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Approximations of Multigrid Methods for Parabolic Quasi Variational Inequalities

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DEDICATION

To my mother who still gives us love, strength and sacrifice.

To my father who had left us a bright history that we live by until now.

To my honorable siblings, each one with his name and his value.

To my best family, my duaghter AMENA(WARDA), my son WAEL, and my wife.

To all Mojahideen and martyrs who liberated my country from occupation.

To the noble Palestinian people who are fighting alone against injustice and tyranny.

To all the free people who still maintain their freedom despite all the humiliation in the world .

I dedicate you this hard and distinguished work .

BAHI MOSTAFA

Acknowledgments

" And Allah has extracted you from the wombs of your mothers not knowing a thing and He made for you hearing and vision and intellect that perhaps you would be grateful(to Allah)" . Verse 78 from surah An-Nahl.

First of all thanks and praise be to the God who enabled us to complete this thesis after hard and perseverance work. I would like to extend my thanks to everyone who helped me to complete this work, professors, friends especially my supervisor Dr. Muhammad Baggas who deservesthe title of ideal supervisor and my co-supervisor Pr.Haiour Mohamed, because they did not skimp on scientific information advice and guidance. I would like to thank the jury members of this thesis: Pr. Said Beloul the jury president of this thesis, Pr. Yahia Djabrane and Pr. Nadjet Doudi and Pr. Bensayah Abdellah for their agreement to be the rapporteurs of this thesis, as well as their comments and suggestions to improve this thesis.I would also like to extend my sincere thanks and appreciation to Dr. Baloufi Al-Said and Dr.Nesba Nour Al-Hoda who both helped me a lot in this research. Fanilly ,A big thanks to my family for providing all the good conditions for completing this thesis, thanks a lot for everybody .

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ملخص باللغة العربية

في هذه الأطروحة، درسنا التقارب المنتظم لطريقة متعددة الشبكات، والتي استعملناها في حل عدة أصناف خطية وغير خطية من المتراجحات شبه التغيراتية المكافئة, حيث قمنا بصياغة كل متراجحة على شكل معادلة HJB، حيث أثبتنا التكافؤ بين المتراجحة التغيراتية ومعادلة HJB . في المسألة المتقطعة للمتراجحة استخدمنا طريقة العناصر المنتهية بالنسبة للفضاء و الفروق المنتهية على الزمن ثم برهنا التقارب المنتظم لطريقة متعدد الشبكات، وتم تدعيم النتائج النظرية بتطبيقات عددية .

الكلمات المفتاحية : المتراجحة التغيراتية المكافئة ، طريقة العناصر المنتهية طريقة الفروق المنتهية ، طريقة متعدد الشبكات ، معادلة HJB .

Abstract in English

In this thesis, we studied the multigrid method for parabolic quasi variational inqualities. For geting the discrete problem, we used the finite element method on the space and the difference method on the time, where the discrete problem is equivalent the continuous problem, after that, we proved the uniform convergence of the multigrid method, for soloving the solution of our problem we trasfometed the parabolic quasi variational inqualities to the HJB equation, because there is an equivalence between the parabolic quasi variational inqualities and the HJB equation. In the applied branch, we gived a numerical applications for each studied problems, through which we confirmed the validity of the results obtained in the theoretical part.

Keywords: Parabolic variational inqualities, Finite element scheme, HJB-equation, Multigrid techinc.

Abstract in French

Dans ce travail, nous avons étutidée certains classe des inéquations quasi-variationnelles paraboliques où le seconde membre dépandent de la solution linéairement et non linéairement en appliquant les méthodes multigrilles. Aprés l'étude mathématiquement du problème continue, pour avoir le problème discret nous avons utilisée la méthode de déffirences finies et la méthode des éléments finis, finalement nous avons obtenus la convergence uniforme des méthodes multigrilles pour notre problèmes, il est important de noter notre problèmes sont équivalent de l'équation HJB. Dans la partie appliquée, nous avons donné des applications numériques pour chaque problème étudié, à travers laquelle nous avons confirmé la validité des résultats obtenus dans la partie théorique.

Mots clés: Inégalité quasi-variationnelle, Méthodes des éléments finis, Equation HJB, Méthode multigrilles.

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Introduction

The multigrid method is one a numerical methods which used to approximate solutions of partial differential equations, by iteratively improving the solution through coarse meshes. This method is based on a mesh algorithm where the approximate solution is iteratively improved in complex relaxation with correction of defects. Historically, the final form of the multigrid method has been appeared in the 1970s, exactly, in Bernad's work see [19], but the first main idea of the multigrid method has been began in the Fedorenko's research and Backvalov's articles in the seventh decade of last century, references can be returned [2, 30]. In the following table, we summarize a history of the development of the multigrid

year	1964	1966	1971	1972	1973	1975	1977	1979	1981	1982	1983	1984	1985
Num	1	1	1	1	1	1	11	22	31	70	93	94	149

Table 1: Published articles about multigrid method.

To learn more about the history of the multigrid method we recommend reading the following [5, 53]. An excellent work has been presented in the last fifth decade on application of the multigrid method to find the approximate solution of the partial differential equations, such as Poisson's equation. The main destination of the multigrid method is finding the approximate solution of the partial differential equations with a few number of iterations and on a faster time than other iterative methods like the relaxation method, recently, multigrid methods has been developed to solve the elliptic variational and quasi varitional inequalities, for example, in article [6]: The elliptic quasi variational inequality was studied with the solution is dependent on the right hand side, and then the multigrid method was applied to solve the approximate solution of the inequality, the study gave very satisfactory results, when we compare it with the other numerical methods like Gauss Seidel method [4, 6, 46], but as far as we know in this subject, this method has not been applied to evolutionary inequalities as the parabolic quasi variational inqualities , wherefore, in this thesis we will study the multigrid method and its application on the parabolic quasi variational inequalities. There are two main reasons for choosing the parabolic quasi variational inequalities for applying it on the multigrid method:

• The scarcity of scientific research then in applying the multigrid method on the parabolic quasi variational inequalities because the most researchs has focused on the partial differential equation and elliptic varitional and quasi variational inequality [5, 34, 35, 38, 45, 46].
• The extreme importance of the parabolic quasi-variational inequalities in the field of physics and economics because many physical and economic phenomena can be transformed into parabolic quasi variational inequalities, for example, the problem of parabolic quasi variational inequalities related to impulse control problems. In this work, we will choose four different types of the parabolic quasi varitional inequalities in order to apply the multigrid method for them the four types that are, the firt type: parabolic quasi variational inequality with the operator noncoercive[18], the second type: parabolic quasi varitional inequality with the obstacle dependent the solution[7], the third type: parabolic quasi variational inequality non linear with the obstacle dependent the solution[16], the fouth type: parabolic quasi variational inequality non linear [16].

The main objective of this thesis is studying the approximations of multigrid method, then we applyed this method on the four different types of parabolic quasi variational inequalities mentioned above. We started by defining the continuous problem of the parabolic quasi variational inequalities, after that, we gave the theorem that shows the existence and uniqueness of the solution [7, 42] for obtain the discrete problem, we used the finite element method P1 on the operator and the theta time scheme and we also gave the theorem that proves the existence and o uniqueness of the discrete solution[12, 18, 7]. To find an approximate solution of the parabolic quasi variational inequatities by using the multigrid method, we defined the HJB equation in the discrete case, becuase there is an equivalence between the parabolic quasi variational inqualities and the HJB equation [9, 10, 39, 43]. We also used Hoppe's method for obtain a linear system[38], to solve the linear system we emploeyed a multigrid method and in the case a non-linear system we applyed Newton's method to obtain a linear system with a Jacobian matrix. Finally, we took a numerical application on each type of the parabolic quasi variational inequalities studied previously, through which we confirmed the results obtained in the theoretical part. The chapters of this thesis can be arranged as follows:
In the chapter zero, we presented the important definitions and symbols that we needed in this research. The first chapter, we explained the four types of the parabolic quasi variational inequalities, with mention an existence and an uniqueness of the solution in the continuous problem, while the discrete problem we defined V_h: the space of finite elements method by using P1 $\big($with P1 is a Lagrange polynomial with a degree $\leqslant 1$ and $V_h \subset H_0^1\big)$, we employed the finite element method with Euler scheme for obtaining the disecrete problem, then, we intreduced the theorem of error estimate between the continuous problem and the discrete problem on uniform norm, where, we did this work on the four types of parabolic quasi variational inequalities, for more explanation you can read and understand the literatures [8, 11, 23, 24, 32].

In the second chapter, before giving a detailed explanation of the multigrid method, we briefly explain the relaxation method and the Newton's method[8, 11], because we need both in our work. We introduced a detailed presentation of the twogrid method, the two grid method goes by three basic steps, step 1: We performed 4 or 5 iterations in the relaxation method, then, we calculated the residus which represents the difference between the two sides of the equation, step 2: We define two operators an operator of prolongation which takes us from coarse mesch to fine mesch and an operator of restriction which returns us from fine mesch to coarse mesch[20, 21, 22, 33], after that, we applyed the multigrid method on Elliptic quasi variational inequalities ,then , we study the uniform convergence of the multigrid method for Eliptic quasi variational inequalities [48, 49, 50]. At the end of the chapter, we provide an applied example of the multigrid method.

Here in the chapter 3, we took a parabolic variational inequality with the opertor non coercive. We know very well, there is an equivalence between the parabolic quasi variational inequalities and the HJB equation [38], in addition, we have proven that equivalence in this chapter then, we trasformeted the parabolic quasi variational inequalities to the HJB equation, where, we invested the method of Hoppe that he adopted to obtain a linear equation equivalent to the studied inequality, before applying the multigrid method to the linear equation, we gave the resulting interations matrix of the mutigrid method, and we also gave all the properties of the uniform convergence, finally we took a numerical application compatible with the data of the theoretical part, and we solved the applied using the multigrid method and relaxation method using the MATLAB program 2018 .
In the chapter 4, we applied the multigrid method (V-cycle and W-cycle) on the parabolic quasi variational inequalities, where, the obstacale is related to the solution and the operator is a coercive. To obtain the resulting equation from the HJB equation, we treated the HJB equation by following the same steps in the Hoppe's method see [38], then we studied the convergence on the multigrid method . As usual, at the end of each chapter, we took a numerical example, and applied two numerical methods to it: the multigrid method (V-cycle, W-cycle) and Gauss Seidel method, we noticed that the multigrid method is more effective and efficient when we compared it to the Gauss Seidel method.

We divided the fifth chapter into two parts, in the first part, we treated the parabolic quasi variational inequalities non linear with the obstacle related by the solution . Using the previous Hoppe's method in [38] we obtained a nonlinear system, so , we must applyed Newton's method to get a linear system with a Jacobian matrix (this method called the linearisation), we made sure that the Jacobian matrix is M-matrix, after that we solved the linear system with the Jacobian matrix using the V-cycle, W- cycle and relaxation method, the seconde part, we followed the same method in the first part, but applied it to the parabolic quasi variational inequalities with the obstacle independent the solotion and gave a numerical example using the multi-grid method. finally we presented a summary of the results obtained in this study, and we also talked about the future work that we will do in this field.

Symbols and basic tools

Before delving into the subject of the thesis, we present some symbols and definitions that we need them in this study.

0.1 Symbols

1. $P.Q.V.I$: Parabolic Quasi Variational Inequalities.
2. $E.Q.V.I$: Elliptic Quasi Variational Inequalities.
3. A^{-1} : The inverse of matrix A.
4. A^t : The transpose of matrix A.
5. I : A saquare identite matrix.
6. P_k : An operator of prolongation.
7. R_k : An operator of restrection.
8. $\rho(A)$: Spectracl redius of matrix$A$.
9. $|.|$: Absolute value.
10. $\|.\|$: The vector norm .
11. $\mathcal{J}$: The Jacobian matrix.
12. φ and Φ : The obstacles.
13. r : Restriction operator.
14. HJB equation : Hamilton-Jacobi-Bellman equation.
15. ∇ : A gradient.
16. $\triangle$: Laplacian.

17. Ω : Smooth boundary domain in $\mathbb{R}^d$, $d \geq 1$.
18. $\triangle x$: The step in the direction x.
19. δt : Step of the time.

0.2 Definitions

Let Ω and Γ be a smooth boundary domain of $\mathbb{R}^d$ and sufficiently smooth boundary respectively, with V is a real vector space [27, 31].

0.2.1 Hilbert space

Definition 1. *A mapping $c : V \times V \longrightarrow \mathbb{R}$ is called scalar product on V, if and only if it is $c(.,.)$ are belinear form, symetric and positive definite satisfying as follow*

- $(i) \quad \forall (x, y) \in V^2, \quad c(x, y) = c(y, x).$
- $(ii) \quad \forall x \in V^2, \quad c(x, x) \geq 0$ *and* $c(x, x) = 0 \Leftrightarrow x = 0$
- $(iii) \quad \forall x \in V, \quad c(x, .)$ *is linear* , *and* $c(., y)$ *is linear,*

we denote it by :$\langle\ ,\ \rangle$.

Example 1. *The standar scalar productor on $\mathbb{R}^d$ is given by $\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i$*

Definition 2. *The pre-Hilbertien space is Hilbert space if and only if it is a completed space under the normed associated with the scalar product .*

In particular, every Hilbert space is a Banach space with respect to the norm in.

Example 2. *The space $\mathbb{R}^n$ with respect to the scalar product is a Hilbert space.*

Definition 3. *Consider H a separable Hilbert space of infinite dimension. A Hilbert basis or orthonormal basis is a sequence $(e_n)_{n\in\mathbb{N}}$ of elements of H that is complete, and such that:*

$$< e_i, e_j >= \delta_{ij}, \quad \textit{with} \quad \delta_{ij} \quad \textit{the Kronecker symbol}$$

Definition 4. *A Banach space is a normed space that is complete with its norm.*

Example 3. *The space $\mathbb{R}^n$ with respect to the following scalar product$\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i$ is a Hilbert space.*

Definition 5. *The following inquality called the Cauchy−Schwarz inequality*

$$\forall u, v \in V, \quad | < u, v > | \leq < u, u >< v, v >$$

Definition 6. *$L^p(\Omega)$ is a Banach space,*

$$L^p(\Omega) = \left\{ g : \Omega \longrightarrow \mathbb{R}^d \quad mesurable \quad and \quad \int_\Omega |g|^p dx < +\infty \right\}$$

where , $1 \leqslant p < +\infty$ and its norm given by

$$\|g\|_{L^p(\Omega)} = \left(\int_\Omega |g|^p dx \right)^{\frac{1}{p}}$$

In particular

- *$p = 2$ the space $L^2(\Omega)$ is a Hilbert space*
- *$p = +\infty$*

$$L^\infty(\Omega) = \left\{ g : \Omega \longrightarrow \mathbb{R}^d \quad mesurable \quad and \quad \exists C > 0 \quad where \quad |g(x)| < C \quad p.p \quad on \quad \Omega \right\}$$

and its norm defined by

$$\|g\|_{L^\infty(\Omega)} = inf \left\{ C, \quad \text{where} \quad |g(x)| < C \quad p.p \quad on \quad \Omega \right\}$$

0.2.2 Sobolev space

Definition 7. *We call the space $H^1(\Omega)$ a Sobolev space of degree 1, we can express it as follow*

$$H^1(\Omega) = \left\{ u \in L^2(\Omega), \quad \partial_{x_i} u \in L^2(\Omega), \quad 1 \leqslant i \leqslant d \right\}$$

the space $H^1(\Omega)$ associated by a product scalar and a norm respectevly are

$$< u, v >_{H^1} = \int_\Omega \left[uv + \sum_{1 \leqslant i \leqslant d} \partial_{x_i} u \partial_{x_i} v \right] dx$$

and its norm is

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \sum_{1 \leqslant i \leqslant d} \|\partial_{x_i} u\|_{L^2(\Omega)}$$

Definition 8. *We can define the Sobolev space where, $1 \leqslant p \leqslant +\infty$, for even, $m \in \mathbb{N}^\star$ as*

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega), \quad \partial^\alpha u \in L^p(\Omega), \quad \forall \alpha \in \mathbb{N}^\star, | \alpha | \leqslant m \right\}$$

the norm associated from Sobelov space given by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leqslant m} \int_\Omega |\partial^\alpha u|^p dx \right)^{\frac{1}{p}}$$

All Sobolev space is Banach space, in particular,

- If $p = 2$, we have, $W^{m,2}(\Omega) = H^m(\Omega)$
- If $m = 0$, we have, $W^{0,p}(\Omega) = L^p(\Omega)$

0.3 Matrix generalities

The properties below are a group of important definitions on matrix , consider $M = (a_{ij})_{ij\in\mathbb{N}} \in \mathbb{R}$ is a sequare matrix .

1. **M-matrix**: the matrix M called M-matrix if M^{-1} exists, in addition to

$$a_{ji} \leq 0 \quad and \quad a_{ii} > 0 \quad (i \neq j).$$

2. **Triangular matrix** is a square matrix in which elements below and/or above the diagonal are all zeros

$$a_{ij} = \begin{cases} 0 & if \quad i > j \\ \neq 0 & if \quad i \leqslant o \end{cases} \tag{1}$$

3. **Tridiagonal matrix** square matrix with nonzero elements only on the diagonal and slots horizontally or vertically adjacent the diagonal (i.e. along the subdiagonal and superdiagonal),

5. **Regular matrix** we called M is a regular marix if M^{-1}exist, (M^{-1} is the inverse matrix).

6. **Monotone matrix** the matrix M is monotone if M^{-1} exists and all the coefficients of are positive or nuls .

7. **Positive definite matrix** we call quadratic form associated with the matrix M the folowing form

$$q(x) = xMx^t$$

If $xMx^t > 0$, we called M is monotone matrix.

9. **Diagonally dominant matrix** the matrix M is said to be diagonally dominant if

$$|a_{ii}| \geq \sum_{|i=1,\dots,N} |a_{ij}|, \quad i \neq j.$$

Chapter 1

Generalities of parabolic quasi variational inequalities

We organize this chapter as follows: the first part, we will presnt two important numerical methods: a finite element method and a finite difference method, we will use them in all work, in the second part, we will offer four different types of Parabolic Quasi Variational Inqualities :

- P.V.I with the operator is noncoercive .
- P.Q.V.I with the obstacle related of the solution .
- P.Q.V.I with the right hand side and the obstacle both related to the solution .
- P.Q.V.I with the right hand side non linear .

1.1 Finite element method

A finite element scheme is one of numerical methods that it have been used for soloving the aprroximation solution for study the problems in this work. The basic steps involved in the FEM as: discretization the domain, derivative, assembly and the approximate solution of the equation, see[31].

1.1.1 Triangulation

Definition 9. *(Mesch), Let* $\Omega \subset \mathbb{R}^2$ *we called a mesh of* Ω *all* T_h *a partition of* Ω *into subintervals* k_i *where* $T_h = \{k_i, i = 1 ... N_h\}$*, we also called* T_h *triangulation and* k_i *elemmet of him, and the element diameters via is :*

$$h_i = diam(k_i), \quad i = 1, ..., N_h \quad \textit{the value} \quad h = max(h_i)$$

h it is called the mesh size and N_h it is called the noeds .

1.1.2 Proprities of the element k_i

The intersection of two elements k and $k^{'}$ verfient the following proprieties:

$$k \cap k^{'} = \begin{cases} \phi, \\ \textit{one sommet}, \\ \textit{one cote}. \end{cases} \tag{1.1}$$

Example 4.

The example below is a discretization for the domain Ω

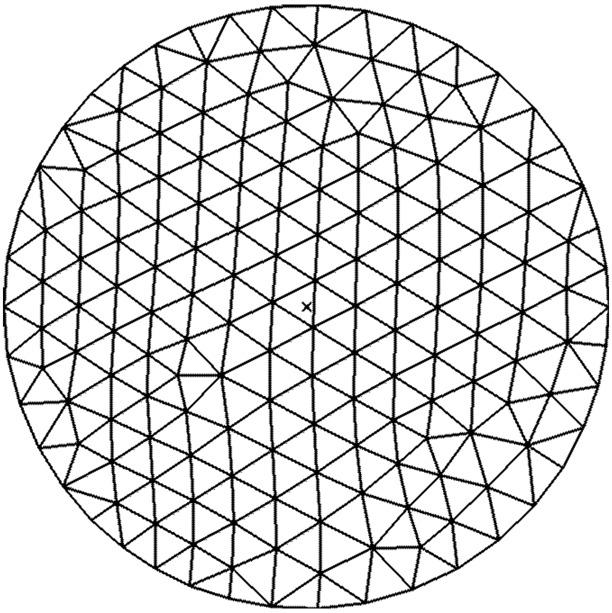


Figure 1.1: Triangulation of the domain Ω

1.1.3 The finite element space

X_h^1 is the functions continuous spaces afines of triangulations elements i.e

$$X_h^1 = \left\{ v_h \in C(\Omega),\ v_h|_k \in P_1,\ \text{for all } k \in T_h \right\}$$

P_1 represents the Lagrange polynomial of degree less than or equal to 1.

Theorem 1. *Let $V_h = X_h^1$, V_h is a subspace of $H_0^1(\Omega)$.*

1.2 Finite difference method

Here, we show the main principe of finite difference method, because us we need it in our work, in the beginning, the finite difference methods are one of numericals method that used to find the solution approximation, it depends to replace the partial derivatives by feffirence divises thant combinaisoons of ponctuells of the function on descrets points of finite number or noudes of mesh [28]. Discretze on the time, we use the Euler scheme as

$$\partial u_t = \frac{u^n - u^{n-1}}{\delta t} \quad ; \quad n = 0, 1, 2, ..., m+1 \quad ; \quad \delta t = \frac{T}{m+1} \quad ; \quad t_n = n\delta t$$

To discretze on the space , we use the θ-scheme as follows

$$\triangle u = \partial^2 u_{xx} + \partial^2 u_{yy}$$

$$\partial^2 u_{xx} = \left[\theta\left(\frac{u^{n+1}_{i+1,j} - 2u^{n+1}_{i,j} + u^{n+1}_{i-1,j}}{h^2}\right) + (1-\theta)\left(\frac{u^{n}_{i,j+1} - 2u^{n}_{i,j} + u^{n}_{i,j-1}}{h^2}\right)\right]$$

$$\partial^2 u_{yy} = \left[\theta\left(\frac{u^{n+1}_{i+1,j} - 2u^{n+1}_{i,j} + u^{n+1}_{i-1,j}}{h^2}\right) + (1-\theta)\left(\frac{u^{n}_{i,j+1} - 2u^{n}_{i,j} + u^{n}_{i,j-1}}{h^2}\right)\right]$$

where

$$\triangle x = \triangle y = h \quad ; \quad 0 \le \theta \le 1$$

$$x_i = ih \quad ; \quad y_j = jh \quad \forall i, j \in \{0, 1, ..., N+1\}$$

Remark 1

- $\theta = 0$ gives the Forward Euler scheme.
- $\theta = 1$ gives the Backward Euler scheme.
- $\theta = \frac{1}{2}$ gives the Crank Nicolson scheme .

