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Equilibrium equations for a class of elastic materials occupying a thin region

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Table des matières

Introduction **v**

1 Constitutive equations of continuum mechanics **2**

1.0.1 The equation of conservation of momentum. 2

1.0.2 The energy conservation equation 3

1.0.3 Linear behavior laws of elastic materials. 3

1.0.4 Non-isothermal case 5

2 Asymptotic analysis of a stationary problem of linear elasticity with friction **8**

2.1 Introduction and problem statement 9

2.2 Variational Formulation of the Problem 12

2.2.1 Existence and uniqueness of the solution 15

2.3 Asymptotic analysis of the problem 16

2.3.1 Variational formulation of the problem in Ω 17

2.3.2 Prior Estimation 18

2.3.3 Convergence Results and Limit Problem 21

3 Asymptotic Analysis of a Stationary Non-Isothermal Linear Elasticity Problem **28**

3.1 Introduction and Problem Statement 29

3.2 Variational Formulation of the Problem 30

3.3 Asymptotic Analysis of the Problem 33
3.3.1 A Priori Estimation 34
3.3.2 Convergence Results and Limit Problem 40

Introduction

In daily life, we find many materials occupying a thin area. Take, for example, cloth, paper, spare parts, car bodies and tires, etc., which, despite their differences, most of them share the elastic property of mechanical behavior.

This topic explores the laws of behavior and differential equations that govern elastic bodies that occupy a thin three-dimensional region whose thickness is assumed to be very small in relation to the other two dimensions.

The starting point is a system of partial differential equations defined on a three-dimensional region. These systems agree with the assumptions of continuity mechanics, so that the system can be described by general conservation equations represented by the laws of motion, the energy law, as well as the constitutive laws associated with each substance, such as Hooke's law for linear elasticity, and Kelvin-Voag's law for viscous elasticity. See [24] for more details on the constituves laws.

From a mathematical perspective, for several decades researchers have presented mathematical models consisting of differential equations with partial derivatives that provide descriptions of various mechanics problems, see $[19-23]$. The choice of boundary conditions is essential from both an engineering and mathematical point of view.

If the body is fixed at one end, Direchlet conditions are usually used [25], but if the boundaries are free, there are several models adopted in many sources, among which we mention $[5, 7, 25]$. If the area is in contact with sliding on a solid base material, then friction or frictional conditions are used as laws defined at the contact edge. Sometimes Newman's terms make the problem technically and mathematically consistent. For more details we refer the reader to Reference $[5, 7, 8, 23]$.

The thesis aims to define an asymptotic model for the problem of linear elasticity in a three-dimensional medium when the dimension in one direction is very negligible relative to the other two dimensions. The asymptotic technique is based on the steps:

i)- The change in scale, in relation to the thickness,

ii)- Estimates in a "fixed" domain,

iii)- Passage to the limit.

Thin material structure problem was built on the basis of sequences related to the weak film thickness, taking appropriate measurements on all assumed system data regarding the film thickness, see e.g. [2–5, 7–10], so it was absolutely necessary to make a priori estimates of stress and displacement, moreover, an estimate of temperature. All these estimates must be independent of the film thickness and also independent of time, in order to pass from the initial to the final problem.

This work that is being presented consists of :

The first chapter is devoted to recalling the main notions of the theory of continuous media and necessary functional analysis and the results used throughout this work. Then, we present the system of partial differential equations which model the non isothermal evolution of an elastic homogeneous body.

In the second chapter, we first prove the existence and uniqueness result of a weak solution, for linear elasticity equations in a bounded three-dimensional domain with friction conditions on part of the boundary and Dirichlet on the other part. In the second step, we study the asymptotic analysis when one of the dimensions of the thin region tends to zero.

In the third chapter, we are interested in the study of the asymptotic analysis of a problem of linear elasticity but this time non-isothermal, that is to say the elastic modulus μ depends on the temperature. We couple the equation for the conservation of momentum with the equation for the conservation of energy deduced from Fourier's law.

Notations

$\mathbb{R}$ set of real numbers,

$\mathbb{R}^d$ represents a d-dimensional Euclidean space with real-number coordinates $(d = 2, 3)$,

$\bar{\Omega}$ the adherence of Ω,

Ω^ε scaled three-dimensional region ,

Γ the boundary of Ω, assumed to be smooth, partitioned into three pairwise disjoint measurable parts.

Γ^ε the boundaries of Ω^ε,

Γ_i^ε the parts of the boundaries of $\Gamma^\varepsilon, (i = 1, 2, 3)$,

v the outgoing unit normal to Γ,

$L^\infty(\Omega)$ the space of Lebesgue-measurable functions on Ω such that $\exists c > 0$ $|u(x)| \le c$,a.e on Ω

$L^2(\Omega^\varepsilon)$ the space of measurable functions u^ε on Ω^ε such that $\int_{\Omega^\varepsilon} |u^\varepsilon|^2 dx < +\infty$

$C^1(\mathbb{R})$ the set of continuously differentiable functions on the real space $\mathbb{R}$,

$H^1(\Omega)$ the Sobolev space of order 1 sur Ω,

$H_0^1(\Omega)$ the adherence of $D(\Omega)$ in $H^1(\Omega)$,

$L^2(\Omega)^d$ the space $\{u = (u_i)/u_i \in L^2(\Omega), i = \overline{1, d}\}$,

$H^1(\Omega)^d$ the space $\{u = (u_i)/u_i \in H^1(\Omega), i = \overline{1, d}\}$,

δ_{ij} the symbol of Kronecker.

$a.e$ almost everywhere,

C a strictly positive generic constant,

(u, v) the inner product of vectors u and v,

$Div\ \sigma$ the divergence of the tensor σ

Chapitre 1

Constitutive equations of continuum mechanics

Consider a continuous medium which occupies a bounded domain Ω of $\mathbb{R}^3$ during a time interval $[0, T]$. Let $u(x, t)$ be the displacement field at time $t \in [0, T]$ of points $x = (x_1, x_2, x_3) \in \Omega$ of the continuous medium in motion relative to the orthogonal frame, and let $\sigma(x, t)$ be the stress tensor with components $\sigma_{ij}(i, j = 1, 2, 3)$, representing the tensor of stresses.

1.0.1 The equation of conservation of momentum.

The fundamental law of continuum mechanics expressing the equivalence between the tensor of external forces and the tensor of accelerations for any material system leads to the following equation of motion :

$$\rho \frac{d^2 u}{dt^2} = Div\ \sigma + f\ , \tag{1.0.1}$$

Where the vector f , with components $f_i(i = 1, 2, 3)$, represents a mass density of external forces, ρ is the mass density, and Div denotes the divergence operator.

$$Div\ \sigma = \left(\frac{\partial \sigma_{ij}}{\partial x_j}\right)_{i=1,2,3} .$$

1.0.2 The energy conservation equation

The general expression of the first law of thermodynamics is written as :

$$\rho \frac{de}{dt} = \sigma : D(u) - \operatorname{div} q + r, \tag{1.0.2}$$

where

- e is a scalar representing the specific internal energy of the continuous medium.

- r is a scalar representing the energy input per unit mass and time.

- q, with components q_i, is the energy transport vector.

- $D(u)$ is the rate of strain tensor, with components.

$$d_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \leq i, j \leq 3.$$

-$\sigma : D(u)$ is the product of two tensors σ and $D(u)$ defined by the expression.

$$\sigma : D(u) = \sum_{i,j=1}^{3} \sigma_{ij} d_{ij}(u).$$

From a mathematical standpoint, we have too many unknowns compared to the number of equations. Therefore, the laws of conservation of momentum and energy are insufficient to describe the movements of continuous media; they must be supplemented by other relationships called constitutive laws. Below, we present the constitutive laws of linear elasticity addressed in this thesis.

1.0.3 Linear behavior laws of elastic materials.

This law linking the stress tensor σ and the linearized strain tensor $D(u)$ is:

$$\sigma(u) = E \ : D(u) \qquad (\ \sigma_{ij}(u) = E_{ijkl}\ d_{kl}(u)\,). \tag{1.0.3}$$

Where $E = (E_{ijkl})$ is the fourth-order symmetric elasticity tensor. Its components E_{ijkl} are called elasticity coefficients and are independent of the strain tensor. In the case of a *homogeneous* material, the coefficients E_{ijkl} are constant (independent at point x in Ω). In practice, this law is often too general, and it is possible to simplify it by considering an *isotropic* material (a material that behaves the same in all directions). It is then called *Hooke's law* and is valid for a wide range of materials. In the isotropic case, the elasticity tensor E is defined by

$$E_{\ ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}. \tag{1.0.4}$$

Where λ and μ are the Lame coefficients that are specific to each material. δij denotes the Kronecker symbol.

Using equations (1.0.3) and (1.0.4), one can demonstrate that the stress tensor σ of *a homogeneous elastic and isotropic body* is:

$$\sigma(u) = 2\mu D\left(u\right) + \lambda TrD\left(u\right) I,$$

Where

- $D(u)$ is the strain rate tensor.

- I is the 3rd order identity matrix.

- $TrD(u)$ denotes the trace of $D(u)$ defined by

$$TrD\left(u\right) = \sum_{k=1}^{3} d_{kk}\left(u\right).$$

As the elastic body is considered *non-isothermal*, its coefficient μ is a function of temperature T.

$$\mu = \mu(T)$$

The constitutive law of *linear elasticity for a homogeneous and non-isothermal body* is therefore

$$\sigma(u) = 2\mu(T)D\left(u\right) + \lambda TrD\left(u\right) I. \tag{1.0.5}$$

From these laws, we will seek the partial differential equations modeling the evolution of a linear elastic homogeneous body in a bounded open domain Ω of $\mathbb{R}^3$ over a time interval $[0, T]$.

1.0.4 Non-isothermal case

Let's consider a homogeneous, elastic, isotropic, and heat-conducting body occupying a domain Ω in $\mathbb{R}^3$ over a time interval$[0, T]$.

If the environment is homogeneous, the density at point x in Ω remains constant over time, thus becoming increasingly independent of spatial variables. This is what we will assume.

$$\rho = \rho_0 = \text{constant}$$

It is even possible to set $\rho = 1$, which is done later, and simply amounts to choosing the unit of mass density.

Using the constitutive law (1.0.5), the equation of motion (1.0.1) becomes :

$$\frac{d^2u}{dt^2} = Div\,(2\mu(T)D\,(u) + \lambda TrD\,(u)\,I) + f\ .$$

we have therefore

$$\frac{d^2u}{dt^2} = Div\ (2\mu(T)D\,(u)\,) + \lambda\,\text{div}\,u + f\ . \tag{1.0.6}$$

We assume that

- e the specific internal energy is given by the law

$$\frac{de}{dt} = C_v(T)\frac{dT}{dt},$$

Where C_v represents the specific heat at constant volume.

- The energy input per unit mass r depends on the temperature T:

$$r = r(T)$$

The heat flux vector q through diffusion is given by Fourier's law:

$$q = -K\nabla T$$

Where K is the thermal conductivity.

Under these conditions, the energy equation (1.0.2) is written as:

$$C_v(T)\frac{dT}{dt} = \sigma : D(u) + \operatorname{div}(K\nabla T) + r(T). \tag{1.0.7}$$

Finally, the general equations for the dynamic problem of non-isothermal linear elasticity are given by the system:

$$\begin{cases} \dfrac{d^2u}{dt^2} = Div(2\mu(T)D\,(u)\,) + \lambda \operatorname{div} u + f, \\ C_v(T)\dfrac{dT}{dt} = \sigma : D(u) + \operatorname{div}(K\nabla T) + r(T). \end{cases} \tag{1.0.8}$$

In the steady-state case, we obtain the system:

$$\begin{cases} -Div\ (2\mu(T)D\,(u)\,) = \lambda \operatorname{div} u + f, \\ -\operatorname{div}(K\nabla T) = \sigma : D(u) + r(T). \end{cases} \tag{1.0.9}$$

Then, we assume that there is a tangential force on a part of the boundary denoted by ω, indicating frictional contact. We need to introduce a friction law that relates this tangential component to the other variables of the system. In this thesis, we will consider the friction law known as the Tresca type.

This friction law exhibits a fixed friction threshold K^ε when the solid and the foundation are in contact. The foundation applies a tangential force on the solid that does not exceed a certain threshold.

$$|\sigma_\tau^\varepsilon| \leq k^\varepsilon \quad \text{on} \quad \omega \times]0, T[\,.$$

As long as the tangential stress has not reached the threshold, the continuous medium cannot move relative to the obstacle, resulting in a state of blockage, which is expressed as:

$$|\sigma_\tau^\varepsilon| < k^\varepsilon \Longrightarrow \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau = 0 \quad \text{on} \quad \omega \times]0, T[\,.$$

When this threshold is reached, the solid can move tangentially relative to the foundation, resulting in sliding. The tangential stress opposes the velocity.

So, we have:

$$|\sigma_\tau^\varepsilon| = k^\varepsilon \Longrightarrow \text{there exists } \lambda \geq 0 \text{ such that } \sigma_\tau^\varepsilon = -\lambda \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau \quad \text{on} \quad \omega \times]0, T[\,.$$

In conclusion, the Tresca-type friction boundary conditions are written as follows:

$$\begin{cases} |\sigma_\tau^\varepsilon| \leq k^\varepsilon, \\ \text{If } \ |\sigma_\tau^\varepsilon| < k^\varepsilon \ \text{ then } \left(\dfrac{\partial u^\varepsilon}{\partial t}\right)_\tau = 0, \\ \text{If } \ |\sigma_\tau^\varepsilon| = k^\varepsilon \ \text{ then } \sigma_\tau^\varepsilon = -\lambda \left(\dfrac{\partial u^\varepsilon}{\partial t}\right)_\tau \ \text{ with } \lambda \geq 0. \end{cases} \quad \text{on} \quad \omega \times]0, T[\,, \tag{1.0.10}$$

Remarque 1.0.1 *When* $k^\varepsilon = 0$, *we obtain* $\sigma_\tau^\varepsilon = 0$ *on* $\omega \times \,]0, T[$. *This corresponds to the case without friction.*

Chapitre 2

Asymptotic analysis of a stationary problem of linear elasticity with friction

Abstract. In this chapter, we focus on the asymptotic analysis of a stationary problem for linear isothermal elasticity in a bounded domain of $\mathbb{R}^3$ with nonlinear Tresca friction conditions on a portion of the boundary and Dirichlet conditions on the other portion.

Content.

