrt{A}) = \sqrt{r(A)}$.

Proof.

1. First, according to (Theorem 1.5.3), we have

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\} .$$

Then,

$$\sup_{\lambda \in \sigma(AB) \cup \{0\}} \mid \lambda \mid = \sup_{\lambda \in \sigma(BA) \cup \{0\}} \mid \lambda \mid .$$

However,

$$\sup_{\lambda \in \sigma(AB) \cup \{0\}} \mid \lambda \mid = \sup_{\lambda \in \sigma(AB)} \mid \lambda \mid$$

and

$$\sup_{\lambda \in \sigma(BA) \cup \{0\}} \mid \lambda \mid = \sup_{\lambda \in \sigma(BA)} \mid \lambda \mid .$$

So,

$$\sup_{\lambda \in \sigma(AB)} \mid \lambda \mid = \sup_{\lambda \in \sigma(BA)} \mid \lambda \mid .$$

Thus,

$$r(AB) = r(BA).$$

2. We have

$$r(\alpha A) = \sup_{\lambda \in \sigma(\alpha A)} \mid \lambda \mid .$$

However,

$$\sup_{\lambda \in \sigma(\alpha A)} \mid \lambda \mid = \sup_{\lambda \in \sigma(A)} \mid \alpha\lambda \mid$$

and

$$\sup_{\lambda \in \sigma(A)} \mid \alpha\lambda \mid = \mid \alpha \mid \sup_{\lambda \in \sigma(A)} \mid \lambda \mid .$$

So,

$$\mid \alpha \mid \sup_{\lambda \in \sigma(A)} \mid \lambda \mid = \mid \alpha \mid r(A).$$

Thus,

$$r(\alpha A) = \mid \alpha \mid r(A).$$

3. First, according to (Corollary 1.5.2), we have

$$\sigma(A^n) = (\sigma(A))^n.$$

Then,

$$r(A^n) = \sup_{\lambda \in \sigma(A^n)} \mid \lambda \mid .$$

However,

$$\sup_{\lambda \in \sigma(A^n)} \mid \lambda \mid = \sup_{\lambda \in \sigma(A)} \mid \lambda \mid^n$$

and

$$\sup_{\lambda \in \sigma(A)} \mid \lambda \mid^n = (\sup_{\lambda \in \sigma(A)} \mid \lambda \mid)^n.$$

So,

$$(\sup_{\lambda \in \sigma(A)} \mid \lambda \mid)^n = r^n(A).$$

Thus,

$$r(A^n) = r^n(A)$$

4. First, according to (Lemma 1.5.1), we have

$$\sigma(A^*) = \left\{ \bar{\lambda} : \lambda \in \sigma(A) \right\}.$$

Then,

$$r(A^*) = \sup_{\lambda \in \sigma(A^*)} \mid \lambda \mid .$$

However,

$$\sup_{\lambda \in \sigma(A^*)} \mid \lambda \mid = \sup_{\lambda \in \sigma(A)} \mid \bar{\lambda} \mid$$

and

$$\sup_{\lambda \in \sigma(A)} \mid \bar{\lambda} \mid = \sup_{\lambda \in \sigma(A)} \mid \lambda \mid .$$

So,

$$\sup_{\lambda \in \sigma(A)} \mid \lambda \mid = r(A).$$

Thus,

$$r(A^*) = r(A)$$

5. First, according to (Corollary 1.5.3), we have

$$\sigma(\sqrt{A}) = \sqrt{\sigma(A)}.$$

Then,

$$r(\sqrt{A}) = \sup_{\lambda \in \sigma(\sqrt{A})} \lambda.$$

However,

$$\sup_{\lambda \in \sigma(\sqrt{A})} \lambda = \sup_{\lambda \in \sigma(A)} \sqrt{\lambda}$$

and

$$\sup_{\lambda \in \sigma(A)} \sqrt{\lambda} = \sqrt{\sup_{\lambda \in \sigma(A)} \lambda}.$$

So,

$$\sqrt{\sup_{\lambda \in \sigma(A)} \lambda} = \sqrt{r(A)}.$$

Thus,

$$r(\sqrt{A}) = \sqrt{r(A)}.$$

$\square$

Remark 2.1.1 *$A = 0$, then $r(A) = 0$.*

But the converse is not true, that is if $r(A) = 0$, this does not imply that $A = 0$.

consides $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have that

$$\sigma(A) = \sigma_p(A) = \{0\}.$$

Then, $r(A) = 0$ but $A \neq 0$

Lemma 2.1.2 *Let $A \in \mathcal{L}(\mathcal{H})$, then*

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \textit{ exists and equals } \inf_{n\in\mathbb{N}} \|A^n\|^{\frac{1}{n}}.$$

Proof. Any $n \in \mathbb{N}$ can be written as $n = p^m + q$ where $0 \leqslant q \leqslant m$ and $p \in \mathbb{N}$, then

$$\begin{aligned} \|A^n\|^{\frac{1}{n}} = \|A^{p^m+q}\|^{\frac{1}{n}} &\leqslant \|A^m\|^{\frac{q}{n}} \\ &\leqslant (\alpha+\varepsilon)^{\frac{p^m}{n}} \|A\|^{\frac{q}{n}}. \end{aligned}$$

Since $\frac{p^m}{n} \underset{n\to\infty}{\longrightarrow} 1$, and $\frac{q}{n} \underset{n\to\infty}{\longrightarrow} 0$, it follows that

$$\sup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leqslant \alpha + \varepsilon.$$

As ε arbitrary, we have

$$\lim_{n\to\infty} \sup \|A^n\|^{\frac{1}{n}} \leqslant \alpha.$$

Since, $\forall n \in \mathbb{N} \quad \alpha \leqslant \|A^n\|^{\frac{1}{n}}$ and

$$\alpha \leqslant \lim_{n\to\infty} \inf \|A^n\|^{\frac{1}{n}}.$$

Then,

$$\lim_{n\to\infty} \inf \|A^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \sup \|A^n\|^{\frac{1}{n}} = \alpha.$$

Thus,

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \text{ exists, and } \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf \|A^n\|^{\frac{1}{n}}.$$

$\square$

Theorem 2.1.1 *(Glfand's formula)*
Let $A \in \mathcal{L}(\mathcal{H})$, then

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|A^n\|^{\frac{1}{n}}.$$

Proof. We have $\forall n \in \mathbb{N} \quad r^n(A) = r(A^n) \leqslant \| A^n \|$.

Then,

$$\forall n \in \mathbb{N} \quad r(A) \leqslant \| A^n \|^{\frac{1}{n}} .$$

Thus,

$$r(A) \leqslant \lim_{n\to\infty} \| A^n \|^{\frac{1}{n}} .$$

Let $| \lambda |> r(A)$ then, $\lambda \in \rho(A)$, recall that if $| \lambda |>\| \lambda \|$.

Then,

$$R(\lambda) = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} A^n.$$

Let $f \in \mathcal{L}(\mathcal{H})^{'}$, by the continuity of f, we have

$$f(R(\lambda)) = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} f(A^{-n}) \quad \forall \mid \lambda \mid > \| A \|$$

$\lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} f(A^{-n})$ is a Laurent expansion of $f(R(\lambda))$ about the origin when $\lambda \in \rho(A)$ wherer $| \lambda |>\| A \|$. □

Corollary 2.1.1 *Let A is normal and $A \in \mathcal{L}(\mathcal{H})$, then*

$$r(A) =\| A \| .$$

Proof. We have $\forall n \in \mathbb{N},\|A^n\| = \|A\|^n$

Then,

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \|A\|.$$

So,

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Thus,

$$r(A) = \|A\|.$$

□

Theorem 2.1.2 *Let $A, B \in \mathcal{L}(\mathcal{H})$ and, $AB = BA$ then*

1. $r(A+B) \leqslant r(A) + r(B)$.

2. $r(AB) \leqslant r(A)r(B)$.

Proof.

1. Let $\varepsilon > 0$, then there exists $m \in \mathbb{N}$ lary enough such that

$$\| A^n \|^{\frac{1}{n}} \leqslant r(A) + \varepsilon$$

and

$$\| B^n \|^{\frac{1}{n}} \leqslant r(B) + \varepsilon \quad \forall n \geqslant m.$$

Set $S = r(A) + \varepsilon$ and $A = r(B) + \varepsilon$, then $\| A^n \| \leqslant A$ and $\| B^n \| \leqslant S^n$ for all $n \geqslant m$.

