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Classification of Kac-Moody Lie Algebras

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الإهداء

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كان الطريق إلى النور شائكاً، محفوفاً بالتعب والسهر والإحباط. وها قد منَّ الله عليا بأن شارفتُ على الوصول، وبهذه المناسبة الطيبة أهدي ثمرة جهدي :

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Contents

Notations

Symbol	**Meaning**
L	A Lie algebra.
$[xy]$	Lie product of elements.
$[HK]$	Lie product of subspaces.
$\mathfrak{gl}_n(k)$	The general linear Lie algebra of degree n over k.
$\operatorname{ad} x$	The adjoint map.
L^n	Power of the Lie algebra L.
$L^{(n)}$	A power of the Lie algebra L.
$N(H)$	Normaliser of a subalgebra H.
H	A Cartan subalgebra of a Lie algebra.
$L_{0,x}$	The null component of x in L.
Φ	The root system of a finite dimensional Lie algebra.
L_α	A root space of L.
$\langle , \rangle$	The Killing form on a finite dimensional Lie algebra.
$M^\perp$	The orthogonal subspace of a subspace M.
$\{ , \}$	A symmetric scalar product.
$H_\mathbb{R}$	A real vector space in H.
$H_\mathbb{R}^*$	The dual space of $H_\mathbb{R}$.
Φ^+	Set of positive roots.
Φ^-	Set of negative roots.
Π	Fundamental roots of a semisimple Lie algebra.
s_α	Reflection.
W	The Weyl group of a semisimple Lie algebra.
ht_α	The height of a root α.
s_i	Fundamental reflection.
$\alpha_1 \dots \alpha_l$	Fundamental roots of a semisimple Lie algebra.
$l(w)$	The length of w.

$Q(x_1, \ldots, x_l)$	Quadratic form.
h_i	A fundamental coroot.
h_α	The coroot of the root α.
Δ	The Dynkin diagram of a semisimple Lie algebra.

Introduction

Lie theory has its roots in the work of Sophus Lie, who studied certain transformation groups that are now called Lie groups. His work led to the discovery of Lie algebras.However, Lie algebras also proved to be of interest in their own right. The finite dimensional simple Lie algebras over the complex field were investigated independently by Elie Cartan and Wlihelm Killing and the classification of such algebras was achieved during the decade 1890/1900. Basic ideas on the structure and representation theory of these Lie algebras were also contributed at a later stage by Hermann Weyl. Since then the theory of finite dimensional simple Lie algebras has found many and varied applications both in mathematics and in mathematical physics, to the extent that it is now generally regarded as one of the classical branches of mathematics. [8]

Lie algebras found numerous applications in various branches of mathematics, including differential geometry, representation theory, and algebraic geometry. In physics, they became essential tools for describing symmetries in quantum mechanics, gauge theories in particle physics, and differential equations arising in mathematical physics.

Wilhelm Killing's classification of semisimple Lie algebras in the late 19th century was a significant achievement. He provided a systematic way to understand and categorize these algebras. Élie Cartan's contributions, particularly his work on Cartan subalgebras and root systems, further deepened the understanding of the structure of Lie algebras.

In recent decades, Lie algebras have continued to be an active area of research. Mathematicians have explored extensions of the classical theory, such as loop algebras, quantum groups, and Kac-Moody algebras. These developments have deepened our understanding of algebraic structures and their applications in mathematics and physic.

Lie algebras continued to evolve with contributions from mathematicians and physicists worldwide. The classification of simple Lie algebras, initiated by Killing and completed by Cartan, has been extended to the classification of semisimple Lie algebras over arbitrary fields. Moreover, Lie algebras have found applications in areas such as quantum field theory, string theory, and geometric Langlands correspondence.

In this thesis, we embark on a journey to explore the classification of Lie algebras. We will delve into the foundational concepts and techniques involved in the classification process, surveying key results and insights that have shaped our understanding of these algebraic structures. Our aim is to provide a comprehensive overview of the classification theory of Lie algebras. Through Lie algebras theory, the root system and the Weyl group, Dynkin diagrams and the classification of semisimple Lie algebras, last thing an overview of the Kac-Moody Lie algebras.

This thesis is divided into four chapters in the following manner:
First chapter: in this chapter we presented some basic concepts of linear algebras, and the

basic definitions of Lie algebras, their subalgebras, representations and modules, the standard results are proved on the representation theory of soluble and nilpotent Lie algebras.
Second chapter: the main notions presented here are Cartan subalgebre. The the existence and conjugacy of Cartan subalgebras are proved. The Killing form is introduced and used to describe the Cartan decomposition of a semisimple Lie algebra into root spaces with respect to a Cartan subalgebra.
Third chapter: we present here the notion of positive systems and fundamental systems of roots and the Weyl group is introduced and shown to be a Coxeter group. This leads on to the definition of the Cartan matrix and the Dynkin diagram. The possible Dynkin diagrams and Cartan matrices are classified.
Fourth chapter: the existence and uniqueness of a semisimple Lie algebra with a given Cartan matrix are proved. The finite dimensional simple Lie algebras are discussed individually and their root systems determined.
In Appendix: an overview of the Kac-Moody Lie algebras is presented.

Chapter 1

Preliminaries

This chapter provides preliminaries where we present some basic concepts of linear algebra, including the fundamental definitions of Lie algebras, their subalgebras, and ideals, as well as representations and modules. We prove standard results on the representation theory of soluble and nilpotent Lie algebras, focusing particularly on representations of nilpotent Lie algebras.

1.1 Linear algebra

We recall here some preliminaries form linear algebra. Our main reference here is [1].

A **ring** A is a set with two binary operations (addition and multiplication) such that

1) A is an abelian group.
2) Multiplication is associative ($(xy)z = x(yz)$) and distributive over addition ($x(y+z) = xy + xz, (y+z)x = yx + zx$).
3) We say A is a commutative ring if $xy = yx$ for every $x, y \in A$.
4) We say A is a unital ring if there is $1 \in A$ such that $x1 = 1x = x$ for every $x \in A$.

Let $(A, +, \cdot)$ be a commutative ring. An A-**module** is an abelain group M for an operation $+$ equipped with an operation of A on M, denoted as $(\cdot)$, such that for a, b in A ,x, y in M we have:

i) $a \cdot (x + y) = a \cdot x + a \cdot y$
ii) $(a + b) \cdot x = a \cdot x + b \cdot x$
iii) $a \cdot (b \cdot x) = (a \cdot b) \cdot x$
iv) $1 \cdot x = x$

Definition 1.1. *Let k be a field.*
A k-algebra is an ordered 4-tuple$(A, +, \cdot, *)$ satisfying the following axioms:

i) $(A, +, \cdot)$ is a ring (here: assuned to be associative)

ii) $(A, +, *)$ is a k-vector space

iii) $\lambda * (x \cdot y) = x \cdot (\lambda \cdot y) = (\lambda \cdot x) \cdot y \qquad \forall \lambda \in k, \forall x, y \in A.$

Definition 1.2. *A map $f : A \to B$ between two k-algebras A and B is called a k-algebra homomorphism if:*

i) f is k-linear

ii) f is a homomorphism of rings, i.e.
$f(x \cdot y) = f(x) \cdot f(y) \quad \forall x, y \in A.$

Examples 1.1.

1) the field k itself is a k-algebra.

2) $\forall\ n \in \mathbb{Z}_{\geq 1}$ the $n \times n$-square matrices $M_n(k)$ form a k-algebra.

3) $k[x_1, \cdot\cdot, x_n]$ is a k-algebra.

Let k be a field. A k**-vector space** V is a set with two operations

$$\begin{array}{ccc} V \times V & \to & V \\ (v, w) & \mapsto & v + w \end{array} \quad \text{and} \quad \begin{array}{ccc} k \times V & \to & V \\ (\lambda, w) & \mapsto & \lambda \cdot w \end{array}$$

called **addition** and **multiplication by scalars** with the usual axioms.

Let V be a k-vector space. A non-empty subset $W \subseteq V$ is called a **subspace**, if

$$u + v \in W \qquad \text{and} \qquad \lambda u \in W \qquad \text{for all } u, v \in W \text{ and all } \lambda \in k.$$

In particular, a subspace is itself a k-vector space. In fact, every subspace W is the span of some vectors $v_1, \ldots, v_k$ for some k, and every such span is a subspace.

1.2 Lie algebras

Our main references here are [2, 8] and [12].

1.2.1 Elementary properties of Lie algebras

A **Lie algebra** is a vector space L over a field k on which a multiplication

$$L \times L \to L$$

$$(x, y) \to [xy]$$

is defined satisfying the following axioms:

(i) $(x, y) \to [xy]$ is linear in x and in y ;

(ii) $[xx] = 0$ for all $x \in L$;

(iii) $[[xy]z] + [[yz]x] + [[zx]y] = 0$ for all $x, y, z \in L$.

Axiom (iii) is called the **Jacobi identity**.

Lemma 1.1. $[yx] = -[xy]$ *for all $x, y \in L$.*

Proof. Since $[x+y, x+y] = 0$ we have $[xx]+[xy]+[yx]+[yy] = 0$. It follows that $[xy]+[yx] = 0$, that is $[yx] = -[xy]$. □

The following lemma asserts that multiplication in a Lie algebra is anticommutative.

Let H and K be subspaces of a Lie algebra L. Then $[HK]$ is defined as the subspace spanned by all products $[xy]$ with $x \in H$ and $y \in K$. Each element of $[HK]$ is a finite sum

$$[x_1y_1] + \cdots + [x_ry_r]$$

with $x_i \in H, y_i \in K$.

Proposition 1.1. $[HK] = [KH]$ *for all subspaces* H, K *of* L.

Proof. Let $x \in H, y \in K$.Then $[xy] = [-y, x] \in [KH]$. This shows that $[HK] \subset [KH]$. Similarly we have $[KH] \subset [HK]$ and so we have equality. □

This result asserts that multiplication of subspaces in a Lie algebra is commutative.

Definition 1.3. ***(Lie subalgebras and ideals)***
Let L be a Lie algebra over k.
A subset H of L is called a ***subalgebra*** *of L if H is a subspace of L and $[HH] \subset H$. Thus H is itself a Lie algebra under the same operations as L.*
A subset I of L is called an ***ideal*** *of L if I is a subspace of L and $[IL] \subset I$.*
The latter condition can be expressed as equivalent to $[LI] \subset I$. Thus there is no distinction between left ideals and right ideals in the theory of Lie algebras. Every ideal is two-sided.

Note that, every ideal is a subalgebra.

Proposition 1.2.

i) If H, K are subalgebras of L so is $H \cap K$ a subalgebra of L.

ii) If H, K are ideals of L so is $H \cap K$ ideal of L.

iii) If H is an ideal of L and K a subalgebra of L then $H + K$ is a subalgebra of L.

iv) If H, K are ideals of L then $H + K$ is an ideal of L.

Proof. i) $H \cap K$ is a subspace of L and $[H \cap K, H \cap K] \subset [HH] \cap [KK] \subset H \cap K$. Thus $H \cap K$ is a subalgebra.

ii) This time we have $[H \cap K, L] \subset [HL] \cap [KL] \subset H \cap K$. Thus $H \cap K$ is an ideal of L

iii) $H + K$ is a subspace of L. Also $[H + K, H + K] \subset [HH] + [HK] + [KH] + [KK] \subset H + K$, since $[HH] \subset H, [HK] \subset H, [KK] \subset K$. Thus $H + K$ is a subalgebra.

iv) This time we have $[H + K, L] \subset [HL] + [KL] \subset H + K$. Thus $H + K$ is an ideal of L. □

Next, we have factor algebra. Let I be an ideal of a Lie algebra L. Then I is in particular a subspace of L and so we can form the factor space L/I whose elements are the cosets $I + x$ for $x \in L$. The set $I + x$ is a subset of L and comprises all elements $y + x$ for $y \in I$.

Proposition 1.3. *Let I be an ideal of L. Then the factor space L/I can be made into a Lie algebra by defining*

$$[I + x, I + y] = I + [xy] \quad \textit{for all } x, y \in L$$

Proof. We need to demonstrate initially that this definition is unambiguous, that is if.
$I + x = I + x'$ and $I + y = I + y'$ then $I + [xy] = I + [x'y']$.
$I + x = I + x'$ implies that $x = x' + i_1$ for some $i_1 \in I$.
In a similar manner $I + y = I + y'$ implies $y = y' + i_2$ for some $i_2 \in I$. Thus

$$\begin{aligned} I + [xy] &= I + [x' + i_1, y' + i_2] \\ &= I + [i_1 y'] + [x' i_2] + [i_1 i_2] + [x'y'] \\ &= I + [x'y'], \end{aligned}$$

since $[i_1 y'], [x' i_2], [i_1 i_2]$ all lie in I. Thus our multiplication is well defined.
And we have as well

$$[I + x, I + x] = I + [xx] = I,$$

and the Jacobi identity in L/I clearly follows from the Jacobi identity in L.

□

Definition 1.4. ***(Homomorphisms, Isomorphisms)***
Let L_1 and L_2 be Lie algebras over the same field k. A ***homomorphism of Lie algebras*** *from L_1 to L_2 is a linear map $\theta : L_1 \to L_2$, such that*

$$\theta[xy] = [\theta x, \theta y] \qquad \textit{for all} \quad x, y \in L_1.$$

If θ is bijective , then it is called an ***isomorphism of Lie algebras****.*
The Lie algebras L_1, L_2 are said to be ***isomorphic*** *if there exists an isomorphism $\theta : L_1 \to L_2$.*

Proposition 1.4. *Let $\theta : L_1 \to L_2$ be a homomorphism of Lie algebras. Then the image of θ is a subalgebra of L_2, the kernel of θ is an ideal of L_1 and $L_1/\ker\theta$ is isomorphic to Im θ.*

Proof. Im θ is a subspace of L_2. Moreover for x, y in L_1 we have

$$[\theta(x), \theta(y)] = \theta[xy] \in \text{Im } \theta$$

Hence Im θ is a subalgebra of L_2.
Now $\ker\theta$ is a subspace of L_1. Let $x \in \ker\theta$ and $y \in L_1$. Then

$$\theta[xy] = [\theta(x), \theta(y)] = [0, \theta(y)] = 0.$$

Hence $[xy] \in \ker\theta$ and so $\ker\theta$ is an ideal of L_1. Now let $x, y \in L_1$. We consider when $\theta(x)$ is equal to $\theta(y)$. We have

$$\begin{aligned} \theta(x) = \theta(y) \Leftrightarrow \theta(x - y) = 0 &\Leftrightarrow x - y \in \ker\theta \\ &\Leftrightarrow \ker\theta + x = \ker\theta + y. \end{aligned}$$

This shows that there is a bijective map $\theta(x) \mapsto \ker\theta + x$ between Im θ and $L_1/\ker\theta$. We show this bijection is an isomorphism of Lie algebras. It is clearly linear. Moreover given $x, y, z \in L_1$ we have

$$\begin{aligned} [\theta(x), \theta(y)] = \theta(z) &\Leftrightarrow \theta[xy] = \theta(z) \\ &\Leftrightarrow \ker\theta + [xy] = \ker\theta + z \\ &\Leftrightarrow [\ker\theta + x, \ker\theta + y] = \ker\theta + z. \end{aligned}$$

Thus the bijection preserves Lie multiplication, so it is an isomorphism of Lie algebras. □

Proposition 1.5. *Let I be an ideal of L and H a subalgebra of L. Then*

(i) I is an ideal of $I + H$.

(ii) $I \cap H$ is an ideal of H.

(iii) $(I + H)/I$ is isomorphic to $H/(I \cap H)$.

Proof. We recall from Proposition 1.2 that $I \cap H$ and $I + H$ are subalgebras.
We have $[I, I+H] \subset [IL] \subset I$, thus I is an ideal of $I+H$. Also $[I\cap H, H] \subset [IH]\cap[HH] \subset I\cap H$, thus $I \cap H$ is an ideal of H.

Let $\theta : H \to (I + H)/I$ be defined by $\theta(x) = I + x$. This is clearly a linear map, and is also evidently a homomorphism of Lie algebras. It is surjective since each element of $(I + H)/I$ has form $I + x$ for some $x \in H$. Finally its kernel is the set of $x \in H$ for which $I + x = I$, that is $I \cap H$. Thus $(I + H)/I$ is isomorphic to $H/(I \cap H)$ by Proposition 1.4 □

1.2.2 Representations and modules

Consider $M_n(k)$ as the associative algebra consisting of all $n\times n$ matrices over the field k, and denote $[M_n(k)]$ as the corresponding Lie algebra. This is commonly referred to as the **general linear Lie algebra** of degree n over k and we write .

$$\mathfrak{gl}_n(k) = [M_n(k)] .$$

We have $\dim \mathfrak{gl}_n(k) = n^2$.

Definition 1.5. *Let L be a Lie algebra over the field k.*

- *A **representation** of L is a Lie algebra homomorphism*

$$\rho : L \to \mathfrak{gl}_n(k)$$

for some n, and ρ is called a representation of degree n. This means nothing but: ρ is a linear map and

$$\rho([xy]) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x)$$

for all $x, y \in L$.
*Two representations ρ, ρ' of degree n are called **equivalent** if there exists a non-singular $n \times n$ matrix T such that*

$$\rho'(x) = T^{-1}\rho(x)T \quad \text{for all } x \in L.$$

- *An L-**module** is a finite-dimensional k-vector space V together with an action*

$$\begin{aligned} L \times V &\to V \\ (x, v) &\to xv \end{aligned}$$

such that

(i) $(v + w)x = vx + wx$ *and* $(\lambda v)x = \lambda(vx)$ *(the action is linear),*

(ii) $v(\lambda x + y) = \lambda \cdot (vx) + vy$*, and*

(iii) $[xy]v = x(yv)-y(xv)$.

For all $v, w \in V$ and all $x, y \in L$ and all $\lambda \in k$ respectively.

Suppose V is a finite dimensional L-module. For $x \in L$, let $e_1, \dots, e_n$ be a basis of V. Let

$$xe_j = \sum_i \rho_{ij}(x)e_i$$

with $\rho_{ij}(x) \in k$ and let $\rho(x) = (\rho_{ij}(x))_{ij}$. Then ρ is a representation of L. For we have

$$
\begin{aligned}
[xy]e_j &= x\,(ye_j) - y\,(xe_j) \\
&= x\left(\sum_k \rho_{kj}(y)e_k\right) - y\left(\sum_k \rho_{kj}(x)e_k\right) \\
&= \sum_k \rho_{kj}(y)xe_k - \sum_k \rho_{kj}(x)ye_k \\
&= \sum_k \rho_{kj}(y)\left(\sum_i \rho_{ik}(x)e_i\right) - \sum_k \rho_{kj}(x)\left(\sum_i \rho_{ik}(y)e_i\right) \\
&= \sum_i\left(\sum_k \left(\rho_{ik}(x)\rho_{kj}(y) - \rho_{ik}(y)\rho_{kj}(x)\right)\right)e_i \\
&= \sum_i (\rho(x)\rho(y) - \rho(y)\rho(x))_{ij}e_i.
\end{aligned}
$$

Thus $\rho[xy] = \rho(x)\rho(y) - \rho(y)\rho(x) = [\rho(x), \rho(y)]$ and ρ is a representation of L.