2.1 Introduction and problem statement;

2.2. Variational formulation of the problem;

2.2.1. Existence and uniqueness of the solution;

2.3. Asymptotic analysis of the problem;

2.3.1. Variational formulation of the problem in Ω;

2.3.2. Priori estimate;

2.3.3. Convergence results and limit problem.

2.1 Introduction and problem statement

We consider a problem involving deformations of a homogeneous elastic and isotropic body in a steady state within a thin domain Ω^ε, where ε is a positive real number belonging to $]0,1[$ and tends to zero.

The boundary of Ω^ε will be denoted by $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$, with:

*Γ_1^ε is the upper boundary defined by the equation $x_3 = \varepsilon h(x_1, x_2)$;

*Γ_L^ε is the lateral boundary;

*ω is a bounded domain in $\mathbb{R}^3$ defined by the equation $x_3 = 0$, which constitutes the lower boundary of the domain Ω^ε.

Let $x^{'} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $x = (x_1, x_2) \in \mathbb{R}^2$.

The domain Ω^ε is given by :

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3, (x, 0) \in \omega, \quad 0 < x_3 < \varepsilon h\,(x)\},$$

where h is a C^1 function defined on ω such that:

$$0 < h_* \leq h\,(x) \leq h^*, \ \forall\,(x, 0) \in \omega.$$

For given external forces f^ε , we assume that the deformations of an elastic body are governed by the following equations:

The law of conservation of momentum :

$$\frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0. \tag{2.1.1}$$

Here $\sigma^\varepsilon = (\sigma_{ij})^\varepsilon_{1\leq i,j\leq 3}$ represents the stress tensor, and $D = (d_{ij)_{1\leq i,j\leq 3}}$ represents the strain rate tensor:

$$d_{ij}\,(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right),\ 1 \leq i, j \leq 3.$$

We assume that the material behavior follows Hooke's law.

$$\sigma_{ij}^\varepsilon\,(u^\varepsilon) = 2\mu d_{ij}\,(u^\varepsilon) + \lambda d_{kk}\,(u^\varepsilon)\,\delta_{ij}, \tag{2.1.2}$$

Where:

- μ, λ are the Lamé coefficients, and δ_{ij} is the Kronecker delta symbol ;

- $u^\varepsilon(x)$ is the displacement velocity of the elastic body at point x.

The unit exterior normal vector to Γ^ε will be denoted as $n = (n_1, n_2, n_3)$.

The unit exterior normal to ω is the vector $(0, 0, -1)$.

We define the normal and tangential components u_n^ε and $u_T^\varepsilon = (u_{T_i}^\varepsilon)_{1\leq i\leq 3} \in \mathbb{R}^3$ of the velocity by

$$u_n^\varepsilon = u^\varepsilon.n,$$
$$u_{T_i}^\varepsilon = u_i^\varepsilon - u_n^\varepsilon.n_i.$$

Similarly, the normal and tangential components σ_n^ε and $\sigma_T^\varepsilon = \left(\sigma_{T_i}^\varepsilon\right)_{1\leq i\leq 3} \in \mathbb{R}^3$ of the stress tensor are defined by

$$\sigma_n^\varepsilon = (\sigma^\varepsilon.n_i)\,.n_j,$$
$$\sigma_{T_i}^\varepsilon = \sigma_{ij}^\varepsilon.n_j - (\sigma_n^\varepsilon)\,.n_i$$

To describe the boundary conditions, we first introduce the function $g = (g_1, g_2, g_3)$ such that:

$$\int_{\Gamma_L^\varepsilon} g.nds = 0. \tag{2.1.3}$$

The velocity on the boundary is given in terms of g except on ω.

$$u^\varepsilon = g = 0 \qquad\qquad on\ \Gamma_1^\varepsilon, \tag{2.1.4}$$

$$u^\varepsilon = g\ with\ \ g_3 = 0 \quad on\ \Gamma_L^\varepsilon. \tag{2.1.5}$$

On ω, the velocity is assumed unknown and it satisfies the following condition:

$$u.n = 0 \ \ on\ \omega. \tag{2.1.6}$$

Condition (2.1.6) is called the non-penetration condition.

Condition (2.1.3) is equivalent to the existence of a lifting $G^\varepsilon \in H^1(\Omega^\varepsilon)$ of g on Ω^ε verifying:

$$G^\varepsilon = g \text{ on } \Gamma_L^\varepsilon,\ G^\varepsilon = 0\ on\ \ \Gamma_1^\varepsilon,\ G^\varepsilon.n = 0\ on\ \omega$$

We also assume the existence of friction on ω,this friction is modeled by the nonlinear Tresca law:

$$\left.\begin{array}{c} |\sigma_T^\varepsilon| < k^\varepsilon \implies u_T^\varepsilon = s \\ |\sigma_T^\varepsilon| = k^\varepsilon \implies \exists \beta \geq 0 \text{ such that } u_T^\varepsilon = s - \beta\sigma_T^\varepsilon \end{array}\right\} \quad on\ \omega \tag{2.1.7}$$

where $|.|$ denotes the Euclidean norm of $\mathbb{R}^2$, s is the shear velocity, k^ε is the friction threshold.

Remark 2.1.1. The third component of the velocity satisfies:

$$u_3^\epsilon = 0 \qquad on\ \Gamma^\epsilon \tag{2.1.8}$$

Indeed, according to condition (2.1.6), we have :

$$u^\epsilon n = u_1^\epsilon n_1 + u_2^\epsilon n_2 + u_3^\epsilon n_3 = 0 \qquad on\ \ \omega$$

where, $n = (0, 0, -1)$ is the unit exterior normal vector to ω hence,

$$u_3^\epsilon = 0 \text{ implies that } u_3^\epsilon = 0 \ \ on\ \omega.$$

According to conditions (2.1.4) and (2.1.5) and the fact that $g_3 = 0$ on Γ_L^ϵ, $u_3^\epsilon = 0$ on $\Gamma_1^\epsilon \cup \Gamma_L^\epsilon$ because $u^\epsilon = 0$ on Γ_1^ϵ, thus: $u_3^\epsilon = 0$ on Γ^ε.
The complete problem consists of finding a velocity field u^ε satisfying the following equations and boundary conditions:

$$(Pb^\varepsilon)\begin{cases} \begin{array}{l} \dfrac{\partial\sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0 \\ \sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + \lambda d_{kk}(u^\varepsilon)\delta_{ij}, \ \ 1 \leq i, j \leq 3 \end{array} & in\ \ \Omega^\varepsilon, \\ u^\varepsilon = 0 & on\ \ \Gamma_1^\varepsilon, \\ u^\varepsilon = g & on\ \ \Gamma_L^\varepsilon, \\ u^\varepsilon.n = 0 & on\ \ \omega, \\ \left.\begin{array}{c} |\sigma_T^\varepsilon| < k^\varepsilon \implies u_T^\varepsilon = s \\ |\sigma_T^\varepsilon| = k^\varepsilon \implies \exists \beta \geq 0 \text{ such that } u_T^\varepsilon = s - \beta\sigma_T^\varepsilon \end{array}\right\} & on\ \ \omega. \end{cases}$$

To provide the variational formulation of problem (pb^ε), we will establish the following lemma :

Lemma2.1.1. Condition (2.1.7) is equivalent to the following relation :

$$(u_T^\epsilon - s)\,\sigma_T^\epsilon + k^\epsilon |u_T^\epsilon - s| = 0 \qquad on\ \omega.$$

Proof. Suppose that $(u_T^\varepsilon - s)\,\sigma_T^\varepsilon + k^\varepsilon |u_T^\varepsilon - s| = 0.$

▷ If $|\sigma_T^\varepsilon| = k^\varepsilon$, then

$$(u_T^\epsilon - s)\,\sigma_T^\epsilon = -\,|\sigma^\varepsilon|\,|u_T^\epsilon - s|,$$

which implies the existence of a $\beta \geq 0$ such that

$$u_T^\epsilon - s = -\beta\sigma_T^\varepsilon\ .$$

▷ If $|\sigma^\varepsilon| < k^\varepsilon$,then

$$\begin{aligned}(u_T^\epsilon - s)\,\sigma_T^\epsilon + k^\varepsilon |u_T^\epsilon - s| \ &= 0 \geq -\,|u_T^\epsilon - s|\,.\,|\sigma_T^\epsilon| + k^\varepsilon |u_T^\epsilon - s| \\ &\geq\ |u_T^\epsilon - s|\,.\,(-\,|\sigma_T^\epsilon| + k^\varepsilon)\end{aligned}$$

and since $-\,|\sigma_T^\epsilon| + k^\varepsilon > 0$, then $u_T^\epsilon = s$.

Conversely, suppose that u^ε satisfies the Tresca boundary condition.

▷ If $|\sigma_T^\varepsilon| < k^\varepsilon$, then $u_T^\varepsilon = s$

$$(u_T^\epsilon - s)\,\sigma_T^\epsilon + k^\epsilon |u_T^\epsilon - s| = (s - s)\,\sigma_T^\epsilon + k^\epsilon |s - s| = 0\ .$$

▷ If $|\sigma_T^\varepsilon| = k^\varepsilon$, then there exists $\beta \geq 0$ such that $u_T^\varepsilon = s - \beta\sigma_T^\varepsilon$.

Hence

$$(u_T^\epsilon - s)\,\sigma_T^\epsilon + k^\epsilon |u_T^\epsilon - s| = -\beta\,|\sigma_T^\epsilon|^2 + \beta\,|\sigma_T^\varepsilon|^2 = 0.$$

2.2 Variational Formulation of the Problem

For the open set Ω^ε , we define the following space and set:

$$H^1(\Omega^\varepsilon)^3 = \left\{ v \in \left(L^2(\Omega^\varepsilon)\right)^3 : \frac{\partial v_i}{\partial x_j} \in L^2(\Omega^\varepsilon)\,,\ \forall i,j = 1,...,3 \right\},$$

the Sobolev space equipped with the norm $||.||_{1.\Omega^\varepsilon}$,where the norm of $\left(L^2(\Omega^\varepsilon)\right)^3$will be denoted $||.||_{0.\Omega^\varepsilon}$

We define the non-empty closed convex set of $H^1(\Omega^\varepsilon)^3$

$$K^\varepsilon = \left\{ v \in H^\varepsilon(\Omega^\varepsilon)^3 : v = 0 \ on\ \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon,\ v.n = 0 \ on\ \omega \right\}.$$

To simplify the writing, we denote:

$$a\left(u,v\right)=2\mu\int_{\Omega^{\varepsilon}}d_{ij}\left(u\right)d_{ij}\left(v\right)dx'+\lambda\int_{\Omega^{\varepsilon}}\operatorname{div}\left(u\right)\operatorname{div}\left(v\right)dx'.$$

For $v\in H^{1}(\Omega^{\varepsilon})^{3}$ we define the functional J^{ε} as :

$$J^{\varepsilon}\left(v\right)=\int_{\omega}k^{\varepsilon}|v-s|dx.$$

Finally, we denote:

$$\left(f,v\right)=\int_{\Omega^{\varepsilon}}f_{i}v_{i}dx',\ \forall v\in H^{1}\left(\Omega^{\varepsilon}\right).$$

Lemma2.2.1. *if u^{ε} is a solution of problem* (Pb^{ε}), then it satisfies the following variational problem:

$$(PV^{\varepsilon})\begin{cases}\text{find } u^{\varepsilon}\in K^{\varepsilon}\text{, such as}\\ a\left(u^{\varepsilon},\varphi-u^{\varepsilon}\right)+J^{\varepsilon}\left(\varphi\right)-J^{\varepsilon}\left(u^{\varepsilon}\right)\geq\left(f^{\varepsilon},\varphi-u^{\varepsilon}\right),\quad\forall\varphi\in K^{\varepsilon}.\end{cases}$$

Proof. By multiplying equation (2.1.1) by $(\varphi-u^{\epsilon})$, where $\varphi\in K^{\epsilon}$, and using the Green's formula, we obtain:

$$\int_{\Omega^{\epsilon}}\sigma_{ij}^{\varepsilon}\frac{\partial}{\partial x_{j}}\left(\varphi_{i}-u_{i}^{\epsilon}\right)dxdx_{3}-\int_{\Gamma^{\epsilon}}\sigma_{ij}^{\varepsilon}n_{j}\left(\varphi_{i}-u_{i}^{\epsilon}\right)d\sigma=\int_{\Omega^{\epsilon}}f_{i}^{\epsilon}\left(\varphi_{i}-u_{i}^{\epsilon}\right)dxdx_{3}.$$

The boundary conditions (2.1.4) and (2.1.5) imply that:

$$\int_{\Gamma^{\epsilon}}\sigma_{ij}^{\epsilon}n_{j}\left(\varphi_{i}-u_{i}^{\epsilon}\right)d\sigma=\int_{\omega}\sigma_{ij}^{\epsilon}n_{j}\left(\varphi_{i}-u_{i}^{\epsilon}\right)dx.$$

But $\sigma_{ij}^{\varepsilon}n_{j}=\sigma_{T_{i}}^{\epsilon}+\sigma_{n}^{\epsilon}n_{i}$, we find :

$$\int_{\Gamma^{\epsilon}}\sigma_{ij}^{\epsilon}n_{j}\left(\varphi_{i}-u_{i}^{\epsilon}\right)d\sigma=\int_{\omega}\sigma_{T_{i}}^{\epsilon}\left(\varphi_{i}-u_{i}^{\epsilon}\right)dx+\int_{\omega}\sigma_{n}^{\varepsilon}n_{i}\left(\varphi_{i}-u_{i}^{\epsilon}\right)dx$$

and since $\left(\varphi_{i}-u_{i}^{\epsilon}\right)n_{i}=0$, we have :

$$\int_{\Gamma^{\epsilon}}\sigma_{ij}^{\epsilon}n_{j}\left(\varphi_{i}-u_{i}^{\epsilon}\right)d\sigma=\int_{\omega}\sigma_{T_{i}}^{\epsilon}\left(\varphi_{i}-u_{i}^{\epsilon}\right)dx.$$

So :

$$\int_{\Omega^{\epsilon}}\sigma_{ij}^{\epsilon}\frac{\partial}{\partial x_{j}}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)dxdx_{3}-\int_{\omega}\sigma_{T_{i}}^{\varepsilon}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)dx=\int_{\Omega^{\epsilon}}f_{i}^{\varepsilon}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)dxdx_{3}\ \forall\varphi\in K^{\varepsilon}.\quad(2.2.9)$$

in (2.2.9) we add and subtract the term $\int_\omega k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx$, we obtain: :

$$\int_{\Omega^\epsilon} \sigma^\epsilon_{ij} \frac{\partial}{\partial x_j} \left(\varphi_i - u^\varepsilon_i\right) dx dx_3 - \int_\omega \sigma^\varepsilon_{T_i} \left(\varphi_i - u^\varepsilon_i\right) dx + \int_\omega k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx -$$

$$- \int_\omega k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx = \int_{\Omega^\epsilon} f^\varepsilon_i \left(\varphi_i - u^\varepsilon_i\right) dx dx_3.$$