For $n \geqslant 2m$, and using the fact that $AB = BA$, we have the following

$$\begin{aligned} \| (A+B)^n \| &= \| \sum_{k=0}^{n} C_k^n A^k B^{n-k} \| \\ &\leqslant \sum_{k=0}^{n} C_k^n \| A^k \| \| B^{n-k} \| \\ &= \sum_{k=0}^{n-1} C_k^n \| A^k \| \| B^{n-k} \| + \sum_{k=m}^{n-m} C_k^n \| A^k \| \| B^{n-k} \| + \sum_{k=n-m+1}^{n} C_k^n \| A^k \| \| B^{n-k} \| \end{aligned}$$

when $0 \leqslant k \leqslant m-1$, then $n-k \geqslant m$ (recall that $n \geqslant 2m$), thus $\| B^{n-k} \| \leqslant S^{n-k}$ for all $0 \leqslant k \leqslant m-1$.

when $m \leqslant k \leqslant n-m$, then $k \geqslant m$ and $-n+m \leqslant -k \leqslant -m$,

So $m \leqslant n-k \leqslant n-m$, thus $n-k \geqslant m$ for all $m \leqslant k \leqslant n-m$

therefore $\| A^k \| \leqslant T^k$ and $\| B^{n-k} \| \leqslant S^{n-k}$ for all $m \leqslant k \leqslant n-m$

when $n-m+1 \leqslant k \leqslant n$, then $k \geqslant m$ (recall that $n \geqslant 2m$),

thus $\| A^k \| \leqslant T^k$ for all $n-m+1 \leqslant k \leqslant n$.

Then,

$$\begin{aligned} \| (A+B)^n \| &\leqslant \sum_{k=0}^{m-1} C_k^n \| A^k \| S^{n-k} + \sum_{k=m}^{n-m} C_k^n T^k S^{n-k} + \sum_{k=n-m+1} C_k^n T^k \| B^{n-k} \| \\ &\leqslant \sum_{k=0}^{m-1} C_k^n \| A^k \| S^{n-k} + \sum_{k=0}^{n} C_k^n T^k S^{n-k} + \sum_{k=0}^{n-1} C_k^n T^{n-k} \| B^k \| \\ &= p(n) S^n + (T+S)^n + q(n) T^n \end{aligned}$$

when $p(n)$ and $q(n)$ are polynomial in n of degree $m-1$ give by
$p(n) = \sum_{k=0}^{m-1} C_k^n \parallel A^k \parallel S^{-k}$ and $q(n) = \sum_{k=0}^{m-1} C_k^n T^{-k} \parallel B^k \parallel$. For example
$p(n) = 1 + \parallel A \parallel S^{-1}n + 2^{-1} \parallel A^2 \parallel S^{-2}n(n-1) + ... + m!^{-1} \parallel A^{m-1} \parallel S^{-m+1}n(n-1)....(n-m+2)$.

Then,

$$\begin{array}{rcl} \lim_{n\to\infty} \parallel (A+B)^n \parallel^{\frac{1}{n}} & \leqslant & \lim_{n\to\infty} (p(n)S^n + (T+S)^n + q(n)T^n)^{\frac{1}{n}} \\ & = & (T+S) \lim_{n\to\infty} (1 + p(n)(\frac{S}{T+S})^n + q(n)(\frac{T}{T+S})^n)^{\frac{1}{n}} \end{array}.$$

Let's prove that

$$\lim_{n\to\infty} (1 + p(n)(\frac{S}{T+S})^n + q(n)(\frac{T}{T+S})^n)^{\frac{1}{n}} \leqslant 1.$$

Since

$$1 + p(n) + q(n) = \alpha_0 + \alpha_1 n + ... + \alpha_{m-1} n^{m-1}.$$

Then,

$$\begin{array}{rcl} (1 + p(n) + q(n))^{\frac{1}{n}} & = & \exp(\frac{1}{n} \ln(1 + p(n) + q(n))) \\ & = & \exp(\frac{1}{n} \ln(\alpha_0 + \alpha_1 n + ... + \alpha_{m-1} n^{m-1})) \underset{n\to\infty}{\longrightarrow} 1 \end{array}$$

and

$$(1 + p(n) + q(n))^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} 1.$$

So,

$$\lim_{n\to\infty} (1 + p(n)(\frac{S}{T+S})^n + q(n)(\frac{T}{T+S})^n)^{\frac{1}{n}} \leqslant 1$$

Thowever,

$$\begin{array}{rcl} \lim_{n\to\infty} \parallel (A+B)^n \parallel^{\frac{1}{n}} & \leqslant & (T+S) \lim_{n\to\infty} (p(n)S^n + (T+S)^n + q(n)T^n)^{\frac{1}{n}} \\ & \leqslant & A+B \end{array}.$$

Then,

$$\lim_{n\to\infty} \parallel (A+B)^n \parallel^{\frac{1}{n}} \leqslant T+S.$$

So,

$$r(A+B) \leqslant r(A) + r(A) + 2\varepsilon \quad \forall \varepsilon > 0.$$

Thus,

$$r(A+B) \leqslant r(A) + r(B)$$

2. we have $r(AB) = \lim_{n\to\infty} \| (AB)^n \|^{\frac{1}{n}}$, since $AB = BA$, then

$$(AB)^n = A^n B^n.$$

Therefore,

$$\begin{aligned} r(AB) &= \lim_{n\to\infty} \| A^n B^n \|^{\frac{1}{n}} \\ &\leqslant \lim_{n\to\infty} \| A^n \|^{\frac{1}{n}} \| B^n \|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \| A^n \|^{\frac{1}{n}} \lim_{n\to\infty} \| B^n \|^{\frac{1}{n}} \\ &= r(A)r(B). \end{aligned}$$

Thus,

$$r(AB) \leqslant r(A)r(B).$$

□

2.2 Spectral Radius of Matrix

Definition 2.2.1 *Let $A \in \mathcal{L}(\mathcal{H})$, we define the spectral radius $r(A)$ of A. By*

$$r(A) := \{\max \mid \lambda \mid, \, \lambda \in \sigma(A)\}.$$

Proposition 2.2.1 *The sum of the Eigen values of a matrix is equal to the sum of the elements of the principal (main) diagonal.*

Proof. Let A be a square matrix of order n.
The characteristic equation of A is $\det(A - \lambda I) = 0$

$$(i.e.) \quad \lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - ... + (-1)S_n = 0 \tag{2.2.1}$$

where

$$S_1 = \text{Sum of the diagonal elements of } A.$$

$$...$$

$$...$$

$$...$$

$$S_n = \text{determinant of } A.$$

We know the roots of the characteristic equation are called Eigen values of the given matrix.

Solving (2.2.1) we get n roots.

Let the n be $\lambda_1, \lambda_2, ...\lambda_n$.

i.e.,$\lambda_1, \lambda_2, ...\lambda_n$ are the Eignvalues of A.

We know already,

$$\begin{aligned} \lambda^n &- (Sum\ of\ the\ roots\lambda^{n-1} + [sum\ of\ the\ product\ of\ the\ roots\ taken\ two\ at\ a\ time\ ]\lambda^{n-2} \\ &- ... + (-1)^n(Product\ of\ the\ roots)) = 0 \end{aligned} \tag{2.2.2}$$

Sum of the roots $= S_1$ by (2.2.1) and (2.2.2)

(i.e.) $\lambda_1 + \lambda_2 + ... + \lambda_n = S_1$

(i.e.) $\lambda_1 + \lambda_2 + ... + \lambda_n = tr(A)$

Sum of the Eigen values = Sum of the main diagonal elements

Product of the roots $= S_n$ by (2.2.1) and (2.2.2)

(i.e.) $\lambda_1\lambda_2...\lambda_n = \det(A)$

Product of the Eigenvalues $= \det(A)$. □

Proposition 2.2.2 *A square matrix A and its transpose A^T have the same Eigenvalues.*

Proof. Let A be a square matrix of order n.

The characteristic equation of A and A_T are

$$\mid A - \lambda I \mid = 0 \tag{2.2.3}$$

and

$$\mid A^T - \lambda I \mid = 0 \tag{2.2.4}$$

Since, the determinant value is unaltered by the interchange of rows and columns.

We know $\mid A \mid = \mid A^T \mid$

Hence, (2.2.3) and (2.2.4) are identical.

