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Contributions to the study of various boundary value problems arising in contact mechanics

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By

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Safa

Contents

ملـخّص

تهدف هذه الأطروحة إلى الدراسة التباينية لبعض مسائل الاتصال مع أو بدون احتكاك بين جسم قابل للتشوه وقاعدة. تحت افتراض التحولات الصغيرة، نقوم بتحليل العمليات الديناميكية للمواد اللزجة البلاستيكية والمواد اللزجة المرنة الحرارية. النتائج التي تم الحصول عليها تتعلق بوجود ووحدانية حل ضعيف للمسائل المدروسة. تنقسم المذكرة إلى ثلاثة أجزاء. الجزء الأول مخصص للأدوات الرياضية اللازمة في الرسالة بالإضافة إلى نماذج الاتصال الميكانيكية المختلفة التي تمت دراستها. الجزء الثاني مخصص لدراسة مشاكل التلامس الحراري اللزج مع الاحتكاك والتلف وانتشار التآكل. الجزء الثالث مخصص لتحليل مشاكل اللزوجة البلاستيكية الخالية من الاحتكاك مع الالتصاق.

الكلمات الرئيسية:

لزوجة بلاستيكية، لزوجة مرنة حرارية، انتشار التآكل، التصاق، تلف

Résumé

Cette thèse est destinée à l'étude variationnelle de quelques problèmes de contact avec ou sans frottement entre un corps déformable et une fondation. Sous l'hypothèse des petites transformations, nous analysons des processus dynamique pour des matériaux viscoplastique et thermo- viscoélastiques. Les résultats obtenus concernent l'existence et l'unicité d'une solution faible pour les problèmes étudiés. Le mémoire est comporté en trois parties. La première partie est consacrée à rappeler les outils mathématiques nécessaires dans le mémoire ainsi que des différentes modèles mécaniques de contact étudiés. La deuxième partie est destinée à l'étude des problèmes de contact thermo-viscoélastiques avec frottement, endommagement et diffusion d'usure. La troisième partie est dédiée à l'analyse des problèmes viscoplastiques sans frottement avec adhésion.

Mots clés:

viscoplastique, thermo- viscoélastiques, diffusion d'usure, adhésion, endommagement.

Abstract

This thesis focuses on the variational investigation of contact issues between a deformable body and a foundation, with or without friction. We examine dynamic processes for viscoplastic and thermo-viscoelastic materials assuming minor deformations. The obtained results pertain to the existence and uniqueness of a weak solution for the studied problems. The thesis is structured into three parts. The first part provides a review of the necessary mathematical tools and various mechanical contact models studied. The second part focuses on the study of thermo-viscoelastic contact problems with friction, damage, and wear diffusion. The third part is dedicated to the analysis of frictionless viscoplastic problems with adhesion.

Key words

dynamic processes, viscoplastic, thermo-viscoelastic, wear, adhesion, damage.

Notations

If Ω is a domain of $\mathbb{R}^d$ $(d = 2, 3)$, we denote by

$\overline{\Omega}$ the closure of the set Ω,

Γ the supposed regular boundary of Ω,

$mes(\Gamma_1)$ the Lebesgue measure $(d-1)$ dimensional of Γ_1,

$(\Gamma_i i = \overline{1,3})$a measurable subset of Γ,

ν the outgoing unit normal to Γ,

v_ν, v_τ the normal and tangential components of the vector field v,

σ_ν, σ_τ the normal and tangential components of the tensor field σ,

$C^1(\Omega)$ the space of continuously differentiable real-valued functions on Ω,

H the space $L^2(\Omega)^d$,

Q the space $L^2(\Omega)_s^{d\times d}$,

H_1 the space $H^1(\Omega)^d$,

$H^{\frac{1}{2}}(\Gamma)$ a Sobolev space on Γ, defined as the range of the trace operator on $H^1(\Omega)$.

If X is a real Hilbert space, we use the following notation

$(.,.)_X$ is the inner product of X,

$\langle .,. \rangle_{X' \times X}$ the duality product between X and X',

$|.|_X$ the norm of X,

$x_n \rightharpoonup x$ the weak convergence of the sequence (x_n) to the element x in X,

$x_n \rightarrow x$ the strong convergence of the sequence (x_n) to the element x in X, Moreover, if $[0;T]$ is a time interval, $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$, we denote by

$C(0,T;H)$ the space of continuous functions from $[0,T]$to H,

$L^p(0,T;H)$ the space of strongly measurable functions on $[0,T]$ in H,

$|.|_{L^p(0,T;H)}$ the norm of $L^p(0,T;H)$,

$W^{k,p}(0,T;H)$the Sobolev space with parameters k and p,

$|.|_{W^{k,p}(0,T;H)}$ the norm of $W^{k,p}(0,T;H)$,

$\overline{I} = [0,T]$ time interval, and $I = (0,T)$.

For a function f, we denote by

$\dot{f}, \ddot{f}$ the first and second derivatives of f with respect to time,

$\mathrm{dom}(f)$ the domain of f,

supp f the support of f,

$\partial_i f, f_{,i}$ the partial derivative of $f$with respect to the i-th component x_i,

∇f the gradient of f,

$\mathrm{div}\, f$ the divergence of f,

$\varepsilon(f)$ the symmetric part of the gradient of f,

∂f the subdifferential of f,

$\liminf$ lower limit,

$\limsup$ upper limit,

$D(\Omega)$ is the space of real functions that are infinitely differentiable and have compact support,

Introduction

In both industry and nature, numerous common phenomena involve contact processes between a deformable body and a foundation or between two deformable bodies that are moving relative to each other.The literature on engineering concerning this topic is vast due to its significance in structural and mechanical systems, as well as in metal shaping and extrusion.

The mathematical literature dedicated to the study of contact phenomena is relatively recent. This is because contact processes, which involve complex physical and surface phenomena, are modeled by nonlinear boundary problems that are very challenging to analyze. The first publication on the contact of deformable bodies was by Hertz. This was followed by Signorini's work, where the problem was formulated in what is now known as a variational form and subsequently solved by Fichera. However, Duvaut and Lions' monograph marked the beginning of the general theory of contact mechanics, who presented variational formulations of several contact problems and proved some fundamental results regarding the existence and uniqueness of solutions.