1.3 P.V.I with the operator noncoercive

1.3.1 Continuous problem

The following problem is called as: parabolic variational inquality with the operator noncoercive given by, find u $\in L^2(H_0^1(\Omega);[0,T])$:

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial t} + \mathfrak{L}\mathrm{u} \leq f \quad ; \quad \mathrm{u} - \varphi \leq 0, \\ \\ \left(\dfrac{\partial \mathrm{u}}{\partial t} + \mathfrak{L}\mathrm{u} - f\right)(\mathrm{u} - \varphi) = 0 \ \text{ in } Q_t, \\ \\ \mathrm{u}(x,0) = \mathrm{u}_0 \qquad in \quad \Omega, \\ \\ \mathrm{u}(x,t) = 0 \qquad in \quad \sum, \end{cases} \tag{1.2}$$

where Q_T is defined as $Q_T = \Omega\times[0,T]$, $\Sigma = \Gamma\times[0,T]$, $T < \infty$, and φ realization as $\varphi \in W^{2,\infty}(\Omega)$ such that $\varphi \geq 0$. Let $\mathfrak{L}$ an operator noncoercive defined by:

$$\mathfrak{L} = -\sum_{1\leq i,j\leq d}^{d} \ell_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} \ell_j(x)\frac{\partial}{\partial x_j} + \ell_0(x). \tag{1.3}$$

The coefficients $\ell_{ij}(x),\ \ell_j(x),\ \ell_0(x) \in L^\infty(\Omega) \cap C^2(\bar{\Omega})$ accept the conditions below

$$\ell_{ij}(x) = \ell_{ji}(x), \quad \ell_0(x) \geq \beta > 0, \quad \beta\, is\ a\ constant. \quad x \in \bar{\Omega}, \tag{1.4}$$

$$\sum_{1\leq i,j\leq d}^{d} \ell_{ij}(x)\zeta_i\zeta_j \geq \gamma|\zeta|^2; \quad \forall \zeta \in \mathbb{R}^d \quad \gamma > 0.$$

$\ell(.,.)$ is associated with the operator $\mathfrak{L}$, and $\ell(.,.)$ is a continuous and noncoercive bilinear form, where

$$\ell(\mathrm{u},\mathrm{v}) = \int_\Omega \left[\sum_{1\leq i,j\leq d}^{d} \ell_{ij}(x)\frac{\partial \mathrm{u}}{\partial x_i}\frac{\partial \mathrm{v}}{d x_j} + \sum_{j=1}^{d} \ell_j(x)\frac{\partial \mathrm{u}}{\partial x_j} + \ell_0(x)\mathrm{u}\mathrm{v} \right] dx. \tag{1.5}$$

$$\ell(\mathrm{v},\mathrm{v}) + \lambda\|\mathrm{v}\|^2_{L^2(\Omega)} \geq \gamma\|\mathrm{v}\|^2_{H_0^1(\Omega)}, \quad \gamma > 0 \quad , \quad \lambda > 0, \quad \forall \mathrm{v} \in H_0^1(\Omega) \tag{1.6}$$

in the problem (1.7), we choose λ is sufficiently large

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial t} + \mathfrak{L}\mathrm{u} + \lambda \mathrm{u} \leq f + \lambda \mathrm{u} \quad ; \quad \mathrm{u} - \varphi \leq 0, \\ \left(\dfrac{\partial \mathrm{u}}{\partial t} + \mathfrak{L}\mathrm{u} - f \right) (\mathrm{u} - \varphi) = 0 \ \text{ in } \mathbf{Q}_t, \\ \mathrm{u}(x,0) = \mathrm{u}_0 \qquad in \quad \Omega, \\ \mathrm{u}(x,t) = 0 \qquad in \quad \Sigma, \end{cases} \tag{1.7}$$

clearly, the problem (1.7) equivalent the problem (1.8)

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial t} + \mathfrak{C}\mathrm{u} \leq F \quad ; \quad \mathrm{u} - \varphi \leq 0, \\ \left(\dfrac{\partial \mathrm{u}}{\partial t} + \mathfrak{C}\mathrm{u} - F \right) (\mathrm{u} - \varphi) = 0 \ \text{ in } Q_t, \\ \mathrm{u}(x,t) = \mathrm{u}_0 \qquad in \quad \Omega, \\ \mathrm{u}(x,t) = 0 \qquad in \quad \Sigma, \end{cases} \tag{1.8}$$

so that $\mathfrak{C} = \mathfrak{L} + \lambda I$ are strongly coercive on $H^1(\Omega)$ (see [25]).
Additionally, we consider f a second member as following: $f \in L^\infty(\Omega) \cap C^1(\Omega), \quad f \geq 0$.
After applying Green's formulation on the problem (1.8) , we obtain the weak formulation of our proposed problem, v in $H_0^1(\Omega)$ and $\mathrm{v} \leqslant \varphi$, find u in $L^2(H_0^1(\Omega); [0,T])$ a solution to the following:

$$\begin{cases} \left(\dfrac{\partial \mathrm{u}}{\partial t}, \mathrm{v} - \mathrm{u} \right) + \mathrm{c}(\mathrm{u}, \mathrm{v} - \mathrm{u}) \geq (F, \mathrm{v} - \mathrm{u}), \\ \mathrm{u} \leq \varphi, \\ \mathrm{u}(x,t) = \mathrm{u}_0 \qquad in \quad \Omega, \\ \mathrm{u}(x,t) = 0 \quad in \quad \Sigma, \end{cases} \tag{1.9}$$

we put

$$\mathrm{c}(\mathrm{u}, \mathrm{v} - \mathrm{u}) = \ell(\mathrm{u}, \mathrm{v} - \mathrm{u}) + \lambda(\mathrm{u}, \mathrm{v} - \mathrm{u}) \quad \text{and} \quad F = f + \lambda u$$

The theorem below give us the existence and uniqueness of solution for the problem $\left(1.9\right)$

Theorem 2. [7, 16] *Under all the previous assumptions and hypotheses , the problem* $\left(1.9\right)$ *has an unique solution* u *in* $L^2(H_0^1(\Omega);[0,T])$.

1.3.2 Discrete problem

Now, we apply the finite element scheme on the space and the finite difference scheme on the time $\left(\text{Euler scheme}\right)$, $\{h_k\}_{k=0}^{m}$ mesh size paramater and k is a level mesh in the multigrid method such that

$$0 < h_k < h_{k+1} < 1, \quad 0 < k \leq m, \quad m \in \mathbb{N}^\star.$$

Let $\{\Upsilon_h\}$ is a family of uniform triangulations where $\Omega_h = \bigcup\limits_{Y\in\Upsilon_h} \Upsilon$, and for all Υ, we have:

$$\Omega_h \subset \Omega_{h+1} \subset \Omega, \quad dist(\Gamma_h,\Gamma) \leq ch_k^2, \quad h_{k+1}h_k \leq c_1.$$

We associate each h_k with the following symbols to facilitate our work

$$\Omega_{h_k} = \Omega_h \quad , \quad \mathfrak{L}_{h_k} = \mathfrak{L}_h.$$

We consider ϕ_h^i, $i = 1,2,...,m(h)$ as the conventional basis for affine functions defined as follows:

$$\phi_h^i(K_h^j) = \delta_{ij},$$

K_h^j denote a vertex of the triangulation Υ_h. Let $\mathfrak{U}_h = R^{mh}$, and V_h denote the discrete space of FEM $\left(\text{the order on } V_h \text{ will be that induced by } \mathbb{R}^{m(h)}\right)$, as given by

$$V_h = \{v_h \in L^2(H_0^1(\Omega);[0,T]) \cap C^1(H_0^1(\overline{\Omega});[0,T])\, v_h(.,0) = v_{h_0} \ \in \ \Omega \,|\, v_{h|\Omega_h} \in P_1,\ v_h \leq \varphi v_h\}$$

we consider the restriction operator defined as:

$$r_h : \mathfrak{U}_h \longrightarrow V_h,$$

$$r_h\mathrm{v}(x) = \sum_{i=1}^{m(h)} V(K_h^i)\phi_h^i(x), \tag{1.10}$$

we provide a definition for the following adjoint operator $r_h^* : V_h \longrightarrow \mathfrak{U}_h$ satisfies :

$$\langle r_h\mathrm{u},\mathrm{u}\rangle_{L^2} = \left\langle \mathrm{u}, r_h^*\mathrm{v}\right\rangle, \forall \mathrm{u} \in \mathfrak{U}_h, \mathrm{v} \in V_h,$$

the norm $\|.\|_\infty$ $\left(\text{on } \mathfrak{U}_k\right)$ and the norm $\|.\|_{L^\infty}$ $\left(\text{on } V_k\right)$ are equivalent, which are indicated by $\|.\|_\infty$.

To discretize problem $\left(1.9\right)$ with respect to time, we use the semi-implicit Euler scheme , subsequently we are looking for a series of items $\mathrm{u}_h^n \in V_h$ that approaches $\mathrm{u}(t_n)$, $t_n = n\delta t$, with initial data u_0 and $\dfrac{\partial \mathrm{u}_h}{\partial t} = \frac{\mathrm{u}_h^n - \mathrm{u}_h^{n-1}}{\delta t}$, thus, we have for $n = 1, \ldots, N$, and $\forall \mathrm{v}_h \in V_h$

$$\begin{cases} (\dfrac{\mathrm{u}_h^n - \mathrm{u}_h^{n-1}}{\delta t}, \mathrm{v}_h - \mathrm{u}_h^n) + \mathrm{c}(\mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n) \geq (F_h^n, \mathrm{v}_h - \mathrm{u}_h^n), \\ \mathrm{u}_h^n \leq r_h\varphi; \quad \mathrm{v}_h^n \leq r_h\varphi \end{cases} \tag{1.11}$$

we can write the problem (1.11) as: $\forall \mathrm{v}_h \in V_h$

$$\begin{cases} \mathrm{c}(\mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n) + \dfrac{1}{\delta t}(\mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n) \geq (F_h^n + \dfrac{\mathrm{u}_h^{n-1}}{\delta t}, \mathrm{v}_h - \mathrm{u}_h^n), \\ \mathrm{u}_h^n \leq r_h\varphi; \quad \mathrm{v}_h^n \leq r_h\varphi \end{cases} \tag{1.12}$$

when we put $\xi = \dfrac{1}{\delta t}$ and $F_h^{n-1} = F_h^n + \omega \mathrm{u}_h^{n-1}$, we obtain

$$\begin{cases} \xi(\mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n) + \mathrm{c}(\mathrm{u}_h^n, \mathrm{v}_h - u_h^n) \geq (F_h^{n-1}, \mathrm{v}_h - \mathrm{u}_h^n), \\ \\ \mathrm{u}_h^n \leq r_h\varphi. \end{cases} \tag{1.13}$$

We can express the problem (1.13) in the following form, find $\mathrm{u}_h^n \in V_h, \forall \mathrm{v}_h \in V_h$

$$\begin{cases} \mathrm{d}(\mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n) \geq (F_h^{n-1}, \mathrm{v}_h - \mathrm{u}_h^n), \\ \\ \mathrm{u}_h^n \leq r_h\varphi, \\ \\ \mathrm{u}_h(x,0) = \mathrm{u}_h(x) \qquad in \quad \Omega_h, \\ \\ \mathrm{u}(x,t) = 0 \quad in \quad \Sigma_h, \end{cases} \tag{1.14}$$

where

$$\mathrm{d}(\mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n) = \xi(\mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n) + \mathrm{c}(\mathrm{u}_h^n, \mathrm{u}_h - \mathrm{u}_h^n).$$

The bilinear form $\mathrm{d}(.,.)$ exhibits strong coerciveness. Consequently, the formulation (1.14) denotes a coercive and continuous issue concerning elliptic variational inequalities (see[45]). Denote by $\mathcal{D}_h$ the outcome obtained by addressing problem (1.14) through a finite element method, resulting in the solution to the subsequent problem ascertain $\mathrm{u}_h^n \in V_h$ such that:

$$\begin{cases} \left\langle \mathcal{D}_h \mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n \right\rangle \ \geq \ \left\langle F_h^{n-1}, \mathrm{v}_h - \mathrm{u}_h^n \right\rangle, \\ \\ \mathrm{u}_h^n \leq r_h \varphi, \\ \\ \mathrm{u}_h(x) = \mathrm{u}_h(x,0) \qquad in \quad \Omega_h, \\ \\ \mathrm{u}(x,t) = 0 \quad in \quad \Sigma_h, \end{cases} \tag{1.15}$$

where $(\mathcal{D}_h)_{i,j}$ denotes the finite element matrix defined by:

$$(\mathcal{D}_h)_{i,j} = \mathrm{d}(\phi_i, \phi_j) = \mathrm{c}(\phi_i, \phi_j) + \xi(\phi_i, \phi_j)$$

and $\{\phi_i\}$ is basis of V_h, $\quad 1 \leq i, 1 \leq j$. The discretization matrices $\mathcal{D}_h$ and the generic coefficient matrices $\mathrm{c}(\phi_h^i, \phi_h^j)$ are introduced in a natural progression, where the customary basic functions are denoted by $\phi_h = 1, 2, \ldots, m(h_h)$. Now that these definitions have been established, we may state the discrete problem as follows, determine $\mathrm{u}_h^n \in V_h$, representing the solution for

$$\begin{cases} \left\langle \mathcal{D}_h \mathrm{u}_h^n, \mathrm{v}_h - \mathrm{u}_h^n \right\rangle \ \geq \ \left\langle F_h^{n-1}, \mathrm{v}_h - \mathrm{u}_h^n \right\rangle, \\ \\ \mathrm{u}_h^n \leq r_h \varphi, \\ \\ \mathrm{u}_h(x) = \mathrm{u}_h(x,0) \qquad in \quad \Omega_h, \\ \\ \mathrm{u}(x,t) = 0 \quad in \quad \Sigma_h, \end{cases} \tag{1.16}$$

the matrix $(\mathcal{D}_h)$ with coefficients $\mathrm{d}(\phi_i, \phi_j)$ are an M-matrix (see[23, 44]).

Theorem 3. [7, 15] *The problem (1.16) has a unique solution (as is well know).*

The following theorem proves the L^∞ convergence of the discrete solution u_h^N to the continuous solution u^∞, (the estimation result) .

Theorem 4. [15] *Let u^∞ and u_h^N be the solutions of (1.9) and (1.16), respectively*

$$\|\mathrm{u}_h^N - \mathrm{u}^\infty\|_\infty \leq C \left[h^2 |log(h)|^2 + \left(\frac{1}{1+\beta\delta t} \right)^N \right] \tag{1.17}$$

The asymptotic solution of the problem (1.9) is denoted by u^{∞}, the discrete solution calculated at the moment $T = n\delta t$ is represented by u_h^n for more details please refer to [16, 23].

1.4 P.Q.V.I with the obstacle related of the solution

1.4.1 Continuous problem

The problem below is P.Q.V.I with the obstacle dependent of the solution called : P.Q.V.I arising from stochastic inventory problems with impulse control see [7],
find $u \in \mathcal{V}, (\mathcal{V} = L^2(H_0^1(\Omega); [0,T]))$:

$$\begin{cases} (\dfrac{\partial u}{\partial t} + Au \leq f) \quad , \quad (u \leq Mu) \quad \text{in } \Omega \times [0,T] \\[2ex] \left(\dfrac{\partial u}{\partial t} + Au - f\right)(u - Mu) = 0 \;\; \text{in } \Omega \times [0,T] \\[2ex] u(x,0) = u_0 \;\; \text{in } \Omega, \\[2ex] u(0,t) = 0 \;\; \text{in } \Gamma \times [0,T], \end{cases} \tag{1.18}$$

we consider the elliptic linear operator A given by

$$A = \sum_{i,j=1}^{d} \mathrm{a}_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} \mathrm{a}_j(x)\frac{\partial}{\partial x_j} + \mathrm{a}_0(x), \quad 1 \leq i,j \leq d, \tag{1.19}$$

the functions $\mathrm{a}_{ij}(x),\ \mathrm{a}_j(x),\ \mathrm{a}_0(x) \in L^{\infty}(\Omega) \cap C^2(\bar{\Omega})$ meet the following conditions:

$$\mathrm{a}_{ij}(x) = \mathrm{a}_{ji}(x), \quad \mathrm{a}_0(x) \geq \beta > 0, \quad \beta \, \textit{is a constant} \quad x \in \bar{\Omega}, \tag{1.20}$$

$$\sum_{1 \leq i,j \leq d}^{d} \mathrm{a}_{ij}(x)\zeta_i\zeta_j \geq \gamma|\zeta|^2; \quad \zeta \in \mathbb{R}^d \quad \gamma > 0,$$

and $\mathrm{a}(.,.)$ is resulted with the operator A, where $\mathrm{a}(.,.)$ is a continuous and coercive bilinear form, as defined by

$$\mathrm{a}(\mathrm{u},\mathrm{v}) = \int_{\Omega} \left[\sum_{1 \leq i,j \leq d}^{d} \mathrm{a}_{ij}(x)\frac{\partial \mathrm{u}}{\partial x_i}\frac{\partial \mathrm{v}}{dx_j} + \sum_{j=1}^{d} \mathrm{a}_j(x)\frac{\partial \mathrm{u}}{\partial x_j} + \mathrm{a}_0(x)\mathrm{u}\mathrm{v} \right] dx, \tag{1.21}$$

$$\mathrm{a}(\mathrm{v},\mathrm{v}) + \omega\|\mathrm{v}\|^2_{L^2(\Omega)} \geq \gamma\|\mathrm{v}\|^2_{H_0^1(\Omega)}, \quad \gamma > 0 \quad , \quad \omega > 0, \quad \forall \mathrm{v} \in H_0^1(\Omega), \tag{1.22}$$

we put f a right hand side as following

$$f \in L^{\infty}(\Omega) \cap C^{1}(\Omega) \quad , \quad f \geq 0 \tag{1.23}$$

the obstacle of impulse control M given by

$$M : \mathcal{V} \cap L^{\infty}(H_0^1(\Omega); [0,T]) \longrightarrow \mathcal{V} \cap L^{\infty}((H_0^1(\Omega); [0,T]),$$

$$Mu = \sigma + \inf_{x+\zeta \in \bar{\Omega}, \zeta \geq 0} u(x+\zeta), \quad \sigma > 0. \tag{1.24}$$

Lemma 1. [3] *The operator M satsfies*

$$M(v+\psi) = M(v) + \psi, \quad \textit{for all } \psi \in \mathbb{R},$$

$$v \leq w \Rightarrow M(v) \leqslant M(w),$$

$$\|M(v) - M(w)\|_{\infty} \leqslant \|v - w\|_{\infty} \textit{ for all } v, w \in L^{\infty}(\Omega),$$

We employ the Green's formulation as the problem (1.18), we get the following weak formulation , for all $v \in L^2(H_0^1(\Omega); [0,T])$, find $u \in L^2(H_0^1(\Omega); [0,T])$ such that

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v - u\right), + \mathsf{a}(u, v-u) \geq (f, v-u) \ \text{ in } Q_T, \\ \\ u \leq Mu \ \text{ in } Q_T, \\ \\ u(x,0) = u_0 \ \text{ in } \Omega, \\ \\ u(x,t) = 0 \text{ in } \Sigma. \end{cases} \tag{1.25}$$

The existence and uniqueness of the problem (1.25) get it in the theorem following

Theorem 5. [7] *Under all the previous assumptions and hypotheses , the problem (1.25) has an unique solution u in $L^2(H_0^1(\Omega); [0,T])$.*

1.4.2 Discrete problem

Let V_h denote the discrete space of FEM as given by

$$V_h = \{v_h \in L^2(H_0^1(\Omega); [0,T]) \cap C^1(H_0^1(\overline{\Omega}); [0,T]) \, v_h(.,0) = v_{h_0} \ \in \ \Omega \,|\, v_{h|\Omega_h} \in P_1, \ v_h \leq Mv_h\}$$

we consider φ_h^i as the usual basis of affine functions defined as

$$\varphi_h^i(y_h^j) = \delta_{ij}, \quad i = 1, 2, ..., m(h)$$

where y_h^j denote a vertex of the triangulation T_h. Let $U_h = \mathbb{R}^{m_h}$, we consider the restriction operator defined as

$$r_h : U_h \longrightarrow V_h$$

$$r_h v(y) = \sum_{i=1}^{m(h)} v(y_h^i)\varphi_h^i(y) \tag{1.26}$$

we give the definition of the following adjoint operator

$$r_h^* : V_h \longrightarrow U_h \quad \text{which satisfies}$$

$$\langle r_h u, v\rangle_{L^2} = \left\langle u, r_h^* v\right\rangle, \ \forall u \in U_h, \ v \in V_h$$

Now, we discretize the following problem to find $u_h \in V_h$

$$\begin{cases} \dfrac{\partial u_h}{\partial t} + a(u_h, v_h - u_h) \geq (f_h, v_h - u_h), \ \forall v_h \in V_h \\ \\ u_h \leq M_h u_h \\ \\ u_{h_0}(x) = u_h(x, 0) \\ \\ u(0, t) = 0, \end{cases} \tag{1.27}$$

for any $\theta \in]0, 1]$, and for the discretization of problem (1.27), we employ the θ-time scheme and semi-discretization, where

$$\frac{\partial u_h}{\partial t} = \frac{u_h^n - u_h^{n-1}}{\delta t} \quad , \quad T = n\delta t$$

$$u_h^{\theta,n} = \theta u_h^n + (1-\theta)u_h^{n-1}$$

$$f_h^{\theta,n} = \theta f_h^n + (1-\theta)f_h^{n-1}$$

we obtain

$$\begin{cases} \mu(u_h^{\theta,n}, v_h - u_h^{\theta,n}) + a(u_h^{\theta,n}, v_h - u_h^{\theta,n}) \geq (f^{\theta,n} + \mu u_h^{\theta,n-1}, v_h - u_h^{\theta,n}) \\ u_h^{\theta,n} \leq M_h u_h^{\theta,n} \\ \mu = \dfrac{1}{\theta \delta t} \end{cases} \tag{1.28}$$

We can formulate problem (1.28) in the following form as follows, find $u_h^{\theta,n} \in H_0^1(\Omega)$ such that

$$\begin{cases} b(u_h^{\theta,n}, v_h - u_h^{\theta,n}) \geq (F^{\theta,n}, v_h - u_h^{\theta,n}), \quad \forall v_h \in H_0^1(\Omega) \\ u_h^{\theta,n} \leq M_h u_h^{\theta,n} \\ u_{h_0}(x) = u_h(x,0) \\ u(0,t) = 0 \end{cases} \tag{1.29}$$

the problems (1.28) and (1.29) are equivalent, we put

$$F^{\theta,n} = f^{\theta,n} + \mu u^{\theta,n-1}$$

$$M_h u_h = \sigma + \inf_{x+\zeta \in \bar{\Omega}, \zeta \geq 0} u_h(x+\zeta), \ \sigma > 0$$

$$b(u_h^{\theta,n}, v_h - u_h^{\theta,n}) = \mu(u_h^{\theta,n}, v_h - u_h^{\theta,n}) + a(u_h^{\theta,n}, v_h - u_h^{\theta,n})$$

Let $\mathbf{B}_h$ result from solving problem (1.29) using the finite element method, leading to the solution of the following problem, find $u_h^{\theta,n} \in V_h$ such that:

$$\begin{cases} \left\langle \mathbf{B}_h u_h^{\theta,n}, v_h - u_h^{\theta,n} \right\rangle \quad \geq \quad \left\langle F^{\theta,n}, v_h - u_h^{\theta,n} \right\rangle \\ u_h^{\theta,n} \leq M_h u_h^{\theta,n} \\ u_{h_0}(x) = u_h(x,0) \\ u(0,t) = 0 \end{cases} \tag{1.30}$$

where $(\mathbf{B}_h)_{i,j}$ denotes the finite elements matrix defined by

$$(\mathbf{B}_h)_{i,j} = b(\varphi_i, \varphi_j) = a(\varphi_i, \varphi_j) + \mu(\varphi_i, \varphi_j),\ 1 \leq i, j \leq m(h)$$

and $\{\varphi_i\}$ is basis of V_h, $1 \leq i, 1 \leq j$. The matrix $(\mathbf{B}_h)_{i,j}$ whose coefficients $b(\varphi i, \varphi_j)$ are M-matrix entries (as defined in [23]).

Theorem 6. [12, 15] *The problem (1.30) has a unique solution (as is well known).*

The following theorem gives us the approximation of the discrete solution $u_h^{\theta,n}$ to the continuous solution u^∞ .

Theorem 7. [12, 15] *Let u^∞ and $u_h^{\theta,n}$ be the solutions of (1.25) and (1.30) respectively, we have for all $\theta \in]0\ ;\ \frac{1}{2}[$*

$$\|u_h^{\theta,n} - u^\infty\|_\infty \leq C \left[h^2 |log(h)|^2 + \left(\frac{2}{2 + \theta(1-2\theta)\rho(\boldsymbol{B}_h)}\right)^n\right] \tag{1.31}$$

and for all $\theta \in [\frac{1}{2}\ ;\ 1]$

$$\|u_h^{\theta,n} - u^\infty\|_\infty \leq C \left[h^2 |log(h)|^2 + \left(\frac{1}{1 + \beta\theta\delta t}\right)^n\right] \tag{1.32}$$

The asymptotic solution of the problem (1.25) is denoted by u^∞, the discrete solution calculated at the moment $T = n\delta t$ is represented by $u_h^{\theta,n}$ and $\rho(\mathbf{B}_h)$ is the spectral radius of $(\mathbf{B}_h)_{ij}$ where $C > 0$, for more details please refer to [12].

1.5 P.Q.V.I nonlinear with the obstacle related of the solution

1.5.1 Continuous problem

The problem that will be studied in this section is characterized by: the operator J is ceorcive and the right hand side $g(\mathrm{v})$ is non linear, the problem is find $\mathrm{v} \in \mathcal{V}$

$$\begin{cases} \left(\dfrac{\partial \mathrm{v}}{\partial t} + J\mathrm{v} \leq g(\mathrm{v})\right) \quad , \quad (\mathrm{v} \leq \mathcal{M}\mathrm{v}) \quad \text{in } \Omega \times [0,T] \\ \\ \left(\dfrac{\partial \mathrm{v}}{\partial t} + J\mathrm{v} - g(\mathrm{v})\right)(\mathrm{v} - \mathcal{M}\mathrm{v}) = 0 \ \text{ in } \Omega \times [0,T] \\ \\ \mathrm{v}(x,0) = \mathrm{v}_0 \ \text{ in } \Omega, \\ \\ \mathrm{v}(0,t) = 0 \ \text{ in } \Sigma, \end{cases} \tag{1.33}$$

$\Sigma = \Gamma \times [0,T]$ and Ω is an open domain in $\mathbb{R}^d$. Let $\mathcal{M}$ the obstacle of impulse control given

$$\mathcal{M} : \mathcal{V} \cap L^\infty(H_0^1(\Omega); [0,T]) \longrightarrow \mathcal{V} \cap L^\infty((H_0^1(\Omega); [0,T]),$$

$$\mathcal{M}\mathrm{v} = \sigma + \inf_{x+\zeta \in \bar{\Omega}, \zeta \geq 0} \mathrm{v}(x+\zeta), \quad \sigma > 0.$$

From transform the problem (1.33) to the weak formulation , we use the Green's formulation, we get afer that as the following problem, find v in $\mathcal{V}$ a solution to the following

$$\begin{cases} \left(\dfrac{\partial \mathrm{v}}{\partial t}, \mathrm{w} - \mathrm{v}\right) + \mathrm{j}(\mathrm{v}, \mathrm{w} - \mathrm{v}) \geq (g(\mathrm{v}), \mathrm{w} - \mathrm{v}), \\ \\ \mathrm{v} \leq \mathcal{M}\mathrm{v}, \\ \\ \mathrm{v}(x,0) = \mathrm{v}_0 \qquad in \quad \Omega, \\ \\ \mathrm{v}(x,t) = 0 \quad in \quad \Sigma, \end{cases} \tag{1.34}$$

$\mathrm{j}(.,.)$ is a continuous and coercive bilinear form related to the elliptic operator J defined as

$$\mathrm{j}(\mathrm{v}, \mathrm{w}) = \int_\Omega \left(\sum_{i,j=1}^d \mathrm{j}_{ij}(x) \frac{\partial \mathrm{v}}{\partial x_i} \frac{\partial \mathrm{w}}{d x_j} + \sum_{i=1}^d \mathrm{j}_i(x) \frac{\partial \mathrm{v}}{\partial x_i} + \mathrm{j}_0 \mathrm{v}\mathrm{w} \right) dx \tag{1.35}$$

$$\mathrm{j}_{ij}(x) = \mathrm{j}_{ji}(x), \quad \mathrm{j}_0(x) \geq \alpha > 0, \quad \alpha \ is\ a\ constant \quad x \in \bar{\Omega}, \tag{1.36}$$

the operator J defined by:

$$J = \sum_{i,j=1}^d J_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^d J_j(x) \frac{\partial}{\partial x_j} + J_0(x), \quad 1 \leq i,j \leq d, \tag{1.37}$$

we suppose that the function $g(.)$ is nonlinear, continuous and $c-$ Lipschitz,

$$|g(\mathrm{y}) - g(\mathrm{z})| \leq c\, |\mathrm{y} - \mathrm{z}| \quad \text{for all } \mathrm{y}, \mathrm{z} \in \mathbb{R}, \quad c\ > 0.$$

Theorem 8. [13, 17] *The problem* (1.34) *clearly has a unique solution, overmore* v *in* $L^2(H_0^1(\Omega); [0, T])$.