The Eigenvalues of A and A_T are the same. □

Proposition 2.2.3 *The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.*

Proof. Let us consider the triangular matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Characteristic equation of is matrix. $\mid A - \lambda I \mid= 0$

i.e.,

$$\begin{vmatrix} a_{11}-\lambda & 0 & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

On expansion it gives $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$

i.e.,$\lambda = a_{11}, a_{22}, a_{33}$

which are diagonal elements of the matrix A. □

Proposition 2.2.4 *If λ is an Eigenvalue of a matrix A, then $\frac{1}{\lambda}$, $(\lambda \neq 0)$ is the Eignvalue of A^{-1}.*

Proof. If X be the Eigenvector corresponding to λ, then

$$Ax = \lambda x \tag{2.2.5}$$

Pre multiplying both sides by A^{-1}, we get

$$\begin{aligned} A^{-1}AX &= A^{-1}\lambda X \\ (2.2.5) \Rightarrow X &= \lambda A^{-1}X \\ X &= \lambda A^{-1}X \\ \Rightarrow \tfrac{1}{\lambda}X &= A^{-1}X \\ (i.e.) \quad A^{-1}X &= \tfrac{1}{\lambda}X \end{aligned}$$

This being of the same form as (2.2.5), shows that $\dfrac{1}{\lambda}$ is an Eigenvalue of the inverse matrix A^{-1}. □

Definition 2.2.2 *Orthogonal matrix.*

A square matrix A is said to be orthogonal if $AA^T = A^T A = I$.

i.e., $A^T = A^{-1}$.

Proposition 2.2.5 *If λ is an Eigenvalue of an orthogonal matrix, then $\frac{1}{\lambda}$ is an Eigenvalue*

Proof. Let A be an orthogonal matrix.
Given λ is an Eigenvalue of A.
$\Rightarrow \frac{1}{\lambda}$ is an Eigenvalue of A^{-1}.
Since, $A^T = A^{-1}$
$\frac{1}{\lambda}$ is an Eigenvalue of A^T
But, the matrices A and A^T have the same Eigenvalues, since the determinants $\mid A - \lambda I \mid$ and $\mid A^T - \lambda I \mid$ are the same.
Hence, $\frac{1}{\lambda}$ is also an Eigenvalue of A. □

2.3 Relation between Spectral Radius and Numerical Radius

Proposition 2.3.1 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

$$w(A) \leqslant \parallel A \parallel \leqslant 2w(A).$$

Proof. Let $x \in \mathcal{H}$ such that $\parallel x \parallel = 1$, then

$$\begin{aligned} \mid \langle Ax, x\rangle \mid &\leqslant \parallel Ax \parallel \parallel x \parallel \\ &\leqslant \parallel A \parallel \parallel x \parallel^2 \\ &= \parallel A \parallel . \end{aligned}$$

Thus $\sup\limits_{\parallel x \parallel = 1} \mid \langle Ax, x\rangle \mid \leqslant \parallel A \parallel$ therefore $w(A) \leqslant \parallel A \parallel$.
Let $x, y \in \mathcal{H}$, then

$$\mid \langle A\frac{x}{\parallel x \parallel}, \frac{x}{\parallel x \parallel}\rangle \mid \leqslant \parallel A \parallel .$$

Then,

$$\mid \langle Ax, x\rangle \mid \leqslant w(A) \parallel x \parallel^2$$

we have that
$\langle Ax, y\rangle = \frac{1}{4}\left(\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + i\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle\right)$

$$\begin{aligned}
|\langle Ax, y\rangle| &\leqslant \tfrac{1}{4}\left(\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + i\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle\right)\\
&\leqslant \tfrac{w(A)}{4}\left(\| x+y \|^2 + \| x-y \|^2 + \| x+iy \|^2 + \| x-iy \|^2\right)\\
&= w(A)\left(\| x \|^2 + \| y \|^2\right).
\end{aligned}$$

Thus $|\langle Ax, y\rangle| \leqslant w(A)\left(\| x \|^2 + \| y \|^2\right) \quad x, y \in H$, then

$$\begin{aligned}
\| A \| &= \sup_{\|x\|=\|y\|=1} |\langle Ax, x\rangle|\\
&\leqslant \sup_{\|x\|=\|y\|=1} w(A)\left(\| x \|^2 + \| y \|^2\right)\\
&= 2w(A).
\end{aligned}$$

Hence $\| A \| \leqslant 2w(A)$. □

Remark 2.3.1 *We deduce that usual norm on $\mathcal{L}(\mathcal{H})$ and the numerical radius w are equivalent norms.*

Lemma 2.3.1 *Let $A \in \mathcal{L}(\mathcal{H})$, then*

$$r(A) \leqslant w(A).$$

Proof. We have $\sigma(A) \subset \overline{w(A)}$. then

$$\begin{aligned}
r(A) &= \sup_{\lambda\in\sigma(A)} |\lambda|\\
&\leqslant \sup_{\lambda\in\overline{w(A)}} |\lambda|\\
&= \sup_{\lambda\in w(A)} |\lambda|\\
&= w(A).
\end{aligned}$$

Thus,

$$r(A) \leqslant w(A).$$

□

Proposition 2.3.2 *Let $A \in \mathcal{L}(\mathcal{H})$. If $r(A) = \| A \|$, then*

$$w(A) = \| A \|.$$

Proof. Since $r(A) = \| A \|$.

And we have that $r(A) \leqslant w(A) \leqslant \| A \|$.

Thus,

$$w(A) = r(A) = \| A \| .$$

$\square$

Corollary 2.3.1 *Let $A \in \mathcal{L}(\mathcal{H})$ be normal.*

Then,

$$w(A) = r(A) = \| A \| .$$

Theorem 2.3.1 *Let $A \in \mathcal{L}(\mathcal{H})$, if $w(A) = \| A \|$ then $r(A) = \| A \|$.*

Proof. Since $w(A) = \| A \|$, then $\exists (x_n) \subset \mathcal{H}$

where $\| x_n \| = 1$ such that $\lim\limits_{n\to\infty} | \langle Ax_n, x_n \rangle | = \| A \|$.

Set $\lambda = \lim\limits_{n\to\infty} \langle Ax_n, x_n \rangle$, so $| \lambda | = \| A \|$, then

$$\begin{aligned} | \langle Ax_n, x_n \rangle | &\leqslant \| Ax_n \| \| x_n \| \\ &= \| Ax_n \| \end{aligned}$$

and $\| Ax_n \| \leqslant \| A \|$.

Thus $\lim\limits_{n\to\infty} \| Ax_n \| = \| A \|$, therefore

$$\begin{aligned} \| (A - \lambda I)x_n \|^2 &= \langle (A - \lambda I)x_n, (A - \lambda I)x_n \rangle \\ &= \| Ax_n \|^2 + | \lambda |^2 + \overline{\lambda} \langle Ax_n, x_n \rangle - \lambda \langle x_n, Ax_n \rangle . \end{aligned}$$

So,

$$\| Ax_n \|^2 \underset{n\to\infty}{\longrightarrow} 2 \| A \|^2 - 2 | \lambda |^2 = 0.$$

Thus,

$$\lim_{n\to\infty} \| (A - \lambda I)x_n \| = 0.$$

Therefore $\lambda \in \sigma_{ap}(A)$, then $\lambda \in \sigma(A)$, and since $r(A) \leqslant \| A \|$ and $| \lambda | = \| A \|$.

Thus,

$$r(A) = \| A \| .$$

$\square$

Proposition 2.3.3 *Let $A \in \mathcal{L}(\mathcal{H})$, if $\lambda \in w(A)$ such that $\mid \lambda \mid = \parallel A \parallel$, then $\lambda \in \sigma_p(A)$.*

Proof. Since $\lambda \in w(A)$, then $\exists x \in \mathcal{H}$ where $\parallel x \parallel = 1$

$\lambda = \langle Ax, x \rangle$,$\parallel A \parallel = \mid \langle Ax, x \rangle \mid$, then

$$\begin{aligned} \mid \lambda \mid &= \mid \langle Ax, x \rangle \mid \\ &\leqslant \parallel Ax \parallel \parallel x \parallel \\ &\leqslant \parallel A \parallel \\ &= \mid \langle Ax, x \rangle \mid . \end{aligned}$$

Then,

$$\mid \langle Ax, x \rangle \mid = \parallel Ax \parallel \parallel x \parallel .$$

Then Ax and x are linearly dependent i.e

$\exists \mu \in \mathbb{C}$ such that $Ax = \mu x$, then

$$\begin{aligned} \lambda &= \langle Ax, x \rangle \\ &= \langle \mu x, x \rangle \\ &= \mu. \end{aligned}$$

Then $\lambda = \mu$ so, $Ax = \lambda x$.

