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Representation Theory of Finite Groups

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إِهْدَاء

«الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ الَّذِي بِنِعْمَتِهِ تَتِمُّ الصَّالِحَاتُ»

بِسْمِ اللَّهِ وَبِحَمْدِهِ، أُهْدِي ثِمَارَ هَذَا الْعَمَلِ الْمُتَوَاضِعِ:

إِلَى وَالِدَيَّ الْحَبِيبَيْنِ: صَانِعَيْ أَحْلَامِي وَرَاعِيَيْ آمَالِي، اللَّذَيْنِ ضَحَّيَا بِالْغَالِي وَالنَّفِيسِ لِأَكُونَ فِي هَذِهِ الْمَرْتَبَةِ، فَلَنْ أَسْتَطِيعَ مُقَابَلَةَ تَضْحِيَاتِهِمَا إِلَّا بِالدُّعَاءِ لَهُمَا بِالرِّضَا وَالْمَغْفِرَةِ.

إِلَى إِخْوَتِي وَأَخَوَاتِي الَّذِينَ كَانُوا لِي سَنَدًا وَسَتَرًا وَأُنْسًا فِي كُلِّ الْأَوْقَاتِ.

إِلَى صَدِيقَاتِي الْغَالِيَاتِ اللَّوَاتِي كُنَّ خَيْرَ مُعِينٍ فِي رِحْلَتِي الْعِلْمِيَّةِ.

إِلَى مُشْرِفِي الْفَاضِلِ **الدُّكْتُور حَسَن زَلَاسِي**، لِمَا قَدَّمَهُ مِنْ عِلْمٍ غَزِيرٍ وَتَوْجِيهٍ قَيِّمٍ، وَلِصَبْرِهِ عَلَى تَقْوِيمِ أَعْوَجَاجِي، فَجَزَاهُ اللَّهُ خَيْرَ الْجَزَاءِ.

«رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَى وَالِدَيَّ»

إِكْرَام رَزِيق

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ملخص

تدرس هذه المذكرة نظرية تمثيل الزمر المنتهية، بدءاً بأساسيات نظرية الزمر كالزمر الجزئية والزمر الطبيعية، ثم مفاهيم الجبر الخطي مثل فضاءات الجداء الداخلي والمؤثرات الوحدوية. يركز الجزء الرئيسي على التمثيلات الخطية وعدم القابلية للاختزال، مع نظرية ماشكي التي تحلل التمثيلات إلى مكونات غير قابلة للاختزال. كما يغطي نظرية المحارف وعلاقات التعامد، وتمثيلات الزمر المتناظرة باستخدام جداول يونج. وتوضح الأمثلة العملية لجداول المحارف وصيغة فروبنيوس هذه المفاهيم.

Abstract

This thesis explores the representation theory of finite groups, focusing on their structure and applications. It begins with group theory fundamentals, including subgroups and normal subgroups, then transitions to linear algebra concepts like inner product spaces and unitary operators. The core discusses linear representations, irreducibility, and Maschke's theorem, which decomposes representations into irreducible components. Character theory and orthogonality relations are analyzed, along with representations of symmetric groups using Young tableaux. Practical examples, such as character tables and the Frobenius formula, illustrate key ideas.

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Notations

Symbol	**Meaning**
$\mathbb{Z}_n$	The set of integers modulo n
$\mathbb{Z}_n^\times$	The group of invertible elements in $\mathbb{Z}_n$
G	A group
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
G/H	Quotient group of G by normal subgroup H
S_n	Symmetric group on n elements
A_n	Alternating group on n elements
D_n	Dihedral group of order $2n$
$\mathrm{GL}(V)$	General linear group of invertible maps on V
$\mathrm{GL}_n(\mathbb{C})$	Group of invertible $n \times n$ matrices over $\mathbb{C}$
$\mathrm{U}(V)$	Group of unitary operators on V
$\mathrm{U}_n(\mathbb{C})$	Group of unitary $n \times n$ matrices over $\mathbb{C}$
$\mathrm{Hom}(V, W)$	Space of linear maps from V to W
$\mathrm{End}(V)$	Endomorphism space of V (maps from V to itself)
$\mathrm{Tr}(A)$	Trace of matrix A
$\det(A)$	Determinant of matrix A
χ_ρ	Character of the representation ρ
V^G	Subspace of G-invariant vectors in V
$V \oplus W$	Direct sum of vector spaces V and W
$V \otimes W$	Tensor product of vector spaces V and W
$\mathbb{C}[G]$	Group algebra of G over $\mathbb{C}$
$\mathrm{Z}(\mathbb{C}[G])$	Center of the group algebra $\mathbb{C}[G]$
$\mathrm{Hom}_G(V, W)$	G-equivariant linear maps from V to W
$\mathrm{End}_G(V)$	G-equivariant endomorphisms of V
I_n	Identity matrix of size $n \times n$
A^T	Transpose of matrix A
A^*	Conjugate transpose (Hermitian adjoint) of A
$\overline{A}$	Entry-wise complex conjugate of matrix A
$\langle \cdot, \cdot \rangle$	Inner product on a vector space
$\lVert v \rVert$	Norm (length) of vector v
$W^\perp$	Orthogonal complement of subspace W
$\lambda \vdash n$	λ is a partition of n
$[\lambda]$	Young diagram of partition λ
C_t	Column stabilizer of tableau t
P_λ, Q_λ	Row/column subgroup
δ_x	Delta function at point x
$\mathrm{sgn}(\sigma)$	Sign of permutation σ
e_t	Polytabloid associated to t
S^λ	Specht Representation for partition λ

Introduction

Representation theory is one of the fundamental branches of modern mathematics, offering a unified theoretical framework for studying algebraic structures by representing their elements as linear transformations on vector spaces. The theory of representations of finite groups lies at the heart of this field, forging deep connections between group theory, linear algebra, and harmonic analysis. It also provides profound insights into the structure of groups and has wide-ranging applications in areas such as theoretical physics, number theory, and computer science.

This thesis aims to provide a comprehensive introduction to the representation theory of finite groups, with a focus on the core concepts and central theorems that form the foundation of the discipline. The work is organized into four main chapters, each addressing a key aspect of the theory:

Chapter 1: Preliminaries This chapter lays the mathematical groundwork necessary for understanding representation theory. It reviews essential concepts in group theory, inner product spaces, and unitary operators, along with advanced topics in linear algebra.

Chapter 2: Group Representations Here, we introduce the definition of linear representations of groups and examine their algebraic properties. We cover fundamental operations such as restriction, induction, and direct sum. This chapter also explores fixed spaces and irreducible representations, concluding with a proof of Maschke's theorem, which ensures complete reducibility in complex vector spaces.

Chapter 3: Orthogonality Relations and Character Theory This chapter develops the tools of character theory to classify irreducible representations. We present Schur's orthogonality relations, which reveal orthogonality between matrix coefficients of irreducible representations. The regular representation, character tables, and the representations of abelian groups are also discussed in detail.

Chapter 4: Representations of Symmetric Groups We devote this chapter to the study of symmetric group representations, which serve as fundamental examples in representation theory. We explore the relationship between integer partitions and irreducible representations through Young diagrams and Young tableaux and conclude with Frobenius's formula for computing irreducible representation characters.

The material is presented with mathematical rigor, supplemented by illustrative examples and practical applications, making this thesis a comprehensive reference for students and researchers in representation theory.

Chapter 1

Preliminaries

This chapter presents foundational concepts in linear algebra, bilinear algebra, and group theory. Unless otherwise stated, all vector spaces discussed herein are assumed to be finite-dimensional over the field of complex numbers, denoted by $\mathbb{C}$. The exposition draws upon the following references [4, 6, 1, 8, 5].

1.1 Introduction to Group theory

Definition 1.1.1. *[4] A group is a triple* $(G, \cdot, e)$*, where* G *is a nonempty set,* $\cdot : G \times G \to G$ *is a binary operation, and* $e \in G$ *is called the identity element. These must satisfy the following axioms for all* $a, b, c \in G$*:*

- ***Associativity:*** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- ***Identity Element:*** $a \cdot e = e \cdot a = a$
- ***Existence of Inverses:*** *For each* $a \in G$*, there exists* $b \in G$ *such that* $a \cdot b = b \cdot a = e$

If, in addition, the operation is commutative, i.e. $a \cdot b = b \cdot a$ *for all* $a, b \in G$*, then the group is called abelian.*

Example 1.1.1. *Example of groups:*

$$(GL_n(\mathbb{Q}), \cdot, I), \quad (GL_n(\mathbb{R}), \cdot, I), \quad (GL_n(\mathbb{C}), \cdot, I) \text{ etc.}$$

Lemma 1.1.1. *[4] If* G *is a group, the identity element in the group is unique. Moreover, for each element* $a \in G$*, there exists a unique inverse, denoted by* a^{-1}

A group G is called **cyclic**[4] if there exists an element $g \in G$ such that every element of G can be written as a power of g. In other words:

$$G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$$

Here:

- g is called a **generator** of G.
- The notation $\langle g \rangle$ denotes the group generated by g.
- g^n represents the group operation applied n times:

Definition 1.1.2. *[4] The* ***order of a group*** G*, denoted* $|G|$*, is the number of elements in* G*.*

The ***order of an element*** $g \in G$*, denoted* $|g|$*, is the smallest positive integer* n *such that:*

$$g^n = e \quad \text{(where } e \text{ is the identity element).}$$

Example 1.1.2.

In the multiplicative group $(\mathbb{Z}_5^\times, \cdot)$*, the element* 2 *has order* 4 *since:*

$$2^1 \equiv 2,\ 2^2 \equiv 4,\ 2^3 \equiv 3,\ 2^4 \equiv 1 \pmod 5.$$

Definition 1.1.3 (Homorphisms)**.** *[6] Let* $(G, \cdot)$ *and* $(H, \circ)$ *be groups. A function* $\varphi : G \to H$ *is a homomorphism if, for all* $a, b \in G$*,* $\varphi(a \cdot b) = \varphi(a) \circ \varphi(b)$*.*

- *A homomorphism* $\varphi : G \to G$ *is called an* ***endomorphism*** *if* $G = H$*. In other words, it is a homomorphism from a group to itself.*
- *A homomorphism* $\varphi : G \to H$ *is called an* ***isomorphism*** *if it is* ***bijective*** *(both injective and surjective). denoted by* $G \cong H$
- *A homomorphism* $\varphi : G \to G$ *is called an* ***automorphism*** *if it is both an* ***endomorphism*** *and an* ***isomorphism****.*

Theorem 1.1.1. *[6] Let* $\varphi : (G, \cdot) \to (G', \circ)$ *be a homomorphism.*

(i) $\varphi(e) = e'$*, where* e' *is the identity in* G'*.*

(ii) If $a \in G$*, then* $\varphi(a^{-1}) = \varphi(a)^{-1}$*.*

(iii) If $a \in G$ *and* $n \in \mathbb{Z}$*, then* $\varphi(a^n) = \varphi(a)^n$*.*

Let $(G, \cdot, e)$ be a group

Definition 1.1.4 (Subgroup)**.** *[6] A subset* H *of* G *is called a* **subgroup** *if it satisfies the following condition:*

1. $e \in H$*.*
2. *for all* $a, b \in H$ *we have* $a \cdot b \in H$*.*
3. *for all* $a \in H$ *we have* $a^{-1} \in H$*.*

Notation*:* $H \leq G$*.*

Proposition 1.1.1. *[6] A subset* H *of a group* G *is a subgroup if and only if* $e \in H$ *and* $a, b \in H$ *imply* $a.b^{-1} \in H$

Example 1.1.3. *Let* $\varphi : G_1 \to G_2$ *be a homomorphism, and define*

$$\ker \varphi = \{a \in G_1 : \varphi(a) = e_2\}.$$

and

$$\operatorname{im} \varphi = \{h \in G_2 : h = \varphi(a) \textit{ for some } a \in G_1\}.$$

Then $\ker \varphi$ *is a subgroup of* G_1 *and* $\operatorname{im} \varphi$ *is a subgroup of* G_2*.*

Theorem 1.1.2. *[4] Every subgroup of a cyclic group is cyclic.*

Definition 1.1.5 (Coset)**.** *[6] Let* H *be a subgroup of a group* G*. For any element* $g \in G$*:*

1. *The* ***left coset*** *of* H *in* G *with respect to* g *is the set:*

$$Hg = \{hg \mid h \in H\}.$$

2. *The* ***right coset*** *of* H *in* G *with respect to* g *is the set:*

$$gH = \{gh \mid h \in H\}.$$

Let $H \leq G$ be a subgroup of a group G, and let $a, b \in G$. Then the left cosets Ha and Hb are equal if and only if $ab^{-1} \in H$; equivalently, $aH = bH$ if and only if $b^{-1}a \in H$. Moreover, for any $g, g' \in G$, the left cosets Hg and Hg' are either equal or disjoint; that is, $Hg = Hg'$ or $Hg \cap Hg' = \emptyset$. The same statement holds for right cosets: $gH = g'H$ or $gH \cap g'H = \emptyset$.

The right cosets of a subgroup H form a disjoint partition of G. This corresponds to an equivalence relation on G: $a \sim b$ if $ab^{-1} \in H$, whose equivalence classes are the right cosets themselves [6] .

Notation: $H \setminus G$ the set of all left coset and G/H the set of all right coset.

Theorem 1.1.3. *[4] If $H \leq G$, then the number of right coset of H in G is equal to the number of left coset of H in G. and also If G is finite, then*

$$|G/H| = |H \setminus G| = \frac{|G|}{|H|}.$$

Example 1.1.4. *Consider $G = \mathbb{Z} = (\mathbb{Z}, +, 0)$ and let $H = 2\mathbb{Z} = \{2i \mid i \in \mathbb{Z}\}$ be the subgroup of all even numbers.*

Then we have two left coset of $\mathbb{Z}$ modulo $2\mathbb{Z}$, namely:

- *the coset $2\mathbb{Z}$ consisting of all even numbers;*
- *the coset $1 + 2\mathbb{Z}$ consisting of all odd numbers.*

In group theory, a normal subgroup is a special type of subgroup that remains invariant under conjugation by any group element

Definition 1.1.6 (Normal subgroups)**.** *[6]*

*A subgroup H of G is called **normal**, provided that*

$$ghg^{-1} \in H, (gHg^{-1} = H) \quad \text{for all } g \in G \text{ and } h \in H \iff gH = Hg, \text{for all } g \in G.$$

Notation. $H \lhd G$.

Remark 1.1.1. *[6] if G is abelian then every subgroup of G is normal in G*

Proof. G is abelian and let $H \leq G$ then for all $g \in G$ and $h \in H$

$$ghg^{-1} = gg^{-1}h = eh = h \in H,$$

so $ghg^{-1} \in H$. □

Example 1.1.5. *The kernel K of a homomorphism $\varphi : G \to H$ is a normal subgroup: if $a \in K$, then $\varphi(a) = e$; if $g \in G$, then*

$$\varphi(gag^{-1}) = \varphi(g)\varphi(a)\varphi(g)^{-1} = \varphi(g)e\varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = e,$$

so $gag^{-1} \in K$. for all $g \in G$, and so $K \lhd G$.

Theorem 1.1.4 (quotient group)**.** *[6] When H is a normal subgroup of G (denoted $H \lhd G$), the set of cosets G/H forms a group called the **quotient group**, where the group operation is defined by:*

$$(Ha)(Hb) = H(ab) \quad \text{for all } a, b \in G.$$

Proof.

$$HaHb = Ha(a^{-1}Ha)b \quad \text{(because } H \text{ is normal)}$$
$$= H(aa^{-1})Hab = HHab = Hab \quad \text{(because } H \leq G\text{)}.$$

□

Remark 1.1.2. *we have* $(Ha)^{-1} = Ha^{-1}$ *and* $He = H$.

Theorem 1.1.5. *(The first isomorphism theorem)[6]*
Let $G_1 = (G_1, \cdot_1, e_1)$ *and* $G_2 = (G_2, \cdot_2, e_2)$ *be two groups and* $\varphi : G_1 \to G_2$ *a homomorphism.*

- $\ker(\varphi) \lhd G_1$.
- $\mathrm{Im}(\varphi) \leq G_2$.
- $G_1 / \ker(\varphi) \cong \mathrm{Im}(\varphi)$.

Proof. ($\ker(\varphi) \lhd G_1$ and $\mathrm{Im}(\varphi) \leq G_2$) are clear.
Let $\varphi : G \to H$ be a group homomorphism, and let $g, x \in G$. Then:

$$\varphi(x) = \varphi(g) \iff \varphi(g)^{-1}\varphi(x) = e_H \iff \varphi(g^{-1}x) = e_H \iff g^{-1}x \in \ker\varphi \iff x \in g\ker\varphi.$$

Thus, the set $\{x \in G \mid \varphi(x) = \varphi(g)\}$ is equal to the coset $g\ker\varphi$.
Now, let $h \in \mathrm{Im}(\varphi)$. Then there exists some $g \in G$ such that $\varphi(g) = h$, and we have:

$$\varphi^{-1}(h) = \{x \in G \mid \varphi(x) = h\} = g\ker\varphi.$$

This shows that the preimage of h under φ is the coset $g\ker\varphi$. Consequently, we can write:

$$g\ker\varphi \in G/\ker\varphi.$$

Next, consider the map $\varphi^{-1} : \mathrm{Im}(\varphi) \to G/\ker\varphi$ defined by $\varphi^{-1}(h) = g\ker\varphi$ for $h = \varphi(g)$. We claim that φ^{-1} is a homomorphism. To verify this, let $h_1, h_2 \in \mathrm{Im}(\varphi)$, and let $g_1, g_2 \in G$ such that $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then:

$$\varphi^{-1}(h_1h_2) = \varphi^{-1}(\varphi(g_1)\varphi(g_2)) = \varphi^{-1}(\varphi(g_1g_2)) = g_1g_2\ker\varphi = (g_1\ker\varphi)(g_2\ker\varphi) = \varphi^{-1}(h_1)\varphi^{-1}(h_2).$$

Thus, φ^{-1} is indeed a homomorphism.
To show that φ^{-1} is injective, suppose $\varphi^{-1}(h_1) = \varphi^{-1}(h_2)$. Then $g_1\ker\varphi = g_2\ker\varphi$, which implies $g_1^{-1}g_2 \in \ker\varphi$. Consequently, $\varphi(g_1) = \varphi(g_2)$, so $h_1 = h_2$. Hence, φ^{-1} is injective.
Finally, φ^{-1} is surjective because for any $g\ker\varphi \in G/\ker\varphi$, we have $\varphi^{-1}(\varphi(g)) = g\ker\varphi$. This establishes the isomorphism:

$$\mathrm{Im}(\varphi) \cong G/\ker\varphi.$$

□

Definition 1.1.7 (The direct product). *[6]*
Let $G_1 = (G_1, \cdot, e_1)$ *and* $G_2 = (G_2, *, e_2)$ *be two groups.*
The direct product $G_1 \times G_2$ *is the group defined as follows:*

- $G_1 \times G_2 := \{(g_1, g_2) \mid g_1 \in G_1 \text{ and } g_2 \in G_2\}$;
- $(g_1, g_2) \cdot (g_1', g_2') := (g_1 \cdot g_1', g_2 * g_2')$;
- $e := (e_1, e_2)$.

If p is a prime number, an elementary abelian p-group is defined as a finite group G that is isomorphic to the direct product of finitely many copies of the cyclic group $\mathbb{Z}_p$.

Definition 1.1.8 (Group action). *[6] An action of a group* G *on a set* M *is a map:* $G \times M \to M$*, written as* $(g, m) \mapsto g \cdot m$*, satisfying the following two properties:*

1. ***Identity:*** *For all* $m \in M$*, the identity element* $e \in G$ *acts trivially:* $e \cdot m = m$.