Suppose we consider a different basis $f_1, \dots, f_n$ for vector space V. Let ρ' be the representation of L obtained from this basis. Then ρ' is equivalent to ρ. There exists a non-singular $n \times n$ matrix T such that.

$$f_j = \sum_i T_{ij}e_i.$$

Thus we have

$$xf_j = \sum_k T_{kj}xe_k = \sum_k T_{kj}\left(\sum_i \rho_{ik}(x)e_i\right) = \sum_i\left(\sum_k \rho_{ik}(x)T_{kj}\right)e_i.$$

However,

$$xf_j = \sum_k \rho'_{kj}(x)f_k = \sum_k \rho'_{kj}(x)\left(\sum_i T_{ik}e_i\right) = \sum_i\left(\sum_k T_{ik}\rho'_{kj}(x)\right)e_i.$$

This leads to $\rho(x)T = T\rho'(x)$, meaning $\rho'(x) = T^{-1}\rho(x)T$ for all $x \in L$. Consequently, the representation ρ' is equivalent to ρ.

Examples 1.2. *L is itself a L-module.*

The action of L on L is defined as $x \cdot y = [xy]$. Then we have

$$[[xy]z] = [x[yz]] - [y[xz]].$$

This result stems from the Jacobi identity, demonstrating that L is indeed an L-module. This particular module is known as the **adjoint module**. We define $\operatorname{ad} x : L \to L$ by

$$\operatorname{ad} x \cdot y = [xy] \qquad \text{for all } x, y \in L.$$

Then we have

$$\operatorname{ad}[xy] = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x.$$

Let V be an L-module, U be a subspace of V and H be a subspace of L. We define HU as the subspace of V generated by all elements of the form xu for $x \in H$ and $u \in U$.

Remarks

- A **submodule** of V is a subspace U of V such that $LU \subset U$. In particular V is a submodule of V and the zero subspace $0 = \{0\}$ is a submodule of V.
- A **proper submodule** of V is a submodule distinct from V and 0.
- An L-module V is called **irreducible** if it has no proper submodules.
- V is called **completely reducible** if it is a direct sum of irreducible submodules.
- V is called **indecomposable** if V can not be written as a direct sum of two proper submodules.
- Of course every irreducible L-module is indecomposable, but the converse need not be true.

1.2.3 Abelian, nilpotent and soluble Lie algebras

A Lie algebra L is **abelian** if $[LL] = 0$. Thus $[xy] = 0$ for all $x, y \in L$ when L is abelian.

Given any Lie algebra L we define the powers of L by

$$L^1 = L, \quad L^{n+1} = [L^n L] \qquad \text{for } n \geq 1.$$

Thus L is abelian if and only if $L^2 = 0$.

Proposition 1.6. *L^n is an ideal of L. Also*

$$L = L^1 \supset L^2 \supset L^3 \supset \cdots .$$

Proof. first,let us consider if I,J are ideals of L then $[IJ]$ is also an ideal of L.
For let $x \in I, y \in J, z \in L$.Then

$$[[xy]z] = [x[yz]] - [y[xz]] \in [IJ].$$

It follows that L^n is an ideal of L for each $n > 0$.Thus we have

$$L^{n+1} = [L^n L] \subset L^n.$$

□

A Lie algebra L is called **nilpotent** if $L^n = 0$ for some $n \geq 0$. Therefore, every abelian Lie algebra is nilpotent. It's evident that every subalgebra and factor algebra of a nilpotent Lie algebra are also nilpotent.

We now consider a different kind of powers of L. We define

$$L^{(0)} = L, \quad L^{(n+1)} = [L^{(n)} L^{(n)}] \qquad \text{for } n \geq 0.$$

Proposition 1.7. *$L^{(n)}$ is an ideal of L. Also*

$$L = L^{(0)} \supset L^{(1)} \supset L^{(2)} \supset \cdots.$$

Proof. $L^{(n)}$ is an ideal of L because the product of two ideals is itself an ideal. Additionally

$$L^{(n+1)} = [L^{(n)} L^{(n)}] \subset L^{(n)}.$$

□

A Lie algebra L is called **soluble** if $L^{(n)} = 0$ for some $n \geq 0$.

Proposition 1.8.

(a) $[L^m L^n] \subset L^{m+n}$ *for all* $m, n \geq 1$.

(b) $L^{(n)} \subset L^{2n}$ *for all* $n \geq 0$.

(c) Every nilpotent Lie algebra is soluble.

Proof. (a). We use induction on n. The result is clear if $n = 1$. Suppose it is true for $n = r$. Then

$$\begin{aligned}[L^m L^{r+1}] &= [L^m[L^r L]] = [[L^r L]L^m] \\ &\subset [[LL^m]L^r] + [L^m L^r]L] \qquad \text{by the Jacobi identity} \\ &\subset [L^m LL^r] + [[L^m L^r L] \\ &\subset L^{m+r+1} \qquad \text{by inductive hypothesis.}\end{aligned}$$

Thus the result holds for $n = r + 1$, so for all n.

(b) We again use induction on n. The result is clear if $n = 1$. Suppose it is true for $n = r$. Then

$$L^{(r+1)} = [L^{(r)} L^{(r)}] \subset [L^{2r} L^{2r}] \subset L^{2r+1}$$

by (a). Thus the result holds for $n = r + 1$, so for all n.

(c) Suppose L is nilpotent. Then $L^{2^n} = 0$ for n sufficiently large. Hence $L^{(n)} = 0$ by (b) and so L is soluble. □

It's evident that every subalgebra and factor algebra of a soluble Lie algebra is also soluable.

Proposition 1.9. *Suppose* I *is an ideal of* L *and both* I *and* L/I *are soluble. Then* L *is soluble.*

Proof. Since L/I is soluble we have $(L/I)^{(n)} = 0$ for some n. This implies $L^{(n)} \subset I$. Since I is soluble we have $I^{(m)} = 0$ for some m. Hence

$$L^{(n+m)} = (L^{(n)})^{(m)} \subset I^{(m)} = 0$$

and so L is soluble. □

Proposition 1.10. *In every finite-dimensional Lie algebra* L*, there exists a unique maximal soluble ideal* R*. Additionally,* L/R *does not contain any non-zero soluble ideals.*

Proof. Let I, J be soluble ideals of L. Then $I + J$ is also an ideal of L and $(I + J)/I$ is isomorphic to $J/(I \cap J)$. Now J is soluble, thus $J/(I \cap J)$is soluble and so $(I + J)/I$ is soluble. Since I is soluble we see that $I + J$ is soluble. Thus the sum of two soluble ideals of L is a soluble ideal. It follows that L has a unique maximal soluble ideal R.

If I/R is a soluble ideal of L/R then I is a soluble ideal of L. Hence $I = R$ and $I/R = 0$. □

Remarks

The ideal R is referred to as the **soluble radical** of L. A Lie algebra L is called **semisimple** if $R = 0$, meaning L is semisimple if and only if it has no non-zero soluble ideals.

A Lie algebra L is labeled as **simple** if it lacks any proper ideals, meaning it has no ideals other than itself and 0.

1.2.4 Representations of soluble and nilpotent Lie algebras

Representations of soluble Lie algebras

We will now and in the subsequent discussion consider the base field k to be the field of complex numbers.

Let's delve into 1-dimensional representations of a Lie algebra L. A 1-dimensional representation is a linear map $\rho : L \to \mathbb{C}$ such that $\rho[xy] = [\rho(x), \rho(y)] \quad$ for all $x, y \in L$.

Lemma 1.2. *A linear map $\rho : L \to \mathbb{C}$ is a 1-dimensional representation of L if and only if ρ vanishes on L^2.*

Proof. Suppose ρ is a representation. Then for $x, y \in L$ we have

$$\rho[xy] = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x) = 0.$$

Hence vanishes on L^2.

Conversely suppose that ρ vanishes on L^2. Then

$$\rho[xy] = 0 = [\rho(x), \rho(y)]$$

and so ρ is a representation of L. □

We shall now prove a theorem of Lie which shows that any irreducible representation of a soluble Lie algebra is 1-dimensional.

Theorem 1.1. *(Lie's theorem). Let L be a soluble Lie algebra and V be a finite dimensional irreducible L-module. Then* $\dim V = 1$.

Proof. Since L is soluble we have $L^2 \neq L$. Let I be a subspace of L such that $I \supset L^2$ and $\dim I = \dim L - 1$. Then I is an ideal of L since

$$[IL] \subset [LL] = L^2 \subset I.$$

Thus I is an ideal of L of codimension 1.

We shall prove Lie's theorem by induction on $\dim L$. Suppose $\dim L = 1$ and V be an irreducible L-module. Let $L = \mathbb{C}x$ and v be an eigenvector of x in V. Then $\mathbb{C}v$ is an L-submodule of V. Since V is irreducible we have $V = \mathbb{C}v$ and $\dim V = 1$.

Now suppose $\dim L > 1$ and V is an irreducible L-module. We may regard V as an I-module. Then V contains an irreducible I-submodule W and we may assume $\dim W = 1$ by induction. Let w be a non-zero vector in W. Then

$$yw = \lambda(y)w \quad \text{for all } y \in I$$

where λ is the 1-dimensional representation of I given by W. Let

$$U = \{u \in V; yu = \lambda(y)u \quad \text{for all } y \in I\}.$$

Then we have

$$0 \neq W \subset U \subset V.$$

We shall show that U is an L-submodule of V. Let $u \in U$, $x \in L$. Then

$$y(xu) = x(yu) - [xy]u = \lambda(y)xu - \lambda([xy]u)$$

since $[xy] \in I$. We shall show $\lambda([xy)] = 0$. Once we know this we have $xu \in U$ and so U is an L-submodule. Since V is irreducible we have $U = V$. Hence

$$yv = \lambda(y)v \quad \text{for all } v \in V, y \in I.$$

Since $\dim I = \dim L - 1$ we can write $L = I \oplus \mathbb{C}x$, a direct sum of subspaces. Let v be an eigenvector for x on V. Then $\mathbb{C}v$ is an L-submodule of V, being invariant under the action of both I and x. Since V is irreducible we have $V = \mathbb{C}v$ and so $\dim V = 1$.

In order to complete the proof we must show that $\lambda([xy]) = 0$ for all $x \in L$, $y \in I$. In fact it is sufficient to prove this for the element x chosen above such that $L = I \oplus \mathbb{C}x$.

Let u be any non-zero element of U. We write

$$v_0 = u, \quad v_1 = xu, \quad v_2 = x(xu), \ldots$$

We have $v_0, v_1, v_2, \ldots \in V$ and so there exists $p \geq 0$ such that $v_0, v_1, \ldots v_p$ are linearly independent and v_{p+1} is a linear combination of these. Consider the subspace $\langle v_0, v_1 \ldots v_p \rangle$ of V spanned by these vectors. This subspace is invariant under the action of x. We consider the effect on this subspace of elements $y \in I$. We have

$$yv_0 = yu = \lambda(y)u = \lambda(y)v_0.$$

We shall show

$$yv_i = \lambda(y)v_i + \text{a linear combination of } v_0, \ldots, v_{i-1}.$$

This is true for $i = 0$. Assuming it for v_{i-1} we have

$$\begin{aligned} yv_i =& y\left(xv_{i-1}\right) = x\left(yv_{i-1}\right) - [xy]v_{i-1} \\ =& x\left(\lambda(y)v_{i-1} + \text{ a linear combination of } v_0, \ldots, v_{i-2}\right) \\ & - \left(\text{ a linear combination of } v_0, \ldots, v_{i-1}\right) \\ =& \lambda(y)v_i + \text{ a linear combination of } v_0, \ldots, v_{i-1}. \end{aligned}$$

Thus the subspace $\langle v_0, v_1, \ldots, v_p \rangle$ is invariant under the action of y for all $y \in I$, as well as being invariant under x. Hence it is an L-submodule of V. Since V is irreducible we have

$$V = \langle v_0, v_1, \ldots, v_p \rangle .$$

Now $[xy] \in I$ and we see from the above description of the action of I that

$$\operatorname{trace}_V [xy] = (p+1)\lambda([xy]).$$

Thus we have $(p+1)\lambda([xy]) = \operatorname{trace}_V [xy] = \operatorname{trace}_V xy - \operatorname{trace}_V yx = 0$, since $\operatorname{trace}_V xy = \operatorname{trace}_V yx$. Hence $\lambda([xy]) = 0$ and the proof is complete. □

Corollary 1.1. *Let L be soluble and V be a finite dimensional L-module. Then a basis can be chosen for V with respect to which we obtain a matrix representation ρ of L of the form*

$$\rho(x) = \begin{pmatrix} * & & & & & \\ 0 & * & & & * & \\ 0 & & \cdot & & & \\ \cdot & & 0 & \cdot & & \\ \cdot & & & & * & \\ 0 & \cdot & \cdot & 0 & 0 & * \end{pmatrix} \qquad \text{for all } x \in L.$$

Thus the matrices representing elements of L are all of triangular form.

Corollary 1.2. *Let L be a soluble Lie algebra with $\dim L = n$. Then L has a chain of ideals*

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = L$$

with $\dim I_r = r$.

Proof. We apply theorem 1.1 to the adjoint L-module L. The submodules of L are the ideals of L. By taking a maximal chain of submodules we obtain ideals of L with the required property. □

Representations of nilpotent Lie algebras

When L is a nilpotent Lie algebra, we can derive even more powerful results about its representations. These results are particularly crucial for understanding semisimple Lie algebras, which we will discuss later. To begin, we recall some results from linear algebra related to the Jordan canonical form. Any $n \times n$ matrix over $\mathbb{C}$ is similar to a diagonal sum of Jordan block matrices of the form

$$\begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & 0 & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & 1 & \\ & 0 & & & & \lambda & 1 \\ & & & & & & \lambda \end{pmatrix}.$$

In a similar way any linear transformation $\theta : V \to V$ on a finite dimensional vector space V over $\mathbb{C}$ gives rise to a decomposition of V as in the following proposition.

Proposition 1.11. *Let $\theta : V \to V$ be a linear map with characteristic polynomial*

$$\chi(t) = (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \ldots (t - \lambda_r)^{m_r}$$

where $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of θ and $m_1, \ldots, m_r$ are their multiplicities. Let V_i be the set of all $v \in V$ annihilated by some power of $\theta - \lambda_i 1$. Then we have

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r.$$

Moreover $\dim V_i = m_i, \theta(V_i) \subset V_i$ and the characteristic polynomial of θ on V_i is $(t - \lambda_i)^{m_i}$.

The subspace V_i is called the **generalised eigenspace** of V with eigenvalue λ_i. Thus the ordinary eigenspace of λ_i lies in the generalised eigenspace. It is not in general true that V is the direct sum of its eigenspaces with respect to its different eigenvalues, but Proposition 1.11 shows that this result is true if the eigenspaces are replaced by the generalised eigenspaces.

The relevance of the decomposition into generalised eigenspaces for the representations of nilpotent Lie algebras is shown by the following theorem.

Theorem 1.2. *Let L be a nilpotent Lie algebra and V be an L-module. Let $y \in L$ and $\rho(y) : V \to V$ be the map $v \to yv$. Then the generalised eigenspaces V_i of V associated with $\rho(y)$ are all submodules of V.*

Proposition 1.12. *Let L be a Lie algebra and V be an L-module. Let $v \in V$, $x, y \in L$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$(\rho(y) - (\alpha + \beta)1)^n xv = \sum_{i=0}^{n} \binom{n}{i} ((\operatorname{ad} y - \beta 1)^i x)((\rho(y) - \alpha 1)^{n-i} v).$$

Corollary 1.3. *Let L be a nilpotent Lie algebra and V a finite dimensional indecomposable $L-$module. Then a basis can be chosen for V with respect to which we obtain a matrix representation ρ of L of the form*

$$\rho(x) = \begin{pmatrix} \lambda(x) & & & & \\ & \cdot & * & & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & \lambda(x) \end{pmatrix} \quad \textit{for all } x \in L.$$

Proof. We can choose a basis as in Corollary 1.1 with respect to which each $\rho(x)$ is triangular. The generalised eigenspaces of V with respect to $\rho(x)$ are all submodules of V by Theorem 1.2 and V is their direct sum. Since V is indecomposable only one of the generalised eigenspaces is non-zero. Thus all the eigenvalues of $\rho(x)$ on V are equal. Let this eigenvalue be $\lambda(x)$. Then the diagonal entries of the triangular matrix $\rho(x)$ are all equal to $\lambda(x)$. □

We observe that the map $x \to \lambda(x)$ is a 1-dimensional representation of L, as it arises from a 1-dimensional submodule of V.

We have seen from Proposition 1.11 and Theorem 1.2 how to obtain a direct decomposition of V into submodules for any element $y \in L$. We may use this result to obtain a direct decomposition of V into submodules which does not depend on the choice of any particular element of L.

Theorem 1.3. *Let L be a nilpotent Lie algebra and V a finite dimensional L-module. For any 1 -dimensional representation λ of L we define*

$$V_\lambda = \{v \in V; \textit{ for each } x \in L \textit{ there exists } N(x) \textit{ such that } (\rho(x) - \lambda(x)1)^{N(x)}v = 0\}.$$

Then

$$V = \bigoplus_\lambda V_\lambda$$

and each V_λ is a submodule of L.

Proof. We first express V as a direct sum of indecomposable L-modules. Each of these defines a 1-dimensional representation λ of L as in Corollary 1.3. Let W_λ be the direct sum of all indecomposable components giving rise to λ. Then we have

$$V = \bigoplus_\lambda W_\lambda.$$

We shall show that $W_\lambda = V_\lambda$ and so that W_λ is independent of the decomposition chosen into indecomposable components. It is clear that $W_\lambda \subset V_\lambda$ by Corollary 1.3. Suppose if possible that $W_\lambda \neq V_\lambda$. Then there exists $v \in V_\lambda \cap \bigoplus_{\mu\neq\lambda} W_\mu$ with $v \neq 0$. We write $v = \sum_{\mu\in S} w_\mu$ with $w_\mu \in W_\mu$, where the set S is finite. Since $w_\mu \in W_\mu$ there exists N_μ such that $(\rho(x)-\mu(x)1)^{N_\mu}w_\mu = 0$. Hence

$$\prod_{\mu\in S}(\rho(x) - \mu(x)1)^{N_\mu}v = 0.$$

However, we also have $(\rho(x) - \lambda(x)1)^{N_\lambda}v = 0$.

We recall from Lemma 1.2 that the 1-dimensional representations of L are in bijective correspondence with linear maps $L/L^2 \to \mathbb{C}$. The vector space L/L^2 over $\mathbb{C}$ can not be expressed as the union of finitely many proper subspaces. For each $\mu \in S$ the set of x satisfying

$\lambda(x) = \mu(x)$ is a proper subspace. Thus there exists $x \in L$ such that $\lambda(x) \neq \mu(x)$ for all $\mu \in S$. Thus the polynomials

$$\prod_{\mu \in S} (t - \mu(x))^{N_\mu}, \quad (t - \lambda(x))^{N_\lambda}$$

are coprime. Thus there exist polynomials $a(t), b(t) \in \mathbb{C}[t]$ such that

$$a(t) \prod_{\mu \in S} (t - \mu(x))^{N_\mu} + b(t)(t - \lambda(x))^{N_\lambda} = 1.$$

Hence

$$a(\rho(x)) \prod_{\mu \in S} (\rho(x) - \mu(x)1)^{N_\mu} v + b(\rho(x))(\rho(x) - \lambda(x)1)^{N_\lambda} v = v.$$

The left-hand side of this expression is zero, as we have seen above. Thus $v = 0$, a contradiction. Hence $V_\lambda = W_\lambda, V = \bigoplus_\lambda V_\lambda$ and each V_λ is a submodule of V. □

A 1-dimensional representation λ of L is called a **weight** of V if $V_\lambda \neq 0$, and V_λ is called the **weight space** of λ. The decomposition $V = \bigoplus_\lambda V_\lambda$ is called the **weight space decomposition** of V. It follows from Corollary 1.3 that a basis can be chosen for V_λ with respect to which the matrix representation of L on V_λ has form

$$\rho(x) = \begin{pmatrix} \lambda(x) & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & \lambda(x) \end{pmatrix} \quad \text{for each } x \in L.$$

We shall make frequent use of the weight space decomposition in subsequent chapters.

