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Analytical and numerical solutions for volterra equations

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Abstract

The purpose of this note is to study the existence and uniqueness of a non-trivial solution to the linear Volterra integral equations. Where we touched on the classification of these integrative equations and some methods for finding solutions to the Volterra integrative equation with its first and second classification, then using Matlab we programmed algorithms to find numerical solutions and estimate the error by comparing the known exact solution in advance with the numerical solution.

Résumé

Le but de cette paper est d'étudier l'existence et l'unicité d'une solution non triviale aux équations intégrales linéaires de Volterra. Là où nous avons abordé la classification de ces équations intégratives et quelques moyens de trouver des solutions à l'équation intégrative de Volterra avec sa première et sa deuxième classification, puis en utilisant Matlab nous avons programmé des algorithmes pour trouver des solutions numériques et estimer l'erreur en comparant la solution exacte connue précédemment la solution numérique.

ملخص

الغرض من هذه المذكرة هو دراسة وجود وتفرد حل غير تافه لمعادلات فولتيرا التكاملية الخطية. حيث تطرقنا إلى تصنيف هذه المعادلات التكاملية وبعض الطرق لإيجاد حلول لمعادلة فولتيرا التكاملية بتصنيفها الأول والثاني، ثم باستخدام Matlab قمنا ببرمجة الخوارزميات لإيجاد الحلول العددية وتقدير الخطأ بمقارنة الحل الدقيق المعروف مسبقًا مع الحل العددي.

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INTRODUCTION

Scientists began to study natural phenomena in all their forms: physical - chemical - biological - engineering... So they came to the interpretation of these phenomena and finding different solutions to them analytically or numerically by means of equations Integration of all kinds. The integrative equations have been characterized by their important role in many theoretical and applied research due to the possibility of expressing them as a continuous or non-continuous integrative influence and thus modeling some issues in research that accept the integrative influence as a model. Hence, we see that these equations have a key role in mathematical modeling with integrative effects - calculating transformations in the field of electronics - analytical mechanics and in spectral relationships - Direkhleh issues in static electricity and particle transfer issues in astrophysics. In mathematics, its importance appears in: Partial and Ordinary Differential Equations - Functional Analysis - Probability Theory. Although most problems can be formulated and analyzed using differential equations, we often replace these equations with integral equations because they can be solved more easily. In recent years, interest in integrative equations has increased, especially Volterra equations. In the mid-sixties of the last century, the integrative equations began to be solved numerically when the nucleus was weakly anomalous or not. And at the beginning of the eighties, numerical methods began to impose themselves when it was not possible to find analytical solutions, especially when the nucleus was weak anomalies. There are many reasons that increased the importance of numerical methods:

- Many problems cannot be solved using analytical methods.
- It is difficult to solve analytical problems computationally.
- There is always a need for a new method for solving integral equations due to the lack of a capable method to solve all equations.

CHAPTER 1

INTRODUCTION OF VOLTERRA EQUATIONS

1.1 Introduction

The classification of integral equations was adopted Based on it Either linear integral equations Or non linear integral equations. The classification of integral equations It is somewhat similar to the classification of ordinary differential equations Or (partial differential equations). Generally speaking the most prominent integrative equations The most common and used are the two integral equations for Volterra and Fredholm, But there are two other types The integral differential equation and Singular integral equations.

1.2 Classification of Volterra Equations

The general form of the integral equation:

$$f(x) - \lambda \int_{m(x)}^{h(x)} K\left(x, t, f(t)\right) dt = g(x) \qquad a \leqslant x \leqslant T \tag{1.1}$$

where $m(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t,f)$ is a known function, of two variables x and t, called the kernel The unknown function $f(x)$ that will be determined appears inside the integral sign. In many other cases, functions $g(x)$ and $K(x,t,f)$ are given in advance. It is to be noted that the limits of integration $m(x)$ and $h(x)$ may be both variables, constants, or mixed. Integral equations appear in many forms. Two distinct ways that depend on the limits of integration are used to characterize integral equations.

Integrative equations usually appear in physical, chemical, and biological studies, and engineering applications mathematical models for it are described in elementary or boundary value problems.

If at least one limit is a variable, the equation is called a Volterra integral.

General form of the Volterra equation:

$$f(x) - \int_a^x K\left(x,t,f(t)\right) dt = g(x) \qquad a \leqslant x \leqslant T \tag{1.2}$$

Equation (1.2) is one of several forms in which a Volterra equation can be written.

Generally you can think of the shape.

$$F\left(f(x), \int_a^x K\left(x,t,f(t)\right) dt, g(x)\right) = 0 \qquad a \leqslant x \leqslant T \tag{1.3}$$

But well take care of our study in the form of equation (1.2).or our purposes we assume that T is finite. In many practical applications,The behavior of the solution on the field of integration is important, but finite T. In numerical computations it is necessary. For notational simplicity we can, without loss of generality, choose the range of the independent variable so that the lower limit is zero and consider only the equation.

$$f(x) - \int_0^x K\left(x,t,f(t)\right) dt = g(x) \qquad 0 \leqslant x \leqslant T < \infty \tag{1.4}$$

Remark 1.1. *if it was* $g(x) = 0$*:*

$$f(x) - \int_0^x K\left(x,t,f(t)\right) dt = 0 \qquad 0 \leqslant x \leqslant T < \infty \tag{1.5}$$

Called the equation (1.5) *Homogeneous integral equation.*

Exemple 1.1. *Examples of a homogeneous integral equation:*

1. $f(x) = \int\limits_0^x (x + t^2) f(t) dt$

2. $f(x) = 5 \int\limits_0^x (2x + \cos t) f^2(t) dt$

1.2.1 Volterra integral equation of the first order linear

Let the integral equation:

$$\int\limits_0^x K(x, t, f(t))\, dt = g(x) \tag{1.6}$$

$k(x, t, f)$ Linear with respect to the f function,and Where the unknown is $f(x)$. integral equation (1.6) We say it is Volterra integral equation of the first order linear.

Exemple 1.2. *The following integral equations, she Volterra integral equation of the first order linear:*

1. $\int\limits_0^x \cos(x - t) f(t) dt = x^2 \cos(x)$

2. $\exp(x) + \lambda \int\limits_0^x e^{t-x} f(t) dt = 0$

1.2.2 Volterra integral equation of the second order linear

We say about an equation written from the figure:

$$f(x) - \int\limits_0^x K(x, t, f(t))\, dt = g(x) \tag{1.7}$$

Where $k(x, t, f)$ and $g(x)$ are assumed to be known,$k(x, t, f)$ Linear with respect to the f function, And the unknown $f(x)$. the equation (1.7) it's called Volterra integral equation of the second order linear.

Exemple 1.3. *some examples on Volterra integral equation of the second order linear:*

1. $f(x) - 2 \int\limits_0^x (2x - t) f(t) dt = x^2 + 2$

2. $f(x)+\int\limits_0^x (t\sin 2x)f(t)dt = 3\cos x$

3. $f(x)=\beta\int\limits_0^x e^{x+t}f(t)dt+e^x$

1.2.3 Volterra-Fredholm integral equations

The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra. The standard form of the Volterra-Fredholm integral equation reads:

$$f(x)-\int\limits_0^x K_1(x,t,f(t))dt-\int\limits_a^b K_2(x,t,f(t))dt = g(x) \tag{1.8}$$

a and b real numbers, f are the unknown function.

Exemple 1.4. 1. $f(x)=\int\limits_0^x \cos(x-t)f(t)dt+\int\limits_1^2 \sin(x+t)f(t)dt+\tan(x)$

2. $f(x)+2\int\limits_0^x e^{tx}f(t)dt=\int\limits_1^2 e^t f(t)dt+e^{2x}$

1.2.4 Volterra Integro Differential Equations

The standard form of the Volterra integro differential equation reads.

$$f^{(n)}(x)-\int\limits_0^x K\left(x,t,f(t)\right)dt = g(x) \qquad f^{(i)}(0)=\alpha_i \quad : i=\overline{0,n-1} \tag{1.9}$$

Where f are the unknown, n natural number. α_i real numbers

Exemple 1.5. 1. $f'(x)-\int\limits_0^x f(t)dt = x^2+1 \qquad f(0)=0$

2. $f''(x)=2\int\limits_0^x \frac{1}{t-x+1}f(t)dt+\ln(x+2) \qquad f(0)=-1, f'(0)=1$

1.2.5 Equation of Volterra non-linear

$$f(x)-\int\limits_0^x K\left(x,t,f(t)\right)dt = g(x) \tag{1.10}$$

We say about the equation (1.10) It is non-linear if it was $K(x,t,f(t))$ non-linear for me f.

Exemple 1.6. *1.* $f(x) - 2\int\limits_0^x f^2(t)dt = 3x$

2. $f(x) = \int\limits_0^x \frac{t}{f(t)}dt + \frac{1}{x}$

1.3 Connection Between Volterra Equations And Initial Value Problems

1.3.1 Converting IVP to Volterra Integral Equation

We will need when you make this conversion to use the following formula integrative conversion in which they can convert any integration multiply to a single definite integral.

$$\underbrace{\int\limits_0^x \int\limits_0^{x_1} \int\limits_0^{x_2} \cdots \int\limits_0^{x_{n-1}}}_{\text{n time}} f(x_n)dx_n \cdots dx_2 dx_1 = \frac{1}{(n-1)!}\int\limits_0^x (x-t)^{n-1} f(t)dt \tag{1.11}$$

Proof. The proof will be established using mathematical Induction.