1.5.2 Discrete problem

Here, we take the same finite element space V_h and applying the finite element scheme and the semi-implicit Euler to discretize problem (1.34) with respect to time. Consequently, we are in seek of a sequence of elements $\mathrm{w}_h^n \in V_h$ that approaches $\mathrm{w}(t_n)$, $t_n = n\delta t$, with initial data v_0 and $\dfrac{\partial \mathrm{v}_h}{\partial t} = \dfrac{\mathrm{v}_h^n - \mathrm{v}_h^{n-1}}{\delta t}$, thus, we have for $n = 1, \ldots., N$, and for all $\mathrm{w}_h \in V_h$,

$$\begin{cases} (\dfrac{\mathrm{v}_h^n - \mathrm{v}_h^{n-1}}{\delta t}, \mathrm{w}_h - \mathrm{v}_h^n) + \mathrm{j}(\mathrm{v}_h^n, \mathrm{w}_h - \mathrm{v}_h^n) \geq (g(\mathrm{v}_h^n), \mathrm{w}_h - \mathrm{v}_h^n), \\ \mathrm{v}_h^n \leq r_h(\mathcal{M}\mathrm{v}_h^n), \end{cases} \tag{1.38}$$

we can write the problem (1.38) as for all $\mathrm{w}_h \in \mathbb{V}_h$

$$\begin{cases} \mathrm{j}(\mathrm{v}_h^n, \mathrm{w}_h - \mathrm{v}_h^n) + \dfrac{1}{\delta t}(\mathrm{v}_h^n, \mathrm{w}_h - \mathrm{v}_h^n) \geq (g(\mathrm{v}_h^n) + \dfrac{\mathrm{v}_h^{n-1}}{\delta t}, \mathrm{w}_h - \mathrm{v}_h^n), \\ \mathrm{v}_h^n \leq r_h(\mathcal{M}\mathrm{v}_h^n), \end{cases} \tag{1.39}$$

when, we put $\xi = \dfrac{1}{\delta t}$ and $G(\mathrm{v}_h^n) = g(\mathrm{v}_h^n) + \omega \mathrm{v}_h^{n-1}$, we obtain

$$\begin{cases} \omega(\mathrm{v}_h^n, \mathrm{w}_h - \mathrm{v}_h^n) + \mathrm{j}(\mathrm{v}_h^n, \mathrm{w}_h - u_h^n) \geq (G_h^n(\mathrm{v}_h^n), \mathrm{w}_h - \mathrm{v}_h^n), \\ \\ \mathrm{v}_h^n \leq r_h(\mathcal{M}\mathrm{v}_h^n), \end{cases} \tag{1.40}$$

we can express the problem (1.40) in the following form ,find $\mathrm{v}_h^n \in V_h$, for all $\mathrm{w}_h \in V_h$

$$\begin{cases} \mathrm{q}(\mathrm{v}_h^n, \mathrm{w}_h - \mathrm{v}_h^n) \geq (G_h^n(\mathrm{v}_h^n), \mathrm{w}_h - \mathrm{v}_h^n), \\ \\ \mathrm{v}_h^n \leq r_h(\mathcal{M}\mathrm{v}_h^n), \\ \\ \mathrm{v}_h(x) = \mathrm{v}_h(x,t)\mathrm{v}(x,t) = \mathrm{v}_0 \qquad in \quad \Omega_h, \\ \\ \mathrm{v}(x,t) = 0 \quad in \quad \Sigma_h, \end{cases} \tag{1.41}$$

where

$$\mathtt{q}(\mathtt{v}_h^n, \mathtt{w}_h - \mathtt{v}_h^n) = \xi(\mathtt{v}_h^n, \mathtt{w}_h - \mathtt{v}_h^n) + \mathtt{j}(\mathtt{w}_h^n, \mathtt{w}_h - \mathtt{v}_h^n).$$

The bilinear form $\mathtt{q}(.,.)$ exhibits strong coerciveness. Consequently, the formulation (1.41) denotes a coercive and continuous issue concerning elliptic quasi-variational inequalities [1, 13]. Denote by $\mathbb{Q}_h$ the outcome obtained by addressing problem (1.41) through a finite element method, resulting in the solution to the subsequent problem ascertain $\mathtt{v}_h^n \in V_h$ such that

$$\begin{cases} \left\langle \mathbb{Q}_h \mathtt{v}_h^n, \mathtt{w}_h - \mathtt{v}_h^n \right\rangle \ \geq \ \left\langle G^n(\mathtt{v}_h^n), \mathtt{w}_h - \mathtt{v}_h^n \right\rangle, \\ \\ \mathtt{v}_h^n \leq r_h(\mathcal{M}\mathtt{v}_h^n), \\ \\ \mathtt{v}_h(x) = \mathtt{v}_h(x,t) \qquad in \quad \Omega_h, \\ \\ \mathtt{v}(x,t) = 0. \quad in \quad \Sigma_h, \end{cases} \tag{1.42}$$

where $(\mathbb{Q}_h)_{i,j}$ denotes the finite element matrix defined by

$$(\mathbb{Q}_h)_{i,j} = \mathtt{q}(\phi_i, \phi_j) = \mathtt{j}(\phi_i, \phi_j) + \xi(\phi_i, \phi_j),$$

and $\{\phi_i\}$ is basis of $\mathbb{V}_h$, $\quad 1 \leq i, 1 \leq j$.

The discretization matrices $\mathbb{Q}_h$ and the generic coefficient matrices $\mathtt{a}(\phi_h^i, \phi_h^j)$ are introduced in a natural progression, where the customary basic functions are denoted by $\phi_h = 1, 2, \ldots, m(h_h)$, now that these definitions have been established, we may state the discrete problem as follows: Determine $\mathtt{v}_h^n \in V_h$, representing the solution for

$$\begin{cases} \left\langle \mathbb{Q}_h \mathtt{v}_h^n, \mathtt{w}_h - \mathtt{v}_h^n \right\rangle \ \geq \ \left\langle G^n(\mathtt{v}_h^n), \mathtt{w}_h - \mathtt{v}_h^n \right\rangle, \\ \\ \mathtt{v}_h^n \leq r_h(\mathcal{M}\mathtt{v}_h^n), \\ \\ \mathtt{v}_h(x) = \mathtt{v}_h(x,t) \qquad in \quad \Omega_h, \\ \\ \mathtt{v}(x,t) = 0 \quad in \quad \Sigma_h. \end{cases} \tag{1.43}$$

The matrix $(\mathbb{Q}_h)$ with coefficients $\mathtt{q}(\phi_i, \phi_j)$ are M-matrix [36].

Theorem 9. *[12, 17] The problem (1.43) has a unique solution $\mathtt{v}_h^n \in V_h$ for more details [12, 17].*

The theorem below represents the approximation result of our problem in this chapter.

Theorem 10. [17] *Let* $\mathtt{v}^{\infty}$ *and* $\mathtt{v}_h^N$ *be the solutions of* (1.34) *and* (1.43) *respectively, then*

$$\|\mathtt{v}_h^N - \mathtt{v}^{\infty}\|_{\infty} \leq C\left[h_k^2 |log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right], \tag{1.44}$$

where $\mathtt{v}^{\infty}$ is the asymptotic solution of the problem (1.34) ,and $\mathtt{v}_h^N$ the discrete solution calculatedat the moment $T = n\delta t$, $c >\ 0$, additionlly $c \leq \alpha$.

1.6 P.Q.V.I non linear with the right hand side dependent the solution

1.6.1 Continuous problem

When we remplace the obstacle $\mathcal{M}\mathtt{v}$ by the obstacle Φ in the problems (1.34) and (1.42), we find the following continuous problem

$$\begin{cases} \left(\dfrac{\partial \mathtt{v}}{\partial t}, \mathtt{w} - \mathtt{v}\right) + \mathtt{j}(\mathtt{v}, \mathtt{w} - \mathtt{v}) \geq (g(\mathtt{v}), \mathtt{w} - \mathtt{v}), \\ \\ \mathtt{v} \leq \Phi, \\ \\ \mathtt{v}(x,0) = \mathtt{v}_0 \qquad in \quad \Omega, \\ \\ \mathtt{v}(x,t) = 0 \quad in \quad \Sigma, \end{cases} \tag{1.45}$$

with Φ realization as $\Phi \in W^{2,\infty}$ such that $\Phi \geq 0$,

1.6.2 Discrete problem

We get the following discrete problem, find $\mathtt{v}_h^n$ in V_h

$$\begin{cases} \left\langle \mathtt{C}_h \mathtt{v}_h^n, \mathtt{w}_h - \mathtt{v}_h^n \right\rangle \ \geq \ \left\langle G^n(\mathtt{v}_h^n), \mathtt{w}_h - \mathtt{v}_h^n \right\rangle, \\ \\ \mathtt{v}_h^n \leq r_h \Phi, \\ \\ \mathtt{v}_h(x) = \mathtt{v}_h(x,0) \qquad in \quad \Omega_h, \\ \\ \mathtt{v}_h(x,t) = 0. \quad in \quad \Sigma_h, \end{cases} \tag{1.46}$$

where $(\mathtt{C}_h)_{i,j}$ denotes the finite element matrix defined by

$$(\mathtt{C}_h)_{i,j} = \mathtt{c}(\phi_i, \phi_j) = \mathtt{j}(\phi_i, \phi_j) + \xi(\phi_i, \phi_j),$$

the problem (1.45) has an unique solution $\mathtt{v}$ in $L^2(H_0^1(\Omega); [0,T])$ see [17], and the problem (1.46) also has an an unique solution $\mathtt{v}_h^n$ in V_h see [12]. To obtain the result of the regular convergence between the problem and the discrete problem, we apply the theorem 10.

Chapter 2

Description of multigrid technic

Here, we will present the multigrid method, then apply it to elliptic quasi variational inequalities, in addition to give a brief explanation of the relaxation method and Newton's method [29].

2.1 Iterative method

For soloving the following linear system $(Ax = B)$ with $A \in M_{n\times n}(\mathbb{R})$ is a symetric and definte postive marix and $B \in \mathbb{R}^n$ is a vector , we use the iterative method . The principle of iteration method is written the matrix A like this $A = (M - N)$, where M is inversible and diagonal or tridiagonal matrix, the system $Ax = B$ is equivalent to $x = M^{-1}(Nx + B)$, to approximate the solution of the system $(Ax = B)$
we apply the suite $x^{(k+1)}$ below

$$x^{(k+1)} = M^{-1}(Nx^{(k)} + B)$$

2.1.1 Relaxation method

For finding the approximate solution of the system $(Ax = B)$, we divide the matrix A as

$$A = (\frac{1}{\omega}D - E) - (\frac{1-\omega}{\omega}D + F)$$

where D is a diagonal matrix, E is a infereur tridaigonal matrix, F is a supreur tridiagonal matrix
An iteration matrix of relaxation method given by :

$$L_\omega = \left[(I - \omega D^{-1}E)^{-1}((1-\omega)I + \omega D^{-1}F)\right], \quad \omega \in \mathbb{R}^*.$$

Lemma 2. *Let $A \in M_{n\times n}(\mathbb{R})$ symetric and definte postive matrix , the relaxation method is convergente if and only if $\omega \in]0, 2[$.*

2.1.2 Gauss Siedel method

If $\omega = 0$ in the relaxation method , we get an other iterative method called: Gauss Siedel method, the iteration matrix of Gaus Siedel method is that :

$$L_1 = (D - E)^{-1} F$$

for geting the approximate solution of the linear system $(Ax = B)$, we apply Gauss Siedel mehtod and the foulowing suite of approximate

$$(D - E)^{-1} x^{(k+1)} = F x^k + B \quad \text{with} \quad x^{(0)} \quad \text{given}$$

2.2 Newton method

We supose that $f \in C^2([a, b])$, the equation $f(x) = 0$ accept a unique solution $r \in [a, b]$, the idea of Newton s method depends on replace the equation $f(x) = 0$ by the equation $T_1(x) = 0$, where T_1 is a polynome Taylor of adegree 1 on the point x_1.
The suite of approximate for Newton s method given by

$$\text{for,} \quad k > 0 \quad x^{(k+1)} = x^{(k)} + \frac{f(x^k)}{f'(x^k)} \quad \text{where} \quad x^1 \text{given.}$$

2.3 Multigrid method

We want to solve the following equation:

$$\mathcal{A}_k \mathrm{u}_k = f_k, \tag{2.1}$$

where $\mathcal{A}_k$ is the resulting matrix of replace the paratial derivates under the conditions of limites on the born and f_k is the seconde member. Usually, for solving the problem we apply the iteratives methods like the relaxation methods or Newton's methods, but in some cases, the iteratives methods don't give us the prefect solution, so, we use an other itertives methods called as: multigrid methods, this method use it when other iterative methods are not available or in the case of equations of large size,and it is characterized by quick access to the solution and the fewest number of iterations.

2.3.1 Two grids method

For soloving the solution of the problem (2.1) by using twogrids method, now, we pass to the stages of towgrids method :

Stage 01:

Let be Ω_k and Ω_{k-1} are towgrids, adding $\Omega \supset \Omega_k \supset \Omega_{k-1}$,we rise a few iterations by Gauss Siedel method$\big($4 or 5 iterations$\big)$.
Pre-smoothing let $\check{U}$ is the first approximation of the solution $\check{U} = L(\mathcal{A}_h, F_h, U_h, v = 5)$.
The residu d_k: We denote the residu by:

$$d_k = f_k - \mathcal{A}_k U_k$$

The error e_k: We symboly the error by

$$e_k = U_k^\star - \check{U}$$

and we have

$$d_k = \mathcal{A}_k e_k$$

Stage 02:

To calculate the error in the croase mesch and then return to the smooth mesch, we know two important operators called: prolongomation and instrection oporators, which allow us both to move between the croase mesch and the smooth mesch.

Prologation operator For transfer from coarse to fine mesch we define an operator called as interpolation operator P and given by

$$P : V_{2h} \mapsto V_h$$

$$v_{2h} \mapsto v_h$$

$$P_h = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N\div 2)-1}$$

Intersection oporator For return from fine mesch to corse mesch we define an oprator called as intersection operator R and given by

$$R : V_h \mapsto V_{2h}$$

$$v_h \mapsto v_{2h}$$

$$R_h = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N\div 2)-1}$$

Remark 2 We have gave R_h and P_h for $N = 8$, in addition to

$$R_h = \frac{1}{2}P_h^T$$

$$\mathcal{A}_{2h}^v = R_h\mathcal{A}_h^v \quad , \quad e_{2h}^v = R_h e_h^v \quad , \quad d_{2h}^v = R_h d_h^v$$

$$\mathcal{A}_h^v = P_h\mathcal{A}_{2h}^v \quad , \quad e_h^v = P_h e_{2h}^v \quad , \quad d_h^v = P_h d_{2h}^v$$

Stage 03:

now, we calculated the error on coarse mesch, so using the restrection operotor R like this

$$e_{2h} = Re_h$$

2.4 Kinds of multigrid method

We will propose three important kinds of multigrid scheme.

2.4.1 V-cycle scheme

Definition 10. [44] *The V-cycle scheme of multigrid method depends on replacing the operator in the fine mesch to reach the coarse mesch, and it is the most used in solving equations in the numerical analysis of large-scale problems. The figure bleow give the diagram of the The V-cycle scheme*

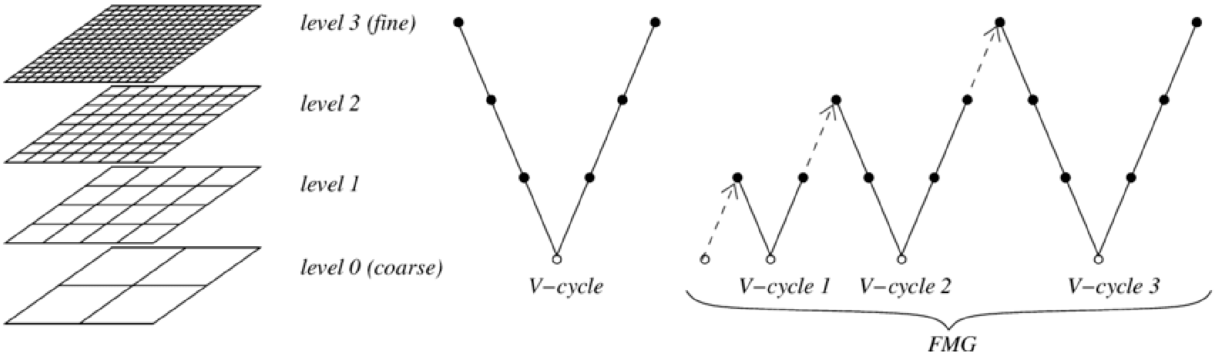


Figure 2.1: V-cycle scheme of multigrid method.

2.4.2 W-cycle scheme

Definition 11. *The W-cycle scheme goes through many operations from the large grid to the coarse grid, before correcting on the fine grid, and is considered a less used method in the field of numerical analysis due to the high cost of its use.*

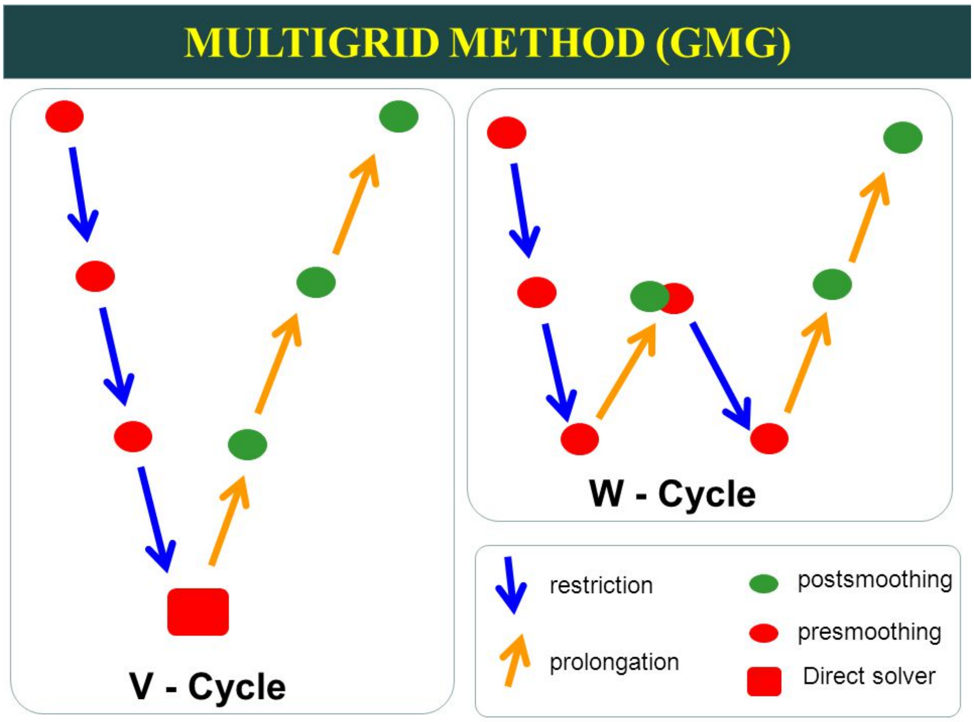


Figure 2.2: W-cycle scheme of multigrid method.

2.4.3 F-cycle scheme

Definition 12. *When the v-cycle and w-cycle don't give us a solution to the problem, we use another method called full-cycle scheme , it is the process of merging the tow methods together, which we summarize in the algorithm below :*

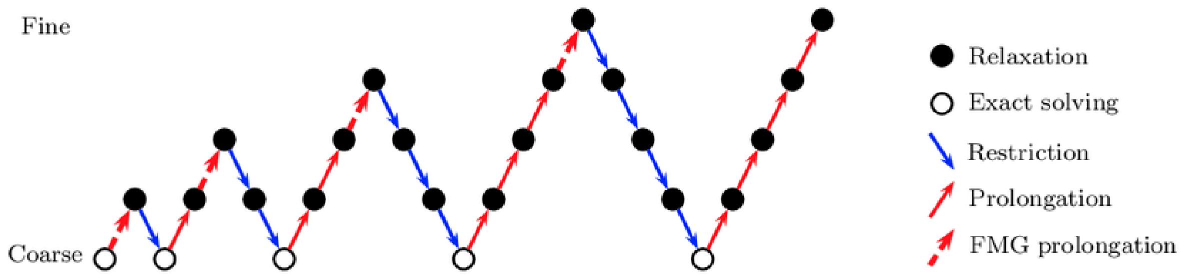


Figure 2.3: F-cycle scheme of multigrid method.

2.5 A solution of the E.Q.V.I by using the multigrid techic

For studing the L^∞-convergence of multi grid techic, we give our problem in the case continuous and descrete as :

2.5.1 Continuous problem

for all $v \in W^{2,p}(\Omega)$, find $w \in W^{2,p}(\Omega)$ such that

$$\begin{cases} a(w, v - w) \geq (f, v - w), \\ \\ w \leq Mw, \end{cases} \tag{2.2}$$

We take $a(.,.)$ is bilinear form, coercive and continuous, f linear function with $f \in L^\infty(\Omega)$, and M the obstacle is defined in Lemma 1.

Theorem 11. [5] *under the hypotheses* (1.19) *to* (1.24) *, the problem* (2.2) *has an unique solution* w *in* $W^{2,p}(\Omega)$*, with* $2 \leqslant p < \infty$ *.*

2.5.2 Discrete problem

for all $v_h^n \in V^h$, find $w_h^n \in V_h$ such that:

$$\begin{cases} \left\langle \mathcal{A}_h w_h^n, v_h - w_h^n \right\rangle \quad \geq \quad \left\langle f^n, v_h - w_h^n \right\rangle \\ \\ w_h^n \leq M_h w_h^n \end{cases} \tag{2.3}$$

where, $V_h = \{v_h \in C(\Omega) \cap H_0^1(\Omega) \ : \mid v_{h|\Omega_h} \in P_1, \ v_h \leq \mathbf{M} v_h\}$ and $(\mathcal{A}_h)_{i,j}$ denotes the finite element matrix defined by $(\mathcal{A}_h)_{i,j} = \mathsf{a}(\phi_i, \phi_j)$, with $\{\phi_i\}$ is basis of V_h, $\quad 1 \leq i, 1 \leq j$.

Theorem 12. [4] *under the hypotheses and notation precedente, the problem* (2.3) *has an unique solution* w_h *in* $W^{2,p}(\Omega)$*,* $2 \leqslant p < \infty$ *.*

To study L^{∞}-convergence of the multigrid method , we need the Theorem 13 and and the Lemma 3.

Theorem 13. [4] *Let w and w_h be the solutions of (2.2) and (2.3) respectively, then there exists a constant C independent of h such that*

$$\|w_h - w\|_{\infty} \leq C\left[h^2 |log(h)|^2\right] \tag{2.4}$$

for more details please refer to [4, 6].

Lemma 3. [50] *There exists η_1 and η_2 independent of h where*

1. $\|r_h(w)\|_{\infty} = \|w\|_{\infty}, \ \forall w \in U_h$
2. $\eta_1 \|v\|_{\infty} \leq \|r^*(v)\|_{\infty} \leq \eta_2 \|v\|_{\infty}, \ \forall v \in V_h$

the norm $\|.\|_{\infty}$ (on U_k) and the norm $\|.\|_{L^{\infty}}$ (on V_k) are equivalent, which are indicated by $\|.\|_{\infty}$.

