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By:

Atallah Djabaria

Ben Amara Chahira

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Dr.Nis Khadija	**MCB**	**President**	**Univ. d'El-Oued**
Dr. Said Beloul	**Prof**	**Supervisor**	**Univ. d'El-Oued**
Dr.Dahda Bashir	**MCB**	**Examiner**	**Univ. d'El-Oued**

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Dedication

Praise be to God who guided us to this, and we would not have been guided had it not been for God who guided us. And prayers and peace be upon my beloved and the apple of my eyes, **Muhammad bin Abdullah**, the best of mankind.
I bring you peace. If it was raised to the sky, it would be a bright moon... And if it came down to the earth, it would cover it with silk and silk... And if it mixed with salty sea water, it would make it sweet in a way.
To those who gave me their life and showered me with their love and pledged to educate me and in my noble name,**my mother and father** raised me
To the flowers that bloom every morning, **my brothers and sisters**. To the one who taught me, "There is no life with despair, and there is no despair with life." My honorable teacher,**Said Beloul**.
To the husbands of my brothers and sisters and to my uncles and aunts and uncles and dear aunts and their children
To everyone who accompanied me on my academic journey
To all the teachers and students of the Mathematics Department, especially the second year master's students

Ben amara Chahira

Dedication

Praise be to God, who enlightened my path and gave me the best help. To the most precious thing I have in this world. To the one who was the reason for my existence on this earth. To the one who placed heaven under her feet To the one to whom I bowed with all respect and appreciation to the one I hope I may have attained her satisfaction, **my mother** May God prolong her life.

To the one to whom I owe my life,to the one who supported me and was a candle burning to light my way. To whom I have feelings of appreciation, respect and gratitude, **my father**, may God prolong his life.

To all my family members, especially **my sisters**.

To the source of tenderness Overflowing to the life partner and the light of life, **my dear husband**.

And to all **my friends** without exception.

To all the professors who gave us a helping hand, To all of them, I dedicate this humble work, and I ask God Almighty to help us to do what is good for us and for our country. He is the best Lord and the best spporter.

Atallah Djabaria

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Contents

Notations

The most frequently used notations, symbols and abbreviations are listed below:

$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{R}$	:	The set of real numbers.
$\mathbb{R}_+$	:	The set of positive real numbers.
$C([a,b],\mathbb{R})$	:	The space continuous functions from $[a,b]$ to $\mathbb{R}$.
$d(x,A)$	:	The distance between a point x and a set A.
$e(A,B)$	:	Designate the excess of a set A to a set B.
$H(A,B)$	:	The Hausdorff distance between A and B.
$\mathcal{P}(X)$	:	The family of non-empty subsets of X.
$CL(X)$	:	The set of all closed subsets of X.
$CB(X)$	:	The set of all closed and bounded subsets of X.
$K(X)$	:	The set of all compact subsets of X.
$T : X \longrightarrow \mathcal{P}(Y)$	:	T is a multivalued map from X to $\mathcal{P}(Y)$.
$Dom(T)$	:	The domain of T.
$Graph(T)$	:	The graph of T.
$Im(T)$	:	The image of T.
$\Gamma(t)$	:	The Gamma function.
I^α	:	The fractional integral.

Introduction

The fractional equation differential in their general form, linear and nonlinear, are considered as a generalization of the theory of differential equations regular to optional.
Differential equations appeared more frequently in various disciplines such as engineering, physics and chemistry, ... etc., where it has played an active and distinguished role in recent years, and this is attributed to the great development of the theory of fractional arithmetic in addition to its multiple applications in biology, economics, signal and image processing, biophysics, blood flow phenomena, electrodynamics of complex media, aerodynamics, meteorology and other various sciences, fractional derivatives also provide us it is an excellent tool for describing the memory and hereditary properties of many substances, processes, and more that they constitute important mathematical tools in the interpretation of some natural phenomena.
The concept of derivation appeared more than a 300 year ago in an inquiry directed by Opital to Leibniz in 1695 , which includes what is the meaning of the derivative $d^n y/dx^n$ if $n = \frac{1}{2}$? who in turn answered him that such can be expressed as an infinite string which result that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$. subsequently, Leibniz referred to derivatives of a more general order in his correspondence with Bernoulli.
In 1697 Leibniz stated that differential calculus could be used to find the infinite result of in correspondence with John Wallace.
Since that time, fractional arithmetic has attracted the attention of many mathematicians In 1823, Abel was the first to apply fractional arithmetic to solve the integral equation that appears in formulate the problem of standardization of time.
After this, the subject of fractional arithmetic remained dormant for about a decade, until

the work of Liouville, galvanized by each among the works of Abel and Fourier's integral formula, he published three memoirs in 1832 and many publications and succeeded in applying his definitions to questions of latency theory.
This led to the deduction of the Caputo derivative, which is related to Riemann-Liouville.
Memory included three chapters: **chapter1**: We touched on some concepts of metric space, the Pompeiu -Hausdorff metric*[2]* and multivalued maps *[3]* with some examples.
chapter2 and chapter 3: we used the fixed point theorem of Nadler and Leray-Schauder to demonstrate the existence of a solution for a boundary value problem.

Chapter 1

Preliminaries

In this chapter, we will discuss some information related to our topic, which is represented in some definitions, basic concepts, and theories used the proposed study.

1.1 The Pompeiu -Hausdorff metric

The objective of this section is to present the definition of the Pompeiu-Hausdorff. This notion will allow us to define the notion of a multivalued contraction.

Firstly, let us first introduce the distance from a point to a set.

Definition 1.1.1. [2] *Let (X, d) be a metric space, $x \in X$ and A a subset of X. The distance from point x to A noted $d(x, A)$ is defined by:*

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

By convention $d(x, \emptyset) = \infty$. If on the contrary, A is not empty, then for all $\varepsilon > 0$, it exists an element $y \in A$ such that $d(x, y) \leq d(x, A) + \varepsilon$.

Theorem 1.1.1. *Let $x \in X, A \subset K(X)$, so it exists $x_0 \in A$ such that $d(x, A) = d(x, x_0)$.*

Proof. A function $x \mapsto d(x, y)$ is continuous, with real values. It reaches its lower limit on any compact. □

Lemma 1.1.2. *Let (X, d) a metric space. For each $B \in CL(X)$and $x \in X$, we have*

$$d(x, B) = 0 \Leftrightarrow x \in B.$$

Proof. By the definition of $d(a,B)$, for each $n_0 \in \mathbb{N}$, it exists a sequence $\{b_n\} \in B$, such that $d(a,b_{n_0}) < \frac{1}{n_0}$. Now, we have a sequence $\{b_n\}$ in B such that $d(a,b_n) < \frac{1}{n}$, for all $n \in \mathbb{N}$. Next, we show that $\lim\limits_{n\longrightarrow\infty} d(a,b_n) = 0$. Let $\varepsilon > 0$ by property Archimede it exist $N \in \mathbb{N}$, such that $\frac{1}{N} < \varepsilon$, where for all $n \in \mathbb{N}$ we find $\frac{1}{n} \leq \frac{1}{N}$. So $d(a,b_n) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$, then $\lim\limits_{n\longrightarrow\infty} d(a,b_n) = 0$, so $\lim\limits_{n\longrightarrow\infty} b_n = a$. As B is closed we have $a \in B$. $\square$

Since we are working in the field of multivalued analysis, we need to define a distance between two sets. A notion of this type was proposed by Pompeiu [21].

In this thesis, we will also use the Hausdorff metric [11] constructed from this function.

Definition 1.1.2. [21] *Let (X,d) be a metric space and let $e : CB(X) \times CB(X) \to \mathbb{R}_+$ be a function defined by:*

$$e(A,B) = \sup\{d(a,B) : a \in A\}.$$

For $A = \emptyset$ and $B \neq \emptyset$, we have: $e(\emptyset,B) = 0$.
For $A = B = \emptyset$, we have: $e(\emptyset,\emptyset) = +\infty$
and $e(A,\emptyset) = +\infty$ for each $A \subset X$ with $A \neq \emptyset$.

Example 1.1.1. *Let $X = \mathbb{R}$, $A = [0,1]$, $B = [2,5]$*
$e(A,B) = \sup\{d(a,B), \quad a \in A\} = 2$
$e(B,A) = \sup\{d(b,A), \quad b \in B\} = 4$

Now, we present some properties:

Proposition 1.1.1. [2] *Let (X,d) a metric space and $A,B,C \in CB(X)$. Then we have:*

(a) $e(A,B) = 0 \Leftrightarrow A \subset B$.

(b) $B \subset C \Rightarrow e(A,C) \leq e(A,B)$.

(c) $e(A \cup B, C) = \max\{e(A,C), e(B,C)\}$.

(d) $e(A,B) \leq e(A,C) + e(C,B)$.

Proof. (a) By the definition of the function e, we have:

$$\begin{aligned} e(A,B) &= 0 \Leftrightarrow \sup_{x\in A} d(x,B) = 0 \\ &\Leftrightarrow d(x,B) = 0 \text{ for each } \ x \in A. \end{aligned}$$

Since B is closed in X,

$$d(x, B) = 0 \Leftrightarrow x \in B.$$

So,

$$e(A, B) = 0 \Leftrightarrow A \subset B.$$

(b) We have

$$B \subset C \Rightarrow d(x, C) \leq d(x, B) \text{ for each } x \in X.$$

$$B \subset C \Rightarrow \sup_{x \in A} d(x, C) \leq \sup_{x \in A} d(x, B).$$

So

$$B \subset C \Rightarrow e(A, C) \leq e(A, B).$$

(c) We have

$$e(A \cup B, C) = \sup_{x \in A \cup B} d(x, C) = \max\{\sup_{x \in A} d(x, C), \sup_{x \in B} d(x, C)\}.$$

So

$$e(A \cup B, C) = \max\{e(A, C), e(B, C)\}.$$

(d) Let $a \in A, b \in B$ and $c \in C$. So

$$d(a, B) \leq d(a, b) \leq d(a, c) + d(c, b),$$

$$d(a, B) \leq d(a, c) + \inf_{b \in B} d(c, b).$$

Then

$$d(a, B) \leq d(a, c) + d(c, B) \leq d(a, c) + e(C, B).$$

This implies

$$d(a, B) \leq d(a, c) + e(C, B),$$

We have

$$d(a, B) \leq \inf_{c \in C} d(a, c) + e(C, B).$$

Then

$$d(a, B) \leq d(a, C) + e(C, B).$$

We have

$$\sup_{x \in A} d(a, B) \leq \sup_{x \in A} d(a, C) + e(C, B).$$

So

$$e(A,B) \leq e(A,C) + e(C,B).$$

□

Definition 1.1.3. [11]

Let (X,d) be a metric space, and let $CB(X)$ be a non empty, closed and bounded subsets of X, the Pompeiu -Hausdorff metric is defined as:

$$H(A,B) = \max\{e(A,B), e(B,A)\} = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

for all $A, B \in CB(X)$.

Remark 1.1.1. *if $A = \{a\}$ (singleton) and $B = \{b\}$, then $H(A,B) = d(a,b)$.*

Example 1.1.2. *Let $X = \mathbb{R}$, $A = [3,5]$,$B = [7,10]$*
So, $\sup_{a\in A} d(a,B) = 4$ and $\sup_{b\in B} d(b,A) = 5$
Then, $H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} = 5$.

Proposition 1.1.2.

1. *For $A, B \in CB(X)$, $H(A,B) = 0 \Longleftrightarrow A = B$.*
2. *$H(A,B) = H(B,A)$, for each $A, B \in CB(X)$.*
3. *$H(A,B) \leq H(A,C) + H(C,B)$, for each $A, B, C \in CB(X)$.*

Proposition 1.1.3. *If (X,d) is a metric space, then H is a metric on $CB(X)$.*

Proof. (1) By the definition of H, we have $H(A,B) \geqslant 0$, and

$$\begin{aligned} H(A,B) &= 0 \Leftrightarrow \max\{e(A,B), e(B,A)\} = 0 \\ &\Leftrightarrow e(A,B) = 0 \text{ and } e(B,A) = 0 \\ &\Leftrightarrow A \subset B \text{ and } B \subset A \\ &\Leftrightarrow A = B. \end{aligned}$$

(2) From the properties of max, we find that

$$\begin{aligned} H(A,B) &= \max\{e(A,B), e(B,A)\} \\ &= \max\{e(B,A), e(A,B)\} \\ &= H(B,A). \end{aligned}$$

(3) Using the proposition (1.1.1), we get

$$\begin{aligned} H(A,B) &= \max\{e(A,B), e(B,A)\} \\ &\leqslant \max\{e(A,C)+e(C,B), e(B,C)+e(C,A)\} \\ &\leq \max\{e(A,C),(C,A)\}+\max\{e(B,C),e(C,B)\} \\ &= H(A,C)+H(C,B). \end{aligned}$$

□

Remark 1.1.2. *It is easy to ensure that the completeness of (X,d) implies the completeness of $(CB(X),H)$ and $(K(X),H)$.*

Proposition 1.1.4. *If (X,d) is a complete metric space, then $K(X)$ is a closed set in metric space $(CB(X),H)$.*

Lemma 1.1.3. [19] *Let $A,B \in CB(X)$ and $a \in A$. Then for $\varepsilon > 0$. There exists $b \in B$ such that:*

$$d(a,b) \leq H(A,B)+\varepsilon$$

Lemma 1.1.4. [19] *Let (X,d) be a metric space and $A,B \in CB(X)$ with $H(A,B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that*

$$d(a,b) < hH(A,B). \tag{1.1}$$

Proof. Let $H(A,B)=0$ then $a \in B$ and the above inequality holds for $b=a$.
Now, suppose that $H(A,B) > 0$ and we choose

$$\varepsilon = (h-1)H(A,B) > 0. \tag{1.2}$$

Then, using the definition of $d(a,B)$ and $H(A,B)$ it follows that, for all $\varepsilon > 0$, it exists $b \in B$ such as

$$d(a,b) \leq d(a,B)+\varepsilon \leq H(A,B)+\varepsilon. \tag{1.3}$$

Finally, replacing (1.2) in (1.3), we get the inequality (1.1).