2. ***Compatibility**: For all $g, h \in G$ and $m \in M$, $g \cdot (h \cdot m) = (g \cdot h) \cdot m$.*

Definition 1.1.9 (conjugate)**.** *[6] Let G be a group and $x \in G$. An element $y \in G$ is called a **conjugate** of x if there exists an element $a \in G$ such that: $y = axa^{-1}$.*

Definition 1.1.10 (Self-Action by Conjugation)**.** *[5] Let G be a group. The **conjugation action** of G on itself is the map:*

$$G \times G \to G, \quad (a, x) \mapsto a \cdot x := axa^{-1}.$$

*This defines a **group action** of G on itself, where:*

1. *Each $\phi_a(x) = axa^{-1}$ is an automorphism of G (called an inner automorphism).*
2. *The map $a \mapsto \phi_a$ is a homomorphism from G to $\mathrm{Aut}(G)$.*

Theorem 1.1.6. *[6] Every homomorphism $\phi : G \to S_M$ defines a group action of G on the set M.*

1.2 Inner Product Spaces

Definition 1.2.1. *[1]*

An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

such that, for $v, w, v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{C}$:

1. **Linearity** *$\langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle$;*
2. **conjugate symmetry***$\langle w, v \rangle = \overline{\langle v, w \rangle}$;*
3. **Positive Definiteness***$\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.*

Example 1.2.1. *[1]*

- *The standard inner product in the complex space $\mathbb{C}^n$ is defined by*

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

- *If $c_1, c_2, \ldots, c_n$ are positive real numbers, then an inner product can be defined on the vector space $\mathbb{C}^n$ as follows:*

$$\langle (w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n) \rangle = c_1 w_1 \overline{z_1} + c_2 w_2 \overline{z_2} + \cdots + c_n w_n \overline{z_n},$$

  for all vectors $(w_1, w_2, \ldots, w_n)$ and $(z_1, z_2, \ldots, z_n)$ in $\mathbb{C}^n$.

Definition 1.2.2. *[1] When a vector space is equipped with an inner product, it is called an* **inner product space** *and the norm $\|v\|$ of a vector v in an inner product space is defined by $\|v\| = \sqrt{\langle v, v \rangle}$.*

1.2.1 Orthonormality

Definition 1.2.3 (**Orthonormal**)**.** *In an inner product space V, two vectors v and w are said to be orthogonal if their inner product satisfies $\langle v, w \rangle = 0$.*

A collection of vectors in V is termed an **orthogonal** set if every pair of vectors in the set is orthogonal

Theorem 1.2.1. *A set of vectors is called orthonormal if each vector in the set has a norm of 1 and is orthogonal to all other vectors in the set.*

We have the following proposition

Proposition 1.2.1. *Any orthogonal set $M = \{v_1, v_2, \dots, v_n\}$ composed of non-zero vectors is guaranteed to be linearly independent.*

This property particularly applies to orthonormal sets, which consequently form a basis for the subspace spanned by M.

Remark 1.2.1. *[1] Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $v \in V$. Then v can be written*

(a) $v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$,

(b) $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$.

Example 1.2.2. *[8] For a finite set X, the set $\mathbb{C}^X = \{f : X \to \mathbb{C}\}$ is a vector space with pointwise operations. Namely, one defines*

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x)$$

for functions $f, g \in \mathbb{C}^X$ and scalar $c \in \mathbb{C}$.

For each $x \in X$, define a function $\delta_x : X \to \mathbb{C}$ by

$$\delta_x(y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

There is a natural inner product on $\mathbb{C}^X$ given by

$$\langle f, g\rangle = \sum_{x \in X} f(x)\overline{g(x)}.$$

The set $\{\delta_x \mid x \in X\}$ is an orthonormal basis with respect to this inner product. If $f \in \mathbb{C}^X$, then its unique expression as a linear combination of the δ_x is given by $f = \sum_{x \in X} f(x)\delta_x$

Consequently, $\dim \mathbb{C}^X = cardX$

Definition 1.2.4 (orthogonal complement). *[8] if W is a subspace of V, then the set of all vectors in V that are orthogonal to every vector in W is denoted $W^\perp$ and called the orthogonal complement of W.*

$$W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in W\}.$$

Proposition 1.2.1. *Let V be an inner product space and $W \leq V$ (i.e., W is a subspace of V). Then there exists a direct sum decomposition:*

$$V = W \oplus W^\perp,$$

where $W^\perp$ denotes the orthogonal complement of W.

Proof. First, if $w \in W \cap W^\perp$, then $\langle w, w\rangle = 0$ implies $w = 0$; hence, $W \cap W^\perp = \{0\}$.

Let $v \in V$, and suppose that $\{e_1, \dots, e_m\}$ is an orthonormal basis for W. Define

$$u = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m,$$

and let $z = v - u$. Then $u \in W$. We claim that $z \in W^\perp$. To prove this, it suffices to show that $\langle z, e_i\rangle = 0$ for all $i = 1, \dots, m$. To this end, we compute:

$$\langle z, e_i\rangle = \langle v, e_i\rangle - \langle u, e_i\rangle = \langle v, e_i\rangle - \langle v, e_i\rangle = 0,$$

because $\{e_1, \dots, e_m\}$ is an orthonormal set. Since $v = u + z$, it follows that $V = W + W^\perp$. This completes the proof. ☐

1.2.2 Unitary operator

We continue to assume that V is an inner product space.

Definition 1.2.5 (Unitary operator). *[8] A linear operator $U \in GL(V)$ is called* ***unitary*** *if it preserves the inner product, meaning:*

$$\langle Uv, Uw\rangle = \langle v, w\rangle$$

for all vectors $v, w \in V$.

Let $U = (u_{ij})$ be a matrix. The conjugate transpose of U, denoted by U^*, is given by $U^* = (\overline{u_{ji}})$. For the standard inner product on $\mathbb{C}^n$, a matrix $U \in \mathrm{GL}_n(\mathbb{C})$ is said to be **unitary** if and only if its inverse equals its conjugate transpose, i.e., $U^{-1} = U^*$. The set of all $n \times n$ unitary matrices is denoted by $\mathrm{U}_n(\mathbb{C})$. Additionally, a matrix $A \in \mathrm{M}_n(\mathbb{C})$ is called **self-adjoint** if it satisfies $A = A^*$. A matrix A is symmetric if $A^T = A$. If A has real entries, then A is self-adjoint if and only if A is symmetric.

Example 1.2.3. *[1] Let $\theta \in \mathbb{R}$, and let A be the operator on $\mathbb{R}^2$ whose matrix with respect to the standard basis is:*

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This is a ***rotation matrix*** *that rotates vectors in $\mathbb{R}^2$ by an angle θ. Since A is real, $A^* = A^T$ (its transpose). So:*

$$A^T A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

so A is ***unitary****.*

1.3 Further Explorations in Linear Algebra

Let $X \subseteq \mathrm{End}(V)$ and $W \subseteq V$. Then W is called X**-invariant** if, for any $A \in X$ and any $w \in W$, one has $Aw \in W$, i.e., $XW \subseteq W$ (A is matrix)[8] .

A key example comes from the theory of eigenvalues and eigenvectors. Recall that $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathrm{End}(V)$ if $\lambda I - A$ is not invertible; in other words, if $Av = \lambda v$ for some $v \neq 0$. The eigen-space corresponding to λ is the set:

$$V_\lambda = \{v \in V \mid Av = \lambda v\},$$

which is a subspace of V. Note that if $v \in V_\lambda$, then $A(Av) = A(\lambda v) = \lambda Av$, so $Av \in V_\lambda$. Thus, V_λ is A-invariant.

Conversely, if $W \subseteq V$ is A-invariant with $\dim W = 1$ (that is, W is a line), then $W \subseteq V_\lambda$ for some $\lambda \in \mathbb{C}$. In fact, if $w \in W \setminus \{0\}$, then w is a basis for W. Since $Aw \in W$, we have $Aw = \lambda w$ for some $\lambda \in \mathbb{C}$. Thus, w is an eigenvector with eigenvalue λ, and so $w \in V_\lambda$; hence $W \subseteq V_\lambda$.

Definition 1.3.1 (Characteristic Polynomial). *Let A be a linear operator on an n-dimensional vector space V. The* ***characteristic polynomial*** *of A, denoted by $p_A(x)$, is defined as:*

$$p_A(x) = \det(xI - A),$$

where I is the identity operator on V, and det *denotes the determinant.*

Definition 1.3.2. *[8] A polynomial is called* ***monic*** *if its leading coefficient (the coefficient of the highest power of x) is* 1

Theorem 1.3.1 (Cayley–Hamilton). *[8] Let $p_A(x)$ be the characteristic polynomial of A. Then, $p_A(A) = 0$. For $A \in End(V)$, the* ***minimal polynomial*** *of A, denoted $m_A(x)$, is the unique monic polynomial of smallest degree $f(x)$ such that $f(A) = 0$.*

[1]

Proposition 1.3.1. *If $q(A) = 0$ for some polynomial $q(x)$, then the minimal polynomial $m_A(x)$ divides $q(x)$, i.e., $m_A(x) \mid q(x)$.*

Proof. [1] Let m denote the minimal polynomial of A. Suppose q is a polynomial that is a multiple of m. Then, there exists a polynomial $s \in \mathbb{P}(\mathbb{C})$ such that $q = m \cdot s$. Consequently, we have:

$$q(A) = m(A) \cdot s(A) = 0.$$

Proof of the other direction:

Suppose $q(A) = 0$. By the Division Algorithm for Polynomials, there exist polynomials $s, r \in \mathbb{P}(\mathbb{C})$ such that:

$$q = m \cdot s + r$$

and $\deg(r) < \deg(m)$. Applying A to both sides, we get:

$$0 = q(A) = m(A) \cdot s(A) + r(A).$$

Since $m(A) = 0$ (by definition of the minimal polynomial), this simplifies to:

$$0 = r(A).$$

Because $\deg(r) < \deg(m)$ and m is the minimal polynomial of A, the only possibility is $r = 0$. Thus:

$$q = m \cdot s,$$

which shows that q is a multiple of m, as desired. ☐

Corollary 1.3.1. *[8] If $p_A(x)$ is the characteristic polynomial of A,then $m_A(x)$ divides $p_A(x)$.*

The relevance of the minimal polynomial is that it provides a criterion for the diagonalizability of a matrix, amongst other things

Theorem 1.3.2. *[8] A matrix A is diagonalizable if and only if every root of its minimal polynomial $m_A(x)$ is a simple root (i.e., it has no multiplicity greater than 1).*

Example 1.3.1. *For the matrix*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the minimal polynomial is $m_A(x) = (x-1)(x-2)$, whereas the characteristic polynomial is $p_A(x) = (x-1)^2(x-2)$. On the other hand, the matrix

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*has minimal polynomial $m_B(x) = (x-1)^2$ and characteristic polynomial $p_B(x) = (x-1)^2$. Since the minimal polynomial has a repeated root, B is **not diagonalizable**.*

*One of the main results from linear algebra is the **spectral theorem** for matrices.*

Theorem 1.3.3. *(Spectral Theorem)[8] Let $A \in M_n(\mathbb{C})$ be a **self-adjoint** matrix.Then, there exists a **unitary matrix** $U \in U_n(\mathbb{C})$ such that UAU^* is diagonal. Moreover, the eigenvalues of A are **real**.*

Proof. First, we verify the eigenvalues are real: for any eigenvalue λ with eigenvector v, the equality $\langle Av, v\rangle = \langle v, Av\rangle$ implies $\lambda\langle v, v\rangle = \overline{\lambda}\langle v, v\rangle$, forcing $\lambda = \overline{\lambda}$ since $\langle v, v\rangle > 0$.

To show diagonalizability, we proceed by induction on n. The base case $n = 1$ is trivial. For $n > 1$, take any eigenvalue λ with eigenspace V_λ. If $V_\lambda = \mathbb{C}^n$, A is already diagonal. Otherwise, $\mathbb{C}^n$ decomposes orthogonally as $V_\lambda \oplus V_\lambda^\perp$, where $V_\lambda^\perp$ is A-invariant because $\langle Aw, v\rangle = \langle w, Av\rangle = \lambda\langle w, v\rangle = 0$ for all $v \in V_\lambda$, $w \in V_\lambda^\perp$.

The restricted operator $A|_{V_\lambda^\perp}$ remains self-adjoint on this smaller-dimensional space, so by induction $V_\lambda^\perp$ admits an orthonormal eigenbasis $\mathcal{B}'$. Combining $\mathcal{B}'$ with any orthonormal basis $\mathcal{B}$ for V_λ yields a full orthonormal eigenbasis for $\mathbb{C}^n$, which when arranged as columns forms the unitary matrix U diagonalizing A as $U^*AU = D$.

□

The trace of a matrix $A = (a_{ij})$ is defined by:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Some fundamental properties of the trace function $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$ are that Tr is linear and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. As a consequence,

$$\mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(P^{-1}PA) = \mathrm{Tr}(A).$$

In particular, this shows that $\mathrm{Tr}(A)$ is independent of the choice of basis. Therefore, if $T \in \mathrm{End}(V)$, then $\mathrm{Tr}(T)$ is well-defined: choose any basis and compute the trace of the associated matrix. Similar observations apply to the determinant.

Chapter 2

Group Representations

In this chapter, we will present fundamental definitions of representation theory along with key illustrative examples. We will explore several important methods for constructing group representations. A central result we will prove is Maschke's Theorem, which establishes that every representation of a finite group over the complex numbers $\mathbb{C}$ decomposes as a direct sum of irreducible representations. This theorem serves as the foundation for understanding the complete reducibility of group representations in the complex case. In this chapter, we will rely on the following references: [8, 10, 3, 7].

2.1 Preliminaries

A linear representation of a group G is a way to study the structure of G through linear transformations on a vector space V over $\mathbb{C}$.

Definition 2.1.1. *[10]*

*Let G be a finite group and V a vector space over $\mathbb{C}$. A **linear action** (or **linear representation**) of G on V is a map:*

$$G \times V \to V, \quad (g, v) \mapsto g \cdot v,$$

satisfying the following conditions:

1. *For each fixed $g \in G$, the map $v \mapsto g \cdot v$ is a linear on V.*
2. *The action preserves the group structure, i.e.:*
    - *$e \cdot v = v$ for the identity element $e \in G$ and for all $v \in V$.*
    - *$(g_1 g_2) \cdot v = g_1 \cdot (g_2 \cdot v)$ for all $g_1, g_2 \in G$ and $v \in V$.*

This representation can be expressed in several equivalent ways:[10]

- As a representation $\rho : G \to \mathrm{GL}(V)$:here, G acts on V via linear transformations.
- As a G-module: here, V is a vector space equipped with a scalar multiplication by G that satisfies the conditions of a group action.

Proposition 2.1.1. *[3]*

Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a group G on a vector space V. Then, for all $g, h \in G$ and $v, w \in V$, the following hold:

1. *Homomorphism Property: $\rho_g \rho_h = \rho_{gh}$.*

2. *Identity Action:* $\rho_e = \mathrm{Id}_V$.

3. *Inverse Action:* $\rho_{g^{-1}} = (\rho_g)^{-1}$.

4. *Linearity:* $\rho_g(av + bw) = a\rho_g v + b\rho_g w \quad$ *(for scalars* a, b*).*

Example 2.1.1. *[8] Let* $G \cong \mathbb{Z}/n\mathbb{Z}$ *be a cyclic group of order* n *generated by* σ*. We define a linear action of* G *on* $\mathbb{C}^2$ *where the generator* σ *acts as rotation about the origin* $(0, 0)$ *by angle* $\theta = \frac{2\pi}{n}$*.*

The group action is given by:

$$\sigma \cdot \begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix}$$

where R_θ *is the rotation matrix:*

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

.

Example 2.1.2. *[10] The dihedral group* D_n *of order* $2n$*, for a non-zero natural number* n*, is the group of symmetries of a regular* n *sided polygon. It consists of* n *rotations and* n *reflections.*

$$D_n = \{\sigma^2 = 1, \quad \tau^n = 1, \quad \sigma\tau\sigma^{-1} = \tau^{-1}\}$$

We can define a representation of the dihedral group D_n *in the following way:*

$$\rho : D_n \to \mathrm{GL}_2(\mathbb{C})$$

with

$$\rho(\tau) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} \quad \textit{and} \quad \rho(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Example 2.1.3. *[8] Let* S_n *be the symmetric group on* n *elements. The* ***standard representation*** *is a homomorphism:*

$$\phi : S_n \to GL_n(\mathbb{C}),$$

defined as, For each permutation $\sigma \in S_n$*,* ϕ_σ *acts on the standard basis* $\{e_1, e_2, \ldots, e_n\}$ *of* $\mathbb{C}^n$ *by:*

$$\phi_\sigma(e_i) = e_{\sigma(i)}.$$

Definition 2.1.2. *[8] The* ***degree*** *or* ***dimension*** *of the representation* ρ *is the dimension of* V*.*

Example 2.1.4. *(One-Dimensional Representation)[8] The trivial representation of a group* G *is the homomorphism:*

$$\rho : G \to GL(\mathbb{C}) = \mathbb{C}^*,$$

defined by:

$$\rho(g) = 1 \quad \textit{for all } g \in G.$$

Here, $\mathbb{C}^*$ *denotes the multiplicative group of nonzero complex numbers. This representation assigns the value* 1 *to every element* $g \in G$*, making it the simplest possible representation of* G*.*

Definition 2.1.3 (regular representation)**.** *[7] Let* $X = \{x_1, \ldots, x_n\}$ *be a finite set equipped with an action of a group* G*. We define the complex vector space* $\mathbb{C}X$ *as the set of all formal linear combinations of the elements of* X *with coefficients in* $\mathbb{C}$*. Explicitly,*

$$\mathbb{C}X = \left\{ \sum_{i=1}^{n} a_i e_{x_i} \;\middle|\; a_i \in \mathbb{C} \right\},$$

where e_{x_i} *denotes the basis vector corresponding to the element* $x_i \in X$*.*

The ***regular representation*** *of G on $\mathbb{C}X$ is defined by the left action of G on X. This action induces a linear representation:*

$$\rho : G \to GL(\mathbb{C}X),$$

where for each $g \in G$, the map $\rho(g) : \mathbb{C}X \to \mathbb{C}X$ is given by:

$$\rho(g)\left(\sum_{i=1}^{n} a_i e_{x_i}\right) = \sum_{i=1}^{n} a_i e_{gx_i}.$$

In other words, $\rho(g)$ acts on the basis vectors by permuting them according to the action of g on X, i.e.,

$$\rho(g)(e_{x_i}) = e_{gx_i}.$$

Definition 2.1.4. *[3] Two representations of a group G, $\rho : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$, are said to be* ***equivalent*** *(or* ***isomorphic****) if there exists an* ***invertible linear map*** *$T : V \to W$ such that:*

$$\psi_g = T \circ \rho_g \circ T^{-1} \quad \textit{for all} \quad g \in G.$$

Equivalently, this means that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd}
V \arrow[r, "\rho_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\
W \arrow[r, "\psi_g"] & W.
\end{tikzcd}$$

In other words, T intertwines the actions of ρ_g and ψ_g for all $g \in G$. When this condition holds, we write $\rho \sim \psi$.

2.2 Operations on representations

Before we begin studying operations on representations, it's important to note that these operations help us construct new representations and understand complex ones through simpler components.

Let ρ be a representation of G on V.

- **Restriction:**[3] if is H subgroup of G, the restriction of a representation ρ to H, denoted $\mathrm{Res}_H\rho$, is a representation of H obtained by restricting the homomorphism of the group ρ to the subgroup H. Specifically:

  The restricted representation $\mathrm{Res}_H\rho : H \to GL(V)$ is defined by:

  $$(\mathrm{Res}_H\rho)(h) = \rho(h) \quad \forall h \in H.$$

  In other words, $\mathrm{Res}_H\rho$ is simply the original representation ρ, but applied only to elements of H.