Thus $\lambda \in \sigma_p(A)$. □

Theorem 2.3.2 *[13] Let $A \in \mathcal{L}(\mathcal{H})$. Then*

$$w(A) \leqslant \frac{1}{2} \parallel \mid A \mid + \mid A^* \mid \parallel .$$

Proof. Let $\parallel x \parallel = 1$ we get

$$\begin{aligned} \mid \langle Ax, x \rangle \mid &\leqslant \sqrt{\langle \mid A \mid x, x \rangle} \sqrt{\langle \mid A^* \mid x, x \rangle} \\ &\leqslant \tfrac{1}{2}(\langle \mid A \mid x, x \rangle + \langle \mid A^* \mid x, x \rangle) \\ &= \tfrac{1}{2}\langle (\mid A \mid + \mid A^* \mid) x, x \rangle \\ &\leqslant \tfrac{1}{2} \parallel \mid A \mid + \mid A^* \mid \parallel . \end{aligned}$$

Thus,

$$w(A) \leqslant \frac{1}{2} \parallel \mid A \mid + \mid A^* \mid \parallel .$$

□

Remark 2.3.2 *Let $A \in \mathcal{L}(\mathcal{H})$*

$$\begin{aligned}\tfrac{1}{2} \parallel\mid A \mid + \mid A^* \mid\parallel &\leqslant \tfrac{1}{2}(\parallel\mid A \mid\parallel + \parallel\mid A^* \mid\parallel)\\ &= \tfrac{1}{2}(\parallel A \parallel + \parallel A \parallel)\\ &= \parallel A \parallel .\end{aligned}$$

Lemma 2.3.2 *Let $A, B \in \mathcal{L}(\mathcal{H})$ be positive operators. Then*

$$r(AB) = \parallel \sqrt{A}\sqrt{B} \parallel^2 .$$

Proof. By the commutativity of the spectral radius r, and using the fact that the spectral radius of any self-adjoint operator is equal its the norm, we obtain

$$\begin{aligned}r(AB) = r(\sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B}) &= r(\sqrt{A}\sqrt{B}\sqrt{B}\sqrt{A})\\ &= r(\sqrt{A}\sqrt{B}(\sqrt{A}\sqrt{B})^*)\\ &= \parallel \sqrt{A}\sqrt{B}(\sqrt{A}\sqrt{B})^* \parallel\\ &= \parallel \sqrt{A}\sqrt{B} \parallel^2 .\end{aligned}$$

Thus,

$$r(AB) = \parallel \sqrt{A}\sqrt{B} \parallel^2 .$$

□

Corollary 2.3.2 *Let $A, B \in \mathcal{L}(\mathcal{H})$ be positive operators. Then*

$$\parallel \sqrt{A}\sqrt{B} \parallel \leqslant \sqrt{\parallel AB \parallel}.$$

Proof. We have $\parallel \sqrt{A}\sqrt{B} \parallel = \sqrt{r(AB)} \leqslant \sqrt{\parallel AB \parallel}$
as desired. □

Proposition 2.3.4 *[4] Let $A \in \mathcal{L}(\mathcal{H})$. Then*

$$w(A) \leqslant \frac{1}{2}(\parallel A \parallel + \sqrt{\parallel A^2 \parallel})$$

.

Proof. We have $w(A) \leqslant \frac{1}{2} \||| A | + | A^* |||$ and we have

$$\||| A | + | A^* |||\leqslant \max\{\||| A |||, \||| A^* |||\} + \| \sqrt{| A |} + \sqrt{| A^* |} \|$$

since $\||| A |||=\||| A^* |||=\| A \|$, then $\max\{\||| A |||, \||| A^* |||\} =\| A \|$.
We have $\| \sqrt{| A |}\sqrt{| A^* |} \|\leqslant \sqrt{\||| A || A^* |||}$ and $\||| A || A^* |||=\| A^2 \|$.
Then $\| \sqrt{| A |}\sqrt{| A^* |} \|\leqslant \sqrt{\| A^2 \|}$, therefore

$$\||| A | + | A^* |||\leqslant\| A \| +\sqrt{\| A^2 \|}.$$

Hence $w(A) \leqslant \frac{1}{2}(\| A \| +\sqrt{\| A^2 \|})$. □

Corollary 2.3.3 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

1. *If $A^2 = 0$, then $w(A) = \frac{1}{2} \| A \|$.*

2. *If $w(A) =\| A \|\Rightarrow\| A^2 \|=\| A \|^2$.*

Proof.

1. We have $w(A) \leqslant \frac{1}{2}(\| A \| +\sqrt{\| A^2 \|})$
since $A^2 = 0$, then $w(A) \leqslant \frac{1}{2} \| A \|$, and $\frac{1}{2} \| A \|\leqslant w(A)$ is always true.
Hence $w(A) = \frac{1}{2} \| A \|$.

2. Since $w(A) \leqslant \frac{1}{2}(\| A \| +\sqrt{\| A^2 \|}) \leqslant\| A \|$, and we have that $w(A) =\| A \|$.
Then $\frac{1}{2}(\| A \| +\sqrt{\| A^2 \|}) =\| A \|\Rightarrow \sqrt{\| A^2 \|} =\| A \|$.
Thus $\| A^2 \|=\| A \|^2$.

□

Lemma 2.3.3 *[5] Let $A, B \in \mathcal{L}(\mathcal{H})$ such that $| A | B = B^* | A |$, then*

$$| \langle ABx, y\rangle |\leqslant r(A) \| \sqrt{| A |}x \|\| \sqrt{| A^* |}y \| \quad \forall x, y \in \mathcal{H}.$$

Proof. By induction Let's prove that
$| \langle ABx, y\rangle |^{2^n}\leqslant \langle| A | B^{2^n}x, x\rangle\langle| A | x, x\rangle^{2^{n-1}-1}\langle| A^* | y, y\rangle^{2^{n-1}} \quad \forall n \in \mathbb{N}$ and $x, y \in \mathcal{H}$
for $n = 1$, Let's prove that $| \langle ABx, y\rangle |^2\leqslant \langle| A | B^2x, x\rangle\langle| A^* | y, y\rangle$

using $B^* \mid A \mid = \mid A \mid B$, we get

$$\begin{aligned} \langle ABx, y\rangle^2 &\leqslant \langle \mid A \mid Bx, Bx\rangle\langle \mid A^* \mid y, y\rangle \\ &= \langle B^* \mid A \mid Bx, x\rangle\langle \mid A^* \mid y, y\rangle. \end{aligned}$$

Then $\langle ABx, y\rangle^2 \leqslant \langle \mid A \mid B^2 x, x\rangle\langle \mid A^* \mid y, y\rangle$

assume that $\mid \langle ABx, y\rangle \mid^{2^n} \leqslant \langle \mid A \mid B^{2^n} x, x\rangle\langle \mid A \mid x, x\rangle^{2^{n-1}-1}\langle \mid A^* \mid y, y\rangle^{2^{n-1}}$ is true

and Let's prove that

$\mid \langle ABx, y\rangle \mid^{2^{n+1}} \leqslant \langle \mid A \mid B^{2^{n+1}} x, x\rangle\langle \mid A \mid x, x\rangle^{2^n-1}\langle \mid A^* \mid y, y\rangle^{2^n}$ is also true then

$$\begin{aligned} \mid \langle ABx, y\rangle \mid^{2^{n+1}} &= (\mid \langle ABx, y\rangle \mid^{2^n})^2 \\ &\leqslant (\langle \mid A \mid B^{2^n} x, x\rangle\langle \mid A \mid x, x\rangle^{2^{n-1}-1}\langle \mid A^* \mid y, y\rangle^{2^{n-1}})^2 \\ &= \langle \mid A \mid B^{2^n} x, x\rangle^2\langle \mid A \mid x, x\rangle^{2^n-2}\langle \mid A^* \mid y, y\rangle^{2^n} \end{aligned}$$

we have $\langle \mid A \mid B^{2^n} x, x\rangle^2 \leqslant \langle \mid A \mid B^{2^n} x, B^{2^n} x\rangle\langle \mid A \mid x, x\rangle$.