- for $n = 1$:

$$\begin{aligned} \int\limits_0^x f(x_1)dx_1 &= \frac{1}{(1-1)!}\int\limits_0^x (x-t)^{1-1} f(t)dt \\ \Rightarrow \quad \int\limits_0^x f(x_1)dx_1 &= \int\limits_0^x f(t)dt \\ \Rightarrow \quad \int\limits_0^x f(t)dt &= \int\limits_0^x f(t)dt \end{aligned}$$

- Now, suppose result is true for n, and let us prove it for $n+1$. Apply the induction hypothesis and switching the order of integration,

$$\underbrace{\int_0^x \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n}}_{\text{n+1 time}} f(x_{n+1}) dx_{n+1} \cdots dx_2 dx_1 = \underbrace{\int_0^x \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_{n-1}}}_{\text{n time}} \int_0^{x_n} f(x_{n+1}) dx_{n+1} dx_n \cdots dx_2 dx_1 =$$

$$= \underbrace{\int_0^x \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_{n-1}}}_{\text{n time}} g(x_n) dx_n \cdots dx_2 dx_1 \qquad : g(x_n) = \int_0^{x_n} f(x_{n+1}) dx_{n+1}$$

$$= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \times g(t) dt \qquad \text{Integration by parts}$$

$$= \frac{1}{(n-1)!} \left(\overbrace{\left[\frac{-1}{n} (x-t)^n g(t) \right]_0^x}^{0} - \int_0^x \frac{-1}{n} (x-t)^n f(t) \right) = \frac{1}{n!} \int_0^x (x-t)^n f(t)$$

so

$$\underbrace{\int_0^x \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n}}_{\text{n+1 time}} f(x_{n+1}) dx_{n+1} \cdots dx_2 dx_1 = \frac{1}{n!} \int_0^x (x-t)^n f(t) dt$$

$\square$

Let the following equation:

$$\begin{cases} y^{(n)}(x) + a_{n-1}(x) y^{n-1}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = b(x) \\ \\ y(0) = c_0, y'(0) = c_1, \cdots, y^{(n-1)}(0) = c_{n-1} \end{cases} \tag{1.12}$$

equation (1.12) the question of the value of primary linear differential equation of rank n, Transfer (1.12) to volterra Integral equation assume that the derivation of the highest ranking in (1.12) Is the function f that is:

$$y^n(x) = f(x) \tag{1.13}$$

and therefore we have to find an appropriate function for formulas y and its derivatives in the form of a single definite integral containing the function f and we get simply by Integral both ends the equation (1.13) between the two extremes 0 and x to obtain the integral formula for the derivative $y^{n-1}(x)$. then we complete the resulting relationship again and and we continue with that until we get an integral formula for the function y.

Finally, all the complementary formulas compensate for that have obtained for the function y and its derivatives in the given differential equation let's get the required integral equation.

Exemple 1.7. *Determine the equivalent integral equation for the following elementary value problem:*

$$\begin{cases} y'''(x) + 2xy'(x) + 3y(x) = 5x^2 + 1 \\ \\ y(0) = -1, y'(0) = 0, y''(0) = 1 \end{cases} \tag{1.14}$$

- *Assume that*

$$y'''(x) = f(x) \tag{1.15}$$

- *Integral* (1.15) *between the two extremes* 0 *and* x *we find that:*

$$y'''(x) = f(x) \Rightarrow y'''(t) = f(t) \Rightarrow \int_0^x y'''(t)dt = \int_0^x f(t)dt \Rightarrow y''(x) - y''(0) = \int_0^x f(t)dt \Rightarrow$$

$$y''(x) = 1 + \int_0^x f(t)dt \tag{1.16}$$

Integral (1.16) *between the two extremes* 0 *and* x *we find that:*

$$y''(x) = 1 + \int_0^x f(t)dt \Rightarrow y''(t_1) = 1 + \int_0^{t_1} f(t)dt \Rightarrow \int_0^x y''(t_1)dt_1 = \int_0^x \left[1 + \int_0^{t_1} f(t)dt\right] dt_1$$

$$\Rightarrow y'(x) - y(0) = x + \int_0^x \int_0^{t_1} f(t)dtdt_1 \Rightarrow y'(x) = x + \int_0^x \int_0^{t_1} f(t)dtdt_1$$

$$y'(x) = x + \int_0^x \int_0^{t_1} f(t)dtdt_1 \tag{1.17}$$

Using the equation (1.11) *find:*

$$y'(x) = x + \int_0^x (x-t)f(t)dt \tag{1.18}$$

by replacing the variable x *with the variable* t_2 *in equation* (1.17) *and between the two* *extremes* 0 *and* x *we find that:*

$$y'(t_2) = t_2 + \int_0^{t_2}\int_0^{t_1} f(t)dtdt_1 \Rightarrow \int_0^{x} y'(t_2)dt_2 = \int_0^{x}\left[t_2 + \int_0^{t_2}\int_0^{t_1} f(t)dtdt_1\right]dt_2$$

$$\Rightarrow y(x) - y(0) = \frac{1}{2}x^2 + \int_0^{x}\int_0^{t_2}\int_0^{t_1} f(t)dtdt_1dt_2 \Rightarrow y(x) = -1 + \frac{1}{2}x^2 + \int_0^{x}\int_0^{t_2}\int_0^{t_1} f(t)dtdt_1dt_2$$

Using the equation (1.11) *find:*

$$y(x) = \frac{1}{2}x^2 - 1 + \frac{1}{2}\int_0^{x}(x-t)^2 f(t)dt \tag{1.19}$$

- *Compensation equations* (1.19),(1.18),(1.16) *and* (1.15) *In the equation* (1.14) *we find:*

$$(f(x)) + 2x\left(x + \int_0^{x}(x-t)f(t)dt\right) + 3\left(\frac{1}{2}x^2 - 1 + \frac{1}{2}\int_0^{x}(x-t)^2 f(t)dt\right) = 5x^2 + 1$$

$$\Rightarrow f(x) + 2x^2 + 2x\int_0^{x}(x-t)f(t)dt + \frac{3}{2}x^2 - 3 + \frac{3}{2}\int_0^{x}(x-t)^2 f(t)dt = 5x^2 + 1$$

$$\Rightarrow f(x) - \int_0^{x}\left(2x(t-x) - \frac{3}{2}(x-t)^2\right)f(t)dt = \frac{3}{2}x^2 + 4$$

so the equivalent integral equation for elementary value problem (1.14)*:*

$$f(x) - \int_0^{x}\left(2x(t-x) - \frac{3}{2}(x-t)^2\right)f(t)dt = \frac{3}{2}x^2 + 4$$

1.3.2 Converting Volterra Equation to an ODE

We will need when you make this conversion to use law Leibniz next used to derive integrals:

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,t)dt = f(x,\beta(x))\beta'(x) - f(x,\alpha(x))\alpha'(x) + \int_{\alpha(x)}^{\beta(x)} \frac{df}{dx}dt \tag{1.20}$$

$$f(x) = \int_0^{x} K\left(x,t,f(t)\right)dt + g(x) \tag{1.21}$$

To convert the Volterra equation (1.21) into a differential equation, Derivation of both sides of the Volterra equation times consecutive even get a pure differential equation (do not contain integrals). And to obtain initial conditions we make compensation $x = 0$ In the resulting equations, let's get the initial values of the function f and its derivatives.

Exemple 1.8. *Determine the problem the equivalent initial value of the following integral equation:*

$$f(x) = e^x - \int_0^x (x-t)^2 f(t)dt \tag{1.22}$$

- $$\begin{aligned}
&\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(e^x - \int_0^x (x-t)^2 f(t)dt\right) \\
\Rightarrow &f'(x) = e^x - \left[(x-x)^2 f(x) \times 1 - (\cdots) \times 0 + \int_0^x 2(x-t)f(t)dt\right] \\
\Rightarrow &\frac{d}{dx}(f'(x)) = e^x - \frac{d}{dx}\left(\int_0^x 2(x-t)f(t)dt\right) \Rightarrow f''(x) = e^x - 2\int_0^x f(t)dt \\
\Rightarrow &f'''(x) = e^x - 2f(x)
\end{aligned}$$

- $$\begin{cases} f(0) = e^0 - \int_0^0 (0-t)^2 f(t)dt \\ f'(0) = e^0 - \int_0^0 2(x-t)f(t)dt \\ f''(0) = e^0 - 2\int_0^0 f(t)dt \end{cases} \Rightarrow \begin{cases} f(0) = 1 \\ f'(0) = 1 \\ f''(0) = 1 \end{cases}$$

$$f(x) = e^x - \int_0^x (x-t)^2 f(t)dt \Leftrightarrow \begin{cases} y'''(x) = e^x - 2y(x) \\ y(0) = 1, y'(0) = 1, y''(0) = 1 \end{cases}$$

CHAPTER 2

NUMERICAL TREATMENT OF THE VOLTERRA INTEGRAL EQUATION

2.1 Introduction

In this chapter we will be concerned with the nonhomogeneous Volterra integral equation of the second kind of the form:

$$f(x) = g(x) + \lambda \int_{0}^{x} K(x,t) f(t) dt \tag{2.1}$$

where $K(x,t)$ is the kernel of the integral equation, and λ is a parameter. As indicated earlier the limits of integration for the Volterra integral equations are functions of x. considered a separable kernel as discussed before in the previous chapter. Our concern will be on applying various methods to determine the solution $f(x)$ of (2.1) and not on the abstract theorems related to the existence, uniqueness of the solution or the convergence concept. These important concepts can be found in a variety of integral equations texts.

In the following we will discuss several methods that handle successfully the linear Volterra integral equations There is a variety on analytic and numerical techniques, traditional and new, that are usually used in studying Volterra integral equations. Accordingly we will first start with the recent methods.