2.5.3 HJB eqution of the discrete problem

We can express the E.Q.V.I as the HJB equation below,

$$\max_{1 \leq i \leq N} \left(\mathcal{A}_{k,i} w_{k,i}^{v} - f_{k,i}^{v};\ w_{k,i}^{v} - M_k w_{k,i}^{v-1}\right) = 0 \tag{2.5}$$

We suppose that w^v is an unique solution of the HJB equation 2.5. Let w^o given , for foloving the following system2.6,we calculate w^{v+1}

$$\mathcal{A}_k^v w_k^{v+1} - f_k^v = 0 \tag{2.6}$$

where

$$\mathcal{A}_k^v = \begin{cases} \mathcal{A}_{k,i} & if \quad (\mathcal{A}_{k,i} w_{k,i}^v - f_k^v) \ > \ w_{k,i}^v - M w_{k,i}^{v-1} \\ I_{k,i} & if \quad (\mathcal{A}_{k,i} w_{k,i}^v - f_k^v) \ \leq \ w_{k,i}^v - M w_{k,i}^{v-1} \end{cases} \tag{2.7}$$

$$f_k^v = \begin{cases} f_k^{v,i} & if \quad (\mathcal{A}_{k,i} w_{k,i}^v - f_{k,i}^v) \ > \ w_{k,i}^v - M w_{k,i}^{v-1} \\ \psi_{k,i} & if \quad (\mathcal{A}_{k,i} w_{k,i}^v - f_{k,i}^v) \ \leqslant w_{k,i}^v - M w_{k,i}^{v-1} \end{cases} \tag{2.8}$$

$\mathcal{A}_{k,i}$ is the i^{th} row of the discretization matrix $\mathcal{A}_k$, $f_{k,i}$ is the i^{th} component of the right-hand side f_k of our discrete problem, and $I_{k,i}$ is the i^{th} row of the identity matrix I_k. In the following theorem 14, we will prove the equivalence between the E.Q.V.I and HJB equation.

Theorem 14. [6] *Let w_k^v be the iterate obtained by the previous iterative scheme, satisfying the HJB-equation above. Additionally, we suppose that $\mathcal{A}_k$ is monotone. Then, the sequence w_k^v, where $(v \geq 0)$, exhibits monotonically decreasing convergence toward the unique solution w_k^* of (2.3).*

2.6 Steps of multigrid techic

Let (w_k^v) be an iterate and $\bar{w_k^v}$ be the solution of below equation (2.9)

$$\bar{w_k^v} = D_k^v(w_k^v) \tag{2.9}$$

D_k^v is the iteration matrix of smoothing. Starting at the error $e_k^v = \bar{w_k^v} - w_k^*$
Where w_k^* is the solution of (2.9) and the residual d_k^v is given by

$$d_k^v = f_k^v - \mathcal{A}_k^v \bar{w_k^v} \tag{2.10}$$

We can write (2.10) as

$$d_k^v = \mathcal{A}_k^v e_k^v \tag{2.11}$$

In order to compute e_k^v, it is imperative to possess the e_{k-1}^v values derived from the $k-1$ solution within the coarse grid equation

$$d_{k-1}^v = \mathcal{A}_{k-1}^v e_{k-1}^v \tag{2.12}$$

We have

$$\mathcal{A}_{k-1}^v = R_k \mathcal{A}_k^v \quad , \quad e_{k-1}^v = R_k e_k^v \quad , \quad d_{k-1}^v = R_k d_k^v$$

$$\mathcal{A}_k^v = P_k \mathcal{A}_{k-1}^v \quad , \quad e_k^v = P_k e_{k-1}^v \quad , \quad d_k^v = P_k d_{k-1}^v$$

P_k and R_k are the prolongation matrix and the restriction matrix respectively.
Let Ψ be the identity operator given by:

$$\Psi : V_{k-1} \longrightarrow V_k$$

$$\Psi v = v$$

To define the prolongation and the restriction operators i.e.

$$R_k = P_k^t \qquad P_k = R_k^{-1} R_{k-1}$$

2.7 Uniform convergence of multigrid techic

In this section, we will give the uniform convergence of multigrid techic

2.7.1 The iteration matrix of two-grids

Let α_1 and α_2 be presmoothing and postsmoothing iterations respectively, and let $TG_k(\alpha_1, \alpha_2)$ be the iteration matrix of two-grid methods given by

$$TG_k(\alpha_1, \alpha_2) = D_k^{\alpha_2}\ ((\mathcal{A}_k^v)^{-1} - P_k\ (\mathcal{A}_{k-1}^v)R_k)\mathcal{A}_k^v\ D_k^{\alpha_1}, \quad k = 1, 2, ... \tag{2.13}$$

Theorem 15. [51] *The multigrid method is a linear with iteration matrix MG_k given by*

$$MG_k = TG_k + D_k^{\alpha_2}\, MG_{k-1}(\mathcal{A}_k^{\nu})^{-1}\, R_k\, \mathcal{A}_k^{\nu}\, (D_k^{\alpha_1}) \quad k = 1, 2, ... \tag{2.14}$$

2.7.2 Approximation property

The following approximation property based by Theorem 13 and Lemma 3 is demonstrated. The proof of the theorem is an adaptation of the one provided in [36] originally presented for the problem of variational inequality.

Theorem 16. *Let Yth iteration matrix of multigrid given by*

$$Y = \left[(\mathcal{A}_k^{\nu})^{-1} - P_k\, (\mathcal{A}_{k-1}^{\nu})^{-1}\, R_k\right] \tag{2.15}$$

Under the previous assumptions the matrix Y satisfies the following approximation properties:

$$\|Y\|_{\infty} \leq C\left[h^2|log(h)|^2\right] \tag{2.16}$$

Proof. Let w_k^* and w_{k-1}^* be solutions to problem (2.3), we then apply Theorem 13
$\|w_k^* - w_{k-1}^*\|_{\infty} \leq \|w_k^* - w^{\infty}\|_{\infty} + \|w_{k-1}^* - w^{\infty}\|_{\infty} \leq C_1\left[h^2|log(h)|^2 +\right] + C_2\left[h^2|log(h)|^2\right]$
$\leq C\left[h^2|log(h)|^2\right]$
where $C = (C_1 + C_2)$.
Then there exist a right hand side $F \in U_k$ such that:

$$\forall v_k \in V_k \quad b(r_k.(\mathcal{A}_k^{\nu})^{-1}\, f\ ;\ v_k) = \left\langle (r_k^*)^{-1}\, f\ ;\ v_k \right\rangle_{L^2(\Omega)}$$

$$\forall v_{k-1} \in V_{k-1} \quad b((r_{k-1})^{-1}\, (\mathcal{A}_{k-1}^{\nu})^{-1} R_k\, f\ ;\ v_{k-1}) = \left\langle (r_k^*)^{-1}\, f\ ;\ v_{k-1} \right\rangle_{L^2(\Omega)}$$

Using the theorem 13 and the lemma 3

$$\|r_k^{-1}\, (\mathcal{A}_k^{\nu})^{-1}\, f - r_{k-1}^{-1}\, (\mathcal{A}_{k-1}^{\nu})^{-1}\, R_k\, f\|_{\infty} \leq C\left[h^2|log(h)|^2\right]$$

Then

$$\|(\mathcal{A}_k^{\nu})^{-1} - r_k^{-1} r_{k-1}\, (\mathcal{A}_{k-1}^{\nu})^{-1}\, R_k\|_{\infty} \|f\|_{\infty} \leq C\left[h^2|log(h)|^2\right] \|f\|_{\infty}$$

This completes the proof:

$$\|(\mathcal{A}_k^{\nu})^{-1} - P_k\, (\mathcal{A}_{k-1}^{\nu})^{-1}\, R_k\|_{\infty} \leq C\left[h^2|log(h)|^2\right]$$

and

$$\|Y\|_{\infty} \leq C\left[h^2|log(h)|^2\right]$$

□

2.7.3 Smoothing property

We composed the matrix $\left(\mathcal{A}_k^\nu = L_k - N_k\right)$ and used the following assumptions: L_k is regular where

$$\|L_k^{-1}\, N_k\|_\infty \leq 1 \qquad \text{for all k} \tag{2.17}$$

$$\|L_k^{-1}\|_\infty \leq \frac{C}{h^2}, \qquad \text{for all k, with C independent a k} \tag{2.18}$$

As a smoother we employed a relaxation method with iteration matrix

$$D_k = I_k - \omega L_k^{-1} N_k \qquad \omega \in [0;1]$$

Theorem 17. [50] *Assuming that the previous assumptions and symbols are satisfied, then there exists a constant C independent of k and such that the following smoothing properties hold*

$$\|\mathcal{A}_k^\nu\, D_k^\alpha\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h^2} \tag{2.19}$$

2.7.4 L^∞-convergence of multigrid method

To prove L^∞-convergence of the multigrid method, we have to establish the following stability bound.

$$\|D_k^\alpha\|_\infty \leq C_s, \qquad \text{for all k and } \alpha \tag{2.20}$$

The convergence analysis will be based on the following splitting of the two-grid iteration matrix, with $\alpha_2 = 0$:

$$\|TG_k(\alpha_1, 0)\|_\infty = \|\left[(\mathcal{A}_k^\nu)^{-1} - P_k\, (\mathcal{A}_{k-1}^\nu)^{-1}\, R_k\right]\, \mathcal{A}_k^\nu\, D_k^\alpha\, \|_\infty \leq \|(\mathcal{A}_k^\nu)^{-1} - P_k\, (\mathcal{A}_{k-1}^\nu)^{-1}\, R_k\|_\infty\, \|\mathcal{A}_k^\nu\, D_k^{\alpha_1}\|_\infty$$

Theorem 18. *Let k and $k-1$ be two grids, suppose the iterate $(w_k^\nu), \nu \geq 0$ follows the previous assumptions and symbols, then we have, exists $C > 0$*

$$\|w_k^{\nu+1} - w_k^*\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h^2}\left[h^2 |log(h)|^2\right]\, \|(w_k^\nu - w_k^*)\|_\infty \tag{2.21}$$

Proof.

$$\begin{aligned}
\|w_k^{\nu+1} - w_k^*\|_\infty &= \|(\mathcal{A}_k^\nu\, D_k^\alpha)(I_k - P_k(I_k - MG_{k-1})(\mathcal{A}_{k-1}^\nu)^{-1})R_k(w_k^\nu - w_k^*)\|_\infty \\
&\leq \|(\mathcal{A}_k^\nu\, D_k^\alpha)\|_\infty \|(I_k - P_k(I_k - MG_{K-1})(\mathcal{A}_{k-1}^\nu)^{-1})R_k\|_\infty \|(w_k^\nu - w_k^*)\|_\infty \\
&\leq \frac{C_1}{\sqrt{\alpha}\, h^2} C_2 \left[h^2 |log(h)|^2\right]\, \|(w_k^\nu - w_k^*)\|_\infty \\
&\leq \frac{C}{\sqrt{\alpha}\, h^2}\left[h^2 |log(h)|^2\right]\, \|(w_k^\nu - w_k^\bullet)\|_\infty
\end{aligned}$$

where $C = C_1 \times C_2$ □

Now, the result of L^∞-convergence of multigrid method can be easily demonstrated using theorem 19.

Theorem 19. [51]:*Under the previous assumptions and symbols, Y is the iteration matrix (given in (4.18)), for $\alpha_1 = \alpha,\ \alpha_2 = 0,\ \forall\varepsilon \in [0,1],\ \exists\ \alpha^* \leq \alpha$:*

$$\|MG_k\|_\infty \leq \varepsilon.$$

2.8 Numerical example

Example 5. *The problem 2.22 represents a numerical example that matches the data of the theoretical part, where we search for the solution using two numerical methods (multi grid and Gauss Siedel methods), and then compare betzeen them.*

$$\begin{cases} \mathcal{A}\mathrm{v} \leq f \quad in\ \Omega \\ \\ \langle \mathcal{A}\mathrm{v} - f\ ;\mathrm{v}\ \rangle = 0, \\ \\ \mathrm{v}^{\nu} \leqslant k + \mathrm{v}^{\nu-1} \\ \mathrm{v} = 0 \quad in\ \Gamma \end{cases} \tag{2.22}$$

where

$$\Omega = \{(x, y|x^2 + y^2 \leq 1)\}$$

$$\mathcal{A}\mathrm{u} = -\Delta$$

$$f(x) = 2x_1 + x_2$$

$$k = 1$$

For discretization in space we have used PDE toolbox in Matlab (r2018) to generate the mesh, and semi discretization on time, and startiterate $u_k^0 = (0, .., 0)^t \in \mathbb{R}^{550}$.

Iterations number	M-G method	Gauss Siedel method
1	$8.03e-03$	$1.14\ e-02$
5	$6.88e-15$	$4.57e-03$
10	$6.88e-15$	$1.03e-04$
15	$6.88e-15$	$2.86e-07$

Table 2.1: The error table.

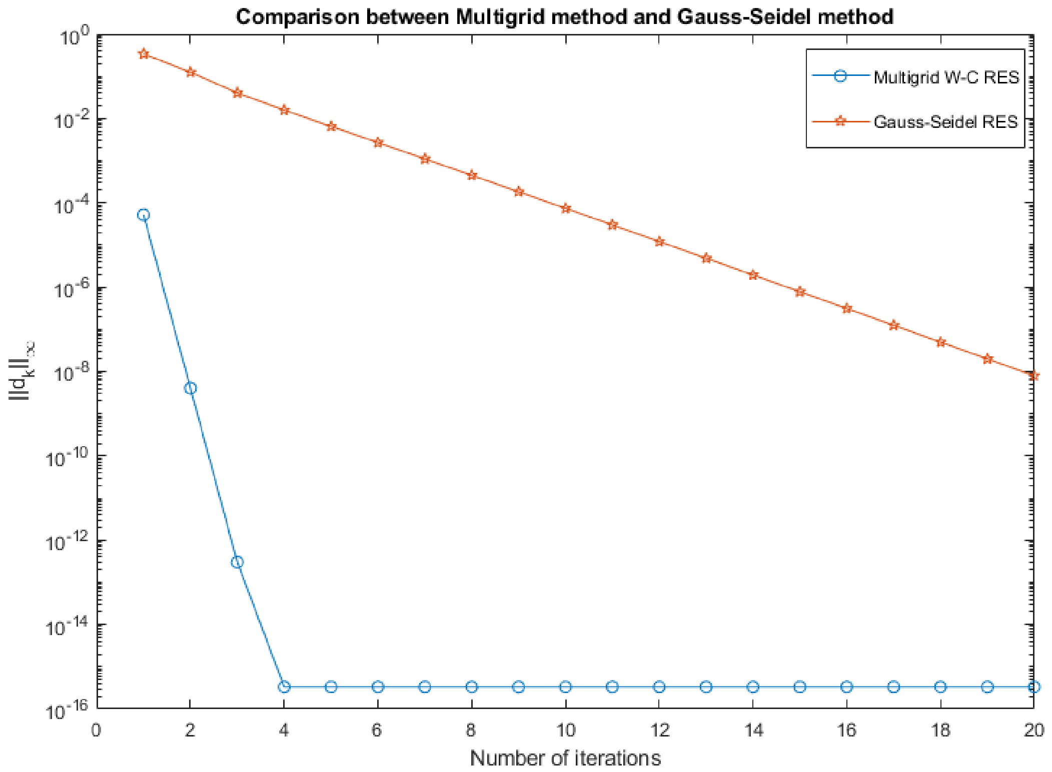


Figure 2.4: Comparison the convrgence of M-G method and G-S methods.

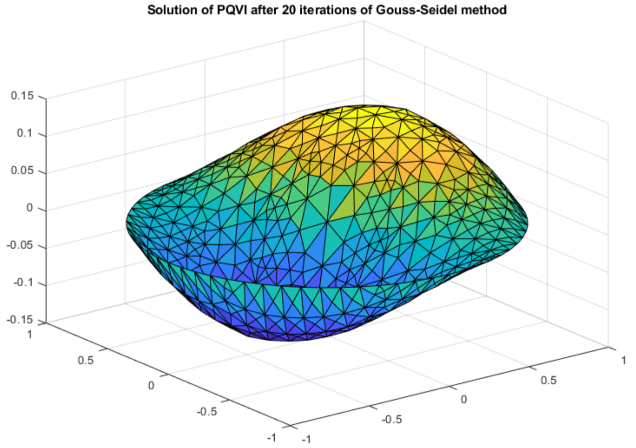


Figure 2.5: Solution of E.Q.V.I after 20 iterations of Gauss Siedel method.

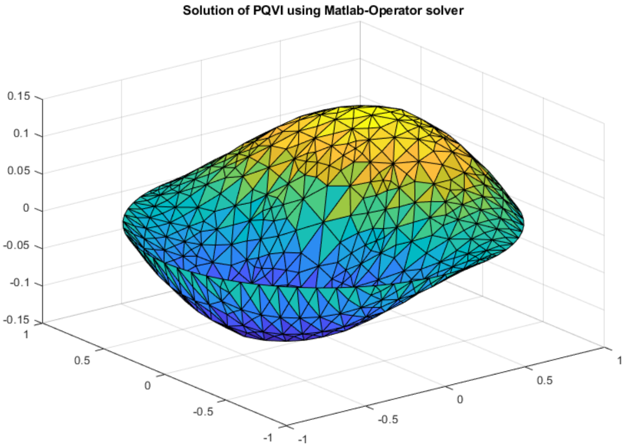


Figure 2.6: Solution of E.Q.V.I after 20 iterations of Gauss Siedel method.

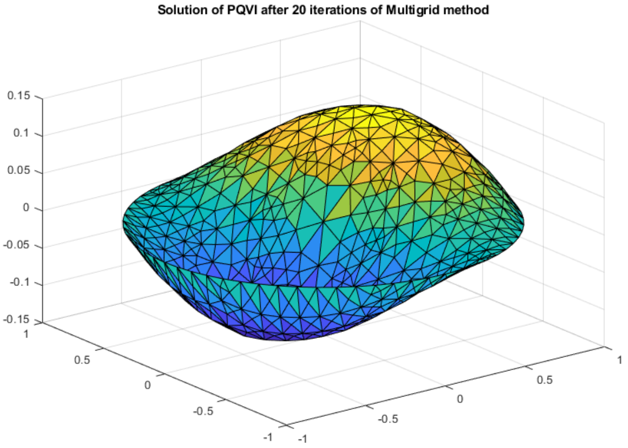


Figure 2.7: Solution of E.Q.V.I after 3 iterations of multigrid method.

Chapter 3

Multigrid methods for P.V.I with the operator noncoercive

Here, our work is focused around proof of the uniform convergence of the multigrid method for P.Q.V.I with a noncoercive operator and its numerical solution , we use two numerical methods for studying the uniform convergence of the multigrid method for P.Q.V.I with a noncoercive operator. To discretize the problem, we utilize a finite element method for the operator and Euler scheme for the time. To obtain the system discretization of the problem, we reformulate the parabolic variational inequality as a Hamilton-Jacobi-Bellman equation. On the smooth grid, we apply the multigrid method as an interior iteration on the linear system. Finally, we provide a proof of the uniform convergence of the multigrid method for P.Q.V.I with a noncoercive operator, with giving an numerical application of this problem.

3.1 HJB-formulation of discrete problem

Before commencing work in this section, we will use the following symbols:

$$\mathcal{D}_h = \mathcal{D}_k^v, \quad \mathrm{u}_h^n = \mathrm{u}_k^*, \quad \mathfrak{U}_h = U_k, \quad V_h = V_k.$$

The problem (1.16) can be expressed as the following HJB equation. Let u_k^v denote the unique solution of the discrete HJB equation

$$\max_{1\le i\le N}\left(\mathcal{D}_{k,i}\mathrm{u}_{k,i}^v - F_{k,i}^v;\ \mathrm{u}_{k,i}^v - r_k\varphi\right) = 0. \tag{3.1}$$

we put

$$\mathrm{u}_{k,i}^v - r_k\varphi = \mathrm{a}_{k,i}^v$$

Iterative steps:

Step 1: We are selecting an initial vector $\mathrm{u}_k^0 \in \mathbb{R}^{h_k}$.

Step 2: Let $\mathrm{u}_k^v \in \mathbb{R}^{h_k}$, $v \ge 0$, and we compute $\mathrm{u}_k^{(v+1)}$, where $\mathrm{u}_k^{(v+1)}$ is a solution of the following equation

$$\mathcal{D}_k^v \mathtt{u}_k^{v+1} - F_k^v = 0 \tag{3.2}$$

where

$$\mathcal{D}_k^v = \begin{cases} \mathcal{D}_{k,i} & if \quad (\mathcal{D}_{k,i}\mathtt{u}_{k,i}^v - F_k^v) \ > \ \mathtt{a}_{k,i}^v \\ I_{k,i} & if \quad (\mathcal{D}_{k,i}\mathtt{u}_{k,i}^v - F_k^v) \ \leq \ \mathtt{a}_{k,i}^v \end{cases} \tag{3.3}$$

$$F_k^v = \begin{cases} F_k^{v,i} & if \quad (\mathcal{D}_{k,i}\mathtt{u}_{k,i}^v - F_k^v) \ > \ \mathtt{a}_{k,i}^v \\ \varphi_{k,i} & if \quad (\mathcal{D}_{k,i}\mathtt{u}_{k,i}^v - F_k^v) \ \leqslant \mathtt{a}_{k,i}^v \end{cases} \tag{3.4}$$

Consider $\mathtt{u}_k^*$ as the unique solution to the discrete HJB equation.

$$\max_{1\leq i\leq N} \left(\mathcal{D}_{k,i}\mathtt{u}_{k,i}^* - F_{k,i}^v;\ \mathtt{u}_{k,i}^* - \varphi_k\right) = 0$$

The following theorem represents the equivalence between HJB equation and the parabolic variational inequality by using finite elements method, and we will rely in its proof on the work done by Hoppe on the elliptic quasi variational inequality using finite differences method.

Theorem 20. [38] *Let $\mathtt{u}_k^v$ be the iterate obtained by the previous iterative scheme, satisfying the HJB-equation above. Additionally, we suppose that $\mathcal{D}_k$ is monotone. Then, the sequence $\mathtt{u}_k^v$, where $(v \geq 0)$, exhibits monotonically decreasing convergence toward the unique solution $\mathtt{u}_k^*$ of (1.16), with startiterate $(\mathtt{u}_k^0)$ satisfying*

$$\max(\mathcal{D}_k^0\mathtt{u}_k^0 - F_k^0, \mathtt{u}_k^0 - \varphi_k) \geq 0, \tag{3.5}$$

Proof. Let $\mathtt{u}_k^v$ represents an iteration, and $\mathtt{u}_k^*$ denote a solution of the HJB equation. To prove that $\mathtt{u}_k^v$ converges towards $\mathtt{u}_k^*$, it is adequate to show that $(\mathtt{u}_k^v, \mathtt{u} \geq 0)$ consistently decreases towards $\mathtt{u}_k^*$, as outlined below:

$$\mathtt{u}_k^* \leq \mathtt{u}_k^{v+1} \leq \mathtt{u}_k^v.$$

To begin, we use (3.5), let us $\mathtt{u}_k^0$ satisfying, we have

$$\max(\mathcal{D}_k^0\mathtt{u}_k^0 - F_k^0, \mathtt{u}_k^0 - \varphi_k) \geq 0,$$

for all $v \geq 0$,

$$\max(\mathcal{D}_k^u\mathtt{u}_k^v - F_k^v, \mathtt{u}_k^v - \varphi_k \geq (\mathcal{D}_k^v\mathtt{u}_k^v - F_k^v) \geq 0.$$

Using (3.2), we get

$$(\mathcal{D}_k^v\mathtt{u}_k^v - \mathcal{D}_k^v u_k^{v+1}) \geq 0,$$

We know that $\mathcal{D}_k^u$ which is linear and monotone,

$$\mathcal{D}_k^v(\mathtt{u}_k^v - \mathtt{u}_k^{v+1}) \geq 0.$$

Then,

$$(\mathrm{u}_k^{v} - \mathrm{u}_k^{v+1}) \geq 0,$$

$$\mathrm{u}_k^{v+1} \leq \mathrm{u}_k^{v}. \tag{3.6}$$

We know that u_k^* is a solution of (1.16),we have

$$\max(\mathcal{D}_k^v \mathrm{u}_k^* - F_k, \mathrm{u}_k^* - \varphi_k) = 0,$$

again, using (3.2)

$$\mathcal{D}_k^v \mathrm{u}_k^* - \mathcal{D}_k^u \mathrm{u}_k^{v+1} \leq 0,$$

$$\mathcal{D}_k^v \mathrm{u}_k^* \leq \mathcal{D}_k^v \mathrm{u}_k^{v+1}.$$

We have $\mathcal{D}_k^v$ is a monotone matrix; that is $\mathcal{D}_k^v$ is an invertible matrix $\big((\mathcal{D}_k^v)^{-1}$ exists$\big)$. Thus

$$(\mathcal{D}_k^v)^{-1} \mathcal{D}_k^v \mathrm{u}_k^* \leq (\mathbb{B}_k^v)^{-1} \mathcal{D}_k^v \mathrm{u}_k^{v+1},$$

Then,

$$\mathrm{u}_k^* \leq \mathrm{u}_k^{v+1}. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\mathrm{u}_k^* \leq \mathrm{u}_k^{v+1} \leq \mathrm{u}_k^{v}.$$

The uniqueness: To demonstrate uniqueness, we suppose that there are two solutions u_1^* and u_2^* of (1.16).