We next prove a theorem of Engel which gives a useful characterisation of nilpotent Lie algebras in terms of the adjoint representation.

Theorem 1.4. *(Engel's theorem). A Lie algebra L is nilpotent if and only if* $\operatorname{ad} x : L \to L$ *is nilpotent for each $x \in L$.*

Proof. Suppose L is nilpotent. Then $L^n = 0$ for some n. Let $y \in L$. Then we have

$$\operatorname{ad} x \cdot y \in L^2, \quad (\operatorname{ad} x)^2 \cdot y \in L^3, \quad \ldots$$

and so $(\operatorname{ad} x)^{n-1} y = 0$ for each $y \in$ L. Thus $(\operatorname{ad} x)^{n-1} = 0$ and so ad x is a nilpotent linear map.

Now suppose conversely that ad x is a nilpotent linear map for each $x \in L$. We wish to show L is nilpotent. We suppose if possible that this is false and let H be a maximal nilpotent subalgebra of L. Thus H is nilpotent but any subalgebra properly containing H is not nilpotent. We may regard L as an H-module. Then H is an H-submodule of L and we can find an H-submodule M of L containing H such that M/H is an irreducible H-module. We have

$$\dim(M/H) = 1 \quad \text{by Theorem 1.1.}$$

Moreover the 1-dimensional representation of H afforded by M/H is the zero representation, as otherwise ad x would fail to be nilpotent for some $x \in H$. Hence we have $[HM] \subset H$. Now there exists $x \in M$ such that

$$M = H \oplus \mathbb{C}x.$$

We have

$$[MM] \subset [HH] + [Hx] \subset H.$$

Thus M is a subalgebra of L and H is an ideal of M.

We shall show that for each positive integer i there exists a positive integer $e(i)$ such that

$$M^{e(i)} \subset H^i$$

This is true for $i = 1$ since $M^2 \subset H$. We prove it by induction on i. Assume that $M^{e(r)} \subset H^r$. Then

$$M^{e(r)+1} = [M^{e(r)}, H + \mathbb{C}x] \subset H^{r+1} + [M^{e(r)}, x].$$

Hence $M^{e(r)+1} \subset H^{r+1}$+ad $x.M^{e(r)}$.

We shall show that

$$M^{e(r)+j} \subset H^{r+1} + (\operatorname{ad} x)^j \cdot M^{e(r)}$$

for each positive integer j. This is true for $j = 1$. Assuming it inductively for j we have

$$\begin{aligned} M^{e(r)+j+1} &\subset \left[H^{r+1} + (\operatorname{ad} x)^j \cdot M^{e(r)}, M\right] \\ &\subset H^{r+1} + \left[(\operatorname{ad} x)^j M^{e(r)}, H + \mathbb{C}x\right] \\ &\subset H^{r+1} + (\operatorname{ad} x)^{j+1} M^{e(r)} \end{aligned}$$

since H^{r+1} is an ideal of M and $(\operatorname{ad} x)^j M^{e(r)} \subset H^r$. Thus we have shown

$$M^{e(r)+j} \subset H^{r+1} + (\operatorname{ad} x)^j M^{e(r)} \quad \text{for all } j.$$

Now we know that $(\operatorname{ad} x)^j = 0$ when j is sufficiently large. For such j we have

$$M^{e(r)+j} \subset H^{r+1}.$$

Thus we define $e(r + 1) = e(r) + j$ and then $M^{e(r+1)} \subset H^{r+1}$ as required.

Now H is nilpotent so $H^i = 0$ for i sufficiently large. For such i we have $M^{e(i)} = 0$. Thus M is nilpotent. But this contradicts the maximality of H.
Thus our initial assumption was incorrect and so L must be nilpotent. □

Corollary 1.4. *A Lie algebra L is nilpotent if and only if L has a basis with respect to which the adjoint representation of L has form*

$$\rho(x) = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & \cdot & * & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix} \quad \textit{for all } x \in L.$$

Proof. Suppose L is nilpotent. Then L has a series of ideals

$$L \supset L^2 \supset L^3 \supset \cdots \supset L^r = 0 \quad \text{for some } r.$$

We refine this series by choosing a sequence of subspaces between consecutive terms, each of codimension 1 in its predecessor. Such subspaces are automatically ideals of L since if $L^i \supset I \supset L^{i+1}$ we have

$$[IL] \subset [L^i L] = L^{i+1} \subset I.$$

Thus we have a chain of ideals

$$L = I_n \supset I_{n-1} \supset \cdots \supset I_1 \supset I_0 = 0$$

with $\dim I_k = k$ and $[LI_k] \subset I_{k-1}$. By choosing a basis of L adapted to this chain of ideals the map $\operatorname{ad} x : L \to L$ is represented by a matrix $\rho(x)$ of zerotriangular form (i.e. triangular with zeros on the diagonal).

Conversely if L has a basis with respect to which $\operatorname{ad} x$ is represented by a zero-triangular matrix $\rho(x)$ for all $x \in L$, we have $\rho(x)$ nilpotent and so $\operatorname{ad} x$ is nilpotent. Thus L must be a nilpotent Lie algebra by Engel's theorem. □

Chapter 2

Cartan subalgebras

The main notions presented here focus on Cartan subalgebras. We prove the existence and conjugacy of Cartan subalgebras. The Killing form is introduced and utilized to describe the Cartan decomposition of a semisimple Lie algebra into root spaces with respect to a Cartan subalgebra.

2.1 Cartan subalgebras

Our main references here are [4, 5] and [6].

2.1.1 Existence of Cartan subalgebras

If H is subalgebra of a Lie algebra L then the **normaliser** $N(H)$ of H is the set of $x \in L$ such that $[xh] \in H$, for all $h \in H$.

It is immediate that $N(H)$ is a subalgebra of L containing H, and H is an ideal in $N(H)$. In fact, as in group theory, $N(H)$ is the largest subalgebra of L in which H is contained as ideal. We now give the following.

Definition 2.1. *A subalgebra H of a Lie algebra L is called a* ***Cartan subalgebra*** *if it satisfies the following two conditions.*

(1) H is nilpotent.

(2) H is its own normaliser in L.

Cartan subalgebras play a very important role in the theory of semisimple Lie algebras. Our aim in this section is to show that L contains a Cartan subalgebra.
If H is a Cartan subalgebra of a semisimple Lie algebra L, then H is abelian (see theorem 2.6) and the normaliser of H is H. It follows that H is a maximal abelian subalgebra of L.

Let us take an element $x \in L$ and consider the linear map ad $x : L \to L$. Let $L_{0,x}$ be the generalised eigenspace of ad x with eigenvalue 0. Thus $L_{0,x} = \{y \in L$; there exists n such that $(\text{ad } x)^n y = 0\}$, and $L_{0,x}$ will be called the **null component** of L with respect to x.

Definition 2.2. *An element $x \in L$ is called* ***regular*** *if the dimension of the null component of L is as small as possible. If this dimensionality is l, Then $n - l$, where $n = \dim L$, is called the* ***rank*** *of L.*
The Lie algebra L certainly contains regular elements.

Theorem 2.1. *Let x be a regular element of L. Then the null component $L_{0,x}$ is a Cartan subalgebra of L.*

Proof. Let $H = L_{0,x}$. We must show that H is a subalgebra of L, that H is nilpotent, and that $H = N(H)$.

We first show that H is a subalgebra. Let $y, z \in H$. We must show that $[yz] \in H$. By Proposition 1.12 we have

$$(\text{ad } x)^n[yz] = \textstyle\sum_{i=0}^{n} \binom{n}{i}[(\text{ad}\, x)^i y, (\text{ad}\, x)^{n-i} z].$$

(We take $V = L, \alpha = \beta = 0$ in Proposition 1.12 to obtain this). Since $y \in H$ we have

$$(\text{ad } x)^i y = 0 \quad \text{ if } i \text{ is sufficiently large.}$$

Since $z \in H$

$$(\text{ad}\, x)^{n-i} z = 0 \quad \text{ if } n{-}i \text{ is sufficiently large.}$$

Hence $(\text{ad}\, x)^n[yz] = 0$ if n is sufficiently large. Thus $[yz] \in H$ and H is a subalgebra of L.

We next show that H is nilpotent. To do this we shall prove that all the matrices in the adjoint representation of H are nilpotent and use Engel's theorem (Theorem 1.4). Let $\dim H = l$ and $b_1, \cdots, b_l$ be a basis for H. Let

$$y = \lambda_1 b_1 + \cdots + \lambda_l b_l \in H \qquad \lambda_1, \cdots, \lambda_l \in \mathbb{C}.$$

Consider the linear map $\text{ad}\, y : L \to L$. We have ad $y : H \to H$ since H is a subalgebra and we obtain an induced map $\text{ad}\, y : L/H \to L/H$.

Let $\chi(t)$ be the characteristic polynomial of $\text{ad}\, y$ on L, $\chi_1(t)$ be its characteristic polynomial on H and $\chi_2(t)$ be its characteristic polynomial on L/H. Then we have

$$\chi(t) = \chi_1(t)\chi_2(t).$$

Since $\chi(t) = \det(t1{-}\text{ad } y)$ and y depends linearly on $\lambda_1, \cdots, \lambda_l$ we see that the coefficients of $\chi(t)$ are polynomial functions of $\lambda_1, \cdots, \lambda_l$. The same applies to $\chi_1(t)$ and $\chi_2(t)$.Let

$$\chi_2(t) = d_0 + d_1 t + d_2 t^2 + \cdots$$

where $d_0, d_1, d_2, \cdots$ are polynomial functions of $\lambda_1, \cdots, \lambda_l$. We claim that d_0 is not the zero polynomial. For in the special case when $y = x$ we know that all eigenvalues of $\text{ad}\, y$ on L/H are non-zero, so $\chi_2(t)$ has non-zero constant term. Let

$$\chi_1(t) = t^m(c_0 + c_1 t + c_2 t^2 + \cdots)$$

where $c_0, c_1, c_2, \cdots$ are polynomial functions of $\lambda_1, \cdots, \lambda_l$ and c_0 is not the zero polynomial. We have

$$m \leq l = deg\chi_1(t).$$

We then have

$$\chi(t) = t^m(c_0 d_0 + \text{ terms involving positive powers of } t).$$

Now $c_0 d_0$ is not the zero polynomial so we can choose $\lambda_1, \cdots, \lambda_l \in \mathbb{C}$ to make $c_0 d_0$ non-zero. For such an element $y \in H$ we have

$$\dim\, L_{0,y} = m.$$

Since x is regular and $\dim L_{0,x} = l$ we have $m \geq l$. Since we also know $m \geq l$ we have $m = l$. Now $\chi_1(t)$ has degree l and is divisible by t^l, hence

$$\chi_1(t) = t^l.$$

It follows by the Cayley–Hamilton theorem that $(\operatorname{ad} y)^l : H \to H$ is zero. Hence by Engel's theorem we deduce that H is nilpotent.

Finally we show that $H = N(H)$. It is certainly true that $H \subset N(H)$. So let $z \in N(H)$. Then $[xz] \in H$. Thus

$$(\text{ad } x)^n[xz] = 0 \quad \text{for some } n.$$

But then $(\operatorname{ad} x)^{n+1} z = 0$ and so $z \in H$. Thus $H = N(H)$ and we have show that H is a Cartan subalgebra of L. □

2.1.2 Derivations and automorphisms

Derivations

Let L be a Lie algebra. A **derivation** of L is a linear map $d : L \to L$ satisfying

$$d[xy] = [dx, y] + [x, dy] \quad \text{for all } x, y \in L.$$

For later use we record the so-called Leibniz formula. Let d be derivation of the a Lie algebra L. Then

$$d^n[xy] = \sum_{k=0}^{n} \binom{n}{k} [d^k x, d^{n-k} y] \quad \text{(Leibniz formula)},$$

which is proved by induction on n.

Lemma 2.1. *Let L be a Lie algebra. Then we claim that for $x \in L$, The map $\operatorname{ad} x : L \to L$ is a derivation of L.*

Proof. For $y, z \in L$, we have

$$\begin{aligned} \operatorname{ad} x[yz] &= [x, [yz]] \\ &= [[xy], z] + [y, [xz]] \\ &= [\operatorname{ad} x \cdot y, z] + [y, \operatorname{ad} x \cdot z] \end{aligned}$$

since by the Jacobi identity. □

Derivation of the form $\operatorname{ad} x$ are called inner. On the other hand, if a derivation d of L is not of this form, then d is said to be an outer derivation.

Automorphisms

Let L be a Lie algebra over $\mathbb{C}$. An **automorphism** of L is an isomorphism $\theta : L \to L$. Since composition of automorphisms are automorphisms and inverses of automorphisms are automorphisms, the automorphisms of L form a group. This group is called the automorphism group of L; it is denoted by $\operatorname{Aut} L$.

Now let the ground over $\mathbb{C}$ be of characteristic 0. If d is a nilpotent derivation of L, i.e., $d^n = 0$ for some integer $n \geq 0$, then we can define its exponential:

$$\exp d = 1 + d + \frac{1}{2!} d^2 + \cdots + \frac{1}{(n-1)!} d^{n-1}.$$

Proposition 2.1. *Let d be a nilpotent derivation of L. Then $\exp d$ is an automorphism of L.*

Proof. Since d is nilpotent we have $d^n = 0$ for some n. Then we have

$$\exp d = \sum_{r=0}^{n-1} \frac{d^r}{r!}$$

The map $\exp d : L \to L$ is clearly linear. Let $x, y \in L$. Then

$$d[xy] = [dx, y] + [x, dy]$$
$$d^r[xy] = \sum_{i=0}^{r} \binom{r}{i} \left[d^i x, d^{r-i} y\right]$$

as is easily seen by induction on r. Hence

$$\exp d \cdot [xy] = \sum_{r\geq 0} \sum_{i=0}^{r} \frac{1}{r!} \binom{r}{i} \left[d^i x, d^{r-i} y\right] = \sum_{i\geq 0} \sum_{j\geq 0} \frac{1}{i!j!} \left[d^i x, d^j y\right]$$
$$= \left[\sum_{i\geq 0} \frac{1}{i!} d^i x, \sum_{j\geq 0} \frac{1}{j!} d^j y\right] = [\exp d \cdot x, \exp d \cdot y].$$

So $\exp d : L \to L$ is a homomorphism.

The inverse of $\exp d$ is the map

$$\exp(-d) = \sum_{j=0}^{n-1} (-1)^j \frac{1}{j!} d^j.$$

It follows that $\exp d$ is a bijective homomorphism of Lie algebras; i.e., it is an automorphism. □

In particular, if $\operatorname{ad} x$ is nilpotent then $\exp \operatorname{ad} x$ is an automorphism of L. An automorphism of this form is called inner. The subgroup of Aut L generated by all inner automorphisms is called the **inner automorphism group**. It denoted by $\operatorname{Inn} L$. Every element of $\operatorname{Inn} L$ has form

$$\exp \operatorname{ad} x_1 \cdot \exp \operatorname{ad} x_2 \cdot \cdots \cdot \exp \operatorname{ad} x_r$$

where $x_1, \ldots, x_r \in L$ and ad $x_1, \ldots,$ ad x_r are all nilpotent.

Lemma 2.2. *Inn L is a normal subgroup of Aut L.*

Proof. Let $\theta \in$ Aut L. It is sufficient to show that $\theta(\exp \operatorname{ad} x)\theta^{-1} \in \operatorname{Inn} L$ for all $x \in L$ with $\operatorname{ad} x$ nilpotent. Now we have

$$\theta(\operatorname{ad} x)\theta^{-1} y = \theta[x, \theta^{-1} y] = [\theta x, y] = (\operatorname{ad} \theta x) \cdot y$$

for all $y \in L$. Hence

$$\theta(\operatorname{ad} x)\theta^{-1} = \operatorname{ad} \theta x.$$

It follows that

$$\theta(\exp \operatorname{ad} x)\theta^{-1} = \exp \operatorname{ad}(\theta x) \in \operatorname{Inn} L.$$

Thus $\operatorname{Inn} L$ is normal in Aut L. □

Definition 2.3. *Two subalgebras M_1, M_2 of L are called* ***conjugate*** *in L if there exists $\theta \in \operatorname{Inn} L$ such that $\theta(M_1) = M_2$.*

We will show that any two Cartan subalgebras of L are conjugate in L.

2.1.3 Conjugacy of Cartan subalgebras

In this part, we will show any Cartan subalgebra is the null component of some regular element. We shall then prove that, given two regular elements, their null components are conjugate in L.

Proposition 2.2. *Let H be a Cartan subalgebra of L. Then there exists a regular element $x \in L$ such that $H = L_{0,x}$.*

Theorem 2.2. *Any two Cartan subalgebras of L are conjugate.*

Proof. Let H, H' be Cartan subalgebras of L. We regard L as an H-module and decompose L into weight spaces with respect to H. We have seen in the proof of Proposition 2.2 that $H = L_{0,x}$. Let the weight space decomposition be

$$L = H \oplus L_{\lambda_1} \oplus \cdots \oplus L_{\lambda_r} \qquad \lambda_1, \ldots, \lambda_r \neq 0.$$

For each $x \in L$ we have

$$x = x_0 + x_1 + \cdots + x_r$$

with $x_0 \in H$ and $x_i \in L_{\lambda_i}$ for $i \neq 0$.

Now for each $x_0 \in H$ we have $L_{0,x_0} \supset H$ and for some $x_0 \in H$ we have $L_{0,x_0} = H$ since H is a Cartan subalgebra. An element $x_0 \in H$ is regular if and only if $L_{0,x_0} = H$. This is equivalent to the condition

$$\lambda_1(x_0)\lambda_2(x_0)\cdots\lambda_r(x_0) \neq 0.$$

We now consider the polynomial function $f : L \to L$ defined by

$$f(x) = \exp \operatorname{ad} x_1 \cdot \exp \operatorname{ad} x_2 \cdot \cdots \cdot \exp \operatorname{ad} x_r \cdot x_0.$$

Let $p : L \to \mathbb{C}$ be the function given by

$$p(x) = \lambda_1(x_0)\lambda_2(x_0)\cdots\lambda_r(x_0)$$

where p is a polynomial function on L which is not identically zero, since $p(x)$ is non-zero when x_0 is a regular element of H. There exists a non-zero polynomial function $q : L \to \mathbb{C}$ such that $f(L_p) \supset L_q$.

We now start with the second Cartan subalgebra H. We can define a corresponding function $f' : L \to L$ and a corresponding function $p' : L \to \mathbb{C}$. There exists a non-zero polynomial function $q' : L \to \mathbb{C}$ such that $f'(L_{p'}) \supset L_{q'}$.

Now $L_q \cap L_{q'} = \{x \in L; (qq')(x) \neq 0\}$. Thus $L_q \cap L_{q'}$ is non-empty. We choose $z \in L_q \cap L_{q'}$. Thus $z \in f(L_p) \cap f'(L_{p'})$. Thus there exists $x \in L$ with $z = f(x)$ and $p(x) \neq 0$. Similarly there exists $x' \in L$ with $z = f'(x')$ and $p'(x) \neq 0$. Thus

$$z = \exp \operatorname{ad} x_1 \cdot \exp \operatorname{ad} x_2 \cdot \cdots \cdot \exp \operatorname{ad} x_r \cdot x_0$$

and so z is conjugate to x_0. Since $p(x) \neq 0 x_0$ is regular. Similarly z is conjugate to x'_0 and x'_0 is regular. Thus we have found regular elements $x_0 \in H$ and $x'_0 \in H'$ such that x_0, x'_0 are conjugate in L.