2.2 Existence and uniqueness of the solution

The classical approach to proving the existence and uniqueness of the solution of (2.1) is the method of successive approximation, also called the Heard method. This consists of the simple iteration

$$f_n(x) = g(x) + \lambda \int\limits_0^x K(x,t) f_{n-1}(t) dt \qquad n \geqslant 1 \tag{2.2}$$

with

$$f_0 = g(x)$$

For ease of manipulation it is convenient to introduce

$$\varphi_n(x) = f_n(x) - f_{n-1}(x) \qquad n \geqslant 1 \tag{2.3}$$

with

$$\varphi_0(x) = g(x)$$

On subtracting from (2.2) the same equation with n replaced by $n-1$, we see that

$$\varphi_n(x) = \int_0^x k(x,t)\varphi_{n-1}(t)dt, \qquad n \geqslant 1. \tag{2.4}$$

Also, from (2.3),

$$f_n(x) = \sum_{i=0}^{n} \varphi_i(x). \tag{2.5}$$

The first theorem uses this iteration to prove the existence and uniqueness of the solution under quite restrictive conditions, namely that $k(x,t)$ and $g(x)$ are continuous.

Theorem 2.1. *If $k(x,t)$ is continuous in $0 \leqslant t \leqslant x \leqslant T$ and $g(x)$ is continuous in $0 \leqslant x \leqslant T$, then the integral equation* (2.1) *possesses a unique continuous solution for $0 \leqslant x \leqslant T$.*

Proof. Choose G and K such that

$$|g(x)| \leqslant G, \qquad 0 \leqslant x \leqslant T,$$

$$|k(x,t)| \leqslant K, \quad 0 \leqslant t \leqslant x \leqslant T.$$

We first prove by induction that

$$|\varphi_n(x)| \leqslant \frac{G(Kx)^n}{n!}, \quad 0 \leqslant x \leqslant T, \qquad n \geqslant 1 \tag{2.6}$$

If we assume that (2.6) is true for $n-1$, then from (2.4)

$$|\varphi_n(x)| \leqslant \frac{GK^n}{(n-1)!} \int_0^x t^{n-1} dt = \frac{GK^n x^n}{n!}$$

Since (2.6) is obviously true for $n=0$, it holds for all n. This bound makes it obvious that the sequence $f_n(x)$ in (2.5) converges and we can write

$$f(x) = \sum_{i=0}^{\infty} \varphi_i(x) \tag{2.7}$$

We now show that this $f(x)$ satisfies equation (2.1). The series (2.7) is uniformly convergent since the terms $\varphi_i(x)$ are dominated by $G(KT)^i/i!$. Hence we can interchange the order of integration and summation in the following expression to obtain

$$\begin{aligned}\int_0^x k(x,t) \sum_{i=0}^{\infty} \varphi_i(t) dt &= \sum_{i=0}^{\infty} \int_0^x k(x,t) \varphi_i(t) dt \\ &= \sum_{i=0}^{\infty} \varphi_{i+1}(x) = \sum_{i=0}^{\infty} \varphi_i(x) - g(x).\end{aligned}$$

This proves that $f(x)$ defined by (2.7) satisfies equation (2.1). Each of the $\varphi_i(x)$ is clearly continuous. Therefore $f(x)$ is continuous, since it is the limit of a uniformly convergent sequence of continuous functions.

To show that $f(x)$ is the only continuous solution, suppose there exists another continuous solution $\tilde{f}(x)$ of equaton (2.1). Then

$$f(x) - \tilde{f}(x) = \int_0^x k(x,t) \left(f(t) - \tilde{f}(t) \right) dt \tag{2.8}$$

Since $f(x)$ and $\tilde{f}(x)$ are both continuous, there exists a constant B such that

$$|f(x) - \tilde{f}(x)| \leqslant B, \quad 0 \leqslant x \leqslant T$$

Substituting this into (2.8)

$$|f(x) - \tilde{f}(x)| \leqslant KBx, \quad 0 \leqslant x \leqslant T$$

and repeating the step shows that

$$|f(x) - \tilde{f}(x)| \leqslant \frac{B(Kx)^n}{n!}, \quad 0 \leqslant x \leqslant T$$

$\square$

Note that although there is only one continuous solution, there may be solutions which are not continuous. One well-known example is the equation

$$f(x) = \int_0^x t^{x-t} f(t) dt \tag{2.9}$$

which has one obvious solution $f(x) = 0$. The kernel t^{x-t} is continuous, hut it can be verified that

$$f(x) = cx^{x-1} \tag{2.10}$$

is a solution of (2.9) for any c. For $c \neq 0$ this solution is discontinuous at $x = 0$. It is not difficult to show that any discontinuous solution of (2.1) must be nonintegrable and this is true of (2.10).

2.3 The Adomian Decomposition Method

As stated earlier, Adomian developed the *Adomian decomposition method* or simply the decomposition method that proved to work for all types of differential, integral and integro-differential equations, linear or nonlinear. The method was introduced by Adomian in his books [3, 2] and other related research papers such as [1, 4]. The focus of the two books was mainly on ordinary and partial differential equations. The approach we will In this method, thesolution $f(x)$ will be decomposed into an infinite series of components, that will be determined, given by the series form:

$$f(x) = \sum_{n=0}^{+\infty} f_n(x) \tag{2.11}$$

where the components $f_n(x)$, $n \geqslant 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components f_0,f_1,f_2,... individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.To establish the recurrence relation, we substitute (2.11) into the Volterra integral equation (2.1) to obtain:

$$\sum_{n=0}^{+\infty} f_n(x) = g(x) + \lambda \int_0^x K(x,t) \sum_{n=0}^{+\infty} f_n(t)dt \tag{2.12}$$

or equivalently:

$$f_0(x) + f_1(x) + f_2(x) + \cdots = g(x) + \lambda \int_0^x K(x,t) \left[f_0(t) + f_1(t) + f_2(t) + \cdots\right] dt \tag{2.13}$$

$$\begin{aligned} f_0(x) + f_1(x) + f_2(x) + \cdots = g(x) &+ \lambda \int_0^x K(x,t) f_0(t)dt \\ &+ \lambda \int_0^x K(x,t) f_1(t)dt \\ &+ \lambda \int_0^x K(x,t) f_2(t)dt \\ &+ \cdots \end{aligned}$$

The zeroth component $f_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $f_i(x), i \geqslant 1$ of the unknown function $f(x)$ are completely determined by setting the recurrence relation that is equivalent to:

$$\begin{cases} f_0(x) = g(x) \\ f_i(x) = \lambda \int_0^x K(x,t) f_{i-1}(t)dt, \quad i \geqslant 1 \end{cases} \tag{2.14}$$

In view of (2.14), the components $f_i(x)$, $i \geqslant 0$, follow immediately upon integrating the easily computable integrals. With these components determined, the solution $f(x)$ of (2.1) is readily determined in a series form upon using (2.11). As discussed before, the series obtained for $f(x)$ frequently provides the exact solution in a closed form if an exact solution exists as will be illustrated later. However, for concrete problems, where (2.11) cannot be evaluated, a truncated series is usually used to approximate the solution $f(x)$. It is to be noted here that, for numerical purposes, few terms of the obtained series usually provide the higher accuracy level of the approximation of the solution if compared with the existing numerical techniques.

It is interesting to recall that the decomposition method provides the solution of any style of equations in the form of a power series with easily computable components. In addition, applications have shown a very fast convergence of the series solution. The convergence concept of the decomposition technique was addressed extensively in [5] and by others, but it will not be discussed in this text.

The following illustrative examples will be discussed to explain the above outlined decomposition method.

Exemple 2.1. *The Volterra integral equation:*

$$f(x) = x + \int_0^x f(t)dt \tag{2.15}$$

We notice that $g(x) = x$, $\lambda = 1$, $K(x,t) = 1$. *Recall that the solution* $f(x)$ *is assumed to have a series form given in* (2.11). *Substituting the decomposition series* (2.11) *into both sides of* (2.15) *gives*

$$\sum_{n=0}^{+\infty} f_n(x) = x + \int_0^x \sum_{n=0}^{+\infty} f_n(t)dt$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$\begin{cases} f_0(x) = x \\ f_i(x) = \displaystyle\int_0^x f_{i-1}(t)dt, \quad i \geqslant 0 \end{cases}$$

so that

$$\begin{aligned}
f_0(x) &= x \\
f_1(x) &= \int_0^x f_0(t)dt = \int_0^x tdt = \frac{1}{2}x^2 \\
f_2(x) &= \int_0^x f_1(t)dt = \int_0^x \frac{1}{2}t^2dt = \frac{1}{6}x^3 \\
f_3(x) &= \int_0^x f_2(t)dt = \int_0^x \frac{1}{6}t^3dt = \frac{1}{24}x^4 \\
f_4(x) &= \int_0^x f_3(t)dt = \int_0^x \frac{1}{24}t^4dt = \frac{1}{120}x^5
\end{aligned}$$

and so on. Using (2.11) *gives the series solution:*

$$\begin{aligned}
f(x) =& x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots \\
=& 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots - 1
\end{aligned}$$

that converges to the closed form solution:

$$f(x) = e^x - 1$$

obtained upon using the Taylor expansion for e^x*.*

Exemple 2.2. *We finally solve the Volterra integral equation:*

$$f(x) = 1 - \int_0^x (x-t)f(t)dt$$

We notice that $g(x) = 1$*,* $\lambda = -1$*,* $K(x,t) = x - t$*. Recall that the solution* $f(x)$ *is assumed to have a series form given in* (2.11)*. Substituting the decomposition series* (2.11) *into both sides gives*

$$\sum_{n=0}^{+\infty} f_n(x) = 1 - \int_0^x (x-t)\sum_{n=0}^{+\infty} f_n(t)dt$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$\begin{cases} f_0(x) = 1 \\ f_i(x) = -\int\limits_0^x (x-t) f_{i-1}(t)dt, \quad i \geqslant 0 \end{cases}$$