- **Induced Representations via Group Homomorphisms:**[3] Let $p : G \to H$ be a group homomorphism, and let $\rho : H \to GL(V)$ be a representation of H. The composition $\rho \circ p : G \to GL(V)$ defines a representation of G on V, where each $g \in G$ acts on V via $\rho(p(g))$. This construction is particularly useful when G has a normal subgroup N, and p is the natural projection $G \to G/N$. In this case, p is surjective, and every representation of the quotient group G/N can be lifted to a representation of G. Even if p is not surjective, the composition $\rho \circ p$ still provides a way to study representations of G through those of H. This approach is widely used to simplify the analysis of group representations, especially when H has a more tractable structure than G.

- **Direct sum:**[8] Let $\rho_1 : G \to GL(V)$ and $\rho_2 : G \to GL(W)$ be two representations of a group G. The direct sum $\rho_1 \oplus \rho_2$ is a representation of G on the vector space $V \oplus W$, defined by:

  $$(\rho_1 \oplus \rho_2)(g)(v, w) = (\rho_1(g)v, \rho_2(g)w) \quad \forall g \in G,\, v \in V,\, w \in W.$$

  This means that G acts on $V \oplus W$ by acting separately on V and W. Let $\rho^{(1)} : G \to GL_m(\mathbb{C})$ and $\rho^{(2)} : G \to GL_n(\mathbb{C})$ be two representations of a group G. The **direct sum** $\rho^{(1)} \oplus \rho^{(2)} : G \to GL_{m+n}(\mathbb{C})$ is defined in terms of block matrices as:

  $$(\rho^{(1)} \oplus \rho^{(2)})(g) = \begin{pmatrix} \rho^{(1)}(g) & 0 \\ 0 & \rho^{(2)}(g) \end{pmatrix}.$$

  This means that G acts on $\mathbb{C}^{m+n}$ by acting separately on $\mathbb{C}^m$ and $\mathbb{C}^n$.

  Remark 2.2.1. *If $n > 1$, the representation $\rho : G \to GL_n(\mathbb{C})$ defined by $\rho(g) = I_n$ for all $g \in G$ (where I_n is the $n \times n$ identity matrix) is not equivalent to the trivial representation. Instead, it is equivalent to the direct sum of n copies of the trivial representation.*

  Example 2.2.1. *Let $\rho : S_3 \to GL_2(\mathbb{C})$ be a representation of the symmetric group S_3, defined on the generators $(1\,2)$ and $(1\,2\,3)$ as follows:*

  $$\rho(1\,2) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \rho(1\,2\,3) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

  *Additionally, let $\psi : S_3 \to \mathbb{C}^*$ be the **trivial representation**, defined by:*

  $$\psi_\sigma = 1 \quad \textit{for all } \sigma \in S_3.$$

  Now, consider the direct sum representation $\rho \oplus \psi$, which combines ρ and ψ into a 3-dimensional representation. The action of $\rho \oplus \psi$ on the generators $(1\,2)$ and $(1\,2\,3)$ is given by the block-diagonal matrices:

  1. *For $(1\,2)$:*

     $$(\rho \oplus \psi)(1\,2) = \begin{pmatrix} \rho(1\,2) & 0 \\ 0 & \psi(1\,2) \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

  2. *For $(1\,2\,3)$:*

     $$(\rho \oplus \psi)(1\,2\,3) = \begin{pmatrix} \rho(1\,2\,3) & 0 \\ 0 & \psi(1\,2\,3) \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- **Tensor product:**[3] Given two representations $\rho\colon G \to \mathrm{GL}(V)$ and $\psi\colon G \to \mathrm{GL}(W)$ of a group G on complex vector spaces V and W, their tensor product $\rho \otimes \psi\colon G \to \mathrm{GL}(V \otimes_{\mathbb{C}} W)$ is a new representation acting on the tensor product space $V \otimes_{\mathbb{C}} W$ (the vector space generated by elementary tensors $v \otimes w$) via the group action $(\rho \otimes \psi)_g(v \otimes w) = \rho_g(v) \otimes \psi_g(w)$ for all $g \in G$, $v \in V$, and $w \in W$, which extends linearly to arbitrary tensors in $V \otimes W$.

  This construction preserves the group structure while combining the two representations into a higher-dimensional representation on the tensor product space.

- **Dual representation:**[3] Given a representation $\rho : G \to GL(V)$ of a group G, the dual representation $\rho^* : G \to GL(V^*)$ is defined as follows:

  For each $g \in G$ and $\phi \in V^*$, the action of $\rho^*(g)$ on ϕ is given by:

  $$\langle \rho^*(g)\phi, v \rangle = \langle \phi, \rho(g^{-1})v \rangle \quad \forall v \in V.$$

In other words, $\rho^*(g)\phi$ is the functional that maps v to $\phi(\rho(g^{-1})v)$.

Let V be a finite-dimensional representation of a group G, and fix a basis for V. For each $g \in G$, let A_g be the matrix representing ρ_g in this basis. Then, the matrix representing the dual representation ρ_g^* in the dual basis of V^* is given by the inverse transpose of A_g, i.e., $(A_g^t)^{-1}$. More generally, if $\rho : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are two representations of a group G, then we can naturally define a representation τ of G on the space of linear maps $\mathrm{Hom}(V, W)$ using the formula:

$$\tau_g(\phi) = \psi_g \circ \phi \circ \rho_{g^{-1}},$$

- **Intertwining maps**

  A linear map $T : V \to W$ is said to be an *intertwining map* if it satisfies the condition:

  $$T \circ \rho_g = \psi_g \circ T \quad \text{for all } g \in G,$$

  where:

  - $\rho : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are representations of the group G on the vector spaces V and W, respectively.
  - ρ_g and ψ_g denote the actions of $g \in G$ on V and W.

  The collection of all such intertwining maps is denoted by $\mathrm{Hom}_G(V, W)$. This set forms a **vector space** under the usual operations of addition and scalar multiplication.

  Furthermore, if the representations ρ and ψ are identical (i.e., $\rho = \psi$), then the space $\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$ has the structure of an **associative k-algebra**. In this algebra, the multiplication operation is defined by the composition of operators.

2.3 Invariant subspaces and irreducibility

Definition 2.3.1 (G-invariant subspace)**.** *[3] Let $\rho : G \to GL(V)$ be a representation. A subspace $W \leq V$ is called G-**invariant** if, for all $g \in G$ and $w \in W$, the following holds: $\rho_g(w) \in W$.*

In other words, the action of ρ_g on any vector $w \in W$ keeps w within the subspace W. This means W is preserved under the group action defined by ρ.

Definition 2.3.2 (Subrepresentation)**.** *[7] Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. If $W \subseteq V$ is a G-invariant subspace, then we can define the subrepresentation $\rho|_W : G \to \mathrm{GL}(W)$ by restricting ρ to W. Explicitly, for each $g \in G$, the linear map $(\rho|_W)_g$ is given by:*

$$(\rho|_W)_g(w) = \rho_g(w) \quad \text{for all } w \in W.$$

This construction ensures that the group action on W is well-defined and preserves the subspace structure, allowing us to study the representation ρ in terms of its restriction to W.

Example 2.3.1. *[7] Consider V as the regular representation of the group G. Define a one-dimensional subspace $W \subset V$ generated by the vector*

$$x = \sum_{s \in G} e_s.$$

One can verify that for every $s \in G$, the action $\rho_s(x) = x$, meaning that x remains fixed under the action of all elements in G. Therefore, W is a G-invariant subspace of V, and thus a subrepresentation. Moreover, W is isomorphic to the trivial representation.

Lemma 2.3.1 (Quotient Representation)**.** *Let V be a G-representation and $W \subseteq V$ a G-invariant subspace. Then the quotient space V/W inherits a natural G-representation structure via the action:*

$$g \cdot (v + W) := g \cdot v + W \quad \text{for all } g \in G, v \in V.$$

Proof. This follows from the definitions since W is invariant under the action of each $g \in G$.

For example, here is the check that the action is well-defined:

$$g((v+w)+W) \overset{\text{def}}{=} g(v+w)+W$$

$$\overset{g \text{ is linear}}{=} g(v)+g(w)+W$$

$$\overset{g(w)\in W}{=} g(v)+W.$$

$\square$

Remark 2.3.1. *The quotient V/W is called the quotient representation of V by W. The natural projection $\pi : V \to V/W$ is a G-intertwining maps with kernel W.*

Remark 2.3.2. *[8] Sometimes, $\rho|_W$ is referred to as a subrepresentation of ρ. If V_1 and V_2 are G-invariant subspaces of V such that $V = V_1 \oplus V_2$, then it can be easily verified that ρ is equivalent to the (external) direct sum $\rho|_{V_1} \oplus \rho|_{V_2}$.*

In mathematical structures, one frequently encounters the fundamental phenomenon of *unique factorization* into prime or irreducible elements. This principle manifests particularly in the framework of representation theory, where the concept of an *irreducible representation* is formally analogous to the notion of a simple group in group theory.

Definition 2.3.3 (irreducible representation)**.** *[8] A representation $\rho : G \to GL(V)$ of a group G on a vector space V is called* ***irreducible*** *if there are no non-trivial subspaces of V that are invariant under the action of G.*

Any one-dimensional representation $\rho : G \to \mathbb{C}^*$ is **irreducible**, because $\mathbb{C}$ has no proper non-zero subspaces.

Proposition 2.3.1. *[8] If $\rho : G \to GL(V)$ is a 2-dimensional representation (i.e., $\dim(V) = 2$), then ρ is irreducible* ***if and only if*** *no single non-zero vector $v \in V$ is an eigenvector for every linear transformation ρ_g with $g \in G$.*

Note: The eigenvector trick applies only to degree 2 representations and, for degree 3, requires G to be finite.

Example 2.3.2. *[8] The representation $\rho : S_3 \to GL_2(\mathbb{C})$ from Example 2.2.1 is irreducible.*

Since $\dim(\mathbb{C}^2) = 2$, any non-zero proper S_3-invariant subspace W must be one-dimensional. Let v be a non-zero vector in W, so $W = \mathbb{C}v$. For any $\sigma \in S_3$, the S_3-invariance of W implies:

$$\rho_\sigma(v) \in W = \mathbb{C}v.$$

Thus, $\rho_\sigma(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, meaning v is an eigenvector for all ρ_σ with $\sigma \in S_3$.

Claim: *$\rho_{(1\,2)}$ and $\rho_{(1\,2\,3)}$ do not share a common eigenvector.*

Proof.

1. A direct computation shows that $\rho_{(1\,2)}$ has eigenvalues 1 and -1, with corresponding eigenspaces:

$$V_{-1} = \mathbb{C}e_1 \quad \text{and} \quad V_1 = \mathbb{C}\begin{pmatrix}-1\\2\end{pmatrix}.$$

2. The vector $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ is not an eigenvector of $\rho_{(1\,2\,3)}$, since:

$$\rho_{(1\,2\,3)}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}.$$

3. Similarly, the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is not an eigenvector of $\rho_{(1\,2\,3)}$, as:

$$\rho_{(1\,2\,3)} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

□

Since $\rho_{(1\,2)}$ and $\rho_{(1\,2\,3)}$ have no common eigenvector, the representation ρ is irreducible by the earlier discussion.

2.3.1 Completely reducible

Our ultimate objective is to demonstrate that every linear representation can be expressed as a direct sum of irreducible representations. To this end, we begin by introducing some fundamental definitions and concepts relevant to this framework.

Definition 2.3.4. *[8]*

Let G be a group. A representation $\rho : G \to GL(V)$ is called completely reducible if the vector space V can be decomposed into a direct sum of G-invariant subspaces $V_1, V_2, \ldots, V_n$, such that the restriction of ρ to each subspace V_i (denoted $\rho|_{V_i}$) is an irreducible representation for all $i = 1, 2, \ldots, n$.

Equivalently, ρ is completely reducible if it is ***isomorphic*** *to a direct sum of irreducible representations. That is, there exist irreducible representations $\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)}$ such that:*

$$\rho \sim \rho^{(1)} \oplus \rho^{(2)} \oplus \cdots \oplus \rho^{(n)}.$$

Definition 2.3.5 (Decomposable representation)**.** *[8]*

A non-zero representation $\phi : G \to GL(V)$ of a group G is said to be decomposable if the vector space V can be expressed as a direct sum of two non-zero G-invariant subspaces V_1 and V_2. That is:

$$V = V_1 \oplus V_2,$$

If no such decomposition exists, the representation ϕ is called ***indecomposable****.*

Lemma 2.3.2. *If the representation $\varphi : G \to GL(V)$ is equivalent to a decomposable representation, then φ itself is decomposable.*

Proof. Let $\psi : G \to GL(W)$ be a decomposable representation such that $\psi \sim \varphi$, and let $T : V \to W$ be a vector space isomorphism satisfying $\varphi_g = T^{-1}\psi_g T$ for all $g \in G$. Suppose W_1 and W_2 are non-zero G-invariant subspaces of W such that $W = W_1 \oplus W_2$. Since T is an equivalence, the diagram commutes, meaning $T\varphi_g = \psi_g T$ for all $g \in G$. Define $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. We claim that $V = V_1 \oplus V_2$. Indeed, if $v \in V_1 \cap V_2$, then $T(v) \in W_1 \cap W_2 = \{0\}$, so $T(v) = 0$, and since T is injective, $v = 0$. Moreover, for any $v \in V$, we can write $T(v) = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$, so $v = T^{-1}(w_1) + T^{-1}(w_2)$, implying $V = V_1 + V_2$. Thus, $V = V_1 \oplus V_2$. Next, we show that V_1 and V_2 are G-invariant: for $v \in V_i$ (where $i = 1, 2$), we have $\varphi_g(v) = T^{-1}\psi_g T(v)$, and since $T(v) \in W_i$ and W_i is G-invariant, $\psi_g T(v) \in W_i$, thus $\varphi_g(v) \in T^{-1}(W_i) = V_i$. Therefore, both V_1 and V_2 are G-invariant, and we conclude that φ is decomposable as V decomposes into the G-invariant subspaces V_1 and V_2. □

We obtain analogous theoretical formulations for alternative representation classes, the detailed proofs of which we shall omit for conciseness.

Lemma 2.3.3. *Let $\varphi : G \to GL(V)$ be a representation.*

- *If φ is equivalent to an irreducible representation, then φ itself is irreducible.*
- *If φ is equivalent to a completely reducible representation, then φ itself is completely reducible.*

2.4 Maschke's Theorem and Complete Reducibility

To perform direct sum decompositions of representations, we utilize the tools of inner products and orthogonal decompositions

Definition 2.4.1 (Unitary representation)**.** *[8] A representation $\phi\colon G \to \mathrm{GL}(V)$ on an inner product space V is* ***unitary*** *if for every $g \in G$, the linear map ϕ_g preserves the inner product:*

$$\langle \phi_g(v), \phi_g(w) \rangle = \langle v, w \rangle \quad \forall v, w \in V.$$

for all $v, w \in W$. In other words, we may view ϕ as a map:

$$\phi\colon G \to \mathrm{U}(V)$$

where $\mathrm{U}(V)$ denotes the group of unitary operators on V.(see 1.2.5)

Proposition 2.4.1. *[8] Let $\varphi\colon G \to \mathrm{U}(V)$ be a unitary representation of a group G on a complex Hilbert space V. Then:*

1. *φ is* ***irreducible*** *(has no non-trivial invariant subspaces), or*
2. *φ is* ***completely decomposable*** *into an orthogonal direct sum:*

$$V = V_1 \oplus \cdots \oplus V_n$$

where each restriction $\varphi|_{V_i} : G \to \mathrm{U}(V_i)$ is an irreducible unitary subrepresentation.

Proof. Suppose φ is not irreducible. Then there exists a nonzero proper G-invariant subspace W of V. The orthogonal complement $W^\perp$ is also nonzero, and we can write $V = W \oplus W^\perp$. Thus, it remains to prove that $W^\perp$ is G-invariant. Let $v \in W^\perp$ and $w \in W$. We compute:

$$\begin{aligned} \langle \varphi_g(v), w \rangle &= \\ \langle \varphi_{g^{-1}}\varphi_g(v), \varphi_{g^{-1}}(w) \rangle &= \\ \langle v, \varphi_{g^{-1}}(w) \rangle &= 0. \end{aligned}$$

Thus, $\varphi_g(v) \in W^\perp$, proving that $W^\perp$ is G-invariant. Consequently, φ is decomposable. □

Theorem 2.4.1. *[8] For any finite group G and any finite-dimensional complex representation (ρ, V) of G, there exists an equivalent unitary representation (ρ', V).*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G, with $\dim V = n$. Choose a basis B of V, and define an isomorphism $T : V \to \mathbb{C}^n$ mapping each vector to its coordinate vector with respect to B. We define a new representation $\rho' : G \to \mathrm{GL}_n(\mathbb{C})$ by

$$\rho'_g = T \circ \rho_g \circ T^{-1}, \quad \text{for all } g \in G.$$

Then ρ' is equivalent to ρ.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^n$. We define a new inner product $(\cdot, \cdot)$ on $\mathbb{C}^n$ by averaging over the group:

$$(v, w) = \sum_{g \in G} \langle \rho'_g v, \rho'_g w \rangle, \quad \text{for all } v, w \in \mathbb{C}^n.$$

1. **Linearity:** For $v_1, v_2, w \in \mathbb{C}^n$ and scalars $c_1, c_2 \in \mathbb{C}$,

$$(c_1 v_1 + c_2 v_2, w) = \sum_{g \in G} \langle \rho'_g (c_1 v_1 + c_2 v_2), \rho'_g w \rangle = c_1 (v_1, w) + c_2 (v_2, w).$$

2. Symmetry:

$$(w,v)=\sum_{g\in G}\langle\rho'_g w,\rho'_g v\rangle=\sum_{g\in G}\overline{\langle\rho'_g v,\rho'_g w\rangle}=\overline{(v,w)}.$$

3. Positive-definiteness:

$$(v,v)=\sum_{g\in G}\langle\rho'_g v,\rho'_g v\rangle\geq 0.$$

Equality holds if and only if $\rho'_g v=0$ for all $g\in G$, which implies $v=0$. Therefore, $(\cdot,\cdot)$ is an inner product.

4. Unitarity: We now verify that ρ' is unitary with respect to this new inner product. Let $h\in G$. Then

$$(\rho'_h v,\rho'_h w)=\sum_{g\in G}\langle\rho'_g(\rho'_h v),\rho'_g(\rho'_h w)\rangle=\sum_{g\in G}\langle\rho'_{gh}v,\rho'_{gh}w\rangle.$$

Let $x=gh$. As g ranges over G, so does x. Thus,

$$(\rho'_h v,\rho'_h w)=\sum_{x\in G}\langle\rho'_x v,\rho'_x w\rangle=(v,w).$$

Therefore, ρ'_h preserves the inner product, and ρ' is a unitary representation.

□

Corollary 2.4.1. *[8] For any finite-dimensional representation $\varphi\colon G\to\mathrm{GL}(V)$ of a finite group G:*

- *Irreducible: V has no proper G-invariant subspaces,* or
- *Decomposable: $V=V_1\oplus V_2$ with V_1,V_2 nontrivial G-invariant subspaces.*

Proof. The proof follows directly from Proposition 2.4.1 and Theorem 2.4.1. □

The following example illustrates that Corollary 2.4.1 does not remain valid in the context of infinite groups. Consequently, Proposition 2.4.1 also fails to hold in the infinite setting.

Example 2.4.1. *We present an example of an indecomposable but reducible representation of $\mathbb{Z}$. Define a homomorphism $\varphi:\mathbb{Z}\to\mathrm{GL}_2(\mathbb{C})$ by*

$$\varphi(n)=\begin{pmatrix}1&n\\0&1\end{pmatrix}.$$

It is clear that φ is a group homomorphism. The vector e_1 is an eigenvector for all $\varphi(n)$, so the subspace $\mathbb{C}e_1$ is invariant under $\mathbb{Z}$, implying that φ is reducible. However, since $\varphi(1)$ is not diagonalizable (see Example 1.3.1), the representation cannot be written as a direct sum of one-dimensional representations. Thus, φ is indecomposable.