Let :

$$B = \int_\omega \sigma^\varepsilon_{T_i} \left(\varphi_i - u^\varepsilon_i\right) dx + \int_\omega k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx.$$

So :

$$\begin{aligned} B &= \int_\omega \sigma^\varepsilon_{T_i} \left(\varphi_i - u^\varepsilon_i + s - s\right) dx + \int_\omega k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx \\ &= \int_\omega \sigma^\varepsilon_{T_i} \left(\varphi_i - s\right) dx - \int_\omega \sigma^\varepsilon_{T_i} \left(u^\varepsilon_i - s\right) dx + \int_\omega k^\varepsilon |\varphi - s| dx - \int_\omega k^\varepsilon |u^\varepsilon - s| dx. \end{aligned}$$

Using Lemma (2.1.1) we get :

$$B = \int_\omega \sigma^\varepsilon_{T_i} \left(\varphi_i - s\right) dx + \int_\omega k^\varepsilon |\varphi - s| dx.$$

Then :

$$\sigma^\varepsilon_T \left(\varphi - s\right) \geq -|\sigma^\varepsilon_T||\varphi - s| \geq -k^\varepsilon |\varphi - s| \qquad on \ \ \omega$$

and

$$B = \int_\omega \sigma^\varepsilon_{T_i} \left(\varphi_i - s\right) dx + \int_\omega k^\varepsilon |\varphi - s| dx \geq 0.$$

This implies the following inequality:

$$\int_{\Omega^\epsilon} \sigma^\varepsilon_{ij} \frac{\partial}{\partial x_j} \left(\varphi_i - u^\varepsilon_i\right) dx dx_3 + \int_\omega k^\varepsilon |\varphi - s| dx - \int_\omega k^\varepsilon |u^\varepsilon - s| dx \geq \int_{\Omega^\epsilon} f^\varepsilon_i \left(\varphi_i - u^\varepsilon_i\right) dx dx_3.$$

And thus, we obtain the variational problem in velocity:

$$\begin{cases} \text{Find } u^\varepsilon \in K^\varepsilon, \text{such that} \\ a\left(u^\varepsilon, \varphi - u^\varepsilon\right) + J^\varepsilon\left(\varphi\right) - J^\varepsilon\left(u^\varepsilon\right) \geq \left(f^\varepsilon, \varphi - u^\varepsilon\right), \ \ \forall \varphi \in K^\varepsilon. \end{cases} \tag{2.2.10}$$

2.2.1 Existence and uniqueness of the solution

We recall a theorem of existence and uniqueness for second-kind variational inequalities that we will apply to study the variational problem (PV^{ε}).

Theorem 2.2.1. *Let* A be a non-empty closed convex set in a Hilbert space$V$, *equipped with the norm* $\|.\|_V$, $a(.,.)$ *be a continuous and coercive bilinear form from* $A \times A$ *to* $\mathbb{R}$ *, and* J *be a functional from* A *to* $\bar{\mathbb{R}}$ that is convex , *lower semi-continuous, and proper. Then, for any linear form,* $\mathcal{L}$ defined on V, *there exists a unique* u *in*V that solves the variational inequality:

$$a(u, v-u) + J(v) - J(u) \geq \mathcal{L}(v-u).$$

Lemma 2.2.2. Suppose $f^{\varepsilon} \in \left(L^2(\Omega^{\varepsilon})\right)^3$ and $k^{\varepsilon} \in L^{\infty}(\omega)$, *where* $k^{\varepsilon} \geq 0$ *almost everywhere in* ω. *Then, there exists one and only one* u^{ε} *in* K^{ε} *satisfying the variational inequality* (2.2.10).

Proof. The bilinear form $a(.,.)$ is coercive on $K^{\varepsilon} \times K^{\varepsilon}$.

Indeed, let v be an element of K^{ε}. By the Korn inequality, we have

$$a(v,v) \geq 2\mu Ck \sum_{i,j=1}^{3} \|d_{ij}(v)\|^2_{L^2(\Omega^{\varepsilon})} \geq 2\mu C_k \|v\|^2_{H^1(\Omega^{\varepsilon})},$$

where $C_k > 0$, is the Korn constant.

The bilinear form $a(.,.)$ is continuous on $K^{\varepsilon} \times K^{\varepsilon}$.

Indeed, let v and u be two elements of K^{ε}.Using the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
|a(u^{\varepsilon}, v^{\varepsilon})| \leq\ & 2\mu \left(\sum_{i,j=1}^{3} \|d_{ij}(u^{\varepsilon})\|^2_{L^2(\Omega^{\varepsilon})}\right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{3} \|d_{ij}(v^{\varepsilon})\|^2_{L^2(\Omega^{\varepsilon})}\right)^{\frac{1}{2}} + \\
& + \lambda \|\mathrm{div}(u^{\varepsilon})\|_{L^2(\Omega^{\varepsilon})} \cdot \|\mathrm{div}(v^{\varepsilon})\|_{L^2(\Omega^{\varepsilon})} \\
& \leq 2\mu \|u^{\varepsilon}\|_{H^1(\Omega^{\varepsilon})} \|v^{\varepsilon}\|_{H^1(\Omega^{\varepsilon})} + \lambda \|u^{\varepsilon}\|_{H^1(\Omega^{\varepsilon})} \|v^{\varepsilon}\|_{H^1(\Omega^{\varepsilon})} \\
& \leq C \|u^{\varepsilon}\|_{H^1(\Omega^{\varepsilon})} \|v^{\varepsilon}\|_{H^1(\Omega^{\varepsilon})},
\end{aligned}$$

where

$$C = 2\mu + \lambda$$

The linear form $(f^{\varepsilon}, .)$ is continuous on K^{ε}.

Indeed, let v be an element of K^{ε},By using the Cauchy-Schwarz inequality, we obtain:

$$|(f^{\varepsilon}, v)| \leq \|f^{\varepsilon}\|_{L^{\varepsilon}(\Omega^{\varepsilon})} \|v^{\varepsilon}\|_{L^{\varepsilon}(\Omega^{\varepsilon})}.$$

And since J is a convex functional, lower semi-continuous, and proper, the previous theorem demonstrates the existence and uniqueness of the solution to (2.2.10).

2.3 Asymptotic analysis of the problem

For the asymptotic analysis of our problem, we use the scaling $z = \dfrac{x_3}{\varepsilon}$.
Thus, the domain Ω^{ε} transforms into a domain Ω independent of ε.
where

$$\Omega = \{(x, z) \in \mathbb{R}^3, (x, 0) \in \omega \ et \ \ 0 < z < h(x)\}.$$

We denote $\Gamma = \bar{\omega} \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_L$ its boundary.

Now, we define new unknowns on Ω .

$$\begin{cases} \hat{u}_i^{\varepsilon}(x, z) = u_i^{\varepsilon}(x, x_3), \\ \hat{u}_3^{\varepsilon}(x, z) = \varepsilon^{-1} u_3^{\varepsilon}(x, x_3). \end{cases} \tag{2.3.1}$$

For the data of problem (Pb^{ε}), we assume that they depend on ε as follows:

$$\begin{cases} \hat{f}(x, z) = \varepsilon^2 f^{\varepsilon}(x, x_3), \\ \hat{k} = \varepsilon k^{\varepsilon}, \\ \hat{g}(x, z) = g^{\varepsilon}(x, x_3), \end{cases} \tag{2.3.2}$$

where $\hat{f}$, $\hat{k}$ and $\hat{g}$ do not depend on ε.
Let $\hat{G}(x, z)$ be such that :

$$\hat{G} = \hat{g} \ \text{ on } \Gamma.$$

The vector G^{ε} introduced previously will be defined as follows:

$$\begin{cases} \hat{G}_i(x, z) = G_i^{\varepsilon}(x, x_3), \\ \hat{G}_3(x, z) = \varepsilon^{-1} G_3^{\varepsilon}(x, x_3). \end{cases} \tag{2.3.3}$$

2.3.1 Variational formulation of the problem in Ω

Now we introduce the functional framework on Ω as follows :

$$K = \left\{\varphi \in H^1(\Omega)^3 : \varphi = 0 \ on \ \Gamma_L \cup \Gamma_1 \text{ and } \varphi.n = 0 \ on \ \omega\right\}.$$

$$V_z = \left\{v = (v_1, v_2) \in L^2(\Omega)^2 : \frac{\partial v_i}{\partial z} \in L^2(\Omega), i = 1, 2 \text{ and } \ v = 0 \ on \ \Gamma_1\right\}.$$

$$\Pi(k) = \left\{\varphi \in H^1(\Omega)^2 : \varphi = (\varphi_1, \varphi_2), \varphi_i = 0 \ on \ \Gamma_L \cup \Gamma_1 \ for \ i = 1, 2\right\}.$$

V_z is a Banach space with the norm :

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2} \|v_i\|^2_{L^2(\Omega)} + \left\|\frac{\partial v_i}{\partial z}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

By multiplying (2.2.10) by ε and passing to the fixed domain Ω we show that the variational problem is equivalent to the following problem:

$$\begin{cases} Find \ \hat{u}^\varepsilon \in K, \text{ such that:} \\ a(\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) + J(\hat{\varphi}) - J(\hat{u}^\varepsilon) \geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - \hat{u}^\varepsilon_i\right) + \varepsilon\left(\hat{f}_3, \hat{\varphi}_3 - \hat{u}^\varepsilon_3\right), \quad \forall \hat{\varphi} \in K \end{cases} \tag{2.3.4}$$

where

$$J(\hat{\varphi}) = \int_\omega \hat{k}\,|\hat{\varphi} - s|\,dx$$

and

$$\begin{aligned} a(\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) = {}& \mu\varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial x_j} + \frac{\partial \hat{u}^\varepsilon_j}{\partial x_i}\right) \frac{\partial}{\partial x_j}(\hat{\varphi}_i - \hat{u}^\varepsilon_i)\,dxdz + \\ & + \mu \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}^\varepsilon_3}{\partial x_i}\right)\left[\frac{\partial}{\partial z}(\hat{\varphi}_i - \hat{u}^\varepsilon_i) + \varepsilon^2 \frac{\partial}{\partial x_i}(\hat{\varphi}_3 - \hat{u}^\varepsilon_3)\right] dxdz + \\ & + 2\mu\varepsilon^2 \int_\Omega \frac{\partial \hat{u}^\varepsilon_3}{\partial z}.\frac{\partial}{\partial z}(\hat{\varphi}_3 - \hat{u}^\varepsilon_3)\,dxdz + \lambda\varepsilon^2 \int_\Omega \operatorname{div}(\hat{u}^\varepsilon).\operatorname{div}(\hat{\varphi} - \hat{u}^\varepsilon)\,dxdz. \end{aligned}$$

2.3.2 Prior Estimation

Let's delve into studying the prior estimates on $\hat{u}^{\varepsilon}$.For this purpose, we need to establish the following lemma:

Lemma 2.3.1. *(Poincaré Inequality). We recall that* $0 < h(x) < h^*(x)$, $\forall x \in \omega$, we have the following inequality:

$$\|\hat{u}_i^{\varepsilon}\|_{L^2(\Omega)} \leq h^* \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}.$$

Proof. Let $0 < z < h(x)$, we have

$$\hat{u}_i^{\varepsilon}(x,z) = -\int_z^{h(x)} \frac{\partial \hat{u}_i^{\varepsilon}}{\partial \zeta}(x,\zeta)\, d\zeta + u_i^{\varepsilon}(x,h(x)), \quad i = 1,2,3$$

since $u_i^{\varepsilon}(x,h(x)) = 0$, then :

$$\hat{u}_i^{\varepsilon}(x,z) = -\int_z^{h(x)} \frac{\partial \hat{u}_i^{\varepsilon}}{\partial \zeta}(x,\zeta)\, d\zeta.$$

By the Cauchy-Schwartz inequality, we get :

$$|\hat{u}_i^{\varepsilon}(x,z)|^2 \leq (h^*) \int_0^{h(x)} \left| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial \zeta}(x,\zeta) \right|^2 d\zeta. \tag{2.3.5}$$

Integrating (2.3.5) over z from 0 to $h(x)$, we obtain :

$$\int_0^{h(x)} |\hat{u}_i^{\varepsilon}(x,z)|^2 dz \leq (h^*)^2 \int_0^{h(x)} \left| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial \zeta}(x,\zeta) \right|^2 d\zeta,$$

By integrating the above inequality over ω, we find

$$\int_\omega \int_0^{h(x)} |\hat{u}_i^{\varepsilon}(x,z)|^2 dz dx \leq (h^*)^2 \int_\omega \int_0^{h(x)} \left| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial \zeta}(x,\zeta) \right|^2 d\zeta dx$$

$$\|\hat{u}_i^{\varepsilon}\|_{L^2(\Omega)} \leq h^* \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}.$$

Lemma 2.3.2. (Korn Inequality). *for any* $\varphi \in K^{\varepsilon}$, *we have :*

$$\|\nabla \varphi\|_{L^2(\Omega^{\varepsilon})} \leq C \|D(\varphi)\|_{L^2(\Omega^{\varepsilon})},$$

where C *is a positive constant independent of* ε *and* φ.