so that

$$\begin{aligned}
&f_0(x) = 1 \\
&f_1(x) = -\int\limits_0^x (x-t) f_0(t)dt = -\int\limits_0^x (x-t)dt = -\left[xt - \frac{1}{2}t^2\right]_0^x = -\frac{1}{2}x^2 \\
&f_2(x) = -\int\limits_0^x (x-t) f_1(t)dt = +\frac{1}{2}\int\limits_0^x (x-t)t^2dt = \frac{1}{2}\left[\frac{1}{3}xt^3 - \frac{1}{4}t^4\right]_0^x = \frac{1}{4!}x^4 \\
&f_3(x) = -\int\limits_0^x (x-t) f_2(t)dt = -\frac{1}{4!}\int\limits_0^x (x-t)t^4dt = -\frac{1}{4!}\left[\frac{1}{5}xt^5 - \frac{1}{6}t^6\right]_0^x = -\frac{1}{6!}x^6 \\
&f_4(x) = -\int\limits_0^x (x-t) f_3(t)dt = \frac{1}{6!}\int\limits_0^x (x-t)t^6dt = \frac{1}{6!}\left[\frac{1}{7}xt^7 - \frac{1}{8}t^8\right]_0^x = \frac{1}{8!}x^8
\end{aligned}$$

and so on. Using (2.11) *gives the series solution:*

$$f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

that converges to the closed form solution:

$$f(x) = \cos x$$

obtained upon using the Taylor expansion for $\cos x$*.*

Exemple 2.3. *We consider here the Volterra integral equation:*

$$f(x) = x + x^2 + \int\limits_0^x t f(t)dt$$

Applying the decomposition technique as discussed before we find

$$
\begin{aligned}
f_0(x) &= x + x^2 \\
f_1(x) &= \int_0^x t f_0(t)dt = \int_0^x t(t+t^2)dt = \left[\frac{1}{3}t^3 + \frac{1}{4}t^4\right]_0^x = \frac{1}{3}x^3 + \frac{1}{4}x^4 \\
f_2(x) &= \int_0^x t f_1(t)dt = \int_0^x t\left(\frac{1}{3}t^3 + \frac{1}{4}t^4\right)dt = \left[\frac{1}{15}t^5 + \frac{1}{24}t^6\right]_0^x = \frac{1}{15}x^5 + \frac{1}{24}x^6 \\
f_3(x) &= \int_0^x t f_2(t)dt = \int_0^x t\left(\frac{1}{15}t^5 + \frac{1}{24}t^6\right)dt = \left[\frac{1}{105}t^7 + \frac{1}{192}t^8\right]_0^x = \frac{1}{105}x^7 + \frac{1}{192}x^8
\end{aligned}
$$

and so on. gives the series solution:

$$f(x) = x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \frac{1}{105}x^7 + \frac{1}{192}x^8 + \cdots$$

2.4 The Modified Decomposition Method

$$f(x) = g(x) + \lambda \int_0^x K(x,t)f(t)dt$$

If the inhomogeneous term $g(x)$ in the Volterra integral equation consists of a polynomial, trigonometric functions, hyperbolic functions, and others, the evaluation of the components $f_i, i \geqslant 0$ requires cumbersome work. A reliable modification of the Adomian decomposition method was developed by Wazwaz and presented in [8, 7]. The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution. The modified decomposition method will be applied, wherever it is appropriate, to all integral equations and differential equations of any order. It is interesting to note that the modified decomposition method depends mainly on splitting the function $g(x)$ into two parts, therefore it cannot be used if the function $g(x)$ consists of only one term. The modified decomposition method will be outlined and employed.

To give a clear description of the technique, we recall that the standard Adomian decomposition method admits the use of the recurrence relation we decompose the function $g(x)$ into two parts such as

$$g(x) = g_1(x) + g_2(x) \tag{2.16}$$

where $g_1(x)$ consists of one term only, or if needed of more terms in fewer other cases, and $g_2(x)$ includes the remaining terms of $g(x)$. Accordingly, (2.16) becomes

$$f(x) = g_1(x) + g_2(x) + \lambda \int_0^x K(x,t) f(t) dt$$

where the solution $f(x)$ is expressed by an infinite sum of components defined before by

$$f(x) = \sum_{n=0}^{+\infty} f_n(x)$$

$$\begin{aligned} f_0(x) + f_1(x) + f_2(x) + \cdots = g_1(x) + g_2(x) &+ \lambda \int_0^x K(x,t) f_0(t) dt \\ &+ \lambda \int_0^x K(x,t) f_1(t) dt \\ &+ \lambda \int_0^x K(x,t) f_2(t) dt \\ &+ \cdots \end{aligned}$$

The zeroth component $f_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $f_i(x), i \geqslant 1$ of the unknown function $f(x)$ are completely determined by setting the recurrence relation that is equivalent to:

$$\begin{cases} f_0(x) = g_1(x) \\ f_1(x) = g_2(x) + \lambda \displaystyle\int_0^x K(x,t) f_0(t) dt \\ f_i(x) = \lambda \displaystyle\int_0^x K(x,t) f_{i-1}(t) dt, \qquad i \geqslant 2 \end{cases} \tag{2.17}$$

This shows that the difference between the standard recurrence relation (2.14) and the modified recurrence relation (2.17) rests only in the formation of the first two components $f_0(x)$ and $f_1(x)$ only. The other components $f_i, i \geqslant 2$ remain the same in the two recurrence relations. Although this variation in the formation of $f_0(x)$ and $f_1(x)$ is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of computational work. Moreover, reducing the number of terms in $g_1(x)$ affects not only the component $f_1(x)$, but also the other components as well. This result was confirmed by several research works.

Two important remarks related to the modified method [8, 7] can be made here. First, by proper selection of the functions $g_1(x)$ and $g_2(x)$, the exact solution $f(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the proper choice of $g_1(x)$ and $g_2(x)$, and this can be made through trials only. A rule that may help for the proper choice of $g_1(x)$ and $g_2(x)$ could not be found yet. Second, if $g(x)$ consists of one term only, the standard decomposition method can be used in this case.

It is worth mentioning that the modified decomposition method will beused for Volterra and Fredholm integral equations, linear and nonlinear equations. The modified decomposition method will be illustrated by discussing the following examples.

Exemple 2.4. *Solve the Volterra integral equation by using the modified decomposition method:*

$$f(x) = \cos x - 2\sin x + 2\int_0^x f(t)dt$$

We first split $g(x)$ given by

$$g(x) = g_1(x) + g_2(x) = (\cos x) + (-2\sin x)$$

We next use the modified recurrence formula (2.17) *to obtain*

$$\begin{cases} f_0(x) = \cos x \\ f_1(x) = -2\sin x + 2\int_0^x f_0(t)dt \\ f_i(x) = 2\int_0^x f_{i-1}(t)dt, \qquad i \geqslant 2 \end{cases}$$

$$f_0(x) = \cos x$$

$$f_1(x) = -2\sin x + 2\int_0^x f_0(t)dt = -2\sin x + 2\int_0^x \cos t dt = -2\sin x + 2\left[\sin t\right]_0^x = 0$$

$$f_i(x) = 2\int_0^x f_{i-1}(t)dt = 2\int_0^x 0dt = 0, \qquad i \geqslant 2$$

This in turn gives the exact solution by

$$f(x) = \cos x$$

Exemple 2.5. *Solve the Volterra integral equation by using the modified decomposition method:*

$$f(t) = 2x - 1 + e^{-x^2} + \int_0^x e^{-x^2+t^2} f(t)dt$$

We first split $g(x)$ given by

$$g(x) = g_1(x) + g_2(x) = (2x) + (e^{-x^2} - 1)$$

We next use the modified recurrence formula (2.17) to obtain

$$\begin{cases} f_0(x) = 2x \\ f_1(x) = e^{-x^2} - 1 + \int_0^x e^{-x^2+t^2} f_0(t)dt \\ f_i(x) = \int_0^x e^{-x^2+t^2} f_{i-1}(t)dt, \qquad i \geqslant 2 \end{cases}$$

$$f_0(x) = 2x$$

$$f_1(x) = e^{-x^2} - 1 + \int_0^x e^{-x^2+t^2} 2tdt = e^{-x^2} - 1 + \left[e^{-x^2+t^2}\right]_0^x = e^{-x^2} - 1 + 1 - e^{-x^2} = 0$$

$$f_i(x) = \int_0^x e^{-x^2+t^2} f_{i-1}(t)dt = \int_0^x 0dt = 0, \qquad i \geqslant 2$$

This in turn gives the exact solution by

$$f(x) = 2x$$

2.5 The Variational Iteration Method

In this section we will study the newly developed variational iteration method that proved to be effective and reliable for analytic and numerical purposes. The variational iteration method (VIM) established by Ji-Huan He [9, 6] is now used to handle a wide variety of linear and non-linear, homogeneous and in homogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems. Moreover, the method gives the solution in a series form that converges to the closed form solution if an exact solution exists. The obtained series can be employed for numerical purposes if exact solution is not obtainable. In what follows, we present the main steps of the method.

Consider the differential equation:

$$u(t) = Lf + Nf \tag{2.18}$$

where L are linear operator and N are nonlinear operator, and $u(t)$ is the source in homogeneous term.

The variational iteration method presents a correction functional for equation (2.18) in the form:

$$f_{n+1}(x) = f_n(x) + \int_0^x \lambda(\xi)(Lf_n(\xi) + N\tilde{f}_n(\xi) - u(\xi))d\xi \tag{2.19}$$

where λ is a general Lagranges multiplier, noting that in this method λ may be a constant or a function, and $\tilde{f}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{f}_n = 0$, where δ is the variational derivative. The Lagrange multiplier λ can be identified optimally via the variational theory as will be seen later.

For a complete use of the variational iteration method, we should follow two steps, namely:

1. the determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally, and
2. with λ determined, we substitute the result into (2.19) where the restrictions should be omitted.