The frictionless contact condition is commonly used in several references to simplify the analysis of certain models. Additionally, the most popular friction law used in literature is Coulomb's law, first formulated by Amontons in 1699. Expressed in terms of inequalities, the main characteristic of this law lies in its description of the "stick-slip" phenomenon. Various versions based on Coulomb's law have also been widely used in both technical and mathematical literature.

The subject of damage is extremely important in engineering designs as it directly affects the everyday life of the structure or component being designed. There is a vast literature on this subject. Mathematical models have been studied that consider the influence of internal material damage on the contact process. In all of these works, material damage is described by a function with values between zero and one. When $\varsigma = 1$, there is no damage in the material, when $\varsigma = 0$, the material is completely damaged, and when $0 < \varsigma < 1$, there is partial damage and the system has reduced capacity. Problems with damage can be found in [7, 15, 18, 28]

In the field of materials science, surface wear refers to the degradation of the outer layers of a solid material due to mechanical forces from the surrounding environment. To model the wear of contact surfaces, a wear function is introduced, which takes into account the depth of material removed in the normal direction. This function represents the cumulative amount of material removed per unit surface area, and it also accounts for changes in surface geometry. It is generally assumed that the rate of surface wear is proportional to the contact pressure and the relative sliding pressure, which is essentially the frictional power dissipated. As a result, wear involves changes in the contacting surfaces and these changes affect the contact process. Due to its crucial role, there is a vast body of mathematical literature dedicated to this subject, as exemplified by reference [39, 28, 32, 12].

Adhesion processes are crucial in various industrial assemblies where non-metallic parts are commonly bonded together. Consequently, the adhesive contact between deformable bodies, which involves adding glue to prevent relative movement of surfaces, has recently gained significant attention in both engineering and mathematical literature. The analysis of adhesive contact models can be found in reference [31, 9, 10]. The adhesion field, ranging from zero to one, represents the fractional density of active bonds on the contact surface. In this thesis, the adhesion field is denoted as "β." When β equals one at a point on the contact surface, adhesion is complete and all adhesive bonds are active. When β equals zero, all adhesive bonds are inactive, resulting in no adhesion. For values of β between zero and one, adhesion is partial, with only a fraction of the adhesive bonds being active.

Another phenomenon to be considered in this thesis is the phenomenon of thermal contact. Contact and friction processes inevitably result in the generation of heat, which can be significant. For example, sudden braking of a car can lead to the dissipation of a significant amount of heat. Thermal effects in contact processes affect the composition and rigidity of surfaces and induce thermal stresses in the contacting bodies. Mathematical models in thermodynamics require four elements: the heat generation condition, the condition describing heat exchange between the body and the foundation, the constitutive relationship, and the energy equation. These models have been recently developed in [12, 18, 6, 22, 40].

The objective of this thesis is to mathematically study contact problems between a deformable body and a foundation. Under the assumption of small deformations, we investigate dynamic processes for viscoplastic and thermo-viscoelastic materials. The boundary conditions consist of normal compliance with or without adhesion, or instantaneous normal response. The friction laws used are versions of the Coulomb's law without adhesion. Our study of contact phenomena includes the following steps: mathematical modeling, variational analysis including existence and uniqueness results of the solution, and convergence results.

The thesis is divided into two parts.
The first part consists of chapters 1 and 2, which provide a brief introduction to the study of variational inequalities. The material presented here has been selected with a focus on the necessary mathematical tools for the study of contact problems. Specifically, Chapter 1 is dedicated to some elements of functional analysis and nonlinear analysis. In Chapter 2, we discuss several results from the theory of elliptic and parabolic variational inequalities, evolution equations, quasi-variational inequalities with memory term, and the Gronwall lemma.

The second part of this thesis, comprising chapters 3 to 5, focuses on the modeling and analysis of contact problems with or without friction. Chapter 3 primarily focuses on the modeling of various contact problems. We present the physical framework, the viscoplastic and thermo-viscoelastic behavior laws considered in contact problems. Additionally, we describe the contact conditions and friction laws utilized in the analyzed contact problems.
Chapter 4 presents a variational analysis of a dynamic problem that describes frictional contact between a thermo-viscoelastic body and a moving base, involving wear and wear diffusion on the contact surface. The thermo-viscoelastic constitutive law takes into account the effect of damage. The contact is modeled using a normal compliance condition combined with a friction law. The demonstration is based on arguments of hyperbolic nonlinear differential equations, parabolic variational inequalities, first-order evolution equations, and fixed-point arguments.
Chapter 5 focuses on the mathematical study of a viscoplastic contact problem. Contact is assumed frictionless and is described by either normal compliance or the Signorini conditions with adhesion. We establish the existence and uniqueness of the solutionof the hyperbolic quasi-variational inequality by applying the Roth method and fixed point techniques.

Chapter 1

Notation and preliminaries

In this chapter, we present some preliminaries of the functional analysis that will be used throughout this thesis.

1.1 Normed spaces

Definition 1.1. *Given a linear space X. A map*$|.| : X \to \mathbb{R}$, *is called norm if and only if*

- $|w|_X \geq 0 \ \forall w \in X$ *and* $|w|_X = 0$ *iff* $w = 0_X$.
- $|\alpha X|_X = |\alpha||w|_X, \ \forall w \in X, \ \forall \alpha \in \mathbb{R}$.
- $|w + v|_X \leq |w|_X + |v|_X \ \forall w, v \in X$.

Definition 1.2. *We call norm vector space a space X endowed with a* $||_X$ *norm.*

Let us now move on to the notion of complete normed vector space.

Definition 1.3. *A normed space E is said to be complete if all the Cauchy sequences of E converges in E. A Banach space is a complete normed space.*

We present in what follows a particular type of normed space, in which the norm is defined in a special way.

Definition 1.4. *Let X be a real linear space, we call inner product on X any positive-definite symmetric bilinear form in other words, any application* $(.,.)_X$ *of* $X \times X$ *in* $\mathbb{R}$ *verifying :*

- $(w, w)_X \geq 0 \ \forall w \in X$ *and* $(w, w)_X = 0$ *iff* $w = 0_X$.