Now we have $H = L_{0,x_0}$ and $H' = L_{0,x'_0}$ since x_0, x'_0 are regular. Thus an inner automorphism of L which transforms x_0 to x'_0 will transform H to H'. Hence H, H' are conjuga in L. □

Definition 2.4. *The dimension of the Cartan subalgebras of L will be called the **rank** of L.*

2.2 The Cartan decomposition

Our main references here are [2, 7] and [8].

2.2.1 Some properties of root spaces

Let L be a Lie algebra and H be a Cartan subalgebra of L. We see that there are distinct λ forms H, and subspace L_λ of L such that

$$L = \bigoplus_{\lambda} L_\lambda$$

as in Theorem 1.3, where

$$L_\lambda = \{x \in L;\ \text{for each}\ h \in H\ \text{there exists}\ n\ \text{such that}\ (\operatorname{ad} h - \lambda(h)1)^n x = 0\}.$$

Proposition 2.3. $L_0 = H$.

Proof. The algebra H is contained in L_0 by Corollary 1.3. Suppose that $H \neq L_0$. Then L_0/H is an H-module, and this module contains a 1-dimensional submodule M/H on which H acts with weight 0. Hence $[HM] \subset H$ and so $M \subset N(H)$. This implies $H \neq N(H)$, a contradiction. □

The direct sum decomposition of L into weight spaces for H may be written as

$$L = H \oplus \left(\bigoplus_{\alpha \in \Phi} L_\alpha \right),$$

where Φ is the set of $\alpha : H \to k$ representation of dim $= 1$ such that $\alpha \neq 0$ and $L_\alpha \neq 0$. Since L is finite dimensional, Φ is finite.

If $\alpha \in \Phi$, then we say that α is a **root** of L and L_α is the associated **root space**. The direct sum decomposition above is the **Cartan decomposition** of L with respect to H. It should be noted that the roots and root spaces depend on the choice of Cartan subalgebra H.

Proposition 2.4. *$[L_\lambda, L_\mu]$ is contained in the $(\lambda(h)+\mu(h))$ eigenspace of $\operatorname{ad} h$ for each $h \in H$. Since $L_{\lambda+\mu}$ is the intersection of all such spaces, we have*

$$[L_\lambda, L_\mu] \subset L_{\lambda+\mu}.$$

Proof. Let $y \in L_\lambda$, $z \in L_\mu$. We show that $[yz] \in L_{\lambda+\mu}$. Let $x \in H$. Then by Proposition 1.12 we have

$$(\operatorname{ad} x - \lambda(x)1 - \mu(x)1)^n [yz] = \sum_{i=0}^{n} \binom{n}{i} [(\operatorname{ad} x - \lambda(x)1)^i y, (\operatorname{ad} x - \mu(x)1)^{n-i} z].$$

Since $y \in L_\lambda$ $(\operatorname{ad} x - \lambda(x)1)^i y = 0$ if i is sufficiently large. Since $z \in L_\mu$ $(\operatorname{ad} x - \mu(x)1)^{n-i} z = 0$ if $n - i$ is sufficiently large. Hence

$$(\operatorname{ad} x - \lambda(x)1 - \mu(x)1)^n [yz] = 0$$

if n is sufficiently large. This shows that $[yz] \in L_{\lambda+\mu}$. □

Corollary 2.1. *Let $\alpha, \beta \in \Phi$ be roots of L with respect to H. Then*

$$[L_\alpha, L_\beta] \subset L_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Phi$$

$$[L_\alpha, L_\beta] \subset H \quad \text{if } \beta = -\alpha$$

$$[L_\alpha, L_\beta] = 0 \quad \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \in \Phi.$$

Proof. This follows from Proposition 2.4 and the fact that $L_0 = H$. □

2.2.2 The Killing form

To further advance our understanding of the cartan decomposition of L, we introduce a bilinear form on L known as the Killing form. We define a map

$$\begin{gathered} L \times L \to \mathbb{C} \\ x, y \to \langle x, y\rangle \end{gathered}$$

given by

$$\langle x, y\rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y).$$

We have $\operatorname{ad} x\colon L \to L$, $\operatorname{ad} y\colon L \to L$ and $\operatorname{ad} x \operatorname{ad} y\colon L \to L$, so $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) \in \mathbb{C}$.

The Killing form is bilinear because ad is linear, the composition of maps is bilinear, and tr is linear. It is symmetric because tr $ab =$ tr ba for linear maps a and b. Another very important property of the Killing form is its associativity, which states that for all $x, y, z \in L$ we have

$$\langle [xy], z\rangle = \langle x, [yz]\rangle \quad \text{for all} \quad x, y, z \in L.$$

Proposition 2.5. *Let I be an ideal of L and $x, y \in I$. Then*

$$\langle x, y\rangle_I = \langle x, y\rangle_L.$$

Thus the Killing form of L restricted to I is the Killing form of I.

Proof. Take a basis for I and extend it to a basis of L. If $x \in I$, then $\operatorname{ad} x$ maps L into I, so the matrix of $\operatorname{ad} x$ in this basis is of the form

$$\begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix}$$

where A_x is the matrix of $\operatorname{ad} x$ restricted to I.
If $y \in I$, then $\operatorname{ad} x \operatorname{ad} y$ has matrix

$$\begin{pmatrix} A_x A_y & A_x B_y \\ 0 & 0 \end{pmatrix}$$

where $A_x A_y$ is the matrix of $\operatorname{ad} x \circ \operatorname{ad} y$ restricted to I. Only the block $A_x A_y$ contributes to the trace of this matrix, so

$$\langle x, y\rangle_L = \operatorname{tr}\langle A_x A_y\rangle = \langle x, y\rangle_I.$$

□

For any subspace M of L we define $M^\perp$ by

$$M^\perp = \{x \in L; \langle x, y\rangle = 0 \quad \text{for all } y \in M\}.$$

$M^\perp$ is also a subspace of L.

Lemma 2.3. *If I is an ideal of L, then $I^\perp$ is also an ideal of L.*

Proof. Let $x \in I^\perp$ and $y \in L$. We must show that $[xy] \in I^\perp$. So let $z \in I$. Then

$$\langle [xy], z\rangle = \langle x, [yz]\rangle = 0$$

since $[yz] \in I$ and $x \in I^\perp$. Thus $[xy] \in I^\perp$ and $I^\perp$ is an ideal of L. □

Remarks

- We see in particular that $L^{\perp}$ is an ideal of L.
- The Killing form of L is said to be **non-degenerate** if $L^{\perp} = 0$. This is equivalent to the condition that if $\langle x, y \rangle = 0$ for all $y \in L$ then $x = 0$.
- The Killing form of L is **identically zero** if $L^{\perp} = L$. This means that $\langle x, y \rangle = 0$ for all $x, y \in L$.
- **The radical** of a symmetric bilinear form $\beta : V \times V \to k$ on a k-vector space V is defined by

$$\text{rad}(\beta)=\{x \in V; \beta(x, \cdot) = 0\}.$$

Non-degenaracy of β is then equivalent to $\text{rad}(\beta) = 0$. Note that the radical of an associative symmetric bilinear form on a Lie algebra is an ideal.
We will now demonstrate more profound result concering the Killing form, which will prove to be extremely useful later on.

Theorem 2.3. *(Cartan's First Criterion)*
If the Killing form of L is identically zero then L is soluble.

Theorem 2.4. *(Cartan's Second Criterion)*
The Killing form of L is non-degenerate if and only if L is semisimple.

This result is called "Cartan's Second Criterion" is an extremely powerful characterisation of semisimplicity. In our first application, we shall show that a semisimple Lie algebra is a direct sum of simple Lie algebras; this finally justifies the name semisimple which we have been using. The following lemma contains the main idea needed.

Lemma 2.4. *If I is a non-trivial proper ideal in a semisimple Lie algebra L, then $L = I \oplus I^{\perp}$. The ideal I is a semisimple Lie algebra in its own right.*

Proof. The restriction of Killing form on L to $I \cap I^{\perp}$ is identically zero, so by Cartan's First Criterion, $I \cap I^{\perp} = 0$. It now follows by dimension counting that $L = I \oplus I^{\perp}$.

We shall show that I is semisimple using Cartan's Second Criterion. Suppose that I has a non-zero soluble ideal. By the "only if" direction of Cartan's Second Criterion, the Killing form on I is degenerate. We have seen that the Killing form on I is given by restricting the Killing form on L, so there exists $a \in I$ such that $\langle a, x \rangle = 0$ for all $x \in I$. But as $a \in I, \langle a, y \rangle = 0$ for all $y \in I^{\perp}$ as well. Since $L = I \oplus I^{\perp}$, this shows that the killing form is degenerate, a contradiction. □

We can now prove the following theorem.

Theorem 2.5. *A Lie algebra L is semisimple if and only if there are simple ideals $L_1, \ldots, L_r$ of L such that $L = L_1 \oplus L_2 \oplus \ldots \oplus L_r$.*

Proof. We begin with the "only if" direction, working by induction on $\dim L$. Let I be an ideal in L of the smallest possible non-zero dimension. If $I = L$, we are done.

Otherwise I is a proper simple ideal of L. (It cannot be abelian as by hypothesis L has no non-zero abelian ideals). By the preceding lemma, $L = I \oplus I^{\perp}$, where, as an ideal of $L, I^{\perp}$ is a semisimple Lie algebra of smaller dimension than L.

So, by induction, $I^\perp$ is a direct sum of simple ideals,

$$I^\perp = L_2 \oplus \ldots \oplus L_r.$$

Each L_i is also an ideal of L, as $[I, L_i] \subseteq I \cap I^\perp = 0$, so putting $L_1 = I$, we get the required decomposition.

Now for the "if" direction. Suppose that $L = L_1 \oplus \ldots \oplus L_r$, where the L_r are simple ideals. Let $I = \operatorname{rad} L$; our aim is to show that $I = 0$. For each ideal L_i,$[I, L_i] \subseteq I \cap L_i$ is a soluble ideal of L_i.But the L_i are simple, so

$$[I, L] \subseteq [I, L_1] \oplus \ldots \oplus [I, L_r] = 0.$$

This shows that I is contained in $Z(L)$, where $Z(L)$ is the centre of L, defined by

$$Z(L) := \{x \in L : [xy] = 0 \text{ for all } y \in L\}.$$

So we get

$$Z(L) = Z(L_1) \oplus \ldots \oplus Z(L_r).$$

We know that $Z(L_1) = \ldots Z(L_r) = 0$ as the L_i are simple ideals, so $Z(L) = 0$ and $I = 0$. □

utilizing analogous ideas we can establish the following

Lemma 2.5. *If L is a semisimple Lie algebra and I is an ideal of L, then L/I is semisimple.*

Proof. We have noted that $L = I \oplus I^\perp$, so L/I is isomorphic to $I^\perp$, which we have seen is a semisimple Lie algebra in its own right. □

2.2.3 The Cartan decomposition of a semisimple Lie algebra

When L is semisimple, we can provide more detailed insights into its Cartan decomposition compared to the general case. Let's delve into this Cartan decomposition in detail.

Consider L as semisimple, where H represents a Cartan subalgebra of L. Let $L = L \bigoplus L_\lambda$ be the Cartan decomposition of L with respect to H; we have $L_0 = H$.

Proposition 2.6. *The subspaces L_λ and L_μ are orthogonal concerning the Killing form, assuming that $\mu \neq -\lambda$.*

Proof. Let $x \in L_\lambda, y \in L_\mu$. We assume $\lambda + \mu \neq 0$ and must show that $\langle x, y\rangle = 0$.
For any weight space L_ν we have

$$\operatorname{ad} x \operatorname{ad} y \ L_\nu \subset L_{\lambda+\mu+\nu}.$$

We select a basis for L that aligns with the Cartan decomposition. In this basis, the operators $\operatorname{ad} x$ and $\operatorname{ad} y$ will be represented by a block matrix in the form of:

$$\begin{pmatrix} 0 & & & & & \\ & 0 & & & * & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & * & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

since $\lambda + \mu + \nu \neq \nu$. Therefore, we have

$$\langle x, y\rangle = \operatorname{tr}(\operatorname{ad} x \ \operatorname{ad}\ y) = 0.$$

□

Proposition 2.7. *If α is a root of L with respect to H then $-\alpha$ is also a root.*

Proof. We recall that α is a root if $\alpha \neq 0$ and $L_\alpha \neq 0$. Assume, if possible that $-\alpha$ is not a root. Since $-\alpha \neq 0$, we have $L_{-\alpha} = 0$. According to Proposition 2.6, L_α is orthogonal to all L_λ, thus $L_\alpha \subset L^\perp$. However, since L is semisimple we have $L^\perp = 0$. Consequently, $L_\alpha = 0$, contradicting the fact that α is a root. □

Proposition 2.8. *The Killing form of L remains non-degenerate when restricted to H. Therefore, if $x \in H$ satisfies $\langle x, y\rangle = 0$ for all $y \in H$, then x must be equal to 0.*

Proof. Let $x \in H$ and suppose $\langle x, y\rangle = 0$ for all $y \in H$. We also have $\langle x, y\rangle = 0$ for all $y \in L_\alpha$ where $\alpha \neq 0$, by Proposition 2.6. Thus $\langle x, y\rangle = 0$ for all $y \in L$ and so $x \in L^\perp$. Since L is semisimple $L^\perp = 0$, hence $x = 0$ as required. □

It's worth noting that the Killing form of L, when restricted to H, differs from the Killing form of H. The latter is degenerate because H is not semisimple.

Theorem 2.6. $[HH] = 0$. *Thus the Cartan subalgebras of a semisimple Lie algebra are abelian.*

Proof. See [8, Theorem 4.15] □

Let $H^* = \mathrm{Hom}(H, \mathbb{C})$ denote the dual space of H. We have $\dim H^* = \dim H$.

We define a map $H \to H^*$ using the Killing form of L. Given $h \in H$ we define $h^* \in H^*$ by

$$h^*(x) = \langle h, x\rangle \quad \text{for all } x \in H.$$

Lemma 2.6. *The map $h \to h^*$ is an isomorphism of vector spaces between H and H^*.*

Proof. The map is certainly linear. Suppose $h \in H$ lies in the kernel. Then $\langle h, x\rangle = 0$ for all $x \in H$. This implies $h = 0$ by Proposition 2.8. Thus the kernel is 0. Hence the image must be the whole of H^*, since $\dim H^* = \dim H$. Hence our map is bijective. □

Now, consider a finite subset $\Phi \subset H^*$, which is the set of roots of L with respect to H. For each $\alpha \in \Phi$, there exists a unique element h'_α such that

$$\alpha(x) = \langle h'_\alpha, x\rangle \quad \text{for all } x \in H.$$

Proposition 2.9. *The vectors h'_a for $\alpha \in \Phi$ span H.*

Proof. Suppose if possible that the h'_α lie in a proper subspace of H. Then there exists an element $x \in H$ with $x \neq 0$ and $\langle h'_\alpha, x\rangle = 0$ for all $\alpha \in \Phi$. Thus $\alpha(x) = 0$ for all $\alpha \in \Phi$. Let $y \in H$. Then we have

$$\langle x, y\rangle = \mathrm{tr}(\mathrm{ad}\, x \ \mathrm{ad}\ y) = \sum_\lambda \dim L_\lambda \lambda(x)\lambda(y) = 0$$

since $\lambda(x) = 0$ for all weights λ. Thus $\langle x, y\rangle = 0$ for all $y \in H$. This implies $x = 0$ by Proposition 2.8, a contradiction. □

Proposition 2.10. $h'_\alpha \in [L_\alpha L_{-\alpha}]$ *for all $\alpha \in \Phi$.*

Proof. See [8, Proposition 4.18]. □

Proposition 2.11. $\langle h'_\alpha, h'_\alpha\rangle \neq 0$ *for all $\alpha \in \Phi$.*

Proof. Suppose $\langle h'_\alpha, h'_\alpha \rangle = 0$ for some $\alpha \in \Phi$, let β be any element of Φ. there exists a number $r_{\beta,\alpha} \in \mathbb{Q}$ such that $\beta = r_{\beta,\alpha}\alpha$ on $[L_\alpha L_{-\alpha}]$.
Now $h'_\alpha \in [L_\alpha L_{-\alpha}]$. Thus

$$\beta\left(h'_\alpha\right) = r_{\beta,\alpha}\alpha\left(h'_\alpha\right),$$

that is, $\left\langle h'_\beta, h'_\alpha \right\rangle = r_{\beta,\alpha} \left\langle h'_\alpha, h'_\alpha \right\rangle = 0$.

This holds for all $\beta \in \Phi$. But the vectors h'_β for $\beta \in \Phi$ span H, and so $\langle x, h'_\alpha \rangle = 0$ for all $x \in H$.
Since the Killing form of L restricted to H is non-degenerate, this implies $h'_\alpha = 0$. This $\alpha = 0$, contradicting the fact that $\alpha \in \Phi$. □

After deriving several results on the Cartan decomposition of a semisimple Lie algebra, each building upon previous findings, we are now poised to uncover one of the most critical properties of the Cartan decomposition.

Theorem 2.7. $\dim L_\alpha = 1$ *for all* $\alpha \in \Phi$.

Proof. See [8, Theorem 4.20]. □

It's worth noting that while all the root spaces L_α are 1-dimensional, the space $H = L_0$ is not necessarily 1-dimensional.

Proposition 2.12. *If* $\alpha \in \Phi$ *and* $r\alpha \in \Phi$ *where* $r \in \mathbb{Z}$ *then* $r = 1$ *or* $r = -1$.

Proof. From the above, we have $\dim L_{-r\alpha} = 0$ for all $r \geq 2$, that is, $-r\alpha$ is not a root.
Now $r\alpha \in \Phi$ if and only if $-r\alpha \in \Phi$. Thus only α and $-\alpha$ can be roots. □

We are now ready to examine some stronger properties of the set Φ of roots.

Let $\alpha, \beta \in \Phi$ be roots such that $\beta \neq \alpha$ and $\beta \neq -\alpha$. Then β is not an integer multiple of α. There do, however, exist integers $p \geq 0, q \geq 0$ such that the elements

$$-p\alpha + \beta, \ldots, -\alpha + \beta, \beta, \alpha + \beta, \ldots, q\alpha + \beta$$

all lie in Φ but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ do not.
The set of roots

$$-p\alpha + \beta, \ldots, q\alpha + \beta$$

is called the α**-chain** of roots through β.

Proposition 2.13. *Let* α, β *be roots such that* $\beta \neq \alpha$ *and* $\beta \neq -\alpha$. *Let*

$$-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta$$

be the α*-chain of roots through* β. *Then*

$$2\frac{\left\langle h'_\alpha, h'_\beta \right\rangle}{\left\langle h'_\alpha, h'_\alpha \right\rangle} = p - q.$$

This result leads to several useful corollaries.

Proposition 2.14. $\langle h'_\alpha, h'_\beta \rangle \in \mathbb{Q}$ *for all* $\alpha, \beta \in \Phi$.