Taking the variation of (2.19) with respect to the independent variable f_n we find

$$\frac{\delta f_{n+1}}{\delta f_n} = 1 + \frac{\delta}{\delta f_n}\left(\int_0^x \lambda(\xi)(Lf_n(\xi) + N\tilde{f}_n(\xi) - u(\xi))d\xi\right)$$

or equivalently

$$\delta f_{n+1} = \delta f_n + \delta\left(\int_0^x \lambda(\xi)(Lf_n(\xi)d\xi\right) \tag{2.20}$$

Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words we can use

$$\begin{aligned}
\int_0^x \lambda(\xi)f_n'(\xi)d\xi &= \lambda(\xi)f_n(\xi) - \int_0^x \lambda'(\xi)f_n(\xi)d\xi \\
\int_0^x \lambda(\xi)f_n''(\xi)d\xi &= \lambda(\xi)f_n'(\xi) - \lambda'(\xi)f_n(\xi) + \int_0^x \lambda''(\xi)f_n(\xi)d\xi \\
\int_0^x \lambda(\xi)f_n'''(\xi)d\xi &= \lambda(\xi)f_n''(\xi) - \lambda'(\xi)f_n'(\xi) + \lambda''(\xi)f_n(\xi) - \int_0^x \lambda'''(\xi)f_n(\xi)d\xi \\
\int_0^x \lambda(\xi)f_n^{(4)}(\xi)d\xi &= \lambda(\xi)f_n'''(\xi) - \lambda'(\xi)f_n''(\xi) + \lambda''(\xi)f_n'(\xi) - \lambda'''f_n(\xi) + \int_0^x \lambda^{(4)}(\xi)f_n(\xi)d\xi
\end{aligned} \tag{2.21}$$

and so on. These identities are obtained by integrating by parts.

For example, if $Lf_n(\xi) = f_n'(\xi)$ in (2.20), then (2.20) becomes

$$\delta f_{n+1} = \delta f_n + \delta\left(\int_0^x \lambda(\xi)(Lf_n(\xi)d\xi\right)$$

Integrating the integral of (2.20) by parts using (2.21) we obtain

$$\delta f_{n+1} = \delta f_n + \delta\lambda(\xi)f_n(\xi) - \int_0^x \lambda'(\xi)\delta f_n(\xi)d\xi$$

or equivalently

$$\delta f_{n+1} = \delta f_n(\xi)(1 + \lambda|_{\xi=x}) - \int_0^x \lambda'\delta f_n d\xi \tag{2.22}$$

The extremum condition of f_{n+1} requires that $\delta f_{n+1} = 0$. This means that the left hand side of (2.22) is zero, and as a result the right hand side should be 0 as well. This yields the stationary conditions:

$$1 + \lambda|_{\xi=x} = 0, \qquad \lambda'|_{\xi=x} = 0$$

This in turn gives

$$\lambda = -1$$

As a second example, if $Lf_n(\xi) = f_n''(\xi)$ in (2.20), then (2.20) becomes

$$\delta f_{n+1} = \delta f_n + \delta \left(\int_0^x \lambda(\xi)(Lf_n(\xi)d\xi \right) \tag{2.23}$$

Integrating the integral of (2.23) by parts using (2.21) we obtain

$$\delta f_{n+1} = \delta f_n + \delta\lambda((f_n)')_0^x - (\lambda'\delta f_n)_0^x + \int_0^x \lambda''\delta f_n d\xi$$

or equivalently

$$\delta f_{n+1} = \delta f_n(\xi)(1 - \lambda'|_{\xi=x}) + \delta\lambda((f_n)'|_{\xi=x}) + \int_0^x \lambda''\delta f_n d\xi \tag{2.24}$$

The extremum condition of f_{n+1} requires that $\delta f_{n+1} = 0$. This means that the left hand side of (2.24) is zero, and as a result the right hand side should be 0 as well. This yields the stationary conditions:

$$1 - \lambda'|_{\xi=x} = x, \qquad \lambda|_{\xi=x} = 0, \qquad \lambda''|_{\xi=x}$$

This in turn gives

$$\lambda = \xi - x$$

Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximations $f_{n+1}, n \geqslant 0$, of the solution $f(x)$ will be readily obtained upon using selective function $f_0(x)$. However, for fast convergence, the function $f_0(x)$ should be selected by using the initial conditions as follows:

$$\begin{aligned} f_0(x) =& f(0), \text{for first order} f_n' \\ f_0(x) =& f(0) + xf'(0), \text{for second order} f_n'' \\ f_0(x) =& f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0), \text{for third order} f_n''' \\ &\vdots \end{aligned}$$

and so on. Consequently, the solution

$$f(x) = \lim_{n\to+\infty} f_n(x)$$

In other words, the correction functional (2.19) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

The determination of the Lagrange multiplier plays a major role in the determination of the solution of the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers, and its correction functional respectively:

$$(1)\begin{cases} f' + h\left(f(\xi), f'(\xi)\right) = 0, \lambda = -1 \\ f_{n+1} = f_n - \int\limits_0^x \left[f_n' + h\left(f_n, f_n'\right)\right] d\xi \end{cases}$$

$$(2)\begin{cases} f'' + h\left(f(\xi), f'(\xi), f''(\xi)\right) = 0, \lambda = (\xi - x) \\ f_{n+1} = f_n + \int\limits_0^x (\xi - x)\left[f_n'' + h\left(f_n, f_n', f_n''\right)\right] d\xi \end{cases}$$

$$(3)\begin{cases} f''' + h\left(f(\xi), f'(\xi), f''(\xi), f'''(\xi)\right) = 0, \lambda = -\frac{1}{2!}(\xi - x)^2 \\ f_{n+1} = f_n - \int\limits_0^x \frac{1}{2!}(\xi - x)^2\left[f_n''' + h\left(f_n, \ldots, f_n'''\right)\right] d\xi \end{cases}$$

$$(4)\begin{cases} f^{(4)} + h\left(f(\xi), f'(\xi), f''(\xi), f'''(\xi), f^{(4)}(\xi)\right) = 0, \lambda = \frac{1}{3!}(\xi - x)^3 \\ f_{n+1} = f_n + \int\limits_0^x \frac{1}{3!}(\xi - x)^3\left[f_n''' + h\left(f_n, f_n', \ldots, f_n^{(4)}\right)\right] d\xi \end{cases}$$

and generally

$$(5)\begin{cases} f^{(n)} + h\left(f(\xi), f'(\xi), \cdots, f^{n}(\xi)\right) = 0, \lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{n-1} \\ f_{n+1} = f_n + (-1)^n \int\limits_0^x \frac{1}{(n-1)!}(\xi - x)^{n-1}\left[f_n''' + h\left(f_n, \cdots, f_n^{(n)}\right)\right] d\xi \end{cases}$$

for $n \geqslant 1$

To use the variational iteration method for solving Volterra integral equations, it is necessary to convert the integral equation to an equivalent initial value problem or to an equivalent integro-differential equation. As defined before, integro-differential equation is an equation that contains differential and integral operators in the same equation. we will examine the obtained initial value problem by two methods, namely, standard methods used for solving ODEs, and by using the variational iteration method as will be seen by the following examples.

Exemple 2.6. *Solve the Volterra integral equation by using the variational iteration method*

$$f(x) = x + \int_0^x (x-t)f(t)dt \tag{2.25}$$

Using Leibnitz (1.20) *rule to differentiate both sides of* (2.25) *gives*

$$f'(x) = 1 + \int_0^x f(t)dt \tag{2.26}$$

Substituting $x = 0$ *into* (2.25) *gives the initial condition* $f(0) = 0$.

Using the variational iteration method

The correction functional for equation (2.26) *is*

$$f_{n+1}(x) = f_n(x) + \int_0^x \lambda(\xi)(f_n'(\xi) - \tilde{f}_n(\xi))d\xi \tag{2.27}$$

$$\tilde{f}_n(x) = 1 + \int_0^x f_n(t)dt$$

Using the given above leads to

$$\lambda = -1$$

So

$$f_{n+1}(x) = f_n(x) - \int_0^x \left(f_n'(\xi) - 1 - \int_0^\xi f_n(t)dt \right) d\xi \tag{2.28}$$

As stated before, we can use the initial condition to select $f_0(x) = f(0) = 0$. *Using this selection into* (2.28) *gives the following successive approximations:*

$$f_0(x) = 0$$

$$f_1(x) = 0 - \int_0^x \left(0 - 1 - \int_0^\xi f_0(t)dt \right) d\xi = x$$

$$f_2(x) = x - \int_0^x \left(1 - 1 - \int_0^\xi tdt \right) d\xi = x + \frac{1}{3!}x^3$$

$$f_3(x) = x + \frac{1}{3!}x^3 - \int_0^x \left(1 + \frac{1}{2}\xi^2 - 1 - \int_0^\xi t + \frac{1}{3!}t^3 dt \right) d\xi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$f_4(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \int_0^x \left(1 + \frac{1}{2}\xi^2 + \frac{1}{4!}\xi^4 - 1 - \int_0^\xi t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 dt \right) d\xi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7$$

and so on. The VIM admits the use of

$$f(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots$$

and this gives the exact solution by

$$f(x) = sh(x)$$

Exemple 2.7. *Solve the Volterra integral equation by using the variational iteration method*

$$f(x) = 5 + 2x^2 - \int_0^x (x-t)f(t)dt$$

Substituting $x = 0$ *gives the initial condition* $f(0) = 5$.