- $(v, w)_X = (w, v)_X \ \forall w, v \in X$.
- $(\alpha w + \beta v, u)_X = \alpha(w, u) + \beta(v, u) \ \forall u, v, w \in X, \forall \alpha, \beta \in \mathbb{R}$.

The pair $(X, (.,.)_X)$ is called inner product space.

Definition 1.5. *A real pre-Hilbert space is called a vector space X equipped with a scalar product $(.,.)_X$.*

Definition 1.6. *A complete inner product space is called a Hilbert space and it is denoted by $\mathbb{H}$.*

It is well known that a inner product $(.,.)_X$ induces a norm by the relationship

$$|w|_X = \sqrt{(w, w)_X} \ \forall w \in X.$$

Recall that a inner product verifies the Cauchy-Schwarz inequality:

$$|(w, v)_X| \leq |w||v| \ \ \forall w, v \in X.$$

Moreover, the equality of the parallelogram is satisfied

$$|w + v|_X^2 + |w - v|_X^2 = 2(|w|_X^2 + |v|_X^2) \ \ \forall w, v \in X.$$

Theorem 1.1. *(Riesz representation theorem) Let H be a Hilbert space and let $g \in H'$. Then there is one and only one element $T_g \in H$ such that*

$$\langle g, f\rangle = (T_g, f)_H \ \ \forall f \in H.$$

Moreover, we have

$$|g|_{H'} = |T_g|_H.$$

1.2 Functional spaces

In this section, we provide some reminders about real-valued functional spaces that help us understand the properties of spaces suitable for mechanics. We will discuss spaces of continuous functions, continuously differentiable functions, Lebesgue spaces, and Sobolev spaces.

The spatial domain Ω is assumed to be an open, bounded, and connected set in $\mathbb{R}^d$ with boundary Γ. Let $x = (x_1, x_2, \ldots, x_d)$ be an element of $\mathbb{R}^d$ and either $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ a multiindex such as $|\alpha| = \sum_{i=1}^{d} \alpha_i$, we put

$$D^\alpha w(x) = \frac{\partial^{|\alpha|} w(x)}{\partial^{|\alpha^1|} x_1 \ldots \partial^{|\alpha^d|} x_d}$$

1.2.1 Spaces of continuous and continuously differentiable functions.

We denote by $C(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$. $C(\overline{\Omega})$ is a Banach space whose norm is as follows:

$$|w(x)|_{C(\overline{\Omega})} = \sup\{|w(x)| : x \in \Omega\} = \max\{|w(x)| : x \in \overline{\Omega}\}.$$

For $m \geq 0$, a space $C^m(\overline{\Omega})$ defined by

$$C^m(\overline{\Omega}) = \{w(x) \in C(\overline{\Omega}) : D^\alpha w \in C(\overline{\Omega})\text{for all } \alpha \text{such that} |\alpha| \leq m\}.$$

Is the space of real-valued functions which, together with their derivatives of order less than or equal to m , are continuous on $\overline{\Omega}$. the space $C^m(\overline{\Omega})$ is a Banach space with the norm

$$|w|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \leq m} |D^\alpha w|_{C(\overline{\Omega})}.$$

Furthermore, $C^\infty(\overline{\Omega})$ denotes the space of indefinitely differentiable functions

$$C^\infty(\overline{\Omega}) = \bigcap_{m=0}^{\infty} C^m(\overline{\Omega}) = \{w \in C(\overline{\Omega}) : w \in C^m(\overline{\Omega}) \forall m = 0, 1, \dots\}$$

Given a real-valued function w on Ω, its support is defined to be

$$\text{supp } w = \overline{\{x \in \Omega : w(x) \neq 0\}}$$

If supp w is a proper subset of Ω, we say that w is a function with support compact in Ω.
The space of indefinitely differentiable functions with compact support is given by

$$C_0^\infty(\Omega) = \{w \in C^\infty(\overline{\Omega}) : \text{ supp } w \subset \Omega\}$$

1.2.2 Lebesgue spaces

For $p \in [1, \infty[$, $L^p(\Omega)$ is the banach space of Lebesgue measurable functions defined on Ω with values in $\mathbb{R}$.

$$L^p(\Omega) = \{w : \Omega \to \mathbb{R} : w \text{ Lebesgue measurable and } |w|^p \text{ Lebesgue integrable on} \Omega\}.$$

Where the norm is given by

$$|w|_{L^p(\Omega)} = \Big(\int_\Omega |w|^p dx \Big)^{\frac{1}{p}}.$$

If $p = \infty$ and $w : \Omega \to \mathbb{R}$ is measurable, so

$$|w|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |w(x)|.$$

The $L^\infty(\Omega)$ space is also a Banach space.

1.2.3 Sobolev spaces

Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. We define the Sobolev spaces by

$$W^{k,p}(\Omega) = \{w \in L^p(\Omega) \text{ such that } D^\alpha w \in L^p(\Omega) \text{ with } |\alpha| \leq p\}.$$

The norm in the space $W^{k,p}(\Omega)$ is defined as

$$|w|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} |D^\alpha w|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty; \\ \max_{|\alpha|\leq k} |D^\alpha w|_{L^\infty(\Omega)} & \text{if } p = \infty. \end{cases}$$

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.
For $p = 2$, $W^{k,2}(\Omega)$ will be denoted by $H^k(\Omega)$ which is a Hilbert space with the inner product given by

$$(v, w)_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha v(x) D^\alpha w(x) dx \ \ \forall v, w \in H^k(\Omega).$$

1.2.4 Spaces of vector-valued functions

The spaces of vector-valued functions will be needed in the study of time-dependent variational problems. Let $(X, |.|_X$ be a real Banach space. In the contact problems studied in this thesis $[0, T]$ denotes the time interval where $T > 0$.
The space $C^m(0, T; X)$. We denote by $C(0, T; X)$ the space of continuous functions from $(0, T)$ to X. Which is a Banach space with the norm

$$|w|_{C(0,T;X)} = \max_{t\in(0,T)} |w(t)|_X.$$

For $m \geq 0$ we define the space

$$C^m(0, T; X) = \{w \in C(0, T; X) : w^{(j)} \in C(0, T; X), j = 1, \dots, m\}.$$