Lemma 2.3.3. *given* $f \in L^2(\Omega)^3$ *and* k *a positive function in* $L^{\infty}(\omega)$, *there exists a constant* $c > 0$ *independent of* ε *such that :*

$$\varepsilon^2 \sum_{i,j=1}^{2} \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j} \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}^2 + \varepsilon^4 \left\| \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right) + \varepsilon^2 \left\| \frac{\partial \hat{u}_3^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}^2 \leq c$$

Proof. Let u^ε be the solution of problem (2.2.10), hence

$$a\left(u^\varepsilon,\varphi-u^\varepsilon\right)+J^\varepsilon\left(\varphi\right)-J^\varepsilon\left(u^\varepsilon\right)\geq\left(f^\varepsilon,\varphi-u^\varepsilon\right)$$

This implies:

$$a\left(u^\varepsilon,u^\varepsilon\right)\leq a\left(u^\varepsilon,\varphi\right)+J^\varepsilon\left(\varphi\right)-J^\varepsilon\left(u^\varepsilon\right)+\left(f^\varepsilon,u^\varepsilon\right)-\left(f^\varepsilon,\varphi\right)\qquad\forall\varphi\in K^\varepsilon.$$

And since $J^\varepsilon\left(u^\varepsilon\right)$ is positive (because $k>0$),we have :

$$a\left(u^\varepsilon,u^\varepsilon\right)\leq a\left(u^\varepsilon,\varphi\right)+J^\varepsilon\left(\varphi\right)+\left(f^\varepsilon,u^\varepsilon\right)-\left(f^\varepsilon,\varphi\right)\qquad\forall\varphi\in K^\varepsilon$$

but

$$\begin{aligned}a\left(u^\varepsilon,u^\varepsilon\right)&=2\mu\sum_{i,j=1}^{2}\int_{\Omega^\varepsilon}d_{ij}\left(u^\varepsilon\right)d_{ij}\left(u^\varepsilon\right)dxdx_3+\lambda\int_{\Omega^\varepsilon}\operatorname{div}\left(u^\varepsilon\right).\operatorname{div}\left(u^\varepsilon\right)dxdx_3\\&=2\mu\sum_{i,j=1}^{2}\int_{\Omega^\varepsilon}\left|d_{ij}\left(u^\varepsilon\right)\right|^2dxdx_3+\lambda\int_{\Omega^\varepsilon}\left|\operatorname{div}\left(u^\varepsilon\right)\right|^2dxdx_3.\end{aligned}$$

Due to the coercivity of $a\left(.,.\right)$, there exists C_K independent of ε such that

$$a\left(u^\varepsilon,u^\varepsilon\right)\geq2\mu C_K\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega)}.\tag{2.3.6}$$

Applying Ho[redacted]lder's and Young's inequalities, we obtain the following upper bound:

$$\begin{aligned}a\left(u^\varepsilon,\varphi\right)&\leq\int_{\Omega^\varepsilon}2\mu\left|d_{ij}\left(u^\varepsilon\right)\right|\left|d_{ij}\left(\varphi\right)\right|dxdx_3+\lambda\int_{\Omega^\varepsilon}\left|\operatorname{div}\left(u^\varepsilon\right)\right|\left|\operatorname{div}\left(\varphi\right)\right|dxdx_3\\&\leq\left(\int_{\Omega^\varepsilon}\sqrt{\frac{\mu C_K}{2}}\left|d_{ij}\left(u^\varepsilon\right)\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega^\varepsilon}\frac{2\sqrt{2\mu}}{\sqrt{C_K}}\left|d_{ij}\left(\varphi\right)\right|^2dxdx_3\right)^{\frac{1}{2}}+\\&+\left(\int_{\Omega^\varepsilon}\frac{\sqrt{\mu C_K}}{2}\left|\operatorname{div}\left(u^\varepsilon\right)\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega^\varepsilon}\frac{2\lambda}{\sqrt{\mu C_K}}\left|\operatorname{div}\left(\varphi\right)\right|^2dxdx_3\right)^{\frac{1}{2}}\\&\leq\frac{\mu C_K}{4}\left\|d_{ij}\left(u^\varepsilon\right)\right\|^2_{L^2(\Omega^\varepsilon)}+\frac{4\mu}{C_K}\left\|d_{ij}\left(\varphi\right)\right\|^2_{L^2(\Omega^\varepsilon)}+\\&+\frac{\mu C_K}{8}\int_{\Omega^\varepsilon}\left|\operatorname{div}\left(u^\varepsilon\right)\right|^2dxdx_3+\frac{2\lambda^2}{\mu C_K}\int_{\Omega^\varepsilon}\left|\operatorname{div}\left(\varphi\right)\right|^2dxdx_3.\end{aligned}$$

On the one hand, we know that:

$$\sum_{i,j=1}^{2}\left|d_{ij}\left(u^\varepsilon\right)\right|^2\leq\left|\nabla u^\varepsilon\right|^2\qquad et\quad\left|\operatorname{div}\left(u^\varepsilon\right)\right|^2\leq\left|\nabla u^\varepsilon\right|^2$$

thus we have:

$$a\left(u^{\varepsilon},\varphi\right) \leq \frac{\mu C_K}{4}\left\|\nabla u^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{4\mu}{C_K}\left\|\nabla \varphi\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu C_K}{8}\left\|\nabla u^{\varepsilon}\right\|^2_{L^{\varepsilon}(\Omega^{\varepsilon})} + \frac{2\lambda^2}{\mu C_K}\left\|\nabla \varphi\right\|^2_{L^{\varepsilon}(\Omega^{\varepsilon})}. \tag{2.3.7}$$

Using the Poincaré inequality, we get:

$$\left\|u^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})} \leq \varepsilon h^* \left\|\nabla u^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})}.$$

Analogously to (2.3.7), we get:

$$\begin{aligned}
\left|\left(f^{\varepsilon},u^{\varepsilon}\right)\right| &\leq \left\|f^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})}\left\|u^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})} \qquad \text{(Cauchy- Schwartz)} \\
&\leq \varepsilon h^*\left\|\nabla u^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})}\left\|f^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})} \\
&\leq \frac{\mu C_K}{2}\left\|\nabla u^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\left(\varepsilon h^*\right)^2}{2\mu C_K}\left\|\nabla f^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} \qquad \text{(Yong inequality)}.
\end{aligned} \tag{2.3.8}$$

Likewise:

$$\left|\left(f^{\varepsilon},\varphi\right)\right| \leq \frac{\mu C_K}{2}\left\|\nabla \varphi\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\left(\varepsilon h^*\right)^2}{2\mu C_K}\left\|\nabla f^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})}. \tag{2.3.9}$$

Using (2.3.6) $-$ (2.3.9) ,and choosing $\varphi = G^{\varepsilon}$, we have:

$$\begin{aligned}
2\mu C_K\left\|\nabla u^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} \leq a\left(u^{\varepsilon},u^{\varepsilon}\right) \leq \frac{\mu C_K}{4}\left\|\nabla u^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{4\mu}{C_K}\left\|\nabla G^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu C_K}{8}\left\|\nabla u^{\varepsilon}\right\|^2_{L^{\varepsilon}(\Omega^{\varepsilon})} + \\
+ \frac{2\lambda^2}{\mu C_K}\left\|\nabla G^{\varepsilon}\right\|^2_{L^{\varepsilon}(\Omega^{\varepsilon})} + \frac{\mu C_K}{2}\left\|\nabla u^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\left(\varepsilon h^*\right)^2}{2\mu C_K}\left\|\nabla f^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \\
+ \frac{\mu C_K}{2}\left\|\nabla G^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\left(\varepsilon h^*\right)^2}{2\mu C_K}\left\|\nabla f^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})}.
\end{aligned}$$

This implies :

$$\frac{9}{8}\mu C_K\left\|\nabla u^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} \leq \frac{\left(\varepsilon h^*\right)^2}{\mu C_K}\left\|\nabla f^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} + \left(\frac{2\lambda^2}{\mu C_K} + \frac{\mu C_K}{2} + \frac{4\mu}{C_K}\right)\left\|\nabla G^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})}. \tag{2.3.10}$$

But :

$$\varepsilon^2\left\|f^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} = \varepsilon^{-1}\left\|\hat{f}\right\|^2_{L^2(\Omega)} \quad \text{and} \quad \left\|\frac{\partial u_3^{\varepsilon}}{\partial x_3}\right\|^2_{L^2(\Omega^{\varepsilon})} = \varepsilon^{-1}\left\|\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}\right\|^2_{L^2(\Omega)},$$

thus we obtain:

$$\varepsilon\left\|\nabla u^{\varepsilon}\right\|^2_{L^2(\Omega^{\varepsilon})} = \varepsilon^2\sum_{i,j=1}^{2}\left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z}\right\|^2_{L^2(\Omega)} + \varepsilon^4\sum_{j=1}^{2}\left\|\frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_j}\right\|^2_{L^2(\Omega)} + \varepsilon^2\left\|\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}\right\|^2_{L^2(\Omega)}.$$

Multiplying (2.3.10) by ε, we find :

$$\begin{aligned}\frac{9}{8}\mu C_K\ \varepsilon\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} &\le \frac{\varepsilon^3 h^{*2}}{\mu C_K}\left\|\nabla f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{2\lambda^2}{\mu C_K}+\frac{\mu C_K}{2}+\frac{4\mu}{C_K}\right)\varepsilon\left\|\nabla G^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)}\\ &\le \frac{h^{*2}}{\mu C_K}\left\|\nabla \hat{f}\right\|^2_{L^2(\Omega)} + \left(\frac{2\lambda^2}{\mu C_K}+\frac{\mu C_K}{2}+\frac{4\mu}{C_K}\right)\varepsilon\left\|\nabla \hat{G}^\varepsilon\right\|^2_{L^2(\Omega)}.\end{aligned}$$

For $0<\varepsilon<1$,we see that :

$$\frac{9}{8}\mu C_K\varepsilon\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} \le \frac{h^{*2}}{\mu C_K}\left\|\nabla \hat{f}\right\|^2_{L^2(\Omega)} + \left(\frac{2\lambda^2}{\mu C_K}+\frac{\mu C_K}{2}+\frac{4\mu}{C_K}\right)\left\|\nabla \hat{G}^\varepsilon\right\|^2_{L^2(\Omega)}$$

So:

$$\varepsilon\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega)} \le c,$$

where

$$\begin{aligned}c &= \frac{8}{9\mu C_K}c_0;\\ c_0 &= \frac{h^{*2}}{\mu C_K}\left\|\nabla \hat{f}\right\|^2_{L^2(\Omega)} + \left(\frac{2\lambda^2}{\mu C_K}+\frac{\mu C_K}{2}+\frac{4\mu}{C_K}\right)\left\|\nabla \hat{G}^\varepsilon\right\|^2_{L^2(\Omega)}.\end{aligned}$$

This concludes the proof of Lemma (2.3.3).

2.3.3 Convergence Results and Limit Problem

Theorem 2.3.1. Under the assumptions of Lemma *2.3.3, there exist* $\hat{u}_i^* \in L^2(V_z)$, $i=1,2$ such that for any subsequence of $\hat{u}^\varepsilon$ *still denoted* $\hat{u}^\varepsilon$ we have the following convergence results:

$$\hat{u}_i^\varepsilon \rightharpoonup \hat{u}_i^* \quad i=1,2 \text{ weakly in } L^2(V_z) \tag{2.3.11}$$

$$\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \qquad i,j=1,2\ \text{ weakly in } L^2(V_z) \tag{2.3.12}$$

$$\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \qquad i=1,2\ \text{ weakly in } L^2(V_z). \tag{2.3.13}$$

Proof. According to Lemma 2.3.3, there exists a constant C independent of ε such that:

$$\left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^2 \leq C, \qquad i=1,2.$$

Using this estimate with Poincaré's inequality, we obtain:

$$\|\hat{u}_i^{\varepsilon}\|_{L^2(\Omega)} \leq h^* \left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}, \ i=1,2,3$$

Which means that u_i^{ε} is bounded in V_z for $i=1,2$, This implies the existence of u_i^* in V_z such that u_i^{ε} converges weakly to u_i^* in $L^2(V_z)$.

Thanks to Lemma 2.3.3 we have:

$$\varepsilon \left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}\right\|_{L^2(\Omega)} \leq c$$

Therefore $\varepsilon \dfrac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}$ converges to $\dfrac{\partial \hat{u}_i^*}{\partial x_j}$. since $\|\hat{u}_i^{\varepsilon}\|_{L^2(\Omega)} \leq c$,then $\dfrac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}$ converges to $\dfrac{\partial \hat{u}_i^*}{\partial x_j}$weakly, which implies the weak convergence of $\dfrac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}$ to 0 in $L^2(V_z)$.

Similarly, thanks to the inequality: : $\varepsilon^2 \left\|\dfrac{\partial \hat{u}_3^{\varepsilon}}{\partial x_j}\right\|_{L^2(\Omega)} \leq c$, we have the convergence:

$$\varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \rightharpoonup \frac{\partial \hat{u}_3^*}{\partial x_i}, \ and \qquad \varepsilon \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_j} \rightharpoonup \frac{\partial \hat{u}_3^*}{\partial x_j}.$$

This shows that $\varepsilon^2 \dfrac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i}$ converges weakly to 0 in $L^2(V_z)$.

Theorem 2.3.2. *With the same assumptions as in Theorem 2.3.1,* $\hat{u}^*$ satisfies :

$$-\mu \frac{\partial^2 \hat{u}_i^*}{\partial z^2} = \hat{f}_i \qquad for\ i=1,2\ \ in\ L^2(\Omega) \tag{2.3.14}$$

$$\mu \sum_{i=1}^{2} \int_{\Omega} \frac{\partial \hat{u}_i^*}{\partial z} . \frac{\partial}{\partial z}(\hat{\varphi}_i - \hat{u}_i^*)\, dxdz + J(\hat{\varphi}) - J(\hat{u}_i^*) \geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - \hat{u}_i^*\right), \ \ \forall \varphi \in \Pi(K) \tag{2.3.15}$$

Proof. The variational inequality (2.3.4) can be written as :

$$\sum_{i=1}^{4} I_i(\varepsilon) + \lambda \varepsilon^2 \int_{\Omega} \operatorname{div}(\hat{u}^{\varepsilon}) . \operatorname{div}(\hat{\varphi} - \hat{u}^{\varepsilon})\, dxdz + \int_{\omega} \hat{k}\, |\hat{\varphi} - s|\, dx -$$

$$- \int_{\omega} \hat{k}\, |\hat{u}^{\varepsilon} - s|\, dx \geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - \hat{u}_i^{\varepsilon}\right) + \varepsilon \left(\hat{f}_3, \hat{\varphi}_3 - \hat{u}_3^{\varepsilon}\right)$$

with

$$I_1 = \mu\varepsilon^2 \sum_{i,j=1}^{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial x_j} \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon \right) dxdz;$$
$$I_2 = \mu \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial z} \left(\hat{\varphi}_i + \hat{u}_i^\varepsilon \right) dxdz;$$
$$I_3 = \mu \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial x_i} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) dxdz;$$
$$I_4 = 2\mu \int_\Omega \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial z} . \frac{\partial}{\partial z} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) dxdz.$$

Using the convergence results from Theorem 2.3.1, we obtain

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{4} I_i(\varepsilon) = \mu \sum_{i=1}^{2} \int_\Omega \frac{\partial \hat{u}_i^*}{\partial z} . \frac{\partial}{\partial z} \left(\hat{\varphi}_i - \hat{u}_i^* \right) dxdz$$
$$\lim_{\varepsilon \to 0} \int_\Omega \varepsilon \hat{f}_3 \varphi dxdz = 0.$$

And since J is convex and lower semi-continuous
$\left(\lim_{\varepsilon \to 0} \left(\inf \int_\omega \hat{k} \left| \hat{u}^\varepsilon - s \right| dx \right) \geq \int_\omega \hat{k} \left| \hat{u}^* - s \right| dx \right)$,
so we have:

$$\mu \sum_{i=1}^{2} \int_\Omega \frac{\partial \hat{u}_i^*}{\partial z} . \frac{\partial}{\partial z} \left(\hat{\varphi}_i - \hat{u}_i^* \right) dxdz + J(\hat{\varphi}) - J(\hat{u}_i^*) \geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - \hat{u}_i^* \right). \tag{2.3.16}$$

Now, let's choose in the variational inequality (2.3.16)

$$\hat{\varphi}_i = \hat{u}_i^* \pm \psi_i, \ \psi_i \in H_0^1(\Omega) \quad i = 1, 2 \ \text{ and } \ \hat{\varphi}_3 = \hat{u}_3^*,$$

we find:

$$\mu \sum_{i=1}^{2} \int_\Omega \frac{\partial \hat{u}_i^*}{\partial z} \frac{\partial \psi_i}{\partial z} dxdz = \sum_{i=1}^{2} \int_\Omega \hat{f}_i \psi_i dxdz.$$

Using Green's formula and choosing $\hat{\varphi}_1 = 0$ and $\hat{\varphi}_2 \in H_0^1(\Omega)$, then $\hat{\varphi}_2 = 0$ and $\hat{\varphi}_1 \in H_0^1(\Omega)$, we get :

$$-\int_\Omega \mu \frac{\partial}{\partial z} \left(\frac{\partial \hat{u}_i^*}{\partial z} \right) \psi_i dxdz = \int_\Omega \hat{f}_i \psi_i dxdz.$$

So

$$-\mu\frac{\partial^2\hat{u}_i^*}{\partial z^2} = \hat{f}_i \qquad for\ i = 1,2\ in\ H^{-1}(\Omega) \tag{2.3.17}$$

And since $\hat{f} \in L^2(\Omega)$, then (2.3.17) is valid in $L^2(\Omega)$.