Using Leibnitz gives

$$f'(x) = 4x - \int_0^x f(t)dt$$

The correction functional is

$$f_{n+1}(x) = f_n(x) + \int_0^x \lambda(\xi)(f_n'(\xi) - \tilde{f}_n(\xi))d\xi$$

$$\tilde{f}_n(x) = 4x - \int_0^x f_n(t)dt$$

Using the given above leads to $\lambda = -1$

$$f_{n+1}(x) = f_n(x) - \int_0^x \left(f_n'(\xi) - 4\xi + \int_0^\xi f_n(t)dt \right) d\xi \tag{2.29}$$

As stated before, we can use the initial condition to select $f_0(x) = f(0) = 5$. *Using this selection into* (2.29) *gives the following successive approximations:*

$$f_0(x) = 5$$

$$f_1(x) = 5 - \int_0^x \left(0 - 4\xi + \int_0^\xi 5dt \right) d\xi = 5 - \frac{1}{2}x^2$$

$$f_2(x) = 5 - \frac{1}{2}x^2 - \int_0^x \left(-\xi - 4\xi + \int_0^\xi (5 - \frac{1}{2}t^2)dt \right) d\xi = 5 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$$

$$f_3(x) = 5 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \int_0^x \left(-\xi + \frac{1}{3!}\xi^3 - 4\xi + \int_0^\xi 5 - \frac{1}{2}t^2 + \frac{1}{4!}t^4dt \right) d\xi = 5 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

and so on. The VIM admits the use of

$$f(x) = 4 + \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)$$

and this gives the exact solution by

$$f(x) = 4 + \cos x$$

2.6 The Successive Approximations Method

The *successive approximations method*, also called the Picard [10] iteration method provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, $f_0(x)$ called the zeroth approximation which can be any real-valued function that will be used in a recurrence relation to determine the other approximations.

most commonly selected functions for $f_0(x)$ are 0,1 or x.

The successive approximations method introduces the recurrence relation:

$$\begin{cases} f_{n+1}(x) = g(x) + \lambda \int\limits_0^x K(x,t) f_n(t)dt \qquad n \geqslant 0 \\ f_0(x) = 0 \end{cases} \tag{2.30}$$

The question of convergence of $f_n(x)$ is justified by noting the following theorem.

Theorem 2.2 ([10]). *If $g(x)$ in (2.30) is continuous for the interval $0 \leqslant x \leqslant a$, and the kernel $K(x,t)$ is also continuous in the triangle $0 \leqslant x \leqslant a, 0 \leqslant t \leqslant x$, the sequence of successive approximations $f_n(x), n \geqslant 0$ converges to the solution $f(x)$ of the integral equation under discussion.*

It is interesting to point out that the variational iteration method admits the use of the iteration formula:

$$f_{n+1}(x) = f_n(x) + \int\limits_0^x \lambda(\xi)\left(\frac{\partial f_n(\xi)}{\partial \xi} - \tilde{f}_n(\xi)\right) d\xi$$

whereas the successive approximations method uses the iteration formula

$$f_{n+1}(x) = g(x) + \lambda \int\limits_0^x K(x,t) f_n(t)dt, n \geqslant 0$$

The difference between the two formulas can be summarized as follows:

1. The first formula contains the Lagrange multiplier λ that should be determined first before applying the formula. The successive approximations formula does not require the use of λ.

2. The first variational iteration formula allows the use of the restriction $\tilde{f}_n(\xi)$ where $\delta\tilde{f}_n(\xi) = 0$. The second formula does require this restriction.

3. The first formula is applied to an equivalent ODE of the integral equation, whereas the second formula is applied directly to the iteration formula of the integral equation itself.

The successive approximations method, or the Picard iteration method will be illustrated by the following examples.

Exemple 2.8. *Solve the following equation by successive approximation*

$$f(x) = x + \int_0^x (x-t)f(t)dt$$

For the zeroth approximation $f_0(x)$, we can select

$$f_0(x) = 0$$

The method of successive approximations admits the use of the iteration formula

$$f_{n+1}(x) = x + \int_0^x (x-t)f_n(t)dt, \qquad n \geqslant 0$$

And from him

$$\begin{aligned}
f_1(x) =& x + \int_0^x (x-t)f_0(t)dt = x \\
f_2(x) =& x + \int_0^x (x-t)(t)dt = x + \left[\frac{1}{2}xt^2 - \frac{1}{3}t^3\right]_0^x = x + \frac{1}{3!}x^3 \\
f_3(x) =& x + \int_0^x (x-t)(t + \frac{1}{3!}t^3)dt = x + \left[\frac{1}{2}xt^2 - \frac{1}{3}t^3 + \frac{1}{3\times 4}xt^4 - \frac{1}{3!\times 5}t^5\right]_0^x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \\
f_4(x) =& x + \int_0^x (x-t)(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5)dt = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 \\
&\vdots
\end{aligned}$$

$f_n(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$
Consequently, the solution $f(x)$ is given by

$$\begin{aligned} f(x) &= \lim_{n\to+\infty} f_n(x) \\ &= \lim_{n\to+\infty} \left(\sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} \right) \\ &= \arcsin x \end{aligned}$$

Exemple 2.9. *Solve the following equation by successive approximation*

$$f(x) = 2 - x + \int_0^x f(t)dt$$

For the zeroth approximation $f_0(x)$, we can select

$$f_0(x) = 1$$

The method of successive approximations admits the use of the iteration formula

$$f_{n+1}(x) = 2 - x + \int_0^x f_n(t)dt, \qquad n \geqslant 0$$

And from him

$$\begin{aligned} f_1(x) =& 2 - x + \int_0^x f_0(t)dt = 2 - x + \int_0^x 1dt = 2 - x + [t]_0^x = 2 - x + x = 2 \\ f_2(x) =& 2 - x + \int_0^x 2dt = 2 + x \\ f_3(x) =& 2 - x + \int_0^x (2+t)dt = 2 - x + \left[2t + \frac{1}{2}t^2\right]_0^x = 2 + x + \frac{1}{2}x^2 \\ f_4(x) =& 2 - x + \int_0^x (2 + t + \frac{1}{2}t^2)dt = 2 - x + \left[2t + \frac{1}{2}t^2 + \frac{1}{3!}t^3\right]_0^x = 2 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \\ &\vdots \\ f_n(x) =& 1 + \sum_{k=0}^{n} \frac{x^k}{k!} \end{aligned}$$

Consequently, the solution $f(x)$ is given by

$$\begin{aligned} f(x) &= \lim_{n\to+\infty} f_n(x) \\ &= \lim_{n\to+\infty}\left(1+\sum_{k=0}^{n}\frac{x^k}{k!}\right) \\ &=1+\exp(x) \end{aligned}$$

2.7 The Series Solution Method

Let the following Volterra equation be

$$f(x) = g(x) + \lambda\int_0^x K(x,t)f(t)dt \tag{2.31}$$

We will consider the solution $f(x)$ written from the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \tag{2.32}$$

where is found a_k by following these steps. We replace (2.32) by (2.31) Thereby creating.

$$\sum_{k=0}^{\infty} a_k t^k = T(g(x)) + \lambda\int_0^x K(x,t)\left(\sum_{k=0}^{\infty} a_k t^k dt\right) \tag{2.33}$$

where $T(g(x))$ is the Taylor series for $g(x)$.

We first integrate the right side of the integral in (2.33), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_i, i \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_i, i \geqslant 0$. Having determined the coefficients $a_i, i \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (2.32). The exact solution may be obtained if such an exact solution exists. series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve. as will be seen by the following examples.

Exemple 2.10. *Let the following Volterra integral equation be*

$$f(x) = x + \int_0^x (x-t)\, f(t)dt \tag{2.34}$$

Suppose the solution is written in the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

produce

$$\sum_{k=0}^{\infty} a_k x^k = x + \int_0^x (x-t) \left(\sum_{k=0}^{\infty} a_k t^k \right) dt$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = x + \int_0^x (x-t) \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots \right) dt$$

$$\Rightarrow a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_k x^k + \cdots = 0 + (1)x + \left(a_0 - \frac{1}{2} a_0 \right) x^2 + \left(\frac{1}{2} a_1 - \frac{1}{3} a_1 \right) x^3 +$$
$$+ \left(\frac{1}{3} a_2 - \frac{1}{4} a_2 \right) x^4 + \left(\frac{1}{4} a_3 - \frac{1}{5} a_3 \right) x^5 + \cdots + \left(\frac{1}{k-1} a_{k-2} - \frac{1}{k} a_{k-2} \right) x^k + \cdots$$
$$\Rightarrow a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_k x^k + \cdots = 0 + x + \left(\frac{1}{2} \right) a_0 x^2 + \left(\frac{1}{6} \right) a_1 x^3 + \left(\frac{1}{12} \right) a_2 x^4 +$$
$$+ \left(\frac{1}{20} \right) a_3 x^5 + \cdots + \left(\frac{1}{(k-1)(k)} \right) a_{k-2} x^k + \cdots$$

Equating the coefficients of like powers of x in equation yields

$$a_0 = 0$$
$$a_1 = 1$$
$$a_2 = \frac{1}{2} a_0 = 0$$
$$a_3 = \frac{1}{6} a_1 = \frac{1}{3!}$$
$$a_4 = \frac{1}{12} a_2 = 0$$
$$a_5 = \frac{1}{20} a_3 = \frac{1}{5!}$$
$$\vdots$$
$$a_{k-1} = \frac{1}{(k-1)(k-2)} a_{k-3} = 0$$
$$a_k = \frac{1}{(k)(k-1)} a_{k-2} = \frac{1}{k!}$$
$$\vdots$$

This result can be combined to obtain

$$a_{2k} = 0 \quad , \quad a_{2k+1} = \frac{1}{(2k+1)!} \quad k \geqslant 0$$

The solution in a series form is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$$

that converges to the exact solution

$$f(x) = \sinh(x)$$

2.8 The Series Solution Method Equation the First Kind

form of the Volterra integral equations of the first kind is given (2.35)

$$g(x) = \int_0^x K(x,t) f(t) dt \tag{2.35}$$

we will consider the solution $f(x)$ to be analytic, of the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \tag{2.36}$$

where the coefficients a_n will be determined recurrently. Substituting (2.36) into (2.35) gives

$$T(g(x)) = \int_0^x K(x,t) \left(\sum_{k=0}^{\infty} a_k t^k dt \right)$$

$T(g(x))$ is the Taylor series for $g(x)$.

$$T(g(x)) = \int_0^x K(x,t) \left(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots \right) dt \tag{2.37}$$

We first integrate the right side of the integral (2.37), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_i, i \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_i, i \geqslant 0$. Having determined the coefficients $a_i, i \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into (2.36). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve. This method will be illustrated by discussing the following examples.