Therefore

$$\begin{aligned} \mid \langle ABx, y\rangle \mid^{2^{n+1}} &\leqslant \langle \mid A \mid B^{2^n} x, B^{2^n} x\rangle\langle \mid A \mid x, x\rangle\langle \mid A \mid x, x\rangle^{2^n-2}\langle \mid A^* \mid y, y\rangle^{2^n} \\ &= \langle (B^{2^n})^* \mid A \mid B^{2^n} x, x\rangle\langle \mid A \mid x, x\rangle^{2^n-1}\langle \mid A^* \mid y, y\rangle^{2^n} \\ &= \langle \mid A \mid B^{2^n} B^{2^n} x, x\rangle\langle \mid A \mid x, x\rangle^{2^n-1}\langle \mid A^* \mid y, y\rangle^{2^n} \\ &= \langle \mid A \mid B^{2^{n+1}} x, x\rangle\langle \mid A \mid x, x\rangle^{2^n-1}\langle \mid A^* \mid y, y\rangle^{2^n}. \end{aligned}$$

Hence $\mid \langle ABx, y\rangle \mid^{2^{n+1}} \leqslant \langle \mid A \mid B^{2^{n+1}} x, x\rangle\langle \mid A \mid x, x\rangle^{2^n-1}\langle \mid A^* \mid y, y\rangle^{2^n}$.

Then $\mid \langle ABx, y\rangle \mid^{2^n} \leqslant \langle \mid A \mid B^{2^n} x, x\rangle\langle \mid A \mid x, x\rangle^{2^{n-1}-1}\langle \mid A^* \mid y, y\rangle^{2^{n-1}} \quad n \in \mathbb{N} \quad x, y \in \mathcal{H}$.

Therefore

$$\mid \langle ABx, y\rangle \mid^{2^n} \leqslant \parallel \mid A \mid \parallel \parallel B^{2^n} \parallel \parallel x \parallel^2 \langle \mid A \mid x, x\rangle^{2^{n-1}-1}\langle \mid A^* \mid y, y\rangle^{2^{n-1}}$$

$$\mid \langle ABx, y\rangle \mid \leqslant \parallel \mid A \mid \parallel^{2^{-n}} \parallel B^{2^n} \parallel^{2^{-n}} \parallel x \parallel^{2^{1-n}} \langle \mid A \mid x, x\rangle^{2^{-1}-2^{-n}}\langle \mid A^* \mid y, y\rangle^{2^{-1}}$$

since

$\lim\limits_{n\to\infty} \parallel \mid A \mid \parallel^{2^{-n}} \parallel B^{2^n} \parallel^{2^{-n}} \parallel x \parallel^{2^{1-n}} \langle \mid A \mid x, x\rangle^{2^{-1}-2^{-n}}\langle \mid A^* \mid y, y\rangle^{2^{-1}} = r(B)\sqrt{\langle \mid A \mid x, x\rangle}\sqrt{\langle \mid A^* \mid y, y\rangle}$.

Then

$$\begin{aligned} \mid \langle ABx, y\rangle \mid &\leqslant r(B)\sqrt{\langle \mid A \mid x, x\rangle}\sqrt{\langle \mid A^* \mid y, y\rangle} \\ &= r(B) \parallel \sqrt{\mid A \mid} x \parallel \parallel \sqrt{\mid A^* \mid} y \parallel. \end{aligned}$$

Hence $\mid \langle ABx, y\rangle \mid \leqslant r(B) \parallel \sqrt{\mid A \mid} x \parallel \parallel \sqrt{\mid A^* \mid} y \parallel \quad \forall x, y \in \mathcal{H}$. $\square$

Theorem 2.3.3 *[15] Let $A, B \in \mathcal{L}(\mathcal{H})$ such that $\mid A \mid B = B^* \mid A \mid$. Then*

$$w(AB) \leqslant \frac{r(B)}{2}(\| A \| + \sqrt{r(\mid A \mid\mid A^* \mid)}).$$

Proof. Let $x \in \mathcal{H}$ such that $\| x \| = 1$, we obtain

$$\begin{aligned}
\mid \langle ABx, x \rangle \mid &\leqslant r(B) \| \sqrt{\mid A \mid} x \| \| \sqrt{\mid A^* \mid} x \| \\
&\leqslant \tfrac{r(B)}{2}(\| \sqrt{\mid A \mid} x \|^2 + \| \sqrt{\mid A^* \mid} x \|^2) \\
&\leqslant \tfrac{r(B)}{2}(\langle \mid A \mid x, x \rangle + \langle \mid A^* \mid x, x \rangle) \\
&= \tfrac{r(B)}{2}(\langle (\mid A \mid + \mid A^* \mid) x, x \rangle) \\
&\leqslant \tfrac{r(B)}{2} \| \mid A \mid + \mid A^* \mid \| \\
&\leqslant \tfrac{r(B)}{2}(\| A \| + \| \sqrt{\mid A \mid} \sqrt{\mid A^* \mid} \|) \\
&= \tfrac{r(B)}{2}(\| A \| + \sqrt{r(\mid A \mid\mid A^* \mid)}).
\end{aligned}$$

Hence $w(AB) \leqslant \frac{r(B)}{2}(\| A \| + \sqrt{r(\mid A \mid\mid A^* \mid)})$. □

Chapter 3

Applications

3.1 Weighted Shifts

We recall some preliminary definitions and results.

Definition 3.1.1 *Let* $X = \ell^p(\mathbb{N})$, $1 \leqslant p < \infty$ *or* $X = c_0(\mathbb{N})$.
Let $w = \{w_n\}_{n\in\mathbb{N}}$ *be a bounded sequence of scalars, called weight sequence. Then*

1. *The unilateral weighted forward shift* $F_w : X \Rightarrow X$ *is defined by*

$$F_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n-1}x_{n-1}\}_{n\in\mathbb{N}}$$

meaning

$$F_w(\{x_1, x_2, ...\}) = \{0, w_1x_1, w_2x_2, ...\}.$$

2. *The unilateral weighted backward shift* $B_w : X \Rightarrow X$ *is defined by*

$$B_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n+1}x_{n+1}\}_{n\in\mathbb{N}}$$

meaning

$$B_w(\{x_1, x_2, ...\}) = \{w_2x_2, w_3x_3, ...\}.$$

If, instead of $\mathbb{N}$, *we consider* $\mathbb{Z}$, *the shift is called bilateral.*

Clearly, a weighted shift (unilateral or bilateral) is injective if and only if none of the weights is zero, and, a bilateral weighted shift is invertible if and only if $\inf_{n\in\mathbb{Z}} \mid w_n \mid> 0$. *Of course, a unilateral weighted shift is never invertible.*

Remark 3.1.1 *Let $w = \{w_n\}_{n\in\mathbb{Z}}$ be a weight sequence with $\inf_{n\in\mathbb{Z}} \mid w_n \mid > 0$.*
Obviously, $B_w^{-1} = F_{\tilde{w}}$ and $F_w^{-1} = B_{\tilde{w}}$, where $\tilde{w} = \{\frac{1}{w_n}\}_{n\in\mathbb{Z}}$.

Remark 3.1.2 *Let $w = \{w_n\}_{n\in\mathbb{Z}}$, be a weight sequence.*
Let A be the bilateral weighted shift F_w or B_w on $X = \ell^p(\mathbb{Z})$, $1 \leqslant p < \infty$ or $X = c_0(\mathbb{Z})$.
It will be useful for the sequel to note that, for every $n \in \mathbb{N}$

1. $\| A^n \| = \sup\limits_{k\in\mathbb{Z}} \mid w_k w_{k+1} \dots w_{k+n+1} \mid$.
2. *If A is invertible, then* $\| A^{-n} \| = \sup\limits_{k\in\mathbb{Z}} \mid w_k w_{k+1} \dots w_{k+n+1} \mid^{-1}$.