Proof. We know from the outset that $\left\langle h'_\alpha, h'_\beta \right\rangle \in \mathbb{C}$. We also have

$$2\frac{\left\langle h'_\alpha, h'_\beta \right\rangle}{\left\langle h'_\alpha, h'_\alpha \right\rangle} \in \mathbb{Z}. \quad \text{Proposition 2.13}$$

Thus $\frac{\left\langle h'_\alpha, h'_\beta \right\rangle}{\left\langle h'_\alpha, h'_\alpha \right\rangle} \in \mathbb{Q}$. It is therefore sufficient to show that $\langle h'_\alpha, h'_\alpha \rangle \in \mathbb{Q}$.
We have

$$\langle h'_\alpha, h'_\alpha \rangle = \operatorname{tr}\left(\operatorname{ad} h'_\alpha \ \operatorname{ad}\ h'_\alpha\right) = \sum_{\beta \in \Phi} \left(\beta\left(h'_\alpha\right)\right)^2 = \sum_{\beta \in \Phi} \left\langle h'_\alpha, h'_\beta \right\rangle^2.$$

Dividing by $\langle h'_\alpha, h'_\alpha \rangle^2$, this yields

$$\frac{1}{\langle h'_\alpha, h'_\alpha \rangle} = \sum_{\beta \in \Phi} \left(\frac{\left\langle h'_\alpha, h'_\beta \right\rangle}{\langle h'_\alpha, h'_\alpha \rangle} \right)^2 \in \mathbb{Q}.$$

Hence $\langle h'_\alpha, h'_\alpha \rangle \in \mathbb{Q}$ completing the proof. □

Chapter 3

The root system and the Weyl group and the dynkin diagram

In this chapter we present the notion of Positive systems and fundamental systems of roots and the Weyl group is introduced and shown to be a Coxeter group. This leads on to the definition of the Cartan matrix and the Dynkin diagram. The possible Dynkin diagrams and Cartan matrices are classified.

3.1 The root system and the Weyl group

Our main reference here is [8].

3.1.1 Positive systems and fundamental systems of roots

As earlier, let L be a semisimple Lie algebra and H be a Cartan subalgebra. Let Φ be the set of roots of L with respect to H. We know that the elements $h'_\alpha, \alpha \in \Phi$, span$H$. Therefore, we can identify a subset that form a basis of H. Let $h'_{\alpha_1}, \dots, h'_{\alpha_l}$ form a basis of H.

Proposition 3.1. *Let $\alpha \in \Phi$. Then $h'_\alpha = \sum_{i=1}^{l} \mu_i h'_{\alpha_i}$ where each μ_i lies in $\mathbb{Q}$.*

We define by $H_{\mathbb{Q}}$ the set of all elements of the form $\sum_{i=1}^{l} \mu_i h'_{\alpha_i}$ for $\mu_i \in \mathbb{Q}$ and $H_{\mathbb{R}}$ the set of all such elements with $\mu_i \in \mathbb{R}$. The spaces $H_{\mathbb{Q}}$ and $H_{\mathbb{R}}$ are independent of the choice of the basis $\{h'_{\alpha_i}\}$.

Next, we demonstrate that the Killing form of L behaves in a favourable manner when restricted to $H_{\mathbb{R}}$.

Proposition 3.2. *Let $x \in H_{\mathbb{R}}$. Then $\langle x, x\rangle \in \mathbb{R}$ and $\langle x, x\rangle \geq 0$. If $\langle x, x\rangle = 0$ then $x = 0$.*

Remarks

- This proposition shows that the Killing form restricted to $H_{\mathbb{R}}$ is a symmetric positive definite bilinear form.
- The vector space $H_{\mathbb{R}}$ endowed with this positive definite form is a Euclidean space, this Euclidean space contains all vectors h'_α for $\alpha \in \Phi$.
- We recall that. We have an isomorphism $h \to h^*$ from H to H^* given by $h^* = \langle h, x\rangle$, we define $H^*_{\mathbb{R}}$ as the image of $H_{\mathbb{R}}$ under this isomorphism. $H^*_{\mathbb{R}}$ is the real subspace of H^* spanned by Φ.

- We can also define a symmetric positive definite bilinear form on H^* by

$$\langle h_1^*, h_2^* \rangle = \langle h_1, h_2 \rangle \in \mathbb{R}.$$

- Thus H^* becomes a Euclidean space containing all the roots $\alpha \in \Phi$.

We will investigate the configuration formed by the roots in the Euclidean space $H_{\mathbb{R}}^*$. We shall denoted it as $H_{\mathbb{R}}^* = V$ temporarily.

Definition 3.1. *A* ***total ordering*** *on V is a relation $<$ on V satisfying the following axioms:*

(i) $\lambda < \mu$ and $\mu < \nu$ implies $\lambda < \nu$.

(ii) For each pair of elements $\lambda, \mu \in V$ just one of the conditions $\lambda < \mu$, $\lambda = \mu, \mu < \lambda$ holds.

(iii) If $\lambda < \mu$ then $\lambda + \nu < \mu + \nu$.

(iv) If $\lambda < \mu$ and $\xi \in \mathbb{R}$ with $\xi > 0$ then $\xi\lambda < \xi\mu$, and if $\xi < 0$ then $\xi\mu < \xi\lambda$.

Every real vector space possesses total orderings. If $\nu_1, \dots, \nu_l$ are a basis of V and $\lambda = \sum \lambda_i v_i, \mu = \sum \mu_i v_i$ with $\lambda \neq \mu$ then we may define $\lambda < \mu$ if the first non-zero coefficient $\mu_i - \lambda_i$ is positive. This gives us a total ordering on V.

Definition 3.2. *A* ***positive system*** *$\Phi^+ \subset \Phi$ is the set of all roots $\alpha \in \Phi$ satisfying $\alpha > 0$ for some total ordering on V.*

Definition 3.3. *The* ***fundamental system*** *$\Pi \subset \Phi^+$ is defined as follows: $\alpha \in \Pi \Leftrightarrow \alpha \in \Phi^+$ and α can not be expressed as the sum of two elements of Φ^+. Φ^- the corresponding set of negative roots.*

Proposition 3.3. *Every root in Φ^+ is a sum of roots in Π.*

Proof. Let $\alpha \in \Phi^+$. Then either $\alpha \in \Pi$ or $\alpha = \beta + \gamma$ where $\beta, \gamma \in \Phi^+$ and $\beta < \alpha, \gamma < \alpha$. We continue this process, which must eventually terminate since Φ^+ is finite. Thus α is a sum of elements of Π. □

Proposition 3.4. *Let $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$.Then $\langle \alpha, \beta \rangle \leq 0$.*

Proof. We first observe that $\alpha - \beta \notin \Phi$. If $\alpha - \beta \in \Phi$ we would have either $\alpha - \beta \in \Phi^+$or $\beta - \alpha \in \Phi^+$. If $\alpha - \beta \in \Phi^+$ then $\alpha = (\alpha - \beta) + \beta$ which contradicts $\alpha \in \Pi$. If $\beta - \alpha \in \Phi^+$ then $\beta = (\beta - \alpha) + \alpha$ which contradicts $\alpha - \beta \notin \Phi$. Now we consider the α-chain of roots through β. This has form

$$\beta, \alpha + \beta, \dots, q\alpha + \beta$$

since $-\alpha + \beta \notin \Phi$. We deduce

$$2\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} = -q.$$

However, $\langle h'_\alpha, h'_\alpha \rangle > 0$, hence $\langle h'_\alpha, h'_\beta \rangle \leq 0$. It follows that $\langle \alpha, \beta \rangle \leq 0$. □

Hence, any two distinct roots in the fundamental system Π are inclined at an obtuse angle.

Our next result highlights the importance of the concept of a fundamental system of roots.

Theorem 3.1. *A fundamental system Π forms a basis of $V = H_{\mathbb{R}}^*$.*

Remark :

We see in particular that $|\Pi| = l = \dim H$. Thus the number of roots in a fundamental system is equal to the rank of the Lie algebra L.

Corollary 3.1. *Let Π be a fundamental system of roots. Then each $\alpha \in \Phi$ can be expressed in the form $\alpha = \sum n_i\alpha_i$ where $\alpha_i \in \Pi$, $n_i \in \mathbb{Z}$ and either $n_i \geq 0$ for all i or $n_i \leq 0$ for all i.*

Proof. The roots $\alpha \in \Phi^+$ have all $n_i \geq 0$ and the roots $\alpha \in \Phi^-$ have all $n_i \leq 0$. □

3.1.2 The Weyl group

Within the root system Φ, a positive system Φ^+ can be selected in numerous manners. Nonetheless, we will illustrate that any two positive systems in Φ can be converted into each other by an element of a specific finite group W, which operates on Φ.
For each $\alpha \in \Phi$ we define a linear map $s_\alpha : V \to V$ by

$$s_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha \quad \text{for all } x \in V.$$

As before, $V = H^*_{\mathbb{R}}$. This map s_α satisfies

$$s_\alpha(\alpha) = -\alpha.$$
$$s_\alpha(x) = x \quad \text{if } \langle \alpha, x\rangle = 0.$$

There is a unique linear map satisfying these conditions, it's the reflection across the hyperplane of V orthogonal to α. Thus s_α is this reflection.

Definition 3.4. *The group W of all non-singular linear maps on V generated by the s_α for all $\alpha \in \Phi$ is called the **Weyl group**.*

This group plays an important role in the Lie theory. It is a group of isometries of V, that is we have

$$\langle w(x), w(y)\rangle = \langle x, y\rangle \quad \text{for all } x, y \in V.$$

Proposition 3.5. *The Weyl group W is finite.*

Proof. W permutes Φ and Φ is finite. If two elements of W induce the same permutation of Φ they must be equal, since Φ spans V. Since there are only finitely many permutations of Φ, W must be finite. □

Now, let's assume Φ^+ that is a positive system in Φ, and that Π is the corresponding fundamental system.

Lemma 3.1. *Let $\alpha \in \Pi$. If $\beta \in \Phi^+$ and $\beta \neq \alpha$ then $s_\alpha(\beta) \in \Phi^+$.*

Theorem 3.2. *Let Φ_1^+,Φ_2^+ be two positive systems in Φ. Then there exists $w \in W$ such that $w(\Phi_1^+) = \Phi_2^+$.*

Proof. Let $m = \left|\Phi_1^+ \cap \Phi_2^-\right|$. We shall use induction on m. If $m = 0$ we have $\Phi_1^+ = \Phi_2^+$and so $w = 1$ has the required property. Thus we may assume $m > 0$.

Let Π_1 be the fundamental system in Φ_1^+. We cannot have $\Pi_1 \subset \Phi_2^+$as this would imply $\Phi_1^+ \subset \Phi_2^+$, contrary to $m > 0$. Thus there exists $\alpha \in \Pi_1 \cap \Phi_2^-$. We consider $s_\alpha\left(\Phi_1^+\right)$. This is also a positive system in Φ. By Lemma 3.1 $s_\alpha\left(\Phi_1^+\right)$contains all roots in Φ_1^+ except α, together with $-\alpha$. Thus we have

$$\left|s_\alpha\left(\Phi_1^+\right) \cap \Phi_2^-\right| = m - 1.$$

By induction there exists $w' \in W$ such that $w's_\alpha\left(\Phi_1^+\right) = \Phi_2^+$. Let $w = w's_\alpha$. Then $w\left(\Phi_1^+\right) = \Phi_2^+$ as required. □

Corollary 3.2. *Let Π_1, Π_2 be two fundamental systems in Φ. Then there exists $w \in W$ such that $w(\Pi_1) = \Pi_2$.*

Proof. Let Φ_1^+, Φ_2^+ be positive systems containing Π_1, Π_2 respectively. Let $\Phi_2^+ = w(\Phi_1^+)$. Then $w(\Pi_1)$ is a fundamental system contained in Φ_2^+, so $w(\Pi_1) = \Pi_2$. □

Proposition 3.6. *Let Π be a fundamental system in Φ. Then for each $\alpha \in \Phi$ there exist $\alpha_i \in \Pi$ and $w \in W$ with $\alpha = w(\alpha_i)$.*

Proof. Let Φ^+ be the positive system with fundamental system Π. First suppose $\alpha \in \Phi^+$. Then we have

$$\alpha = \sum_i n_i \alpha_i \quad \alpha_i \in \Pi, \quad n_i \in \mathbb{Z}, \quad n_i \geq 0$$

by Corollary 3.1. We define the **height** of α by

$$\text{ht } \alpha = \sum_i n_i.$$

We shall argue by induction on ht α. If ht $\alpha = 1$ then $\alpha = \alpha_i$ for some i and $\alpha \in \Pi$. The result is obvious in this case. Thus suppose ht $\alpha > 1$. Then we have $n_i > 0$ for at least two values of i by Proposition 2.12. Now

$$\langle \alpha, \alpha \rangle = \sum_i n_i \langle \alpha, \alpha_i \rangle .$$

Since $\langle \alpha, \alpha \rangle > 0$ and each $n_i \geq 0$ there exist $\alpha_i \in \Pi$ with $\langle \alpha, \alpha_i \rangle > 0$. Let $s_i(\alpha) = \beta$. Then $\beta \in \Phi$ and

$$\beta = \alpha - 2\frac{\langle \alpha_i, \alpha \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i.$$

Since $\langle \alpha_i, \alpha \rangle > 0$ we see that ht $\beta <$ ht α. On the other hand $\beta \in \Phi^+$ since only one coefficient n_i is changed in passing from α to β, thus at least one coefficient remains positive in β. By Corollary 3.1 this is sufficient to show that $\beta \in \Phi^+$. By induction there exist $\alpha_j \in \Pi$ and $w' \in W$ such that $\beta = w'(\alpha_j)$. Then

$$\alpha = s_i(\beta) = c$$

as required.
Finally we suppose that $\alpha \in \Phi^-$. Then $\alpha = s_\alpha(-\alpha)$ and $-\alpha \in \Phi^+$. Thus $-\alpha = w'(\alpha_i)$ for some $w' \in W$, $\alpha_i \in \Pi$. Hence $\alpha = s_\alpha w'(\alpha_i)$ as required. □

Therefore, every root corresponds to a fundamental root transformed by an element of the Weyl group.

Next, we demonstrate that W is generated by the reflections corresponding to roots in a given fundamental system.

Theorem 3.3. *Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a fundamental system in Φ. Then the corresponding fundamental reflections $s_{\alpha_1}, \ldots, s_{\alpha_l}$ generate W.*

We're interested in delving deeper into how the Weyl group W is formed by a set of its fundamental reflections. As previously, we let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a fundamental system of roots and examine the corresponding set of fundamental reflections. To simplify, we shall write

$$s_1 = s_{\alpha_1}, \quad s_2 = s_{\alpha_2}, \quad \ldots, \quad s_l = s_{\alpha_l}.$$

Then every element of W can be expressed as a product of elements s_i.

Remarks

- For each $w \in W$ we define $l(w)$ to be the minimal value of m such that w can be expressed as a product of m fundamental reflections s_i, $l(w)$ is called the **length** of w. It's clear that $l(1) = 0$ and $l(s_i) = 1$.
- An expression of w as a product of fundamental reflections s_i with $l(w)$ terms is called a **reduced expression** for w.

Let's recall that each element $w \in W$ permutes the elements of Φ. We define $n(w)$ to be the number of roots $\alpha \in \Phi^+$ for which $w(\alpha) \in \Phi^-$. Thus $n(w)$ is the number of positive roots made negative by w. We aim to show that $l(w) = n(w)$.

Proposition 3.7. *$n(w) \leq l(w)$ for all $w \in W$.*

Proof. We shall first compare $n(w)$ with $n(ws_i)$. We recall from Lemma 3.1 that s_i transforms α_i to $-\alpha_i$ and all positive roots other than α_i to positive roots. It follows that

$$n(ws_i) = n(w) \pm 1.$$

In order to determine the sign we consider the effect of w and ws_i on α_i. If $w(\alpha_i) \in \Phi^+$ then w transforms α_i to a positive root and ws_i transforms α_i to a negative root. Hence $n(ws_i) = n(w) + 1$. On the other hand if $w(\alpha_i) \in \Phi^-$ then we get the reverse situation and $n(ws_i) = n(w) - 1$.

Now let us take a reduced expression

$$w = s_{i_1} s_{i_2} \dots s_{i_r} \qquad r = l(w).$$

Then we have

$$n(w) \leq n(s_{i_1} \dots s_{i_{r-1}}) + 1 \leq n(s_{i_1} \dots s_{i_{r-2}}) + 2 \leq \dots \leq r.$$

Thus $n(w) \leq l(w)$ as required. □

To establish the converse result $l(w) \leq n(w)$, we give the following result known as the **deletion condition**, which has significant importance on its own.

Theorem 3.4. *Let $w = s_{i_1} \dots s_{i_r}$ be any expression of $w \in W$ as a product of fundamental reflections. Suppose $n(w) < r$.*
Then there exist integers j, k with $1 \leq j < k \leq r$ such that

$$w = s_{i_1} \dots \hat{s}_{i_j} \dots \hat{s}_{i_k} \dots s_{i_r}$$

where $\hat{}$ denotes omission.

Corollary 3.3. $n(w) = l(w)$.

Proof. We know from Proposition 3.7 that $n(w) \leq l(w)$. Suppose if possible that $n(w) < l(w)$. Let $w = s_{i_1} \dots s_{i_r}$ be a reduced expression, thus $r = l(w)$. Since $n(w) < r$ we may apply Theorem 3.4 to show that w is a product of $r - 2$ fundamental reflections. This contradicts the definition of $l(w)$. □

So, the length of w equals the count of positive roots that are transformed into negative roots by w.

Proposition 3.8.

(a) The maximal length of any element of W *is* $| \Phi^+ |$.

(b) W *has a unique element* w_0 *with* $l(w_0) =| \Phi^+ |$.

(c) $w_0(\Phi^+) = \Phi^-$.

(d) $w_0^2 = 1$.

In the following theorem we shall give a description of the Weyl group W by means of generators and relations. Let the order of the element $s_i s_j \in W$ be m_{ij} when $i \neq j$.

Theorem 3.5. *W is isomorphic to the abstract group given by generators and relations:*

$$\langle s_1, \ldots, s_l; s_i^2 = 1, \quad (s_i s_j)^{m_{ij}} = 1 \quad \textit{for} \quad i \neq j \rangle.$$

A group defined by generators and relations of this form is called a **Coxeter group**. Thus the theorem asserts that the Weyl group is a Coxeter group.

There is a wonderful proof presnted by *R*. Steinberg, of this rescult. (To see the proof, please refer to [8, Theorem 5.18]).

3.2 The Cartan matrix and the Dynkin diagram

Our main references here are [2, 3, 8] and [9].

3.2.1 The Cartan matrix

Let's delve deeper into the geometry of the root system in the vector space $V = H^*_{\mathbb{R}}$. V is a Euclidean space under the scalar product $\langle , \rangle$. While the roots Φ span V, they are not linearly independent. However, any fundamental system $\Pi \subset \Phi$ serves as a basis for V.

Our initial focus lies on examining the potential angles between pairs of roots α, β within the set Φ, and the relative lengths of the roots α, β. These angles are constrained to fall within the range $0 \leq \theta \leq \pi$.

Proposition 3.9. *Let* $\alpha, \beta \in \Phi$ *be such that* $\beta \neq \pm\alpha$. *Then:*

(i) The angle between α, β *is one of* $\pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6$

(ii) If α, β *are inclined at* $\pi/3$ *or* $2\pi/3$ *then* α, β *have the same length*

(iii) If α, β *are inclined at* $\pi/4$ *or* $3\pi/4$ *then the ratio of their lengths is* $\sqrt{2}$

(iiii) If α, β *are inclined at* $\pi/6$ *or* $5\pi/6$ *then the ratio of their lengths is* $\sqrt{3}$.

Proof. Let θ be the angle between α, β. Then we have

$$\langle \alpha, \beta \rangle = |\alpha||\beta| \cos\theta$$

where $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$. Hence

$$\cos^2\theta = \frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \cdot \frac{\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle}$$

and so

$$4\cos^2\theta = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \cdot 2\frac{\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle}.$$

Now we recall from Proposition 2.13 that $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ and $2\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}$ are integers. Hence $4\cos^2\theta \in \mathbb{Z}$. Since $0 \leq 4\cos^2\theta \leq 4$ and $\beta \neq \pm\alpha$ we have $4\cos^2\theta \in \{0,1,2,3\}$.
We consider in each case the possible factorisations of $4\cos^2\theta$ into the product of two integers.