Exemple 2.11. *Solve the Volterra integral equation by using the series solution method*

$$1 + xe^x - e^x = \int_0^x tf(t)dt \tag{2.38}$$

Using few terms of the Taylor series for $1+xe^x-e^x$ *and for the solution* $f(x)$*, and by integrating the right side we find*

$$1 + xe^x - e^x = 1 + x\sum_{k=0}^{n} \frac{1}{k!}x^k - \sum_{k=0}^{n} \frac{1}{k!}x^k = \frac{1}{2}x^2 + \frac{2}{3!}x^3 + \frac{3}{4!}x^4 + \frac{4}{5!}x^5 + \frac{5}{6!}x^6 + \cdots$$

Equation (2.38) *equivalent*

$$\frac{1}{2}x^2 + \frac{2}{3!}x^3 + \frac{3}{4!}x^4 + \frac{4}{5!}x^5 + \frac{5}{6!}x^6 + \cdots = \int_0^x t\left(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots\right)dt$$

$$\frac{1}{2}x^2 + \frac{2}{3!}x^3 + \frac{3}{4!}x^4 + \frac{4}{5!}x^5 + \frac{5}{6!}x^6 + \cdots = \left[\frac{1}{2}a_0t^2 + \frac{1}{3}a_1t^3 + \frac{1}{4}a_2t^4 + \frac{1}{5}a_3t^5 + \frac{1}{6}a_4t^6 + \cdots\right]_0^x$$

$$\frac{1}{2}x^2 + \frac{2}{3!}x^3 + \frac{3}{4!}x^4 + \frac{4}{5!}x^5 + \frac{5}{6!}x^6 + \cdots = \frac{1}{2}a_0x^2 + \frac{1}{3}a_1x^3 + \frac{1}{4}a_2x^4 + \frac{1}{5}a_3x^5 + \frac{1}{6}a_4x^6 + \cdots$$

Conformity

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2!}, a_3 = \frac{1}{3!}, a_4 = \frac{1}{4!}, \cdots$$

The solution in a series form is given by

$$f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

that converges to the exact solution

$$f(x) = \exp(x)$$

2.9 Conversion of First Kind to Second Kind

The integrated Volterra equation of the first type can be treated simply by converting this equation into the Volterra equation of the second type. By differentiating both sides of the equation(2.35) by using Leibniz rule. We get

$$g'(x) = K(x,x)f(x) + \int_0^x K(x,t)f(t)dt \tag{2.39}$$

If $K(x,x) \neq 0$ in the interval of discussion, then dividing both sides of (2.39) by $K(x,x)$ yields

$$f(x) = \frac{g'(x)}{K(x,x)} - \frac{1}{K(x,x)} \int_0^x K(x,t)f(t)dt \tag{2.40}$$

a Volterra integral equation of the second kind. The case where the kernel $K(x,x) = 0$ leads to a complicated behavior of the problem that will not be investigated here.

To solve (2.40) we select any method that we discussed before. The technique of differentiating both sides of Volterra integral equation of the first kind, verifying that $K(x,x) \neq 0$, reducing to Volterra integral equation of the second kind and solving the resulting equation will be illustrated by discussing the same two examples solved by using the series solution method.

Exemple 2.12. *Let the Volterra integral equation be of the first kind.*

$$xe^x = \int_0^x e^{x-t} f(t)dt$$

By differentiating both sides of the equation by using Leibniz rule

$$f(x) = e^x + xe^x - \int_0^x e^{x-t} f(t)dt$$

using The Adomian Decomposition Method (2.3).

$$f_0(x) = e^x + xe^x$$

$$f_1(x) = -e^x \left(x + \frac{x^2}{2!} \right)$$

$$f_2(x) = e^x \left(\frac{x^2}{2!} + \frac{x^3}{3!} \right)$$

and so on. Using the above results of the components obtained gives

$$f(x) = e^x \left(1 + x - x + \frac{x^2}{2!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^3}{3!} \cdots \right) = e^x$$

the exact solution obtained upon cancelling like terms with opposite signs.

$$f(x) = e^x$$

CHAPTER 3

NUMERICAL RESULTS

3.1 Introduction

In this chapter, we present four numerical examples to illustrate the above methods for solving linear Volterra integral equations of the second kind. The exact solution is known and is used to show that the numerical solution obtained with our methods we used MATLAB

3.2 The Adomian Decomposition Method

Consider the following linear Volterra integral equation of the second kind

$$f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 f(t)dt \tag{3.1}$$

where $g(x) = 1 - \frac{1}{2}x^2$, $\lambda = \frac{1}{6}$, $k(x,t) = (x-t)^3$ and $N = 8$ produce

$$f(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{1}{32!}x^{32} - \frac{1}{34!}x^{34}$$

We know that the correct answer is $\tilde{f}(x) = \cos(x)$

The following MATLAB code solves the equation (3.1) numerically, and plot the absolute errors and the approximate solution of this equation with the exact solution.

```
% ce programe qui calcul la solution approchee de la ...
    probleme
% f(x)=g(x)+\lambda\int(0,x)k(x,t)f(t)dt
clc  ;  clear;
%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%
l=1/6;
N=8;
%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%
syms x t
g=1-(1/2)*x^2;
k=(x-t)^3;
f2=cos(x);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
k=int(k,x,0,x);k=diff(k,x);
k=inline(k,'x','t');
g=int(g,x,0,x);g=diff(g,x);
g=inline(g);
h=g;
f=g;
    for i=1:N
        h=l*int(k(x,t)*h(t),t,0,x)  ;
        h=inline(h);
    f=inline(f(x)+h(x));
end
pretty(expand(f(x)))
f2=int(f2,x,0,x);f2=diff(f2,x);
f2=inline(f2);
```

```
error=inline(abs(f2(x)-f(x)));
x=0:0.01:1;
figure
subplot(3,1,1)
plot(x,f2(x))
title('exact solution')
subplot(3,1,2)
plot(x,f(x))
title('approximate solution')
subplot(3,1,3)
plot(x,error(x))
title('error')
display('fin')
```

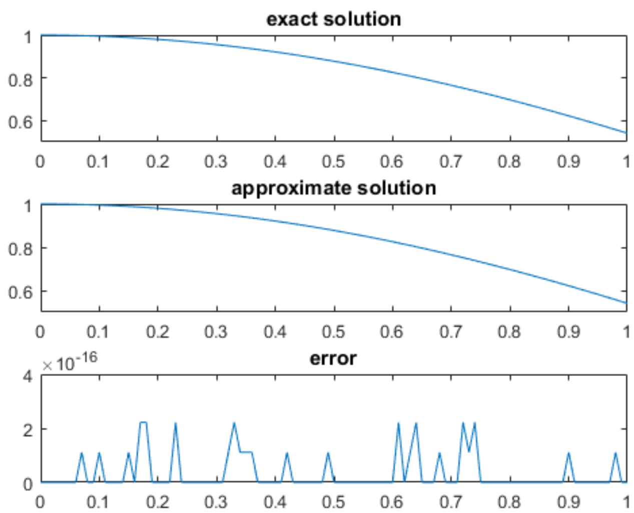

Figure 3.1: Error For N=8

N	$\max\limits_{0\leqslant x\leqslant 1}\left(\lvert\tilde{f}(x)-f(x)\rvert\right)$
1	2.5×10^{-5}
2	2.5×10^{-9}
4	2.5×10^{-16}
8	2.5×10^{-16}

Table 3.1: The numerical error obtained

3.3 The Modified Decomposition Method

Let the Volterra integral linear equation of the second type be the following:

$$f(x)=1+x+\frac{1}{3!}x^3-\int_0^x (x-t)\,f(t)dt \tag{3.2}$$

where $g_1(x)=1+x$, $g_2(x)=\frac{1}{3!}x^3$, $\lambda=-1$,

$k(x,t)=(x-t)$ and $N=16$ approximate solution

$$f(x)=1+x-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots+\frac{1}{30!}x^{30}-\frac{1}{32!}x^{32}$$

We know that the correct answer is $\tilde{f}(x)=x+\cos(x)$

The following MATLAB code solves the equation (3.2) numerically, and plot the absolute errors and the approximate solution of this equation with the exact solution.

```
% ce programe qui calcul la solution approchee de la ...
    probleme
% f(x)=g(x)+\lambda\int(0,x)k(x,t)f(t)dt
clc ; clear;
%%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%%%%%%
```

```
l=-1;
N=16;
%%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%%
syms x t
g1=1+x;
g2=(1/6)*x^3;
k=x-t;
f2=x+cos(x);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
k=int(k,x,0,x);k=diff(k,x);
k=inline(k,'x','t');
g1=int(g1,x,0,x);g1=diff(g1,x);
g1=inline(g1);
g2=int(g2,x,0,x);g2=diff(g2,x);
g2=inline(g2);
f=g1;
h=inline(g2(x)+l*int(k(x,t)*f(t),t,0,x))  ;
f=inline(f(x)+h(x));
      for i=2:N
            h=inline(l*int(k(x,t)*h(t),t,0,x));
            f=inline(f(x)+h(x));
      end
pretty(expand(f(x)))
f2=int(f2,x,0,x);f2=diff(f2,x);
f2=inline(f2);
error=inline(abs(f2(x)-f(x)));
x=0:0.01:1;
figure
```

```
subplot(3,1,1)
plot(x,f2(x))
title('exact solution')
subplot(3,1,2)
plot(x,f(x))
title('approximate solution')
subplot(3,1,3)
plot(x,error(x))
title('error')
display('fin')
```