The following theorem represents the main result of this chapter. Its proof closely parallels the classical arguments used to establish the existence of prime factorizations for integers or the factorization of polynomials into irreducible components.

Theorem 2.4.2 (Maschke). *Any representation of a finite group can be expressed as a direct sum of irreducible subrepresentations.*

Proof. Let $\varphi:G\to GL(V)$ be a representation of a finite group G. We proceed by induction on $\dim V$.

Base Case ($\dim V=1$): The representation φ is irreducible since V has no proper non-zero subspaces.

Inductive Step ($\dim V=n+1$): Assume the statement holds for all representations with $\dim V\leq n$. Consider $\varphi:G\to GL(V)$ with $\dim V=n+1$. If φ is irreducible, we are done. Otherwise, by Corollary 2.4.1, φ is decomposable, and we can write $V=V_1\oplus V_2$, where V_1 and V_2 are non-trivial G-invariant subspaces. Since $\dim V_1,\dim V_2<\dim V$, the induction hypothesis applies to both V_1 and V_2, so we have decompositions $V_1=U_1\oplus\cdots\oplus U_s$ and $V_2=W_1\oplus\cdots\oplus W_r$, with each U_i and W_j being G-invariant and $\varphi|_{U_i}$, $\varphi|_{W_j}$ irreducible. Therefore, V decomposes as $V=U_1\oplus\cdots\oplus U_s\oplus W_1\oplus\cdots\oplus W_r$, completing the proof. □

Remark 2.4.1 (Maschke's Theorem for a General Field k)**.** *Let G be a finite group, and V a vector space over a field k with* $\mathrm{char}(k) \nmid |G|$*. If $W \subseteq V$ is a G-invariant subspace, then there exists a complementary G-invariant subspace $W' \subseteq V$ such that:*

$$V = W \oplus W'.$$

Remark 2.4.2. *A careful examination of the proof reveals that if φ is a unitary matrix representation, then it is unitarily equivalent to a direct sum of irreducible unitary representations; that is, there exists a unitary matrix T such that the equivalence is realized via conjugation by T.*

In summary, given any representation $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ of a finite group, it is always equivalent to a block diagonal representation of the form

$$\varphi \sim \begin{bmatrix} \varphi^{(1)} & 0 & \cdots & 0 \\ 0 & \varphi^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi^{(m)} \end{bmatrix}$$

where each $\varphi^{(i)}$ is an irreducible representation. This decomposition is analogous to the spectral theorem, which asserts that every self-adjoint matrix is diagonalizable.

Chapter 3

The Orthogonality Relations and Character Theory

In this chapter we develop Frobenius-Schur character theory, associating to matrix representations $\phi : G \to \mathrm{GL}_n(\mathbb{C})$ scalar characters $\chi_\phi : G \to \mathbb{C}$. These form an orthonormal system enabling unique irreducible decomposition proofs. The Schur relations establish that the matrix coefficients of irreducible unitary representations form an orthogonal basis for the space of complex-valued functions in G.In this chapter, we will rely on the following references: [8, 3, 7, 6].

3.1 Morphisms of Representations

A fundamental *tenet* of modern mathematics holds that the morphisms between mathematical objects should be accorded the same significance as the objects themselves. Guided by this principle, we now introduce the concept of a *representation morphism*.

Definition 3.1.1 (Morphism)**.** *[8] Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be representations. A* ***morphism of representations*** *from φ to ρ is, by definition, a linear map $T : V \to W$ such that*

$$T\varphi(g) = \rho(g)T \quad \text{for all } g \in G.$$

In other words, the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi(g)"] \arrow[d, "T"] & V \arrow[d, "T"] \\ W \arrow[r, "\rho(g)"] & W. \end{tikzcd}$$

such morphisms are also called intertwining operators and G-morphisms. The set of all morphisms from φ to ρ is denoted by $\mathrm{Hom}_G(\varphi, \rho)$, and we have

$$\mathrm{Hom}_G(\varphi, \rho) \subseteq \mathrm{Hom}(V, W).$$

Remark 3.1.1. *A linear map $T : V \to V$ belongs to $\mathrm{Hom}_G(\varphi, \varphi)$ if and only if it commutes with the action of G via φ; that is,*

$$T\varphi_g = \varphi_g T \quad \text{for all } g \in G.$$

In other words, T lies in the centralizer of the image of $\varphi(G)$ in $\mathrm{End}(V)$. In particular, the identity map $I : V \to V$ always satisfies this condition and is therefore an element of $\mathrm{Hom}_G(\varphi, \varphi)$.

This space is commonly denoted by

$$\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V),$$

and it consists of all G-equivariant endomorphisms of V; that is, all linear maps from V to itself that commute with the group action defined by φ.

As is typical with homomorphisms in algebra, the kernel and image of any morphism between representations are themselves subrepresentations.

Proposition 3.1.1. *Suppose $T : V \to W$ is a G-equivariant linear map, that is, $T \in \mathrm{Hom}_G(\varphi, \rho)$. Then the kernel of T, denoted $\ker T$, is a G-invariant subspace of V, and the image of T, denoted $\mathrm{Im}\, T$, is a G-invariant subspace of W.*

Proof. Let $v \in \ker T$ and $g \in G$. Since $v \in \ker T$, we have

$$T(\varphi_g v) = \rho_g(Tv) = \rho_g(0) = 0,$$

which shows that $\varphi_g v \in \ker T$. Therefore, the kernel of T is stable under the action of G, i.e., it is a G-invariant subspace of V.

Similarly, for any $w \in \mathrm{Im}\, T$, there exists $v \in V$ such that $w = T(v)$. Then,

$$\rho_g(w) = \rho_g(Tv) = T(\varphi_g v) \in \mathrm{Im}\, T,$$

which implies that $\mathrm{Im}\, T$ is also G-invariant. □

Proposition 3.1.2. *For representations $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$, the space $\mathrm{Hom}_G(\varphi, \rho)$ is a subspace of $\mathrm{Hom}(V, W)$.*

A key observation in representation theory, due to I. Schur is that morphisms between irreducible representations are very limited. This relies on the fact that we are working over the field of complex numbers rather than the field of real numbers. Specifically, it uses the fact that every linear operator on a finite-dimensional complex vector space has an eigenvalue, which follows from the fact that every polynomial over $\mathbb{C}$ has a root, including the characteristic polynomial of the operator.

Proposition 3.1.3 (Schur's lemma). *[3] Let $T : V \to W$ be a morphism of irreducible representations. Then either:*

- $\ker(T) = 0$ *and* $\mathrm{Im}(T) = W$ *(so T is invertible), or*
- $T = 0$.

Proof. Suppose $T = 0$; then the statement is trivially satisfied. Assume $T \neq 0$ for the remainder of the proof. Since $\ker(T)$ is a G-invariant subspace (by Proposition 3.1.1), and since φ is irreducible, we have

$$\ker(T) = V \iff T \equiv 0,$$

which contradicts our assumption, thus

$$\ker(T) = 0,$$

meaning that T is injective. Similarly, since $\mathrm{Im}(T)$ is G-invariant (by Proposition 3.1.1), and ρ is irreducible, we have $\mathrm{Im}(T) = 0 \iff T = 0$,

which again contradicts our assumption, thus $\mathrm{Im}(T) = W$, meaning that T is surjective. Therefore,

$$T \text{ is bijective} \iff T \text{ is invertible}.$$

□

Corollary 3.1.1. *[8] Let $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ be irreducible complex representations. Then:*

- *If $\varphi \ncong \rho$, then $Hom_G(V, W) = 0$.*
- *If $\varphi = \rho$, then every morphism $T \in Hom_G(V, V)$ is a scalar multiple of the identity map, i.e., there exists $\lambda \in \mathbb{C}$ such that: $T = \lambda\, id_V$.*
- *If $\varphi \cong \rho$, then:* $\dim Hom_G(V, W) = 1$.

Proof. (1) Assume that $\mathrm{Hom}_G(\varphi, \rho) \neq \{0\}$. Then there exists a nonzero map $T \in \mathrm{Hom}_G(\varphi, \rho)$. By the earlier result, such a nonzero T must be invertible. Therefore, it follows that the representations φ and ρ are isomorphic, i.e.,

$$\varphi \cong \rho.$$

This proves the contrapositive of the statement we want, namely:

$$\varphi \ncong \rho \quad \Rightarrow \quad \mathrm{Hom}_G(\varphi, \rho) = \{0\}.$$

(2) Let $T \in \mathrm{Hom}_G(\varphi, \varphi)$, and suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then, by the definition of eigenvalues, the operator $\lambda I - T$ is not invertible. Since $I \in \mathrm{Hom}_G(\varphi, \varphi)$ and $\mathrm{Hom}_G(\varphi, \varphi)$ is closed under scalar multiplication and addition (by 3.1.2), we have

$$\lambda I - T \in \mathrm{Hom}_G(\varphi, \varphi).$$

Now, from part (1), we know that all nonzero elements of $\mathrm{Hom}_G(\varphi, \varphi)$ are invertible. Hence, the only way $\lambda I - T$ is not invertible is if

$$\lambda I - T = 0 \quad \Rightarrow \quad T = \lambda I.$$

(3) If $\varphi \cong \rho$, then $\mathrm{Hom}_G(V, W) \cong \{\lambda I \mid \lambda \in \mathbb{C}\}$. ☐

At this point, we are ready to describe the irreducible representations of an abelian group.

Corollary 3.1.2. *[7] Let G be an abelian group. Then every irreducible representation of G is one-dimensional.*

Proof. Let $\varphi\colon G \to \mathrm{GL}(V)$ be an irreducible representation of an abelian group G. We will show that $\dim V = 1$.

First, observe that for any $g \in G$ and any $v \in V$, the representation property combined with the abelianness of G gives the following commutative diagram in the group action: for all $h \in G$,

$$\varphi_g(\varphi_h(v)) = \varphi_{gh}(v) = \varphi_{hg}(v) = \varphi_h(\varphi_g(v)).$$

This equality holds precisely because $gh = hg$ in the abelian group G.

The above identity shows that each φ_g commutes with all operators φ_h for $h \in G$. This means that φ_g is not just a linear operator on V, but in fact a G-morphism, i.e., $\varphi_g \in \mathrm{Hom}_G(V, V)$.

By Schur's Lemma (or the preceding corollary mentioned in the original text), since V is irreducible, every G-endomorphism of V must be a scalar multiple of the identity map. Therefore, there exists a complex number $\lambda_g \in \mathbb{C}$ such that $\varphi_g = \lambda_g I$, where I is the identity operator on V.

Now consider any nonzero vector $v \in V$. For every group element $g \in G$, we have

$$\varphi_g(v) = \lambda_g I(v) = \lambda_g v,$$

which shows that $\varphi_g(v)$ is always a scalar multiple of v. This implies that the one-dimensional subspace $\langle v \rangle$ is invariant under the action of G.

Finally, since V is by assumption irreducible and we have found a nonzero G-invariant subspace $\langle v \rangle$, it must be that $V = \langle v \rangle$. This proves that $\dim V = 1$, and therefore every irreducible representation of G is one-dimensional. ☐

We now demonstrate several applications of this result in linear algebra.

Proposition 3.1.4. *[8] For any finite abelian group G and representation $\phi\colon G \to \mathrm{GL}_n(\mathbb{C})$, there exists a simultaneous diagonalizing matrix $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}\phi_g T$ is diagonal for all $g \in G$.*

Proof. Let $\phi\colon G \to \mathrm{GL}(V)$ be a representation of the finite abelian group G.

First, by Maschke's Theorem (which applies since G is finite), ϕ is completely reducible. This gives us an isomorphism of representations:

$$\phi \cong \phi^{(1)} \oplus \phi^{(2)} \oplus \cdots \oplus \phi^{(n)},$$

where each $\phi^{(i)}\colon G \to \mathbb{C}^*$ is an irreducible representation.

Moreover, from the preceding corollary 3.1.2, we conclude that each $\phi^{(i)}$ has degree 1. This one-dimensionality is crucial for what follows.

In matrix terms, this decomposition has the following concrete realization: For every group element $g \in G$, the representation matrix ϕ_g can be expressed as:

$$\phi_g = T^{-1} \begin{pmatrix} \phi_g^{(1)} & 0 & \cdots & 0 \\ 0 & \phi_g^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_g^{(n)} \end{pmatrix} T,$$

where $T \in \mathrm{GL}_n(\mathbb{C})$ is a fixed change-of-basis matrix (independent of the group element g), and each $\phi_g^{(i)} \in \mathbb{C}^*$ is a scalar (reflecting the one-dimensional nature of each irreducible component). The diagonal form of the inner matrix follows immediately from the direct sum decomposition and the scalar nature of the irreducible representations. ☐

This leads directly to the result that any matrix of finite order can be diagonalized.

Corollary 3.1.3. *Let $T \in \mathrm{GL}_m(\mathbb{C})$ be an invertible complex matrix of finite order (i.e., $T^k = I$ for some $k \in \mathbb{N}$). Then T is diagonalizable. Moreover, if $T^n = I$, the eigenvalues of T are n-th roots of unity (i.e., $\lambda^n = 1$).*

Proof. Since T has finite order, its minimal polynomial divides $x^k - 1$, which has no repeated roots over $\mathbb{C}$. Hence, T is diagonalizable.

If $T^n = I$, let λ be an eigenvalue of T with eigenvector v. Then:

$$T^n v = \lambda^n v = Iv = v,$$

which implies $\lambda^n = 1$. Thus, λ is an n-th root of unity. ☐

Remark 3.1.2. *Corollary 3.1.3 may be demonstrated through Proposition 3.1.4.*

We present below an important property concerning one-dimensional representations of groups

Proposition 3.1.5. *[6] Let $\varphi : G \to \mathbb{C}^* = \mathrm{GL}(\mathbb{C})$ be a degree one representation of a group G. Let*

$$G' = \langle [g,h] = ghg^{-1}h^{-1} \mid g,h \in G \rangle$$

be the commutator subgroup of G. Then, there exists a representation $\psi : G/G' \to \mathbb{C}^*$ such that $\varphi = \psi \circ \pi$, where $\pi : G \to G/G'$ is the quotient map.

Proof. Since $\varphi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ is an abelian group, it follows that

$$\varphi([g,h]) = \varphi(ghg^{-1}h^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1} = 1.$$

This induces a group homomorphism, say,

$$\psi : G/G' \to \mathbb{C}^* \quad \text{given by} \quad \psi(gG') = \varphi(g)$$

and

$$\varphi = \psi \circ \pi$$

☐

3.2 The Orthogonality Relations

From this point onward, we assume that the group G is finite. Let $\varphi : G \to GL_n(\mathbb{C})$ be a linear representation of G. Each element $\varphi(g)$, for $g \in G$, is an $n \times n$ complex matrix whose entries $\varphi_{ij}(g)$ define complex-valued functions $\varphi_{ij} : G \to \mathbb{C}$, for $1 \leq i, j \leq n$. Thus, a representation of degree n gives rise to n^2 such functions.

If the representation φ is irreducible and unitary, then these functions φ_{ij} possess a significant structural property: they form an orthogonal set in the space $\mathbb{C}[G]$ of complex-valued functions on G, equipped with the standard inner product. Consequently, the collection $\{\varphi_{ij}\}$ constitutes an orthogonal basis for a subspace of $\mathbb{C}[G]$, reflecting the deep interplay between representation theory and harmonic analysis on finite groups.

Definition 3.2.1 (Group algebra)**.** *[8] Let G be a finite group. We define the group algebra $\mathbb{C}[G]$ as the set of all complex-valued functions on G:*

$$\mathbb{C}[G] := \{f : G \to \mathbb{C}\}$$

equipped with:

1. *Pointwise addition:* $(f_1 + f_2)(g) = f_1(g) + f_2(g)$
2. *Scalar multiplication:* $(c \cdot f)(g) = c \cdot f(g)$
3. *Inner product: defined*

$$\langle -, - \rangle : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$$

by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

Remark 3.2.1. *To make $\mathbb{C}[G]$ a full algebra, we add the convolution product:*

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1}g)$$

This operation ensures algebraic closure and generalizes the natural multiplication in $\mathbb{C}[G]$.

A central objective of this chapter is to establish a fundamental result originally due to . Schur. It is worth recalling that $U_n(\mathbb{C})$ denotes the group of $n \times n$ unitary matrices over the complex numbers.

Theorem 3.2.1 (Schur orthogonality relations)**.** *[8] Suppose that*

$$\varphi : G \to U_n(\mathbb{C}) \quad \textit{and} \quad \rho : G \to U_m(\mathbb{C})$$

be non-equivalent irreducible unitary representations of a finite group G. Then:

1. $\langle \varphi_{ij}, \rho_{k\ell} \rangle = 0$
2. $\langle \varphi_{ij}, \varphi_{k\ell} \rangle = \begin{cases} \frac{1}{n} & \textit{if } i = k \textit{ and } j = \ell \\ 0 & \textit{otherwise} \end{cases}$

The proof of this theorem requires substantial preparation, beginning with our second application of the averaging trick

Proposition 3.2.1. *Let $\varphi : G \to GL(V)$ and $\psi : G \to GL(W)$ be two representations of a group G. Let $T : V \to W$ be a linear map. Then:*

1. $\widetilde{T} := \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \circ T \circ \varphi_g \in Hom_G(V, W)$

2. *If* $T \in Hom_G(V, W)$*, then* $\widetilde{T} = T$

3. *The map* $P : Hom(V, W) \to Hom_G(V, W) \subseteq Hom(V, W)$ *given by* $P(T) = \widetilde{T}$*, satisfies the property* $P^2 = P$

Proof. (1) We aim to show that the map

$$\widetilde{T} = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \circ T \circ \varphi_g$$

is G-equivariant, i.e., $\widetilde{T} \circ \varphi_h = \psi_h \circ \widetilde{T}$ for all $h \in G$.