First suppose $4\cos^2\theta = 0$. Then $\theta = \pi/2$.
Next suppose $4\cos^2\theta = 1$. Then $\cos\theta = \dfrac{1}{2}$ or $\dfrac{-1}{2}$, hence $\theta = \pi/3$ or $2\pi/3$.
The possible factorisations of $4\cos^2\theta$ are

$$1 = 1\cdot 1 \quad \text{or} \quad 1 = -1\cdot -1.$$

In both cases, we have

$$2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = 2\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}$$

and so $\langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle$ and α,β have the same length.

Next suppose $4\cos^2\theta = 2$. Then $\cos\theta = 1/\sqrt{2}$ or $-1/\sqrt{2}$, thus $\theta = \pi/4$ or $3\pi/4$. The possible factorisations of $4\cos^2\theta$ are

$$2 = 1\cdot 2 \quad \text{or} \quad 2 = -1\cdot -2.$$

In either case, by choosing α,β in a suitable order, we have

$$2\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle} = 2\cdot 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle},$$

that is $\langle\alpha,\alpha\rangle = 2\langle\beta,\beta\rangle$ and $|\alpha| = \sqrt{2}|\beta|$. Thus the ratio of the lengths of α,β is $\sqrt{2}$.

Finally suppose that $4\cos^2\theta = 3$. Then $\cos\theta = \sqrt{3}/2$ or $-\sqrt{3}/2$, so $\theta = \pi/6$ or $5\pi/6$. The possible factorisations of $4\cos^2\theta$ are

$$3 = 1\cdot 3 \quad \text{or} \quad 3 = -1\cdot -3$$

In either case, by choosing α,β in a suitable order, we have

$$2\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle} = 3\cdot 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle},$$

that is $\langle\alpha,\alpha\rangle = 3\langle\beta,\beta\rangle$ and $|\alpha| = \sqrt{3}|\beta|$. Thus the ratio of the lengths of α,β is $\sqrt{3}$.

This concludes the proof. No information is gained about the relative lengths of α and β in the case when $\theta = \pi/2$. □

Corollary 3.4. *Let Π be a fundamental system of roots and let $\alpha,\beta \in \Pi$ with $\beta \neq \alpha$. Then the angle between α,β is one of $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$.*

Proof. This result is a consequence of Proposition 3.9, combined with the previously established fact from Proposition 3.4, which states that the angle θ between two distinct fundamental roots falls within the range $\pi/2 \leq \theta \leq \pi$. □

Let $\Pi = \{\alpha_1,\ldots,\alpha_l\}$ be a fundamental system. We incorporate the information about the angles between the α_i and their relative lengths in the form of a matrix.

Definition 3.5. *Let $A_{ij} = 2\frac{\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}$ $\quad i,j = 1,\ldots,l$. Thus $A_{ij} \in \mathbb{Z}$. The $l\times l$ matrix $(A_{ij})_{ij}$ is called the **Cartan matrix**.*

Proposition 3.10. *The Cartan matrix has all entries integers, and the following properties:*

(i) $A_{ii} = 2$ *for all* i.

(ii) If $i \neq j$ *then* $A_{ij} \leq 0$, *and* $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$.

(iii) If $A_{ij} = -2$ *or* -3 *then* $A_{ji} = -1$.

Proof. Properties (i), (ii) are obvious, (iii) follow from the proof of Proposition 3.9. □

Remark

- The Cartan matrix is non-singular.
- The Cartan matrix of L depends only on the numbering of the fundamental roots. It is independent of the choice of Cartan subalgebra H and fundamental system Π.

Let's classify the Cartan matrices. The only 1×1 such matrix is (2), the Cartan matrix of type A_1. There are more possibilities for 2×2 Cartan matrices A. We know that

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

for nonnegative integers a, b. Moreover, $4-ab > 0$, so that $ab \leq 3$. There are four possibilities for A

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

The pair

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

are obtained from one another by reversing the labelling $1, 2$, and so are the pair

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

3.2.2 The Dynkin diagram

Introducing a graph known as the **Dynkin diagram** proves beneficial for determining the possible $l \times l$ Cartan matrices when l is larger. This diagram, determined by the Cartan matrix, consists of vertices labeled from 1 to l. When $i \neq j$, vertices i and j are connected by n_{ij} edges, where

$$n_{ij} = A_{ij} A_{ji}.$$

The Dynkin diagram is uniquely determined by the semisimple Lie algebra L.

The **Dynkin diagram** of a root system Φ is a partially directed Coxeter graphs with directions assigned to certain arrows as follows: if α_i and α_j are connected by two or three edges, then we put a direction on the edges going from the long to the short root.

Examples 3.1. *The Dynkin diagrams of the Cartan matrices of degrees* 1 *and* 2 *are as follows.*

Cartan matrice	***Dynkin diagram***
(2)	
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	
$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	
$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	
$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$	

Proposition 3.11. *$n_{ij} \in \{0, 1, 2, 3\}$ for all $i \neq j$.*

Proof. This follows from Proposition 3.10 and the fact that $n_{ij} = A_{ij}A_{ji}$. □

So, the count of edges connecting any two different vertices in the Dynkin diagram can be $0, 1, 2$ or 3.

Remark

- Any subdiagram of a Dynkin diagram is again a Dynkin diagram.
- Π is indecomposable if and only if the Dynkin diagram is connected.
- Dynkin diagram completely determines the Cartan matrix.

If the Dynkin diagram is disconnected, it can be split into connected components. Assigning consecutive numbers to vertices within each connected component, the Cartan matrix will then split into blocks of the form

$$A = \left(\begin{array}{c|c|c|c} * & O & O & O \\ \hline O & * & O & O \\ \hline O & O & * & O \\ \hline O & O & O & * \end{array}\right).$$

Each connected component of the Dynkin diagram corresponds to a diagonal block in the Cartan matrix, representing the Cartan matrix for that specific component. The set $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ will also be divided into subsets accordingly, ensuring that roots in different subsets are orthogonal to each other.

The possible Dynkin diagrams for semisimple Lie algebras are limited. To identify these potential diagrams, it's helpful to introduce a quadratic form $\mathrm{Q}(x_1, \ldots, x_l)$ defined based on the Dynkin diagram. This quadratic form Q is defined by

$$\mathrm{Q}(x_1, \ldots, x_l) = 2\sum_{i=1}^{l} x_i^2 - \sum_{\substack{i,j=1 \\ i\neq j}}^{l} \sqrt{n_{ij}}\, x_i x_j$$

Let's demonstrate this definition for the cases when $l = 1, 2$.

Dynkin diagram	**Quadratic form**
	$2x_1^2$
	$2x_1^2 + 2x_2^2$
	$2x_1^2 - 2x_1x_2 + 2x_2^2$
	$2x_1^2 - 2\sqrt{2}x_1x_2 + 2x_2^2$
	$2x_1^2 - 2\sqrt{3}x_1x_2 + 2x_2^2$

Proposition 3.12. *The quadratic form* $\mathrm{Q}(x_1, \ldots, x_l)$ *is positive definite.*

Proof. We have, for $i \neq j$,

$$n_{ij} = A_{ij}A_{ji} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \cdot 2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

hence $-\sqrt{n_{ij}} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{|\alpha_i||\alpha_j|}$ since $\langle \alpha_i, \alpha_j \rangle \leq 0$. For $i = j$ we have $\frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i||\alpha_j|} = 2$. Thus the quadratic form may be written

$$\begin{aligned} \mathrm{Q}(x_1, \ldots, x_l) &= \sum_{i,j=1}^{l} \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i| |\alpha_j|} x_i x_j = 2\left\langle \sum_{i=1}^{l} \frac{x_i \alpha_i}{|\alpha_i|}, \sum_{j=1}^{l} \frac{x_j \alpha_j}{|\alpha_j|} \right\rangle \\ &= 2\langle y, y \rangle \quad \text{where } y = \sum_{i=1}^{l} \frac{x_i \alpha_i}{|\alpha_i|}. \end{aligned}$$

So, $\mathrm{Q}(x_1, \ldots, x_l) \geq 0$. Furthermore, if $\mathrm{Q}(x_1, \ldots, x_l) = 0$, then $y = 0$. Since $\alpha_1, \ldots, \alpha_l$ are linearly independent, this implies that $x = 0$ for all i. Therefore, the quadratic form is positive definite. □

The connected parts of the Dynkin diagram for any semisimple Lie algebra must meet the following criteria:

(a) The graph is connected.

(b) Any pair of distinct vertices are joined by $0, 1, 2$ or 3 edges.

(c) The corresponding quadratic form $\mathrm{Q}(x_1, \ldots, x_l)$ is positive definite.

To identify the potential Dynkin diagrams, we will first identify all graphs that meet conditions $(a), (b)$, and (c). Once we have identified these graphs, we will then determine which ones correspond to Dynkin diagrams.

3.2.3 Classification of Dynkin diagrams

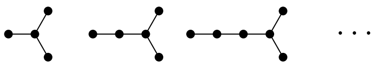
The key result resented in this section is the following.

Theorem 3.6. *The graphs that meet conditions* $(a), (b), (c)$ *as presented in Section* 3.2.2 *are precisely those listed below.*

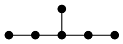
A_n for $n \geq 1$: $\cdots$

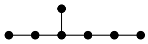
B_n for $n \geq 2$: $\cdots$

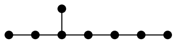
C_n for $n \geq 3$:

D_n for $n \geq 4$: 

where each of the diagrams above has n vertices, or one of the five exceptional diagrams

E_6: 

E_7: 

E_8: 

F_4:

G_2:

The list contains unique elements without repetitions. For instance, C_2 is excluded since it duplicates the B_2 diagram, resulting in isomorphic root systems.

Proposition 3.13. *Let Δ be the Dynkin diagram of a semisimple Lie algebra. Then each connected component of Δ must be one of the graphs*

A_l, $l \geq 1$; B_l, $l \geq 2$; C_l, $l \geq 3$; D_l, $l \geq 4$; E_6; E_7; E_8; F_4; G_2. For the proof and more details we refer to [8, section 6.3].

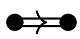
We shall consider later whether all these graphs actually occur as Dynkin diagrams.

3.2.4 Classification of Cartan matrices

We recall that the Dynkin diagram is defined by the Cartan matrix by the property

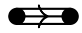
$$n_{ij} = A_{ij}A_{ji} \quad i \neq j.$$

Yet, the Cartan matrix is not always uniquely determined by the Dynkin diagram. Assuming the integers $n_{ij} \in \{0, 1, 2, 3\}$ for all i, j with $i \neq j$ we consider to what extent the A_{ij} are determined. If $n_{ij} = 0$ then we must have $A_{ij} = 0$ and $A_{ji} = 0$, since $A_{ij} = 0$ if and only if $A_{ji} = 0$. If $n_{ij} = 1$ then we must have $A_{ij} = -1$ and $A_{ji} = -1$ since $A_{ij} \in \mathbb{Z}, A_{ji} \in \mathbb{Z}, A_{ij} \leq 0, A_{ji} \leq 0$. However, if $n_{ij} = 2$ there are two possibilities for the factorisation $n_{ij} = A_{ij}A_{ji}$. Either we have $2 = -1 \cdot -2$ or $2 = -2 \cdot -1$. Thus we have either $A_{ij} = -1, A_{ji} = -2$ or $A_{ij} = -2, A_{ji} = -1$. Similarly if $n_{ij} = 3$ we have either $A_{ij} = -1, A_{ji} = -3$ or $A_{ij} = -3, A_{ji} = -1$.
In the connected graphs in Corollary 3.13 the only ones which give rise to such an ambiguity are B_l, $l \geq 2; C_l$, $l \geq 3; F_4$ and G_2. In these graphs we shall place an arrow on the double or triple edges. The direction of the arrow is determined as follows. The arrow points from vertex i to vertex j if and only if $|\alpha_i| > |\alpha_j|$, that is $|A_{ji}| > |A_{ij}|$.
Thus in the situation



i j

we have $|\alpha_i| = \sqrt{2}\,|\alpha_j|\,,\quad A_{ij} = -1,\quad A_{ji} = -2.$ In the situation



$i \quad j$

we have $|\alpha_i| = \sqrt{3}\,|\alpha_j|\,,\quad A_{ij} = -1,\quad A_{ji} = -3.$ The arrow may thus be regarded as an inequality sign on the lengths of the fundamental roots at the vertices.

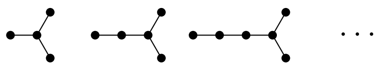
The set of possible connected Dynkin diagrams, including arrows, is shown on the following standard list.

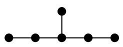
(∗) **Standard list of connected Dynkin diagrams**

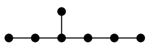
A_n for $n \geq 1$:

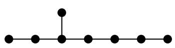
B_n for $n \geq 2$:

C_n for $n \geq 3$:

D_n for $n \geq 4$: 

E_6: 

E_7: 

E_8: 

F_4:

G_2:

The diagrams for types B_2, F_4, and G_2 are symmetric, so arrow direction doesn't matter. All connected components of the Dynkin diagram for any semisimple Lie algebra are on this standard list.
We create a standard list of corresponding Cartan matrices. Two Cartan matrices $(A_{ij})(A'_{ij})$ are considered **equivalent** if they have the same degree l and there is a permutation σ of $1, \cdots, l$ such that

$$A'_{ij} = A_{\sigma(i)\sigma(j)}.$$

Equivalent Cartan matrices come from different labellings of the same Dynkin diagram. For each Dynkin diagram on the standard list (∗), we choose a labelling and obtain a corresponding Cartan matrix which is uniquely determined. These Cartan matrices appear on the following list.

(∗∗) **Standard list of indecomposable Cartan matrices**

$$A_l = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & \cdot & & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{pmatrix} \qquad l \geq 1.$$

$$B_l = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & \cdot & & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -2 & 2 \end{pmatrix} \qquad l \geq 2.$$

$$C_l = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & \cdot & & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -2 \\ & & & & & & & -1 & 2 \end{pmatrix} \qquad l \geq 3.$$

$$D_l = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & \cdot & \cdot & & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & -1 \\ & & & & & & -1 & 2 & 0 \\ & & & & & & -1 & 0 & 2 \end{pmatrix} \qquad l \geq 4.$$

$$E_6 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

$$E_7 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

$$G_2 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

- A Cartan matrix is called **indecomposable** if its Dynkin diagram is connected.
- Any Cartan matrix will determine a set of indecomposable Cartan matrices, unique up to equivalence, whose Dynkin diagrams are the connected components of the Dynkin diagram of the given Cartan matrix.
- If A is the Cartan matrix of any semisimple Lie algebra, each indecomposable component of A will be equivalent to some Cartan matrix from the above standard list.

Proposition 3.14. *If a semisimple Lie algebra L has a connected Dynkin diagram then L is simple.*

Let's explore happens where the Dynkin diagram of L is disconnected.

We first define an action of the Weyl group on H. The Weyl group was introduced as a group of non-singular linear transformations on the real vector space $H^*_{\mathbb{R}}$. The described action can be extended through linearity to yield an action of W on H^* by $\mathbb{C}$-linear transformations. We also define an action of W on H by $h \to w(h)$ where

$$\lambda(w(h)) = \left(w^{-1}(\lambda)\right)(h) \quad \text{for all } h \in H, \lambda \in H^*, w \in W.$$

There is a unique element $w(h) \in H$ satisfying this condition, and

$$w_1\left(w_2(h)\right) = \left(w_1 w_2\right)(h) \quad \text{for all } w_1, w_2 \in W.$$

The actions of W on H^* and H are compatible with the isomorphism $H^* \to H$ given by $\lambda \to h'_\lambda$ where $\lambda(x) = \langle h'_\lambda, x\rangle$ for all $x \in H$. Suppose $w(\lambda) = \mu$ for $\lambda, \mu \in H^*$. Then

$$\begin{aligned}\langle w\left(h'_\lambda\right), x\rangle &= \left\langle h'_\lambda, w^{-1}(x)\right\rangle = \lambda\left(w^{-1}(x)\right) = (w\lambda)x \\ &= \mu(x) = \left\langle h'_\mu, x\right\rangle \qquad \text{for all } x \in H.\end{aligned}$$

Hence $w(\lambda) = \mu$ implies $w\left(h'_\lambda\right) = h'_\mu$.
Since we know that

$$s_\alpha(\lambda) = \lambda - 2\frac{\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle}\alpha \qquad \text{for } \alpha \in \Phi, \lambda \in H^*$$

it follows that

$$s_\alpha(x) = x - 2\frac{\langle h'_\alpha, x\rangle}{\langle h'_\alpha, h'_\alpha\rangle}h'_\alpha \qquad \text{for } x \in H.$$

Proposition 3.15. *Let L be a semisimple Lie algebra whose Dynkin diagram Δ splits into connected components $\Delta_1, \ldots, \Delta_r$. Then we have*

$$L = L_1 \oplus \cdots \oplus L_r$$

a direct sum of Lie algebras, where L_i is a simple Lie algebra with Dynkin diagram Δ_i.

Corollary 3.5. *A semisimple Lie algebra L has a connected Dynkin diagram if and only if L is simple.*

Proof. This follows from Propositions 3.14 and 3.15 □

Chapter 4

The classification theorem

We shall show that both the existence and uniqueness properties hold. In order to do so we shall need some properties of the structure constants of the Lie algebra L. We have obtained a classification of the finite dimensional simple Lie algebras over $\mathbb{C}$. We shall in the present chapter investigate them individually in order to obtain their dimensions and a description of their root systems.

4.1 The existence and uniqueness theorems

Our main reference here is [8].

We have seen that each non-trivial simple Lie algebra L has a Dynkin diagram Δ which appears on the standard list of connected Dynkin diagrams. The following result shows the uniqueness of the Lie algebra associated to a given Dynkin diagram.

Theorem 4.1. *Any two simple Lie algebras with the same Cartan matrix are isomorphic.*

For the proof see [8, Theorem 7.5].

We now turn to the question of the existence of a simple Lie algebra with Cartan matrix on the standard list $(**)$, let

$$L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$$

be the Cartan decomposition. We consider the elements $h_i \in H$ given by

$$h_i = \frac{2h'_{\alpha_i}}{\langle h'_{\alpha_i}, h'_{\alpha_i} \rangle}$$

where $\Pi = \{\alpha_i, \ldots, \alpha_l\}$ is a fundamental system in Φ. We can choose elements $e_i \in L_{\alpha_i}, f_i \in L_{-\alpha_i}$ such that $[e_i f_i] = h_i$.
We shall show that the elements $e_1, \ldots, e_l, h_1, \ldots, h_l, f_1, \ldots, f_l$ generate L.
(Of course this is equivalent to saying that $e_1, \ldots, e_l, f_1 \ldots, f_l$ generate L, but it will be useful to include $h_1, \ldots, h_l$ in the generating set).

Lemma 4.1. *If $\alpha \in \Phi^+$ and $\alpha \notin \Pi$ there exists $\alpha_i \in \Pi$ such that $\alpha - \alpha_i \in \Phi^+$. Thus every positive non-fundamental root is the sum of a fundamental root with a positive root.*

Proof. see [8, Lemma 7.6]. □

Proposition 4.1. *The elements $e_1, \ldots, e_l, h_1, \ldots, h_l, f_1, \ldots, f_l$ generate L.*

Proof. Since $h_1, \dots, h_l$ span H it will be sufficient to show that each L_α for $\alpha \in \Phi^+$ lies in the subalgebra generated by $e_1, \dots, e_l$ and each L_α for $\alpha \in \Phi^-$ lies in the subalgebra generated by $f_1, \dots, f_l$.