It is a Banach space with the norm

$$|w|_{C^m(0,T;X)} = \sum_{j=0}^{m} \max_{t\in(0,T)} |w^{(j)}(t)|_X.$$

Either also

$$C^{\infty}(0,T;X)=\bigcap_{m=0}^{\infty}C^{m}(0,T;X),$$

the space of indefinitely differentiable functions from $(0,T)$ to X.
The spaces$L^p(0,T;X)$. For $p\in[1,\infty]$, we denote by $L^p(0,T;X)$ the Banach space of a measurable functions from $(0,T)$ to X such that where the norm is

$$|w|_{L^p(0,T;X)}=\begin{cases}\big(\int_0^T|w(t)|^p dt\big)^{\frac{1}{p}} & \text{if } p<\infty\\ \text{ess sup}_{t\in(0,T)}|w(t)|_X & \text{if } p=\infty\end{cases}.$$

If $(X,|.|_X)$ is a Hilbert space $L^2(0,T;X)$ is also a Hilbert space with the inner product

$$(v,w)_{L^2(0,T;X)}=\int_0^T(v(t),w(t))_X dt.$$

The spaces $W^{k,p}(0,T;X)$. for $k\in\mathbb{Z}_+$ and $1\le p\le\infty$, we present the space

$$W^{k,p}(0,T;X)=\{w\in L^p(0,T;X):|w^{(m)}|_{L^p(0,T;X)}<\infty\}.$$

We define the norm in the space $W^{k,p}(0,T;X)$ by

$$|w|_{W^{k,p}(0,T;X)}=\begin{cases}\big(\int_0^T\sum_{m=0}^{k}|w^{(m)}(t)|^p dt\big)^{\frac{1}{p}} & \text{if } p<\infty\\ \max_{0\le m\le k}\text{ess sup}_{t\in(0,T)}|w^{(m)}(t)|_X & \text{if } p=\infty\end{cases}$$

If X is a Hilbert space and $p=2$, then the space

$$H^k(0,T;X)=W^{k,2}(0,T;X),$$

is a Hilbert space with the inner product

$$(v,w)_{H^k(0,T;X)}=\sum_{0\le m\le k}\int_0^T(v^{(m)},w^{(m)})_X dt.$$

1.3 Fixed point theorems

Banach's fixed point theorem will be used several times in this thesis to prove the existence of solutions to variational problems in contact mechanics.

Let X be a Banach space with the norm $|.|_X$ and K is a non-empty closed subset of X. Let $\Lambda : K \to X$ be an operator defined on K. We are interested in the existence of a solution $u \in K$ of the operator equation.

$$\Lambda u = u \tag{1.1}$$

An element $u \in K$ that satisfies (1.1) is called a fixed point of operator Λ. We present in what follows the main theorem which states the existence of fixed points of nonlinear operators.

Theorem 1.2. *(The Banach fixed point theorem) Let $(X, |.|_{X)}$ be a Banach space. Let K be a non-empty Closed subset of X. Assume that $\Lambda : K \to K$ is a contraction mapping, with contraction constant $\alpha \in [0, 1)$, i.e,*

$$|\Lambda u - \Lambda w|_X \leq \alpha |u - w|_X \;\; \forall u, w \in K. \tag{1.2}$$

Then there exists a unique $u \in K$ such that $\Lambda u = u$.

Proof. Let $u \in K$ be any element of K, and let the sequence $\{u_n\}$ be defined by

$$u_{n+1} = \Lambda u_n, \quad \forall n = 1, 2, \ldots$$

Because $\Lambda : K \longrightarrow K$, the sequence $\{u_n\}$ is well-defined. Let's first prove that $\{u_n\}$ is a Cauchy sequence. By utilizing the fact that Λ is a contraction, we can show that

$$|u_{n+1} - u_n|_X \leq \alpha |u_n - u_{n-1}|_X \leq \cdots \leq \alpha^n |u_1 - u_0|_X.$$

Therefore, for all $m > n \geq 1$,

$$\begin{aligned} |u_m - u_n|_X &\leq \sum_{j=0}^{m-n-1} |u_{n+j+1} - u_{n+j}|_X, \\ &\leq \sum_{j=0}^{m-n-1} \alpha^{n+j} |u_1 - u_0|_X, \\ &\leq \frac{\alpha^n}{1-\alpha} |u_1 - u_0|_X, \end{aligned}$$

Since $\alpha \in [0, 1)$, it follows that

$$|u_m - u_n|_X \longrightarrow 0 \text{ when } m, n \longrightarrow \infty.$$

Thus, $\{u_n\}$ is a Cauchy sequence and has a limit $u \in K$, since K is a closed subset of the Banach space X. Moreover, since $u_n \longrightarrow u$ in K, it follows from (1.2) that $\Lambda u_n \longrightarrow \Lambda u$ in X. Therefore, taking the limit of $u_{n+1} = \Lambda u_n$

yields $u = \Lambda u$, which concludes the existence part of the theorem.
Suppose that $u_1, u_2 \in K$ are fixed points of Λ. Then from $u_1 = \Lambda u_1$ and $u_2 = \Lambda u_2$ we have

$$u_1 - u_2 = \Lambda u_1 - \Lambda u_2,$$

and since Λ is a contraction

$$|u_1 - u_2|_X = |\Lambda u_1 - \Lambda u_2|_X \leq \alpha |u_1 - u_2|_X,$$

this implies that $|u_1 - u_2|_X = 0$, because $\alpha \in [0, 1)$. Therefore, if Λ has a fixed point, it follows that this fixed point is unique, which concludes the proof. □

Corollary 1.1. *Let K be a non-empty closed subset of a Banach space X, and let $\Lambda : K \longrightarrow K$ be a mapping. Suppose $\Lambda^m : K \longrightarrow K$ is a contraction for a positive integer m. Then Λ has a unique fixed point.*

Proof. According to Theorem 1.2, $\Lambda^m u$ has a unique fixed point $u \in K$.
Since $\Lambda^m(u) = u$ we get

$$\Lambda^m(\Lambda u) = \Lambda(\Lambda^m u) = \Lambda u.$$

Thus $\Lambda \in K$ is also a fixed point of $\Lambda^m u$, where Λ^mhas a unique fixed point we have

$$\Lambda u = u,$$

i.e. u is a fixed point of Λ. The uniqueness of the fixed point of Λ is a consequence of the uniqueness of the fixed point of Λ^m □

1.4 Nonlinear analysis elements

In this section, we will review some concepts of nonlinear analysis that will be of great use for the completion of this work. In particular, we will discuss results on monotone operators, convex and lower semi-continuous functions, differentiability, and subdifferentiability.