Theorem 2.3.3. *Under the same assumptions as in Theorem 2.3.2, we have :*

$$\int_\omega k\,|\psi + s^* - s| - |s^* - s|\,dx - \int_\omega \mu\hat{\tau}^*\psi dx \geq 0 \qquad \forall\psi \in L^2(\omega)^2 \tag{2.3.18}$$

$$\begin{cases} \mu\,|\hat{\tau}^*| < \hat{k} \Rightarrow s^* = s \\ \mu\,|\hat{\tau}^*| = \hat{k} \Rightarrow \exists\beta > 0\ such\ that\ s^* = s + \beta\hat{\tau}^* \end{cases} \tag{2.3.19}$$

with:

$$\hat{\tau}^* = \frac{\partial\hat{u}^*}{\partial z}(x,0) \text{ and } s^*(x) = \hat{u}^*(x,0).$$

Moreover, $\hat{u}^*$ and s^* satisfy the generalized weak Reynolds equation:

$$\int_\omega \left(\tilde{F} - \frac{h}{2}s^* + \int_0^h \hat{u}^*(x,z)\,dz\right)\nabla\psi(x)\,dx = 0,\ \forall\psi \in H^1(\omega), \tag{2.3.20}$$

where

$$\tilde{F}(x) = \frac{1}{\mu}\int_0^h F(x,z)\,dz - \frac{h}{2\mu}F(x,h);$$

$$F(x,z) = \int_0^z \int_0^\zeta \hat{f}_i(x,\alpha)\,d\zeta d\alpha.$$

Proof. The inequality (2.3.4) becomes:

$$\begin{aligned}
&\mu\varepsilon^2\sum_{i,j=1}^2 \int_\Omega \left(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_j}(\hat{\varphi}_i - \hat{u}_i^\varepsilon)\,dxdz+ \\
&+\mu\sum_{i=1}^2 \int_\Omega \left(\frac{\partial\hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right)\left[\frac{\partial}{\partial z}(\hat{\varphi}_i - \hat{u}_i^\varepsilon) + \varepsilon^2\frac{\partial}{\partial x_i}(\hat{\varphi}_3 - \hat{u}_3^\varepsilon)\right]dxdz+ \\
&+2\mu\varepsilon^2\int_\Omega \frac{\partial\hat{u}_3^\varepsilon}{\partial z}.\frac{\partial}{\partial z}(\hat{\varphi}_3 - \hat{u}_3^\varepsilon)\,dxdz + \lambda\varepsilon^2\int_\Omega \operatorname{div}(\hat{u}^\varepsilon).\operatorname{div}(\hat{\varphi} - \hat{u}^\varepsilon)\,dxdz+ \\
&+\int_\omega \hat{k}\,|\hat{\varphi} - s|\,dx - \int_\omega \hat{k}\,|\hat{u}^\varepsilon - s|\,dx \geq \sum_{i=1}^2\left(\hat{f}_i, \hat{\varphi}_i - \hat{u}_i^\varepsilon\right) + \varepsilon\left(\hat{f}_3, \hat{\varphi}_3 - \hat{u}_3^\varepsilon\right).
\end{aligned}$$

By taking the limit and using Green's formula, we obtain:

$$-\sum_{i=1}^{2}\int_{\Omega}\mu\frac{\partial^2\hat{u}_i^*}{\partial z^2}\psi_i dxdz+\int_{\partial\Omega}\mu\frac{\partial\hat{u}_i^*}{\partial z}n\psi_i d\sigma+\int_{\omega}\hat{k}\left|\hat{\varphi}-s\right|dx-\int_{\omega}\hat{k}\left|\hat{u}^*-s\right|dx\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i dxdz.$$

But

$$\int_{\partial\Omega}\mu\frac{\partial\hat{u}_i^*}{\partial z}n\psi_i d\sigma=-\int_{\omega}\mu\frac{\partial\hat{u}_i^*}{\partial z}\left(x,0\right)\psi_i dx.$$

So

$$-\sum_{i=1}^{2}\int_{\Omega}\mu\frac{\partial^2\hat{u}_i^*}{\partial z^2}\psi_i dxdz+\int_{\omega}\hat{k}\left(\left|\psi+s^*-s\right|-\left|s^*-s\right|\right)dx-\int_{\omega}\mu\hat{\tau}_i^*\psi_i dx\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i dxdz.$$

Moreover, using : $-\dfrac{\partial}{\partial z}\left(\mu\dfrac{\partial\hat{u}^*}{\partial z}\right)=\hat{f}_i$, we have :

$$\int_{\omega}\hat{k}\left|\psi+s^*-s\right|-\left|s^*-s\right|dx-\int_{\omega}\mu\hat{\tau}^*\psi dx\geq 0. \tag{2.3.21}$$

For (2.3.19), we use the analogue of [1].

The inequality (2.3.21) also holds for any $\psi\in\left(D\left(\omega\right)\right)^2$ and by density of $D\left(\omega\right)$ in $L^2\left(\omega\right)$,we find (2.3.18).

To prove (2.3.20) by integrating (2.3.14) from 0 to z, we see that :

$$\begin{aligned}\int_0^z\hat{f}_i\left(x,\alpha\right)d\alpha&=-\mu\int_0^z\frac{\partial^2\hat{u}_i^*}{\partial\alpha^2}\left(x,\alpha\right)d\alpha\\&=-\mu\frac{\partial\hat{u}_i^*}{\partial z}\left(x,z\right)+\mu\frac{\partial\hat{u}^*}{\partial z}\left(x,0\right).\end{aligned}$$

By integrating again from 0 to z, we get :

$$\begin{aligned}\int_0^z\int_0^\zeta\hat{f}_i\left(x,\alpha\right)d\alpha d\zeta&=-\mu\int_0^z\frac{\partial\hat{u}_i^*}{\partial\zeta}\left(x,\zeta\right)d\zeta+\mu\int_0^z\hat{\tau}^*d\zeta\\&=-\mu\hat{u}_i^*\left(x,z\right)+\mu\hat{u}_i^*\left(x,0\right)+\mu z\hat{\tau}^*\end{aligned}$$

$$\Rightarrow\hat{u}_i^*\left(x,z\right)=s^*\left(x,0\right)+z\hat{\tau}^*-\frac{1}{\mu}\int_0^z\int_0^\zeta\hat{f}_i\left(x,\alpha\right)d\alpha d\zeta \tag{2.3.22}$$

and since $\hat{u}_i^*\left(x,h\left(x\right)\right)=0$, we have :

$$s^*\left(x,0\right)+h\hat{\tau}^*=\frac{1}{\mu}\int_0^z\hat{f}_i\left(x,\alpha\right)d\alpha d\zeta. \tag{2.3.23}$$

Integrating (2.3.22) from 0 to h, we get :

$$-\mu\int_0^h\hat{u}_i^*\left(x,z\right)dz+\mu\int_0^h\hat{u}_i^*\left(x,0\right)dz+\mu\int_0^h z\hat{\tau}^*dz=\int_0^h\int_0^z\int_0^\zeta\hat{f}_i\left(x,\alpha\right)d\alpha d\zeta dz$$

$$\int_0^h \hat{u}_i^*(x,z)\,dz = hs^*(x,0) + \frac{1}{2}h^2\hat{\tau}^* - \frac{1}{\mu}\int_0^h \int_0^z \int_0^\zeta \hat{f}_i(x,\alpha)\,d\alpha d\zeta dz. \tag{2.3.24}$$

From (2.3.23) and (2.3.24), we deduce that

$$\int_0^h \hat{u}_i^*(x,z)\,dz - \frac{h}{2}s^* + \tilde{F} = 0,$$

with

$$\tilde{F} = \int_0^h F(x,z)\,dz;$$
$$F(x,z) = \int_0^z \int_0^\zeta \hat{f}_i(x,\alpha)\,d\zeta d\alpha.$$

Then

$$\int_\omega \left(\int_0^h \hat{u}^*(x,y)\,dy - \frac{h}{2}s^* + \tilde{F} \right) \nabla\psi = 0.$$

Theorem 2.3.4. The solution $\hat{u}^*$ of the limit problem (2.3.14), (2.3.15) *is unique in* $L^2(V_z)$.

Proof. Suppose there exist two solutions $\hat{u}^1$ and $\hat{u}^2$ of the variational inequality (2.3.15), we have :

$$\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial \hat{u}_i^1}{\partial z}.\frac{\partial}{\partial z}\left(\varphi_i - \hat{u}_i^1\right)dxdz + J(\varphi) - J\left(\hat{u}_i^1\right) \geq \sum_{i=1}^{2}\left(\hat{f}_i, \varphi_i - \hat{u}_i^1\right) \tag{2.3.25}$$

and

$$\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial \hat{u}_i^2}{\partial z}.\frac{\partial}{\partial z}\left(\varphi_i - \hat{u}_i^2\right)dxdz + J(\varphi) - J\left(\hat{u}_i^2\right) \geq \sum_{i=1}^{2}\left(\hat{f}_i, \varphi_i - \hat{u}_i^2\right). \tag{2.3.26}$$

Taking $\varphi = \hat{u}^2$ in (2.3.25) and then $\varphi = \hat{u}^1$ in (2.3.26):

$$\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial \hat{u}_i^1}{\partial z}.\frac{\partial}{\partial z}\left(\hat{u}_i^2 - \hat{u}_i^1\right)dxdz + J\left(\hat{u}^2\right) - J\left(\hat{u}_i^1\right) \geq \sum_{i=1}^{2}\left(\hat{f}_i, \hat{u}_i^2 - \hat{u}_i^1\right) \tag{2.3.27}$$

$$\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial \hat{u}_i^2}{\partial z}.\frac{\partial}{\partial z}\left(\hat{u}_i^1 - \hat{u}_i^2\right)dxdz + J\left(\hat{u}^1\right) - J\left(\hat{u}_i^2\right) \geq \sum_{i=1}^{2}\left(\hat{f}_i, \hat{u}_i^1 - \hat{u}_i^2\right). \tag{2.3.28}$$

Summing up inequalities (2.3.28) and (2.3.27), we obtain :

$$\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial \hat{u}_i^1}{\partial z}.\frac{\partial}{\partial z}\left(\hat{u}_i^2 - \hat{u}_i^1\right)dxdz + \mu\sum_{i=1}^{2}\int_\Omega \frac{\partial \hat{u}_i^2}{\partial z}.\frac{\partial}{\partial z}\left(\hat{u}_i^1 - \hat{u}_i^2\right)dxdz \geq 0$$

and

$$\mu \sum_{i=1}^{2} \int_{\Omega} \frac{\partial}{\partial z} \left(\hat{u}_i^1 - \hat{u}_i^2 \right) . \frac{\partial}{\partial z} \left(\hat{u}_i^1 - \hat{u}_i^2 \right) dxdz \leq 0,$$

which implies

$$\mu \left\| \frac{\partial}{\partial z} \left(\hat{u}_i^1 - \hat{u}_i^2 \right) \right\|_{L^2(\Omega)}^2 = 0.$$

By Poincaré, we find:

$$\left\| \hat{u}_i^1 - \hat{u}_i^2 \right\|_{V_z} \leq c \left\| \frac{\partial}{\partial z} \left(\hat{u}_i^1 - \hat{u}_i^2 \right) \right\|_{L^2(\Omega)}^2 = 0,$$

so

$$\hat{u}^1 = \hat{u}^2.$$

Chapitre 3

Asymptotic Analysis of a Stationary Non-Isothermal Linear Elasticity Problem

Abstract: In this chapter, we focus on the asymptotic analysis of a stationary problem for non-isothermal linear elasticity (i.e., the coefficient μ depends on temperature), with nonlinear Tresca-type friction conditions. We couple the equation of conservation of momentum with the energy conservation equation derived from Fourier's law.

Contents:

3.1 Introduction and Problem Statement;

3.2 Variational Formulation of the Problem;

3.3 Asymptotic Analysis of the Problem;

3.3.1 A Priori Estimation;

3.3.2 Convergence Results and Limit Problem

3.1 Introduction and Problem Statement

In this chapter Ω^ε will denote the same thin domain considered in the second chapter .

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3, (x, 0) \in \omega, \quad 0 < x_3 < \varepsilon h(x)\}.$$

We recall that Γ^ε is its boundary :

$$\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon.$$

We consider a problem of deformations of a homogeneous, elastic, and non-isothermal body in a stationary regime.

For given external forces f^ε , we assume that the deformations of an elastic body are governed by the following equations :

The law of conservation of momentum:

$$\frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0. \tag{3.1.1}$$

Here $\sigma^\varepsilon = (\sigma_{ij})_{1\le i,j\le 3}^\varepsilon$ is the stress tensor, and $D = (d_{ij})_{1\le i,j\le 3}$ is the strain rate tensor :

$$d_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \ 1 \le i, j \le 3. \tag{3.1.2}$$

We assume that the constitutive law follows Hooke's law:

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu(T)\, d_{ij}(u^\varepsilon) + \lambda d_{kk}(u^\varepsilon)\, \delta_{ij}, \tag{3.1.3}$$

where

$*u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ is the displacement velocity of the elastic body;

$*$ λ and μ are Lamé coefficients;

$*$ T^ε is its temperature.