Proof. We only show (1) for $A = F_w$, as the rest follows in a similar fashion.
Given $A = F_w$ on $X = \ell^p(\mathbb{Z})$, $1 \leqslant p < \infty$, then

$$\begin{aligned}
\| F_w^n(\{x_k\}_{k\in\mathbb{Z}}) \|_p &= \| \{w_{k-n} \dots w_{k-1} x_{k-n}\} \|_p \\
&= \left(\sum_{k\in\mathbb{Z}} \mid w_{k-n} \dots w_{k-1} x_{k-n} \mid^p \right)^{\frac{1}{p}} \\
&\leqslant \sup_{k\in\mathbb{Z}} \mid w_{k-n} \dots w_{k-1} \mid \left(\sum_{k\in\mathbb{Z}} \mid x_{k-n} \mid^p \right)^{\frac{1}{p}} \\
&= \sup_{k\in\mathbb{Z}} \mid w_k \dots w_{k+n-1} \mid \left(\sum_{k\in\mathbb{Z}} \mid x_k \mid^p \right)^{\frac{1}{p}} \\
&= \sup_{k\in\mathbb{Z}} \mid w_k \dots w_{k+n-1} \mid \| \{x_k\}_{n\in\mathbb{Z}} \|_p
\end{aligned}$$

and on $X = c_0(\mathbb{Z})$,

$$\begin{aligned}
\| F_w^n(\{x_k\}_{k\in\mathbb{Z}}) \|_\infty &= \| \{w_{k-n} \dots w_{k-1} x_{k-n}\} \|_\infty \\
&= \sup_{k\in\mathbb{Z}} \mid w_{k-n} \dots w_{k-1} x_{k-n} \mid \\
&\leqslant \sup_{k\in\mathbb{Z}} \mid w_{k-n} \dots w_{k-1} \mid \sup_{k\in\mathbb{Z}} \mid x_{k-n} \mid \\
&= \sup_{k\in\mathbb{Z}} \mid w_k \dots w_{k+n-1} \mid \sup_{k\in\mathbb{Z}} \mid x_k \mid \\
&= \sup_{k\in\mathbb{Z}} \mid w_k \dots w_{k+n-1} \mid \| \{x_k\}_{n\in\mathbb{Z}} \|_\infty
\end{aligned}$$

Hence,

$$\begin{aligned}
\| F_w^n \| &= \inf_{\{x_k\}\in X} \{c \geqslant 0 : \| F_w^n(\{x_k\}_{k\in\mathbb{Z}}) \|_X \leqslant c \| \{x_k\}_{k\in\mathbb{Z}} \|_X\} \\
&\leqslant \sup_{k\in\mathbb{Z}} \mid w_k \dots w_{k+n-1} \mid .
\end{aligned}$$

By computing F_w^n at $e_k = \{...0, 0, 1, 0, 0, ...\}$, he reverse of the above inequality is obtained. Hence

$$\| F_w^n \| = \sup_{k\in\mathbb{Z}} | w_k w_{k+1} ... w_{k+n-1} | .$$

Moreover, if F_w is invertible, an analogous computation gives

$$\| F_w^{-n} \| = \sup_{k\in\mathbb{Z}} | w_k w_{k+1} ... w_{k+n-1} |^{-1} .$$

□

Remark 3.1.3 *By replacing $\mathbb{Z}$ by $\mathbb{N}$ in* (1) *of Remark 3.1.2, an analogous argument gives the norms of F_w and B_w in the unilateral case.*

Proposition 3.1.1 *([1], Theorem 2.1); ([2], Theorem 5); ([14], Remark 35)*
Let $X = \ell^p(\mathbb{N})$, $1 \leqslant p < \infty$ or $X = c_0(\mathbb{N})$.
Let A be the bilateral weighted shift F_w or B_w, on X. The followings hold.

1. *If A is non-invertible, then its spectrum is the disk*

$$\sigma(A) = \{\lambda \in \mathbb{C} : | \lambda | \leqslant r(A)\}.$$

2. *If T is invertible, then its spectrum is the annulus*

$$\sigma(A) = \{\lambda \in \mathbb{C} : \frac{1}{r(A^{-1})} \leqslant \| \lambda \| \leqslant r(A)\}.$$

The followings are well-known results which will be useful in the sequel.

Proposition 3.1.2 *([11], Exercise 5.2.10), ([9], Proposition 1.6.14)([2], Proposition 4)*
Let $X = \ell^p, 1 \leqslant p < \infty$, or $X = c_0$. Let A be the unilateral (resp. bilateral) weighted F_w or B_w. Then, A is compact if and only if $\lim_{n\to\infty} w_n = 0$(resp. $\lim_{|n|\to\infty} w_n = 0$).

When the weights $w = \{w_n\}_{n\in A}$, with $A = N$ or $A = Z$, are such that $w_n = 1$ for each $n \in A$, then F_w and B_w reduce to the (unweighted) forward and backward shifts denoted by F and B, respectively.
Clearly, $\| F \| = \| B \| = 1$, and for this simple case the spectrum and its part were completely analyzed, as partially summarized in the following result (for adetailed description of all the parts of the spectrum, see [3] and ([17], Proposition 2.M], for the case $p = 2$; and ([16], Corollary 3.2); ([10], Example 3.7.7); ([8], Proposition 10.2.8), for the case $1 \leqslant p < \infty$).

Proposition 3.1.3 *Let B and F be the unilateral backward and the unilateral forward shift, respectively. The followings hold.*

1. $\sigma_p(F) = \emptyset; \sigma_e(F) = \mathbb{T}$.

2. $\sigma_p(B) = \mathbb{D}; \sigma_e(B) = \mathbb{T}$.

3. $\sigma(F) = \sigma(B) = \{\lambda \in \mathbb{C} :\mid \lambda \mid \leqslant 1\}$.

If B and F denote the bilateral backward and the bilateral forward shift, respectively, then

1. $\sigma_p(F) = \emptyset; \sigma_e(F) = \mathbb{T}$.

2. $\sigma_p(B) = \emptyset; \sigma_e(B) = \mathbb{T}$.

3. $\sigma(F) = \sigma(B) = \mathbb{T}$.

The definition of an adjoint operator is given, in general, for operators defined on a Hilbert space. If one tries to transfer this definition to operators on Banach spaces, it immediately appears an obstacle: the absence of an inner product. Hence, given a Banach space X, it is necessary to introduce the dual space X^*, and to define a new product on $X \times X^*$, $\langle .,. \rangle$, as in the following definition.

Definition 3.1.2 *Let X be a Banach space and let X^* be its dual. Let $x^* \in X^*$. Then, for each $x \in X$, we define*

$$\langle x, x^* \rangle = x^*(x).$$

Example 3.1.1 *(Adjoint operator)*
Let X be a Banach space and let $A \in \mathcal{L}(X)$. The operator $A^* \in \mathcal{L}(X^*)$ defined by $A^*x^* = x^* \circ A$, that is

$$\langle x, A^*x^* \rangle = \langle Ax, x^* \rangle \quad x \in X, x^* \in X^*$$

s the adjoint of A. The operator A is said to be unitary if $A^* = A^{-1}$.

The following example will be useful in the sequel.

$$\begin{aligned} x^*(x) &= \langle x, x^* \rangle \\ &= \sum_{n \in A} x_n \overline{y_n} \end{aligned}$$

with $y = \{y_n\}_{n\in A} \in \ell^p(A)$.

Now, let $A = \mathbb{Z}$ *and let* Fw *be a bilateral weighted forward shift on* $\ell^p(\mathbb{Z}), 1 \leqslant p < \infty$. *Then, according to Definition 3.1.2, its adjoint is a bilateral weighted backward shift on* $\ell^p(\mathbb{Z})$ *given by*

$$F_w^*(e_n) = \overline{w_{n-1}}e_{n-1}$$

for each $n \in \mathbb{Z}$ *(see, for instance, [18]). In fact, given any* $x \in \ell^p(\mathbb{Z})$ *and* $x^* \in \ell^p(\mathbb{Z})$, *it is*

$$\begin{aligned}
\langle x, F_w^* x^* \rangle &= \langle F_w x, x^* \rangle \\
&= \sum_{n=\infty}^{+\infty} (F_w x)_n \overline{y_n} \\
&= \sum_{n=\infty}^{+\infty} w_{n-1} x_{n-1} \overline{y_n} \\
&= \sum_{n=\infty}^{+\infty} w_n x_n \overline{y_{n+1}} \\
&= \sum_{n=\infty}^{+\infty} x_n (\overline{\overline{w_n} y_{n+1}}) \\
&= \sum_{n=\infty}^{+\infty} x_n B_{\tilde{w}}(\overline{y_n}) \\
&= \langle x, B_{\tilde{w}} x^* \rangle
\end{aligned}$$

(where $\tilde{w} = \{\tilde{w_n}\}_n$ *with* $\tilde{w_n} = \overline{w_{n-1}}$*)*

and hence $F_w^*(e_n) = B_{\tilde{w_n}}(e_n) = \tilde{w_n} e_{n-1} = w_{\tilde{n-1}} e_{n-1}$, *for each* $n \in \mathbb{Z}$.

Analogously, if $A = \mathbb{N}$ *and* F_w *is a unilateral weighted forward shift on* $\ell^p(\mathbb{N}), 1 \leqslant p < \infty$, *then its adjoint is the unilateral weighted backward shift on* $\ell^q(\mathbb{N})$ *given by* $F_w^*(e_0) = 0$ *and* $F_w^*(e_n) = \overline{w_{n-1}}e_{n-1}$, *for each* $\geqslant 1$.

Similarly, $B_w^*(e_n) = \overline{w_{n+1}}e_{n+1}$, *for each* $n \geqslant 1$.