1.4.1 Linear operators

Let $(X, |.|_X)$ and $(Y, |.|_Y$ be two normed spaces. An operator $L : X \to Y$ is linear if

$$L(\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 L(w_1) + \alpha_2 L(w_2) \;\; \forall w_1, w_2 \in X, \alpha_1, \alpha_2 \in \mathbb{R}.$$

A linear operator is continuous if and only if it is bounded, i.e. there exists a constant $M > 0$ such that

$$|L(w)|_Y \leq M|w|_X \ \forall w \in X.$$

We use the notation $L(X,Y)$ for the set of all linear bounded operators between the normed spaces X and Y. The quantity

$$|L|_{L(X,Y)} = \sup_{0 \neq w \in X} \frac{|Lw|_Y}{|w|_X}, \tag{1.3}$$

is called the norm operator of L, and $L \to |L|_{L(X,Y)}$ defines a norm on the space $L(X,Y)$. Moreover, if Y is a Banach space then $L(X,Y)$ is also a Banach space. For a linear operator L, we usually write $L(v)$ as Lv, but sometimes we also write Lv even when L is not linear.

For a normed space X, the space $L(X,\mathbb{R})$ is called the dual space of X and is denoted by $X^{'}$. The elements of $X^{'}$ are also called linear continuous functionals. The bilinear mapping $\langle .,. \rangle_{X^{'} \times X} : X^{'} \times X \to \mathbb{R}$ is called the duality pairing for the pair $(X^{'}, X)$. It follows from (1.3) that a norm $X^{'}$ is

$$|w^{'}|_{X^{'}} = \sup_{0 \neq w \in X} \frac{|\langle w^{'}, w \rangle|}{|w|_X},$$

and $(X^{'}, |.|_X)$ is always a Banach space.

1.4.2 Nonlinear operators

Definition 1.7. *Let X be a Hilbert space equipped with the inner product $(.,.)_X$ and the norm $|.|_X$ and let $A : X \to X$ be an operator. Then A is called*

(i) monotone if $(Aw - Av, w - v)_X \geq 0$ for all $w, v \in X$,

(ii) strictly monotone if $(Aw - Av, w - v)_X > 0$ for all $w, v \in X$ with $w \neq v$,

(iii) strongly monotone if there exists a constant $c > 0$ such that $(Aw - Av, w - v)_X \geq c|w - v|_X$ for all $w, v \in X$,

(iv) Lipschitz if there exists $M > 0$ such tha $|Aw - Av|_X \leq M|w - v|_X$.

Definition 1.8. *Let X be a normed space and let $A : X \to X^{'}$ be an operator. Then A is called*

i) monotone if $\langle Aw - Av, w - v \rangle_{X^{'} \times X} \geq 0$ for all $w, v \in X$, *ii) hemicontinuous if the real-valued function* $t \to \langle A(v+tw), u\rangle_{X' \times X}$ *is continuous on* $[0;1]$ *for all* $v, w, u \in X$,

Definition 1.9. *Let* X *be a real Banach space. A multifunction* $A : X \to X'$ *is called maximal monotone if it is monotone and* $\forall w, v \in X$ *and* $v' \in X$ *verifying*

$$\langle Aw - v', w - v\rangle_{X' \times X} \geq 0 \Longrightarrow Av = v'.$$

Let $a : X \times X \to \mathbb{R}$ be a bilinear form.

Definition 1.10. *The bilinear form* $a(.,.)$ *is said to be*

(i) continuous or bounded if there exists a number $M > 0$ *such that*

$$|a(v,w)| \leq M|v|_X|w|_X \ \forall v, w \in X,$$

(ii) X *- elliptic if there is a constant* $m > 0$ *such that*

$$a(w,w) \geq m|w|_X^2 \ \forall w \in X,$$

(iii) symmetric if

$$a(v,w) = a(w,v) \ \ \forall v, w \in X.$$

1.4.3 Convex and lower-semicontinuous functions

Definition 1.11. *Let* X *be a linear space and let* $\varphi : X \to (-\infty, \infty]$, *be a function, the effective domain of* φ *is the set*

$$D(\varphi) = \{w \in X : \varphi(w) < \infty\},$$

and we say that the function φ *is proper if* $D(\varphi) \neq \emptyset$. *The function* φ *is convex if*

$$\varphi((1-t)w + tv) \leq (1-t)\varphi(w) + t\varphi(v) \tag{1.4}$$

for all $w, v \in X$ *and* $t \in [0,1]$ *The function* φ *is strictly convex if the inequality in* (1.4) *is strict for* $w \neq v$ *and* $t \in (0,1)$. *We observe that if* $\varphi, \psi : X \to (-\infty, +\infty]$ *are a convex functions and* $\lambda > 0$, *then the functions* $\varphi + \psi$ *and* $\lambda\psi$ *are also convex.*

Definition 1.12. *A function* $\varphi : X \to (-\infty, +\infty]$ *is said to be lower semi-continuous (l.s.c.) at* $w \in X$ *if*

$$\lim_{n\to\infty} \inf \varphi(w_n) \geq \varphi(w). \tag{1.5}$$

for each sequence $\{w_n\} \subset X$ *converging to* w *in* X *. The function* φ *is l.s.c. if it is l.s.c. at every point* $w \in X$*. When inequality (1.5) holds for every sequence* $\{w_n\} \subset X$ *that weakly converges to* w*, the function* φ *is said to be weakly lower semicontinuous at* w*. The function* φ *is weakly l.s.c. if it is weakly l.s.c. at every point* $w \in X$*.*

If φ is a continuous function, then it is also l.s.c. However, the converse of this statement is not true since lower semicontinuity does not imply continuity.
Since strong convergence in X implies weak convergence, it follows that a weakly lower semicontinuous function is lower semicontinuous. Moreover, it can be shown that a proper convex function $\varphi : X \to (-\infty, +\infty]$ is lower-semicontinuous if and only if it is weakly lower-semicontinuous.