The law of conservation of energy is given by:

$$-\frac{\partial}{\partial x_i}\left(K^\varepsilon \frac{\partial T^\varepsilon}{\partial x_i}\right) = 2\mu^\varepsilon(T^\varepsilon)\, d_{ij}^2(u^\varepsilon) + r^\varepsilon(T^\varepsilon); \tag{3.1.4}$$

where K^ε and r^ε respectively denote the thermal conductivity and the mass heat supply.

We assume that the velocity is known on Γ_1^ε and on Γ_L^ε

$$u^\varepsilon = 0 \qquad \text{on } \Gamma_1^\varepsilon \tag{3.1.5}$$

$$u^\varepsilon = g \qquad \text{on } \Gamma_L^\varepsilon \tag{3.1.6}$$

where, $g = (g_1, g_2, g_3)$ is a function with $g_3 = 0$, such that :

$$\int_{\Gamma^\varepsilon} g.nds = 0.$$

We use the usual notations:

$$\begin{aligned} u_n^\varepsilon &= u^\varepsilon.n = u_i^\varepsilon.n_i, \\ u_{\tau_i}^\varepsilon &= u_i^\varepsilon - u_n^\varepsilon n_i, \\ \sigma_n^\varepsilon &= (\sigma.n)\, n = \sigma_{ij}^\varepsilon n_i n_j, \\ \sigma_{\tau_i}^\varepsilon &= \sigma_{ij}^\varepsilon n_j - \sigma_n^\varepsilon n_i, \end{aligned} \tag{3.1.7}$$

respectively, the normal velocity, the tangential velocity, the normal and tangential component of the stress tensor.

For the temperature, we assume that:

$$T^\varepsilon = 0 \quad on\ \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \quad (\text{ Homogeneous Dirichlet on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon\ ), \tag{3.1.8}$$

$$\frac{\partial T^\varepsilon}{\partial n} = 0 \quad on\ \omega \quad (\text{ Homogeneous Neumann on } \omega)\,. \tag{3.1.9}$$

The complete problem consists of finding the velocity field u^ε and the temperature T^ε that satisfy the following equations and boundary conditions:

$$(pb_1^\varepsilon)\begin{cases} \dfrac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0 & in\ \ \Omega^\varepsilon, \\ -\dfrac{\partial}{\partial x_i}\left(K^\varepsilon \dfrac{\partial T^\varepsilon}{\partial x_i}\right) = 2\mu^\varepsilon\left(T^\varepsilon\right) d_{ij}^2\left(u^\varepsilon\right) + r^\varepsilon\left(T^\varepsilon\right) & in\ \Omega^\varepsilon, \\ u^\varepsilon = 0 & on\ \Gamma_1 \\ T^\varepsilon = 0 & on\ \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \end{cases}$$

3.2 Variational Formulation of the Problem

We introduce the following functional framework :

$$H^1\left(\Omega^\varepsilon\right)^3 = \left\{ v \in \left(L^2\left(\Omega^\varepsilon\right)\right)^3 : \frac{\partial v_i}{\partial x_j} \in L^2\left(\Omega^\varepsilon\right),\ \forall i, j = 1, 2, 3 \right\}.$$

We define the non-empty closed convex set of $H^1\left(\Omega^\varepsilon\right)^3$, by :

$$K^{\varepsilon}=\left\{\varphi\in\left(H^{1}\left(\Omega^{\varepsilon}\right)\right)^{3}:\varphi=0\ on\ \Gamma_{L}^{\varepsilon}\cup\Gamma_{1}^{\varepsilon}\text{ and }\varphi n=0\ on\ \omega\right\}$$

.

We denote by $H^{1}_{\Gamma_{L}^{\varepsilon}\cup\Gamma_{1}^{\varepsilon}}\left(\Omega^{\varepsilon}\right)$ the vector subspace of $H^{1}\left(\Omega^{\varepsilon}\right)$:

$$H^{1}_{\Gamma_{L}^{\varepsilon}\cup\Gamma_{1}^{\varepsilon}}\left(\Omega^{\varepsilon}\right)=\left\{\varphi\in H^{1}\left(\Omega^{\varepsilon}\right):\varphi=0\quad on\ \Gamma_{L}^{\varepsilon}\cup\Gamma_{1}^{\varepsilon}\right\}.$$

We also introduce the following notations:

$$a\left(T,u,v\right)=\sum_{i,j=1}^{3}\int_{\Omega^{\varepsilon}}2\mu^{\varepsilon}\left(T\right)d_{ij}\left(u\right)d_{ij}\left(v\right)dxdx_{3}+\lambda\int_{\Omega^{\varepsilon}}\operatorname{div}\left(u\right).\operatorname{div}\left(v\right)dxdx_{3};\tag{3.2.1}$$

$$\left(f^{\varepsilon},v\right)=\int_{\Omega^{\varepsilon}}f^{\varepsilon}vdxdx_{3}=\sum_{i=1}^{3}\int_{\Omega^{\varepsilon}}f_{i}^{\varepsilon}v_{i}dxdx_{3};\tag{3.2.2}$$

$$b\left(T,\psi\right)=\int_{\Omega^{\varepsilon}}K^{\varepsilon}\frac{\partial T}{\partial x_{i}}.\frac{\partial\psi}{\partial x_{i}}dxdx_{3};\tag{3.2.4}$$

$$C\left(u;T,\psi\right)=\sum_{i,j=1}^{3}\int_{\Omega^{\varepsilon}}2\mu^{\varepsilon}\left(T\right)d_{ij}^{2}\left(u\right)\psi dxdx_{3}+\int_{\Omega^{\varepsilon}}r^{\varepsilon}\left(T\right)\psi dxdx_{3}.\tag{3.2.5}$$

Lemma 3.2.1. If u^{ε} and T^{ε} are solutions of problem $\left(pb_{1}^{\varepsilon}\right)$,then they satisfy the variational problem:

$$\left\{\begin{array}{c}\text{Find } u^{\varepsilon}\in H_{0}^{1}\left(\Omega^{\varepsilon}\right)^{3}\text{ and } T^{\varepsilon}\in H_{0}^{1}\left(\Omega^{\varepsilon}\right)\ \textit{such that}\\ a\left(T^{\varepsilon};u^{\varepsilon},\varphi\right)=\left(f^{\varepsilon},\varphi\right),\ \forall\varphi\in H_{0}^{1}\left(\Omega^{\varepsilon}\right)^{3}\\ b\left(T^{\varepsilon},\psi\right)=C\left(u^{\varepsilon};T^{\varepsilon},\psi\right),\quad\forall\psi\in H_{0}^{1}\left(\Omega^{\varepsilon}\right).\end{array}\right.\tag{3.2.6}$$

Proof. By multiplying equation (3.1.1) by $\left(\varphi_{i}-u_{i}^{\varepsilon}\right)$, where $\varphi\in K^{\varepsilon}$ and using the Green's formula, we obtain :

$$\int_{\Omega^{\varepsilon}}\sigma_{ij}^{\varepsilon}\frac{\partial}{\partial x_{j}}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)dxdx_{3}-\int_{\Gamma^{\varepsilon}}\sigma_{ij}^{\varepsilon}n_{j}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)ds=\int_{\Omega^{\varepsilon}}f_{i}^{\varepsilon}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)dxdx_{3}.$$

According to the boundary conditions, we have:

$$\int_{\Gamma^{\varepsilon}}\sigma_{ij}^{\varepsilon}n_{j}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)ds=0$$

we find :

$$\int_{\Omega^{\varepsilon}}\sigma_{ij}^{\varepsilon}\frac{\partial}{\partial x_{j}}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)dxdx_{3}=\int_{\Omega^{\varepsilon}}f_{i}^{\varepsilon}\left(\varphi_{i}-u_{i}^{\varepsilon}\right)dxdx_{3}.$$

Replacing $\sigma_{ij}^{\varepsilon}$ by the expression (3.1.3), we have

$$a\left(T;u^{\varepsilon},\varphi\right)=\left(f^{\varepsilon},\varphi\right),\ \forall\varphi\in K^{\varepsilon}. \tag{3.2.7}$$

By multiplying equation (3.1.4) by $\psi\in H^{1}_{\Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}}\left(\Omega^{\varepsilon}\right)$ and using the Green's formula, we obtain:

$$\sum_{i=1}^{3}\int_{\Omega^{\varepsilon}}\frac{\partial T^{\varepsilon}}{\partial x_i}\frac{\partial\psi}{\partial x_i}dxdx_3=\sum_{i,j=1}^{3}\int_{\Omega^{\varepsilon}}2\mu^{\varepsilon}\left(T^{\varepsilon}\right)d_{ij}^{2}\left(u^{\varepsilon}\right)\psi dxdx_3+\int_{\Gamma^{\varepsilon}}K^{\varepsilon}\frac{\partial T^{\varepsilon}}{\partial x_i}\psi ds+\int_{\Omega^{\varepsilon}}r^{\varepsilon}\left(T^{\varepsilon}\right)\psi dxdx_3.$$

Now, using the boundary conditions (3.1.8) and (3.1.9), we have :

$$\sum_{i=1}^{3}\int_{\Omega^{\varepsilon}}\frac{\partial T^{\varepsilon}}{\partial x_i}\frac{\partial\psi}{\partial x_i}dxdx_3=\sum_{i,j=1}^{3}\int_{\Omega^{\varepsilon}}2\mu^{\varepsilon}\left(T^{\varepsilon}\right)d_{ij}^{2}\left(u^{\varepsilon}\right)\psi dxdx_3+\int_{\Omega^{\varepsilon}}r^{\varepsilon}\left(T^{\varepsilon}\right)\psi dxdx_3, \tag{3.2.8}$$

which means:

$$b\left(T^{\varepsilon},\psi\right)=C\left(u^{\varepsilon};T^{\varepsilon},\psi\right),\quad\forall\psi\in H^{1}_{\Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}}.$$

3.3 Asymptotic Analysis of the Problem

For the asymptotic analysis of the problem , we employ a similar scaling as in the second chapter by introducing a scale change where $z=\dfrac{x_3}{\varepsilon}$, transforming the domain Ω^ε into a domain Ω independent of ε defined by :

$$\Omega = \left\{(x,z) \in \mathbb{R}^3, (x,0) \in \omega \text{ and } 0 < z < h(x)\right\}.$$

We denote $\Gamma = \bar{\omega} \cup \Gamma_L \cup \Gamma_1$, its boundary.

Following this scale change, we denote by u^ε and T^ε,the unknowns defined on Ω. Furthermore, we define the following functions on Ω :

$$\begin{cases} u_i^\varepsilon(x,x_3) = \hat{u}_i^\varepsilon(x,z), \quad i=1,2 \\ \varepsilon^{-1}u_3^\varepsilon(x,x_3) = \hat{u}_3^\varepsilon(x,z), \\ T^\varepsilon(x,x_3) = \hat{T}^\varepsilon(x,z). \end{cases} \tag{3.3.1}$$

For the data, we have the following relations:

$$\hat{\mu} = \mu^\varepsilon, \quad \hat{f} = \varepsilon^2 f^\varepsilon \;, \hat{K} = K^\varepsilon \text{ and } \hat{r} = \varepsilon^2 r^\varepsilon \tag{3.3.2}$$

with $\hat{\mu}$, $\hat{f}$, $\hat{K}$, $\hat{r}$ independent of ε.

Now we introduce the functional framework on Ω.For this purpose, we note :

$$K = \left\{v \in \left(H^1(\Omega)\right)^3 : v = \hat{G} \text{ on } \Gamma_L \cup \Gamma_1,\ v.n = 0 \text{ on } \omega\right\}.$$

$$\Pi(K) = \left\{\varphi \in H^1(\Omega)^2 : \varphi = (\varphi_1,\varphi_2),\ \varphi_i = \hat{G}_i \text{ on } \Gamma_L \cup \Gamma_1,\ i=1,2\right\}.$$

$$V_z = \left\{v = (v_1,v_2) \in L^2(\Omega)^2 : \frac{\partial v_i}{\partial z} \in L^2(\Omega),\ i=1,2;\ v=0 \text{ on } \Gamma_1\right\}.$$

It is clear that V_z is a Banach space, equipped with the norm:

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2}\|v_i\|^2_{L^2(\Omega)} + \left\|\frac{\partial v_i}{\partial z}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