□

Definition 1.1.4. [12] *Let (X,d) be a metric space, and the sequence $\{A_n\} \in CB(X)$. The sequence $\{A_n\}$ is said*

(1) Convergent if there exists $A \in CB(X)$ such that:

$$H(A_n, A) \longrightarrow 0, when \quad n \longrightarrow \infty,$$

(2) Cauchy if for all $m, n > 0$,

$$H(A_n, A_m) \longrightarrow 0, when \quad n, m \longrightarrow \infty,$$

1.2 Multivalued maps

As we mentioned earlier, in this study we will focus on the concept of multivalued maps. These maps are not necessarily points, but rather groups.

In this part we give somes definitions of the multivalued maps and their properties examples.

Definition 1.2.1. [3] *Let X, Y be two non-empty sets. a multivalued mapping T defines of X in Y is a map which associates to each $x \in X$ with a non-empty subset Tx of Y, which called image or the value of T in X. We denote it by $T : X \longrightarrow \mathcal{P}(Y)$, or $T : X \longrightarrow 2^Y$ where $\mathcal{P}(Y) = 2^Y$ denotes the set of non-empty subsets of Y.*

Remark 1.2.1. *The single-valued maps is also a multivalued maps since for any $x \in X$ singleton $T(x) = y$ is a subset of Y .*

Example 1.2.1. *We present some examples:*

1. *Let $T : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ defines by*

$$Tx = \begin{cases} \{-1\} & \text{if } x < 0, \\ [-1, 1] & \text{if } x = 0, \\ \{1\} & \text{if } x > 0. \end{cases}$$

Then T is multivalued mapping.

2. *The mapping* $T : [0,1] \longrightarrow \mathcal{P}([0,1])$ *defined by*

$$Tx = [0, x^2], \text{ for each } x \in [0,1],$$

is a multivalued mapping.

3. *Let* $T : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$,*and* $n \in \mathbb{Z}$ *defines by*

$$Tx = \begin{cases} [n-1, n], & \text{if } x = n, \\ n; & \text{if } x \in]n, n+1[. \end{cases}$$

So T *is a multivalued mapping.*

4. *Let* $T : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathbb{R})$,*defines by*

$$Tx = [e^{-x}, e^x], \text{ for each } x \in \mathbb{R}_+.$$

T *is a multivalued mapping.*

5. *The mapping* $T : \mathbb{R} \to K(\mathbb{R})$ *defined by* $Tx = [a, b]$, *with* $a, b \in \mathbb{R}$ *such that* $a < b$, *is a constant multivalued mapping.*

In order to work well within the framework of the multivalued analysis, let us now state some definitions and characteristics specific to multivalued mappings. Notions of this part come from [3].

Definition 1.2.2. *Let* X, Y *tow sets,* $T : X \longrightarrow \mathcal{P}(Y)$ *is a multivalued mapping*

1. *We call domain of* T *the set*

$$Dom(T) = \{x \in X : Tx \neq \emptyset\}$$

When Dom(T) $\neq \emptyset$ *(resp. Dom(T)* $= X$*), the multivalued mapping* T *is said to be proper (resp. strict).*

2. *We call image of T the set*

$$Im(T) = \{y \in \mathcal{P}(Y), \exists x \in X, y \in Tx\}.$$

3. *The graph of T is the subset of $X \times Y$ defined by*

$$Graph(T) = \{(x, y) \in X \times Y, y \in Tx\}.$$

Definition 1.2.3. *The inverse of T is a multivalued mapping $T^{-1} : Y \to \mathcal{P}(X)$ such that*

$$x \in T^{-1}y \Leftrightarrow y \in Tx \Leftrightarrow (x, y) \in \textit{Graph(T)}.$$

Thus $\mathrm{Dom}(T^{-1}) = \mathrm{Im(T)}$ and $\mathrm{Im}(T^{-1}) = \mathrm{Dom(T)}$.

Definition 1.2.4. *Let $T : X \to \mathcal{P}(Y)$ is a multivalued mapping, A a subset of X. The image of A by T is defined by:*

$$T(A) := \bigcup_{x \in A} Tx.$$

Definition 1.2.5. *Let $T : X \to \mathcal{P}(Y)$ be a multivalued mapping, and B a subset of Y. The reciprocal image of B by T denoted by: $T^{-1}(B)$ is defined by:*

$$T^{-1}(B) := \{x \in X : Tx \cap B \neq \emptyset\}.$$

Example 1.2.2. *Let $T : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ such that $Tx = [x^2, x^2 + 1]$, for all $x \in \mathbb{R}$ we have:*

$$T([0, 1]) = \bigcup_{x \in [0,1]} Tx = [0, 2], \quad T(\mathbb{R}_+) = \mathbb{R}_+, \quad T(\mathbb{R}_-) = \mathbb{R}_+.$$

Example 1.2.3. *Consider the multivalued mapping $T : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by:*

$$Tx = \begin{cases} \emptyset, & \text{if } x \geq 0, \\ \{-\sqrt{-x}, \sqrt{-x}\}, & \text{if } x < 0. \end{cases}$$

It is clear that T is proper and not strict because $\textit{Dom(T)} = \mathbb{R}_-$.
Here

$$Im\ T = \bigcup_{x \in \mathbb{R}_-} \{-\sqrt{-x}, \sqrt{-x}\} = \mathbb{R},$$

and

$$\textit{Graph(T)} = \{(x, \sqrt{-x}) : x \leqslant 0\} \cup \{(x, -\sqrt{-x}) : x \leqslant 0\}.$$

Moreover, the inverse $T^{-1} : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is given by: $T^{-1}x = \{-x^2\}$.

Definition 1.2.6. *Let A be a subset of X, we denote by $T_{|A}$ the restriction of T on A, defined by:*

$$T_{|A}x = \begin{cases} Tx, & \text{if } x \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have $Dom(T)_{|A} = Dom(T) \cap A$.

1.3 Semi-continuous maps

Definition 1.3.1. [6] *Let X, Y be a metric spaces. Let $T : X \to \mathcal{P}(Y)$ be a multivalued mapping and a point $x_0 \in X$. The mapping T is called*

- *upper semi-continuous on point x_0 if*

$$\lim_{n\to\infty} e(T(x_n), T(x_0)) = 0,$$

for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0.

- *lower semi-continuous on point x_0 if*

$$\lim_{n\to\infty} e(T(x_0), T(x_n)) = 0,$$

for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0.

Example 1.3.1.

1. *The multivalued mapping $T_1 : \mathbb{R} \to \mathbb{R}$ defined by*

$$T_1x = \begin{cases} \{0\} & \text{if } \ x \neq 0, \\ [-1, 1] & \text{if } \ x = 0. \end{cases}$$

is upper semi continuous in 0 but is not lower semi continuous in 0.

2. *The multivalued mapping $T_2 : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by*

$$T_2x = \begin{cases} [-1, 1] & \text{if } \ x \neq 0, \\ \{0\} & \text{if } \ x = 0. \end{cases}$$

is lower semi continuous in 0 *but is not upper semi continuous in* 0.

Lemma 1.3.1. [2] *Let* $T : X \to \mathcal{P}(Y)$ *be a multivalued mapping, then the following statements are equivalent.*

1. T *is upper semi continuous.*

2. $V \subset Y \Rightarrow T^{-1}(\overline{V})$ *is closed in* X.

Lemma 1.3.2. [2] *Let* $T : X \to \mathcal{P}(Y)$ *be a multivalued mapping, then the following statements are equivalent.*

1. T *is lower semi continuous.*

2. $V \subset Y \Rightarrow T^{-1}(int(V))$ *is open in* X.

Definition 1.3.2. [6] *We say that* T *is* ***continuous*** *at* x *if* T *is lower semi continuous and upper semi continuous at* x, *and* T *is continuous if and only if it is continuous at every point* $x \in Dom(T)$.

Now, we will define continuity in the sense of Hausdorff:

Definition 1.3.3. [6] *A multivalued mapping* $T : X \to CB(X)$ *is continuous on* x_0 *with respect to* H *if*

$$\lim_{n \longrightarrow +\infty} H(T(x_n), T(x_0)) = 0$$

for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ *converging to* x_0.

Remark 1.3.1. *Si* $T : X \longrightarrow \mathcal{P}(X)$ *with nonempty and compact values, then* T *is s.c.s if is only if* $\mathrm{Graph}(T)$ *is closed.*

Definition 1.3.4. ***(Carathéodory)*** [3] *A multivocal map* $T : [0,1] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ *is called Carathéodory if:*

1). $t \longrightarrow F(t, x)$ *is measurable for all* $x \in \mathbb{R}$,

2). $x \longrightarrow F(t, x)$ *is continuous for almost all* $t \in [0, 1]$.

Lemma 1.3.3. [16] *Let X be a Banach space. Let $G : [0,T] \times X \to P_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and Γ a linear continuous map from $L^1([0,T],X)$ to $C([0,T],X)$, then the operator*

$$\Gamma \circ S_G : C([0,T],X) \longrightarrow P_{cp,c}(C([0,T],X)), \quad y \longmapsto (\Gamma \circ S_G)(y) = \Gamma(S_{G,y})$$

is a closed graph operator in $C([0,T],X) \times C([0,T],X)$.
Here $S_{G,y} = \{v \in L^1([0,T],X) : v(t) \in G(t,y(t))$ for a.e. $t \in [0,T]\}$.

Definition 1.3.5. [3] *Let (Ω, A) be a measurable space and X a complete separable metric space and $T : \Omega \longrightarrow \mathcal{P}(X)$ a many-to-many mapping with closed image. We say that T is measurable for all $o \in X$ if the inverse image of each open set is a measurable set. for any open set $o \in X$ we have:*

$$T^{-1}(o) = \{\in \Omega \mid T() \cap o \neq \emptyset\} \in A$$

Lemma 1.3.4. [3] *Let Ω be a metric space and (Ω, A, μ) a complete σ-finite space. We assume that A contains all open subsets of Ω. Let X be a complete separable metric space and $T : \Omega \longrightarrow \mathcal{P}(X)$ a many-to-many map with non-empty closed images. If T is upper (or lower) semi-continuous then T is measurable.*

1.3.1 Selection

Definition 1.3.6. ***(Selection)***[3] *Consider a many-to-many mapping $T : X \longrightarrow \mathcal{P}(Y)$ with non-empty images. A mapping $f : X \longrightarrow Y$ is called a selection of T if for all $x \in X, f(x) \in Tx$.*

Theorem 1.3.5. ***(Michael's theorem)***[3] *Let X be a metric space and $T : X \longrightarrow \mathcal{P}(Y)$ a lower semi-continuous multivalued map with convex and closed values of a space Banach Y. Then T has a continuous selection.*

Theorem 1.3.6. ***(Measurable Selection)***[3] *Let X be a metric space and $T : [a,b] \times X \longrightarrow \mathcal{P}(\mathbb{R}^n)$ a measurable map with convex, closed and non-empty images. Then T has a measurable selection f.*

Theorem 1.3.7. ***(Lipschitz selection)***[3] *Consider a Lipschitz multivalued map T from a metric space to convex subsets; closed and nonempty of $\mathbb{R}^n$. Then T admits a Lipschitz selection f.*

1.4 Some fixed point theorems

In this section, we will give definitions on the notion of contractions with certain fixed point theorems in both cases: univocal and multivalued. Additionally, some fixed point theorems for each type.

1.4.1 Fixed point theorems for functions

Definition 1.4.1. [10] *Let $F : X \to X$ is an application from a set X to itself. An element $x \in X$ is said fixed point of the map F if $F(x) = x$*

Example 1.4.1. *Let $X = [0, 2]$, we define the map $F : X \to X$ by:*

$$F(x) = \begin{cases} 0, & si \quad x \in [0, 1], \\ 1, & si \quad x \in]1, 2]. \end{cases}$$

F has a unique fixed point in X because: $F(x) = x$, if $x \in [0, 1] \Longrightarrow F(x) = x = 0$, then $x = 0$ is a fixed point of F.

Definition 1.4.2. [1] *Let (X, d) be a metric space. We say that $F : X \to X$ is Lipschitz, if there exists a real number $k \geq 0$ such that for all $x, y \in X$ we have:*

$$d(F(x), F(y)) \leq kd(x, y)$$

- If $k = 1$ We say that F is non-expansive.

- If $k < 1$ We say that F is contraction.

Example 1.4.2. *Let $(\mathbb{R}^2, d)$ be a metric space, where $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, i.e.*

$F : (\mathbb{R}^2, d) \longrightarrow (\mathbb{R}^2, d)$

$F(x_1, x_2) = \left(\frac{x_1}{2} + b, \frac{x_2}{3} + b\right)$, or $b \in \mathbb{R}$.