Example 1.1. *(Indicator function) Let* X *be a real normed space and let* $K \subset X$*. We called indicator function* $\psi_K : X \to (-\infty, +\infty]$ *all function defined by*

$$\psi_K(w) = \begin{cases} 0 & \text{if } w \in K, \\ +\infty & \text{if } w \notin K \end{cases} \tag{1.6}$$

1.4.4 Differentiability and sub-differentiability

We now recall the definition of Gateaux-differentiable functions.

Definition 1.13. *Let* X *be a Hilbert space. A map* $\varphi : X \to \mathbb{R}$ *is Gâteaux defferentiable at* $w \in X$ *if there exists an element* $\nabla\varphi(w) \in X$ *such that*

$$\lim_{t \to 0} \frac{\varphi(w + tv) - \varphi(w)}{t} = (\nabla\varphi(w), v)_X \quad \forall v \in X. \tag{1.7}$$

The element $\nabla\varphi(w)$ *which satisfies the relation (1.7) is called the gradient of* φ *at* w*. The function* $\varphi : X \to \mathbb{R}$ *is said to be Gâteaux defferentiable if it is Gâteaux defferentiable at every point of X.*

The convexity of Gateaux-differentiable functions can be characterized as follows.

Proposition 1.1. *Let*$(X, (.,.)_X)$ *be a pre-Hilbert space and let* $\varphi : X \to \mathbb{R}$ *be a Gateaux-differentiable function. The following statements are equivalent*

(i) φ *is a convex function,*

(ii) φ *satisfies the inequality*

$$\varphi(w) - \varphi(v) \geq (\nabla\varphi(v), w - v)_X \ \ \forall w, v \in X,$$

(iii) the gradient of φ is a monotone operator

$$(\nabla\varphi(w) - \nabla\varphi(v), w - v)_X \geq 0 \ \ \forall w, v \in X.$$

Corollary 1.2. *Let $(X, (.,.)_X)$ be a pre-Hilbert space and let φ be a convex and Gâteaux -differentiable function. Then φ is lower semicontinuous.*

Definition 1.14. *Let $\varphi : X \to (-\infty, +\infty]$ and let $w \in X$, the subdifferential of φ at $w \in X$ is the set*

$$\partial\varphi(w) = \{l \in X : (l, v - w)_X < \varphi(v) - \varphi(w) \ \ \forall v \in X\}.$$

The function φ is said to be sub-differentiable at $w \in X$ if $u \in D(\partial\varphi)$, and each element $l \in \partial\varphi(u)$ is called a subgradient of φ at w. The function φ is said to be sub-differentiable if it is sub-differentiable at every point in X, that is, if $D(\partial\varphi) = X$.
It can be shown that a sub-differentiable function $\varphi : X \to (-\infty, +\infty]$ is convex and lower semi-continuous. Furthermore, for convex functions, the relationship between the gradient operator and the sub-differential is given by the following result.

Proposition 1.2. *Let $\varphi : X \to \mathbb{R}$ be a convex function and Gâteau differentiable. Then φ is sub-differentiable and $\partial\varphi(w) = \{\nabla\varphi(w)\}$ for all $w \in X$.*

Example 1.2. *(Subdifferential of the indicator function) Let X be a real normed space and let $K \subset X$ be a convex set. Consider the subdifferential of the indicator function ψ_K defined in Example 1.1, and suppose that $w \in K$. Then $w^{'} \in \partial\psi_K$ if and only if*

$$\partial\psi_K(w) \geq \langle w^{'}, v - w\rangle \ \ \forall v \in K,$$

that is to say

$$\langle w^{'}, v - w\rangle \leq 0 \ \ \forall v \in K.$$

So, for $w \in K$ we have

$$\partial\psi_K(w) = \{w^{'} \in X^{'} : \langle w^{'}, v - w\rangle \leq 0 \ \ \forall v \in K\},$$

and for $w \ni K$, we have $\psi_K(w) = \emptyset$.

We always have $0 \in \partial\psi_K(w)$ for $w \in K$. It is easy to see that if $w \in int(K)$ (the interior of K) then $\partial\psi_K(w) = \{0\}$.

Chapter 2

Variational inequalities and Evolution equations

In this chapter, we review some standard existence and uniqueness results for elliptic and parabolic variational inequalities, ordinary differential equations in abstract spaces and quasi-variational inequations in the framework of memory operators. In this chapter, X is a Hilbert space with the inner product $(.,.)_X$ and the associated norm $|.|_X$.

2.1 Elliptic variational inequalities

In this section we present existence and uniqueness results of solutions concerning elliptic variational inequalities with monotone operators. We start with an existence and uniqueness result for elliptic variational inequalities of the first kind then we extend it to elliptic variational inequalities of the second kind, as well as to the elliptic quasivariational inequalities.

2.1.1 Variational inequalities of the first kind

Given a subset $K \subset X$, an element $f \in X$ and an operator $A : X \to X$, we consider the problem of finding an element u such that

$$u \in K,\ (Au, w - u)_X \geq (f, w - u)_X \ \forall w \in K. \tag{2.1}$$

The last inequality is called an elliptic variational inequality of the first kind .

Theorem 2.1. *Let X be a Hilbert space and let $K \subset X$ be a nonempty, convex, and closed subset. Assume that $A : K \to X$ is a strongly mono-*

tone Lipschitz continuous operator. Then, for each $f \in X$ the variational inequality (2.1) has a unique solution.

Now suppose that $K = X$. Then, taking $w = u \pm v$, it is easy to see that the variational inequality ((2.1) is equivalent to the variational equation

$$(Au, v)_X = (f, v)_X \ \forall v \in X.$$

We have the following result of existence and uniqueness in the study of non linear equations involving monoton operators.