With the scale change defined in (3.3.1) and (3.3.2) problem (3.2.6)becomes:

$$\begin{cases} \text{Find the velocity field } \hat{u}^\varepsilon \text{ in } K \text{ and the temperature } \hat{T}^\varepsilon \text{ in } H_0^1(\Omega) \text{ such that} \\ \qquad a\left(\hat{T}^\varepsilon;\hat{u}^\varepsilon,\hat{\varphi}\right) = \left(\hat{f}^\varepsilon,\hat{\varphi}\right), \quad \forall\hat{\varphi} \in K \end{cases} \tag{3.3.5}$$

$$b\left(\hat{T}^{\varepsilon},\hat{\psi}\right)=C\left(\hat{u}^{\varepsilon};\hat{T}^{\varepsilon},\hat{\psi}\right),\quad\forall\hat{\psi}\in H_0^1\left(\Omega\right) \tag{3.3.6}$$

where

$$\begin{aligned}a\left(\hat{T}^{\varepsilon};\hat{u}^{\varepsilon},\hat{\varphi}\right)=\hat{\mu}\left(\hat{T}^{\varepsilon}\right)\varepsilon^2\sum_{i,j=1}^{2}\left(\frac{\partial\hat{u}_i^{\varepsilon}}{\partial x_j}+\frac{\partial\hat{u}_j^{\varepsilon}}{\partial x_i}\right)\frac{\partial}{\partial x_j}\left(\hat{\varphi}_i\right)dxdz+\\ +\mu\sum_{i=1}^{2}\int_{\Omega}\left(\frac{\partial\hat{u}_i^{\varepsilon}}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^{\varepsilon}}{\partial x_i}\right)\left[\frac{\partial}{\partial z}\left(\hat{\varphi}_i\right)+\varepsilon^2\frac{\partial}{\partial x_i}\left(\hat{\varphi}_3\right)\right]dxdz+\\ +2\mu\varepsilon^2\int_{\Omega}\frac{\partial\hat{u}_3^{\varepsilon}}{\partial z}.\frac{\partial}{\partial z}\left(\hat{\varphi}_3\right)dxdxz+\lambda\varepsilon^2\int_{\Omega}\operatorname{div}\left(\hat{u}^{\varepsilon}\right).\operatorname{div}\left(\hat{\varphi}\right)dxdz.\end{aligned}$$

$$\begin{aligned}\left(\hat{f}^{\varepsilon},\hat{\varphi}-\hat{u}^{\varepsilon}\right)&=\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i^{\varepsilon}\left(\hat{\varphi}_i\right)dxdz+\varepsilon\int_{\Omega}\hat{f}^{\varepsilon}\left(\hat{\varphi}_i\right)dxdz.\\ b\left(\hat{T}^{\varepsilon},\hat{\psi}\right)&=\sum_{i=1}^{2}\int_{\Omega}\varepsilon^2\hat{K}\frac{\partial\hat{T}^{\varepsilon}}{\partial x_i}\frac{\partial\hat{\psi}}{\partial x_i}dxdz+\int_{\Omega}\hat{K}\frac{\partial\hat{T}^{\varepsilon}}{\partial z}.\frac{\partial\hat{\psi}}{\partial z}dxdz\\ C\left(\hat{u}^{\varepsilon};\hat{T}^{\varepsilon},\hat{\psi}\right)&=\sum_{i=1}^{2}\frac{1}{2}\int_{\Omega}\varepsilon^2\hat{\mu}\left(\hat{T}^{\varepsilon}\right)\left(\frac{\partial\hat{u}_i^{\varepsilon}}{\partial x_j}+\frac{\partial\hat{u}_j^{\varepsilon}}{\partial x_i}\right)^2\psi dxdz+\\ &+\sum_{i=1}^{2}\frac{1}{2}\int_{\Omega}\hat{\mu}\left(\hat{T}^{\varepsilon}\right)\left(\frac{\partial\hat{u}_i^{\varepsilon}}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^{\varepsilon}}{\partial x_i}\right)^2\psi dxdz+\\ &+\sum_{j=1}^{2}\frac{1}{2}\int_{\Omega}\hat{\mu}\left(\hat{T}^{\varepsilon}\right)\left(\varepsilon^2\frac{\partial\hat{u}_3^{\varepsilon}}{\partial x_i}+\frac{\partial\hat{u}_j^{\varepsilon}}{\partial z}\right)^2\psi dxdz+\\ &+\int_{\Omega}2\varepsilon^2\hat{\mu}\left(\hat{T}^{\varepsilon}\right)\left(\frac{\partial\hat{u}_3^{\varepsilon}}{\partial z}\right)^2\psi dxdz+\int_{\Omega}r\left(\hat{T}^{\varepsilon}\right)\psi dxdz.\end{aligned}$$

3.3.1 A Priori Estimation

We will establish estimates on the velocity u^{ε}, then on the temperature T^{ε} solution of our variational problem.

A priori estimate on the velocity

Lemma 3.3.1. Suppose $f\in\left(L^2\left(\Omega\right)\right)^3$,*the friction coefficient* $k>0$ *in* $L^{\infty}\left(\omega\right)$ *and there exist two constants* μ_* *and* μ^* *such that*

$$0<\mu_*\leq\mu\left(a\right)\leq\mu^*,\quad\forall a\in\mathbb{R}, \tag{3.3.7}$$

then, there exists a strictly positive constant C *independent of* ε *such that* :

$$\sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2+\sum_{i=1}^{2}\left(\left\|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2\right)\leq C. \tag{3.3.8}$$

Proof. Let u^ε be the solution of problem (3.2.6) , i.e.:

$$a\left(T^\varepsilon;u^\varepsilon,\varphi\right)=\left(f^\varepsilon,\varphi\right),$$

choosing $\varphi=u^\varepsilon$, we obtain

$$a\left(T^\varepsilon;u^\varepsilon,u^\varepsilon\right)=\left(f^\varepsilon,u^\varepsilon\right),$$

On the other hand, we have:

$$\begin{aligned}a\left(T^\varepsilon;u^\varepsilon,u^\varepsilon\right)&=2\mu^\varepsilon\left(T^\varepsilon\right)\sum_{i,j=1}^{2}\int_{\Omega^\varepsilon}d_{ij}\left(u^\varepsilon\right)d_{ij}\left(u^\varepsilon\right)dxdx_3+\lambda\int_{\Omega^\varepsilon}\operatorname{div}\left(u^\varepsilon\right).\operatorname{div}\left(u^\varepsilon\right)dxdx_3\\&=2\mu^\varepsilon\left(T^\varepsilon\right)\sum_{i,j=1}^{2}\int_{\Omega^\varepsilon}\left|d_{ij}\left(u^\varepsilon\right)\right|^2dxdx_3+\lambda\int_{\Omega^\varepsilon}\left|\operatorname{div}\left(u^\varepsilon\right)\right|^2dxdx_3.\end{aligned}$$

According to the Korn's inequality, there exists C_K independent of ε such that :

$$a\left(T^\varepsilon;u^\varepsilon,u^\varepsilon\right)\geq 2\mu_*C_K\left\|\nabla u^\varepsilon\right\|_{L^2(\Omega)}^2. \tag{3.3.9}$$

Using the Poincaré inequality, we obtain:

$$\left|\left(f^\varepsilon,u^\varepsilon\right)\right|\leq\left\|f^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}\left\|u^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}$$

$$\left|\left(f^\varepsilon,u^\varepsilon\right)\right|\leq\varepsilon h^*\left\|\nabla u^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}\left\|f^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}\ . \tag{3.3.10}$$

Now, by applying the Young's inequality, we find:

$$\left|\left(f^\varepsilon,u^\varepsilon\right)\right|\leq\frac{\mu_*C_K}{2}\left\|\nabla u^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}^2+\frac{\left(\varepsilon h^*\right)^2}{2\mu_*C_K}\left\|\nabla f^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}^2$$

By applying the Hölder and Young's inequalities, we obtain:

$$\begin{aligned}a\left(T^\varepsilon;u^\varepsilon,\varphi\right)&\leq\int_{\Omega^\varepsilon}2\mu^\varepsilon\left(T^\varepsilon\right)\left|d_{ij}\left(u^\varepsilon\right)\right|.\left|d_{ij}\left(u^\varepsilon\right)\right|dxdx_3+\lambda\int_{\Omega^\varepsilon}\left|\operatorname{div}\left(u^\varepsilon\right)\right|.\left|\operatorname{div}\left(\varphi\right)\right|dxdx_3\\&\leq\int_{\Omega^\varepsilon}2\mu^*\left|d_{ij}\left(u^\varepsilon\right)\right|.\left(\left|d_{ij}\left(\varphi\right)\right|\right)dxdx_3+\int_{\Omega^\varepsilon}\lambda\left|\operatorname{div}\left(u^\varepsilon\right)\right|.\left|\operatorname{div}\left(u^\varepsilon\right)\right|dxdx_3\end{aligned}$$

$$\leq \left(\int_{\Omega^\varepsilon} \sqrt{\frac{\mu_* C_K}{2}} \left| d_{ij}\left(u^\varepsilon\right) \right|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega^\varepsilon} \frac{2\sqrt{2}\mu^*}{\sqrt{\mu_* C_K}} \left| d_{ij}\left(\varphi\right) \right|^2 dx dx_3 \right)^{\frac{1}{2}}$$
$$+ \left(\int_{\Omega^\varepsilon} \frac{\sqrt{\mu_* C_K}}{2} \left| \operatorname{div}\left(u^\varepsilon\right) \right|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega^\varepsilon} \frac{2\lambda}{\sqrt{\mu_* C_K}} \left| \operatorname{div}\left(\varphi\right) \right|^2 dx dx_3 \right)^{\frac{1}{2}}$$

$$\leq \frac{\mu_* C_K}{4} \left\| d_{ij}\left(u^\varepsilon\right) \right\|^2_{L^2(\Omega^\varepsilon)} + \frac{4\left(\mu^*\right)^2}{\mu_* C_K} \left\| d_{ij}\left(\varphi\right) \right\|^2_{L^2(\Omega^\varepsilon)} +$$
$$+ \frac{\mu_* C_K}{8} \int_{\Omega^\varepsilon} \left| \operatorname{div}\left(u^\varepsilon\right) \right|^2 dx dx_3 + \frac{2\lambda^2}{\mu_* C_K} \int_{\Omega^\varepsilon} \left| \operatorname{div}\left(\varphi\right) \right|^2 dx dx_3.$$

However, we know that:

$$\sum_{i,j=1}^{2} \left| d_{ij}\left(u^\varepsilon\right) \right|^2 \leq \left| \nabla u^\varepsilon \right|^2 \quad \text{and} \quad \left| \operatorname{div}\left(u^\varepsilon\right) \right|^2 \leq \left| \nabla u^\varepsilon \right|^2,$$

which yields:

$$a\left(T^\varepsilon; u^\varepsilon, \varphi\right) \leq \frac{\mu_* C_K}{4} \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} + \frac{4\left(\mu^*\right)^2}{\mu_* C_K} \left\| \nabla \varphi \right\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu_* C_K}{8} \left\| \nabla u^\varepsilon \right\|^2_{L^\varepsilon(\Omega^\varepsilon)} + \frac{2\lambda^2}{\mu_* C_K} \left\| \nabla \varphi \right\|^2_{L^\varepsilon(\Omega^\varepsilon)}. \tag{3.3.12}$$

Using (3.3.9) – (3.3.12) and choosing $\varphi = 0$, we obtain :

$$2\mu_* C_K \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} \leq a\left(u^\varepsilon, u^\varepsilon\right) \leq \frac{\mu_* C_K}{4} \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} +$$
$$+ \frac{\mu_* C_K}{8} \left\| \nabla u^\varepsilon \right\|^2_{L^\varepsilon(\Omega^\varepsilon)} + \frac{\mu_* C_K}{2} \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} +$$
$$\frac{\left(\varepsilon h^*\right)^2}{2\mu_* C_K} \left\| \nabla f^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} + + \frac{\left(\varepsilon h^*\right)^2}{2\mu_* C_K} \left\| \nabla f^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)}.$$

$$\frac{9}{8} \mu_* C_K \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} \leq \frac{\left(\varepsilon h^*\right)^2}{\mu C_K} \left\| \nabla f^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} \tag{3.3.13}$$

Using the equalities :

$$\varepsilon^2 \left\| f^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1} \left\| \hat{f} \right\|^2_{L^2(\Omega)} \quad \text{and} \quad \left\| \frac{\partial u_3^\varepsilon}{\partial x_3} \right\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1} \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)},$$

we obtain :

$$\varepsilon \left\|\nabla u^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})}^{2} = \varepsilon^2 \sum_{i,j=1}^{2} \left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}\right\|_{L^2(\Omega)}^{2} + \sum_{i=1}^{2} \left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2} + \varepsilon^4 \sum_{j=1}^{2} \left\|\frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_j}\right\|_{L^2(\Omega)}^{2} + \varepsilon^2 \left\|\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2}.$$

Multiplying (3.3.13) by ε, we find :

$$\frac{9}{8}\mu_* C_K \ \varepsilon \left\|\nabla u^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})}^{2} \leq \frac{(\varepsilon h^*)^2}{\mu_* C_K} \varepsilon \left\|\nabla f^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})}^{2}$$

For $0 < \varepsilon < 1$ we have :

$$\frac{9}{8}\mu_* C_K \varepsilon \left\|\nabla u^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})}^{2} \leq \frac{h^{*2}}{\mu_* C_K} \left\|\nabla \hat{f}\right\|_{L^2(\Omega)}^{2}$$

thus :

$$\varepsilon \left\|\nabla u^{\varepsilon}\right\|_{L^2(\Omega)}^{2} = \varepsilon^2 \sum_{i,j=1}^{2} \left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}\right\|_{L^2(\Omega)}^{2} + \sum_{i=1}^{2} \left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2} + \varepsilon^4 \sum_{j=1}^{2} \left\|\frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_j}\right\|_{L^2(\Omega)}^{2} + \varepsilon^2 \left\|\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2} \leq C\,,$$

where

$$\begin{aligned} c_1 &= \frac{h^{*2}}{\mu_* C_K} \left\|\nabla \hat{f}\right\|_{L^2(\Omega^{\varepsilon})}^{2}; \\ C &= \frac{8}{9\mu_* C_K} c_1. \end{aligned}$$

So, Lemma 3.3.1 is proved.

Estimation on the temperature

In this section, we seek a priori estimates on the temperature $\hat{T}^{\varepsilon}$, For this purpose, we need to establish the following lemma:

Lemma 3.3.2. The temperature $\hat{T}^{\varepsilon}$ is bounded by:

$$\left\|\hat{T}^{\varepsilon}\right\|_{L^2(\Omega)} \leq h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}. \tag{3.3.14}$$

Proof. For the proof of Lemma 3.3.2,we use the same techniques as those used in lemma 2.3.1.