First, we put $\mathrm{u}_1^* = \mathrm{u}_k^*$ and $\mathrm{u}_2^* = \mathrm{u}_k^{v+1}$,

$$\max(\mathcal{D}_k^v \mathrm{u}_1^* - F_k, \mathrm{u}_1^* - \varphi_k) = 0 \geq (\mathcal{D}_k^v \mathrm{u}_1^* - \mathcal{D}_k^v \mathrm{u}_k^{v+1}),$$

$$\mathcal{D}_k^v \mathrm{u}_1^* - \mathcal{D}_k^v \mathrm{u}_2^* \leq 0, \qquad \mathcal{D}_k^v \mathrm{u}_1^* \leq \mathcal{D}_k^v \mathrm{u}_2^*.$$

Therefore

$$\mathrm{u}_1^* - \mathrm{u}_2^* \leq 0, \tag{3.8}$$

secondly, we put $\mathrm{u}_2^* = \mathrm{u}_k^*$ and $\mathrm{u}_1^* = \mathrm{u}_k^{v+1}$

$$\max(\mathcal{D}_k^v \mathrm{u}_2^* - F_k, \mathrm{u}_2^* - \varphi_k) = 0 \geq (\mathcal{D}_k^v \mathrm{u}_2^* - \mathcal{D}_k^v \mathrm{u}_k^{v+1}).$$

$$\mathcal{D}_k^v \mathrm{u}_2^* - \mathcal{D}_k^v \mathrm{u}_1^* \leq 0 \quad then \quad \mathcal{D}_k^v \mathrm{u}_2^* \leq \mathcal{D}_k^v \mathrm{u}_1^*.$$

We have

$$\mathrm{u}_1^* - \mathrm{u}_2^* \geq 0. \tag{3.9}$$

From (3.8) and (3.9) we get

$$\mathrm{u}_1^* = \mathrm{u}_2^*.$$

Then, the solution of (1.16) (u_k^ν), approximates towards the solution of problem (3.1), $\mathrm{u}_k^\nu \mapsto \mathrm{u}_k^*$.

□

3.2 Description of the multigrid method for PQVIs

Now, to analyze uniform convergence of the multigrid method .Let $(\mathrm{u}_k^\nu)_{\nu\geq 0}$ be an iterate and $\bar{\mathrm{u}_k^\nu}$ be the solution of equation (3.2)

$$\bar{\mathrm{u}_k^\nu} = S_k^\nu(\mathrm{u}_k^\nu) \tag{3.10}$$

S_k^ν is the iteration matrix of smoothing. Starting at the error $e_k^\nu = \bar{\mathrm{u}_k^\nu} - \mathrm{u}_k^*$
Where u_k^* is the solution of (5.26) and the residual d_k^ν is given by

$$d_k^\nu = F_k^\nu - \mathcal{D}_k^\nu \bar{\mathrm{u}_k^\nu} \tag{3.11}$$

We can write (3.11) as

$$d_k^\nu = \mathcal{D}_k^\nu e_k^\nu \tag{3.12}$$

In order to compute e_k^ν, it is imperative to possess the e_{k-1}^ν values derived from the $k-1$ solution within the coarse grid equation

$$d_{k-1}^\nu = \mathcal{D}_{k-1}^\nu e_{k-1}^\nu \tag{3.13}$$

We have

$$\mathcal{D}_{k-1}^\nu = R_k \mathcal{D}_k^\nu \quad , \quad e_{k-1}^\nu = R_k e_k^\nu \quad , \quad d_{k-1}^\nu = R_k d_k^\nu$$
$$\mathcal{D}_k^\nu = P_k \mathcal{D}_{k-1}^\nu \quad , \quad e_k^\nu = P_k e_{k-1}^\nu \quad , \quad d_k^\nu = P_k d_{k-1}^\nu$$

P_k and R_k are the prolongation matrix and the restriction matrix respectively.
Let Ψ be the identity operator given by:

$$\Psi : V_{k-1} \longrightarrow V_k$$

$$\Psi \mathrm{v} = \mathrm{v}$$

To define the prolongation and the restriction operators i.e.

$$R_k = P_k^t \qquad P_k = R_k^{-1} R_{k-1}$$

3.2.1 The iteration matrix of two-grids

Let α_1 and α_2 be presmoothing and postsmoothing iterations respectively, and let $TG_k(\alpha_1, \alpha_2)$ be the iteration matrix of two-grid methods given by

$$TG_k(\alpha_1, \alpha_2) = S_k^{\alpha_2} \, ((\mathcal{D}_k^v)^{-1} - P_k \, (\mathcal{D}_{k-1}^v) R_k) \mathcal{D}_k^v \, S_k^{\alpha_1}, \quad k = 1, 2, ... \tag{3.14}$$

Theorem 21. [1] *The multigrid method is a linear with iteration matrix MG_k given by*

$$MG_k = TG_k + S_k^{\alpha_2} \, MG_{k-1} (\mathcal{D}_k^v)^{-1} \, R_k \, \mathcal{D}_k^v \, (S_k^{\alpha_1}) \quad k = 1, 2, ... \tag{3.15}$$

3.3 Approximation property

The following approximation property based on the inequality (1.17) in Theorem 4 and Lemma 3 is demonstrated. The proof of the theorem is an adaptation of the one provided in [6] originally presented for the problem of variational inequality.

Theorem 22. [6, 50] *Let $\mathcal{X}$ the iteration matrix of multigrid given by*

$$\mathcal{X} = \left[(\mathcal{D}_k^v)^{-1} - P_k \, (\mathcal{D}_{k-1}^v)^{-1} \, R_k \right] \tag{3.16}$$

Under the previous assumptions the matrix $\mathcal{X}$ satisfies the following approximation properties:

$$\|\mathcal{X}\|_\infty \leq C \left[h^2 |log(h)|^2 + \left(\frac{1}{1 + \beta \delta t} \right)^N \right] \tag{3.17}$$

Proof. Let u_k^* and u_{k-1}^* be solutions to problem (1.15), we then apply Theorem 4

$$\begin{aligned}
\|u_k^* - u_{k-1}^*\|_\infty &\leq \|u_k^* - u^\infty\|_\infty + \|u_{k-1}^* - u^\infty\|_\infty \\
&\leq C_1 \left[h^2 |log(h)|^2 + \left(\frac{1}{1 + \beta \delta t} \right)^N \right] + C_2 \left[h^2 |log(h)|^2 + \left(\frac{1}{1 + \beta \delta t} \right)^N \right] \\
&\leq C \left[h^2 |log(h)|^2 + \left(\frac{1}{1 + \beta \delta t} \right)^N \right]
\end{aligned}$$

where $C = (C_1 + C_2)$.

Then there exist a right hand side $F \in U_k$ such that:

$$\forall v_k \in V_k \quad b(r_k . (\mathcal{D}_k^v)^{-1} \, F \; ; \; v_k) = \left\langle (r_k^*)^{-1} \, F \; ; \; v_k \right\rangle_{L^2(\Omega)}$$

$$\forall v_{k-1} \in V_{k-1} \quad b((r_{k-1})^{-1} \, (\mathcal{D}_{k-1}^v)^{-1} R_k \, F \; ; \; v_{k-1}) = \left\langle (r_k^*)^{-1} \, F \; ; \; v_{k-1} \right\rangle_{L^2(\Omega)}$$

Using the theorem 4 and the lemma 3

$$\|r_k^{-1}\,(\mathcal{D}_k^v)^{-1}\,F - r_{k-1}^{-1}\,(\mathcal{D}_{k-1}^v)^{-1}\,R_k\,F\|_\infty \le C\left[h^2|log(h)|^2 + \left(\frac{1}{1+\beta\delta t}\right)^N\right]$$

Then

$$\|(\mathcal{D}_k^v)^{-1} - r_k^{-1} r_{k-1}\,(\mathcal{D}_{k-1}^v)^{-1}\,R_k\|_\infty \|F\|_\infty \le C\left[h^2|log(h)|^2 + \left(\frac{1}{1+\beta\delta t}\right)^N\right]\|F\|_\infty$$

This completes the proof:

$$\|(\mathcal{D}_k^v)^{-1} - P_k\,(\mathcal{D}_{k-1}^v)^{-1}\,R_k\|_\infty \le C\left[h^2|log(h)|^2 + \left(\frac{1}{1+\beta\delta t}\right)^N\right]$$

and

$$\|\mathcal{X}\|_\infty \le C\left[h^2|log(h)|^2 + \left(\frac{1}{1+\beta\delta t}\right)^N\right]$$

□

We composed the matrix $\left(\mathcal{D}_k^v = \mathcal{L}_k - N_k\right)$ and used the following assumptions: $\mathcal{L}_k$ is regular where

$$\|\mathcal{L}_k^{-1}\,N_k\|_\infty \le 1 \qquad \text{for all k} \tag{3.18}$$

$$\|\mathcal{L}_k^{-1}\|_\infty \le \frac{C}{h^2}, \qquad \text{for all k, with C independent a k} \tag{3.19}$$

As a smoother we employed a relaxation method with iteration matrix

$$S_k = I_k - \omega\mathcal{L}_k^{-1}N_k \qquad \omega \in [0;1]$$

Theorem 23. [6] *Assuming that the previous assumptions and symbols are satisfied, then there exists a constant C independent of k and such that the following smoothing properties hold*

$$\|\mathcal{D}_k^v\,S_k^\alpha\|_\infty \le \frac{C}{\sqrt{\alpha}\,h^2} \tag{3.20}$$

To prove uniform convergence of the multigrid method, we have to establish the following stability bound.

$$\|S_k^\alpha\|_\infty \le C_s, \qquad \text{for all k and } \alpha \tag{3.21}$$

The convergence analysis will be based on the following splitting of the two-grid iteration matrix, with $\alpha_2 = 0$:

$$\begin{aligned}\|TG_k(\alpha_1, 0)\|_\infty &= \|\left[(\mathcal{D}_k^v)^{-1} - P_k\,(\mathcal{D}_{k-1}^v)^{-1}\,R_k\right]\,\mathcal{D}_k^v\,S_k^\alpha\,\|_\infty\\ &\le \|(\mathcal{D}_k^v)^{-1} - P_k\,(\mathcal{D}_{k-1}^v)^{-1}\,R_k\|_\infty\,\|\mathcal{D}_k^v\,S_k^{\alpha_1}\|_\infty\end{aligned}$$

3.3.1 The main result

Theorem 24. *Let k and $k-1$ be two grids, suppose the iterate $(\mathrm{u}_k^{\nu})_{\nu\geq 0}$, follows the previous assumptions and symbols, then we have, exists $C > 0$*

$$\|\mathrm{u}_k^{\nu+1}-\mathrm{u}_k^{*}\|_{\infty}\leq \frac{C}{\sqrt{\alpha}\, h^2}\left[h^2|log(h)|^2+\left(\frac{1}{1+\beta\delta t}\right)^N\right]\|(\mathrm{u}_k^{\nu}-\mathrm{u}_k^{*})\|_{\infty} \tag{3.22}$$

Proof.

$$\begin{aligned}\|\mathrm{u}_k^{\nu+1}-\mathrm{u}_k^{*}\|_{\infty} &= \|(\mathcal{D}_k^{\nu}\, S_k^{\alpha})(I_k-P_k(I_k-MG_{k-1})(\mathcal{D}_{k-1}^{\nu})^{-1})R_k(\mathrm{u}_k^{\nu}-\mathrm{u}_k^{*})\|_{\infty}\\ &\leq \|(\mathcal{D}_k^{\nu}\, S_k^{\alpha})\|_{\infty}\|(I_k-P_k(I_k-MG_{K-1})(\mathcal{D}_{k-1}^{\nu})^{-1})R_k\|_{\infty}\|(\mathrm{u}_k^{\nu}-\mathrm{u}_k^{*})\|_{\infty}\\ &\leq \frac{C_1}{\sqrt{\alpha}\, h^2}C_2\left[h^2|log(h)|^2+\left(\frac{1}{1+\beta\delta t}\right)^N\right]\|(\mathrm{u}_k^{\nu}-\mathrm{u}_k^{*})\|_{\infty}\\ &\leq \frac{C}{\sqrt{\alpha}\, h^2}\left[h^2|log(h)|^2+\left(\frac{1}{1+\beta\delta t}\right)^N\right]\|(\mathrm{u}_k^{\nu}-\mathrm{u}_k^{*})\|_{\infty}\end{aligned}$$

where $C = C_1 \times C_2$. ☐

Now, the result of uniform convergence of multigrid method can be easily demonstrated using theorem 25.

Theorem 25. [6, 44]:*Under the previous assumptions and symbols, Y is the iteration matrix (given in (3.16)), for $\alpha_1=\alpha,\ \alpha_2=0,\ \forall\varepsilon\in[0,1],\ \exists\ \alpha^*\leq\alpha$:*

$$\|MG_k\|_{\infty}\leq\varepsilon.$$

3.4 Numerical example

Example 6. *The problem (3.23) represents a numerical example that matches the data of the theoretical part, where we search for the solution using two numerical methods (multi grid and Gauss Siedel methods), then compare them.*

$$\begin{cases}\dfrac{\partial \mathrm{u}}{\partial t}+\mathfrak{L}\mathrm{u}\leq f\ in\ [0,1]\times\Omega\\[2ex] \left\langle\dfrac{\partial \mathrm{u}}{\partial t}+\mathfrak{L}\mathrm{u}-f\,;\mathrm{u}-\varphi\right\rangle=0,\\[2ex] \mathrm{u}(x,t)=0,\ \ in\ [0,1]\times\Gamma,\\[2ex] \mathrm{u}(x,0)=0,\ \ in\ \Omega,\end{cases} \tag{3.23}$$

where

$$\Omega = \{(x, y|x^2 + y^2 \leq 1)\}, \quad \delta t = 0.01, \quad \xi = \frac{1}{\delta t}$$

$$\mathfrak{L}\mathrm{u} = -0.01\Delta \mathrm{u} - 0.02\frac{\partial^2 \mathrm{u}}{\partial x \partial y} + 0.15\frac{\partial \mathrm{u}}{\partial x} + 0.15\frac{\partial \mathrm{u}}{\partial y} + (1+\lambda)\mathrm{u}$$

,

$$f(x) = 2x_1 + x_2, \quad F = f(x) + (\lambda + \xi)\mathrm{u}, \quad \lambda = 2, \quad \varphi = 0.$$

For discretization in space we have used PDE toolbox in Matlab (*r2018*) *to generate the mesh, and semi discretization on time, and startiterate* $\mathrm{u}_k^0 = (0, .., 0)^t \in \mathbb{R}^{1024}$.

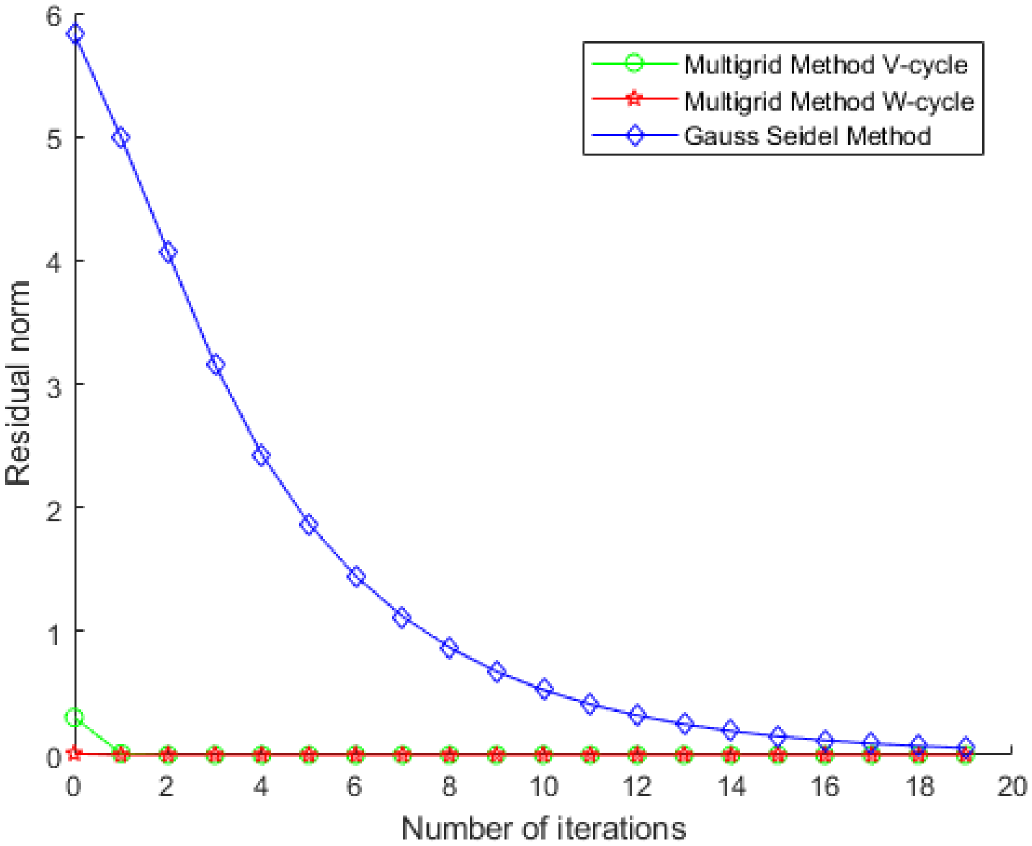


Figure 3.1: Comparison the convrgence of M-G method and G-S methods.

Iterations number	G-S method	W-cycle method	V-cycle method
1	5	1.19	8.77e-14
5	1.93	3.49 e-15	2.36 e-14
10	0.77	0.98 e-16	2.62 e-15
20	0.23 e-15	0.008 e-16	2.62 e-16

Table 3.1: The error table.

We notice from Figure (3.1) that the multi-grid method(V-cycle ,W-cycle) gives us the solu-

tion with the least number of iterations, that is, after 1 iteration, while the Gauss Siedel method after more than 20 iterations. From here we conclude that the multi-grid method gives us the solution to the sets of large linear equations with the least number of iterations.
Figures 2 and 3 represent the solution to the problem using Matlab 2018 using the Gauss Siedel method and the multi-grid method(V-cycle and W-cycle).

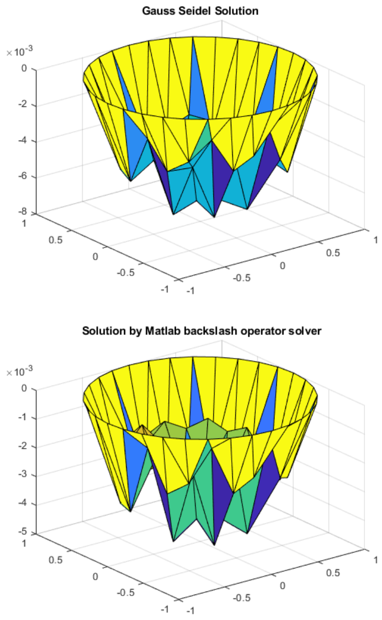


Figure 3.2: Solution of P.Q.V.I after 30 iterations of Gauss Siedel method.

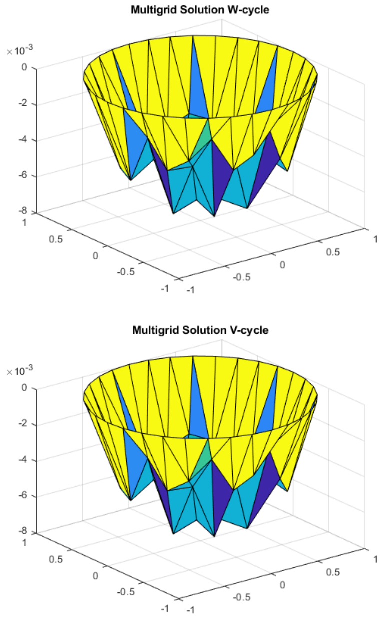


Figure 3.3: Solution of P.Q.V.I after 3 iterations of multigrid method.

Chapter 4

L^∞ convergence multigrid method for P.Q.V.I with the obstacle dependent of the solution

Here, we make some changes to the previous problem (1.2) as follows: we replace the noncoercive operator $\mathfrak{L}$ by the coercive operator A, and we else change the obstacle φ by the obstacle Mu, we get the problem (1.18), then study the multi grid method for the problem (1.18) .

4.1 Multigrid method

We know that, can express the problem is equivalent wih the HJB equation. Here, we will study the uniform convergence, first, we transform $P.Q.V.I$ into HJB equation and we put

$$\mathbf{B}_h = B_k^v \quad , \quad u_h^{\theta,n} = u_k^\bullet \quad , \quad U_h = U_k \quad , \quad V_h = V_k$$

.

4.1.1 HJB formulation of discrete problem

The following HJB equation is equivalent to problem (1.30) (see [39])

$$\max_{1\le i\le N}(B_{k,i}^v u_{k,i}^v - F_{k,i} \ ; \ u_{k,i}^v - M_k u_{k,i}^v) = 0. \tag{4.1}$$

Let u_k^v is a unique solution of the discrete HJB equation. The initial vector is $u_k^0 \in U_k$ and starting from the iterate $(u_k^v)_{v\ge 0} \in U_k$. We can split the group up:

$$L_k = \{1, 2, 3, ..., m(k)\} \text{ with } L_k = \underset{p=1,2}{\cup} L_k^p(u_k^v)$$

as

$$L_k^1(u_k^v) = \{i \in L_k \quad \text{if} \quad (B_{k,i}u_{k,i}^v - F_{k,i}) \quad > \quad (u_{k,i}^v - M_k u_{k,i}^{v-1})\} \tag{4.2}$$

$$L_k^2(u_k^v) = \{i \in L_k \quad \text{if} \quad (B_{k,i}u_{k,i}^v - F_{k,i}) \quad \leq \quad (u_{k,i}^v - M_k u_{k,i}^{v-1})\} \tag{4.3}$$

Suppose that $u_k^{v+1} \in U_k$ is a solution of the equation (4.4):

$$B_k^v u_k^{v+1} = F_k^v \tag{4.4}$$

where

$$B_k^v = \begin{cases} B_{k,i} & if \quad i \in L_k^1(u_k^v) \\ I_{k,i} & if \quad i \in L_k^2(u_k^v) \end{cases} \tag{4.5}$$

$$F_k^v = \begin{cases} F_{k,i} & if \quad i \in L_k^1(u_k^v) \\ M_k u_{k,i}^{v-1} & if \quad i \in L_k^2(u_k^v) \end{cases} \tag{4.6}$$

$B_{k,i}$ is the i^{th} row of the discretization matrix B_k, $F_{k,i}$ is the i^{th} component of the right-hand side F_k of our discrete problem, and $I_{k,i}$ is the i^{th} row of the identity matrix I_k. In the following theorem, we will prove the equivalence between parabolic quasi-variational inequalities and HJB equation.