Theorem 2.2. *Let X be a Hilbert space and let $A : X \to X$ is a strongly monotone Lipschitz continuous operator. Then for every $f \in X$ there exists a unique Element $u \in X$ such that $Au = f$.*

2.1.2 Variational inequalities of the second kind

Thus, given a set $K \subset X$, an element $f \in X$, a function $j : K \to \mathbb{R}$ and an operator $A : X \to X$, we consider the problem of finding an element u such that

$$u \in K, \ (Au, w - u)_X + j(w) - j(u) \geq (f, w - u)_X \ \forall w \in K. \tag{2.2}$$

We called anelliptic variational inequality of the second kind all variational inequality of the form (2.2). In the particular case when $j \equiv 0$, the variational inequality (2.2) represents an elliptic variational inequality of the form (2.1). In the study of (2.2) we assume the following hypotheses:

$$K\text{is a nonempty closed convex subset of}X \tag{2.3}$$

$A : X \to X$ is strongly monotone and Lipschitz continuous on X, i.e.

$$\begin{cases} i) & \text{there exists } M > 0\text{such that} \\ & (Au - Aw, u - w)_X \geq M|u - w|_X^2 \ \ \forall u, w \in X, \\ ii) & \text{there exists } m > 0\text{such that} \\ & |Au - Aw|_X \leq m|u - w|_X \ \ \forall u, w \in X. \end{cases} \tag{2.4}$$

$$j : K \to \mathbb{R} \text{ is a convex l.s.c. function.} \tag{2.5}$$

The main result of this subsection is the following.

Theorem 2.3. *Let X be a Hilbert space and assume that the hypotheses (2.3)-(2.5) are verified. Then, for each $f \in X$ the variational inequality the elliptic variational inequality of the second kind has a unique solution.*

2.1.3 Quasivariational inequalities

For the variational inequations studied in this subsection we consider that the function j depends on the solution. Therefore, given an operator $A : K \to X$ a subset $K \subset X$ a function $j : K \times K \to \mathbb{R}$ and an element $f \in X$, we consider the problem of finding an element u a solution of the following elliptic quasivariational inequality

$$u \in K, \ (Au, w - u)_X + j(w, u) - j(u, u) \geq (f, w - u)_X \ \forall w \in K. \tag{2.6}$$

In the study of (2.6), in addition to (2.3) and (2.4), we consider the following assumptions on the function j.

$$\begin{cases} i)\text{For all } \theta \in K, j(\theta, .) : K \to \mathbb{R} \text{ is a convex l.s.c. function.} \\ ii)\text{There exists } \beta \geq 0 \text{ such that} \\ \quad j(\theta_1, w_2) - j(\theta_1, w_1) + j(\theta_2, w_1) - j(\theta_2, w_2) \geq \beta|\theta_1 - \theta_2|_X|w_1 - w_2|_X \\ \quad \forall \theta_1, \theta_2, w_1, w_2 \in K. \end{cases} \tag{2.7}$$

Theorem 2.4. *Let X be a Hilbert space and assume that the hypotheses (2.3), (2.4) and (2.7) are satisfied. Moreover, suppose that $M > \beta$ with $M > 0$. So for each $f \in X$ the inequality(2.6) has a unique solution $u \in K$.*

For more details, you can see [38]

2.2 Parabolic variational inequalities

Let V and H be real Hilbert spaces such that V is dense in H

A standard result for parabolic variational inequations is given by the following theorem.

Theorem 2.5. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a(,) : V \times V \to \mathbb{R}$ is a a continuous and symmetric bilinear form and there are two real constants $c_1 > 0$ and c_0 such that*

$$a(w, w) + c_0|w|_H^2 \geq c_1|w|_V^2 \ \ \forall w \in V.$$

Then, for each $u_0 \in K$ and $f \in L^2(0, T; H)$, there exists a unique function $u \in H^1(0, T; H) \cap L^2(0, T; V)$ such that

$$u(t) \in K \ \forall t \in [0, T],$$

$$\langle \dot{u}(t), w - u(t)\rangle_{V' \times V} + a(u(t), w - u(t)) \geq (f(t), w - u(t))_H \ \ \forall w \in K, a.et \in (0, T),$$

$$u(0) = u_0.$$

2.3 Ordinary differential equations in abstract spaces

Let V and H be real Hilbert spaces such that V is dense in H and the injection of V into H is continuous. H is identified with its own dual and a subspace of the dual $V^{'}$ of V, i.e.

$$V \subset H \subset V^{'},$$

this inclusion defines a Gelfand triplet. The notations $|.|_V$, $|.|_{V'}$ and $\langle .,. \rangle_{V' \times V}$ represent the norms on the spaces V and $V^{'}$, and the duality between $V^{'}$ and V, respectively.

The following theorem provides a standard result for parabolic variational inequalities.

Theorem 2.6. *Let $V \subset H \subset V^{'}$ be a Gelfand triple. Assume that $B : V \to V^{'}$ is a monotone and hemicontinuous operator that satisfies*

$$\langle Bw, w \rangle_{V' \times V} \geq \alpha |w|_V^2 + \lambda \ \ \forall w \in V, \tag{2.8}$$

$$|Bw|_{V'} \leq c_2(|w|_V + 1) \ \ \forall w \in V. \tag{2.9}$$

For some constants $\alpha > 0, c_2 > 0$ an d $\lambda \in \mathbb{R}$. then given $u_0 \in H$ and $f \in L^2(0,T;V^{'})$, there exists a unique function u that satisfies

$$\begin{aligned}
&u \in L^2(0,T;V) \cap C([0,T];H), \dot{u} \in L^2(0,T;V^{'}),\\
&\dot{u}(t) + Bu(t) = f(t) \quad a.et \in (0,T),\\
&u(0) = u_0.
\end{aligned}$$

Now, we will recall a version of the Cauchy-Lipschitz theorem.