F is a contraction on $\mathbb{R}^2$ because :

$$\begin{aligned}
d\left(F\left(x_1,x_2\right),F\left(y_1,y_2\right)\right) &= d\left(\left(\frac{x_1}{2}+b,\frac{x_2}{3}+b\right),\left(\frac{y_1}{2}+b,\frac{y_2}{3}+b\right)\right)\\
&= \left|\frac{x_1}{2}+b-\frac{y_1}{2}-b\right|+\left|\frac{x_2}{3}+b-\frac{y_2}{3}-b\right|\\
&= \left|\frac{x_1}{2}-\frac{y_1}{2}\right|+\left|\frac{x_2}{3}-\frac{y_2}{3}\right|\\
&= \frac{1}{2}\left|x_1-y_1\right|+\frac{1}{3}\left|x_2-y_2\right|\\
&\leq \frac{1}{2}\left(\left|x_1-y_1\right|+\left|x_2-y_2\right|\right)\\
&= \frac{1}{2}d(x,y).
\end{aligned}$$

Theorem 1.4.1. ***(Banach)*** [1] *Let (X,d) be a complete metric space and $F:X\longrightarrow X$ a contraction map of constant k. Then F admits a unique fixed point $u\in X$. In moreover, for any $x\in X$ we have:*

$$\lim_{n\to\infty}F^n(x)=u$$

with

$$d\left(F^n(x),u\right)\leq\frac{k^n}{1-k}d(x,F(x))$$

Proof. We first show uniqueness. Suppose there exists $x,y\in X$ with $x=F(x)$ and $y=F(y)$. Afterwards

$$\begin{aligned}
d(x,y)=d(F(x),F(y)&\leq kd(x,y)\\
(1-k)d(x,y)&\leq 0\\
0\leq d(x,y)&\leq 0\\
d(x,y)&=0.
\end{aligned}$$

So $x=y$.

To show existence, select $x\in X$. We first show that $F^n(x)$ is a Cauchy sequence. Indeed, for all $n\in\{0,1,...\}$ we have :

$$d\left(F^n(x),F^{n+1}(x)\right)\leq kd\left(F^{n-1}(x),F^nx\right)\leq\cdots\leq k^nd(x,F(x))$$

and for $m > n \in \mathbb{N}$,

$$\begin{aligned}
d\left(F^n(x), F^m(x)\right) &\leq d\left(F^n(x), F^{n+1}(x)\right) + d\left(F^{n+1}(x), F^{n+2}(x)\right) + \cdots + d\left(F^{m-1}(x), F^m(x)\right) \\
&\leq k^n d(x, F(x)) + \cdots + k^{m-1} d(x, F(x)) \\
&\leq k^n \left(1 + k + k^2 + \cdots + k^{m-n-1}\right) d(x, F(x)) \\
&= \frac{k^n}{1-k} d(x, F(x)).
\end{aligned}$$

so:

$$d\left(F^n(x), F^m(x)\right) \leq \frac{k^n}{1-k} d(x, F(x)).$$

$\square$

Therefore $(F_n(x))_{n\geq 0}$ is a Cauchy sequence, since (X, d) is complete then $(F_n(x))_{n\geq 0}$ converges to such a point $u \in X$, with $\lim_{n\to\infty} F^n(x) = u$. Since F is continuous then:

$$u = \lim_{n\to\infty} F^{n+1}(x) = \lim_{n\to\infty} F\left(F^n(x)\right) = F(u)$$

so u is a fixed point of F. Passing to the $m \longrightarrow \infty$ in the previous inequality, we obtain :

$$d\left(F^n(x), u\right) \leq \frac{k^n}{1-k} d(x, F(x))$$

Example 1.4.3. *Let $X = \mathbb{R}_*^+$ equipped with the usual distance $d : X \longrightarrow \mathbb{R}^+$ defined by:*

$$\forall x, y \in X, \quad d(x, y) = |x - y|$$

We define the map $F : X \longrightarrow X$ by $F(x) = \sqrt{x}$. We have (X, d) is a metric space complete. Let us show that F is a contraction:

$$\forall x, y \in X, \quad d(F(x), F(y)) = |\sqrt{x} - \sqrt{y}|$$

Using the mean value theorem, we have, $\forall x, y \in X, \exists c \in]x, y[$,

$$d(F(x), F(y)) \leq \frac{1}{2\sqrt{c}}|x - y| \leq \frac{1}{2}|x - y| = \frac{1}{2} d(x, y)$$

Or $k = \frac{1}{2} < 1$.then F is a contraction according to theorem 1.4.1, F admits a fixed point unique

$$F(x) = x \Longrightarrow \sqrt{x} = x \Longrightarrow x_0 = 0 \quad ou \quad x_1 = 1$$

Since $x_0 \notin \mathbb{R}_*^+$, then x_1 is a unique fixed point of F.

Theorem 1.4.2. [20]*Let $F : X \longrightarrow X$ be a map from a complete metric space X to itself $\exists n \in \mathbb{N}^*$ such that F^n is a contract of X. Then F has a fixed point unique in X.*

Remark 1.4.1. [20] *There are maps that are not contractions, but they have a single fixed point.*

Example 1.4.4. *Let $X = [0, 1]$ (equipped with the usual distance), we define the map $F : X \longrightarrow X$ by:*

$$F(x) = 1 - x$$

F is not a contraction because:

$$|F(x) - F(y)| = |1 - x - 1 + y| = |x - y|$$

But F admits a unique fixed point:

$$F(x) = x \Rightarrow 1 - x = x \Rightarrow x = \frac{1}{2}$$

Theorem 1.4.3. ***(Schauder)*** [7]*Let M be a bounded, closed, convex and not void of a Banach space X on and $F : M \longrightarrow M$ a compact map. So F has a fixed point*

1.4.2 Fixed point theorems for multivalued functions

Definition 1.4.3. [17] *Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$ a map multi-valued, we say that z is a fixed point of T if $z \in Tx$.*

Example 1.4.5. *Let $T : [0.1] \longrightarrow P([0.1])$ to be map defined by:*

$$Tx = \{0\} \cup \{f(x)\}$$

such that :

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & si\ x \in \left[0, \frac{1}{2}\right] \\ \frac{-1}{2}x + 1 & si\ x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

T admits two fixed points $x_0 = 0$ and $x_1 = \frac{2}{3}$ because: if $x_0 = 0 \Rightarrow T0 = \{0\} \cup \{f(0)\} = \{0\} \cup \left\{\frac{1}{2}\right\} = \left\{0, \frac{1}{2}\right\}$ so $0 \in T0$. if $x_1 = \frac{2}{3} \Rightarrow T\frac{2}{3} = \left\{0, \frac{2}{3}\right\}$ so $\frac{2}{3} \in T\frac{2}{3}$.

Definition 1.4.4. [17] *A multivalued map $T : X \longrightarrow CB(X)$ is said to be contraction if there exists a constant $k, 0 \leq k < 1$, such that:*

$$H(Tx, Ty) \leq kd(x, y)$$

Remark 1.4.2. [19] *Let $A, B \in CB(X)$ and $a \in A$.if $\eta > 0$, then it is a simple consequence of the definition of $H(A, B)$ that there exists $b \in B$ such that:*

$$d(a, b) \leq H(A, B) + \eta$$

Lemma 1.4.4. [4] *Let (X, d) be a metric space and $A, B \in CB(X)$. So for each $h > 1$ and for all $a \in A$, there is $b = b(a) \in B$, such that:*

$$d(a, b) \leq hH(A, B) \tag{1.4}$$

Proof. Note that if $H(A, B) = 0$, then $A = B$ and $a \in B$, we conclude that inequality (1.4) holds for $b = a$. On the other hand, if $H(A, B) > 0$ let

$$\epsilon = (h - 1)H(A, B) > 0 \tag{1.5}$$

Using the definition of $d(a, B)$ and $H(A, B)$, it becomes that for every $\epsilon > 0$ there exists $b \in B$ such that:

$$d(a, b) \leq d(a, B) + \epsilon \leq H(A, B) + \epsilon \tag{1.6}$$

On remplace (1.5) dans (1.6) nous trouvons:

$$d(a, b) \leq H(A, B) + (h - 1)H(A, B)$$
$$d(a, b) \leq hH(A, B).$$

□

Theorem 1.4.5. *(Nadler)* [19] *Let (X, d) be a complete metric space. If $T : X \longrightarrow CB(X)$ is a multivocal contraction, then T has a fixed point, i.e. there exists $x \in X$ such that $x\ inTx$.*

Proof. The contractivity of the F map implies

$$\exists k \in]0, 1[, \quad H(Tx, Ty) \leq kd(x, y) \tag{1.7}$$

Let h be a real number such that $h > 1$, and $h < \frac{1}{k}$. we will build a sequence $(x_n)_{n\geq 0}$ as follows, for $x_0 \in X, x_1 \in Tx_0, x_2 \in Tx_1, \ldots x_{n+1} \in Tx_n$, and we will show the convergence of this sequence towards a point z (fixed point). Indeed, according to Lemma 1.4.4 there exists $x_2 \in Tx_1$ such that:

$$d(x_1, x_2) \leq hH(Tx_0, Tx_1)$$

According to (1.7) we have:

$$\begin{aligned} d(x_1, x_2) &\leq hH(Tx_0, Tx_1) \\ &\leq hkd(x_0, x_1) \end{aligned}$$

For $x_2 \in Tx_1$, there is $x_3 \in Tx_2$ such that:

$$d(x_2, x_3) \leq hH(Tx_1, Tx_2)$$

From (1.7), we get:

$$\begin{aligned} d(x_2, x_3) &\leq hH(Tx_1, Tx_2) \\ &\leq hkd(x_1, x_2) \\ &\leq (hk)^2 d(x_0, x_1). \end{aligned}$$

We set $\lambda = hk < 1$, we can generalize the previous result, for $x_n \in Tx_{n-1}$, there exists $x_{n+1} \in Tx_n$ such that:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq hH(Tx_{n-1}, Tx_n) \\ &\leq \lambda^n d(x_0, x_1) \end{aligned}$$

For $n < m \in \mathbb{N}$, we have:

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ cdots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdot + \lambda^{m-1} d(x_0, x_1) \\ &\leq \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right) d(x_0, x_1) \\ &\leq \left(\sum_{i=n}^{m-1} \lambda^i\right) d(x_0, x_1). \end{aligned}$$

$\sum_{i=n}^{m-1} \lambda^i$ This is the rest of a convergent geometric series, so $\lim_{n,m\to\infty} \left(\sum_{i=n}^{m-1} \lambda^i = 0\right)$. Passing to the limit in the previous inequality we find:

$$0 \leq \lim_{n,m\to\infty} d(x_n, x_m) \leq \lim_{n,m\to\infty} \left(\sum_{i=n}^{m-1} \lambda^i\right) d(x_0, x_1)$$

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Therefore $(x_n)_{n\geq 0}$ is a Cauchy sequence, since (X, d) is complete then $(x_n)_{n\geq 0}$ converges to such a point $z \in X$,

$$\begin{aligned} d\left(x_{n+1}, Tz\right) &\leq H\left(Tx_n, Tz\right) \\ &\leq kd\left(x_n, z\right). \end{aligned}$$

Passing to the limit when $n \longrightarrow \infty$ we get

$$d(z, Tz) = 0$$

□

So $z \in \overline{Tz} = Tz$, because Tz is closed. Then z is a fixed point of T.

Theorem 1.4.6. ***(Leray-Schauder nonlinear alternative)*** *Let X be a Banach space and C a convex, closed and non-empty subset of X. Let U be a nonempty convex subset of C with $0 \in U$ and $T : \bar{U} \longrightarrow K(X)$ an upper semicontinuous compact map. So either*

(i). T admits a fixed point in U, or

(ii). there is $x \in \partial U$ and $\lambda \in [0, 1]$ such that $x \in \lambda Tx$.

1.4.3 convexity

Definition 1.4.5. *Let Ube a convex set of $\mathbb{R}$ and $f : U \to \mathbb{R}$ is a function. We say that f is convex if*
$\forall x, y \in U \forall t \in [0, 1] : f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$

Definition 1.4.6. ***(equicontinuous)*** [23] *Let $\mathcal{H}$ be the set of all functions from J to a Banach space E. We say that M is equicontinuous if:*

$$\forall \epsilon > 0, \exists \delta > 0, \forall f(x) \in M, \forall x_1, x_2 \in J, |x_1 - x_2| \leq \delta, \|f\left(x_1\right) - f\left(x_2\right)\| \leq \epsilon$$

Example 1.4.6. *The family of c-Lipschitz maps F for the same constant c is equicontinuous $(X, ||.) \longrightarrow (Y, \|\cdot\|)$. $\forall f \in \mathcal{H}$ we have $\forall x, y \in X$:*

$$\|f(x) - f(y)\| \leq c|x - y|$$

We choose $\delta = \frac{\varepsilon}{c}$ *then we have:*

$$\|f(x) - f(y)\| \leq \frac{\varepsilon}{c} \times c$$
$$\|f(x) - f(y)\| \leq \varepsilon$$

Remark 1.4.3 (8). *It is clear that if* H *is equicontinuous at a point* $x_0 \in X$*, then all* $f \in H$ *is continuous at* x_0*. The converse is generally false. Indeed, for all* $n \geq 0$*, the function* $f_n : x \longmapsto \sin(nx)$ *is continuous on* $\mathbb{R}$*, but the family of functions* $f_n; n \geq 0$ *is not equicontinuous at 0 .*

Theorem 1.4.7. ***(Ascoli-Arzelà theorem)*** [23] *Suppose that* $I = [a, b]$ *is a finite interval,* $C(I)$ *is the set of all continuous functions on I. Then a set* $F \subset C(I)$ *is relatively compact if and only if* F *is uniformly bounded and equicon*tin.

Corollary 1.4.1. [9] *Let* X *be a compact space and* $\mathcal{H}$ *a subset of* $C(X, \mathbb{R}^n)$*. The space* $\mathbb{R}^n$ *is endowed with the Euclidean distance. The following properties are equivalent.*
1). H *is compact in* $(C(X, \mathbb{R}^n), Y)$*.*
2). H *is equicontinuous and for all* $x \in X, \mathcal{H}(x)$ *is bounded in* $\mathbb{R}^n$*.*

1.5 Fractional integral and derivative

In this section, we recall the basic tools and results essential to our work. In particular, we present some fundamental results on the properties of fractional derivation and fractional integration.