3.2 On the Spectrum of Weighted Shifts

Up to now, we have not made assumptions about the sign of the weights. It is well-known, (see ([7], p. 54); ([2], p. 56)), that if a weighted shift has a finite number of zero weights, then it is the direct sum of a finite number of finite dimensional operators and a weighted shift with nonzero weights. Hence, weighted shifts with a finite number of zero weights lead back to weighted shifts with all weights non-zero. For this reason, and by Proposition 2.3.4 and Proposition 2.3.5, from now on we only consider positive weights $\{w_n\}_{n\in A}$.

In the previous section, Proposition 2.2.6 describes the spectrum of a weighted shifts with $w = \{w_n\}_{n\in A}$, $A = \mathbb{N}$ or $A = \mathbb{Z}$, a bounded sequence of scalars. In this section, we will see more detailed results on the spectrum of F_w and B_w, both in the unilateral case and in the bilateral case.

3.2.1 The Unilateral Case

Proposition 3.2.1 *Let A be a unilateral weighted shift F_w or B_w on $\ell^p(\mathbb{N})$, $1 \leqslant p < \infty$, with $\{w_n\}_{n\in\mathbb{N}}$ a bounded sequence of positive reals. Let*

$$\lim_{n\to\infty} w_n = w.$$

Then, the spectral radius of A is

$$r(A) = w.$$

Proof. The proof is showed for $A = F_w$, as small changes provide the case $A = B_w$. Hence, consider $A = F_n$. Let $\varepsilon < 0$ and $\overline{n} \in \mathbb{N}$ such that $\mid w_n - w \mid < \varepsilon$ for each $n \geqslant \overline{n}$. Then, in particular, for each $n > \overline{n}$

$$0 < w_{\overline{n}} ... w_{n-1} < (w+\varepsilon)^{n-\overline{n}}.$$

Therefore, for each $k \in \mathbb{N}$ and for each $n > \overline{n}$

$$0 < w_{k+\overline{n}} ... w_{k+n-1} < (w+\varepsilon)^{n-\overline{n}}$$

and, hence

$$\begin{aligned} 0 &< w_k w_{k+\overline{n}-1} ... w_{k+\overline{n}} \\ &< w_{k+\overline{n}-1} w_{k+\overline{n}} ... (w+\varepsilon)^{n-\overline{n}} \end{aligned}$$

Then, it follows that

$$\begin{aligned}
0 &\leqslant r(F_w) \\
&= \lim_{n\to\infty} \| F_w^n \|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} (\sup_{n\in\mathbb{N}} \mid w_k ... w_{k+n-1} \mid)^{\frac{1}{n}} \\
&\leqslant \lim_{n\to\infty} [\sup_{n\in\mathbb{N}} (w_k ... w_{k+\overline{n}-1})]^{\frac{1}{n}} (w+\varepsilon)^{1-\frac{\overline{n}}{n}} \\
&\leqslant \lim_{n\to\infty} (\lim_{k\in\mathbb{N}} w_k)^{\frac{\overline{n}}{n}} (w+\varepsilon)^{1-\frac{\overline{n}}{n}} \\
&\leqslant \lim_{n\to\infty} M^{\frac{\overline{n}}{n}} (w+\varepsilon)^{1-\frac{\overline{n}}{n}} \quad (M = \max\{\sup_{n\in\mathbb{N}} w_k, w+\varepsilon\}) \\
&= w+\varepsilon.
\end{aligned}$$

If $w = 0$, then, as ε is arbitrary, it follows that $r(F_w) = 0 = w$. Now, assume that $w > 0$. From the previous computation we obtain that $r(F_w) \leqslant w$. We want to show that $r(F_w) \leqslant w$. Let F denote the unilateral forward shift on $\ell^p(\mathbb{N}), 1 \leqslant p < \infty$, i.e.

$$F(\{x_n\}_{n\in\mathbb{N}}) = \{x_{n-1}\}_{n\in\mathbb{N}}.$$

Consider the operator $F_{\tilde{w}} = F_w - wF$ defined by

$$\begin{aligned}
F_{\overline{w}}(\{x_n\}_{n\in\mathbb{N}}) &= (F_w - wF)(\{x_n\}_{n\in\mathbb{N}}) \\
&= \{(w_{n-1} - w)x_{n-1}\}_{n\in\mathbb{N}}.
\end{aligned}$$

Hence, F is the unilateral weighted forward shift on $\ell^p(\mathbb{N}), 1 \leqslant p < \infty$, with weights $\tilde{w} = \{w_n - w\}_{n\in\mathbb{N}}$. In our case, the hypothesis

$$\lim_{n\to\infty} w_n = w$$

implies that

$$\lim_{n\to\infty} \tilde{w_n} = 0$$

which means, by Proposition 2.2.7, that $F_{\tilde{w}}$ is compact. Therefore, from Proposition 2.1.4 it follows that

$$\begin{aligned}
\sigma_e(F_w) &= \sigma_e(wF + (F_w - wF)) \\
&= \sigma_e(wF + F_{\tilde{w}}) \\
&= \sigma_e(wF) \quad (\textit{as } F_{\tilde{w}} \textit{ is compact}) \\
&= w\mathbb{T} \\
&= \{\lambda :\mid \lambda \mid = w\}.
\end{aligned}$$

As $\sigma_e(F_w)$ is a closed subset of $\sigma(F_w)$, then it follows

$$r(F_w) = \sup\{\mid \lambda \mid, \lambda \in \sigma(F_w)\} \geqslant w$$

Hence, the spectral radius of F_w is $r(F_w) = w$. □

Remark 3.2.1 *Let $X = \ell^p(\mathbb{N}), 1 \leqslant p < \infty$, and let A denote a unilateral weighted shift F_w or B_w, on X, with $\{w_n\}_{n\in\mathbb{N}}$ a positive weight sequence. Let*

$$\lim_{n\to\infty} w_n = w.$$

Then, by (1) of Remark 2.2.3 and Proposition 3.1.1, together with the spectral radius formula, we get

$$\begin{aligned} r(A) &= w \\ &= \lim_{n\to\infty} w_n \\ &= \lim_{n\to\infty} \parallel A^n \parallel^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left(\sup_{k\in\mathbb{N}} \mid w_k w_{k+1} \ldots w_{k+n-1} \mid \right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left[\sup_{k\in\mathbb{N}} (w_k w_{k+1} \ldots w_{k+n-1}) \right]^{\frac{1}{n}}. \end{aligned}$$

Take $w_n = 3^{(-1)^n}$. Then

$$\lim_{n\to\infty} \left[\sup_{k\in\mathbb{N}} (w_k w_{k+1} \ldots w_{k+n-1}) \right]^{\frac{1}{n}} = 1$$

but, clearly, $\{w_n\}_{n\in\mathbb{N}}$ is not regular, with $\overline{\lim_{n\to\infty}} w_n = 3$ and $\underline{\lim_{n\to\infty}} w_n = \frac{1}{3}$.
In particular, Proposition 3.1.1 does not hold anymore if we replace $\lim$ with $\overline{\lim}$.

We point out that, in the Hilbert case $p = 2$, Proposition 3.1.1 is a consequence of ([2], Proposition 15) and ([7], Proposition 6.8(a)). As in ([12], Remark 1.2) and ([2], Theorem 8) for the Hilbert case $p = 2$, the point spectrum of a unilateral forward weighted shift on $\ell^p(\mathbb{N}), 1 \leqslant p < \infty$, turns out to be empty, and the point spectrum of a unilateral backward weighted shift on $\ell^p(\mathbb{N}), 1 \leqslant p < \infty$, satisfies

$$\{0\} \cup \{\lambda \in \mathbb{C} : \mid \lambda \mid < \tilde{r}\} \subseteq \sigma_p(B_w) \subseteq \{\lambda \in \mathbb{C} : \mid \lambda \mid \leqslant \tilde{r}\}$$

where $\tilde{r} = \underline{\lim_{n\to\infty}} (w_2 \ldots w_{n+1})^{\frac{1}{n}}$. The proofs for the general case $1 \leqslant p < \infty$ are analogous to the ones for $p = 2$.

Proposition 3.2.2 *Let $F_w : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}), 1 \leqslant p < \infty$, be a unilateral weighted forward shift with $\{w_n\}_{n \in N}$ a bounded positive weight sequence.*
Then $\sigma_p(F_w) = \emptyset$.