Lemma 3.3.3. Suppose the assumptions of Lemma *3.3.1* are met, and additionally, let's assume there exist:

- *two strictly positive constants K_* and K^*, such that*

$$0 < K_* \leq K(x,z) \leq K^*, \quad \forall (x,z) \in \Omega \tag{3.3.15}$$

- *a positive constant $\hat{r}^*$, such that*

$$\hat{r}(a) \leq \hat{r}^*. \tag{3.3.16}$$

Then, there exists a positive constant C_2 *independent of* ε, *such that* :

$$\varepsilon^2 \sum_{i=1}^{2} \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq C_2. \tag{3.3.17}$$

Proof. In the variational inequality (3.2.8), choosing $\varphi = \hat{T}^\varepsilon$, we obtain :

$$\sum_{i=1}^{2} I_i = \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \frac{\partial \hat{T}^\varepsilon}{\partial x_i} dxdz + \int_\Omega \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial z} \frac{\partial \hat{T}^\varepsilon}{\partial z} dxdz,$$

with

$$\begin{aligned} I_1 = & \sum_{i=1}^{2} \frac{1}{2} \int_\Omega \varepsilon^2 \hat{\mu}\left(\hat{T}^\varepsilon\right) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right)^2 \hat{T}^\varepsilon dxdz + \\ & + \sum_{i=1}^{2} \frac{1}{2} \int_\Omega \hat{\mu}\left(\hat{T}^\varepsilon\right) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right)^2 \hat{T}^\varepsilon dxdz + \\ & + \sum_{j=1}^{2} \frac{1}{2} \int_\Omega \hat{\mu}\left(\hat{T}^\varepsilon\right) \left(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} + \frac{\partial \hat{u}_j^\varepsilon}{\partial z} \right)^2 \hat{T}^\varepsilon dxdz + \\ & + \int_\Omega 2\varepsilon^2 \hat{\mu}\left(\hat{T}^\varepsilon\right) \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right)^2 \hat{T}^\varepsilon dxdz; \end{aligned}$$

$$I_2 = \int_\Omega r\left(\hat{T}^\varepsilon\right) \hat{T}^\varepsilon dxdz.$$

Applying Cauchy-Schwarz inequality yields:

$$\begin{aligned} |I_1| \leq & \frac{\varepsilon^2 \hat{\mu}^*}{2} \sum_{i=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)} + \\ & + \frac{\hat{\mu}^*}{2} \sum_{i=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)} + \\ & + \frac{\hat{\mu}^*}{2} \sum_{j=1}^{2} \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} + \frac{\partial \hat{u}_j^\varepsilon}{\partial z} \right\|^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)} + \\ & + 2\varepsilon^2 \hat{\mu}^* \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}. \end{aligned}$$

Using the inequality $\|a+b\|^2 \leq 2\left(\|a\|^2+\|b\|^2\right)$, we find :

$$|I_1| \leq 2\varepsilon^2\hat{\mu}^* \sum_{i,j=1}^{2} \left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2 \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)} + 2\hat{\mu}^* \sum_{i=1}^{2} \left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)} + $$
$$+ 2\hat{\mu}^*\varepsilon^4 \sum_{i=1}^{2} \left\|\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2 \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)} + 2\varepsilon^2\hat{\mu}^* \left\|\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right\|^2 \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)}.$$

Now, inequality (3.3.8), gives us :

$$|I_1| \leq 2\mu^* C \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)}.$$

Therefore, using lemma (3.3.2), we obtain:

$$|I_1| \leq 2\mu^* h^* C \left\|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}. \tag{3.3.18}$$

Similarly, by applying the Cauchy-Schwartz inequality and lemma (3.3.2) we get :

$$|I_2| \leq r^* \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)} \leq r^* h^* \left\|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right\|_{L^2(\Omega)} \tag{3.3.19}$$

Moreover, hypotheses (3.3.15) and (3.3.16), lead us to :

$$\begin{aligned} b\left(T^\varepsilon, T^\varepsilon\right) &= \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \frac{\partial \hat{T}^\varepsilon}{\partial x_i} dxdz + \int_\Omega \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial z}.\frac{\partial \hat{T}^\varepsilon}{\partial z} dxdz \\ &= \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{K} \left|\frac{\partial \hat{T}^\varepsilon}{\partial x_i}\right|^2 dxdz + \int_\Omega \hat{K} \left|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right|^2 dxdz. \end{aligned}$$

which implies :

$$\begin{aligned} \hat{K}_*\varepsilon^2 \left\|\frac{\partial \hat{T}^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2 + \hat{K}_* \left\|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 &\leq b\left(\hat{T}^\varepsilon, \hat{T}^\varepsilon\right) \\ &\leq 2\hat{\mu}^* h^* C \left\|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right\|_{L^2(\Omega)} + \hat{r}^* h^* \left\|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}, \end{aligned}$$

which implies :

$$K_*\varepsilon^2 \left\|\frac{\partial \hat{T}^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2 + \hat{K}_* \left\|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 \leq C_1 \left\|\frac{\partial \hat{T}^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}, \tag{3.3.20}$$

where , C_1 is a constant independent of ε given by :

$$C_1 = 2\hat{\mu}^* h^* C + \hat{r}^* h^*$$

thus :

$$\left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \leq \hat{K}_*^{-1} C_1. \tag{3.3.21}$$

Additionally, by plugging this last estimation into (3.3.20) we obtain :

$$\varepsilon^2 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \leq C_2,$$

where

$$C_2 = \hat{K}_*^{-1} C_1^2.$$

3.3.2 Convergence Results and Limit Problem

In this section, we will state the convergence theorem.

Theorem 3.3.1. Under the same hypotheses of lemmas (3.3.1) *and*(3.3.3), *there exists* $\hat{u}^* = (\hat{u}_1^*, \hat{u}_2^*)$ *in* V_z *and*$\hat{T}^*$ *in* V_z, *such that for subsequences of* $\hat{u}^\varepsilon$ $\Big($ resp $\hat{T}^\varepsilon \Big)$ *still denoted* $\hat{u}^\varepsilon$ $\Big($ resp. $\hat{T}^\varepsilon \Big)$, *we have* the following convergence results:

$$\hat{u}_i^\varepsilon \rightharpoonup \hat{u}_i^* \quad 1 \leq i \leq 2 \;\; \text{weakly in } V_z \tag{3.3.22}$$

$$\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad 1 \leq i, j \leq 2 \;\; \text{weakly in } V_z \tag{3.3.23}$$

$$\varepsilon \frac{\hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \;\; \text{weakly in } V_z \tag{3.3.24}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad 1 \leq i \leq 2 \;\; \text{weakly in } V_z \tag{3.3.25}$$

$$\hat{T}^\varepsilon \rightharpoonup \hat{T}^* \;\; \text{weakly in } V_z \tag{3.3.26}$$

$$\varepsilon \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \rightharpoonup 0 \;\; \text{weakly in } V_z \tag{3.3.27}$$

Proof. Following (3.3.8), we obtain (3.2.22) – (3.3.25) as in Theorem 2.2.2 of the second chapter.

Thanks to the estimation (3.3.17) , there exists a positive constante C independent of ε such that :

$$\left\| \frac{\partial \hat{T}^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)} \leq C.$$

Using lemma (3.3.2), we deduce that :

$$\left\| \hat{T}^{\varepsilon} \right\|_{L^2(\Omega)} \leq h^* \left\| \frac{\partial \hat{T}^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)} \leq h^* C,$$

thus $\hat{T}^{\varepsilon}$ is bounded in V_z, implying the existence of an element $\hat{T}^*$ in V_z such that $\hat{T}^{\varepsilon}$ weakly converges to $\hat{T}^*$ in V_z.

Furthermore, the estimation (3.3.17), shows that $\varepsilon \left\| \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i} \right\|_{L^2(\Omega)} \leq C$, thus $\varepsilon \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i}$ converges to $\frac{\partial \hat{T}^*}{\partial x_i}$, and since $\hat{T}^{\varepsilon}$ converges to $\hat{T}^*$ in V_z, then $\varepsilon \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i}$ converges to 0 in V_z.

Theorem 3.3.2. Under the hypotheses of Theorem *3.3.1 the solutions* $\hat{u}^*$ *and* $\hat{T}^*$ satisfy the following relations:

$$\sum_{i=1}^{2} \int_{\Omega} \hat{\mu} \left(\hat{T}^* \right) \frac{\partial \hat{u}_i^*}{\partial z} . \frac{\partial}{\partial z} \left(\hat{\varphi}_i \right) dxdz = \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i \right); \tag{3.3.28}$$

$$-\frac{\partial}{\partial z} \left(\hat{\mu} \left(\hat{T}^* \right) \frac{\partial \hat{u}_i^*}{\partial z} \right) = \hat{f}_i \qquad for\ i = 1, 2 \ \ in\ L^2 \left(\Omega \right); \tag{3.3.29}$$

$$-\frac{\partial}{\partial z} \left(\hat{K} \frac{\partial \hat{T}^*}{\partial z} \right) = \sum_{i=1}^{2} \hat{\mu} \left(\hat{T}^* \right) \left(\frac{\partial \hat{u}_i^*}{\partial z} \right)^2 + \hat{r} \left(\hat{T}^* \right) \ in\ L^2 \left(\Omega \right). \tag{3.3.30}$$

Proof. The variational inequality (3.3.5) can be written as :

$$\sum_{i=1}^{4} I_i \left(\varepsilon \right) + \lambda \varepsilon^2 \int_{\Omega} \operatorname{div} \left(\hat{u}^{\varepsilon} \right) . \operatorname{div} \left(\hat{\varphi} \right) dxdz = \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i \right) + \varepsilon \left(\hat{f}_3, \hat{\varphi}_3 \right),$$

with

$$I_1 = \hat{\mu} \left(\hat{T}^{\varepsilon} \right) \varepsilon^2 \sum_{i,j=1}^{2} \left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j} + \frac{\partial \hat{u}_j^{\varepsilon}}{\partial x_i} \right) \frac{\partial}{\partial x_j} \left(\hat{\varphi}_i \right) dxdz;$$

$$I_2 = \hat{\mu} \left(\hat{T}^{\varepsilon} \right) \sum_{i=1}^{2} \int_{\Omega} \left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \right) \frac{\partial}{\partial z} \left(\hat{\varphi}_i \right) dxdz;$$

$$I_3 = \hat{\mu}\left(\hat{T}^{\varepsilon}\right)\sum_{i=1}^{2}\int_{\Omega}\varepsilon^2\left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z}+\varepsilon^2\frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i}\right)\frac{\partial}{\partial x_i}\left(\hat{\varphi}_3\right)dxdz;$$

$$I_4 = 2\hat{\mu}\left(\hat{T}^{\varepsilon}\right)\int_{\Omega}\varepsilon^2\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}.\frac{\partial}{\partial z}\left(\hat{\varphi}_3\right)dxdz.$$

Passing to the limit and using the convergence results of Theorem 3.3.1, there exists a subsequence $\hat{T}^{\varepsilon}$ that converges almost everywhere to $\hat{T}^{*}$, and since $\mu\left(\hat{T}^{\varepsilon}\right)$ is continuous, then $\mu\left(\hat{T}^{\varepsilon}\right)$converges almost everywhere to $\hat{\mu}\left(\hat{T}^{*}\right)$.

then

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(\hat{T}^{*}\right)\frac{\partial \hat{u}_i^{*}}{\partial z}.\frac{\partial}{\partial z}\left(\hat{\varphi}_i\right)dxdz = \sum_{i=1}^{2}\left(\hat{f}_i,\hat{\varphi}_i\right). \tag{3.3.31}$$

Using the Green's formula, we have

$$-\hat{\mu}\left(\hat{T}^{*}\right)\frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_i^{*}}{\partial z}\right) = \hat{f}_i \quad \text{for } i=1,2 \quad in\ H^{-1}\left(\Omega\right) \tag{3.3.32}$$

And since $\hat{f}_i \in L^2\left(\Omega\right)$, then (3.3.32) holds true in $L^2\left(\Omega\right)$.

Furthermore, passing to the limit in (3.3.6) and using the convergence results of Theorem 3.3.1, we obtain :

$$\begin{aligned}\int_{\Omega}\hat{K}\frac{\partial \hat{T}^{*}}{\partial z}.\frac{\partial \hat{\psi}}{\partial z}dxdz &= \sum_{i=1}^{2}\frac{1}{2}\int_{\Omega}\hat{\mu}\left(\hat{T}^{*}\right)\left(\frac{\partial \hat{u}_i^{*}}{\partial z}\right)^2\psi dxdz + \sum_{j=1}^{2}\frac{1}{2}\int_{\Omega}\hat{\mu}\left(\hat{T}^{\varepsilon}\right)\left(\frac{\partial \hat{u}_j^{*}}{\partial z}\right)^2\psi dxdz+\\ &+\int_{\Omega}\hat{r}\left(\hat{T}^{*}\right)\psi dxdz\\ &= \sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(\hat{T}^{*}\right)\left(\frac{\partial \hat{u}_i^{*}}{\partial z}\right)^2\psi dxdz + \int_{\Omega}\hat{r}\left(\hat{T}^{*}\right)\psi dxdz, \quad \forall\psi \in H^1_{\Gamma_L\cup\Gamma_1}\left(\Omega\right).\end{aligned}$$

Now, using the Green's formula, we have:

$$-\int_{\Omega}\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial \hat{T}^{*}}{\partial z}\right).\psi dxdz = \sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(\hat{T}^{*}\right)\left(\frac{\partial \hat{u}_i^{*}}{\partial z}\right)^2\psi dxdz+\int_{\Omega}\hat{r}\left(\hat{T}^{*}\right)\psi dxdz.\forall\psi \in H^1_{\Gamma_L\cup\Gamma_1}\left(\Omega\right).$$

Therefore :

$$-\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial \hat{T}^{*}}{\partial z}\right) = \sum_{i=1}^{2}\hat{\mu}\left(\hat{T}^{*}\right)\left(\frac{\partial \hat{u}_i^{*}}{\partial z}\right)^2 + \hat{r}\left(\hat{T}^{*}\right) \text{ in } H^{-1}_{\Gamma_L\cup\Gamma_1}\left(\Omega\right). \tag{3.3.33}$$

The formula (3.3.33) holds true in $L^2\left(\Omega\right)$, , since $\hat{\mu}$ and $\hat{r}$ sare two bounded functions on $\mathbb{R}$ and $\left(\frac{\partial u_i^{*}}{\partial z}\right)^2$ is an element of $L^2\left(\Omega\right)$.

Theorem 3.3.4. The solution *$(\hat{u}^*, \hat{T}^*)$ of the limit problem* (3.3.28) *and* (3.3.29) is unique.

Proof. The proof of this theorem is analogous to that of Theorem 2.3.4.

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[1] Some texts in this study were translated using the ChatGPT model.

Résumé

Le but de ce mémoire est l'analyse asymptotique d'un problème d' l'élasticité linéaire dans un domaine borné à trois dimensions avec des conditions de frottement sur une partie de la frontière et Dirichlet sur l'autre. Ici nous considérons dans cette étude l'élasticité isotherme *et non-isotherme.*

Mots-clés : *Analyse Asymptotique, Condition de Treska, Elasticité, Espace de Sobolev, Equations de Reynolds, Estimation a priori, Stokes.*

AMS Classification : *35R35, 76F10, 78M35.*

Abstract

The aim of this memory is the asymptotic analysis of a linear elasticity system in a bounded domain in three dimensions with friction conditions on part of the boundary and Dirichlet on the other. Here we consider in this study the isothermal and non-isothermal elasticity.

Keywords : *Asymptotic Analysis, Treska Conditions, Elasticity, Sobolev Space, Reynolds Equations, A priori estimates, Stokes.*

AMS Subject Classification : *35R35, 76F10, 78M35.*

ملخــــــــــــــــــــص

الهدف من هده المذكرة هو التحليل التقاربي لمسألة حدية حركية تتحكم فيها جملة المرونة في ميدان محدود ثلاثي الأبعاد مع شروط الأحتكاك على طرف من الحدودية و ديريشليه علي باقي الجبهية. هنا نعتبر في هده الدراسة المرونة لا تتأثر بالحرارة بينما في الدراسة الثانية تتأثر بالحرارة.

كلمات المفاتيح : التحليل *التقاربي، شروط الأحتكاك تريسكا، المرونة، فضاء صوبوليف، معادلة رينولدس، متراجحات قبلية، ستوكس.*