1.5.1 The Gamma function

We call Eulerian Gamma function (or Euclidean integral of the second kind) the function noted Γ defined by:

$$\Gamma(q) = \int_0^{+\infty} t^{q-1} e^{-t} dt.$$

where q is any complex number such that $Re(q) > 0$.

Proposition 1.5.1. *We have the following properties:*

1.
$$\Gamma(q+1) = q\Gamma(q).$$

2.
$$\Gamma(q+1) = n!.$$

3. *For* $q \in \mathbb{C} \setminus \{0, 1, 2, 3, ...\}$
$$\Gamma(q-1) = -q\Gamma(-q).$$

4. *For* $n \in \mathbb{N}$
$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad (2n-1)!! = 1 \times 3...(2n-1).$$

1.5.2 Fractional integral

Definition 1.5.1. [14] *The fractional integral of the function* $f \in C([a, b])$ *of order* $q \in \mathbb{R}_+$ *is defined by*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(t) dt.$$

Proposition 1.5.2. *We have the following properties:*

i) $I^0 f(t) = f(t)$,

ii) $I^q I^p f(t) = I^{q+p} f(t)$,

iii) the integral operator I^q *is linear,*

$$I^q(\lambda f(t) + g(t)) = \lambda I^q f(t) + I^q g(t), \quad q \in \mathbb{R}_+, \quad \lambda \in \mathbb{C},$$

iv) $\frac{d}{dt}(I^q f)(t) = I^{q-1} f(t)$.

Example 1.5.1. *Let $f(t) = t^{\mu}$ where $\mu > -1$ and $t > 0$ let's find the fractional integral of the function f of order $q > 0$:*

$$\begin{aligned}
I^q f(t) = I^q t^{\mu} &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{\mu} ds \\
&= \frac{1}{\Gamma(q)} \int_0^t t^{q-1} \left(1 - \frac{s}{t}\right)^{q-1} s^{\mu} ds \\
&= \frac{t^{q-1} \cdot t^{\mu}}{\Gamma(q)} \int_0^t \left(1 - \frac{s}{t}\right)^{q-1} \left(\frac{s}{t}\right)^{\mu} ds
\end{aligned}$$

And by making a change in the variable $u = \frac{s}{t}$,$ds = t.du$ is obtained from it:

$$\begin{aligned}
I^q f(t) &= \frac{t^{\mu+q-1}}{\Gamma(q)} \int_0^1 (1-u)^{q-1} u^{\mu} \cdot t du \\
&= \frac{t^{\mu+q}}{\Gamma(q)} \int_0^1 (1-u)^{q-1} u^{\mu} du \\
&= \frac{t^{\mu+q}}{\Gamma(q)} \cdot \beta(\mu+1, q) \\
&= \frac{t^{\mu+q}}{\Gamma(q)} \cdot \frac{\Gamma(\mu+1)\Gamma(q)}{\Gamma(q+\mu+1)} = \frac{\Gamma(\mu+1)}{\Gamma(q+\mu+1)} t^{\mu+q}
\end{aligned}$$

1.5.3 Fractional derivative

There are several approaches for fractional differentiation, in this part we will give some definitions and proprieties of fractional derivative.

Definition 1.5.2. [14] *Let f be an integrable function on $[a, b]$, then the derivative fractional order q (with $n - 1 \leq q < n$) in the sense of **Riemann-Liouville** is defined by:*

$$\begin{aligned}
{}^{RL}D^q f(t) &= \tfrac{1}{\Gamma(n-q)} \tfrac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} f(s) ds \\
&= \tfrac{d^n}{dt^n} (I^{n-q} f(t)) = D^n I^{n-q} f(t).
\end{aligned}$$

here $n = [q] + 1$ and $[q]$ denoting the integer part of q.

Proposition 1.5.3. *We have the following properties:*

1. *The fractional differentiation operator in the sense of Riemann-Liouville is a left inverse*

$$ {}^{RL}D^r \left(I^r f(t)\right) = f(t), $$

in general we have

$$^{RL}D^{r_1}\left(I^{r_2}f(t)\right) =^{RL} D^{r_1-r_2}f(t)$$

and if $r_1 - r_2 < 0, ^{RL}D^{r_1-r_2}f(t) = I^{r_2-r_1}f(t)$.

2. *In general derivation and fractional integration do not commute*

$$^{RL}D^{-r_1}\left({}_a^{RL}D_t^{r_2}f(t)\right) =^{RL} D^{r_2-r_1}f(t) - \sum_{k=1}^{m}\left[{}^{RL}D_t^{r_2-k}f(t)\right]_{t=a}\frac{(t-a)^{r_1-k}}{\Gamma\left(r_1-k+1\right)}$$

with $m-1 \leq r_2 < m$

3. *if :*$f^{(k)}(a) = 0$ *for all* $k = 0, 1, 2, \ldots, n-1$

$$\frac{d^n}{dt^n}\left({}^{RL}D^r f(t)\right) =^{RL} D^{n+r}f(t),$$

but

$$^{RL}D^r\left(\frac{d^n}{dt^n}f(t)\right) =^{RL} D^{n+r}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k-r-n}}{\Gamma(k-r-n+1)}.$$

4. *For the Riemann-Liouville fractional differential operator, the corresponding interpolation property reads*

$$\lim_{q\to n}{}^{RL}D^q f(t) = f^{(n)}(t).$$
$$\lim_{\alpha\to n}{}^{RL}D^q f(t) = f^{(n-1)}(t)$$

5. *is a linear operator* $q \in \mathbb{R}_+, \lambda \in \mathbb{C}$

$$^{RL}D^q(\lambda f(t) + g(t)) = \lambda^{RL}D^q f(t) +^{RL} D^r g(t).$$

Definition 1.5.3. [14] *Let* f *be an integrable function on* $[a, b]$, *then the derivative fractional order* q *(with* $n-1 \leq q < n$*) in the sense of* ***Caputo*** *is defined by:*

$$^{c}D^q f(t) = \frac{1}{\Gamma(n-q)}\int_a^t (t-s)^{n-q-1}f^{(n)}(s)ds,$$

here $n = [q]+1$ *and* $[q]$ *denoting the integer part of* q.

Proposition 1.5.4. *We have the following proprieties:*

1. *The relationship with the Riemann-Liouville derivative.*

   Suppose that $n-1<q<n,\ m,n\in\mathbb{N},\ q\in\mathbb{R}$ *and let the function* $f(t)$ *be such that* ${}^{c}Df(t)$ *and* ${}^{RL}Df(t)$ *exist, then:*

$$^{c}Df(t) =^{RL} Df(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k-q}}{\Gamma(k-q+1)}.$$

   We deduce that if $f^{(k)}(a)=0$*, for* $k=0,1,2,...,n-1$*, we will have:*

$$^{c}D^{q}f(t) =^{RL} D^{q}f(t).$$

2. *Suppose that* $n-1<q<n,\ m,n\in\mathbb{N},\ q\in\mathbb{R}$ *and let the function f(t) be such that* ${}^{c}Df(t)$ *exists, then:*

$$^{c}D^{q}D^{m}f(t) = {}^{c}D^{q+m}f(t)\neq D^{m\,c}D^{q}f(t).$$

3. *For the Caputo fractional differential operator, the corresponding interpolation property reads*

$$\lim_{q\to n}{}^{c}D^{q}f(t)=f(t).$$

$$\lim_{q\to n}{}^{c}D^{q}f(t)=f^{(n-1)}(t)-f^{n-1}(0)$$

4. *it is a linear operator* $q\in\mathbb{R}_{+},\lambda\in\mathbb{C}$

$$^{c}D^{q}(\lambda f(t)+g(t))=\lambda^{c}D^{q}f(t)+^{c}D^{q}g(t).$$

Lemma 1.5.1. [14] *Let* $q>0$*, then*

$$I^{q\,c}D^{q}f(t)=f(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$$

for $c_i\in\mathbb{R}, i=0,1,2,\ldots,n-1, n=[q]+1,$

Chapter 2

Existence of solution for a fractional boundry value problem of differntial inclusions with integral boundry conditons

in this chapter,we will propose an issue for a study related to the title of our research represented in On the existence of solutions for Caputo fractional boundary problems of differential inclusions

2.1 Position of the problem

consider the problem

$$\begin{cases} D^q x(t) \in F(t, x(t)), 0 \leq t \leq 1, 1 < q \leq 2 \\ x(0) = 0 \\ x(1) = \lambda \int_0^1 x(s)ds, \lambda \in \mathbb{R} \end{cases} \tag{2.1}$$

where D^q denotes the Caputo fractional derivative of order $q, F : [0,1]\times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$. the problem, that is the equation $\mathrm{D}^q x(t) = v(t)$, with the boundary conditions $\left(x(0) = 0, x(1) = \lambda \int_0^1 x(s)ds, \lambda \in \mathbb{R}\right)$ we establish the existence of solutions for the problem(2.1), when the right hand side is convex as well as nonconvex valued. the first result relies on the nonlinear alternative of Schauder type.in the second result, we shall combine the nonlinear alternative of LeraySchauder type for multivalued maps with a selection theorem for lower semicontinuous multivalued maps with nonempty closed values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Nadler.

Definition 2.1.1. *For each $x \in C([0,1],\mathbb{R})$, define the set of selections of F by:*

$$S_{F,x} := \left\{ v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1] \right\}.$$

A function x is said to be a solution of problem(2.1) if there exists a function $v \in L^1([0,1],\mathbb{R})$ with $v(t) \in F(t,x(t))$ for all $t \in [0,1]$ such that $D^q x(t) = v(t)$, and the function x satisfies the boundary conditions of problem (2.1).

To show the existence of solutions to the problem (2.1), we will transform the problem to the fixed point problem, indeed we have the following lemma:

Lemma 2.1.1. *A function x is a solution of the fractional integral equation*

$$x(t) = \int_0^1 G(t,s)v(s)ds + t\lambda \int_0^1 x(s)ds \tag{2.2}$$

if and only if x is a solution of the following limit fractional problem:

$$\begin{cases} D^q x(t) = v(t), 0 \leq t \leq 1, 1 < q \leq 2 \\ x(0) = 0 \\ x(1) = \lambda \int_0^1 x(s)ds, \lambda \in \mathbb{R}. \end{cases} \tag{2.3}$$

where G is the Green function given by:

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} + t(1-s)^{q-1}, 0 < s < t \\ (1-s)^{q-1}, t < s < 1. \end{cases} \tag{2.4}$$

.

Proof. For some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = \int_0^t (t-s)^{q-1} v(s)ds + c_0 + c_1 t.$$

Using the boundary conditions on (2.2), we get

$$x(0) = 0, \quad x(1) = \lambda \int_0^1 x(s)ds.$$

Therefore

$$x(0) = 0 \Rightarrow c_0 = 0$$

$$x(1) = c_1 + \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds = \lambda \int_0^1 x(s)ds \Rightarrow c_1 = \lambda \int_0^1 x(s)ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds.$$

It means that

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + t \left[\lambda \int_0^1 x(s)ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds \right]. \\ &= \int_0^1 G(t,s)v(s)ds + t\lambda \int_0^1 x(s)ds. \end{aligned}$$

in which

$$(\int_0^1 (1-s)^{q-1} v(s)ds = \int_0^t (1-s)^{q-1} v(s)ds + \int_t^1 (1-s)^{q-1} v(s)ds).$$

On the other hand, by applying the Caputo derivative to equation (2.2), we get:

$$D^q x(t) = D^q(I^q v(t)) + D^q(t)\lambda \int_0^1 x(s)ds - D^q(t) \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds$$

since:

$$D^q(I^q v(t)) = v(t) \text{ and } D^q(t) = 0$$

so:

$$D^q x(t) = v(t).$$

with:

$$x(0) = 0, \quad x(1) = \lambda \int_0^1 x(s)ds.$$

Then a solution of problem (2.1) is given by:

$$x(t) = \int_0^1 G(t,s)v(s)ds + t\lambda \int_0^1 x(s)ds.$$

where

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} + t(1-s)^{q-1}, 0 < s < t \\ (1-s)^{q-1}, t < s < 1. \end{cases}$$

$\square$

2.2 Existence of solutions using Leray-Schauder alternative

Theorem 2.2.1. *Assume that:*

(H_1) $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact and convex values.*

(H_2) *There exists* $k > 0$ *such that:*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leqslant k \textit{ for each } (t,x) \in [0,1]\times\mathbb{R}.$$

(H_3) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{G^*k + |\lambda|A} > 1$$

Or

$$G^* = \sup_{t,s\in[0,1]} |G(t,s)|.$$

And

$$A = \sup_{s\in[0,1]} |x(s)|ds$$

Then the problem (2.1) *has at least one solution on* $[0,1]$.

Proof. Define the operator:

$T : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by:

$Tx(t) = \{h \in C([0,1],\mathbb{R}) : \quad h(t) = \int_0^1 G(t,s)v(s)ds + t\lambda\int_0^1 x(s)ds\}$.

for $v \in S_{F,x}$.

We divide the proof into several cases:

Step1:Tx is convex.

Let $h_1, h_2 \in Tx$ et $\theta \in [0,1]$, then there exists $v_1, v_2 \in S_{F,x}$ such that:

$$\begin{aligned}(\theta h_1 + (1-\theta)h_2)(t) &= \theta\int_0^1 G(t,s)v_1(s)ds + \theta t\lambda\int_0^1 x(s)ds + (1-\theta)\int_0^1 G(t,s)v_2(s)ds \\ &+ (1-\theta)t\lambda\int_0^1 x(s)ds \\ &= t\lambda\int_0^1 x(s)ds + \int_0^1 G(t,s)\left[\theta v_1(s) + (1-\theta)v_2\right]ds.\end{aligned}$$

since $v_3 = v_1 + (1-\theta)v_2 \in S_{F,x}$ $(becauseS_{F,x}$ is convex).