Let us compute $\widetilde{T} \circ \varphi_h$:

$$\widetilde{T} \circ \varphi_h = \frac{1}{|G|} \sum_{g \in G} (\psi_{g^{-1}} \circ T \circ \varphi_g) \circ \varphi_h = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \circ T \circ \varphi_{gh}.$$

Now, we perform a change of variable. Let $x = gh$, which implies $g = xh^{-1}$. Since G is a group, the change of variable is bijective, and the sum becomes:

$$\widetilde{T} \circ \varphi_h = \frac{1}{|G|} \sum_{x \in G} \psi_{(xh^{-1})^{-1}} \circ T \circ \varphi_x = \frac{1}{|G|} \sum_{x \in G} \psi_{hx^{-1}} \circ T \circ \varphi_x.$$

Factoring out ψ_h, we obtain:

$$\widetilde{T} \circ \varphi_h = \psi_h \left(\frac{1}{|G|} \sum_{x \in G} \psi_{x^{-1}} \circ T \circ \varphi_x \right) = \psi_h \circ \widetilde{T}.$$

Thus, we conclude that:

$$\widetilde{T} \in \mathrm{Hom}_G(V, W), \quad \text{for all } h \in G.$$

(2) Let $T \in \mathrm{Hom}_G(V, W)$. By the intertwining property, we have $\psi_{g^{-1}} \circ T \circ \varphi_g = T$ for all $g \in G$. Consequently, the averaging operator yields:

$$\widetilde{T} = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \circ T \circ \varphi_g = \frac{1}{|G|} \sum_{g \in G} T = T.$$

Finally, for (3) we establish linearity by checking
Let $T_1, T_2 \in \mathrm{Hom}(V, W)$ and $c_1, c_2 \in \mathbb{C}$. Then:

$$\begin{aligned} P(c_1 T_1 + c_2 T_2) &= \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \circ (c_1 T_1 + c_2 T_2) \circ \varphi_g \\ &= \frac{1}{|G|} \sum_{g \in G} \big(c_1 \psi_{g^{-1}} \circ T_1 \circ \varphi_g + c_2 \psi_{g^{-1}} \circ T_2 \circ \varphi_g \big) \\ &= c_1 \left(\frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \circ T_1 \circ \varphi_g \right) + c_2 \left(\frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \circ T_2 \circ \varphi_g \right) \\ &= c_1 P(T_1) + c_2 P(T_2). \end{aligned}$$

Thus, the map P is linear.
Now consider the projection map $P : \mathrm{Hom}(V, W) \to \mathrm{Hom}_G(V, W)$ defined by $P(T) = \widetilde{T}$. For any $T \in \mathrm{Hom}(V, W)$, we verify idempotence:

$$P^2(T) = P(\widetilde{T}) = \widetilde{\widetilde{T}} = \widetilde{T} = P(T).$$

Furthermore, when $T \in \mathrm{Hom}_G(V, W)$, we immediately obtain $P(T) = \widetilde{T} = T$, demonstrating that P is indeed a projection onto $\mathrm{Hom}_G(V, W)$. ☐

The version of Schur's Lemma that follows is the one we will most frequently use throughout this work. It relies on a straightforward observation: if I_n denotes the $n \times n$ identity matrix and $\lambda \in \mathbb{C}$, then the trace of λI_n is given by:

$$\mathrm{Tr}(\lambda I_n) = n\lambda.$$

Proposition 3.2.2. *Let $\phi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be irreducible representations of a group G, and let $T : V \to W$ be a linear map. Then:*

1. *Non-isomorphic case: If ϕ and ρ are not isomorphic (i.e., $\phi \ncong \rho$), then $\widetilde{T}$ must be the zero map:*

$$\widetilde{T} = 0$$

2. *Isomorphic case: If ϕ and ρ are isomorphic (i.e., $\phi \cong \rho$), then $\widetilde{T}$ is a scalar multiple of the identity:*

$$\widetilde{T} = \left(\frac{\mathrm{Tr}(T)}{\deg \phi}\right) \cdot \mathrm{Id}_V$$

where Id_V denotes the identity operator on V.

Proof. We prove the result in two cases:

Case 1: $\phi \ncong \rho$. By Schur's Lemma, we have $\mathrm{Hom}_G(\phi, \rho) = 0$,and $\widetilde{T} \in \mathrm{Hom}_G(V, W)$ which immediately implies: $\widetilde{T} = 0$

Case 2: $\phi \cong \rho$. Schur's Lemma shows that any intertwining operator must be scalar:

$$\widetilde{T} = \lambda I \quad \text{for some } \lambda \in \mathbb{C}$$

To determine λ, we compute the trace in two different ways:

1. Through the scalar form:

$$\mathrm{Tr}(\widetilde{T}) = \mathrm{Tr}(\lambda I) = \lambda \mathrm{Tr}(I) = \lambda \dim V = \lambda \deg \phi$$

2. Through the original definition:

$$\mathrm{Tr}(\widetilde{T}) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\phi_{g^{-1}} T \phi_g) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(T) = \mathrm{Tr}(T)$$

where we used the cyclic property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Equating both expressions yields: $\lambda = \frac{\mathrm{Tr}(T)}{\deg \phi}$

Therefore, we conclude: $\widetilde{T} = \frac{\mathrm{Tr}(T)}{\deg \phi} \mathrm{Id}_V$ □

Let $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ and $\rho : G \to \mathrm{GL}_m(\mathbb{C})$ be two representations. Then the space $\mathrm{Hom}(V, W)$ can be naturally identified with the space of complex $m \times n$ matrices, denoted by $\mathrm{M}_{m \times n}(\mathbb{C})$. The subspace $\mathrm{Hom}_G(\varphi, \rho) \subseteq \mathrm{Hom}(V, W)$ is therefore a subspace of $\mathrm{M}_{m \times n}(\mathbb{C})$.

Accordingly, the projection map P, defined in Proposition 3.2.1, can be viewed as a linear operator: $P : \mathrm{M}_{m \times n}(\mathbb{C}) \to \mathrm{M}_{m \times n}(\mathbb{C})$.

It is then natural to consider the matrix of P with respect to the standard basis of $\mathrm{M}_{m \times n}(\mathbb{C})$. Recall that this standard basis consists of the matrices $E_{11}, E_{12}, \ldots, E_{mn}$, where each E_{ij} is the matrix with a 1 in the (i, j) entry and zeros elsewhere. Any matrix $(a_{ij}) \in \mathrm{M}_{m \times n}(\mathbb{C})$ can then be written as:$(a_{ij}) = \sum_{i,j} a_{ij} E_{ij}$.

This lemma follows immediately from a direct application of the matrix multiplication algorithm.

Lemma 3.2.1. *Let $A \in M_{r \times m}(\mathbb{C})$, $B \in M_{n \times s}(\mathbb{C})$, and let $E_{ki} \in M_{m \times n}(\mathbb{C})$. Then the (j, ℓ)-entry of the matrix product $AE_{ki}B$ is given by:*

$$(AE_{ki}B)_{j\ell} = a_{jk} b_{i\ell}$$

where $A = (a_{jq})$ and $B = (b_{p\ell})$.

Proof. By definition of matrix multiplication:

$$(AE_{ki}B)_{j\ell} = \sum_{x=1}^{m}\sum_{y=1}^{n} a_{jx}(E_{ki})_{xy}b_{y\ell}$$

Since $(E_{ki})_{xy} = \delta_{xk}\delta_{yi}$ (Kronecker delta), the only non-zero term occurs when $x = k$ and $y = i$, yielding:

$$(AE_{ki}B)_{j\ell} = a_{jk} \cdot 1 \cdot b_{i\ell} = a_{jk}b_{i\ell} \qquad \square$$

We now proceed to construct the matrix representation of P relative to the standard basis. The final expression is presented in a form suitable for subsequent analysis.

Lemma 3.2.2 (Orthogonality of Matrix Coefficients)**.** *[8] Let $\phi\colon G \to U_n(\mathbb{C})$ and $\rho\colon G \to U_m(\mathbb{C})$ be unitary representations of a finite group G. For an elementary matrix $A = E_{ki} \in M_{m\times n}(\mathbb{C})$, the averaged matrix*

$$\widetilde{A} = \frac{1}{|G|}\sum_{g\in G} \rho_g^{-1}E_{ki}\phi_g$$

satisfies the entry-wise relation:

$$\widetilde{A}_{\ell j} = \langle \phi_{ij}, \rho_{k\ell}\rangle,$$

where $\langle \phi_{ij}, \rho_{k\ell}\rangle$ denotes the inner product of matrix coefficients.

Proof. Since ρ is unitary, $\rho_g^{-1} = \rho_g^*$ for all $g \in G$. This implies:

$$\rho_{k\ell}(g^{-1}) = \overline{\rho_{\ell k}(g)}.$$

$$\begin{aligned}
\widetilde{A}_{\ell j} &= \frac{1}{|G|}\sum_{g\in G}\left(\rho_g^{-1}E_{ki}\phi_g\right)_{\ell j} \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{a=1}^{m}\sum_{b=1}^{n}(\rho_g^{-1})_{\ell a}(E_{ki})_{ab}(\phi_g)_{bj} \\
&= \frac{1}{|G|}\sum_{g\in G}(\rho_g^{-1})_{\ell k}(\phi_g)_{ij} \quad \text{(using } (E_{ki})_{ab} = \delta_{ak}\delta_{bi}\text{)} \\
&= \frac{1}{|G|}\sum_{g\in G}\overline{\rho_{k\ell}(g)}\phi_{ij}(g) \\
&= \langle \phi_{ij}, \rho_{k\ell}\rangle
\end{aligned}$$

$\square$

Remark 3.2.2. *The linear operator $P(T) = \frac{1}{|G|}\sum_{g\in G}\rho_g^{-1}T\phi_g$ induces an $mn \times mn$ matrix B whose entries are the inner products $B_{(k\ell),(ij)} = \langle \phi_{ij}, \rho_{k\ell}\rangle$ of matrix coefficients. This construction directly embeds the orthogonality relations of group representations into the matrix structure - the entries vanish for inequivalent irreducible representations while giving specific values for equivalent ones. The matrix B therefore provides a concrete realization of representation-theoretic orthogonality through linear algebra, where the averaging process systematically captures the underlying representation theory.*

We now have all the necessary elements to establish Schur orthogonality relations (Theorem 3.2.1) in its full generality.

Proof. (1) Take $A = E_{k,i} \in M_{n\times m}(\mathbb{C})$.

We compute $\tilde{A}$ and find:

$$\tilde{A} = 0$$

However, by the preceding proposition, we have:

$$0 = (\tilde{A})_{ij} = \langle \varphi_{ij}, \rho_{kl}\rangle$$

(2) Now take $A := E_{k,i} \in M_n(\mathbb{C})$, then:

$$\tilde{A} = \frac{\text{Trace}(A)}{\deg(\varphi)} \cdot I$$

Thus:

$$\tilde{A} = \frac{\text{Trace}(E_{k,i})}{\deg(\varphi)} \cdot I$$

We observe that:

- If $i \neq k$ or $j \neq l$, then
$$\langle \varphi_{ij}, \varphi_{kl} \rangle = 0$$
- If $i = k$ and $j = l$, then
$$\langle \varphi_{ij}, \varphi_{ij} \rangle = \frac{1}{n}$$

□

Since the group algebra $\mathbb{C}[G]$ is a vector space with dimension equal to the group's order $|G|$, the number of equivalence classes of irreducible representations of G must necessarily be finite. This fundamental limitation arises because Theorem 3.2.1 demonstrates that the matrix coefficients of inequivalent unitary representations form an orthogonal set of non-zero vectors in $\mathbb{C}[G]$. Given that $\mathbb{C}[G]$ has finite dimension ($|G|$), it cannot contain more than $|G|$ linearly independent vectors.

More precisely, if $\{\varphi^{(1)}, \ldots, \varphi^{(s)}\}$ represents a complete set of inequivalent irreducible representations of G, where each $\varphi^{(i)}$ has degree d_i, then the collection of functions $\{\sqrt{d_k}\varphi^{(k)}_{ij} \mid 1 \leq k \leq s, 1 \leq i, j \leq d_k\}$ forms an orthonormal set in $\mathbb{C}[G]$. This immediately implies the key inequality:

$$s \leq d_1^2 + d_2^2 + \cdots + d_s^2 \leq |G|$$

The first inequality holds because each $d_i \geq 1$ (since every irreducible representation has dimension at least 1), while the second inequality reflects the crucial fact that the sum of squares of all irreducible representation dimensions cannot exceed the total dimension of $\mathbb{C}[G]$, which is $|G|$.

Corollary 3.2.1. *Let φ be an irreducible unitary representation of a group G and let d be the degree of φ. Then the n^2 functions $\sqrt{d}\,\varphi_{ij} \quad (1 \leq i, j \leq d)$ form an orthonormal set in $\mathbb{C}[G]$.*

Corollary 3.2.2. *[8]*

1. *Let G be a finite group. Then G has only finitely many inequivalent irreducible representations.*
2. *Let $\varphi^1, \varphi^2, \ldots, \varphi^s$ be representatives of all equivalence classes of irreducible representations of G. Let the degree of φ^i be d_i. Then:*
$$s \leq d_1^2 + d_2^2 + \cdots + d_s^2 \leq |G|.$$

3.3 Characters and Class Functions

In this section, we finally establish the uniqueness of the decomposition of a representation into irreducible representations. The key element in this proof is the association of each representation ϕ with a *characteristic function* $\chi_\phi : G \to \mathbb{C}$, which encodes the complete structure of the representation.

Definition 3.3.1 (Character of a Representation)**.** *[8] Let $\phi\colon G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of a finite group G. The* character *χ_ϕ of ϕ is the function*

$$\chi_\phi\colon G \to \mathbb{C}, \quad \chi_\phi(g) = \mathrm{Tr}(\phi(g)),$$

where Tr *denotes the trace of the matrix $\phi(g) \in \mathrm{GL}_n(\mathbb{C})$.*

More explicitly, if we express ϕ in matrix form with coordinate functions $\phi_{ij}\colon G \to \mathbb{C}$, then:

$$\chi_\phi(g) = \sum_{i=1}^{n} \phi_{ii}(g).$$

In the special case when $\phi\colon G \to \mathbb{C}^*$ is a one-dimensional representation (degree 1), the character coincides with the representation itself:

$$\chi_\phi = \phi.$$

Lemma 3.3.1. *Let $\phi\colon G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of a group G. Then the character value at the identity element satisfies:*

$$\chi_\phi(1_G) = \deg(\phi),$$

where $\deg(\phi) = n$ is the degree of the representation.

Proof. The result follows from the direct computation of the character at the identity element:

$$\begin{aligned}
\chi_\phi(1_G) &= \mathrm{Tr}(\phi(1_G)) && \text{(by definition of the character)}\\
&= \mathrm{Tr}(I_n) && \text{(since ϕ is a group homomorphism)}\\
&= n && \text{(as the trace of the $n \times n$ identity matrix)}\\
&= \deg(\phi). && \text{(by definition of the degree)}
\end{aligned}$$

□

Proposition 3.3.1 (Character Invariance under Equivalence)**.** *Let (ϕ, V) and (ρ, W) be finite-dimensional representations of a group G over $\mathbb{C}$. If ϕ and ρ are equivalent representations then their characters coincide:*

$$\chi_\phi(g) = \chi_\rho(g) \quad \text{for all } g \in G.$$

Proof. Since characters are computed using matrix traces, we may fix a basis and consider both representations as matrix representations:

$$\phi, \rho\colon G \to \mathrm{GL}_n(\mathbb{C}).$$

Given that ϕ and ρ are equivalent representations, there exists an invertible change-of-basis matrix $T \in \mathrm{GL}_n(\mathbb{C})$ such that for every $g \in G$:

$$\phi_g = T\rho_g T^{-1}.$$

The character equality follows through these trace calculations:

$$\begin{aligned}
\chi_\phi(g) &= \mathrm{Tr}(\phi_g)\\
&= \mathrm{Tr}(T\rho_g T^{-1}) && \text{(by equivalence)}\\
&= \mathrm{Tr}(T^{-1}T\rho_g) && \text{(by cyclic property of trace $Tr(AB) = Tr(BA)$)}\\
&= \mathrm{Tr}(I_n\rho_g)\\
&= \mathrm{Tr}(\rho_g)\\
&= \chi_\rho(g).
\end{aligned}$$

Thus, we conclude that $\chi_\phi = \chi_\rho$ as functions on G. □

Definition 3.3.2 (Class function)**.** *[8] Let G be a group. A function $f\colon G \to \mathbb{C}$ is called a class function if it is invariant under conjugation, that is,*

$$f(g) = f(hgh^{-1}) \quad \text{for all } g, h \in G.$$

Equivalently, f is constant on each conjugacy class of G.

Example 3.3.1. *The following are classical examples of class functions:*

1. *Let $G = \mathrm{GL}_n(\mathbb{C})$, the general linear group of complex $n \times n$ matrices. The* ***determinant function***

   $$\mathrm{Det}\colon \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}, \quad \mathrm{Det}(A) = \det A$$

   is a class function because for any $A, B \in \mathrm{GL}_n(\mathbb{C})$,

   $$\mathrm{Det}(B^{-1}AB) = \det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \det(A) = \mathrm{Det}(A).$$

2. *Again for $G = \mathrm{GL}_n(\mathbb{C})$, the* ***trace function***

   $$\mathrm{Tr}\colon \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}, \quad \mathrm{Tr}(A) = \mathrm{tr}\, A$$

   is a class function since for any $A, B \in \mathrm{GL}_n(\mathbb{C})$,

   $$\mathrm{Tr}(B^{-1}AB) = \mathrm{Tr}(B^{-1}AB) = \mathrm{Tr}(ABB^{-1}) = \mathrm{Tr}(A) = \mathrm{Tr}(A).$$

This property demonstrates that every character is constant on conjugacy classes, or equivalently, invariant under group conjugation.

Proposition 3.3.2. *Suppose ϕ is a representation of the group G. Then for any elements $g, h \in G$, the character χ_ϕ assigns the same value to g and its conjugate hgh^{-1}; that is,*

$$\chi_\phi(g) = \chi_\phi(hgh^{-1}).$$

Proof.

$$\begin{aligned}\chi_\phi(hgh^{-1}) &= \mathrm{Tr}(\phi_{hgh^{-1}}) \\ &= \mathrm{Tr}(\phi_h\phi_g\phi_h^{-1}) \\ &= \mathrm{Tr}(\phi_h^{-1}\phi_h\phi_g) \\ &= \mathrm{Tr}(\phi_g) \\ &= \chi_\phi(g)\end{aligned}$$

Hence, χ_ϕ is constant on conjugacy classes. □

Let $Z(\mathbb{C}[G])$ denote the *center of the group algebra* $\mathbb{C}[G]$, consisting of all class functions $f\colon G \to \mathbb{C}$:

$$Z(\mathbb{C}[G]) = \{f \in \mathbb{C}[G] \mid f(hgh^{-1}) = f(g), \forall g, h \in G\}.$$

Proposition 3.3.3. *[8]*

1. *The center $Z(\mathbb{C}[G])$ forms a linear subspace of the class function space $\mathbb{C}[G]$.*
2. *Let $\mathrm{Cl}(G)$ denote the set of conjugacy classes of G. Then*

   $$\dim Z(\mathbb{C}[G]) = |\mathrm{Cl}(G)|$$

Proof. (1)Let f_1, f_2 be class functions on G and $c_1, c_2 \in \mathbb{C}$. For any $g, h \in G$:

$$\begin{aligned}(c_1f_1 + c_2f_2)(hgh^{-1}) &= c_1f_1(hgh^{-1}) + c_2f_2(hgh^{-1}) \\ &= c_1f_1(g) + c_2f_2(g) \\ &= (c_1f_1 + c_2f_2)(g).\end{aligned}$$

Thus $c_1f_1 + c_2f_2$ is a class function.

(2) For each conjugacy class C_i, define its characteristic function:

$$\delta_{C_i}(g) = \begin{cases} 1 & \text{if } g \in C_i \\ 0 & \text{otherwise} \end{cases}$$

These functions satisfy $\delta_{C_i} \in Z(\mathbb{C}[G])$ since they are constant on conjugacy classes.

Any class function $f \in Z(\mathbb{C}[G])$ decomposes uniquely as:

$$f = \sum_{i=1}^{k} f(C_i)\delta_{C_i}$$

This follows because for $g \in C_j$:

$$f(g) = f(C_j) = \sum_{i=1}^{k} f(C_i)\delta_{C_i}(g)$$

To show linear independence of $\mathcal{B} = \{\delta_{C_i}\}_{i=1}^{k}$, we verify orthogonality:

$$\langle \delta_{C_i}, \delta_{C_j} \rangle = \frac{1}{|G|}\sum_{g\in G} \delta_{C_i}(g)\delta_{C_j}(g) = \begin{cases} \frac{|C_i|}{|G|} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This inner product calculation shows:

- Non-degeneracy: $\langle \delta_{C_i}, \delta_{C_i} \rangle = |C_i|/|G| > 0$
- Orthogonality: $\langle \delta_{C_i}, \delta_{C_j} \rangle = 0$ for $i \neq j$

Since $\mathcal{B}$ is an orthogonal set of non-zero vectors with $|\mathcal{B}| = k = |\mathrm{Cl}(G)|$, it forms a basis, proving $\dim Z(\mathbb{C}[G]) = |\mathrm{Cl}(G)|$. ☐

The following theorem represents a key result in the theory of group representations. It establishes that irreducible characters constitute an orthonormal set of class functions. This property plays a central role in proving the uniqueness of how a representation can be broken down into irreducible components, and it also provides a method to precisely determine the number of equivalence classes of irreducible representations.