Let $\alpha \in \Phi^+$. If $\alpha = \alpha_i$ for some i we have $L_\alpha = \mathbb{C}e_i$. If $\alpha \notin \Pi$ we can write $\alpha = \alpha_i + \beta$ for some $\alpha_i \in \Pi$ and some $\beta \in \Phi^+$ by Lemma 4.1. We then have $[L_{\alpha_i} L_\beta] = L_\alpha$. Thus we may choose $e_\alpha = [e_i, e_\beta]$ for some $e_\beta \neq 0$ in L_β. By repeating this process we obtain

$$e_\alpha = [[e_{i_1} e_{i_2}] \dots e_{i_k}]$$

for some sequence $i_1, \dots, i_k$. Thus each L_α for $\alpha \in \Phi^+$ lies in the subalgebra generated by $e_1, \dots, e_l$. Similarly each L_α for $\alpha \in \Phi^-$ lies in the subalgebra generated by $f_1, \dots, f_l$. □

Proposition 4.2. *The generators $e_1, \dots, e_l, h_1, \dots, h_l, f_1, \dots, f_l$ of L satisfy the following relations:*

(a) $[h_i h_j] = 0$

(b) $[h_i e_j] = A_{ij} e_j$

(c) $[h_i f_j] = -A_{ij} f_j$

(d) $[e_i f_i] = h_i$

(e) $[e_i f_j] = 0$

(f) $[e_i [e_i \dots [e_i e_j]]] = 0 \quad$ *if* $i \neq j$
$\leftarrow 1 - A_{ij} \rightarrow$

(g) $[f_i [f_i \dots [f_i f_j]]] = 0 \quad$ *if* $i \neq j$.
$\leftarrow 1 - A_{ij} \rightarrow$

Note that in relations (f), (g) there are $1 - A_{ij}$ occurrences of e_i, f_i respectively. Since $A_{ij} \leq 0$ for $i \neq j$ this number $1 - A_{ij}$ is a positive integer.

We shall construct a Lie algebra $L(A)$ which will be shown to be a finite dimensional simple Lie algebra with Cartan matrix A.

Let A be an $l \times l$ matrix. Consider $\mathfrak{F}$ as the free associative algebra over $\mathbb{C}$, generated by $3l$ elements: $e_1, \dots, e_l, h_1, \dots, h_l, f_1, \dots, f_l$. The set of all monomials formed from these generators forms a basis for $\mathfrak{F}$. Let $[\mathfrak{F}]$ be the Lie algebra obtained from $\mathfrak{F}$ by redefining the multiplication in the usual way and let $\mathfrak{L}$ be the subalgebra of $[\mathfrak{F}]$ generated by the elements $e_1, \dots, e_l, h_1, \dots, h_l, f_1, \dots, f_l$.
Let J be the ideal of $\mathfrak{L}$ generated by the elements

$$\begin{aligned}
&[h_i h_j] \\
&[h_i e_j] - A_{ij} e_j \\
&[h_i f_j] + A_{ij} f_j \\
&[e_i f_i] - h_i \\
&[e_i f_j] \quad \text{for } i \neq j \\
&[e_i [e_i \dots [e_i e_j]]] \quad \text{for } i \neq j \\
&[f_i [f_i \dots [f_i f_j]]] \quad \text{for } i \neq j,
\end{aligned}$$

in the last two elements, the occurrence count of e_i and f_i respectively is $1 - A_{ij}$.

We define $L(A) = \mathfrak{L}/J$, we're now shifting our focus to investigating the Lie algebra $L(A)$. Our goal is to that it is a finite-dimensional simple Lie algebra associated to the Cartan matrix A .

Theorem 4.2. *$L(A)$ is a simple Lie algebra with Cartan matrix A.*

We shall prove that the Cartan matrix of $L(A)$ is precisely A. This implies that $L(A)$ is simple Lie algebra. (For the proof that is semisimple Lie algebra, see [8, Proposition 7.33]).

Proof. Now, we have $L(A) = H \oplus \sum \alpha \in \Phi$ which is the Cartan decomposition of $L(A)$ with respect to H.(by [8, Proposition 7.34]). Thus Φ is the root system of $L(A)$. The Cartan matrix $A' = (A'_{ij})$ of $L(A)$ is determined by the condition

$$s_i(\alpha_j) = \alpha_j - A'_{ij}\alpha_i.$$

However, we have

$$s_i(h_j) = h_j - A_{ji}h_i.$$

Based on the given facts $(s_i\alpha_j)h_k = \alpha_j(s_ih_k)$ and $\alpha_j(h_k) = A_{kj}$, we can conclude that

$$s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i.$$

Thus $A' = A$ and the cartan matrix of $L(A)$ is A.

Given that the Dynkin diagram of A which is assumed connected,hence according to Proposition 3.14, $L(A)$ must be a simple Lie algebra. □

So, we've created a finite-dimensional simple Lie algebra $L(A)$ with Cartan matrix A for each Cartan matrix listed in the standard list $(**)$.

Theorem 4.3. *The finite dimensional non-trivial simple Lie algebras over $\mathbb{C}$ are*

$$\begin{array}{ll} A_l & l \geq 1 \\ B_l & l \geq 2 \\ C_l & l \geq 3 \\ D_l & l \geq 4 \\ E_6, & E_7, E_8 \\ F_4 & \\ G_2 & \end{array}$$

these Lie algebras are pairwise non-isomorphic.

Proof. Each Cartan matrix listed in the standard list $(**)$ corresponds to a finite-dimensional simple Lie algebra, as established in Theorem 4.1, which uniquely determines the algebra up to isomorphism. Simple Lie algebras with distinct Cartan matrices cannot be isomorphic because, the Cartan matrix in the standard list uniquely identifies the associated Lie algebra. □

4.2 The simple Lie algebras

Having obtained a classification of the finite dimensional simple Lie algebras over $\mathbb{C}$ we shall in the present section investigate them individually in order to obtain their dimensions and a description of their root systems. In the case of Lie algebras of type A_l, B_l, C_l or D_l we shall also give a description in terms of Lie algebras of matrices. Our main references here are [10, 11] and [8].

4.2.1 Lie algebras of type A_l

The **special linear algebra** $\mathfrak{sl}_{l+1}(\mathbb{C})$ be the set of all $(l+1)\times(l+1)$ matrices of trace 0 equipped with the multiplication given by $[AB] = AB - BA$.

We say that the root system of $\mathfrak{sl}_{l+1}(\mathbb{C})$ has type A_l and the Dynkin diagram is

A_l

This diagram is connected, so L is simple.

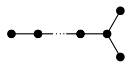
Theorem 4.4. *The simple Lie algebras of type A_l have dimension $l(l+2)$.*

4.2.2 Lie algebras of type D_l

Let $\mathfrak{so}(2l,\mathbb{C})$ be the set of all $2l\times 2l$ matrices X satisfying $X^tM + MX = 0$ where

$$M = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}.$$

The Dynkin diagram is

D_l 

as this diagram is connected, the Lie algebra is simple. When $l = 3$, the Dynkin diagram is the same as A_3, the root system of $\mathfrak{sl}(4,\mathbb{C})$, so we have that $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$. For $l \geq 4$, the root system of $\mathfrak{so}(2l,\mathbb{C})$ is said to have type D_l.

Theorem 4.5. *The simple Lie algebra of type D_l have dimension $l(2l-1)$.*

4.2.3 Lie algebras of type B_l

Lie algebra B_l is given by the **orthogonal algebra** $\mathfrak{so}(2l+1,\mathbb{C})$ be the set of all $(2l+1)\times(2l+1)$ matrices X satisfying $X^tM + MX = 0$ where

$$M = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & & I_l \\ \vdots & & & \\ 0 & I_l & & 0 \end{pmatrix}.$$

This is a simple Lie algebras of type B_l when $l \geq 2$ and the Dynkin diagram is

B_l

Theorem 4.6. *The simple Lie algebra of type B_l have dimension $l(2l+1)$.*

4.2.4 Lie algebras of type C_l

The **symplectic algebra** $\mathfrak{sp}(2l,\mathbb{C})$ be the set of all $2l\times 2l$ matrices X satisfying $X^tM+MX = 0$ where

$$M = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

The Dynkin diagram is

C_l

which is connected, so L is simple. The root systems of $\mathfrak{sp}(2l,\mathbb{C})$ is said to have type C_l when $l \geq 3$

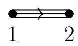
Theorem 4.7. *The simple Lie algebra of type C_l have dimension $l(2l+1)$.*

The Lie algebras of type A_l, B_l, C_l or D_l are called the **simple Lie algebras of classical type**. The remaining simple Lie algebras E_6, E_7, E_8, F_4, G_2 are called **the exceptional simple Lie algebras**.

We now determine the dimensions and root systems of the exceptional Lie algebras.

4.2.5 Lie algebras of type G_2

The Dynkin diagram of type G_2 is



and the corresponding Cartan matrix is

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Let $\{\alpha_1, \alpha_2\}$ be the fundamental roots in a root system of type G_2. Then we have

$$\begin{array}{ll} s_1(\alpha_1) = -\alpha_1 & s_2(\alpha_1) = \alpha_1 + 3\alpha_2 \\ s_1(\alpha_2) = \alpha_1 + \alpha_2 & s_2(\alpha_2) = -\alpha_2 \end{array}$$

and $W = \langle s_1, s_2 \rangle$. Thus each root in Φ is obtained from α_1 or α_2 by applying s_1, s_2 alternately. Now we have

$$\begin{aligned} &\alpha_1 \underset{s_1}{\rightarrow} -\alpha_1 \underset{s_2}{\rightarrow} -\alpha_1 - 3\alpha_2 \underset{s_1}{\rightarrow} -2\alpha_1 - 3\alpha_2 \\ &\alpha_1 \underset{s_2}{\rightarrow} \alpha_1 + 3\alpha_2 \underset{s_1}{\rightarrow} 2\alpha_1 + 3\alpha_2 \\ &\alpha_2 \underset{s_1}{\rightarrow} \alpha_1 + \alpha_2 \underset{s_2}{\rightarrow} \alpha_1 + 2\alpha_2 \\ &\alpha_2 \underset{s_2}{\rightarrow} -\alpha_2 \underset{s_1}{\rightarrow} -\alpha_1 - \alpha_2 \underset{s_2}{\rightarrow} -\alpha_1 - 2\alpha_2 \end{aligned}$$

and

$$\begin{aligned} s_2(-2\alpha_1 - 3\alpha_2) &= -2\alpha_1 - 3\alpha_2 \\ s_2(2\alpha_1 + 3\alpha_2) &= 2\alpha_1 + 3\alpha_2 \\ s_1(\alpha_1 + 2\alpha_2) &= \alpha_1 + 2\alpha_2 \\ s_1(-\alpha_1 - 2\alpha_2) &= -\alpha_1 - 2\alpha_2. \end{aligned}$$

Thus all the vectors in the above sequences are roots, and we do not obtain new vectors by continuing the sequences further. Hence

$$\begin{aligned} \Phi = \{&\alpha_1, \quad \alpha_2, \quad \alpha_1 + \alpha_2, \quad \alpha_1 + 2\alpha_2, \quad \alpha_1 + 3\alpha_2, \quad 2\alpha_1 + 3\alpha_2, \quad -\alpha_1, \\ &-\alpha_2, -\alpha_1 - \alpha_2, \quad -\alpha_1 - 2\alpha_2, \quad -\alpha_1 - 3\alpha_2, \quad -2\alpha_1 - 3\alpha_2\}. \end{aligned}$$

Thus we have $|\Phi| = 12$ and $\dim L = 14$. Hence we have proved

Theorem 4.8. *The simple Lie algebra of type G_2 has dimension* 14 .

Figures 4.1, 4.2 and 4.3 compare the simple root systems of types A_2, B_2 and G_2.

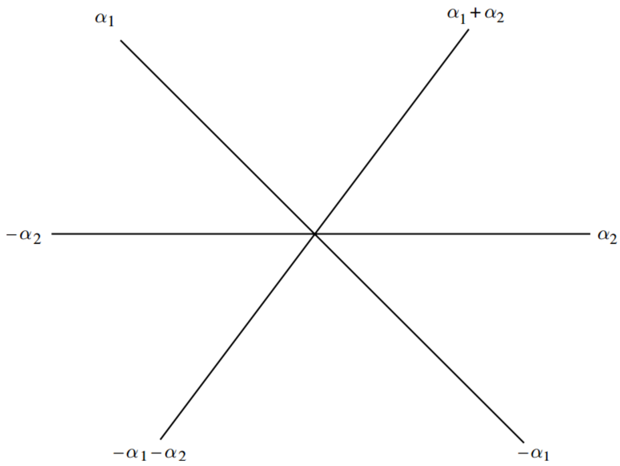


Figure 4.1: Simple root system of type A_2

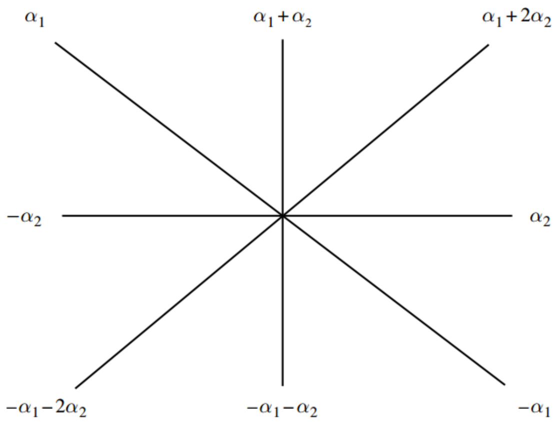


Figure 4.2: Simple root system of type B_2

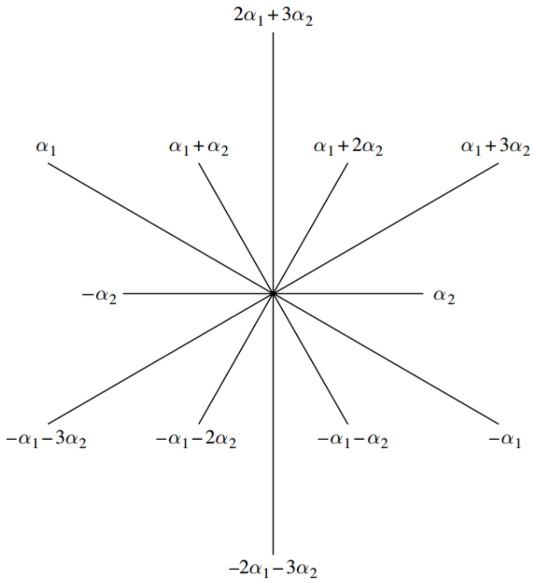
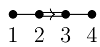


Figure 4.3: Simple root system of type G_2

4.2.6 Lie algebras of type F_4

The Dynkin diagram of type F_4 is



and the corresponding Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let V be a real vector space with $\dim V = 4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ be a basis of V. Let the scalar product $\{,\}$ on V be defined by $\{\beta_i, \beta_j\} = \delta_{ij}$. We define $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V$ by

$$\alpha_1 = \beta_1 - \beta_2 \quad \alpha_2 = \beta_2 - \beta_3 \quad \alpha_3 = \beta_3 \quad \alpha_4 = \frac{1}{2}\left(-\beta_1 - \beta_2 - \beta_3 + \beta_4\right).$$

Then we have

$$\begin{aligned} \{\alpha_1, \alpha_1\} &= \{\alpha_2, \alpha_2\} = 2 \\ \{\alpha_3, \alpha_3\} &= \{\alpha_4, \alpha_4\} = 1 \\ \{\alpha_1, \alpha_2\} &= \{\alpha_2, \alpha_3\} = -1 \\ \{\alpha_3, \alpha_4\} &= -\frac{1}{2} \\ \{\alpha_i, \alpha_j\} &= 0 \quad \text{ if } |i-j| > 1. \end{aligned}$$

It follows that

$$2\frac{\{\alpha_i, \alpha_j\}}{\{\alpha_i, \alpha_i\}} = A_{ij} \quad \text{ for all } i, j.$$

Thus the scalar product $\{,\}$ is a non-zero multiple of the Killing form. We consider the action of the corresponding fundamental reflections s_1, s_2, s_3, s_4.
We have

$$\begin{array}{llll} s_1(\beta_1) = \beta_2, & s_1(\beta_2) = \beta_1, & s_1(\beta_3) = \beta_3, & s_1(\beta_4) = \beta_4 \\ s_2(\beta_1) = \beta_1, & s_2(\beta_2) = \beta_3, & s_2(\beta_3) = \beta_2, & s_2(\beta_4) = \beta_4 \\ s_3(\beta_1) = \beta_1, & s_3(\beta_2) = \beta_2, & s_3(\beta_3) = -\beta_3, & s_3(\beta_4) = \beta_4. \end{array}$$

We consider the subgroup $\langle s_1, s_2, s_3 \rangle$ of the Weyl group W generated by s_1, s_2, s_3. Elements in this subgroup all fix β_4 but act on $\beta_1, \beta_2, \beta_3$ by means of a permutation combined with sign changes. Thus $w(\beta_i) = \varepsilon_i \beta_{\sigma(i)}$ for $i = 1, 2, 3$.

Moreover each permutation σ and each choice of signs ε_i arise in this way. Applying the elements of this subgroup of W to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ we see that the vectors

$$\begin{aligned}
&\pm \beta_i \quad 1 \leq i \leq 3 \\
&\pm \beta_i \pm \beta_j \quad i \neq j \quad 1 \leq i, j \leq 3 \\
&\frac{1}{2}(\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)
\end{aligned}$$

all lie in Φ. We next consider the action of s_4. We have

$$\begin{aligned}
s_4(\beta_1) &= \frac{1}{2}(\beta_1 - \beta_2 - \beta_3 + \beta_4) \\
s_4(\beta_2) &= \frac{1}{2}(-\beta_1 + \beta_2 - \beta_3 + \beta_4) \\
s_4(\beta_3) &= \frac{1}{2}(-\beta_1 - \beta_2 + \beta_3 + \beta_4) \\
s_4(\beta_4) &= \frac{1}{2}(\beta_1 + \beta_2 + \beta_3 + \beta_4).
\end{aligned}$$

Since $s_4^2 = 1$ we have $s_4\left(\frac{1}{2}(\beta_1 + \beta_2 + \beta_3 + \beta_4)\right) = \beta_4$. Hence $\beta_4 \in \Phi$. We also have $s_4(\beta_1 + \beta_2) = -\beta_3 + \beta_4$. Hence $-\beta_3 + \beta_4 \in \Phi$. Thus, applying further elements of the subgroup $\langle s_1, s_2, s_3 \rangle$ we see that the vectors

$$\begin{aligned}
&\pm \beta_i \quad 1 \leq i \leq 4 \\
&\pm \beta_i \pm \beta_j \quad i \neq j \quad 1 \leq i, j \leq 4 \\
&\frac{1}{2}(\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)
\end{aligned}$$

all lie in Φ, where the choice of signs is arbitrary.

We show this set of vectors is the whole of Φ. To do so it is sufficient to show that the set is invariant under s_1, s_2, s_3, s_4. The set is clearly invariant under s_1, s_2, s_3 because of the simple action of these reflections on $\beta_1, \beta_2, \beta_3, \beta_4$ described above. Thus it is sufficient to show the set is invariant under s_4. Now the action of s_4 given above shows that

$$s_4(\pm\beta_i) = \frac{1}{2}(\varepsilon_1\beta_1 + \varepsilon_2\beta_2 + \varepsilon_3\beta_3 + \varepsilon_4\beta_4)$$

where $\varepsilon_i \in \{1, -1\}$ and $\prod \varepsilon_i = 1$. Thus s_4 transforms vectors $\pm\beta_i$ into the given set, giving as images vectors $\frac{1}{2}\sum \varepsilon_i\beta_i$ with $\prod \varepsilon_i = 1$. Since there are eight such vectors they all appear as vectors $s_4(\pm\beta_i)$. Since $s_4^2 = 1$ we deduce that all vectors $\frac{1}{2}\sum \varepsilon_i\beta_i$ with $\prod \varepsilon_i = 1$ are transformed by s_4 into the given set.