So:

$$\begin{aligned}(\theta h_1 + (1-\theta)h_2)(t) &= t\lambda \int_0^1 x(s)ds + \int_0^1 G(t,s)v_3(t)ds \\ &= h_3(t) \in Tx.\end{aligned}$$

Then

$$\theta_1 h_1 + (1-\theta)h_2 \in T_x$$

Step2: T is bounded on $C([0,1],\mathbb{R})$. we will show that all the elements of Tx are bounded. Then, for each $h \in T(x)$, there exists $v \in S_{F,x}$ such that:

$$\begin{aligned}h(t) &= t\lambda \int_0^1 x(s)ds + \int_0^1 G(t,s)v(s)ds. \\ |h(t)| &\leq t|\lambda| \int_0^1 |x(s)|ds + \int_0^1 |G(t,s)||v(s)|ds. \\ &\leq |\lambda| \int_0^1 |x(s)|ds + \int_0^1 |G(t,s)|kds. \\ &\leq G^*k + |\lambda| \int_0^1 |x(s)|ds.\end{aligned}$$

or

$$G^* \;\; = \sup_{\mathrm{t,s}\in[0,1]} |G(t,s)|.$$

So

$$\|x\|_\infty \;\; \leq G^*k + |\lambda| \int_0^1 |x(s)|ds.$$

$\int_0^1 |x(s)|ds$ is a continuous function integral over a compact interval, i.e. it reaches its bounded, then:

$$\int_0^1 |x(s)|ds \;\; \leqslant \sup_{\mathrm{s}\in[0,1]} |x(s)| = A.$$

So:

$$\|x\|_\infty \;\; \leq G^*k + |\lambda|A.$$

so T is bounded.

$$B=\{,x\in C([0,1],\mathbb{R}):\|x\|_\infty\leq R\}$$

is a bounded and closed subset of $C([0,1],\mathbb{R})$.

We will prove that B is convex.

Let $x,y\in B$ and $\lambda\in[0,1]$ we have:

$$\begin{aligned}\|\lambda x+(1-\lambda)y\| &\leq |\lambda|\|x\|+|(1-\lambda\varsigma)|\|y\|\\ &\leq \lambda R+(1-\lambda)R\\ &= R\end{aligned}$$

So $\quad \lambda x+(1-\lambda)y\in B$.

Step3 : Now we show that T maps bounded sets into equicontinuous set of $C([0,1],\mathbb{R})$.

Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in B$, we obtain

$$\begin{aligned}|h(t_2)-h(t_1)| &= |t_2\lambda\int_0^1 x(s)ds+\int_0^1 G(t_2,s)v(s)ds-(t_1\lambda\int_0^1 x(s)ds+\int_0^1 G(t_1,s)v(s)ds)|\\ &\leqslant |t_2-t_1||\lambda|\int_0^1 |x(s)|ds+\int_0^1 |G(t_2,s)-G(t_1,s)||v(s)|ds\\ &\leqslant |t_2-t_1||\lambda|\int_0^1 |x(s)|ds+k\int_0^1 |G(t_2,s)-G(t_1,s)|ds\end{aligned}$$

So:

$$\|h(t_2)-h(t_1)\| \quad \leq |t_2-t_1||\lambda|\int_0^1 |x(s)|ds+k\int_0^1 |G(t_2,s)-G(t_1,s)|ds$$

when $t_1\to t_2$, also G is uniformly continuous, so $|G(t_2,s)-G(t_1,s)|\to 0$ as $t_1\to t_2$

so:

$$\lim_{t_1\to t_2}\|h(t_2)-h(t_1)\|_\infty=0$$

Then by steps 2 and 3 TB is bounded and equicontinuous.

Thus, according to the Arzelà -Ascoli theorem, we can conclude that:

$$T:C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$$

is compact.

(because: $T(B)$ is relatively compact then the operator T is compact).

Step4: In our next step, we show that T has a closed graph. Let $x_n\to x_*, h_n\in T\left(x_n\right)$ and

$h_n \to h_*$. Then we need to show that $h_* \in T(x_*)$. According with $h_n \in T(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = t\lambda \int_0^1 x_n(s)ds + \int_0^1 G(t,s)v_n(s)ds.$$

thus we have to show that there exist $v_\star \in S_F, x$ such that for each $t \in [0,1]$.

$$h_\star(t) = t\lambda \int_0^1 x_\star(s)ds + \int_0^1 G(t,s)v_\star(s)ds.$$

Let us consider the continuous linear operator $\Gamma : \mathbf{L}^1([0,1], R) \to C([0,1], R)$ given by

$$v \mapsto \Gamma(v)(t) = h(t) - t\lambda \int_0^1 x(s)ds = \int_0^1 G(t,s)v(s)ds\ ,$$

observe that

$$\begin{aligned} \|\Gamma(v_n)(t) - \Gamma(v_\star)(t)\| &= \| \int_0^1 G(t,s)v_n(s)ds - \int_0^1 G(t,s)v_\star(s)ds\| \\ &\leq \int_0^1 G(t,s)\|v_n(s) - v_\star(s)ds\|_\infty \end{aligned}$$

as $n \to \infty$, the right-hand side tends to 0.

Thus, it follows by Lemma1.3.3 and remark 1.3.1 that $\Gamma \circ S_F$ is a closed graph operator.

Further, we have $v_n(t) \in \Gamma(S_{F,x_n})$. since $x_n \to x_\star$, therefore, we have

$h_\star(t) = t\lambda \int_0^1 x_\star(s)ds + \int_0^1 G(t,s)v_\star(s)ds$,

for some $v_\star \in S_{F,x_\star}$.

$v_\star \in S_{F,x_\star} \Rightarrow h_\star \in Tx_\star \Rightarrow T$is a closed graph $\Rightarrow T$ is upper semicontinuous.

Step5: A priori bounds on the solutions.

We now show that there exists an open set $U \in X$ with $x \notin \rho Tx$, for $\rho \in [0,1]$ and $x \in \partial U$. Then we have:

$$\begin{aligned} x(t) &= t\lambda \int_0^1 x(s)ds + \rho \int_0^1 G(t,s)v(s)ds. \\ |x(t)| &\leq t|\lambda| \int_0^1 |x(s)|ds + \rho \int_0^1 |G(t,s)||v(s)|ds. \\ &\leq |\lambda| \int_0^1 |x(s)|ds + \int_0^1 |G(t,s)|kds. \\ &\leq G^*k + |\lambda| \int_0^1 |x(s)|ds. \end{aligned}$$

so

$$\|x\|_\infty \ \ \leq G^*k + |\lambda|A.$$

or

$$A = \sup_{s\in[0,1]} |x(s)|ds.$$

so

$$\frac{\|x\|_\infty}{G^*k + |\lambda|A} \leq 1.$$

Then by condition (H_3), there exists $M > 0$ such that:

$$\|x\|_\infty \leq G^*k + |\lambda|A < M.$$

so we choose

$$U = \{x \in C([0,1], \mathbb{R}), \|x\|_\infty < M\}.$$

From our hypothesis, if

$$x \in \partial U \Rightarrow \|x\|_\infty = M.$$

and according to the previous result and our hypothesis, we find

$$M = \|x\|_\infty < M,$$

which is a contradiction.

Note that the operator $T : \bar{U} \to \mathcal{P}(C[0,1], \mathbb{R})$ semicontinuous and compact. From the choice of U, there is no $x \in \partial U$ such that $x \in \rho T x$, for some $\rho \in [0,1]$. As consequence of the Leray-Schauder nonlinear alternative, we deduce that T has a point fixed $x \in \bar{U}$ which is a solution of the problem (2.1) $\square$

Example 2.2.1. *Consider the following strip fractional boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t, x(t)), 0 < t < 1, \\ x(0) = 0, x(1) = \int_0^1 x(s)ds. \end{cases}$$

Here, $q = 3/2, \lambda = 1,$ *and* $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a multivalued map given by*

$$x \to F(t, x(t)) = \left[\frac{|x(t)|^3}{|x(t)|^3 + 3} + 3t^3 + 5, \frac{|x(t)|}{|x(t)| + 1} + t + 1\right].$$

1. *F is from carathéodory:*

a). The map $t \to F(t,x)$ is measurable for all $x \in \mathbb{R}$:

let $t_n \longrightarrow t$ so $x(t_n) \longrightarrow x(t)$ si:

$$\begin{aligned}\lim_{n\to\infty} F(t_n, x(t_n)) &= \lim_{n\to\infty}\left[\frac{|x(t_n)|^3}{|x(t_n)|^3+3}+3t_n^3+5, \frac{|x(t_n)|}{|x(t_n)|+1}+t_n+1\right]\\ &= \left[\frac{|x(t)|^3}{|x(t)|^3+3}+3t^3+5, \frac{|x(t)|}{|x(t)|+1}+t+1\right] = F(t,x(t)).\end{aligned}$$

So :

$$\lim_{n\to\infty} H(F(t_n, x(t_n)), F(t,x(t))) = H(F(t,x(t)), F(t,x(t)))$$

since the distance H is a metric then:

$$\lim_{n-\infty} H(F(t_n, x(t_n)), F(t,x(t))) = 0$$

So the map $t \longrightarrow F(t,x(t))$ is continuous, F is measurable for all $x \in \mathbb{R}$.

b). The map $x \longrightarrow F(t,x(t))$ is continuous for almost all $t \in [0,1]$: let $x_n \longrightarrow x$, for all $t \in [0,1]$ we have

$$\begin{aligned}\lim_{n\to\infty} F(t, x_n(t)) &= \lim_{n\to\infty}\left[\frac{|x_n(t)|^3}{|x_n(t)|^3+3}+3t^3+5, \frac{|x_n(t)|}{|x_n(t)|+1}+t+1\right]\\ &= \left[\frac{|x(t)|^3}{|x(t)|^3+3}+3t^3+5, \frac{|x(t)|}{|x(t)|+1}+t+1\right] = F(t,x(t)).\end{aligned}$$

So:

$$\lim_{n\to\infty} H(F(t, x_n(t)), F(t,x(t))) = H(F(t,x(t)), F(t,x(t))) = 0.$$

then the map $x \to F(t,x(t))$ is continuous

So F is from Carathéodory.

2. *F is compact:*

   To prove that F is compact, we use the Arzelà -Ascoli theorem. we go first show that FB is bounded, then we will prove that is equicontinuous, where $B\{x \in C([0,1]), \|x\|_\infty \leq R\}$.

   a). F is bounded:

   For $f \in F$, we have:

$$|f(t)| \leq \max\left(\frac{|x(t)|^3}{|x(t)|^3+3}+3t^3+5, \frac{|x(t)|}{|x(t)|+1}+t+1\right) \leq 9$$

So:

$$\|F(t,x(t))\|_{\infty} = \sup\{|f(t)| : \quad f \in F(t,x(t))\} \leq 9 = k.$$

b). FB is equicontinuous: let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $f \in F$, we obtain:

$$\frac{|x(t_1)|^3}{|x(t_1)|^3+3} + 3t_1^3 + 5 \leq f(t_1) \leq \frac{|x(t_1)|}{|x(t_1)|+1} + t_1 + 1,$$
$$\frac{|x(t_2)|^3}{|x(t_2)|^3+3} + 3t_2^3 + 5 \leq f(t_2) \leq \frac{|x(t_2)|}{|x(t_2)|+1} + t_2 + 1.$$

So:

$$\begin{aligned}\left|\frac{|x(t_1)|^3}{|x(t_1)|^3+3} + 3t_1^3 - \frac{|x(t_2)|^3}{|x(t_2)|^3+3} - 3t_2^3\right| &\leq |f(t_1) - f(t_2)| \\ &\leq \left|\frac{|x(t_1)|}{|x(t_1)|+1} + t_1 - \frac{|x(t_2)|}{|x(t_2)|+1} - t_2\right|\end{aligned}$$

So:

$$\begin{aligned}\lim_{t_1 \to t_2} \|\frac{|x(t_1)|^3}{|x(t_1)|^3+3} + 3t_1^3 - \frac{|x(t_2)|^3}{|x(t_2)|^3+3} - 3t_2^3\|_{\infty} &\leq \lim_{t_1 \to t_2} \|f(t_1) - f(t_2)\|_{\infty} \\ &\leq \lim_{t_1 \to t_2} \|\frac{|x(t_1)|}{|x(t_1)|+1} + t_1 - \frac{|x(t_2)|}{|x(t_2)|+1} - t_2\|\end{aligned}$$

$$\lim_{t_1 \to t_2} \|f(t_1) - f(t_2)\|_{\infty} = 0.$$

So FB is equicontinuous and bounded, then F is compact.