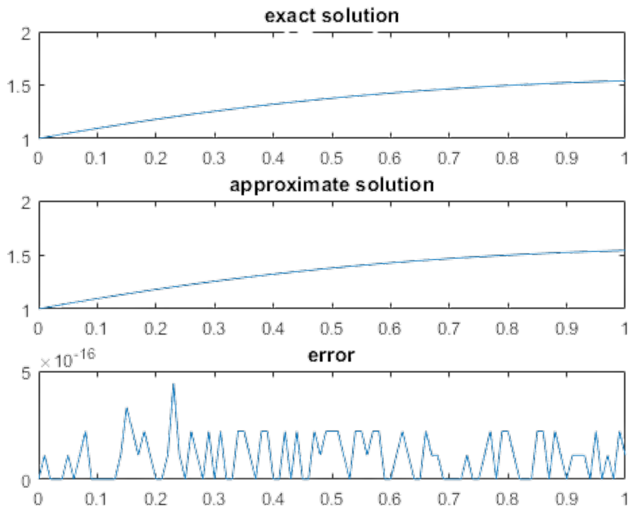

Figure 3.2: Error For N=16

N	$\max\limits_{0\leqslant x\leqslant 1}\left(\lvert\tilde{f}(x)-f(x)\rvert\right)$
2	1.4×10^{-3}
4	3×10^{-7}
8	4.5×10^{-16}
16	4.5×10^{-16}

Table 3.2: The numerical error obtained

3.4 The Variational Iteration Method

Let the Volterra integral linear equation of the second type be the following:

$$f(x)=x+\int_{0}^{x}(x-t)\,f(t)dt \tag{3.3}$$

where $g(x)=x$, $\lambda=1$, $k(x,t)=(x-t)$ and $N=8$ We use the following algorithm to get the approximate solution to the equation

```
% ce programe qui calcul la solution approchee de la ...
    probleme
% f(x)=g(x)+\lambda\int(0,x)k(x,t)f(t)dt
clc ; clear;
%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%
l=1;
N=8;
%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms x t z
g=x;
```

```
k=x-t;
f2=sinh(x);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
k=int(l*k,x,0,x);k=diff(k,x);
k=inline(k,'x','t');
g=int(g,x,0,x);g=diff(g,x);
g=inline(g);
f=g(0);
      for i=1:N
            f=int(f,x,0,x);f=diff(f,x);
            f=inline(f);
            kp=diff(k(x,t),x);
            h=diff(g(x),x)+k(x,x)*f(x)+int(kp*f(t),t,0,x);
            h=int(h,x,0,z);h=diff(h,z);
            fp=diff(f(z),z);
            f=f(x)+int(h-fp,z,0,x);
      end
pretty(expand(f))
```

$$f(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots + \frac{1}{13!}x^{13} + \frac{1}{15!}x^{15}$$

We know that the correct answer is $\tilde{f}(x) = \sinh(x)$, We use the following algorithm to draw the error

```
clc ; clear;
syms x
f=x+(1/6)*x^{3}+(1/factorial(5))*x^{5}+...
(1/factorial(7))*x^{7}+(1/factorial(9))*x^{9}+...
(1/factorial(11))*x^{11}+(1/factorial(13))*x^{13}+...
```

```
(1/factorial(15))*x^{15};
f2=sinh(x);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
f2=int(f2,x,0,x);f2=diff(f2,x);
f2=inline(f2);
error=inline(abs(f2(x)-f));
x=0:0.01:1;
hold on
plot(x,error(x),'k-')
hold off
display('fin')
```

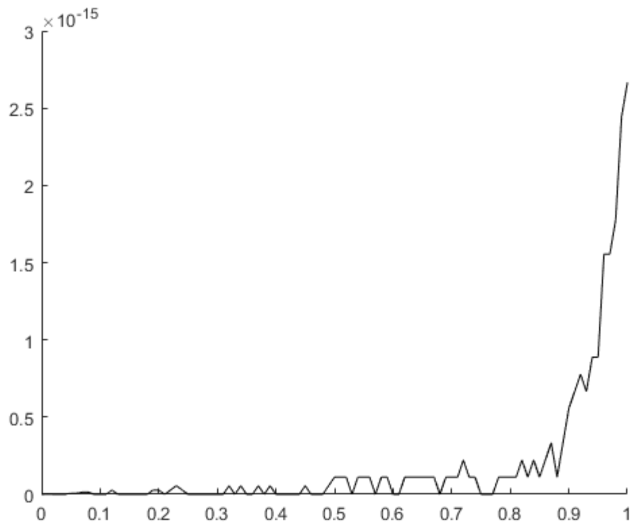

Figure 3.3: Error For N=8

The following table shows the error estimation with a change N

N	$\max\limits_{0\leqslant x\leqslant 1}\left(\lvert\tilde{f}(x)-f(x)\rvert\right)$
2	9×10^{-3}
4	3×10^{-6}
8	3×10^{-15}
16	4.5×10^{-16}

Table 3.3: The numerical error obtained

3.5 The Successive Approximations Method

Let the Volterra integral linear equation of the second type be the following:

$$f(x) = 1 + x(\cos x - \sin x) + \int_0^x t f(t)dt \tag{3.4}$$

where $g(x) = 1 + x(\cos x - \sin x)$, $\lambda = 1$, $f_0(x) = x$, $k(x,t) = t$ and $N = 4$ produce We use the following algorithm to get the approximate solution to the equation

```
% ce programe qui calcul la solution approchee de la ...
    probleme
% f(x)=g(x)+\lambda\int(0,x)k(x,t)f(t)dt
clc ; clear;
%%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%%
l=1;
N=4;
%%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%%
syms x t
g=1+x*(cos(x)-sin(x));
```

```
k=t;
f=x;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
k=int(l*k,x,0,x);k=diff(k,x);
for i=1:N
f=int(f,x,0,t);f=diff(f,t);
f=g+int(k*f,t,0,x);
end
pretty(expand(f))
```

$$f(x) = 45x^2\cos(x) - 104\sin(x) - 104\cos(x) - 10x^3\cos(x) - x^4\cos(x) + 45x^2\sin(x) + 10x^3\sin(x) - x^4\sin(x) + 105x\cos(x) - 105x\sin(x) + \frac{15}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{945}x^9 + 105$$

We know that the correct answer is $\tilde{f}(x) = \sin(x) + \cos(x)$ and use the following algorithm to draw the error

```
clc ; clear;
syms x
f=45*x^2*cos(x)-104*sin(x)-104*cos(x)-10*x^3*cos(x)-...
x^4*cos(x)+45*x^2*sin(x)+10*x^3*sin(x)-x^4*sin(x)+...
105*x*cos(x)-105*x*sin(x)+(15/2)*x^2+(3/8)*x^4+...
(1/48)*x^6+(1/945)*x^9+105
f2=sin(x)+cos(x);
f2=int(f2,x,0,x);f2=diff(f2,x);
f2=inline(f2);
error=inline(abs(f2(x)-f));
x=0:0.01:1;
hold on
```

```
plot(x,error(x),'k-')
hold off
display('fin')
```

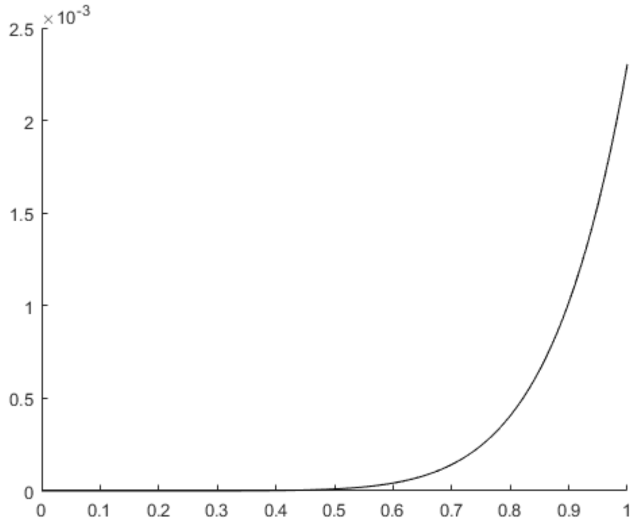

Figure 3.4: Error For N=4

N	$\max\limits_{0\leqslant x\leqslant 1}\left(\lvert\tilde{f}(x)-f(x)\rvert\right)$
2	0.12
4	2.5×10^{-3}
8	3×10^{-7}
9	3×10^{-8}

Table 3.4: The numerical error obtained

3.6 Conversion of First Kind to Second Kind

Let the integrated linear Volterra equation of the first kind be:

$$xe^{-x} = \int_{0}^{x} e^{t-x} f(t)dt$$

We use the following algorithm to convert the integral equation from the first order to the second order

```
% ce programe qui calcul la solution approchee de la ...
    probleme
% g(x)=\lambda\int(0,x)k(x,t)f(t)dt
clc ; clear;
%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%
l=1;
%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%
syms x t
g=x*exp(-x);
k=exp(t-x);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
k=int(l*k,x,0,x);k=diff(k,x);
k=inline(k,'x','t');
disp('g(x)=')
disp(expand(diff(g,x)/k(x,x)))
disp('k(x,t)=')
disp(expand(-1*k(x,t)/k(x,x)))
```

We get

$$f(x) = e^{-x} - xe^{-x} - \int_{0}^{x} e^{t-x} f(t)dt$$

This integral equation is linear and of the second order, it is solved by one of the previous methods.

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