Theorem 3.3.1 (First orthogonality relations)**.** *[8] Let ϕ and ρ be irreducible representations of a finite group G. Then their characters satisfy:*

$$\langle \chi_\phi, \chi_\rho \rangle = \begin{cases} 1 & \text{if } \phi \sim \rho \text{ (isomorphic representations)} \\ 0 & \text{otherwise} \end{cases}$$

*Consequently, the irreducible characters form an **orthonormal system** in the space of class functions on G.*

Proof. By applying Theorem 2.4.1 and Proposition 3.3.1, we can, without loss of generality, consider that $\phi\colon G \to U_n(\mathbb{C})$ and $\rho\colon G \to U_m(\mathbb{C})$ are unitary representations. Let us now evaluate the inner product:

$$\begin{aligned}
\langle \chi_\phi, \chi_\rho \rangle &= \frac{1}{|G|}\sum_{g\in G} \chi_\phi(g)\,\overline{\chi_\rho(g)} \\
&= \frac{1}{|G|}\sum_{g\in G} \left(\sum_{i=1}^{n} \phi_{ii}(g)\right)\left(\sum_{j=1}^{m} \overline{\rho_{jj}(g)}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \left(\frac{1}{|G|}\sum_{g\in G} \phi_{ii}(g)\,\overline{\rho_{jj}(g)}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \phi_{ii}, \rho_{jj} \rangle.
\end{aligned}$$

According to the orthogonality relations provided by Schur's lemma (Theorem3.2.1), we find that $\langle \phi_{ii}, \rho_{jj} \rangle = 0$ whenever $\phi \ncong \rho$, hence $\langle \chi_\phi, \chi_\rho \rangle = 0$ for distinct (non-equivalent) representations.

On the other hand, if $\phi \cong \rho$, then by Proposition3.3.1, we can assume $\phi = \rho$. In such a case, Schur's orthogonality tells us:

$$\langle \phi_{ii}, \phi_{jj} \rangle = \begin{cases} \frac{1}{n} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Consequently, we obtain:

$$\langle \chi_\phi, \chi_\phi \rangle = \sum_{i=1}^{n} \langle \phi_{ii}, \phi_{ii} \rangle = \sum_{i=1}^{n} \frac{1}{n} = 1,$$

which completes the argument. □

Corollary 3.3.1. *The number of equivalence classes of irreducible representations of G is bounded above by the number of conjugacy classes of G. Formally:*

$$|\mathrm{Irr}(G)| \leq |\mathrm{Cl}(G)|,$$

where:

- $\mathrm{Irr}(G)$ *denotes the set of equivalence classes of irreducible representations.*

Proof. First, Theorem 3.3.1 tells us that inequivalent irreducible representations have distinct characters. Moreover, these irreducible characters form an orthonormal set with respect to the standard inner product on the space of class functions on G.

It is also known that the dimension of the center of the group algebra $Z(\mathbb{C}[G])$ is equal to the number of conjugacy classes of G, that is,

$$\dim Z(\mathbb{C}[G]) = |\mathrm{Cl}(G)|.$$

Since orthonormal sets are linearly independent, and each irreducible character is a class function, the number of irreducible characters cannot exceed the dimension of the space of class functions. Therefore, the number of inequivalent irreducible representations of G is less than or equal to the number of conjugacy classes in G. □

Proposition 3.3.4. *[6] $Z(\mathbb{C}[G])$ is the center of the group algebra $\mathbb{C}[G]$.*

Proof. Recall that the multiplication in $\mathbb{C}[G]$ is the convolution given by

$$(f_1 * f_2)(x) = \sum_{y \in G} f_1(xy^{-1}) f_2(y).$$

Let $f \in Z(\mathbb{C}[G])$ and $t \in \mathbb{C}[G]$.
Then

$$(t * f)(x) = \sum_{y \in G} t(y^{-1}x) f(y) = \sum_{y \in G} t(y^{-1}x) f(xyx^{-1}).$$

Setting $z = xy^{-1}$, we see that

$$(t * f)(x) = \sum_{z \in G} t(z) f(xz^{-1}) = \sum_{z \in G} f(xz^{-1}) t(z) = (f * t)(x).$$

□

Let V be a vector space and φ a representation. For any integer $m > 0$, we define:

$$mV = \underbrace{V \oplus \cdots \oplus V}_{m \text{ copies}}, \quad m\varphi = \underbrace{\varphi \oplus \cdots \oplus \varphi}_{m \text{ copies}}.$$

Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of inequivalent irreducible unitary representations of the group G.

Definition 3.3.3. *Let $\rho \sim m_1\varphi^1 \oplus m_2\varphi^2 \oplus \cdots \oplus m_k\varphi^k$ where $m_i \geq 0$, and φ^i are irreducible representations of G.*
If $m_i > 0$, then we say that φ^i is an irreducible constituent of ρ with multiplicity m_i.

Note: *For each $g \in G$, the representation matrix ρ_g is similar to:*

$$\rho_g \sim \begin{bmatrix} \underbrace{\begin{matrix}\varphi^1_g & & \\ & \ddots & \\ & & \varphi^1_g\end{matrix}}_{m_1 \ \textit{times}} & 0 & \cdots & 0 \\ 0 & \underbrace{\begin{matrix}\varphi^2_g & & \\ & \ddots & \\ & & \varphi^2_g\end{matrix}}_{m_2 \ \textit{times}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underbrace{\begin{matrix}\varphi^k_g & & \\ & \ddots & \\ & & \varphi^k_g\end{matrix}}_{m_k \ \textit{times}} \end{bmatrix}$$

Remark 3.3.1. *Suppose that the representation ρ decomposes into irreducible representations as:*

$$\rho \simeq m_1\varphi^{(1)} \oplus m_2\varphi^{(2)} \oplus \cdots \oplus m_s\varphi^{(k)},$$

where:

- *$\varphi^{(i)}$ are pairwise non-isomorphic irreducible representations,*
- *m_i is the multiplicity of $\varphi^{(i)}$ in ρ,*
- *$d_i = \deg(\varphi^{(i)})$ is the degree (dimension) of $\varphi^{(i)}$.*

Then, the degree of ρ is given by:

$$\deg(\rho) = m_1d_1 + m_2d_2 + \cdots + m_kd_k.$$

This follows from the additivity of degrees under direct sums of representations.

Proposition 3.3.5. *[2] Let (ϕ, V) and (ψ, W) be finite-dimensional representations of a group G with characters χ_ϕ and χ_ψ respectively. Then for any $g \in G$:*

1. $\chi_{\phi\oplus\psi}(g) = \chi_\phi(g) + \chi_\psi(g)$
2. $\chi_{\phi\otimes\psi}(g) = \chi_\phi(g) \cdot \chi_\psi(g)$
3. $\chi_{\phi^*}(g) = \overline{\chi_\phi(g)}$
4. $\chi_{\wedge^2\phi}(g) = \frac{1}{2}\left[\chi_\phi(g)^2 - \chi_\phi(g^2)\right]$

Proof. Let $g \in G$ be an element of finite order n. Let:

- (ϕ, V) be a representation of G, with eigenvalues $\{\lambda_i\}_{i=1}^{\dim V}$ for $\phi(g)$.
- (ψ, W) be another representation, with eigenvalues $\{\mu_j\}_{j=1}^{\dim W}$ for $\psi(g)$.

We use the fact that the character χ of a representation is the trace of the matrix representing g, and the trace is equal to the sum of its eigenvalues.

1. Direct Sum: $\chi_{\phi\oplus\psi}(g)$

$$\chi_{\phi\oplus\psi}(g) = \mathrm{Tr}(\phi(g)) + \mathrm{Tr}(\psi(g)) = \chi_\phi(g) + \chi_\psi(g)$$

2. Tensor Product: $\chi_{\phi\otimes\psi}(g)$ The eigenvalues of $\phi(g) \otimes \psi(g)$ are $\lambda_i\mu_j$, so:

$$\chi_{\phi\otimes\psi}(g) = \sum_{i,j} \lambda_i\mu_j = \left(\sum_i \lambda_i\right)\left(\sum_j \mu_j\right) = \chi_\phi(g) \cdot \chi_\psi(g)$$

3. Dual Representation: $\chi_{\phi^*}(g)$ The eigenvalues of $\phi^*(g)$ are $\overline{\lambda_i}$, so:

$$\chi_{\phi^*}(g) = \sum_i \overline{\lambda_i} = \overline{\chi_\phi(g)}$$

4. Second Exterior Power: $\chi_{\wedge^2\phi}(g)$ The eigenvalues of $\wedge^2\phi(g)$ are $\lambda_i\lambda_j$ for $i < j$, hence:

$$\chi_{\wedge^2\phi}(g) = \sum_{i<j} \lambda_i\lambda_j = \frac{1}{2}\left(\left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2\right)$$
$$= \frac{1}{2}\left(\chi_\phi(g)^2 - \chi_\phi(g^2)\right)$$

□

Theorem 3.3.2. *[8]*

Let $\{\varphi^{(1)}, \ldots, \varphi^{(s)}\}$ be a complete set of representatives of the equivalence classes of irreducible representations of a finite group G.

Suppose a representation ρ decomposes as:

$$\rho \cong m_1\varphi^{(1)} \oplus m_2\varphi^{(2)} \oplus \cdots \oplus m_s\varphi^{(s)}$$

where m_i denotes the multiplicity of $\varphi^{(i)}$ in ρ. Then:

1. *The multiplicities are given by the character inner product:*

$$m_i = \langle \chi_\rho, \chi_{\varphi^{(i)}} \rangle$$

2. *The decomposition into irreducible constituents is unique up to equivalence*

3. *ρ is completely determined (up to equivalence) by its character χ_ρ*

Proof. Since the representation ρ is a direct sum of irreducible representations, its character χ_ρ can be expressed as a linear combination of the characters of the irreducible components:

$$\chi_\rho = m_1\chi_{\varphi(1)} + \cdots + m_s\chi_{\varphi(s)}$$

To determine the coefficient m_i, we compute the inner product of χ_ρ with $\chi_{\varphi(i)}$:

$$\langle \chi_\rho, \chi_{\varphi(i)} \rangle = m_1\langle \chi_{\varphi(1)}, \chi_{\varphi(i)} \rangle + \cdots + m_s\langle \chi_{\varphi(s)}, \chi_{\varphi(i)} \rangle$$

By the orthogonality relations of irreducible characters, all inner products are zero unless the indices match, in which case the inner product is 1. Therefore:

$$\langle \chi_\rho, \chi_{\varphi(i)} \rangle = m_i$$

This establishes the formula for the multiplicities. The uniqueness of the decomposition and the fact that the representation is completely determined up to equivalence by its character follow directly from this result and previous propositions. □

Theorem 3.3.2 provides a practical and rigorous criterion for determining whether a given representation is irreducible

Corollary 3.3.2. *Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G with character χ_ρ. Then ρ is* ***irreducible*** *if and only if its character satisfies the orthonormality condition:*

$$\langle \chi_\rho, \chi_\rho \rangle = 1$$

Proof. ($\Rightarrow$) If ρ is irreducible, the first orthogonality relation for characters gives $\langle \chi_\rho, \chi_\rho \rangle = 1$.

($\Leftarrow$) If $\langle \chi_\rho, \chi_\rho \rangle = 1$, suppose ρ were reducible. Then $\rho \cong \bigoplus_{i=1}^{k} \rho_i$ with $k \geq 2$, and by linearity of characters:

$$\langle \chi_\rho, \chi_\rho \rangle = \sum_{i,j} \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle \geq \sum_{i=1}^{k} \langle \chi_{\rho_i}, \chi_{\rho_i} \rangle \geq k \geq 2$$

contradicting our assumption. □

Example 3.3.2. *Consider the representation ρ of the symmetric group S_3 from Example 2.2.1, as introduced earlier. The elements Id, (12), and (123) represent the three conjugacy classes of S_3. To determine whether ρ is irreducible, we compute the inner product $\langle \chi_\rho, \chi_\rho \rangle$, using the known character values:*

$$\chi_\rho(Id) = 2, \quad \chi_\rho((12)) = 0, \quad \chi_\rho((123)) = -1$$

The group S_3 contains:

- *1 identity element,*
- *3 transpositions,*
- *2 3-cycles.*

Using the orthogonality formula for characters:

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{6}\left(1 \cdot |2|^2 + 3 \cdot |0|^2 + 2 \cdot |-1|^2\right) = \frac{1}{6}(4 + 0 + 2) = 1$$

Since the inner product is equal to 1, the representation ρ is irreducible.

In representation theory, we study how groups act on vector spaces. A fundamental concept in this context is the *fixed subspace* V^G, which consists of all vectors that remain unchanged under the action of every group element. This subspace is closely tied to the *trivial character* of G, as $\dim V^G$ counts the multiplicity of the trivial representation in ϕ.

Definition 3.3.4 (Fixed subspace)**.** *[8] For a group representation $\phi \colon G \to \mathrm{GL}(V)$, the* fixed subspace V^G *is defined as the set of all vectors in V that are invariant under the group action:*

$$V^G = \{v \in V \mid \phi_g(v) = v \text{ for every } g \in G\}.$$

Remark 3.3.2. *1. One easily verifies that V^G is a G-invariant subspace.*

2. The subrepresentation $\phi|_{V^G}$ is equivalent to $\dim V^G$ *copies of the trivial representation, that is,*

$$\phi|_{V^G} \cong \mathbf{1}^{\oplus \dim V^G}.$$

Proposition 3.3.6. *Let $\phi \colon G \to \mathrm{GL}(V)$ be a finite-dimensional linear representation of a finite group G, and let χ_1 denote the trivial character of G. Then the inner product of characters satisfies:*

$$\langle \chi_\phi, \chi_1 \rangle = \dim V^G$$

Proof. We can decompose V as a direct sum:

$$V = m_1 V_1 \oplus \cdots \oplus m_s V_s$$

where each V_i is a G-invariant, irreducible subspace, and the corresponding subrepresentations belong to distinct equivalence classes (allowing some $m_i = 0$). Without loss of generality, assume that V_1 corresponds to the trivial representation.

Let $\phi^{(i)}$ denote the restriction of ϕ to V_i. For any $v \in V$, we can write:

$$v = v_1 + \cdots + v_s$$

with $v_i \in m_i V_i$. Applying $\phi(g)$ gives:

$$\phi(g)v = (m_1\phi^{(1)})(g)v_1 + \cdots + (m_s\phi^{(s)})(g)v_s$$

Since $\phi^{(1)}$ is trivial, this simplifies to:

$$\phi(g)v = v_1 + (m_2\phi^{(2)})(g)v_2 + \cdots + (m_s\phi^{(s)})(g)v_s$$

Thus, v is fixed under the action of G if and only if each component satisfies $v_i \in m_i V_i^G$ for all $2 \leq i \leq s$.

Therefore,

$$V^G = m_1 V_1 \oplus m_2 V_2^G \oplus \cdots \oplus m_s V_s^G$$

Now, for each $i \geq 2$, V_i is irreducible and not equivalent to the trivial representation. Since V_i^G is a G-invariant subspace of V_i, and V_i is irreducible and non-trivial, it follows that:

$$V_i^G = 0$$

Thus:

$$V^G = m_1 V_1$$

This shows that the dimension of V^G equals the multiplicity of the trivial representation in ϕ, as required. □

3.4 The Regular Representation

For a finite set X, consider the regular representation. An inner product is given by:

$$\left\langle \sum_{x\in X} a_x x, \sum_{x\in X} b_x x \right\rangle = \sum_{x\in X} a_x \overline{b_x}.$$

Let $X = G$. It is also important to observe that $\mathbb{C}G \cong \mathbb{C}[G]$.

Definition 3.4.1 (Regular representation)**.** *Let G be a finite group. The* ***regular representation*** *of G on $\mathbb{C}G$ is the group homomorphism:*

$$L\colon G \to \mathrm{GL}(\mathbb{C}G),$$

where $\mathbb{C}G$ denotes the group algebra of G over $\mathbb{C}$.

For each $g \in G$, the linear operator $L_g\colon \mathbb{C}G \to \mathbb{C}G$ acts on a generic element $\sum_{h\in G} c_h h \in \mathbb{C}G$ by left multiplication:

$$L_g\left(\sum_{h\in G} c_h h\right) = \sum_{h\in G} c_h (gh).$$

Equivalently, by the change of variables $x = gh$ (so $h = g^{-1}x$), this can be expressed as:

$$L_g\left(\sum_{h\in G} c_h h\right) = \sum_{x\in G} c_{g^{-1}x} x.$$

Remark 3.4.1. *The notation L in L_g stands for* left *multiplication. On a basis element $h \in G$, the action is:*

$$L_g h = gh,$$

i.e., L_g permutes basis elements via left multiplication by g.

Proposition 3.4.1. *The regular representation $L\colon G \to \mathrm{GL}(\mathbb{C}G)$ of a finite group G is* ***unitary****. That is, for every $g \in G$, the operator L_g satisfies:*

$$\langle L_g v, L_g w\rangle = \langle v, w\rangle \quad \forall v, w \in \mathbb{C}G,$$

where the inner product on $\mathbb{C}G$ is defined by:

$$\left\langle \sum_{h\in G} a_h h, \sum_{h\in G} b_h h \right\rangle = \sum_{h\in G} a_h \overline{b_h}.$$

Proof. We demonstrate the unitarity of the regular representation L:

1. Linearity and Homomorphism: For any $g_1, g_2 \in G$ and basis element $h \in G$,

   $$L_{g_1}(L_{g_2}h) = L_{g_1}(g_2 h) = g_1 g_2 h = L_{g_1 g_2} h,$$

   establishing the homomorphism property.
2. Inner Product Preservation: For $v = \sum_{h\in G} c_h h$ and $w = \sum_{h\in G} k_h h$,

   $$\langle L_g v, L_g w\rangle = \sum_{x\in G} c_{g^{-1}x}\overline{k_{g^{-1}x}}.$$

3. Invariance Under Substitution: Letting $y = g^{-1}x$ transforms the sum to

   $$\sum_{y\in G} c_y \overline{k_y} = \langle v, w\rangle.$$

Thus, each L_g is unitary, making L a unitary representation. ☐

Proposition 3.4.2 (Character of the Regular Representation)**.** *[8] Let G be a finite group with regular representation $L : G \to \mathrm{GL}(\mathbb{C}G)$. The character χ_L of L is given by:*

$$\chi_L(g) = \begin{cases} |G| & \text{if } g = 1_G \text{ (the identity element),} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $G = \{g_1, \ldots, g_n\}$ be a finite group of order $n = |G|$, where we fix an ordering of the group elements.

The regular representation $L : G \to \mathrm{GL}(\mathbb{C}G)$ acts by left multiplication:

$$L_g(g_j) = gg_j \quad \text{for all } g_j \in G.$$

Matrix Construction:

1. For each $g \in G$, the matrix $[L_g]$ of the linear operator L_g with respect to the basis G has entries:

   $$[L_g]_{ij} = \begin{cases} 1 & \text{if } g_i = gg_j \quad (\text{or equivalently } g = g_i g_j^{-1}), \\ 0 & \text{otherwise.} \end{cases}$$

   This is because $L_g(g_j) = gg_j$ must be expressed in terms of the basis G.

2. The diagonal entries $[L_g]_{ii}$ are particularly important for computing the trace:

$$[L_g]_{ii} = \begin{cases} 1 & \text{if } gg_i = g_i \quad (\text{i.e., } g = 1_G), \\ 0 & \text{otherwise.} \end{cases}$$

   This follows from setting $i = j$ in the general matrix entry formula.