3. *F is convex:*

   let $f_1, f_2 \in F$ and $\lambda \in [0,1]$, and for all $t \in [0,1]$ we have:

$$\lambda\left(\frac{|x(t)|^3}{|x(t)|^3+3} + 3t^3 + 5\right) \leq \lambda f_1(t) \leq \lambda\left(\frac{|x(t)|}{|x(t)|+1} + t + 1\right)$$
$$(1-\lambda)\left(\frac{|x(t)|^3}{|x(t)|^3+3} + 3t^3 + 5\right) \leq (1-\lambda) f_2(t) \leq (1-\lambda)\left(\frac{|x(t)|}{|x(t)|+1} + t + 1\right)$$

   So:

$$\frac{|x(t)|^3}{|x(t)|^3+3} + 3t^3 + 5 \leq \lambda f_1(t) + (1-\lambda) f_2(t) \leq \frac{|x(t)|}{|x(t)|+1} + t + 1$$
$$\lambda f_1(t) + (1-\lambda) f_2(t) \in \left[\frac{|x(t)|^3}{|x(t)|^3+3} + 3t^3 + 5, \frac{|x(t)|}{|x(t)|+1} + t + 1\right] = F(t,x(t)).$$

4. *there is a constant* $M > 0$ *such that:*

$$\begin{aligned} M &> G^{*}k + |\lambda| A \\ &> \sup_{s\in[0,1]} |x(s)| ds \end{aligned}$$

2.3 Existence of solutions by Nadler theorem

In this section, we will work on the transformation of problem (2.1) into a fixed point problem using Nadler's fixed point theorem, for this we define the operator T as follows:
$T : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$
$Tx(t) = \{h \in C([0,1],\mathbb{R}) : \quad h(t) = \int_0^1 G(t,s)v(s)ds + t\lambda \int_0^1 x(s)ds\}.$
or $v \in S_{F,x}$

Theorem 2.3.1. *We introduce the following assumptions:*
(H_5) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is compact*
and the map $x \to F(t, x(t))$ *is measurable for almost any* $t \in [0,1]$.
(H_6) There exists $m \in L^1([0,1],\mathbb{R})$ *such that:*
$H(F(t,x_1(t)), F(t,x_2(t))) \leq m(t)|x_1(t) - x_2(t)|$ *for each* $x_1, x_2 \in C([0,1],\mathbb{R}$
and $d(0, F(t,0)) \leq m(t)$ *for almost all* $t \in [0,1]$.
Under the assumptions (H_2), (H_3), (H_5), (H_6), (H_7) the problem (2.1) *has at least one solutionknowing that :*

$$[G^{\star}k + |\lambda| A] < 1.$$

Proof. From condition (H_5), the multivalued map $F(t,x)$ is measurable and has convex, closed and non-empty values. Then, by the measurable selection theorem for all $x \in C([0,1],\mathbb{R})$, there is a measurable selection $v(t) \in F(t,x(t))$, so Tx is non-empty i.e. well defined

Step1: Tx is closed for all $C([0,1],\mathbb{R})$.
Let $(h_n)_{n\geq 0} \in Tx$ such that, $h_n \to h$. Then there exists $v_n \in S_{F,x_n}$ converges to v, $\forall t \in [0,1]$ and checked:

$$h_n(t) = t\lambda \int_0^1 x_n(t)ds + \int_0^1 G(t,s)v_n(s)ds$$

we will first show that $v \in S_{F,x}$.

since $F(t,x)$ is compact we can extract a subsequence v_{n_k}, converges in$S_{F,x}$, to the same limit as v_n, then $v \in S_{F,x}$.

Passing to the limit of the previous relation we find:

$$h(t) = t\lambda \int_0^1 x(t)ds + \int_0^1 G(t,s)v(s)ds$$

like $v \in S_{F,x}$, then $h \in Tx$.

Step2: T is bounded.

we will show that there exists a constant $k > 0$, such that$\| h(t) \|$, for all $h \in Tx$.

Indeed, we have:

$$\begin{aligned} |h(t)| &\leq t|\lambda| \int_0^1 |x(t)|ds + \int_0^1 |G(t,s||v(s)|ds \\ &\leqslant |\lambda| \int_0^1 |x(t)|ds + \int_0^1 |G(t,s||v(s)|ds \end{aligned}$$

by (H_2) and (H_3) we have:

$$\leqslant |\lambda| \int_0^1 |x(s)| + G^\star K ds.$$

Where $G^* = \sup \mid G(t,s) \mid$. So:

$$\|h\|_\infty \leqslant |\lambda| \int_0^1 |x(s)|ds + G^\star K.$$

$\int_0^1 |x(s)|ds$ is an integral of a Continuous function on the compact interval, i. e. it reaches its bounded then:

$$\int_0^1 |x(s)|ds \leqslant \sup_{s\in[0,1]} |x(s)|.$$

So:

$$\|h\|_\infty \leqslant |\lambda| \sup |x(s)| + G^* K = L.$$

Where L is a constant.

then h is bounded, that is, T is bounded .

Step3: T is contracting.

We show that there exists $0 < c < 1$ such that,

$$H\left(T\left(x_1(t)\right), T\left(x_2(t)\right) \leqslant c\|x_1(t) - x_2(t)\|_\infty . \forall x_1, x_2 \in C([0,1], \mathbb{R}).$$

Also $x_1, x_2 \in C([0,1],\mathbb{R})$ and $h_1 \in Tx$ and there exists $v_1(t) \in F(t, x(t)$ such that $\forall t \in [0,1]$.

$$h_1(t) = t\lambda \int_0^1 x(s)ds + \int_0^1 G(t,s)v(s)ds.$$

From (H_6) we have :

$$H\left(F\left(t, x_1\right), F\left(t, x_2\right)\right) \leqslant m(t)\left\|x_1 - x_2\right\|_{\infty}.$$

Where:

there exists $w \in F\left(t, x_2\right)$ Such that:

$$|v_1(t) - w| \leqslant m(t)|x_1(t) - x_2(t)|, \forall \in [0,1].$$

We consider $M : [0,1] \to P(\mathbb{R})$ given by:

$$M(t) = \left\{w \in F\left(t, x_2\right) : |v_1(t) - w| \leqslant m(t)\ |x_1(t) - x_2(t)|\right\}.$$

given that the operator $M(t)$ is measurable. there exists a function $v_2(t)$ which is a measurable selection for M. the, $v_2 \in F\left(t, x_2\right)$, and for each $t \in [0,1]$,

$$\left|v_1(t) - v_2(t)\right| \leq m(t)\left|x_1(t) - x_2(t)\right|$$

it exists $h_2 \in Tx_2$.

Let us define for each $t \in [0,1]$.

$$|h_1(t) - h_2(t)| \leq t|\lambda| \int_0^1 \left|x_2(s) - x_2(s)\right| ds + \int_0^1 G(t,s)\left|v_1(s) - v_2(s)\right| ds,$$

Using(H_3) and (H_6)

$$\leqslant t|\lambda| \int_0^1 \mid x_1(s) - x_2(s) \mid ds + G^{\star}(t,s) \int_0^1 m(s)\left|x_1(s) - x_2(s)\right| ds.$$

t is a continuous function of the compact interval i.e. that it reaches its bounded.

So:

$$\left\|h_1(t) - h_2(t)\right\|_{\infty} \leqslant \left[|\lambda| + G^{*}(t,s) \int_0^1 m(s)ds\right] \left\|x_1(t) - x_2(t)\right\|$$

From an analogous relation; obtained by Changing the roles of x_1 and x_2, it follows that:

$$H\left(T\left(x_1\right), T\left(x_2\right)\right) \leqslant c\left\|x_1 - x_2\right\|, \forall x_1, x_2 \in C([0,1],\mathbb{R}).$$

then, T is contractive and thus, by theorem 2.3.1 it admits a fixed point$x$ which is a solution of the problem(2.1).

$\square$

Example 2.3.1.

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) \in F(t, x(t)), 0 < t < 1 \\ x(0) = 0 \\ x(1) = \frac{1}{2}\int_0^1 x(s)ds \end{cases} \tag{2.5}$$

Here, $q = \frac{3}{2}, \lambda = \frac{1}{2}$*and* $F : [0, 1] \times R \to P(\mathbb{R})$ *is a many-to-many map given by:*

$$x \to F(t, x(t)) \in \{\frac{|x|}{3} + 2t^3 + 5N\}, N \in \{0, 1, 2\}.$$

1). *We will show that* F *is compact. First we show that* FB *is bounded, where* $B = \{x \in C([0, 1], \mathbb{R}); \|x\| < R\}$.

Since $X(t)$ *is continuous on a compact, thenit is bounded*

So for all $V \in F, t \in [0, 1]$ *and* $N \in \{0, 1, 2\}$ *we have:*

$$\begin{aligned} |V(t)| &= \left|\frac{|x(t)|}{3} + 2t^3 + 5N\right| \\ &\leqslant \left|\frac{|x(t)|}{3} + 2 + 10\right| \\ &\leqslant \left|\frac{|x(t)|}{3} + 12\right| \end{aligned}$$

So

$$\begin{aligned} \|v(t)\|_\infty &\leqslant \frac{\|x\|_\infty}{3} + 12 \\ \|F(t, x(t))\| &\leqslant \frac{\|x\|_\infty}{3} + 12. \end{aligned}$$

Now we will show that FB *equicontinuous.*

We have $x(t)$ *is Continuous on Compact then Uniformly continuous and for all* $v \in F$ *and* $t_1, t_2 \in [0, 1]$ *we have:*

$$\begin{aligned} |V(t_1) - V(t_2)| &=| \left(\frac{|x(t_1)|}{3} + 2t_1^3 + 5N\right) - \frac{|x(t_2)|}{3} + 2t_2^3 + 5N) \mid \\ &= |(|x(t_1)| - |x(t_2)|) + (2t_1^3 - 2t_2^3) \mid \\ &\leqslant |x(t_1) - x(t_2)| + \left|\left(2t_1^3 - 2t_2^3\right)\right| \end{aligned}$$

then for $t_1 \to t_2$ *we find:*

$$|V(t_1) - V(t_2)| \to 0$$

so by ArzelÃ -Ascoli theorem F*is Compact.*

2).*The map* $x \to F(t, x(t))$ *continuous for almost any* $t \in [0,1]$ *:*

$$\begin{aligned}\lim_{n\to\infty} F(t, x_n) &= \lim_{n\to\infty}\left\{\frac{|x_n|}{3} + 2t^3 + 5N\right\}\\ &= \left\{\frac{|x|}{3} + 2t^3 + 5N\right\} = F(t, x(t))\end{aligned}$$

So:

$$\lim_{n\to\infty} H\left(F\left(t, x_n(t), F(t, x(t)\right)\right) = H(F(t, x(t)), F(t, x(t))) = 0$$

Then the map $x \to F(t, x(t))$ *is Continuous and by Lemma1.3.4* F *is measurable.*

3).*For* $v_1 \in F(t, x_1)$ *and* $v_2 \in F(t, x_2)$ *we have:*

$$\begin{aligned}|v_1(t) - v_2(t)| &= |(\frac{|x_1|}{3} + 2t^3 + 5N) - (\frac{|x_2|}{3} + 2t^3 + 5N)|\\ &= |\frac{|x_1|}{3} - \frac{|x_2|}{3}|\\ &\leq \frac{1}{3}|x_1 - x_2|.\end{aligned}$$

So

$$\| v_1(t) - v_2(t) \|_\infty \leq \tfrac{1}{3} \| x_1(t) - x_2(t) \|$$

So:

$$H(F(t, x_1), F(t, x_2)) \leq \tfrac{1}{3} \| x_1(t) - x_2(t) \|_\infty .$$

4). *We have:*

$$c = [\lambda A + G^\star(t, s)\int_0^1 m(s)ds].$$

So:

$$c = \tfrac{1}{2} < 1$$

Then according to theorem 2.2.1 the problem (2.5) *has at least one solution on* $[0, 1]$

Chapter 3

Study of a Caputo fractional boundary problem of differential inclusions

In this chapter, we will study an issue related to the title of our reseach with different conditions than the previous ames in the second chapter

3.1 Position of the problem

condescend to the problem

$$\begin{cases} D^q x(t) \in F(t, x(t)), \quad 0 \leq t \leq 1, 1 < q \leq 2 \\ x(0) = \int_0^1 g(s, x(s))ds \\ x(1) = \int_0^1 h(s, x(s))ds, \end{cases} \tag{3.1}$$

Where $1 < q \leqslant 2, D^q$ is the standard caputo fractional derivative of order $q, F = [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map, $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Definition 3.1.1. *For each $x \in C([0,1], \mathbb{R})$, define the set of selections of F by:*

$$S_{F,x} := \left\{ v \in L^1([0,1], \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1] \right\}.$$

A function x is said to be a solution of problem(3.1) if there exists a function $v \in L^1([0,1],\mathbb{R})$ with $v(t) \in F(t,x(t))$ for all $t \in [0,1]$ such that $D^q x(t) = v(t)$, and the function x satisfies the boundary conditions of problem (3.1).
To show the existence of solutions to the problem (3.1), we will transform the problem to the fixed point problem, indeed we have the following lemma:

Lemma 3.1.1. *A function x is a solution of the fractional integral equation*

$$x(t) = \int_0^1 G(t,s)v(s)ds + t\int_0^1 h(s,x(s))ds + (1-t)\int_0^1 g(s,x(s))ds \tag{3.2}$$

if and only if x is a solution of the following limit fractional problem:

$$\begin{cases} D^q x(t) = v(t), \quad 0 \le t \le 1, 1 < q \le 2 \\ x(0) = \int_0^1 g(s,x(s))ds \\ x(1) = \int_0^1 h(s,x(s))ds, \end{cases} \tag{3.3}$$

where G is the Green function given by:

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} + t(1-s)^{q-1}, 0 \le s < t \\ (1-s)^{q-1}, t < s \le 1. \end{cases} \tag{3.4}$$

Proof. For Some constants $c_0, c_1 \in \mathbb{R}$, We have

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds + c_0 + c_1 t$$

Wing the boundary Conditions on (3.1), we get

$$x(0) = \int_0^1 g(s,x(s)ds$$
$$x(1) = \int_0^1 h(s,x(s))ds$$

therefore

$$x(0) = c_0 \Rightarrow c_0 = \int_0^1 g(s,x(s))ds$$

$$x(1) = \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}(\ s)ds + c_0 + c_1 = \int_0^1 h(s,x(s))$$

$$\Rightarrow c_1 = \int_0^1 h(s,x(s))ds - \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}v(s)ds - \int_0^1 g(s,x(s))ds$$