The formulae for $s_4(\beta_i)$ also show that, for all $i \neq j$, $s_4(\pm\beta_i \pm \beta_j)$ has form $\pm\beta_k \pm \beta_l$ for certain $k \neq l$. Thus s_4 transforms vectors $\pm\beta_i \pm \beta_j, i \neq j$, into the given set.

It remains to show that s_4 transforms all vectors $\frac{1}{2}\sum \varepsilon_i\beta_i$ with $\prod \varepsilon_i =$ -1 into the given set. We may clearly assume $\varepsilon_4 = 1$. There are four such vectors. One of them is α_4 and we have

$s_4(\alpha_4) = -\alpha_4$. The other three are all orthogonal to α_4 and so are transformed into themselves by s_4.

Thus the given set of vectors is invariant under s_1, s_2, s_3, s_4 so is the whole of Φ. Thus we have

$$\begin{aligned}\Phi = \{&\pm\beta_i \quad 1 \le i \le 4 \\ &\pm \beta_i \pm \beta_j \quad i \neq j \quad 1 \le i, j \le 4 \\ &\frac{1}{2}(\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)\Big\}.\end{aligned}$$

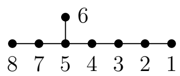
In particular we have $|\Phi| = 48$, hence $\dim L = 52$. Thus we have

Theorem 4.9. *The simple Lie algebra of type F_4 has dimension* 52.

We observe that the roots of F_4 are of two different lengths. There are 24 short roots and 24 long roots. The short roots are $\pm\beta_i$ and $\frac{1}{2}(\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)$. The long roots are $\pm\beta_i \pm \beta_j$.

4.2.7 Lie algebras of types E_6, E_7, E_8

We now consider the simple Lie algebra of type E_8. Its Dynkin diagram is



Let V be a real vector space with $\dim V = 8$ and with basis $\beta_i \quad i = 1, \ldots, 8$. Let the scalar product $\{,\}$ on V be defined by $\{\beta_i, \beta_j\} = \delta_{ij}$. We wish to find a fundamental system of roots of type E_8 in V. We note that if the vertex 8 is removed from the Dynkin diagram we obtain a Dynkin diagram of type D_7. This indicates how the first seven vectors in the fundamental system should be chosen. The last one is chosen to be linearly independent of the others and to satisfy the appropriate conditions relating to the scalar product. Thus we define $\alpha_1, \ldots, \alpha_8 \in V$ by:

$$\begin{aligned}\alpha_i &= \beta_i - \beta_{i+1} \quad 1 \le i \le 6 \\ \alpha_7 &= \beta_6 + \beta_7 \\ \alpha_8 &= -\tfrac{1}{2}\textstyle\sum_{i=1}^{8} \beta_i.\end{aligned}$$

Then we have

$$\begin{aligned}&\{\alpha_i, \alpha_i\} = 2 \quad \text{for } 1 \le i \le 8 \\ &\{\alpha_i, \alpha_{i+1}\} = -1 \quad \text{for } 1 \le i \le 5 \\ &\{\alpha_5, \alpha_7\} = -1 \\ &\{\alpha_7, \alpha_8\} = -1 \\ &\{\alpha_i, \alpha_j\} = 0 \quad \text{for all other pairs } i, j.\end{aligned}$$

It follows that

$$2\frac{\{\alpha_i, \alpha_j\}}{\{\alpha_i, \alpha_i\}} = A_{ij}$$

where $A = (A_{ij})$ is the Cartan matrix of type E_8 on the standard list.
In order to obtain the remaining roots we consider the action of the fundamental reflections $s_1, \ldots, s_8$. We have

$$\begin{aligned}&s_i(\beta_i) = \beta_{i+1} \\ &s_i(\beta_{i+1}) = \beta_i \\ &s_i(\beta_j) = \beta_j \quad \text{for } j \neq i, i+1\end{aligned}$$

when $1 \leq i \leq 6$. Thus the subgroup of the Weyl group W generated by $s_1, \ldots, s_6$ will give all permutations of $\beta_1, \ldots, \beta_7$ and will fix β_8. The fundamental reflection s_7 acts by:

$$\begin{aligned} s_7(\beta_6) &= -\beta_7 \\ s_7(\beta_7) &= -\beta_6 \\ s_7(\beta_i) &= \beta_i \quad i \neq 6, 7. \end{aligned}$$

Thus the subgroup of W generated by $s_1, \ldots, s_7$ will act on $\beta_1, \ldots, \beta_7$ by permutations and sign changes, and will fix β_8. Moreover the number of sign changes will be even, and any permutation of $\beta_1, \ldots, \beta_7$ combined with any even number of sign changes will arise in this way. It is then clear that the vectors

$$\begin{aligned} &\pm\beta_i \pm \beta_j \quad 1 \leq i, j \leq 7 \quad i \neq j \\ &\frac{1}{2}\left(\sum_{i=1}^{8} \varepsilon_i \beta_i\right) \quad \varepsilon_i = \pm 1, \quad \prod \varepsilon_i = 1 \end{aligned}$$

are all in the root system Φ. We also have

$$s_8(\beta_i) = \beta_i - 2\frac{\{\alpha_8, \beta_i\}}{\{\alpha_8, \alpha_8\}}\alpha_8 = \beta_i + \frac{1}{2}\alpha_8$$

for $1 \leq i \leq 8$. Thus

$$s_8(\beta_7 + \beta_8) = \frac{1}{2}(-\beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_5 - \beta_6 + \beta_7 + \beta_8) \in \Phi.$$

Since $s_8^2 = 1$ it follows that $\beta_7 + \beta_8 \in \Phi$. We then see that $\pm\beta_i \pm \beta_8 \in \Phi$ for all i with $1 \leq i \leq 7$. Thus the set of vectors

$$\begin{aligned} &\pm\beta_i \pm \beta_j \qquad 1 \leq i, j \leq 8 \quad i \neq j \\ &\tfrac{1}{2}\left(\textstyle\sum_{i=1}^{8} \varepsilon_i \beta_i\right) \qquad \varepsilon_i = \pm 1, \quad \Pi\varepsilon_i = 1 \end{aligned}$$

lies in Φ. We shall show this is the full root system Φ. In order to do so we must verify that this set is invariant under $s_1, \ldots, s_8$. It is clearly invariant under $s_1, \ldots, s_7$ since these fix β_8 and act by permutations together with an even number of sign changes on $\beta_1, \ldots, \beta_7$. Thus it is sufficient to verify that this set is invariant under s_8. Now we have

$$s_8(\beta_i - \beta_j) = \beta_i - \beta_j \quad \text{for all } i \neq j.$$

Thus the set of vectors of form $\beta_i - \beta_j, i \neq j$, is invariant under s_8. Also

$$s_8(\beta_i + \beta_j) = \beta_i + \beta_j + \alpha_8 \quad \text{for all } i \neq j.$$

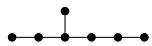
Thus s_8 transforms vectors of form $\beta_i + \beta_j, i \neq j$, into vectors $\frac{1}{2}\left(\sum \varepsilon_i \beta_i\right)$ with two ε_i equal to 1 and six equal to -1 . Moreover all vectors $\frac{1}{2}\left(\sum \varepsilon_i \beta_i\right)$ with this property arise in this way. Similarly such vectors with six ε_i equal to 1 and two equal to -1 have the form $s_8(-\beta_i - \beta_j)$. Thus vectors of form $\beta_i + \beta_j$ or $-\beta_i - \beta_j$ with $i \neq j$ are transformed by s_8 into the given set, and so are vectors $\frac{1}{2}\sum \varepsilon_i \beta_i$ of type $(2, 6)$ or $(6, 2)$. The vectors of this form of type $(0, 8)$ or $(8, 0)$ are α_8 and $-\alpha_8$, which are transformed into one another by s_8. It remains to show that s_8 transforms vectors $\frac{1}{2}\left(\sum \varepsilon_i \beta_i\right)$ of type $(4, 4)$ into the given set. However, since such vectors have four positive signs and four negative signs they are orthogonal to α_8, hence s_8 transforms each such vector into itself. Thus the given set of vectors is invariant under $s_1, \ldots, s_8$ so is the full root system Φ.
There are $4 \cdot \binom{8}{2} = 112$ vectors of form $\pm\beta_i \pm \beta_j$ with $i \neq j$ and $2^7 = 128$ vectors of form $\frac{1}{2}\sum \varepsilon_i \beta_i$ with $\varepsilon_i = \pm 1$ and $\prod \varepsilon_i = 1$. Thus the total number of roots is

$$|\Phi| = 112 + 128 = 240.$$

Finally we have $\dim L = 8 + |\Phi| = 248$. Thus we have proved

Theorem 4.10. *The simple Lie algebra of type E_8 has dimension* 248 .

We now turn to the simple Lie algebra of type E_7. Its Dynkin diagram is



Thus the vectors $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ considered above form a fundamental root system of type E_7. In order to obtain the full root system we must transform these vectors repeatedly by $s_2, \dots, s_8$ until no new vectors are obtained. Now the vectors $\alpha_2, \dots, \alpha_8$ are all orthogonal to $\beta_1 - \beta_8$. Thus all their transforms by $s_2, \dots, s_8$ will also be orthogonal to $\beta_1 - \beta_8$. These transforms are contained in the set of roots of E_8 obtained above.
Now the roots of E_8 orthogonal to $\beta_1 - \beta_8$ are:

$$\begin{aligned}
&\pm \beta_i \pm \beta_j, \quad 2 \le i, j \le 7, \quad i \ne j \\
&\pm (\beta_1 + \beta_8) \\
&\frac{1}{2} \sum \varepsilon_i \beta_i, \quad \varepsilon_i = \pm 1, \quad \prod \varepsilon_i = 1, \quad \varepsilon_1 = \varepsilon_8.
\end{aligned}$$

Thus the required root system of E_7 is contained in this set. We shall show it is the whole of this set.

We first consider the action of the subgroup of the Weyl group of E_7 generated by $s_2, s_3, s_4, s_5, s_6, s_7$. Elements of this subgroup fix β_1 and β_8 and act on $\beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7$ by permutations combined with sign changes with an even number of negative signs. By applying elements of this subgroup to $\alpha_2, \dots, \alpha_8$ we see that the vectors

$$\begin{aligned}
&\pm \beta_i \pm \beta_j, \quad 2 \le i, j \le 7, \quad i \ne j \\
&\frac{1}{2} \sum \varepsilon_i \beta_i, \quad \varepsilon_i = \pm 1, \quad \prod \varepsilon_i = 1, \quad \varepsilon_1 = \varepsilon_8
\end{aligned}$$

are all roots of E_7. It remains to show that $\pm(\beta_1 + \beta_8)$ are also roots of E_7.
However,
$s_8(\beta_1 + \beta_8) = \beta_1 + \beta_8 + \alpha_8 = \frac{1}{2}(\beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_5 - \beta_6 - \beta_7 + \beta_8)$
is a root of E_7, thus so is $\beta_1 + \beta_8$ and $-\beta_1 - \beta_8$.

There are $4\binom{6}{2} = 60$ roots of form $\pm\beta_i \pm \beta_j, \quad 2 \le i, j \le 7, i \ne j$ and $2^6 = 64$ roots of form $\frac{1}{2}\sum \varepsilon_i \beta_i$ with $\varepsilon_i = \pm 1, \prod \varepsilon_i = 1$ and $\varepsilon_1 = \varepsilon_8$. Thus the number of roots of E_7 is given by

$$|\Phi| = 60 + 2 + 64 = 126.$$

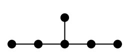
Also we have

$$\dim L = 7 + |\Phi| = 133.$$

Thus we have shown:

Theorem 4.11. *The simple Lie algebra of type E_7 has dimension* 133.

Finally we consider the simple Lie algebra of type E_6. Its Dynkin diagram is



Thus the vectors $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ considered above form a fundamental root system of type E_6. In order to obtain the full root system of E_6 we must transform these vectors successively by the fundamental reflections $s_3, s_4, s_5, s_6, s_7, s_8$.

Now the vectors $\alpha_3, \dots, \alpha_8$ are all orthogonal to both $\beta_1 - \beta_8$ and $\beta_2 - \beta_8$. Thus the full root system of E_6 is orthogonal to $\beta_1 - \beta_8$ and $\beta_2 - \beta_8$.
Now the roots of E_8 orthogonal to both $\beta_1 - \beta_8$ and $\beta_2 - \beta_8$ are:

$$\pm\beta_i \pm \beta_j \quad 3 \le i, j \le 7, \quad i \ne j$$
$$\frac{1}{2}\left(\sum_{i=1}^{8} \varepsilon_i \beta_i\right) \qquad \varepsilon_i = \pm 1, \quad \prod \varepsilon_i = 1, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_8.$$

Thus the required root system of E_6 is contained in this set. We shall show it is equal to this set of vectors.
Consider the action of the subgroup of the Weyl group of type E_6 generated by s_3, s_4, s_5, s_6, s_7. Elements of this subgroup fix $\beta_1, \beta_2, \beta_8$ and act on $\beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$ by permutations combined with sign changes with an even number of negative signs. By applying elements of this subgroup to $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ we can obtain all vectors of form $\pm\beta_i \pm \beta_j$ with

Table 4.1: The simple Lie algebras

L		$\dim H$	$\lvert\Phi\rvert$	$\dim L$
A_l	$l \ge 1$	l	$l(l+1)$	$l(l+2)$
B_l	$l \ge 2$	l	$2l^2$	$l(2l+1)$
C_l	$l \ge 3$	l	$2l^2$	$l(2l+1)$
D_l	$l \ge 4$	l	$2l(l-1)$	$l(2l-1)$
E_6		6	72	78
E_7		7	126	133
E_8		8	240	248
F_4		4	48	52
G_2		2	12	14

$3 \le i, j \le 7, i \ne j$, and (up to sign) all vectors of form $\frac{1}{2}\sum \varepsilon_i e_i$ with $\varepsilon_i = \pm 1$, $\prod \varepsilon_i = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_8$. Hence the vectors in the above set are all roots of E_6. There are $\binom{5}{2} \cdot 4 = 40$ vectors of type $\pm\beta_i \pm \beta_j$ with $3 \le i, j \le 7, i \ne j$, and $2^5 = 32$ vectors of type $\frac{1}{2}\sum \varepsilon_i e_i$ with $\varepsilon_i = \pm 1, \prod \varepsilon_i = 1$ and $\varepsilon_1 = \varepsilon_i = \varepsilon_8$. Thus the total number of roots is

$$|\Phi| = 40 + 32 = 72$$

and we have

$$\dim L = 6 + |\Phi| = 78.$$

Thus:

Theorem 4.12. *The simple Lie algebra of type E_6 has dimension* 78.

We have now determined the dimensions of all the simple Lie algebras. We summarise the information we have obtained in Table 4.1. In this table L is a simple Lie algebra, H is a Cartan subalgebra and Φ the system of roots of L with respect to H.

Appendix

Toward Kac-Moody Lie Algebras

In the pursuit of understanding the broader landscape of Lie algebras, this appendix embarks on an exploration of Kac-Moody Lie algebras. Originating from the seminal work of Victor Kac and Robert Moody in the 1960s, these algebras have garnered significant attention due to their rich structure and diverse applications in various branches of mathematics and theoretical physics.

A Kac–Moody algebra is a Lie algebra, usually infinite-dimensional, that can be defined by generators and relations through a generalized Cartan matrix, these matrices are defined by relaxing only the condition on the rank, i.e. generalized Cartan matrix need not to have nonzero determinant.

Kac-Moody algebras form a generalization of finite-dimensional semisimple Lie algebras that we have pesented so for, and many properties related to the structure of a Lie algebra such as its root system, irreducible representations, and connection to flag manifolds have natural analogues in the Kac–Moody setting.

While finite-dimensional semisimple Lie algebras provide a foundational framework for understanding the algebraic structures underlying symmetries in mathematical and physical systems, Kac-Moody Lie algebras offer a broader perspective by incorporating an infinite-dimensional setting.

One of the defining features of Kac-Moody Lie algebras is their generalized Cartan matrix, which encapsulates essential information about the algebra's structure and representation theory. A class of Kac–Moody algebras called affine Lie algebras is of particular importance in mathematics and theoretical physics, especially two-dimensional conformal field theory and the theory of exactly solvable models.

Moreover, Kac-Moody Lie algebras exhibit remarkable symmetry properties and possess an intricate interplay between algebraic and geometric structures. This interplay has profound implications for diverse areas of mathematics, including algebraic geometry, representation theory, and mathematical physics. Notably, Kac-Moody algebras have found applications in conformal field theory, string theory, and integrable systems, illuminating deep connections between seemingly disparate mathematical concepts.

Despite their complexity, efforts toward understanding Kac-Moody Lie algebras have led to significant progress in recent decades. The classification of finite-dimensional semisimple Lie algebras served as a crucial stepping stone in this endeavor, providing valuable insights and techniques for studying their infinite-dimensional counterparts.

In conclusion, Kac-Moody algebras is natural generalization of finite dimensional Lie algebras that we are looking forward to investegate in the upcoming projects. While the journey toward a comprehensive understanding of these algebras remains ongoing, their profound impact on mathematics and theoretical physics underscores the importance of continued research and exploration in this fascinating area of study.

Conclusion

In this thesis, we have undertaken a detailed investigation into the classification of semisimple Lie algebras, elucidating their intricate structure and foundational properties. By employing root systems and Dynkin diagrams, we provided a clear and systematic framework for understanding the classification process. The Cartan-Killing classification theorem emerged as a central result, enabling us to enumerate all simple Lie algebras over the complex numbers comprehensively.

The methodologies and results presented in this thesis not only reinforce the existing body of knowledge but also offer a solid foundation for future research. Potential areas for further investigation include, as we have presented in the appendix, the theory of Kac-Moody algebras.

In conclusion, this work contributes significantly to the understanding and classification of semisimple Lie algebras. Through rigorous analysis and comprehensive classification, we have deepened our appreciation of the elegant and powerful structure of semisimple Lie algebras.

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Abstract

In this thesis, we present a comprehensive study on the classification of semisimple Lie algebras, a fundamental area in the field of algebraic structures. Semisimple Lie algebras, characterized by their rich structure and profound applications in both mathematics and theoretical physics, are investigated through a systematic approach. The research begins with an in-depth exploration of the basic properties and defining characteristics of Lie algebras, followed by an examination of their representation theory. Central to the classification process is the use of root systems and Dynkin diagrams, which serve as powerful tools for understanding the underlying structure of semisimple Lie algebras. The Cartan-Killing classification theorem is a pivotal result in this study, providing a complete list of simple Lie algebras over the complex numbers.

الملخص

في هذه المذكرة، نقدم دراسة شاملة عن تصنيف جبر لي شبه البسيط، وهو مجال أساسي في مجال الهياكل الجبرية. تم دراسة جبر لي شبه البسيط، الذي يتميز ببنيته الغنية وتطبيقاته العميقة في كل من الرياضيات والفيزياء النظرية، من خلال دراسة اسلوب منهجي. يبدأ البحث باستكشاف متعمق للخصائص الأساسية والخصائص المحددة لجبر لي، يليه فحص نظرية تمثيلها. من الأمور الأساسية في عملية التصنيف استخدام أنظمة الجذر ومخططات Dynkin ، والتي تعمل كأدوات قوية لفهم البنية الأساسية لجبر لي شبه البسيط. تعد نظرية تصنيف كارتان-كيلنج نتيجة محورية في هذه الدراسة، حيث توفر قائمة كاملة من جبر لي البسيط على الأعداد المركبة.