Theorem 26. [39] *Let u_k^v be the iterate obtained by the previous iterative scheme such that it satisfies the HJB equation above. Moreover, suppose that B_k is monotone. Then the sequence $(u_k^v)_{v\geq 0}$, converges monotonically decreasingly towards the unique solution $u_k^\bullet$ of (1.30), with startiterate (u_k^0) satisfying*

$$\max(B_k^0 u_k^0 - F_k^0, u_k^0 M_k u_k^0) \geq 0. \tag{4.7}$$

Proof. We prove that the iterates $(u_k^v)_{v\geq 0}$, monotonically decrease towards $u_k^\bullet$ as follows

$$u_k^\bullet \leq u_k^{v+1} \leq u_k^v$$

from (4.7)we have

$$\max(B_k^0 u_k^0 - F_k, u_k^0 - M_k u_k^0) \geq 0$$

for all $v \geq 0$

$$\max(B_k^v u_k^v - F_k, u_k^v - M_k u_k^v) \geq (B_k^v u_k^v - F_k) \geq 0$$

Using (4.4), we get

$$(B_k^v u_k^v - B_k^v u_k^{v+1}) \geq 0$$

we have B_k^v a linear and monotone function

$$B_k^v(u_k^v - u_k^{v+1}) \geq 0$$

then

$$(u_k^v - u_k^{v+1}) \geq 0$$

$$u_k^{v+1} \leq u_k^v \tag{4.8}$$

We know that $u_k^\bullet$ solution of (4.1)

$$\max(B_k^v u_k^\bullet - F_k, u_k^\bullet - M_k u_k^\bullet) = 0 \geq (B_k^v u_k^\bullet - F_k)$$

seconde time, we use (4.4)

$$B_k^v u_k^\bullet - B_k^v u_k^{v+1} \leq 0$$

$$B_k^v u_k^\bullet \leq B_k^v u_k^{v+1}$$

we have B_k^v is monotone matrix i.e B_k^v is inverse matrix

$$(B_k^v)^{-1} B_k^v u_k^\bullet \leq (B_k^v)^{-1} B_k^v u_k^{v+1}$$

then

$$u_k^\bullet \leq u_k^{v+1} \tag{4.9}$$

from (4.8) and (4.9) we obtain

$$u_k^\bullet \leq u_k^{v+1} \leq u_k^v$$

The uniqueness: To demonstrate the uniqueness we suppose there are two solutions $u_1^\bullet$ and $u_2^\bullet$ of (4.1), first we put $u_1^\bullet = u_k^\bullet$ and $u_2^\bullet = u_k^{v+1}$

$$\max(B_k^v u_1^\bullet - F_k, u_1^\bullet - M_k u_1^\bullet) = 0 \geq (B_k^v u_1^\bullet - B_k^v u_k^{v+1})$$

$$B_k^v u_1^\bullet - B_k^v u_2^\bullet \leq 0 \quad then \quad B_k^v u_1^\bullet \leq B_k^v u_2^\bullet$$

$$u_1^\bullet \leq u_2^\bullet \tag{4.10}$$

secondly we put $u_2^\bullet = u_k^\bullet$ and $u_1^\bullet = u_k^{v+1}$

$$\max(B_k^v u_2^\bullet - F_k, u_2^\bullet - M_k u_2^\bullet) = 0 \geq (B_k^v u_2^\bullet - B_k^v u_k^{v+1})$$

$$B_k^v u_2^\bullet - B_k^v u_1^\bullet \leq \quad then \quad B_k^v u_2^\bullet \leq 0 B_k^v u_1^\bullet$$

$$u_1^\bullet - u_2^\bullet \geq 0 \tag{4.11}$$

from (4.10) and (4.11) we get

$$u_1^\bullet = u_2^\bullet$$

then, the solution (u_k^v) of (4.1) approximates towards the solution of problem (1.30) $u_k^v \mapsto u_k^\bullet$. □

4.2 L^∞-convergence of multigrid

Here, to study L^∞-convergence of the multigrid techic ,we consider $(u_k^v)_{v\geq 0}$ an iterate and $\bar{u}_k^v$ a solution of equation (4.4)

$$\bar{u}_k^v = S_k^v(u_k^v) \tag{4.12}$$

we denote of the iteration matrix on smoothing grid by S_k^v, and starting at the error $e_k^v = \bar{u}_k^v - u_k^\bullet$, where $u_k^\bullet$ is the solution of (4.4) and the residual d_k^v is given by

$$d_k^v = F_k^v - B_k^v \bar{u}_k^v, \tag{4.13}$$

we can write (4.13) as

$$d_k^v = B_k^v e_k^v \tag{4.14}$$

In order to compute e_k^v, it is imperative to possess the e_{k-1}^v values derived from the $k-1$ solution within the coarse grid equation

$$d_{k-1}^v = B_{k-1}^v e_{k-1}^v \tag{4.15}$$

We have

$$\begin{aligned} B_{k-1}^v &= R_k B_k^v \quad , \quad e_{k-1}^v = R_k e_k^v \quad , \quad d_{k-1}^v = R_k d_k^v \\ B_k^v &= P_k B_{k-1}^v \quad , \quad e_k^v = P_k e_{k-1}^v \quad , \quad d_k^v = P_k d_{k-1}^v \end{aligned}$$

P_k and R_k are the prolongation matrix and the restriction matrix respectively.
Let Ψ be the identity operator given by:

$$\Psi : V_{k-1} \longrightarrow V_k$$

$$\Psi v = v$$

To define the prolongation and the restriction operators i.e.

$$R_k = P_k^t \qquad P_k = R_k^{-1} R_{k-1}$$

4.2.1 The iteration matrix of two-grids

Let α_1 and α_2 be presmoothing and postsmoothing iterations respectively, and let $TG_k(\alpha_1, \alpha_2)$ be the iteration matrix of two-grid methods given by

$$TG_k(\alpha_1, \alpha_2) = S_k^{\alpha_2}\,((B_k^v)^{-1} - P_k\,(B_{k-1}^v)R_k)B_k^v\,S_k^{\alpha_1}, \quad k = 1, 2, ... \tag{4.16}$$

Theorem 27. [51] *The multigrid method is a linear with iteration matrix MG_k given by*

$$MG_k = TG_k + S_k^{\alpha_2}\, MG_{k-1}(B_k^v)^{-1}\, R_k\, B_k^v\, (S_k^{\alpha_1}) \quad k = 1, 2, ... \tag{4.17}$$

4.2.2 Approximation property

The following approximation property based on the inequality (1.31) in Theorem 7 and Lemma 3 is demonstrated. The proof of the theorem is an adaptation of the one provided in [36] originally presented for the problem of variational inequality.

Theorem 28. *Let Y the iteration matrix of multigrid given by*

$$Y = \left[(B_k^v)^{-1} - P_k\, (B_{k-1}^v)^{-1}\, R_k\right] \tag{4.18}$$

Under the previous assumptions the matrix Y satisfies the following approximation properties:

$$\|Y\|_{\infty} \leq C \left[h^2 |log(h)|^2 + \left(\frac{1}{1+\beta\theta\delta t}\right)^n\right] \tag{4.19}$$

Proof. See proof of the theorem 16 . □

4.2.3 Smoothing property

We composed the matrix $(B_k^v = L_k - N_k)$ and used the following assumptions: L_k is regular where

$$\|L_k^{-1}\, N_k\|_{\infty} \leq 1 \qquad \text{for all k} \tag{4.20}$$

$$\|L_k^{-1}\|_{\infty} \leq \frac{C}{h^2}, \qquad \text{for all k, with C independent a k} \tag{4.21}$$

As a smoother we employed a relaxation method with iteration matrix

$$S_k = I_k - \omega L_k^{-1} N_k \qquad \omega \in [0; 1]$$

Theorem 29. [50] *Assuming that the previous assumptions and symbols are satisfied, then there exists a constant C independent of k and such that the following smoothing properties hold*

$$\|B_k^v\, S_k^{\alpha}\|_{\infty} \leq \frac{C}{\sqrt{\alpha}\, h^2} \tag{4.22}$$

4.2.4 L^{∞}-convergence of multigrid method

To prove L^{∞}-convergence of the multigrid method, we have to establish the following stability bound.

$$\|S_k^{\alpha}\|_{\infty} \leq C_s, \qquad \text{for all k and } \alpha \tag{4.23}$$

The convergence analysis will be based on the following splitting of the two-grid iteration matrix, with $\alpha_2 = 0$:

$$\|TG_k(\alpha_1, 0)\|_{\infty} = \|\left[(B_k^v)^{-1} - P_k\, (B_{k-1}^v)^{-1}\, R_k\right]\, B_k^v\, S_k^{\alpha}\, \|_{\infty} \leq \|(B_k^v)^{-1} - P_k\, (B_{k-1}^v)^{-1}\, R_k\|_{\infty}\, \|B_k^v\, S_k^{\alpha_1}\|_{\infty}$$

Theorem 30. *Let k and $k-1$ be two grids, suppose the iterate $(u_k^v)_{v\geq 0}$, follows the previous assumptions and symbols, then we have, exists $C > 0$*

$$\|u_k^{v+1} - u_k^\bullet\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h^2}\left[h^2|log(h)|^2 + \left(\frac{1}{1+\beta\theta\delta t}\right)^n\right] \|(u_k^v - u_k^\bullet)\|_\infty \tag{4.24}$$

Proof. See proof of the theorem 18 □

Now, the result of L^∞-convergence of multigrid method can be easily demonstrated using theorem 31.

Theorem 31. [51]:*Under the previous assumptions and symbols, Y is the iteration matrix (given in (4.18)), for $\alpha_1 = \alpha,\ \alpha_2 = 0,\ \forall\varepsilon \in [0,1],\ \exists\ \alpha^* \leq \alpha$:*

$$\|MG_k\|_\infty \leq \varepsilon.$$

4.3 Numerical example

In this numerical example (stochastic inventory problems with impulse control), we find the solution of the problem (4.25) by using both the methods : multigrid method and relaxation method.

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au - f \leq 0 \ \text{ in } Q_T, \\[2ex] \left\langle \dfrac{\partial u}{\partial t} + Au - f\, ;\, u^v - \sigma - u^{v-1} \right\rangle = 0, \\[2ex] u^v \leq \sigma + u^{v-1} \quad ,\sigma > 0, \\[2ex] u(0,t) = 0 \text{ in } [0,1]\times\Gamma, \\[2ex] u(x,0) = 0 \ \text{ in } \Omega, \end{cases} \tag{4.25}$$

where

$$Q_T = [0,1]\times\Omega,\ (\Omega = \{(x_1, x_2 | x_1^2 + x_2^2 \leq 1)\}),\ \Delta t = 0.1,\ \sigma = 1,$$

$$A = -\Delta, \quad f(x) = 2x_1 + x_2.$$

For discretization , we utilized the PDE toolbox in MATLAB (r2018) to generate the mesh and for semi-discretization in time we initialized the iterative process with $u_k^0 = (0,..,0)^t \in \mathbb{R}^{1024}$.

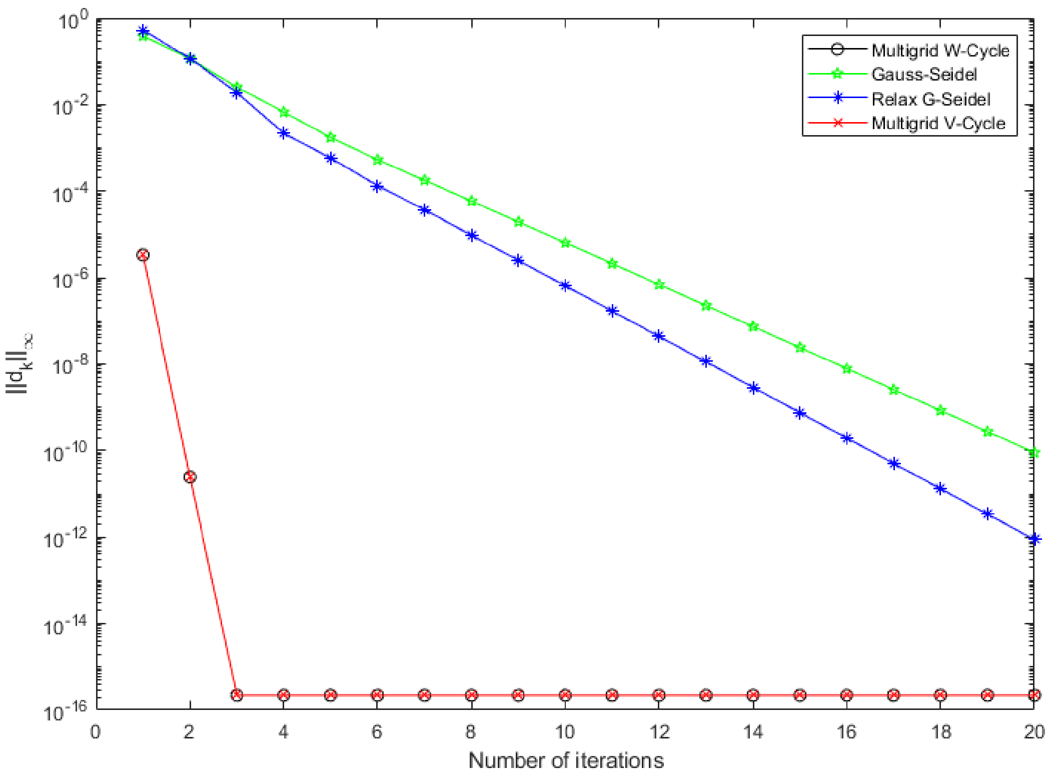


Figure 4.1: Comparison between Multigrid method and relaxation method.

Iterations number	G-S method	Relax method	V-cycle method
1	10^{0}	1.1	1.32×10^{-5}
10	1.5×10^{-02}	1.001×10^{-3}	3.07×10^{-16}
15	1.1×10^{-5}	2.1×10^{-6}	3.07×10^{-16}
20	1.06×10^{-10}	1.07×10^{-12}	3.07×10^{-16}

Table 4.1: The error table.

Figure (4.1) illustrates the comparison of convergence behavior between the multigrid method and the relaxation method. From this figure, we remark that the multigrid method converges after 3 iterations, while the relaxation method still does not exhibit satisfactory convergence even after 20 iterations. The multigrid method is implemented on the fine grid with 1024 nodes and 8 nodes on the coarse grid. Figures (3.3) and (3.4), represent the solutions of (4.25) using MATLAB operator solver,V-cycle method, and relaxation methods (Gauss-Seidel) respectively.

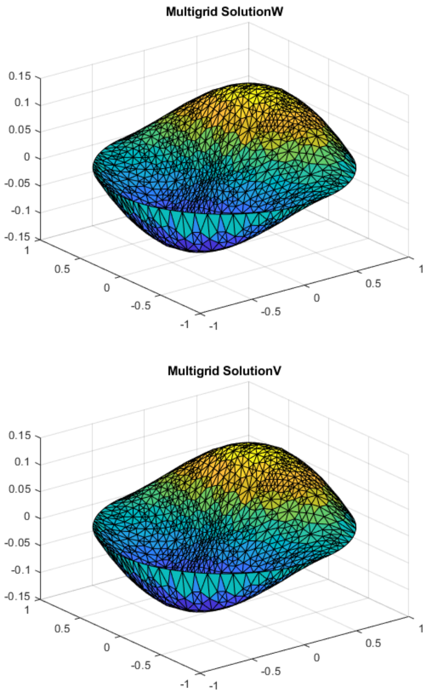


Figure 4.2: Solution of P.Q.V.I after 4 iterations of multigrid method.

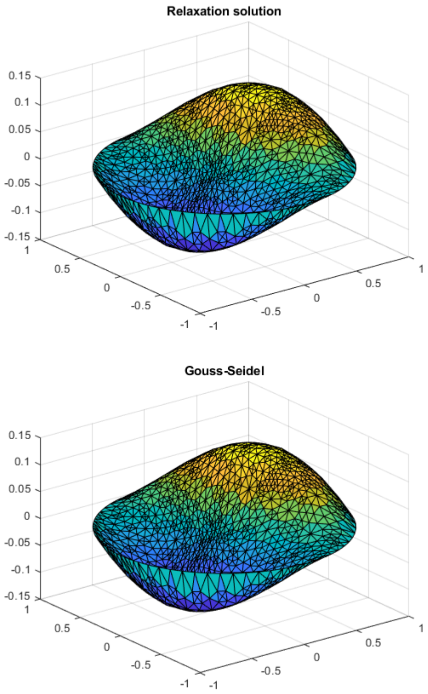


Figure 4.3: Solution of P.Q.V.I after 4 iterations of relaxation method.

Chapter 5

The multigrid methods for P.Q.V.I non linear right hand side

Here, we apply three numerical methods to study the $\mathbb{L}^{\infty}$-convergence of the Newton-Multigrid method for parabolic quasi-variational inequalities with a nonlinear right hand side. To discretize the problem, we utilize a finite element method for the operator and Euler scheme for the time. To obtain the system discretization of the problem, we reformulate the parabolic quasi-variational inequality as a Hamilton-Jacobi-Bellman equation. For linearizing the problem on the coarse grid, we employ Newton's method as an external iteration to obtain the Jacobian system. On the smooth grid, we apply the multigrid method as an interior iteration on the Jacobian system. Finally, we provide a proof of the $\mathbb{L}^{\infty}$-convergence of the Newton-Multigrid method for parabolic quasi-variational inequalities with a nonlinear right hand, with giving an numerical example of this problem.

5.1 HJB-formulation of discrete problem

Here, for to simplify understanding and follow-up, we will use the following symbols:

$$\mathbb{Q}_{\mathrm{h}} = \mathbb{Q}_k^{\nu}, \quad \mathrm{v}_{\mathrm{h}}^{n} = \mathrm{v}_k^{*}, \quad \mathfrak{U}_h = \mathbb{U}_k, \quad V_{\mathrm{h}} = V_k.$$

The problem $\left(1.43\right)$ can be written as the HJB equation below ,in addition to this, we put v_k^{ν} the unique solution of the discrete HJB equation

$$\max_{1 \le i \le N} \left(\mathbb{Q}_{k,i}\mathrm{v}_{k,i}^{\nu} - G(\mathrm{v}_{k,i}^{\nu});\ \mathrm{v}_{k,i}^{\nu} - \mathcal{M}_k \mathrm{v}_{k,i}^{\nu}\right) = 0.$$

and

$$\mathrm{v}_{k,i}^{\nu} - \mathcal{M}_k \mathrm{v}_{k,i}^{\nu} = \lambda_{k,i}^{\nu}$$

Iterative steps:

Step 1: We are selecting an initial vector $\mathrm{v}_k^{0} \in \mathbb{R}^{n_k}$.

Step 2: Let $\mathrm{v}_k^{\nu} \in \mathbb{R}^{n_k}$, $\nu \ge 0$, and we compute $\mathrm{v}_k^{(\nu+1)}$, where $\mathrm{v}_k^{(\nu+1)}$ is a solution of the following equation

$$\mathbb{Q}_k^\nu \mathrm{v}_k^{\nu+1} - G_k^\nu(\mathrm{v}_k^\nu) = 0 \tag{5.1}$$

where

$$\mathbb{Q}_k^\nu(\mathrm{v}_k^\nu) = \begin{cases} \mathbb{Q}_{k,i}(\mathrm{v}_k^\nu) & \text{if} \quad \mathbb{Q}_{k,i}\mathrm{v}_{k,i}^\nu - G(\mathrm{v}_{k,i}^\nu) \; > \; \lambda_{k,i}^\nu \\ I_{k,i} & \text{if} \quad \mathbb{Q}_{k,i}\mathrm{v}_{k,i}^\nu - G(\mathrm{v}_{k,i}^\nu) \; \leq \; \lambda_{k,i}^\nu \end{cases} \tag{5.2}$$

$$G_k^\nu(\mathrm{v}_k^\nu) = \begin{cases} G_{k,i}^\nu(\mathrm{v}_k^\nu) & \text{if} \quad \mathbb{Q}_{k,i}v_{k,i}^\nu - G(v_{k,i}^\nu) \; > \; \lambda_{k,i}^\nu \\ \mathcal{M}_k \mathrm{v}_{k,i}^\nu & \text{if} \quad \mathbb{Q}_{k,i}\mathrm{v}_{k,i}^\nu - G(\mathrm{v}_{k,i}^\nu) \; \leq \lambda_{k,i}^\nu \end{cases} \tag{5.3}$$

Let v_k^* as the unique solution to the discrete HJB equation

$$\max_{1 \leq i \leq N} \left(\mathbb{Q}_{k,i}\mathrm{v}_{k,i}^* - G(\mathrm{v}_{k,i}^*); \; \mathrm{v}_{k,i}^\nu - \mathcal{M}_k \mathrm{v}_{k,i}^\nu \right) = 0 \tag{5.4}$$

The theorem 26 represents the equivalence between HJB equation (5.4) and the parabolic quasi variational inequality (1.43) by applying finite elements scheme, and we will depend in its proof on the work complated by Hoppe on the elliptic quasi variational inequality using finite differences scheme, see 26 and its proof .

5.2 Description of the Newton-multigrid method for PQVIs

In addressing the nonlinear system (5.1), we utilize the Newton-Multigrid method, a hybrid technique that merges Newton's method for nonlinear systems with the Multigrid method for linear systems. The application of the Newton-Multigrid approach involves the following procedural steps:

First step: By utilizing Newton's method, we obtain the nonlinear system, recognizing that f is a Lipschitz function.

Second step: In every linear step of system (5.1), we seek the Jacobian system to derive a linear system connected with the Jacobian matrix.

Third step: We use the Multigrid method to solve the linear system resulting from the Jacobian matrix.

Let v_k^ν be the exact solution of the system (5.1), and $\mathtt{w}_k^\nu$ an approximation to v_k^ν. We define the residual as follows:

$$\Re_k(\mathrm{v}_k^\nu) = G_k^\nu(\mathrm{v}_k^\nu) - \mathbb{Q}_k^\nu \mathtt{w}_k^\nu, \tag{5.5}$$

Substituting the equation (5.1) in the equation(5.5), we find

$$\Re_k(\mathrm{v}_k^\nu) = \mathbb{Q}_k^\nu \mathrm{v}_k^\nu - \mathbb{Q}_k^\nu \mathtt{w}_k^\nu, \tag{5.6}$$

$$\Re_k(\mathrm{v}_k^\nu) = \mathbb{Q}_k^\nu(\mathrm{v}_k^\nu - \mathtt{w}_k^\nu) = \mathbb{Q}_k^\nu(e_k^\nu), \tag{5.7}$$

$\mathtt{e}_k^\nu$ the error, where $\mathtt{e}_k^\nu = \mathtt{v}_k^\nu - \mathtt{w}_k^\nu$. In the context of equation (5.7), determining the error $\mathtt{e}_k^\nu$ for the linear equation (5.7) on the coarse grid, as commonly done in the multigrid method, is not feasible. Consequently, we adopt equation (5.7) as a residual equation, given the non-linear characteristics of f.

To facilitate access to the solution, we choose $\mathcal{H}$ as a nonlinear operator, where:

$$\mathcal{H}_k(\mathtt{v}_k^\nu) = \mathbb{Q}_k^\nu \mathtt{v}_k^\nu - G_k^\nu(\mathtt{v}_k^\nu) = 0. \tag{5.8}$$

The residual in the fine grid can be rewritten as follows:

$$\mathfrak{R}_k(\mathtt{v}_k^\nu) = -\mathcal{H}_k(\mathtt{v}_k^\nu). \tag{5.9}$$

To solve (5.8), we employ the Newton iteration method as follows:

$$\mathtt{v}_k^{\nu+1} = \mathtt{v}_k^\nu + \frac{\mathfrak{R}_k(\mathtt{v}_k^\nu)}{\mathcal{J}_k(\mathtt{v}_k^\nu)}, \tag{5.10}$$

where $\mathcal{J}_k(\mathtt{v}_k^\nu)$ represents the Jacobian matrix of the nonlinear system, and $\mathcal{J}_k(\mathtt{v}_k^\nu) = \mathcal{H}_k^{'}(\mathtt{v}_k^\nu)$. From equation (5.10), we can derive the following Jacobian linear system for $\mathtt{e}_k^\nu$.

$$\mathcal{J}_k(\mathtt{v}_k^\nu)\mathtt{e}_k^\nu = \mathfrak{R}_k(\mathtt{v}_k^\nu). \tag{5.11}$$

where $\mathtt{v}_k^{\nu+1} - \mathtt{v}_k^\nu = \mathtt{e}_k^\nu$

The solution to the linear system (5.11) is utilized as an approach to solving the nonlinear system (5.8). Therefore, to find the solution for the nonlinear system (5.8), we employ the multigrid method to solve the associated linear system (5.11).

5.3 Multigrid technique

In addressing the linear system (5.11), we employ the multigrid technique. By selecting an iteration $\mathtt{e}_k^\nu$, where $\nu > 0$, within the multigrid method, we derive $\overline{\mathtt{e}}_k^\nu$ through the application of an iterative method. This iterative process is utilized to solve the system (5.11), utilizing α as the coefficient expression.

$$\overline{\mathtt{e}}_k^\nu = O_k^\nu(\mathtt{e}_k^\nu), \tag{5.12}$$

Here, O_k^ν stands for the iteration or smoothing operator, and α represents the number of iterations executed. The solution to (5.11) is denoted as $\mathtt{e}^\star$.

The error is defined as $\mathtt{E}_k^\nu = \overline{\mathtt{e}}_k^\nu - \mathtt{e}_k^\star$, and the residual is also considered.

$$\mathtt{d}_k^\nu = \mathfrak{R}_k(\mathtt{v}_k^\nu) - \mathcal{J}_k(\mathtt{v}_k^\nu)\overline{\mathtt{e}}_k^\nu,$$

we can write the equation (5.11) as

$$\mathcal{J}_k(\mathtt{v}_k^\nu)(\overline{\mathtt{e}}_k^\nu + \mathtt{E}_k^\nu) = \Re_k(\mathtt{v}_k^\nu), \tag{5.13}$$

We derive the subsequent residual equation.

$$\mathcal{J}_k(\mathtt{v}_k^\nu)\mathtt{E}_k^\nu = \Re_k(\mathtt{v}_k^\nu) - \mathcal{J}_k(\mathtt{v}_k^\nu)\overline{\mathtt{e}}_k^\nu = \mathtt{d}_k^\nu,$$

$$\mathcal{J}_k(\mathtt{v}_k^\nu)\mathtt{E}_k^\nu = \mathtt{d}_k^\nu.$$

To fully determine $\mathtt{E}_k^\nu$, it is necessary to compute $\mathtt{E}_k^\nu$ at the level $(k-1)$ as the solution to the coarse grid system.

$$\mathcal{J}_{k-1}(\mathtt{e}_{k-1}^\nu)\mathtt{E}_{k-1}^\nu = \mathtt{d}_{k-1}^\nu. \tag{5.14}$$

In the context where $\mathtt{E}_{k-1}^\nu$, $\mathcal{J}_{k-1}(\mathtt{e}_{k-1}^\nu)$, $\mathtt{d}_{k-1}^\nu$ represent approximations at the $(k-1)$ of $\mathtt{E}_k^\nu$, $\mathcal{J}_k(\mathtt{e}_k^\nu)$, $\mathtt{d}_k^\nu$
respectively, we have

$$\mathtt{E}_{k-1}^\nu = \mathtt{R}_k\mathtt{E}_k^\nu,$$

$$\mathtt{d}_{k-1}^\nu = \mathtt{R}_k\mathtt{d}_k^\nu,$$

$$\mathcal{J}_{k-1}^\nu(\mathtt{e}_{k-1}^\nu) = \mathcal{R}_k\mathcal{J}_k^\nu(e_k^\nu)\mathcal{P}_k,$$

$\mathcal{R}_k$ is the restriction matrix and $\mathcal{P}_k$ the prolongation matrix . Owing to the nested structure, we employ the clearly defined identity operator.

$$\Psi : V_{k-1} \longrightarrow V_k.$$

for defining the prolongation and restriction operators, i.e.

$$\mathcal{R}_k = \mathcal{P}_k^t$$

$$\mathcal{P}_k = r_k r_{k-1}^{-1}. \tag{5.15}$$

Note: To solve the linear system (5.11) for the two sequences Ω_k and Ω_{k-1}, we employ the multi-grid technique as an internal iteration. Additionally, to address system (5.8), we utilize Newton's method as an external iteration. This results in the iterative solution of system (5.14), achieved by applying a two-grid iteration repeatedly to the sequence $\Omega_k\{k = 0, ..., m_k\}$ until the iteration process is halted.