It means that

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s) ds + \int_0^1 g(s, x(s)) ds \\ &+ t[\int_0^1 h(s, x(s)) ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s) ds - \int_0^1 g(s, x(s)) ds] \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s) ds + t \int_0^1 h(s, x(s)) ds + (1-t) \int_0^1 g(s, x(s)) ds \\ &- t \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s) ds \\ x(t) &= \int_0^1 G(t,s) v(s) ds + t \int_0^1 h(s, x(s)) ds + (1-t) \int_0^1 g(s, x(s)) ds \end{aligned}$$

On the other hand, by applying the Caputo derivative to equation (3.2), we get:

$$D^q x(t) = D^q(I^q v(t)) + D^q(t) \int_0^1 h(s, x(s)) ds - D^q(t) \int_0^1 g(s, x(s)) ds - D^q(t) \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s) ds$$

since:

$$D^q(t) = 0 \text{ and } D^q(I^q v(t)) = v(t)$$

so:

$$D^q x(t) = v(t).$$

with:

$$x(0) = \int_0^1 g(s, x(s)) ds, \quad x(1) = \int_0^1 h(s, x(s)) ds.$$

Then a solution of problem (3.1) is given by:

$$x(t) = \int_0^1 G(t,s) v(s) ds + (1-t) \int_0^1 g(s, x(s)) ds + t \int_0^1 h(s, x(s)) ds$$

where

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} + t(1-s)^{q-1}, 0 \leq s < t \\ (1-s)^{q-1}, t < s \leq 1. \end{cases}$$

$\square$

3.2 Existence of solutions using Leray-Schauder alternative

In this chapter, we will work on converting problem (3.1) into a fixed point using Leray-Schauder

Theorem 3.2.1. *Assume that:*

(H_1) $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact and convex values;*

(H_2) *there exists* $k_1 > 0$ *such that:*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leqslant k \text{ for each } (t,x) \in [0,1]\times\mathbb{R}.$$

(H_3) *there exists* $k_2 > 0$ *and* $k_3 > 0$ *such that:*

$$|g(t,x_1) - g(t,x_2)| \leq k_2|x_1 - x_2|, \text{for all } t \in [0,1] \text{and all } x_1, x_2 \in C([0,1],\mathbb{R}).$$

$$|h(t,x_1) - h(t,x_2)| \leq k_3|x_1 - x_2|, \text{for all } t \in [0,1] \text{and all } x_1, x_2 \in C([0,1],\mathbb{R}).$$

(H_4) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{G^* k_1 + k_3 A} > 1$$

Or

$$G^* = \sup_{t,s\in[0,1]} |G(t,s)|.$$

And

$$A = \sup_{s\in[0,1]} |x(s)|$$

Then the problem (3.1) *has at least one solution on* $[0,1]$.

***Proof*.** Define the operator:

$T : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by:

$Tx(t) = \{y \in C([0,1],\mathbb{R}) : \ y(t) = \int_0^1 G(t,s)v(s)ds + (1-t)\int_0^1 g(s,x(s))ds + t\int_0^1 h(s,x(s))ds\}$

for $v \in S_{F,x}$.

We divide the proof into several cases:

Step1 :Tx is convex.

Let $h_1, h_2 \in Tx$ et $\theta \in [0,1]$, then there exists $v_1, v_2 \in S_{F,x}$ such that:

$$\begin{aligned}(\theta y_1 + (1-\theta)y_2)(t) &= \theta \int_0^1 G(t,s)v_1(s)ds + \theta(1-t)\int_0^1 g(s,x(s))ds + \theta t \int_0^1 h(s,x(s))ds \\ &+ (1-\theta)\int_0^1 G(t,s)v_2(s)ds + (1-\theta)(1-t)\int_0^1 g(s,x(s))ds \\ &+ (1-\theta)t\int_0^1 h(s,x(s))ds \\ &= (1-t)\int_0^1 g(s,x(s))ds + t\int_0^1 h(s,x(s))ds + \int_0^1 G(t,s)[\theta v_1(s) + (1-\theta)v_2(s)]ds.\end{aligned}$$

since $v_3 = v_1 + (1-\theta)v_2 \in S_{F,x}$ (because $S_{F,x}$ is convex)

So:

$$\begin{aligned}(\theta y_1 + (1-\theta)y_2)(t) &= (1-t)\int_0^1 g(s,x(s))ds + t\int_0^1 h(s,x(s))ds + \int_0^1 G(t,s)v_3(t)ds. \\ &= y_3(t) \in Tx\end{aligned}$$

So:

$$\theta y_1 + (1-\theta)y_2 \in T_x$$

Step2 : T is bounded on $C([0,1],\mathbb{R})$. we will show that all the elements of Tx are bounded. Then, for each $y \in T(x)$, there exists $v \in S_{F,x}$ such that:

$$\begin{aligned}y(t) &= \int_0^1 G(t,s)v(s)ds + (1-t)\int_0^1 g(s,x(s))ds + t\int_0^1 h(s,x(s))ds \\ |y(t)| &\leq \int_0^1 |G(t,s)||v(s)|ds + (1-t)\int_0^1 |g(s,x(s))|ds + t\int_0^1 |h(s,x(s))|ds\end{aligned}$$

By (H_2) and (H_3) we have:

$$\begin{aligned}
|h(s,x(s))| &\leq |h(s,x(s)) - h(s,0) + h(s,0)| \\
&\leq |h(s,x(s)) - h(s,0)| + |h(s,0)\ | \\
&\leq |h(s,x(s)) - h(s,0)| + h_0 \\
&\leq k_3|x(s)| + h_0. \\
|g(s,x(s))| &\leq |g(s,x(s)) - g(s,0) + g(s,0)| \\
&\leq ||g(s,x(s)) - g(s,0)| + |g(s,0)\ | \\
&\leq |g(s,x(s)) - g(s,0)| + g_0 \\
&\leq k_2|x(s)| + g_0.
\end{aligned}$$

Or

$$h_0 = \sup_{s\in[0,1]} |h(s,0)|,$$

and

$$g_0 = \sup_{s\in[0,1]} |g(s,0)|.$$

So:

$$\begin{aligned}
|y(t)| &\leq \int_0^1 |G(t,s)|k_1 ds + \int_0^1 (k_3|x(s)| + h_0)ds \\
&\leq G^\star k_1 + h_0 + k_3 \int_0^1 |x(s)|ds
\end{aligned}$$

or

$$G^* = \sup_{\text{t,s}\in[0,1]} |G(t,s)|.$$

so:

$$\|x\|_\infty \quad \leq G^\star k_1 + h_0 + k_3 \int_0^1 |x(s)|ds.$$

$\int_0^1 |x(s)|ds$ is a continuous function integral over a compact interval, i.e. it reaches its bounded, then:

$$\int_0^1 |x(s)|ds \quad \leqslant \sup_{\text{s}\in[0,1]} |x(s)| = A$$

So:

$$\|x\|_\infty \quad \leq G^\star k_1 + h_0 + k_3 A.$$

so T is bounded.

$$B=\{x\in C([0,1],\mathbb{R}):\|x\|_\infty\leq R\}$$

is a bounded and closed subset of $C([0,1],\mathbb{R})$.

We will prove that B is convex.

Let $x,y\in B$ and $\lambda\in[0,1]$ we have:

$$\begin{aligned}\|\lambda x+(1-\lambda)y\| &\leq |\lambda|\|x\|+|(1-\lambda\varsigma)|\|y\|\\ &\leq \lambda R+(1-\lambda)R\\ &=R\end{aligned}$$

So $\quad \lambda x+(1-\lambda)y\in B.$

Step3 : Now we show that T maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.

Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in B$, we obtain:

$$\begin{aligned}|y(t_2)-y(t_1)|=&|\int_0^1 G(t_2,s)v(s)ds+(1-t_2)\int_0^1 g(s,x(s))ds+t_2\int_0^1 h(s,x(s))ds\\ &-(\int_0^1 G(t_1,s)v(s)ds+(1-t_1)\int_0^1 g(s,x(s))ds+t_1\int_0^1 h(s,x(s))ds)|\\ &\leqslant \int_0^1 |G(t_2,s)-G(t_1,s)||v(s)|ds+|t_1-t_2|\int_0^1 |g(s,x(s))|ds\\ &+|t_2+t_1|\int_0^1 |h(s,x(s)|ds\\ &\leqslant k_1\int_0^1 |G(t_2,s)-G(t_1,s)|ds+|t_1-t_2|g_0+k_2|t_1-t_2|\int_0^1|x(s)|ds\\ &+|t_2-t_1|h_0+k_3|t_2-t_1|\int_0^1|x(s)|ds\end{aligned}$$

So:

$$\begin{aligned}\|y(t_2)-y(t_1)\|_\infty\leq{}& k_1\int_0^1 |G(t_2,s)-G(t_1,s)|ds+|t_1-t_2|g_0+k_2|t_1-t_2|\int_0^1|x(s)|ds\\ &+|t_2-t_1|h_0+k_3|t_2-t_1|\int_0^1|x(s)|ds\end{aligned}$$

when $t_1\to t_2$, also G is uniformly continuous, so $|G(t_2,s)-G(t_1,s)|\to 0$ as $t_1\to t_2$

so:

$$\lim_{t_1\to t_2}\|y(t_2)-y(t_1)\|_\infty=0$$

Then by steps 2 and 3 TB is bounded and equicontinuous.

Thus, according to the Arzelà -Ascoli theorem, we can conclude that:

$$T : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$$

is compact.

(because: $T(B)$ is relatively compact then the operator T is compact).

Step4: In our next step, we show that T has a closed graph. Let $x_n \to x_*, y_n \in T(x_n)$ and $y_n \to y_*$. Then we need to show that $y_* \in T(x_*)$. Associated with $y_n \in T(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$y_n(t) = \int_0^1 G(t,s)v_n(s)ds + (1-t)\int_0^1 g(s,x_n(s))ds + t\int_0^1 h(s,x_n(s))ds\},$$

thus we have to show that there exist $v_\star \in S_F, x$ such that for each $t \in [0,1]$.

$$y_\star(t) = \int_0^1 G(t,s)v_*(s)ds + (1-t)\int_0^1 g(s,x_*(s))ds + t\int_0^1 h(s,x_*(s))ds.$$

Let us consider the continuous linear operator $\Gamma : \mathbf{L}^1([0,1],R) \to C([0,1],R)$ given by

$$v_1 \mapsto \Gamma(v)(t) = \textstyle\int_0^1 G(t,s)v(s)ds.$$

observe that:

$$\begin{aligned}
\|\Gamma(v_n)(t) - \Gamma(v_*)(t)\|_\infty = \| &\int_0^1 G(t,s)v_n(s)ds + (1-t)\int_0^1 g(s,x_n(s))ds + t\int_0^1 h(s,x_n(s))ds \\
&- \int_0^1 G(t,s)v_*(s)ds + (1-t)\int_0^1 g(s,x_*(s))ds + t\int_0^1 h(s,x_*(s))ds\|_\infty \\
&\leq \int_0^1 G(t,s)\|v_n(s) - v_*(s)\|_\infty ds + (1-t)\int_0^1 \|g(s,x_n(s)) - g(s,x_*(s))\|_\infty ds \\
&+ t\int_0^1 \|h(s,x_n(s)) - h(s,x_*(s))\|_\infty ds
\end{aligned}$$

as $n \to \infty$.

thus, it follows by Lemma1.3.3 remark 1.3.1 that $H \circ S_F$ is a closed graph operator. Further, we have $y_n(t) \in \Gamma(S_{F,x_n})$. since $x_n \to x_\star$, therefore, we have

$y_\star(t) = \int_0^1 G(t,s)v_*(s)ds + (1-t)\int_0^1 g(s,x_*(s))ds + t\int_0^1 h(s,x_*(s))ds,$

for some $v_\star \in S_{F,x_\star}$.

$v_\star \in S_{F,x_\star} \Rightarrow y_\star \in Tx_\star \Rightarrow T$is a closed graph $\Rightarrow T$ is upper semicontinuous

Step5: A priori bounds on the solutions.

We now show that there exists an open set $U \in X$ with $x \in \partial U$, such that $x \notin \rho Tx$.