Proof. By contradiction, let $\lambda \in \sigma_p(F_w)$. Then, there exists $\{x_n\}_{n\in\mathbb{N}} \in \ell_p(\mathbb{N}) \setminus \{0\}$ eigenvector of F_w corresponding to the eigenvalue λ,i.e., $F_w(\{x_n\}_{n\in\mathbb{N}}) = \{\lambda x_n\}_{n\in\mathbb{N}}$. By definition, $F_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n-1}x_{n-1}\}_{n\in\mathbb{N}}$ and, hence, the coordinates of $\{x_n\}_{n\in N}$ are such that

$$\lambda x_1 = 0 \textit{ and } w_{n-1}x_{n-1} = \lambda x_n \textit{ for each } n \geqslant 2. \tag{3.2.1}$$

By hypothesis, $w_n > 0$ for each $n \in \mathbb{N}$, therefore F_w is injective and then $\lambda \neq 0$. Hence, it follows from (3.2.1) that $x_n = 0$ for each $n \in \mathbb{N}$. This is a contradiction and so it must be $\sigma_p(F_w) = \emptyset$ □

Proposition 3.2.3 *Let $B_w : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}), 1 \leqslant p < \infty$, eral weighted forward shift with $\{w_n\}_{n\in\mathbb{N}}$ a bounded positive weight sequence.*
Then

$$\{0\} \cup \{\lambda \in \mathbb{C} : \mid \lambda \mid < \tilde{r}\} \subseteq \sigma_p(B_w) \subseteq \{\lambda \in \mathbb{C} : \mid \lambda \mid \leqslant \tilde{r}\}$$

where $\tilde{r} = \varliminf_{n\to\infty} (w_2 \ldots w_{n+1})^{\frac{1}{n}}$.

Proof. Let $\lambda \in \sigma_p(B_w)$. Then, there exists $\{x_n\}_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}) \setminus \{0\}$ eigenvector of B_w corresponding to the eigenvalue λ, i.e., $B_w(\{x_n\}_{n\in\mathbb{N}}) = \{\lambda x_n\}_{n\in\mathbb{N}}$. By definition, $B_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n+1}x_{n+1}\}_{n\in\mathbb{N}}$ and, hence, the coordinates of $\{x_n\}_{n\in\mathbb{N}}$ are such that

$$w_{n+1}x_{n+1} = \lambda x_n \textit{ for each } n > 1 \tag{3.2.2}$$

Therefore, assuming without loss of generality $x_1 = 1$, for each $n \in \mathbb{N}$, we have

$$x_{n+1} = \frac{\lambda^n}{w_2 \ldots w_{n+1}}.$$

Then

$$\begin{aligned} \| x \|_p^p &= \sum_{n=1}^{\infty} \mid x_n \mid^p \\ &= 1 + \sum_{n=2}^{\infty} \left(\frac{\mid \lambda \mid^{n-1}}{w_2 \ldots w_n} \right)^p . \end{aligned}$$

By applying the Cauchy-Hadamard criterion, as

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{|\lambda|^{n-1}}{w_2 \dots w_n}} = \frac{|\lambda|}{\tilde{r}}$$

we obtain the thesis. □

Program Matlab

A Program to calculate the spectral radius of a square matrix.

```
A=input('A')
n=size(A,1); %nombre de lignes
m=size(A,2); %nombre de colonnes
if n==m
    v=eig(A);
    x=max(abs(v))
else
    disp('Reject this case')
end
```

```
A[1 2 0 5 ; 6 -1 6 8 ; 6 2 -3 0 ; 0 1 4 3]

A =

     1     2     0     5
     6    -1     6     8
     6     2    -3     0
     0     1     4     3


x =

    8.8680

A[-5 0 1 ; 6 7 6 ]

A =

    -5     0     1
     6     7     6

Reject this case
>> |
```

Exercise 3.2.1 *D et ermine the spectrum and the radius of the following matrix*

1. $A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$ *over the field of real numbers.*

2. $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ *over the complex of real field .*

Solution 3.2.1 1. *For a given matrix* $A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$, *its eigenvalues are* $\lambda_1 = -2$ *and* $\lambda_2 = 5$ *then the spectrum and the spectral radius of* A *are*

$$\sigma(A) = \{-2, 5\}$$

and

$$\begin{aligned} r(A) &= \max\{\mid -2 \mid, \mid 5 \mid\} \\ &= \{2, 5\} \\ &= 5. \end{aligned}$$

Thus $r(A) = 5$.

2. *Also all the eigenvalues of the given matrix* $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ *over the complex field are* $\lambda_1 = 1 + 2i$ *and* $\lambda_1 = 1 - 2i$

   Then the spectrum and the spectral radius of A *are*

$$\begin{aligned} \sigma(A) &= \{\lambda_1, \lambda_2\} \\ &= \{1 + 2i, 1 - 2i\} \end{aligned}$$

   and

$$\begin{aligned} r(A) &= \max\{\mid 1 + 2i \mid, \mid 1 - 2i \mid\} \\ &= \max\{\sqrt{5}, \sqrt{5}\} \\ &= \sqrt{5}. \end{aligned}$$

   Thus $r(A) = \sqrt{5}$.

Exercise 3.2.2 *The right shift map on the space* ℓ^2 *is given by*

$$\begin{aligned} S : \ell^2 &\to \ell^2 \\ (x_1, x_2, ...) &\mapsto (0, x_1, x_2, ...). \end{aligned}$$

1. *Show that that the map S is a continuous linear operator with norm $\| S \| = 1$.*

2. *Compute the eigenvalues and the spectral radius of S.*

Solution 3.2.2 1. *Let $x \in \ell^2$. By definition of S and the ℓ^2-norm $\| Sx \|_{\ell^2} = \| x \|_{\ell^2}$, which implies $\| S \| = 1$. Being linear and bounded, the map S is continuous.*

2. *Suppose $x = (x_n)_{n \in \mathbb{N}} \in \ell^2$ satisfies $Sx = \lambda x$ for some $\lambda \in \mathbb{R}$. Then*

$$(0, x_1, x_2, ...) = (\lambda x_1, \lambda x_2, \lambda x_3, ...).$$

If $\lambda = 0$, then $x = 0$ is immediate. If $\lambda \neq 0$, then $x = 0$ follows via

$$\begin{aligned} 0 = \lambda x_1 &\Rightarrow 0 = x_1 = \lambda x_2 \\ &\Rightarrow 0 = x_2 = \lambda x_3 \\ &\Rightarrow ... \end{aligned}$$

We conclude that S does not have eigenvalues. Since $\| S^n \| = 1$ for every $n \in \mathbb{N}$ (the solution is as in (1)), the spectral radius of S is

$$r_s = \lim_{n \to \infty} \| S^n \|^{\frac{1}{n}} = 1.$$

Conclusion

In conclusion of this thesis, we reaffirm the importance of studying the inequality in spectral radius and its impacts on technological applications.

Understanding this phenomenon can open new avenues for developing more effective and accurate solution in fields such as wireless communications, biomedical science, and astronomical research.

It is crucial to dedicate further efforts to support future research in this area to enhance our understanding of this phenomenon and its practical applications, thereby contributing to scientific and technological advancements and achieving further progress and innovation.

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Résumé

Nous concentrons ce travail à l'étude des définitions et propriétés du rayon spectral, ainsi que de certaines inégalités du rayon spectral et de ses relation avec le rayon numérique et la norme. Enfin, nous voyons quelques applications, quelques exemples de calcul du rayon spectral de certains opérateurs et du programme Matlab pour calculer le rayon spectral de certaines matrices

Mots clés: Spectre, Rayon spectral, Rayon numérique, Norme.

Abstract

We concern this work to study the spectral radius definitions and properties, also some inequalities of spectral radius and their relation with the numerical radius and norm. Finally, we see some applications, some example of calculate the spectral radius of certain operators and Matlab program to calculate the spectral radius of certain matrix

Keywords: Spectrum, Spectral radius, Numerical radius, Norm.

الـمـلـخص

في هذه المذكرة، قمنا بدراسة نصف القطر الطيفي تعاريف وخواص، حيث خصصنا جزء كبير لتعرف على هذا المفهوم الرياضي وعلاقته بنصف القطر الرقمي والنظيم من خلال عرض بعض المترجحات حول هذا الموضوع، في الأخير قدمنا بعض الأمثلة حول نصف القطر الطيفي لبعض المؤثرات أما المصفوفات فاستعملنا برنامج الماتلاب لحساب نصف القطر الطيفي للمصفوفات.

الكلمات المفتاحية: الطيف، نصف القطر الطيفي، نصف القطر الرقمي، النظيم.