5.4 $\mathbb{L}^\infty$-convergence of multi-grid method

This paragraph explores the assessment of uniform convergence for the multigrid algorithm (inner iteration) and the convergence characteristics of Newton's method (outer iteration). The assumptions employed in these convergence analyses closely resemble those utilized in multigrid methods specifically crafted for addressing nonlinear equations. We now present the main hypotheses:

Hypotheses 1:

$$\exists \mathrm{v}_k^\star \in V_k \quad such\ that \quad \mathcal{H}_k(\mathrm{v}_k^\star)^{-1} = 0$$

Hypotheses 2 (inverse):

$$\mathcal{H}_k^{'}(\mathrm{v}_k^\star)^{-1}\ is\ exist\ and\ \|\mathcal{H}_k^{'}(\mathrm{v}_k^\star)^{-1}\|_\infty \leq k \quad with\ k > 0$$

Hypotheses 3 (continuous):

$$\forall \epsilon > 0, \quad \exists \eta > 0, \quad \|I - \mathcal{H}_k^{'}(\mathrm{u}_k^\star)\mathcal{H}_k^{'}(\mathrm{u}_k)^{-1}\|_\infty \leq \epsilon$$

Hypotheses 4: For any v_k in the neighborhood of $\mathrm{v}_k^\star$, there is a linear mapping denoted as $\mathcal{H}_k^{'}(\mathrm{v}_k)$ such that:
$\forall \epsilon > 0\ \exists \eta > 0$

$$\|\mathcal{H}_k(\mathrm{v}_k) - \mathcal{H}_k(\mathrm{v}_k^\star) - \mathcal{H}_k^{'}(\mathrm{v}_k^\star)\|\|(\mathrm{v}_k - \mathrm{v}_k^\star)\|_\infty \leq \epsilon\|(\mathrm{v}_k - \mathrm{v}_k^\star)\|_\infty$$

hold, whenever

$$\|(\mathrm{v}_k - \mathrm{v}_k^\star)\|_\infty \leq \eta.$$

5.4.1 Matrix associated with the MGHJB algorithm

The two-grid method's iteration matrix, which includes α_1 presmoothing and α_2 postsmoothing at level $k-1$, can be represented as follows:

$$\mathbb{TG}_k(\alpha_1, \alpha_2) = O_k^{\alpha_2}\,(\mathcal{J}_k^{-1} - \mathcal{P}_k\,\mathcal{J}_{k-1}^{-1}\mathcal{R}_k)\mathcal{J}_k\,O_k^{\alpha_1}, k = 1, 2.. \tag{5.16}$$

Theorem 32. [45, 51] *The multigrid approach is a linear iterative technique characterized by the iteration matrix $\mathbb{MG}_k$, which is expressed as:*

$$\mathbb{MG}_0 = 0 \quad if \quad k = 0,$$

$$\mathbb{MG}_k = O_k^{\alpha_2}\left(\mathbb{I}_k - \mathcal{P}_k\,(\mathbb{I}_k - \mathbb{MG}_{k-1})\,(\mathcal{J}_{k-1})^{-1}\mathcal{R}_k\right)\mathcal{J}_k\,O_k^{\alpha_1} \quad if \quad k = 1, 2, \ldots \tag{5.17}$$

$$\begin{aligned}\mathbb{MG}_k &= \mathcal{O}_k^{\alpha_2}\left(\mathbb{I}_k - \mathcal{P}_k\left(\mathbb{I}_k - \mathbb{MG}_{k-1}\right)\left(\mathcal{J}_k^{-1}\mathcal{R}_k\right)\right)\mathcal{J}_k\,\mathcal{O}_k^{\alpha_1}\\ &= \mathbb{TG}_k + \mathcal{O}_k^{\alpha_2}\mathcal{P}_k(\mathbb{MG}_{k-1})\mathcal{J}_k^{-1}\mathcal{R}_k\mathcal{J}_k\mathcal{O}_k^{\alpha_1} \quad if \quad k = 1, 2, \ldots\end{aligned}$$

5.4.2 Approximation property

The following approximation property is based in Theorem 10 and in Lemma 3. The demonstration of the theorem is adapted from the one in[36] , originally presented for the problem of variational inequality.

5.4.3 The main result

The following theorems gives us the convergence of the multi-grid matrix of the parabolic quasi variational, in its proof we will rely on the work done by Haiour [36] in the elliptic quasi variational inquality.

Theorem 33. *$\mathbb{X}$ is the iteration matrix of the multigrid, given by:*

$$\mathbb{X} = \left[\mathcal{J}_k^{-1} - \mathcal{P}_k\,\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\right]. \tag{5.18}$$

Under the previous assumptions, the matrix $\mathbb{X}$ satisfies the following approximation properties:

$$\|\mathbb{X}\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right] \tag{5.19}$$

Proof. According to Theorem 10, let $\mathrm{v} \in H_0^1(\Omega)$, we have

$$\|\mathrm{v}_k^n - \mathrm{v}^\infty\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty.$$

Then

$$\|\mathrm{v}_k^\star - \mathrm{v}^\infty\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty$$

Let $\mathrm{v}_k^\star$ and $\mathrm{v}_{k-1}^\star$ be solutions to the problem (1.42), and we apply Theorem 10

$$\begin{aligned}\|\mathrm{v}_k^* - \mathrm{v}_{k-1}^*\|_\infty &\leq \|\mathrm{v}_k^* - \mathrm{v}^\infty\|_\infty + \|\mathrm{v}_{k-1}^* - \mathrm{v}^\infty\|_\infty\\ &\leq C_1\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty + C_2\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty.\\ &\leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty\end{aligned}$$

we obtain

$$\|\mathtt{v}_k^* - \mathtt{v}_{k-1}^*\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty \tag{5.20}$$

where $C = (C_1 + C_2)$.
We use Galerkin discretization to obtain:

$$\mathtt{q}(r_k\mathtt{v},\ r_k\mathtt{w}) = \left\langle(\mathbb{Q}_k),\ \mathtt{w}\right\rangle_{L^2(\Omega)};\forall \mathtt{v},\mathtt{w}\in\mathbb{U}_k.$$

So

$$(\mathtt{q}(\mathtt{v},\ \mathtt{w}))' = \left\langle(\mathcal{J}_k),\ \mathtt{w}_k\right\rangle_{L^2(\Omega)};\forall \mathtt{v},\mathtt{w}\in\mathbb{U}_k.$$

Also we have, $\forall \mathtt{w}\in V$

$$(\mathtt{q}((r_k)^{-1}\,(\mathcal{J}_k)^{-1}(G_k),\ \mathtt{w}))' = \left\langle(r_{k-1}^*)^{-1}\,\nabla G\ ,\ \mathtt{w}\right\rangle_{L^2(\Omega)}$$

Let $\mathtt{v}_k \in V_k$ and $\mathtt{v}_{k-1}\in V_{k-1}$

$$\mathtt{q}(\mathtt{v}_k,\ \mathtt{w}) = \left\langle\left(r_k^*\right)^{-1}G,\ \mathtt{w}\right\rangle_{L^2(\Omega)},$$

$$\mathtt{q}(\mathtt{v}_{k-1},\ \mathtt{w}) = \left\langle\left(r_{k-1}^*\right)^{-1}G,\ \mathtt{w}\right\rangle_{L^2(\Omega)},$$

this implies $\mathtt{v}_k^\star = r_k^{-1}\mathcal{J}_k^{-1}\nabla(G)$ and $\mathtt{v}_{k-1}^\star = r_{k-1}^{-1}\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\nabla(G)$
Using (5.20),(5.15) and Lemma3 we get

$$\|r_k^{-1}\mathcal{J}_k^{-1}\nabla(G) - r_{k-1}^{-1}\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\nabla(G)\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty$$

Then

$$\|\mathcal{J}_k^{-1} - r_k r_{k-1}^{-1}\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\|_\infty\|\nabla G\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]\|\nabla G\|_\infty$$

The proof is now concluded

$$\|\mathcal{J}_k^{-1} - \mathcal{P}_k\mathcal{J}_{k-1}^{-1}\mathcal{R}_k\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right]$$

$$\|\mathbb{X}\|_\infty \leq C\left[h_k^2|log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right].$$

□

5.4.4 Smoothing property

To establish a smoothing property, we formed the matrix $\mathcal{J}_k^v = \mathcal{O}_k - \mathcal{N}_k$, and relied on the subsequent assumptions: $\mathcal{O}_k$ is considered regular where

$$\|\mathcal{O}_k^{-1}\, \mathcal{N}_k\|_\infty \leq 1 \qquad \text{for all } k. \tag{5.21}$$

$$\|\mathcal{O}_k^{-1}\|_\infty \leq \frac{C}{h^2}, \quad \text{for all } k, \text{ with } C \text{ independent a } k. \tag{5.22}$$

We utilized a relaxation method with an iteration matrix as a smoother.

$$\mathcal{O}_k = I_k - \omega \mathcal{J}_k^{-1} \mathcal{N}_k, \qquad \omega \in [0;1].$$

Theorem 34. [50] *Given that the preceding assumptions and notations are met, there exists a constant C, independent of k, such that the following smoothing property holds:*

$$\|\mathcal{J}_k\, \mathcal{O}_k^{\alpha}\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h_k^2}. \tag{5.23}$$

Following the approximation and smoothing properties, it is essential to establish the subsequent stability bound:

$$\exists C_s \quad \|\mathcal{O}_k^{\alpha}\|_\infty \leq C_s, \qquad \text{for all k and } \alpha. \tag{5.24}$$

The convergence analysis relies on the following decomposition of the iteration matrix for the two-grid method, where $\alpha_2 = 0$.

$$\begin{aligned} \|\mathbb{TG}_k(\alpha_1, 0)\|_\infty &= \|\left[(\mathcal{J}_k)^{-1} - \mathcal{P}_k\, (\mathcal{J}_{k-1})^{-1}\, \mathcal{R}_k\right]\, \mathcal{J}_k\, \mathcal{O}_k^{\alpha_1}\, \|_\infty \\ &\leq \|(\mathcal{J}_k)^{-1} - \mathcal{P}_k\, (\mathcal{J}_{k-1})^{-1}\, \mathcal{R}_k\|_\infty\, \|\mathcal{J}_k\, \mathcal{O}_k^{\alpha_1}\|_\infty. \end{aligned}$$

Now, choosing for a hierarchy comprising more than two grids, we can formulate the matrix $\mathbb{X}$ in (5.43) by recursively utilizing the matrix $\mathbb{MG}_k$ in (5.42) for all levels. Assuming the validity of condition (5.43), convergence outcomes can be easily inferred from the preceding findings.

Theorem 35. [51] *Examine a multigrid approach for a given iterative matrix (5.18). Then, based on the earlier assumption, for the specified parameter value* $\alpha_1 = \alpha, \alpha_2 = 0, \forall \varepsilon \in [0,1], \quad \exists \alpha^* \leq \alpha$:

$$\|\mathbb{MG}_k\|_\infty \leq \varepsilon.$$

Proof. Combining the approximation and smoothness properties with (3.5) allows us to utilize identical parameters, as outlined in the subsequent theorem. This theorem constitutes the primary outcome of our study. □

Theorem 36. *Given the preceding assumptions and notations, the iterated* $(v_k^\nu)_{\nu>0}$, *for two meshes* k *and* $k-1$, *adhere to the following relationship*: $\exists C > 0$:

$$\|\mathrm{v}_k^{\nu+1} - \mathrm{v}_k^*\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h_k^2}\left[h_k^2 |log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right] \|\mathrm{v}_k^\nu - \mathrm{v}_k^*\|_\infty.$$

Proof.

$$\begin{aligned}
\|\mathrm{v}_k^{\nu+1} - \mathrm{v}_k^*\|_\infty &= \|(\mathcal{J}_k\, O_k^\alpha)(I_k - \mathcal{P}_k(I_k - \mathbb{MG}_{k-1})(\mathcal{J}_{k-1}^{-1})\mathcal{R}_k(\mathrm{v}_k^\nu - \mathrm{v}_k^*)\|_\infty \\
&\leq \|(\mathcal{J}_k\, O_k^\alpha)\|_\infty \|(I_k - \mathcal{P}_k(I_k - \mathbb{MG}_{k-1})(\mathcal{J}_{k-1}^{-1})\mathcal{R}_k\|_\infty \\
&\times \|\mathrm{v}_k^\nu - \mathrm{v}_k^*\|_\infty \\
&\leq \frac{C_1 C_2}{\sqrt{\alpha}\, h_k^2} C\left[h_k^2 |log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right] \|\mathrm{v}_k^\nu - \mathrm{v}_k^*\|_\infty \\
&\leq \frac{C}{\sqrt{\alpha}\, h_k^2}\left[h_k^2 |log(h_k)|^2 + \left(\frac{1+c\delta t}{1+\alpha\delta t}\right)^N\right] \|\mathrm{v}_k^\nu - \mathrm{v}_k^*\|_\infty
\end{aligned}$$

where $C = C_1 \times C_2$. □

5.5 Numerical Example

Exemple 1:(the obstacle related of the solution)

In this example, we find the solution of the problem (5.25) by using multigrid method and relaxation method.

$$\begin{cases}
\dfrac{\partial \mathrm{v}}{\partial t} + J\mathrm{v} - g(\mathrm{v}) \leq 0, \quad \text{in } Q_T, \\[2ex]
\left\langle \dfrac{\partial \mathrm{v}}{\partial t} + J\mathrm{v} - g(\mathrm{v})\ ;\ \mathrm{v}^\nu - \sigma - \mathrm{v}^{\nu-1} \right\rangle = 0, \\[2ex]
\mathrm{v}^\nu \leq \sigma + \mathrm{v}^{\nu-1} \quad, \sigma > 0, \\[2ex]
\mathrm{v}(0,t) = 0, \ \text{in } [0,1]\times\Gamma, \\[2ex]
\mathrm{v}(x,0) = 0, \quad \text{in } \Omega,
\end{cases} \tag{5.25}$$

where

$$Q_T = [0,1]\times\Omega, \ (\Omega = \{(x_1, x_2 | x_1^2 + x_2^2 \leq 1)\}),$$

$$\Delta t = 0.1, \quad \sigma = 1, \ J = -\Delta, \ g(\mathrm{v}) = cos(2\mathrm{v}).$$

For discretization in space we have used PDE toolbox in Matlab (r2018) to generate the mesh, and semi discretization on time, and startiterate $\mathrm{v}_k^0 = (0, .., 0)^t \in \mathbb{R}^{1024}$.

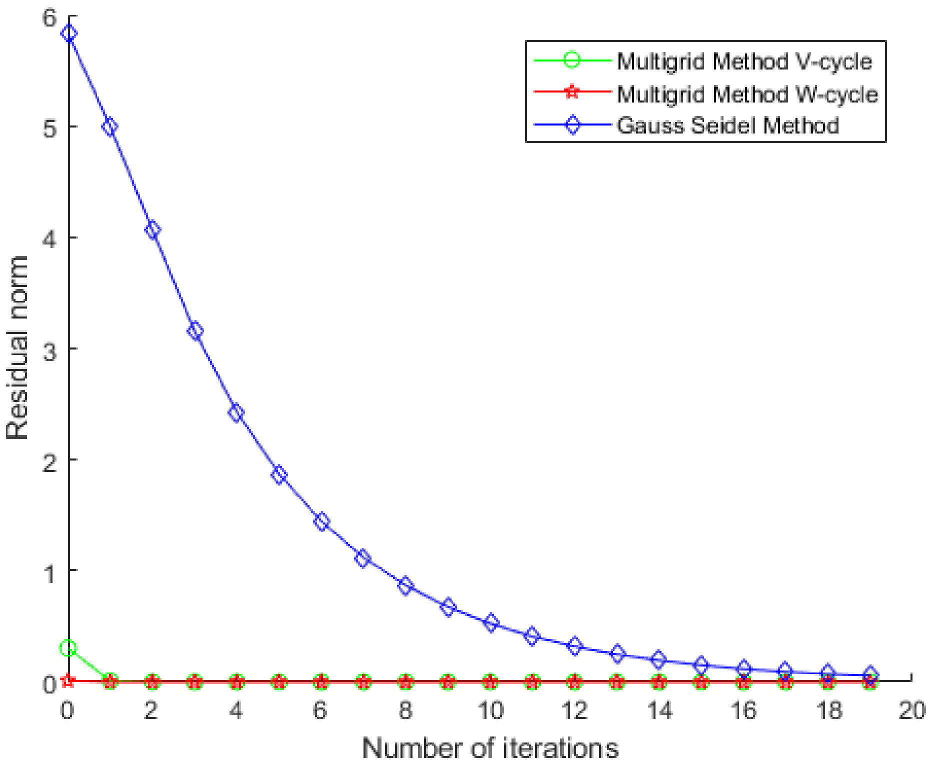
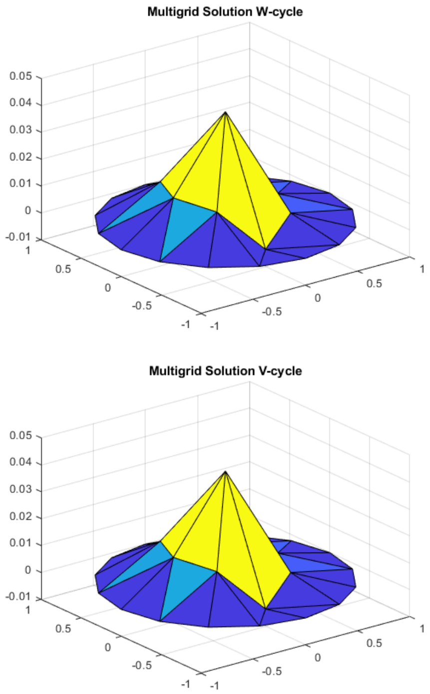


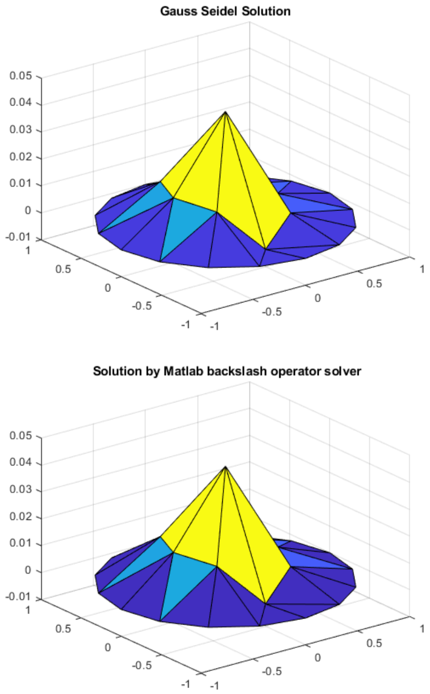
Figure 5.1: Comparison between the convrgence of M-G method and G-S methods.

Figure 5.1 illustrate the comparison of convergence behavior of the multigrid method and relaxation method, from this figure we remark that the multigrid method converge after 3 iterations while relaxation method still not good even after 20 iteretations.The multigrid method is carried out on the fine grid with 1024 nodes and 8 nodes on the coarse grid. Figure 5.2 and 5.3 represents the solution of (5.25) using matlab operator solver, multigrid(V-cycle ,W-cycle) and relaxation methods (Gauss-Seidel) respectively.


Multigrid Solution W-cycle

Multigrid Solution V-cycle

Figure 5.2: Solution of P.Q.V.I after 3 iterations of multigrid method.


Gauss Seidel Solution

Solution by Matlab backslash operator solver

Figure 5.3: Solution of P.Q.V.I after 30 iterations of Gass Siedel method.

5.6 HJB-formulation of discrete problem

Before commencing work in this section, we will use the following symbols:

$$\mathtt{C}_h = \mathtt{C}_k^\nu, \quad \mathrm{v}_h^n = \mathrm{v}_k^*, \quad \mathbb{U}_h = \mathbb{U}_k, \quad \mathbb{V}_h = \mathbb{V}_k.$$

The problem (1.46) can be expressed as the following HJB equation. Let v_k^ν denote the unique solution of the discrete HJB equation,

$$\max_{1\leq i\leq N}\left(\mathtt{C}_{k,i}\mathrm{v}_{k,i}^\nu - G(\mathrm{v}_{k,i}^\nu);\ \mathrm{v}_{k,i}^\nu - r_k\Phi\right) = 0,$$

we put

$$\mathrm{v}_{k,i}^\nu - r_k\Phi = \lambda_{k,i}^\nu.$$

Iterative steps:
Step 1: We select an initial vector $\mathrm{v}_k^0 \in \mathbb{R}^{n_k}$.
Step 2: Let $\mathrm{v}_k^\nu \in \mathbb{R}^{n_k}$, $\nu \geq 0$, and we compute $\mathrm{v}_k^{(\nu+1)}$, where $\mathrm{v}_k^{(\nu+1)}$ is a solution of the following equation:

$$\mathtt{C}_k^\nu \mathrm{v}_k^{\nu+1} - G_k^\nu(\mathrm{v}_k^\nu) = 0, \tag{5.26}$$

where

$$\mathtt{C}_k^\nu = \begin{cases} \mathtt{C}_{k,i}(\mathrm{v}_k^\nu) & if\ \mathtt{C}_{k,i}\mathrm{v}_{k,i}^\nu - G(\mathrm{v}_{k,i}^\nu)\ >\ \lambda_{k,i}^\nu \\ I_{k,i} & if\ \mathtt{C}_{k,i}\mathrm{v}_{k,i}^\nu - G(\mathrm{v}_{k,i}^\nu)\ \leq\ \lambda_{k,i}^\nu \end{cases} \tag{5.27}$$

$$G_k^\nu(\mathrm{v}_k^\nu) = \begin{cases} G_k^{\nu,i}(\mathrm{v}_k^\nu) & if\ \mathtt{C}_{k,i}\mathrm{v}_{k,i}^\nu - G(v_{k,i}^\nu)\ >\ \lambda_{k,i}^\nu \\ \psi_{k,i} & if\ \mathtt{C}_{k,i}\mathrm{v}_{k,i}^\nu - G(\mathrm{v}_{k,i}^\nu)\ \leqslant\ \lambda_{k,i}^\nu \end{cases} \tag{5.28}$$

Consider v_k^* as the unique solution to the discrete HJB equation,

$$\max_{1\leq i\leq N}\left(\mathtt{C}_{k,i}\mathrm{v}_{k,i}^* - G(\mathrm{v}_{k,i}^*);\ \mathrm{v}_{k,i}^* - \Phi_k\right) = 0. \tag{5.29}$$

The theorem 26 represents the equivalence between HJB equation and the parabolic quasi variational inequality by applying finite elements scheme, and we will depend in its proof on the work complated by Hoppe on the elliptic quasi variational inequality using finite differences scheme. see 26 and its proof.

5.7 Description of the Newton-multigrid method for PQVIs

In addressing the nonlinear system (5.26), we utilize the Newton-multigrid method, a hybrid technique that merges Newton's method for nonlinear systems with the multigrid method for linear systems. The application of the Newton-multigrid approach involves the following procedural steps:

First step: By utilizing Newton's method, we obtain the nonlinear system, recognizing that f is a Lipschitz function.

Second step: In every linear step of system (5.26), we seek the Jacobian system to derive a linear system connected with the Jacobian matrix.

Third step: We use the multigrid method to solve the linear system resulting from the Jacobian matrix.

Let $\mathtt{v}_k^{\nu}$ be the exact solution of the system (5.26), and $\mathtt{w}_k^{\nu}$ an approximation to $\mathtt{v}_k^{\nu}$. We define the residual as follows:

$$\mathfrak{R}_k(\mathtt{v}_k^{\nu}) = G_k^{\nu}(\mathtt{v}_k^{\nu}) - \mathtt{C}_k^{\nu}\mathtt{w}_k^{\nu}. \tag{5.30}$$

Substituting (5.26) in (5.30), we find

$$\mathfrak{R}_k(\mathtt{v}_k^{\nu}) = \mathtt{C}_k^{\nu}\mathtt{v}_k^{\nu} - \mathtt{C}_k^{\nu}\mathtt{w}_k^{\nu}, \tag{5.31}$$

$$\mathfrak{R}_k(\mathtt{v}_k^{\nu}) = \mathtt{C}_k^{\nu}(\mathtt{v}_k^{\nu} - \mathtt{w}_k^{\nu}) = \mathtt{C}_k^{\nu}(e_k^{\nu}), \tag{5.32}$$

where $\mathtt{e}_k^{\nu}$ is the error with $\mathtt{e}_k^{\nu} = \mathtt{v}_k^{\nu} - \mathtt{w}_k^{\nu}$. In the context of (5.32), determining the error $\mathtt{e}_k^{\nu}$ for the linear equation (5.32) on the coarse grid, as commonly done in the multigrid method, is not feasible. Consequently, we adopt equation (5.32) as a residual equation, given the non-linear characteristics of f.

To facilitate access to the solution, we choose $\mathcal{H}$ as a nonlinear operator, where:

$$\mathcal{H}_k(\mathtt{v}_k^{\nu}) = \mathtt{C}_k^{\nu}\mathtt{v}_k^{\nu} - G_k^{\nu}(\mathtt{v}_k^{\nu}) = 0. \tag{5.33}$$

The residual in the fine grid can be rewritten as follows:

$$\mathfrak{J}_k(\mathtt{v}_k^{\nu}) = -\mathcal{H}_k(\mathtt{v}_k^{\nu}). \tag{5.34}$$

To solve (5.33), we employ the Newton iteration method as follows:

$$\mathtt{v}_k^{\nu+1} = \mathtt{v}_k^{\nu} + \frac{\mathfrak{R}_k(\mathtt{v}_k^{\nu})}{\mathfrak{J}_k(\mathtt{v}_k^{\nu})}, \tag{5.35}$$

where $\mathfrak{J}_k(\mathtt{v}_k^{\nu})$ represents the Jacobian matrix of the nonlinear system, and $\mathfrak{J}_k(\mathtt{v}_k^{\nu}) = \mathcal{H}_k'(\mathtt{v}_k^{\nu})$.