We have

$$\begin{aligned} &= \int_0^1 G(t,s)v(s)ds + (1-t)\int_0^1 g(s,x(s))ds + t\int_0^1 h(s,x(s))ds \\ |x(t)| &\leq || \int_0^1 |G(t,s)||v(s)|ds + ||(1-t)\int_0^1 |g(s,x(s))|ds + ||t\int_0^1 |h(s,x(s))|ds \\ &\leq \int_0^1 |G(t,s)|k_1 ds + \int_0^1 (k_3|x(s)| + h_0)ds \\ &\leq G^\star k_1 + h_0 + k_3\int_0^1 |x(s)|ds \end{aligned}$$

so

$$\|x\|_\infty \leq G^\star k_1 + h_0 + k_3 A.$$

or

$$A = \sup_{\mathrm{s}\in[0,1]} |x(s)|ds.$$

so

$$\frac{\|x\|_\infty}{G^\star k_1 + h_0 + k_3 A} \leq 1.$$

Then by condition (H_4), there exists $M > 0$ such that:

$$\|x\|_\infty \leq G^\star k_1 + h_0 + k_3 A < M.$$

so

$$\|x\|_\infty < M.$$

We choose

$$U = \{x \in C([0,1],\mathbb{R}), \|x\|_\infty < M\}.$$

Then we have:

$$x \in \partial U \Rightarrow \|x\|_\infty = M.$$

and from the previous result and our hypothesis, we find

$$M = \|x\|_\infty < M,$$

and this is a contradiction.
So the operator $T : \bar{U} \to \mathcal{P}(C[0,1], \mathbb{R})$ semicontinuous and compact. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda T x$, for some $\lambda \in [0,1]$. As consequence of the Leray-Schauder nonlinear alternative, we deduce that T has a point fixed $x \in \bar{U}$ which is a solution of the problem (3.1) □

3.3 Existence of solutions via Nadler theorem

In this section, we will work on the transformation of problem (3.1) into a fixed point problem using Nadler's fixed point theorem, for this we define the operator T as follows:
$T : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$
$Tx(t) = \{y \in C([0,1], \mathbb{R}) : \ y(t) = \int_0^1 G(t,s)v(s)ds + t\int_0^1 h(s, x(s))ds + (1-t)\int_0^1 g(s, x(s))ds\}$.
or $v \in S_{F,x}$

Theorem 3.3.1. *We introduce the following assumptions:*
(H_5) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}$ is compact
and the map $x \to F(t, x(t))$ is measurable for almost any $t \in [0,1]$.
(H_6) ther exists $m \in L^1([0,1], \mathbb{R})$ such that:
$H(F(t, x_1(t)), F(t, x_2(t))) \leq m(t)|x_1(t) - x_2(t)|$ for each $x_1, x_2 \in C([0,1], \mathbb{R})$
and $d(0, F(t,0)) \leq m(t)$ for almost all $t \in [0,1]$.
Under the assumptions (H_2), (H_3), (H_5), (H_6), (H_7) the problem (3.1) has at least one solution knowing that :

$$\left[G^\star \int_0^1 m(s)ds + k_3\right] < 1.$$

Proof. From condition (H_5), the multivalued map $F(t,x)$ is measurable and has convex, closed and non-empty values. Then, by the measurable selection theorem for all $x \in C([0,1], \mathbb{R})$, there is a measurable selection $v(t) \in F(t, x(t))$, so Tx is non-empty i.e. well defined
Step1: Tx is closed for all $C([0,1], \mathbb{R})$.
Let $(y_n)_{n\geq 0} \in Tx$ such that, $y_n \to y$. Then there exists $v_n \in S_{F,x_n}$ converges to v, $\forall t \in [0,1]$ and checked:

$$y_n(t) = \int_0^1 G(t,s)v_n(s)ds + t\int_0^1 h(s,x(s))ds + (1-t)\int_0^1 g(s,x(s))ds$$

we will first show that $v \in S_{F,x}$.

since $F(t,x)$ is compact we can extract a subsequence v_{n_k}, converges in$S_{F,x}$, to the same limit as v_n, then $v \in S_{F,x}$.

Passing to the limit of the previous relation we find:

$$y(t) = \int_0^1 G(t,s)v(s)ds + t\int_0^1 h(s,x(s))ds + (1-t)\int_0^1 g(s,x(s))ds$$

like $v \in S_{F,x}$, then $y \in Tx$.

Step2: T is bounded.

we will show that there exists a constant $k > 0$, such that$\| h(t) \|$, for all $h \in Tx$.

Indeed, we have:

$$\begin{aligned}|y(t)| &= \int_0^1 |G(t,s)||v(s)|ds + t\int_0^1 |h(s,x(s))|ds + (1-t)\int_0^1 |g(s,x(s))|ds \\ &\leq \int_0^1 |G(t,s)||v(s)|ds + \int_0^1 |h(s,x(s))|ds\end{aligned}$$

by (H_2) and (H_3) we have:

$$\leqslant k_3 \int_0^1 |x(s)|ds + h_0 + G^\star k.$$

Where $G^* = \sup \mid G(t,s) \mid$. So:

$$\|y\|_\infty \leqslant k_3 \int_0^1 |x(s)|ds + h_0 + G^\star K.$$

$\int_0^1 |x(s)|ds$ is an integral of a Continuous function on the compact interval, i. e. it reaches its bounded then:

$$\int_0^1 |x(s)|ds \leqslant \sup_{s\in[0,1]} |x(s)|$$

So:

$$\|y\|_\infty \leqslant \sup |x(s)| + G^* K = L.$$

Where L is a constant.

then y is bounded, that is, T is bounded .

Step3: T is contracting.

We show that there exists $0 < c < 1$ such that,

$$H\left(T\left(x_1(t)\right), T\left(x_2(t)\right)\right) \leqslant c\|x_1(t) - x_2(t)\|_{\infty}. \ \forall x_1, x_2 \in C([0,1], \mathbb{R})$$

Also $x_1, x_2 \in C([0,1], \mathbb{R})$ and $y_1 \in Tx$ and there exists $v_1(t) \in F(t, x(t)$ such that $\forall t \in [0,1]$.

$$y_1(t) = \int_0^1 G(t,s)v_1(s)ds + t\int_0^1 h(s,x(s))ds + (1-t)\int_0^1 g(s,x(s))ds$$

From (H_6) we have :

$$H\left(F\left(t, x_1\right), F\left(t, x_2\right)\right) \leqslant m(t)\left\|x_1 - x_2\right\|_{\infty}.$$

Where:

there exists $w \in F\left(t, x_2\right)$ Such that:

$$|v_1(t) - w| \leqslant m(t)|x_1(t) - x_2(t)|, \forall \in [0,1].$$

We consider $M : [0,1] \to P(\mathbb{R})$ given by:

$$M(t) = \left\{w \in F\left(t, x_2\right) : |v_1(t) - w| \leqslant m(t)\ |x_1(t) - x_2(t)|\right\}.$$

given that the operator $M(t)$ is measurable. there exists a function $v_2(t)$ Which is a measurable selection for M. the, $v_2 \in F\left(t, x_2\right)$, and for each $t \in [0,1]$,

$$|v_1(t) - v_2(t)| \leq m(t)\,|x_1(t) - x_2(t)|$$

it exists $y_2 \in Tx_2$

Let us define for each $t \in [0,1]$.

$$\begin{aligned}|y_1(t) - y_2(t)| \leq (1-t)\int_0^1 |g(s,x(s)) - g(s,x(s))| + t\int_0^1 |h(s,x_1(s)) - h(s,x_2(s))|ds \\ + \int_0^1 G(t,s)|v_1(s) - v_2(s)|ds,\end{aligned}$$

Using(H_3) and (H_6)

$$\begin{aligned}&\leqslant (1-t)\int_0^1 k_2 \mid x_1(s) - x_2(s) \mid ds + t\int_0^1 k_3|x_1(s) - x_2(s)|ds \\ &+ G^{\star}(t,s)\int_0^1 m(s)\,|x_1(s) - x_2(s)|\,ds.\end{aligned}$$

t is a continuous function of the compact interval i.e. that it reaches its bounded.
So:

$$\|y_1(t) - y_2(t)\|_\infty \leqslant \left[k_3 + G^*(t,s)\int_0^1 m(s)ds\right] \|x_1(t) - x_2(t)\|_\infty$$

From an analogous relation; obtained by Changing the roles of x_1 and x_2, it follows that:

$$H\left(T\left(x_1\right), T\left(x_2\right)\right) \leqslant c\left\|x_1 - x_2\right\|, \forall x_1, x_2 \in C([0,1], \mathbb{R}).$$

then, T is contractive and thus, by theorem 3.3.1 it admits a fixed point$x$ which is a solution of the problem(3.1).

□

Example 3.3.1.

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) \in F(t, x(t)), \qquad 0 < t < 1 \\ x(0) = \int_0^1 (\ln s + \frac{x}{3})ds \\ x(1) = \int_0^1 (\frac{x}{4} + 2s^2 + 4)ds \end{cases} \tag{3.5}$$

Here, $q = \frac{3}{2}$, and $F : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a map given by:

$$x \to F(t, x(t)) \in \{\frac{|x|}{4} + 2t^3 + 5N\}, N \in \{0,1,2,3,4\}.$$

1). *We will show that F is compact. First we show that FB is bounded, where $B = \{x \in C([0,1], \mathbb{R}); \|x\| < R\}$.*
Since $X(t)$ is continuous on a compact, thenit is bounded
So for all $V \in F, t \in [0,1]$ and $N \in \{0,1,2\}$ we have:

$$\begin{aligned} |V(t)| &= \left|\frac{|x(t)|}{4} + 2t^3 + 5N\right| \\ &\leqslant \left|\frac{|x(t)|}{4} + 2 + 20\right| \\ &\leqslant \left|\frac{|x(t)|}{4} + 22\right| \end{aligned}$$

So

$$\begin{aligned} \|v(t)\|_\infty &\leqslant \frac{\|x\|_\infty}{4} + 22 \\ \|F(t, x(t))\| &\leqslant \frac{\|x\|_\infty}{4} + 22. \end{aligned}$$

Now we will show that FB equicontinuous.

We have $x(t)$ is Continuous on Compact then Uniformly continuous and for all $v \in F$ and $t_1, t_2 \in [0,1]$ we have:

$$\begin{aligned}
|V(t_1) - V(t_2)| &= |\,(\frac{|x(t_1)|}{4} + 2t_1^3 + 5N) - \frac{|x(t_2)|}{4} + 2t_2^3 + 5N)\,| \\
&= \frac{1}{4}|(|x(t_1)| - |x(t_2)|) + (2t_1^3 - 2t_2^3)\,| \\
&\leqslant |x\,(t_1) - x\,(t_2)| + \left|\left(2t_1^3 - 2t_2^3\right)\right|
\end{aligned}$$

then for $t_1 \to t_2$ we find:

$$|V(t_1) - V\,(t_2)| \to 0$$

so by Arzela -Ascoli theorem $F$is Compact.

2).*The map $x \to F(t, x(t))$ continuous for almost any $t \in [0,1]$:*

$$\begin{aligned}
\lim_{n\to\infty} F\,(t, x_n) &= \lim_{n\to\infty} \left\{ \frac{|x_n|}{4} + 2t^3 + 5N \right\} \\
&= \left\{ \frac{|x|}{4} + 2t^3 + 5N \right\} = F(t, x(t))
\end{aligned}$$

So:

$$\lim_{n\to\infty} H\,(F\,(t, x_n(t), F(t, x(t))\,) = H(F(t, x(t)), F(t, x(t))) = 0$$

Then the map $x \to F(t, x(t))$ is Continuous and by Lemma1.3.4 F is measured.

3).*For $v_1 \in F(t, x_1)$ and $v_2 \in F(t, x_2)$ we have:*

$$\begin{aligned}
|v_1(t) - v_2(t)| &= |(\frac{|x_1|}{4} + 2t^3 + 5N) - (\frac{|x_2|}{4} + 2t^3 + 5N)| \\
&= |\frac{|x_1|}{4} - \frac{|x_2|}{4}| \\
&\leq \frac{1}{4}|x_1 - x_2|.
\end{aligned}$$

So

$$\parallel v_1(t) - v_2(t) \parallel_\infty \leq \tfrac{1}{4} \parallel x_1(t) - x_2(t) \parallel$$

So:

$$H(F(t,x_1),F(t,x_2)) \leq \tfrac{1}{4} \parallel x_1(t)-x_2(t) \parallel_\infty .$$

It is observed that:

$$\begin{aligned}|g(t,x(t))-g(t,x(t))| &= |\ln t+\frac{x(t)}{3}-\ln t+\frac{y(t)}{3}\\ &\leq \frac{1}{3}|x(t)-y(t)|\end{aligned}$$

So g is Lipschitz $k_2=\frac{1}{3}$

$$\begin{aligned}|h(t,x(t))-h(t,x(t))| &= |\frac{x(t)}{4}+2t^2+4-\frac{y(t)}{4}-(2t^2+4)|\\ &= |\frac{x(t)}{4}-\frac{y(t)}{4}|\\ &\leq \frac{1}{4}|x(t)-y(t)|.\end{aligned}$$

So h is Lipschitz $k_3=\frac{1}{4}$

4). *We have:*

$$c=[k_3+G^\star(t,s)\int_0^1 m(s)ds]$$

such that:

$$c=\frac{1}{4}<1$$

then according to theorem3.2.1.the problem (3.5) *has at least one solution on* $[0,1]$

Conclusion

In this memory, we discussed the existence of solutions to certain parametric problems of differential equations of fractioal order by Caputo in metric spaces, this study was carried out using the alternative of Leray-Schauder and the fixed point theory of Nadler.
This topic is still among the studies that require more in-depth research, due to the importance of this mathematical branch in various scientific fields such as applied mathematics, physics, biology and other scientific horizons.

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الملخص

تهدف هذه المذكرة الى دراسة مسألة وجود الحل لبعض المسائل الحدية لإحتواءات تفاضلية ذات رتب كسرية وشروط حدية تكاملية، وذلك باستخدام تقنية النقطة الصامدة إذ إستخدمنا في إثبات الوجود مبرهنتي Leray Schaude وNadler، كما دعمنا الجانب النظري بأمثلة توضيحية.

الكلمات المفتاحية: مؤثر متعدد القيم، النقطة الثابتة، الاشتقاق وفق مفهوم كابيتو، احتواءات تفاضلية.

Abstract

In this memory, we give some definitions and properties of multivalued mappings and we study the existence of solutions of some boundary valued problems of fractional differential inclusions in the sense of Caputo, by using fixed point technique, in fact we use LeraySchauder alternative and Nadler theorem.

Some illustrative examples were given to support our results.

Keywords: multivalued operator, fixed point, derivation according to Capito's concept, differential inclusions.

Résume

Dans ce mémoire, nous traitons quelques définitions et propriétés concernant les opérateurs multivoques; ainsi que l'étude le problème d'existence de solutions pour des problèmes aux limites d'inclusions différentielles fractionnaires avec des des conditions aux limites intégrales.

Quelques exemples ont été données pour valider nos résultats.

Mots clés: opérateur multivalué, point fixe, dérivation selon le concept de Capito, inclusions différentielles.