Character Computation: The character $\chi_L(g)$ is the trace of $[L_g]$:

$$\chi_L(g) = \mathrm{Tr}([L_g]) = \sum_{i=1}^{n} [L_g]_{ii}.$$

- If $g = 1_G$, then $[L_{1_G}]_{ii} = 1$ for all i (since $1_G g_i = g_i$), so:

$$\chi_L(1_G) = \sum_{i=1}^{n} 1 = n = |G|.$$

- If $g \neq 1_G$, then $[L_g]_{ii} = 0$ for all i (since $gg_i \neq g_i$), so:

$$\chi_L(g) = \sum_{i=1}^{n} 0 = 0.$$

This completes the proof of the character formula for the regular representation. □

We now decompose the regular representation L into irreducible components. Let $\{\phi^{(1)}, \ldots, \phi^{(s)}\}$ be a complete set of inequivalent irreducible unitary representations of our finite group G, with $d_i = \deg \phi^{(i)}$. For simplicity, we denote $\chi_i = \chi_{\phi^{(i)}}$ for each $i = 1, \ldots, s$.

Theorem 3.4.1. *[8] Let L be the regular representation of a finite group G. Then L decomposes into irreducible representations as:*

$$L \cong d_1\phi^{(1)} \oplus d_2\phi^{(2)} \oplus \cdots \oplus d_s\phi^{(s)}$$

Proof. We compute the inner product between the regular character χ_L and the irreducible character χ_i:

1. Recall the character values:

$$\chi_L(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$$

$$\chi_i(1) = \deg \phi^{(i)} = d_i$$

2. Compute the inner product:

$$\begin{aligned} \langle \chi_L, \chi_i \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_L(g)\overline{\chi_i(g)} \\ &= \frac{1}{|G|} \left[\chi_L(1)\chi_i(1) + \sum_{\substack{g \in G \\ g \neq 1}} \chi_L(g)\overline{\chi_i(g)} \right] \\ &= \frac{1}{|G|} \left[|G| \cdot d_i + \sum_{g \neq 1} 0 \cdot \overline{\chi_i(g)} \right] \\ &= \frac{1}{|G|} (|G| d_i + 0) \\ &= d_i \end{aligned}$$

3. Apply Theorem 3.3.2
   The multiplicity of $\phi^{(i)}$ in L equals $\langle \chi_L, \chi_i \rangle = d_i$. Therefore:

$$L \cong \bigoplus_{i=1}^{s} d_i \phi^{(i)}$$

This completes the proof of the decomposition of the regular representation. □

Corollary 3.4.1. *Let G be a finite group with a complete set of irreducible representations $\{\phi^{(1)}, \ldots, \phi^{(s)}\}$, where each $\phi^{(i)}$ has degree $d_i = \dim \phi^{(i)}$. Then:*

$$|G| = \sum_{i=1}^{s} d_i^2 = d_1^2 + d_2^2 + \cdots + d_s^2$$

Proof. By Theorem 3.4.1, the character of the regular representation decomposes as:

$$\chi_L = \sum_{i=1}^{s} d_i \chi_i \tag{3.1}$$

Evaluating at the identity element $1 \in G$:

$$\begin{aligned} |G| &= \chi_L(1) \\ &= \sum_{i=1}^{s} d_i \chi_i(1) \\ &= \sum_{i=1}^{s} d_i^2 \quad (\text{since } \chi_i(1) = \dim \phi^{(i)} = d_i) \end{aligned}$$

Thus we obtain the fundamental degree sum formula:

$$|G| = \sum_{i=1}^{s} d_i^2 \tag{3.2}$$

□

Theorem 3.4.2. *[8] The set $\{\chi_1, \chi_2, \ldots, \chi_k\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

Proof. In this argument, we continue using the established notation. According to the first orthogonality relations (as stated in Theorem 3.3.1), the irreducible characters constitute an orthonormal collection within the space of class functions. What remains is to demonstrate that these characters also span the center $Z(\mathbb{C}[G])$ of the group algebra.

Consider a function $f \in Z(\mathbb{C}[G])$, the center of the group algebra. As a class function, f satisfies:

$$f(x) = \frac{1}{|G|} \sum_{g \in G} f(g^{-1}xg)$$

From previous results, we can write f as:

$$f = \sum_{i,j,k} c_{ij}^{(k)} \varphi_{ij}^{(k)}$$

where $\varphi_{ij}^{(k)}$ denotes the (i,j)-th matrix element of the irreducible representation $\varphi^{(k)}$, and $c_{ij}^{(k)} \in \mathbb{C}$.

Inserting this into the expression for $f(x)$:

$$f(x) = \sum_{i,j,k} c_{ij}^{(k)} \left(\frac{1}{|G|} \sum_{g \in G} \varphi_{ij}^{(k)}(g^{-1}xg) \right)$$

Since conjugating inside the matrix coefficient corresponds to:

$$\varphi_{ij}^{(k)}(g^{-1}xg) = \left(\varphi^{(k)}(g^{-1})\varphi^{(k)}(x)\varphi^{(k)}(g)\right)_{ij}$$

we can write:

$$f(x) = \sum_{i,j,k} c_{ij}^{(k)} \left[\frac{1}{|G|}\sum_{g\in G}\left(\varphi^{(k)}(g^{-1})\varphi^{(k)}(x)\varphi^{(k)}(g)\right)_{ij}\right]$$

Applying the result from character theory and Schur's lemma, this becomes:

$$f(x) = \sum_{i,j,k} c_{ij}^{(k)} \left[\left(\widetilde{\varphi}^{(k)}(x)\right)\right]_{ij}$$

Which leads to:

$$= \sum_{i,k} c_{ii}^{(k)} \cdot \frac{\mathrm{Tr}(\varphi^{(k)}(x))}{\deg \varphi^{(k)}} \cdot I_{ij}$$

Hence, only diagonal components remain:

$$= \sum_{i,k} c_{ii}^{(k)} \cdot \frac{1}{d_k}\chi_k(x)$$

This confirms that f lies in the span of $\chi_1, \ldots, \chi_s$, completing the proof.

□

Theorem 3.4.3. *The number of equivalence classes of irreducible representations of a finite group is the number of conjugacy classes in the group.*

Proof. By the preceding theorem, the characters of inequivalent irreducible representations of the group G form a basis for $Z(\mathbb{C}[G])$.

But, we showed that

$$\dim Z(\mathbb{C}[G]) = \text{Number of conjugacy classes in } G.$$

Hence, the result follows. □

Corollary 3.4.2. *A finite group G is abelian if and only if it has $|G|$ many equivalence classes of irreducible representations.*

Proof. ($\Rightarrow$) First, assume G is abelian. Then for any $g, h \in G$, conjugation acts trivially: $hgh^{-1} = g$. This means every element $g \in G$ forms its own conjugacy class $\mathrm{Cl}(g) = \{g\}$. Consequently, there are exactly $|G|$ distinct conjugacy classes, proving $|G| = |\mathrm{Cl}(G)|$.

($\Leftarrow$) Conversely, suppose $|G| = |\mathrm{Cl}(G)|$. This equality forces each conjugacy class to contain exactly one element, since the maximum possible number of conjugacy classes is $|G|$ (achieved when all classes are singletons). Therefore, for all $g, h \in G$ we have $hgh^{-1} = g$, which immediately implies commutativity: $hg = gh$. Thus G is abelian. □

3.4.1 Character table

In the representation theory of finite groups, the character table serves as a fundamental data structure that systematically organizes all irreducible character information.

Definition 3.4.2. *Let $\chi_1, \chi_2, \cdots, \chi_k$ The irreducible characters of G and let $C_1, C_2, \ldots, C_k$ be the conjugacy classes of G. Thus, we get a matrix whose (i, j)-th entry is $\chi_i(C_j)$. It's a $k \times k$ matrix, called the character table of G.*

Example 3.4.1. *Let $G = S_3$. Then G has 3 irreducible characters corresponding to the irreducible representations given by*

$$\varphi^1 : G \to \mathbb{C}^\times, \quad \varphi^1_\sigma = 1,$$

$$\varphi^2 : G \to \mathbb{C}^\times, \quad \varphi^2_\sigma = sign(\sigma) = \begin{cases} 1 & \textit{if } \sigma \textit{ is even,} \\ -1 & \textit{if } \sigma \textit{ is odd,} \end{cases}$$

$$\varphi^3 : G \to GL_2(\mathbb{C}) \quad \textit{given by} \quad \varphi^3_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \varphi^3_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then the characters are given by

$$\chi_1 = \varphi^1, \quad \chi_2 = \varphi^2,$$

$$\chi_3(1) = 2, \quad \chi_3(12) = 0, \quad \chi_3(123) = -1.$$

The character table is given by:

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 3.1: Character table of S_3

Example 3.4.2. *[3] Let $G = A_5$. Then G has five irreducible representations over $\mathbb{C}$, and the corresponding character table is given in Table 3.2.*

To construct the irreducible characters, we first consider the permutation representation to obtain χ_2, and then compute the symmetric and exterior square characters of ρ_2 to find χ_{sym} and χ_{alt}. Using orthogonality relations, we deduce:

$$\chi_3 = \chi_{sym} - \chi_1 - \chi_2, \quad \chi_4 + \chi_5 = \chi_{alt}.$$

Applying dimension formulas and inner product computations, we determine:

$$\chi_4((12345)) = \frac{1+\sqrt{5}}{2}, \quad \chi_4((12354)) = \frac{1-\sqrt{5}}{2}, \quad \chi_5((12345)) = \frac{1-\sqrt{5}}{2}, \quad \chi_5((12354)) = \frac{1+\sqrt{5}}{2}.$$

	1	(123)	(12)(34)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Table 3.2: Character table of A_5

The functions δ_C (which indicate conjugacy classes C in the group G) form a basis for the center of the class function algebra $Z(\mathbb{C}[G])$, just like the irreducible characters. Expressing δ_C in terms of irreducible characters leads to the *orthogonality of columns* in the character table, a fundamental result in representation theory.

Theorem 3.4.4 (Second orthogonality relations)**.** *[8] Let C and C' be two conjugacy classes in a finite group G, and let $g \in C$ and $h \in C'$. Then the irreducible characters $\chi_1, \chi_2, \ldots, \chi_s$ of G satisfy:*

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} \dfrac{|G|}{|C|} & \textit{if } C = C', \\ 0 & \textit{if } C \neq C'. \end{cases}$$

This implies that the columns of the character table are pairwise orthogonal, and hence, the character table is invertible.

Proof. Recall that if C is a conjugacy class,then

$$\delta_C(g) = \begin{cases} 1 & \text{if } g \in C, \\ 0 & \text{otherwise.} \end{cases}$$

We use the fact that the indicator function δ_C of a conjugacy class C can be expressed as a linear combination of the irreducible characters. Specifically, we write:

$$\begin{aligned}
\delta_{C'} &= \sum_{i=1}^{s} \langle \delta_{C'}, \chi_i \rangle \chi_i \\
\delta_{C'}(g) &= \sum_{i=1}^{s} \langle \delta_{C'}, \chi_i \rangle \chi_i(g) \\
\langle \delta_{C'}, \chi_i \rangle &= \frac{1}{|G|} \sum_{x \in G} \delta_{C'}(x)\overline{\chi_i(x)} = \frac{1}{|G|} \sum_{x \in C'} \overline{\chi_i(x)} \\
\langle \delta_{C'}, \chi_i \rangle &= \frac{|C'|}{|G|}\overline{\chi_i(h)} \quad \text{for any} \quad h \in C' \\
\delta_{C'}(g) &= \frac{|C'|}{|G|} \sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} \\
\frac{|C'|}{|G|} \sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} &= \begin{cases} 1 & \text{if} \quad g \in C' \\ 0 & \text{otherwise} \end{cases} \\
\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} &= \begin{cases} \frac{|G|}{|C'|} & \text{if} \quad C = C' \\ 0 & \text{if} \quad C \neq C' \end{cases}
\end{aligned}$$

which proves the result. □

Remark 3.4.2.

- *In representation theory, non-isomorphic groups may possess identical character tables. A classical example is the dihedral group D_8 (symmetries of a square) and quaternion group Q_8, both of order 8, which share the same character table despite being fundamentally different groups. This shows that character tables alone cannot fully distinguish group isomorphism classes.*
- *The character table can be viewed as the transpose of the matrix that changes the basis from $\{\chi_1, \ldots, \chi_s\}$ to $\{\delta_C \mid C \in \mathrm{Cl}(G)\}$, which forms a basis for the center $Z(\mathbb{C}[G])$ of the group algebra.*

3.5 Representations of Abelian Groups

In this section, we compute the characters of a finite abelian group. Since every finite abelian group is a direct product of cyclic groups, it suffices to know how to compute the characters of the direct product of cyclic groups. Let us proceed with this task

Proposition 3.5.1. *[7] Let G_1 and G_2 be two abelian groups. Suppose that $\chi_1, \ldots, \chi_m$ are the irreducible characters of G_1, and $\tau_1, \ldots, \tau_n$ are the irreducible characters of G_2. Since both groups are abelian, we have $m = |G_1|$ and $n = |G_2|$.*

Define functions

$$\alpha_{ij} : G_1 \times G_2 \longrightarrow \mathbb{C}^\times \quad by \quad \alpha_{ij}(g_1, g_2) = \chi_i(g_1)\tau_j(g_2)$$

for $1 \le i \le m$ and $1 \le j \le n$.

Then the set $\{\alpha_{ij}\}$ forms a complete list of irreducible characters of the group $G_1 \times G_2$.

Proof.

$$\begin{aligned}
\alpha_{ij}((g_1, g_2)(g_1', g_2')) &= \alpha_{ij}(g_1g_1', g_2g_2') \\
&= \chi_i(g_1g_1')\tau_j(g_2g_2') \\
&= \chi_i(g_1)\chi_i(g_1')\tau_j(g_2)\tau_j(g_2') \quad \text{(since } \chi_i, \tau_j \text{ are homomorphisms)} \\
&= \alpha_{ij}(g_1, g_2)\alpha_{ij}(g_1', g_2')
\end{aligned}$$

□

Remark 3.5.1.

- *If $\alpha_{ij} = \alpha_{k\ell}$, then evaluating at $(g_1, 1)$ and $(1, g_2)$ gives:*

$$\chi_i(g_1) = \alpha_{ij}(g_1, 1) = \alpha_{k\ell}(g_1, 1) = \chi_k(g_1) \Rightarrow i = k$$
$$\tau_j(g_2) = \alpha_{ij}(1, g_2) = \alpha_{k\ell}(1, g_2) = \tau_\ell(g_2) \Rightarrow j = \ell$$

- *Number of α_{ij}: $m \times n$*
- *For finite abelian groups: $|G_1 \times G_2| = mn$*
- *Thus $\{\alpha_{ij}\}$ exhaust all irreducible characters*

Example 3.5.1. *[8] Consider the abelian group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $|G| = 4$ and G is abelian, it has exactly four irreducible characters, all of degree one. The character table of G is given in Table 3.5.1.*

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
α_{11}	1	1	1	1
α_{12}	1	-1	1	-1
α_{21}	1	1	-1	-1
α_{22}	1	-1	-1	1

Table 3.1: Character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Chapter 4

Representation Theory of the Symmetric Group

This chapter develops the essential triad of symmetric group theory: (1) permutations in S_n, (2) integer partitions $\lambda \vdash n$, and (3) their Young diagram realizations. These combinatorial objects, following [9, 8, 2], parametrize the irreducible representations via Specht's construction.

4.1 Some Basic Definitions

Definition 4.1.1. *[9] The **symmetric group** on n letters, denoted by S_n, is the group consisting of all permutations (bijective functions) of the set $\{1, 2, \ldots, n\}$.*

Formally, it is defined as:

$$S_n = \big\{\sigma\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\} \mid \sigma \textit{ is a bijection}\big\}$$

Definition 4.1.2 (Partitions and p-Regularity)**.** *[9] Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a weakly decreasing sequence of non-negative integers (i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$). We say that:*

1. *λ is a **partition of** n (denoted $\lambda \vdash n$) if $\sum_{i=1}^{\infty} \lambda_i = n$.*
2. *For convenience, we truncate trailing zeros and write $\lambda = (\lambda_1, \ldots, \lambda_m)$ when $\lambda_k = 0$ for all $k > m$.*
3. *For a prime number p, the partition λ is:*
    - *p-**regular** if no non-zero part λ_i repeats p or more times (i.e., $\#\{j \mid \lambda_j = k\} < p$ for all $k \geq 1$).*
    - *p-**singular** otherwise (if any non-zero part repeats p or more times).*
4. *The set of all partitions of n is denoted by $\mathscr{P}(n)$.*

Remark 4.1.1. *If the sequence is not arranged in a weakly decreasing order, it is called an improper partition, denoted by $\lambda \models n$.*

Definition 4.1.3. *[9] To each permutation $\sigma \in S_n$, there is naturally associated a partition of n, called the **cycle type** of σ. Specifically,*

$$\operatorname{type}(\sigma) = (\lambda_1, \ldots, \lambda_m),$$

where the λ_i are the lengths of the cycles of σ, arranged in weakly decreasing order (including multiplicities).

It is important to count cycles of length one when determining the cycle type, even though such cycles are typically omitted in the standard cycle notation.

standard theorem in elementary group theory establishes that for permutations $\sigma, \tau \in S_n$, the conjugacy relation $\sigma \sim \tau$ holds precisely when their cycle types coincide[8] .

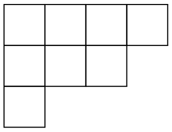
Definition 4.1.4 (Young diagram)**.** *[9]*

*Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, the **diagram** $[\lambda]$ consists of all ordered pairs (i, j) of integers, called **nodes**, such that $1 \leq i \leq m$ and $1 \leq j \leq \lambda_i$.*

The diagram $[\lambda]$ is typically displayed on a two-dimensional grid, where each node (i, j) is represented by a square located in row i and column j.

For example, if $\lambda = (4, 3, 1)$, then

$$[\lambda] = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array} \qquad \textit{or simply:}$$



The conjugate partition λ' is the partition whose diagram $[\lambda']$ consists of the set of nodes $\{(i, j) \mid (j, i) \in [\lambda]\}$. In other words, $[\lambda']$ is obtained by interchanging the rows and columns of $[\lambda]$.

In the study of integer partitions within the set $\mathscr{P}(n)$, two fundamental ordering systems exist: the *lexicographical order* ($\geq$) and the *dominance order* ($\trianglerighteq$). The lexicographical order compares partitions element-wise from left to right, where $\lambda \geq \mu$ if either $\lambda = \mu$ or there exists a first position l where $\lambda_l > \mu_l$ with all preceding elements equal. The dominance order compares cumulative sums, with $\lambda \trianglerighteq \mu$ if $\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i$ for all r. The lexicographical order forms a total order that extends the dominance order, meaning $\lambda \trianglerighteq \mu$ implies $\lambda \geq \mu$, but not conversely. For instance, while $(3, 1) \geq (2, 2)$ lexicographically, these partitions are incomparable under dominance. These ordering systems play crucial roles in representation theory for analyzing symmetric groups, in combinatorics for studying symmetric functions, and in group theory for classifying irreducible representations, providing powerful tools for understanding the structural relationships between various integer partitions [9].