From (5.35), we can derive the following Jacobian linear system for $\mathtt{e}_k^{\nu}$:

$$\mathfrak{J}_k(\mathtt{v}_k^{\nu})\mathtt{e}_k^{\nu} = \mathfrak{R}_k(\mathtt{v}_k^{\nu}). \tag{5.36}$$

where $\mathtt{v}_k^{\nu+1} - \mathtt{v}_k^{\nu} = \mathtt{e}_k^{\nu}$.

The solution to the linear system (5.36) is utilized as an approach to solving the nonlinear system (5.33). Therefore, to find the solution for the nonlinear system (5.33), we employ the multigrid method to solve the associated linear system (5.36).

5.8 Multigrid technique

In addressing the linear system (5.36), we employ the multigrid technique. By selecting an iteration $\mathtt{e}_k^\nu$, where $\nu > 0$, within the multigrid method, we derive $\bar{\mathtt{e}}_k^\nu$ through the application of an iterative method. This iterative process is utilized to solve the system (5.36), utilizing α as the coefficient expression as follows:

$$\bar{\mathtt{e}}_k^\nu = \mathcal{S}_k^\nu(\mathtt{e}_k^\nu), \tag{5.37}$$

Here, $\mathcal{S}_k^\nu$ stands for the iteration or smoothing operator, and α represents the number of iterations executed. The solution to (5.36) is denoted as $\mathtt{e}^\star$.

The error is defined as $\mathtt{E}_k^\nu = \bar{\mathtt{e}}_k^\nu - \mathtt{e}_k^\star$, and the residual is also considered.

$$\mathtt{d}_k^\nu = \mathfrak{R}_k(\mathtt{v}_k^\nu) - \mathfrak{J}_k(\mathtt{v}_k^\nu)\bar{\mathtt{e}}_k^\nu,$$

we can write (5.36) as

$$\mathfrak{J}_k(\mathtt{v}_k^\nu)(\bar{\mathtt{e}}_k^\nu + \mathtt{E}_k^\nu) = \mathfrak{R}_k(\mathtt{v}_k^\nu), \tag{5.38}$$

We derive the subsequent residual equation.

$$\mathfrak{J}_k(\mathtt{v}_k^\nu)\mathtt{E}_k^\nu = \mathfrak{R}_k(\mathtt{v}_k^\nu) - \mathfrak{J}_k(\mathtt{v}_k^\nu)\bar{\mathtt{e}}_k^\nu = \mathtt{d}_k^\nu,$$

$$\mathfrak{J}_k(\mathtt{v}_k^\nu)\mathtt{E}_k^\nu = \mathtt{d}_k^\nu.$$

To fully determine $\mathtt{E}_k^\nu$, it is necessary to compute $\mathtt{E}_k^\nu$ at the level $(k-1)$ as the solution to the coarse grid system.

$$\mathfrak{J}_{k-1}(\mathtt{e}_{k-1}^\nu)\mathtt{E}_{k-1}^\nu = \mathtt{d}_{k-1}^\nu. \tag{5.39}$$

In the context where $\mathtt{E}_{k-1}^\nu$, $\mathfrak{J}_{k-1}(\mathtt{e}_{k-1}^\nu)$, $\mathtt{d}_{k-1}^\nu$ represent approximations at the $(k-1)$ of $\mathtt{E}_k^\nu$, $\mathfrak{J}_k(\mathtt{e}_k^\nu)$, $\mathtt{d}_k^\nu$ respectively.
We have

$$\mathtt{E}_{k-1}^\nu = \mathtt{R}_k\mathtt{E}_k^\nu,$$

$$\mathtt{d}_{k-1}^\nu = \mathtt{R}_k\mathtt{d}_k^\nu,$$

$$\mathfrak{J}_{k-1}^\nu(\mathtt{e}_{k-1}^\nu) = \mathcal{R}_k\mathfrak{J}_k^\nu(e_k^\nu)\mathcal{P}_k,$$

where $\mathcal{R}_k$ is the restriction matrix and $\mathcal{P}_k$ the prolongation matrix . Owing to the nested structure, we employ the clearly defined identity operator.

$$\Psi : \mathbb{V}_{k-1} \longrightarrow \mathbb{V}_k.$$

for defining the prolongation and restriction operators, i.e.

$$\mathcal{R}_k = \mathcal{P}_k^t$$

$$\mathcal{P}_k = r_k r_{k-1}^{-1}. \tag{5.40}$$

Note: To solve the linear system (5.36) for the two sequences Ω_k and Ω_{k-1}, we employ the multi-grid technique as an internal iteration.Additionally, to address system (5.33), we utilize Newton's method as an external iteration. This results in the iterative solution of system (5.39), achieved by applying a two-grid iteration repeatedly to the sequence $\Omega_k\{k = 0, ..., m_k\}$ until the iteration process is halted.

5.9 $\mathbb{L}^{\infty}$-convergence of multi-grid method

This paragraph explores the assessment of uniform convergence for the multigrid algorithm (inner iteration) and the convergence characteristics of Newton's method (outer iteration). The assumptions employed in these convergence analyses closely resemble those utilized in multigrid methods specifically crafted for addressing nonlinear equations. We now present the main hypotheses:

Hypotheses 1:

$$\text{there exists } \mathrm{v}_k^\star \in V_k \quad \textit{such that} \quad \mathcal{H}_k(\mathrm{v}_k^\star)^{-1} = 0$$

Hypotheses 2 (inverse):

$$\mathcal{H}_k^{'}(\mathrm{v}_k^\star)^{-1} \text{ is exist and } \|\mathcal{H}_k^{'}(\mathrm{v}_k^\star)^{-1}\|_\infty \leq k \quad \textit{with } k > 0$$

Hypotheses 3 (continuous):

$$\text{for all } \epsilon > 0, \quad \text{there exists } \eta > 0, \quad \|I - \mathcal{H}_k^{'}(\mathrm{v}_k^\star)\mathcal{H}_k^{'}(\mathrm{v}_k)^{-1}\|_\infty \leq \epsilon$$

whenever

$$\|(\mathrm{v}_k - \mathrm{v}_k^\star)\|_\infty \leq \eta.$$

Hypotheses 4: For any v_k in the neighborhood of $\mathrm{v}_k^\star$, there is a linear mapping denoted as $\mathcal{H}_k^{'}(\mathrm{v}_k)$ such that for all $\epsilon > 0$ there exists $\eta > 0$ such that

$$\|\mathcal{H}_k(\mathrm{v}_k) - \mathcal{H}_k(\mathrm{v}_k^\star) - \mathcal{H}_k^{'}(\mathrm{v}_k^\star)\|\|(\mathrm{v}_k - \mathrm{v}_k^\star)\|_\infty \leq \epsilon\|(\mathrm{v}_k - \mathrm{v}_k^\star)\|_\infty$$

hold, whenever

$$\|(\mathrm{v}_k - \mathrm{v}_k^\star)\|_\infty \leq \eta.$$

5.9.1 Matrix associated with the MGHJB algorithm

The two-grid method's iteration matrix, which includes α_1 presmoothing and α_2 postsmoothing at level $k-1$, can be represented as follows:

$$\mathbb{TG}_k(\alpha_1, \alpha_2) = \mathcal{S}_k^{\alpha_2}\,(\mathfrak{J}_k^{-1} - \mathcal{P}_k\,\mathfrak{J}_{k-1}^{-1}\mathcal{R}_k)\mathfrak{J}_k\,\mathcal{S}_k^{\alpha_1}, k = 1, 2.. \tag{5.41}$$

Theorem 37. [48, 49] *The multigrid approach is a linear iterative technique characterized by the iteration matrix $\mathbb{MG}_k$, which is expressed as*:

$$\mathbb{MG}_0 = 0 \quad if \quad k = 0,$$

$$\mathbb{MG}_k = \mathcal{S}_k^{\alpha_2}\left(\mathbb{I}_k - \mathcal{P}_k\,(\mathbb{I}_k - \mathbb{MG}_{k-1})\,(\mathfrak{J}_{k-1})^{-1}\mathcal{R}_k\right)\mathfrak{J}_k\,\mathcal{S}_k^{\alpha_1} \quad if \quad k = 1, 2, \ldots \tag{5.42}$$

$$\begin{aligned}\mathbb{MG}_k &= \mathcal{S}_k^{\alpha_2}\left(\mathbb{I}_k - \mathcal{P}_k\,(\mathbb{I}_k - \mathbb{MG}_{k-1})\,(\mathfrak{J}_k^{-1}\mathcal{R}_k\right)\mathfrak{J}_k\,\mathcal{S}_k^{\alpha_1} \\ &= \mathbb{TG}_k + \mathcal{S}_k^{\alpha_2}\mathcal{P}_k(\mathbb{MG}_{k-1})\mathfrak{J}_k^{-1}\mathcal{R}_k\mathfrak{J}_k\mathcal{S}_k^{\alpha_1} \quad if \quad k = 1, 2, \ldots\end{aligned}$$

5.9.2 Approximation property

The following approximation property is based on the inequality (1.44) in Theorem 10 and in Lemma **??**. The demonstration of the theorem is adapted from the one in[36] , originally presented for the problem of variational inequality.

5.9.3 The main result

The following theorems gives us the convergence of the multi-grid matrix of the parabolic quasi variational, in its proof we will rely on the work done by Haiour [36] in the elliptic quasi variational inquality.

Theorem 38. *$\mathbb{X}$ is the iteration matrix of the multigrid, given by*

$$\mathbb{X} = \left[\mathfrak{I}_k^{-1} - \mathcal{P}_k \, \mathfrak{I}_{k-1}^{-1} \mathcal{R}_k\right]. \tag{5.43}$$

Under the previous assumptions, the matrix $\mathbb{X}$ satisfies the following approximation properties:

$$\|\mathbb{X}\|_\infty \leq C \left[h_k^2 |log(h_k)|^2 + \left(\frac{1 + c\delta t}{1 + \alpha\delta t} \right)^N \right] \tag{5.44}$$

Proof. See proof of theorem 33

□

5.9.4 Smoothing property

To establish a smoothing property, we form the matrix $\mathfrak{I}_k^v = \mathcal{L}_k - \mathcal{N}_k$, and rely on the subsequent assumptions:
$\mathcal{L}_k$ is considered regular, where

$$\|\mathcal{L}_k^{-1} \, \mathcal{N}_k\|_\infty \leq 1 \qquad \text{for all } k. \tag{5.45}$$

$$\|\mathcal{L}_k^{-1}\|_\infty \leq \frac{C}{h^2}, \quad \text{for all } k, \text{ with } C \text{ independent } k. \tag{5.46}$$

We utilized a relaxation method with an iteration matrix as a smoother,

$$\mathcal{S}_k = I_k - \omega \mathfrak{I}_k^{-1} \mathcal{N}_k, \qquad \omega \in [0; 1].$$

Theorem 39. [6] *Given that the preceding assumptions and notations are met, there exists a constant C, independent of k, such that the following smoothing property holds:*

$$\|\mathfrak{I}_k \, \mathcal{S}_k^\alpha\|_\infty \leq \frac{C}{\sqrt{\alpha} \, h_k^2}. \tag{5.47}$$

Following the approximation and smoothing properties, it is essential to establish the subsequent stability bounds:

$$\text{there exists } C_s \quad \|\mathcal{S}_k^{\alpha}\|_\infty \leq C_s, \qquad \text{for all k and } \alpha. \tag{5.48}$$

The convergence analysis relies on the following decomposition of the iteration matrix for the two-grid method, where $\alpha_2 = 0$.

$$\begin{aligned} \|\mathbb{TG}_k(\alpha_1, 0)\|_\infty &= \left\| \left[(\mathfrak{I}_k)^{-1} - \mathcal{P}_k \, (\mathfrak{I}_{k-1})^{-1} \, \mathcal{R}_k \right] \, \mathcal{J}_k \, \mathcal{S}_k^{\alpha_1} \right\|_\infty \\ &\leq \|(\mathfrak{I}_k)^{-1} - \mathcal{P}_k \, (\mathfrak{I}_{k-1})^{-1} \, \mathcal{R}_k\|_\infty \, \|\mathfrak{I}_k \, \mathcal{S}_k^{\alpha_1}\|_\infty. \end{aligned}$$

Now, choosing more than two grids for a hierarchy comprising, we can formulate the matrix $\mathbb{X}$ in (5.43) by recursively utilizing the matrix $\mathbb{MG}_k$ in (5.42) for all levels. Assuming the validity of condition 5.43, convergence outcomes can be easily inferred from the preceding findings.

Theorem 40. [44] *Examine a multigrid approach for a given iterative matrix* (5.43)*. Then, based on the earlier assumption, for the specified parameter value* $\alpha_1 = \alpha, \alpha_2 = 0$
for all $\varepsilon \in [0, 1]$*, there exists* $\alpha^* \leq \alpha$ *:*

$$\|\mathbb{MG}_k\|_\infty \leq \varepsilon.$$

Combining the approximation and smoothness properties with (3.5) allows us to utilize identical parameters, as outlined in the subsequent theorem. This theorem constitutes the primary outcome of our study.

Theorem 41. *Given the preceding assumptions and notations, the iterated* v_k^ν*,* $\nu > 0$*, for two meshes* k *and* $k-1$*, adhere to the following relationship: There exists* $C > 0$ *:*

$$\|\mathrm{v}_k^{\nu+1} - \mathrm{v}_k^*\|_\infty \leq \frac{C}{\sqrt{\alpha}\, h_k^2} \left[h_k^2 |log(h_k)|^2 + \left(\frac{1 + c\delta t}{1 + \alpha\delta t} \right)^N \right] \|\mathrm{v}_k^\nu - \mathrm{v}_k^*\|_\infty.$$

Proof. See proof of theorem 36 ☐

5.10 Numerical exemple

Example 7. *In this numerical example , we find the solution of the following problem* (5.49) *by using multigrid method and Gauss-Siedel method*:

$$\begin{cases} \dfrac{\partial \mathrm{v}}{\partial t} + A\mathrm{v} \leq g(\mathrm{v}) \ in \ [0,1] \times \Omega, \\ \\ \left\langle \dfrac{\partial \mathrm{v}}{\partial t} + A\mathrm{v} - g(\mathrm{v}) \, ; \mathrm{v} - \Phi \right\rangle = 0, \\ \\ \mathrm{v}(x,t) = 0, \ in \ [0,1] \times \Gamma, \\ \\ \mathrm{v}(x,0) = 0, \ \ in \ \Omega, \end{cases} \tag{5.49}$$

where

$$\Omega = \{(x_1, x_2 | x_1 + x_2 \leq 1)\}, \quad \Delta t = 0.01, \quad J\mathrm{v} = -\Delta \mathrm{v}, \quad g(\mathrm{v}) = cos 2\mathrm{v}, \quad \Phi = 0.$$

For discretization in space we have used PDE toolbox in MATLAB $($*r2018*$)$ *to generate the mesh, and semi discretization on time, and startiterate* $\mathrm{v}_k^0 = (0, .., 0)^t \in R^{1024}$.

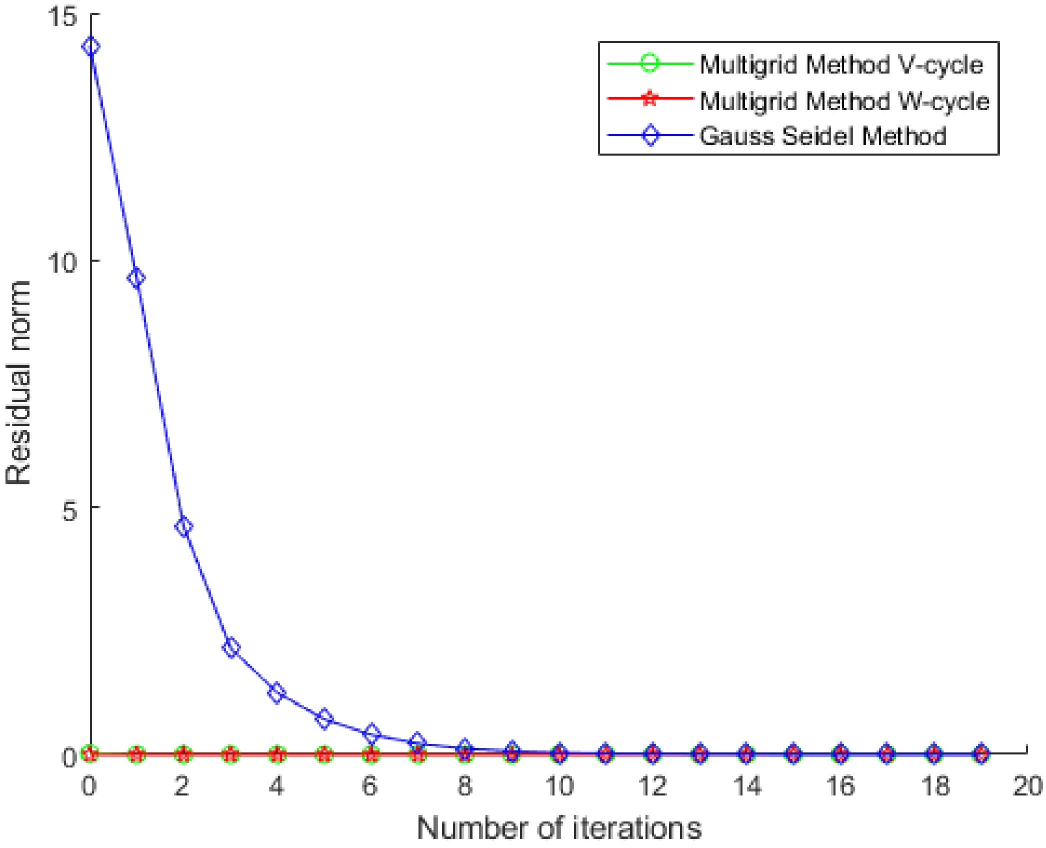


Figure 5.4: Comparison between the convrgence of M-G method and G-S methods.

We note from Figure 5.4 that the multi-grid method(V-cycle ,W-cycle) gives us the solution with the least number of iterations, that is, after 1 iteration, while the Gauss Siedel method after more than 20 iterations. From here we conclude that the multi-grid method gives us the solution to the sets of large linear equations with the least number of iterations.

Figures 2 and 3 represent the solution to the problem using MATLAB 2018 using the Gauss Siedel method and the multi-grid method(V-cycle and W-cycle).

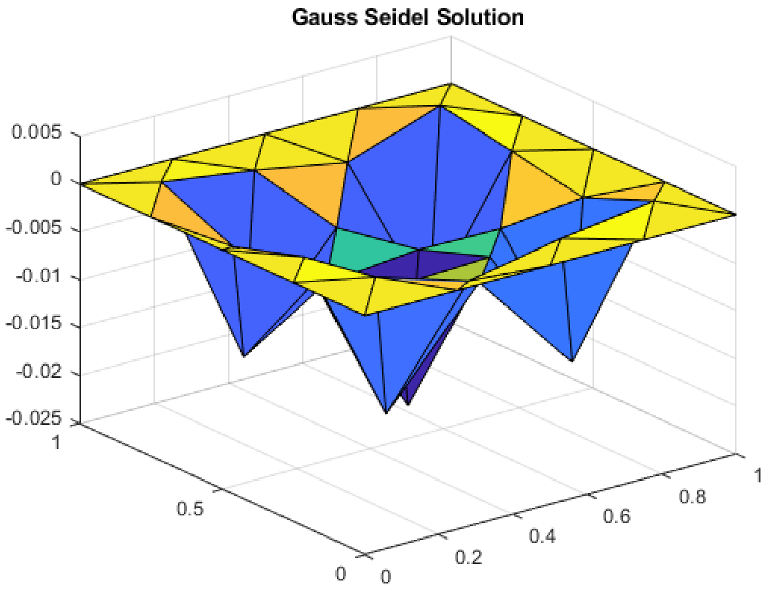
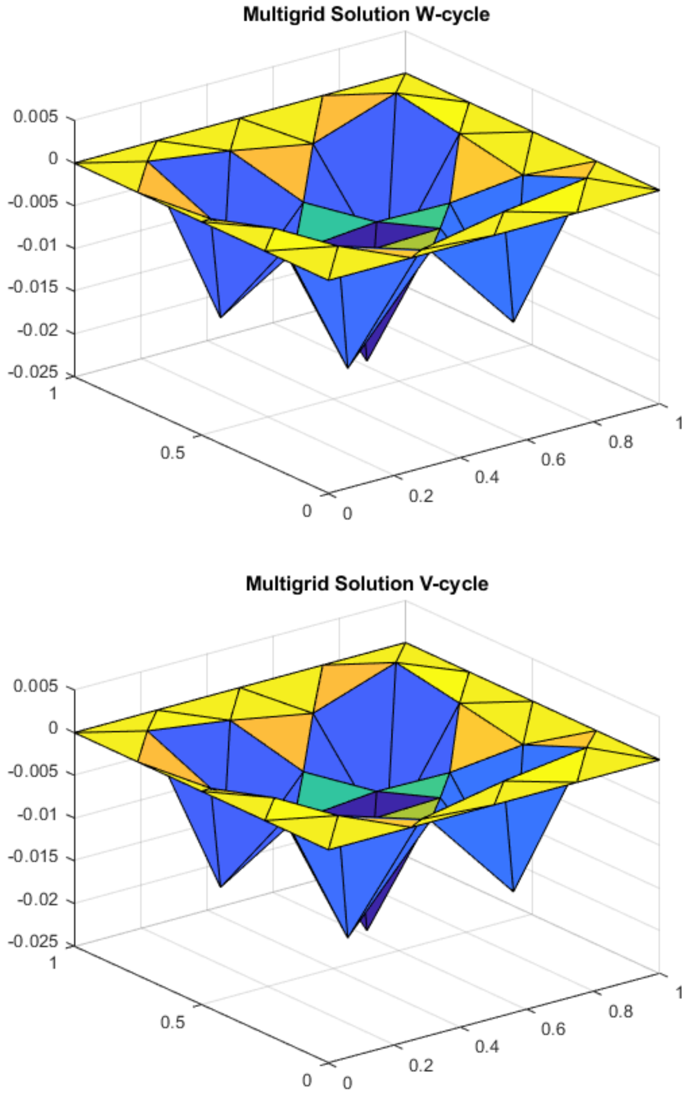

Gauss Seidel Solution

Figure 5.5: Solution of P.Q.V.I using Gauss Siedel method(after 20 iterations).


Multigrid Solution W-cycle

Multigrid Solution V-cycle

Figure 5.6: Solution of P.Q.V.I after 2 iterations of multigrid method.

Conclusion and future works

In this research, we obtain good approximations of the multigrid method which we applied it to four different types of parabolic quasi variational inequalities. First, we combined the finite element method and the finite difference method for getting the discrete problem, then, we reformulated the parabolic quasi variational inequality to the equivalent HGB-equation, we applied the multigrid method and the relaxation method to the system obtained from the parabolic quasi variational inequalities, and we found that the multigrid method is more efficient in reaching the solution compared to the relaxation method, especially in large size linear systems, because it reaches the solution with fewer iterations and in a shorter time.

In the future, we will study approximations of multigrid method for different types quasi variational inequalities with the operator A non linear, and we will take numerical applications using multigrid methods by Matlab program .

List of publications

1. M. Bahi, M. Beggas, N-E, Nesba and A Imtiaz, A numerical solution of Parabolic Quasi Variational inquality non-linear using Newton multigrid method, Iranian Journal of Numerical Analysis and Optimization, Vol. 14, No. 4, 2024, pp 991–1015.
2. M. Bahi, M. Beggas, M. Haiour and S. Boulaaras, Multigrid method for noncoercive parabolic variational inequality, Article number: 38 (2025), doi.org/10.1186/s13660-025-03285-8 , (Published online 20 March 2025).
3. M. Bahi, M. Beggas, M. Haiour and S. Belouafi, An approximate solution of parabolic quasi variational inequality by multigrid technique, Journal of Inequalities and Applications, (in review).

List of presentations

National conferences

1. M. Bahi and M. Beggas. A numerical solution of Parabolic Quasi Variational inquality using multigrid method, National seminar "Mthematics, Mathematics and Society"(MMS2023),Higher School of Teachers, Bousada, Algeria, 25-26 October 2023 .
2. M. Bahi and M. Beggas. A numerical solution of Parabolic Quasi Variational inquality with non linear right hand side using Newton multigrid method, "National Conference: Mathematical Modeling for Dynamic Systems (M2DS)", brothrs Mentouri Constantine University, Algeria, 26-27 June 2024 .
3. M. Bahi and M. Beggas. Uniform confergence of Parabolic Variational inquality non linear and its implementation , National conference " Mathematical modeling for Dynamic Systems"(M2DS2024),University of Annaba Badji Mokhtar,Algeria, 6-8 Octobre 2024.

International conferences

1. M. Bahi and M. Beggas. Approximation solution for Noncoercive Parabolic Variational inequality using multigrid methods ,Université 8 Mai 1945-Guelma, Algeria, November 19-20,2024

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