Definition 4.1.5 (Young Tableaux)**.** *[8] Let $\lambda \vdash n$ be a partition of n. A **Young tableau** (or λ-tableau) of shape λ is an array obtained by filling the boxes of the Young diagram $[\lambda]$ with the integers $\{1, \ldots, n\}$, each appearing exactly once.*

Formally:

1. *The Young diagram $[\lambda]$ consists of left-justified rows with λ_i boxes in row i*
2. *A λ-tableau is a bijective assignment $T : [\lambda] \to \{1, \ldots, n\}$*
3. *The number of distinct λ-tableaux is $n!$ (n factorial)*

Example 4.1.1. *Suppose that $\lambda = (3, 2, 1)$. Then some λ-tableaux are as follows:*

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline 6 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|} \hline 3 & 2 & 6 \\ \hline 1 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Proposition 4.1.1 (Rearrangement Lemma for Young Tableaux)**.** *[8] Let $\lambda \vdash n$ and $\mu \vdash n$ with t_λ a λ-tableau and s_μ a μ-tableau where same-row entries of s_μ occupy distinct columns in t_λ. Then $\exists$ a λ-tableau u_λ where: (1) $\forall j$, column j of u_λ equals column j of t_λ setwise, and (2) $\forall i$, entries from rows 1 to i of s_μ lie in rows 1 to i of u_λ. For $\lambda = (3, 2)$, $\mu = (2, 2, 1)$, if $t_\lambda = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$ and $s_\mu = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$, then $u_\lambda = t_\lambda$ suffices.*

Lemma 4.1.1. *Let λ and μ be partitions of n, and let t_λ and s_μ be tableaux of respective shapes λ and μ. If the integers appearing in the same row of s_μ occupy distinct columns in t_λ, then $\lambda \trianglerighteq \mu$.*

Definition 4.1.6 (Tabloid)**.** *[8] Let $\lambda \vdash n$ be an integer partition. A λ-*tabloid *is an equivalence class of λ-tableaux under the row equivalence relation $\sim$, where two tableaux t and s satisfy $t \sim s$ if they contain identical sets of entries in each corresponding row. We denote the tabloid containing tableau t by $[t]$.The set of all tabloids of shape λ is denoted T^λ.*

4.2 Constructing the Irreducible Representations

Having established the foundational concepts of symmetric group representations, we now turn to the explicit construction of their irreducible representations using combinatorial tools such as Young tableaux and Specht Representations.

Definition 4.2.1 (Column stabilizer)**.** *Let t be a **Young tableau** with n boxes. The **column stabilizer** of t, denoted C_t, is the subgroup of the symmetric group S_n defined by:*

$$C_t := \left\{ \sigma \in S_n \mid \text{for all } i \in \{1, \ldots, n\}, \sigma(i) \text{ lies in the same column as } i \text{ in } t \right\}.$$

The symmetric group S_n acts naturally on the set of all λ-tableaux via permutation of entries.

Proposition 4.2.1. *[8] This action is transitive - for any two λ-tableaux t^λ and s^λ, there exists $\sigma \in S_n$ such that $\sigma t^\lambda = s^\lambda$.*

Let T^λ denote the set of all λ-tabloids (equivalence classes of λ-tableaux under row equivalence).

Definition 4.2.2 (Group Action on Tabloids)**.** *The symmetric group S_n acts on T^λ by*

$$\sigma \cdot [t^\lambda] := [\sigma t^\lambda], \quad \text{for } \sigma \in S_n, [t^\lambda] \in T^\lambda.$$

This action is well-defined.

Proof. Suppose $t^\lambda \sim s^\lambda$. We must show $\sigma t^\lambda \sim \sigma s^\lambda$.

If i, j lie in the same row of σt^λ and σs^λ, then $\sigma^{-1}(i)$ and $\sigma^{-1}(j)$ lie in the same row of both t^λ and s^λ by row equivalence.

Thus the action preserves row equivalence classes and is well-defined on tabloids. ☐

Remark 4.2.1. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$ be a partition of the positive integer n.*

The symmetric group S_n acts transitively on the set of λ-tableaux.

We define the standard λ-tabloid T_λ *as the equivalence class of the tableau with:*

- *Numbers $1, 2, \ldots, \lambda_1$ in the first row.*
- *Numbers $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in the second row.*
- $\vdots$
- *Numbers $\lambda_1 + \cdots + \lambda_{\ell-1} + 1, \ldots, n$ in the ℓ-th row.*

we have:

$$|T^\lambda| = \frac{n!}{\lambda_1!\lambda_2! \cdots \lambda_\ell!}$$

Let $\lambda \vdash n$ be a partition of n, and let T^λ denote the set of all λ-tabloids.

Let $M^\lambda := \mathbb{C}T^\lambda$ be the permutation representation - the $\mathbb{C}$-vector space with basis T^λ. This induces the group homomorphism:

$$\phi^\lambda : S_n \to \mathrm{GL}(M^\lambda),$$

where $\phi^\lambda(\sigma)$ acts as a linear transformation by permuting the basis elements according to the group action.

The representation ϕ^λ is *reducible*, but contains an *irreducible subrepresentation*. We now proceed to explain how to isolate it.

Definition 4.2.3. *[8] Let $\lambda, \mu \vdash n$. Let t^λ be a λ-tableau. Define a linear map $A_{t^\lambda} : M^\mu \to M^\lambda$ by*

$$A_{t^\lambda} = \sum_{\pi \in C_{t^\lambda}} sign(\pi) \phi^\mu_\pi$$

*If $\lambda = \mu$, the element $e_{t^\lambda} = A_{t^\lambda}[t^\lambda] = \sum_{\pi \in C_{t^\lambda}} sign(\pi)\pi[t^\lambda]$ of M^λ is called the **polytabloid** associated to t^λ.*

Let P^λ be the set of all polytabloids corresponding to the partition λ of n.

$$P^\lambda = \{e_{t^\lambda} \mid t^\lambda \text{ is a } \lambda\text{-tableau}\}$$

Proposition 4.2.1. *Let $\sigma \in S_n$ and let t_λ be a λ-tableau. Then,*

$$\phi^\lambda_\sigma(e_{t_\lambda}) = e_{\sigma t_\lambda},$$

where $e_{t_\lambda} = A_{t_\lambda}[t_\lambda] = \sum_{\pi \in C_{t_\lambda}} sign(\pi)\pi[t_\lambda]$ is the ***polytabloid*** *associated with t_λ.*

Proof. We begin by observing that the column-stabilizing subgroup of the tableau σt_λ satisfies the conjugation relation:

$$C_{\sigma t_\lambda} = \sigma C_{t_\lambda} \sigma^{-1}.$$

This follows because any permutation preserving the columns of σt_λ must be conjugate under σ to a permutation preserving the columns of t_λ.

Next, we analyze the action of ϕ^λ_σ on the antisymmetrizer map A_{t_λ}:

$$\phi^\lambda_\sigma A_{t_\lambda} = \sum_{\pi \in C_{t_\lambda}} \text{sign}(\pi)\phi^\lambda_\sigma \phi^\lambda_\pi = \sum_{\pi \in C_{t_\lambda}} \text{sign}(\pi)\phi^\lambda_{\sigma\pi},$$

where we have used the linearity and homomorphism property of ϕ^λ.

By substituting $\tau = \sigma\pi\sigma^{-1}$, we rewrite the sum in terms of $C_{\sigma t_\lambda}$:

$$\phi^\lambda_\sigma A_{t_\lambda} = \sum_{\tau \in C_{\sigma t_\lambda}} \text{sign}(\sigma^{-1}\tau\sigma)\phi^\lambda_\tau \sigma.$$

Since the sign of a permutation is invariant under conjugation (i.e., $\text{sign}(\sigma^{-1}\tau\sigma) = \text{sign}(\tau)$), and because $\phi^\lambda_\tau \sigma = \phi^\lambda_\tau \phi^\lambda_\sigma$, this simplifies to:

$$\phi^\lambda_\sigma A_{t_\lambda} = \left(\sum_{\tau \in C_{\sigma t_\lambda}} \text{sign}(\tau)\phi^\lambda_\tau \right) \phi^\lambda_\sigma = A_{\sigma t_\lambda} \phi^\lambda_\sigma.$$

Finally, applying both sides to the tableau t_λ, we obtain:

$$\phi^\lambda_\sigma(e_{t_\lambda}) = \phi^\lambda_\sigma A_{t_\lambda}[t_\lambda] = A_{\sigma t_\lambda}\phi^\lambda_\sigma[t_\lambda] = A_{\sigma t_\lambda}[\sigma t_\lambda] = e_{\sigma t_\lambda},$$

which completes the proof. □

We can now define our desired subrepresentation.

Definition 4.2.4 (Specht Representation)**.** *[8] For a partition $\lambda \vdash n$, let S^λ be the subspace of M^λ spanned by the collection P^λ of all polytabloids e_{t_λ} associated to λ-tableaux t_λ. By Proposition 4.2.1, S^λ forms an S_n-invariant subspace of M^λ under the representation ϕ^λ. The* Specht representation *corresponding to λ is then defined as the restriction $\psi^\lambda : S_n \to \mathrm{GL}(S^\lambda)$ of ϕ^λ to this invariant subspace S^λ.*

Remark 4.2.2. *The P^λ are not in general linearly independent. See the next example.*

Example 4.2.1. *Let $\lambda = (1^n) \vdash n$ be the partition consisting of all ones. In this special case:*

- *λ-tableaux coincide with λ-tabloids*
- *The representation ϕ^λ becomes the regular representation of S_n*
- *The Young diagram for λ consists of a single column*
- *The column stabilizer of any λ-tableau t^λ is $C_{t^\lambda} = S_n$*

The polytabloid is given by: $e_{t^\lambda} = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\pi[t^\lambda]$. *For any* $\sigma \in S_n$, *we compute*

$$\begin{aligned}\phi^\lambda_\sigma(e_{t^\lambda}) &= \phi^\lambda_\sigma\left(\sum_{\pi \in S_n} \operatorname{sign}(\pi)\pi[t^\lambda]\right)\\ &= \sum_{\pi \in S_n} \operatorname{sign}(\pi)\phi^\lambda_\sigma[\pi t^\lambda]\end{aligned}$$

Since $\phi^\lambda_\sigma(e_{t^\lambda}) = e_{\sigma t^\lambda}$, *we conclude* $e_{\sigma t^\lambda} = \operatorname{sign}(\sigma)e_{t^\lambda}$. *By transitivity of the* S_n *action on tableaux, for any tableau* s^λ *there exists* $\sigma \in S_n$ *with* $e_{s^\lambda} = \operatorname{sign}(\sigma)e_{t^\lambda}$, *showing the Specht module* $S^\lambda = \mathbb{C}e_{t^\lambda}$ *is one-dimensional. Thus, the Specht representation* $\psi^\lambda : S_n \to \mathrm{GL}(S^\lambda) \cong \mathbb{C}^*$ *is precisely the sign representation, with* $\psi^\lambda(\sigma) = \operatorname{sign}(\sigma)$.

The proof that the representations ψ^λ are the irreducible representations of the symmetric group S_n proceeds through a series of lemmas, but we omit the full proof here. For the complete proof, we refer to standard references on representation theory [8]

Lemma 4.2.1. *Let* $\lambda, \mu \vdash n$ *with* t_λ *a* λ*-tableau and* s_μ *a* μ*-tableau. If* $A_{t_\lambda}[s_\mu] \neq 0$, *then:*

1. $\lambda \trianglerighteq \mu$ *(dominance order)*
2. *When* $\lambda = \mu$, $A_{t_\lambda}[s_\lambda] = \pm e_{t_\lambda}$

Lemma 4.2.2. *Let* $\lambda \vdash n$ *and* t^λ *a* λ*-tableau. Then the image of the operator*

$$A_{t^\lambda} : M^\lambda \longrightarrow M^\lambda \text{ is } \mathbb{C}e_{t^\lambda}.$$

To establish that the ψ^λ are precisely the irreducible representations of S_n, the following theorem is fundamental:

Theorem 4.2.1. *[8] Let* $\lambda \vdash n$ *and* V *be an* S_n*-invariant subspace of* M^λ. *Then either* $S^\lambda \subseteq V$ *or* $V \subseteq (S^\lambda)^\perp$.

Corollary 4.2.1. *The Specht Representation* S^λ *is irreducible as a representation of* S_n.

Proof. Let $V \subsetneq S^\lambda$ be a proper S_n-invariant subspace. By Theorem 4.2.1 on the orthogonality of Specht Representation, we have the containment:

$$V \subseteq (S^\lambda)^\perp \cap S^\lambda = \{0\}$$

This forces $V = 0$, proving that S^λ admits no nontrivial proper subrepresentations. The irreducibility follows. ☐

4.2.1 Frobenius Formula and Character Computations

After presenting the theoretical foundations of Specht representations using Young tableaux, we now practically apply them to S_3 through group algebra elements a, b, and c to demonstrate these concepts.

Definition 4.2.5. *[2] For a given Young diagram with n boxes, we define the following two subgroups of the symmetric group* S_n*:*

1. *The* ***row-preserving subgroup****:*

$$P_\lambda = \{\sigma \in S_n \mid \sigma \textit{ preserves the numbers in each row}\}$$

2. *The* ***column-preserving subgroup****:*

$$Q_\lambda = \{\sigma \in S_n \mid \sigma \textit{ preserves the numbers in each column}\}$$

Let λ be a Young diagram of size n. We define three key elements in the group algebra $\mathbb{C}S_n$:

$$\begin{aligned} a_\lambda &= \sum_{\sigma \in P} e_\sigma \\ b_\lambda &= \sum_{\tau \in Q} \operatorname{sgn}(\tau) e_\tau \\ c_\lambda &= a \cdot b \end{aligned}$$

Remarkably, the subspace $\mathbb{C}S_n \cdot c$ constitutes an irreducible representation of the symmetric group S_n, with distinct partitions yielding non-isomorphic irreducible representations. This exemplifies the profound yet rare correspondence between conjugacy classes and irreducible representations in finite groups, which we shall explicitly illustrate through concrete examples.

Example 4.2.2. *[2] To see this in action, let's find all irreducible representations of S_3. There are three partitions of 3, and therefore three corresponding Young diagrams:*

$$\begin{array}{ccc} \lambda = (3) & \mu = (2,1) & \nu = (1,1,1) \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \end{array}$$

In the first diagram, 1, 2 and 3 are all in one row, so any permutation preserves the row. The only permutation preserving columns is the identity. Therefore:

$$\begin{aligned} P_\lambda &= S_3 \\ Q_\lambda &= \{e\} \\ a_\lambda &= e + (12) + (23) + (13) + (123) + (132) \\ b_\lambda &= e \\ c_\lambda &= a_\lambda \cdot e = a_\lambda \end{aligned}$$

The irreducible representation is $\mathbb{C}S_3 \cdot c_\lambda = \mathbb{C}c_\lambda$, which is one-dimensional since multiplying by any group element just rearranges the summands in c_λ.

This subspace corresponds to the trivial representation, as $g \cdot (rc_\lambda) = rc_\lambda$ for all $g \in S_3$ and scalars r, meaning every group element acts as the identity.

For the second diagram $\mu = (2,1)$:

$$\begin{aligned} &P_\mu = \langle (12) \rangle, \quad Q_\mu = \langle (13) \rangle \\ &a_\mu = e + (12) \\ &b_\mu = e - (13) \\ &c_\mu = a_\mu b_\mu = (e + (12))(e - (13)) = e - (13) + (12) - (132) \end{aligned}$$

The irreducible representation is $\mathbb{C}S_3 \cdot c_\mu$. Multiplying c_μ by S_3 basis elements shows the subspace is spanned by:

$$\begin{aligned} v_{1,\mu} &= e - (13) + (12) - (132) \\ v_{2,\mu} &= (13) - e + (123) - (23) \end{aligned}$$

This 2-dimensional subspace corresponds to the standard representation of S_3.

For the third diagram $\nu = (1^3)$:

$$\begin{aligned} P_\nu &= \{e\} \\ Q_\nu &= S_3 \\ a_\nu &= e \\ b_\nu &= e - (12) - (23) - (13) + (123) + (132) \\ c_\nu &= a_\nu b_\nu = e - (12) - (23) - (13) + (123) + (132) \end{aligned}$$

The irreducible representation is $\mathbb{C}S_3 \cdot c_\nu = \mathbb{C}c_\nu$*, which is one-dimensional since:*

$$g \cdot c_\nu = \begin{cases} c_\nu & \textit{if } g \textit{ is even} \\ -c_\nu & \textit{if } g \textit{ is odd} \end{cases}$$

This corresponds to the alternating representation of S_3*, where:*

$$\rho(g) = \begin{cases} +1 & \textit{if } g \textit{ is even permutation} \\ -1 & \textit{if } g \textit{ is odd permutation} \end{cases}$$

We not only have a systematic method for constructing all irreducible representations of S_n, but also an explicit character formula:

For a partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$ of n, let C_i be any conjugacy class of S_n. For each index j from 1 to n, when writing elements of C_i as products of disjoint cycles, we define ℓ_{ij} as the number of cycles of length j.

Consider k independent variables $x_1, x_2, \ldots, x_k$ and define the j-th power sum:

$$P_j(\mathbf{x}) = \sum_{m=1}^{k} x_m^j$$

The discriminant is given by the Vandermonde determinant:

$$\Delta(\mathbf{x}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1} \end{vmatrix} = \prod_{1 \le i < j \le k} (x_i - x_j)$$

Define $\lambda_s = \nu_s + k - s$ for $1 \le s \le k$. For any polynomial $f(\mathbf{x})$ in $x_1, \ldots, x_k$, let $[f(\mathbf{x})]_{(\nu_1, \ldots, \nu_k)}$ denote the coefficient of the monomial:

$$\mathbf{x}^\lambda = \prod_{m=1}^{k} x_m^{\lambda_m}$$

The character value $\chi^\nu(C_i)$ is then given by:

Theorem 4.2.2 (The Frobenius Formula)**.** *[2] The character of the irreducible representation of* S_n *associated with* ν *is given by:*

$$\chi^\nu(C_i) = \left[\Delta(\mathbf{x}) \prod_{j=1}^{n} P_j(\mathbf{x})^{\ell_{ij}} \right]_{(\lambda_1, \lambda_2, \ldots, \lambda_k)}$$

Example 4.2.3. *[2] Let's compute* $\chi^{(2,1)}$ *for the conjugacy class of transpositions in* S_3*.*

Given:

Partition $\nu = (2,1)$ *(so* $k = 2$*), Variables* x_1, x_2*, Cycle type* $\mu = (1^1 2^1)$*:* $\ell_1 = 1$*,* $\ell_2 = 1$*,* $\ell_3 = 0$*, Shifted weights:* $\lambda_1 = 3$*,* $\lambda_2 = 1$

The character formula gives:

$$\begin{aligned} \chi^{(2,1)}(C) &= \left[(x_2 - x_1)(x_1 + x_2)(x_1^2 + x_2^2) \right]_{(3,1)} \\ &= \left[(x_2^2 - x_1^2)(x_1^2 + x_2^2) \right]_{(3,1)} \\ &= \left[x_2^4 - x_1^4 \right]_{(3,1)} \end{aligned}$$

The coefficient of $x_1^3 x_2^1$ *is* 0*, which agrees with the known character value for the standard representation* V *on transpositions:*

$$\chi_V((12)) = 0$$

This value was previously computed using an alternative method when constructing the character table of S_3 *3.1.*

Conclusion

In conclusion, this thesis has provided a comprehensive exploration of the representation theory of finite groups, systematically uncovering its profound mathematical beauty and wide-ranging applications. Through our rigorous investigation, we have demonstrated how abstract algebraic structures find concrete realization via linear representations, creating a powerful synergy between group theory and linear algebra.

The study has successfully achieved its primary objectives by establishing the fundamental principles of group representations and their complete reducibility via Maschke's theorem, developing character theory as both a classification tool and computational framework, and constructing explicit irreducible representations for symmetric groups using combinatorial methods. Our work has effectively bridged theoretical concepts with practical computational techniques.

The examination of character tables and the Frobenius formula has particularly highlighted the remarkable efficiency with which representation theory encodes complex algebraic information. Moreover, the combinatorial aspects emerging from Young tableaux and Specht modules have revealed deep connections between algebra and discrete mathematics.

The interdisciplinary nature of representation theory, as demonstrated throughout this thesis, confirms its central position in modern mathematics. The foundations laid here provide a robust understanding of both theoretical principles and their concrete applications, showcasing the enduring relevance and vitality of representation theory in contemporary mathematical research.

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