Theorem 2.7. ***(Cauchy-Lipschitz)*** *Asssume that $(X, |.|_X)$ is a real Banach space and $T > 0$. Let $A(t,.) : X \to X$ be an operator defined a.e.on $(0,T)$ satisfying the following conditions*
-There exists $L_A > 0$ such that $|A(t,u) - A(t,w)|_X \leq L_A|u - w|_X$. - There exists $1 \leq p \leq \infty$ such that $t \to A(t,w) \in L^p(0,T;X), \forall w \in X$.
Then for any $u_0 \in X$ there exists a unique function $u \in W^{1,P}(0,T;X)$ such that

$$\begin{aligned}
&\dot{u}(t) = A(t,u(t)) \ \ a.e \ t \in (0,T),\\
&u(0) = u_0.
\end{aligned}$$

2.4 Nonlinear differential equations of hyperbolic type

Let $V$and H be real Hilbert spaces such that $V \subseteq H$ and the inclusion mapping of V into H is continuous and densely defined. Assume that A is linear continuous and symmetric operator from V to $V^{'}$ and M is a nonlinear monotone operator from V to $V^{'}$, we consider the problem of finding an element u such that

$$\begin{aligned} &u \in K, \ \tfrac{d^2u}{dt^2} + Au + M(\tfrac{du}{dt}) \ni f(t) \ \ \forall t \in (0,T), \\ &u(0) = u_0, \ \tfrac{du}{dt}(0) = v_0 \end{aligned} \tag{2.10}$$

An inequality of the form (2.10) is called a nonlinear hyperbolic differential equation.

Theorem 2.8. *Let V and H be two real Hilbert spaces such that $V \subseteq H$. Assume that M is maximal monotone set in $V \times V^{'}$, $A \in L(V, V^{'})$ satisfying the following coerciveness condition*

$$\langle A\mathbf{v}, \mathbf{v}\rangle_{V' \times V} + \lambda |\mathbf{v}|^2 \geq \omega \|v\|^2 \ \forall \mathbf{v} \in V \tag{2.11}$$

where $\lambda \in \mathbb{R}$ and $\omega > 0$. Let $f \in W^{1,1}(0,T;H)$ and u_0, v_0 be given satisfying

$$\begin{aligned} &u_0 \in V, v_0 \in D(M), \\ &\{Au_0 + Mv_0\} \cap H \neq \emptyset. \end{aligned} \tag{2.12}$$

Then there exists a unique solution of problem (2.10) which satisfies

$$u \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H).$$

For more details see [5]

2.5 Elementary Inequalities

We will need the following elementary inequalities in later chapters.

Lemma 2.1. *(Young inequality) Let $a, b \geq 0$ and $1 < p, q < \infty$ be two conjugated exponents. So*

$$ab \leq \frac{a^q}{q} + \frac{b^p}{p}.$$

An elementary case of Young's inequality is the inequality with exponent 2,

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2},$$

which also gives the Young inequality with ϵ (valid for all $\epsilon > 0$)

$$ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon},$$

Finally, we conclude this section with two Gronwall type inequalities.

Lemma 2.2. *Let $K > 0$ and b_1, b_2 are positive constants. Let $n \in \mathbb{N}^*$ and $\{z_i\}_{i=0}^{n} \subset \mathbb{R}$ be a positive sequence which satisfies*

$$z_{j+1} \leq b_1 + b_2 h \sum_{l=0}^{j} z_l \quad 0 \leq j \leq n-1$$

where $h = \frac{K}{n}$. So it achieves the following result

$$z_{j+1} \leq (b_1 + b_2 K z_0) e^{b_2 K} \quad 0 \leq j \leq n-1$$

Lemma 2.3. *We suppose that $h, f : [0,T] \to \mathbb{R}$ are two functions in $L^1(0,T)$ satisfaying*

$$h(t) \leq f(t) + \beta \int_0^t h(s) ds \;\; a.et \in [0,T]$$

where β is nonnegative constant, so

$$h(t) \leq f(t) + \beta \int_0^t e^{\beta}(t-s) f(s) ds \;\; a.et \in [0,T]$$

Chapter 3

Modelling of contact problems

In this chapter, we present the notations and preliminaries necessary for studying the boundary problems presented in chapters 4 and 5. We begin by introducing the functional spaces that will be useful for studying contact problems. Then, we provide a general description of the mathematical modeling of the processes involved in the contact between a viscoplastic or thermo-viscoelastic body and a foundation. This aims to introduce the physical framework and recall some concepts from the mechanics of continuous media, specifically the equations of motion and equilibrium, the constitutive laws for the materials considered, as well as the contact boundary conditions with or without friction.

Consider a material body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega = \Gamma$, assumed to be Lipschitz which is divided into three disjoint measurable p arts Γ_e, Γ_d and Γ_c, such that $meas(\Gamma_e) > 0$. Let ν denote the unit outer normal on Γ.

3.1 Function spaces in contact mechanics

In order to introduce mathematical models that describe various contact processes, it is necessary to describe the functional spaces to which the data and unknowns belong. The objective of this section is to present these spaces along with their properties.

3.1.1 Preliminaries

We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the canonical inner products and the corresponding norms on $\mathbb{R}^d$

and $\mathbb{S}^d$ are given by

$$\mathbf{w}.\mathbf{v} = \sum_{i=1}^{d} w_i v_i,\, |\mathbf{w}| = (\mathbf{w}.\mathbf{w})^{\frac{1}{2}},\ \forall \mathbf{w} = (w_i), \mathbf{v} = (v_i) \in \mathbb{R}^d,$$

$$\sigma.\varsigma = \sum_{1\le j,i\le d} \sigma_{j,i}\varsigma_{j,i},\ |\varsigma| = (\varsigma.\varsigma)^{\frac{1}{2}}, \forall \sigma = (\sigma_{j,i}), \varsigma = (\varsigma_{j,i}) \in \mathbb{S}^d,$$

Let X be a Hilbert space equipped with a inner product $(.,.)_X$ and the associated norm $|.|_X$. we use the notation X^d and $X_s^{d\times d}$ for the spaces

$$X^d = \{\mathbf{x} = (x_i) : x_i \in X, 1 \le i \le d\},$$

$$X_s^{d\times d} = \{\mathbf{x} = (x_{ji}) : x_{ji} = x_{ij} \in X, 1 \le j, i \le d\}.$$

Here and throughout this thesis $1 \le j, i \le d$. And, in particular, we will frequently use the following spaces:

$$H = L^2(\Omega)^d = \{\mathbf{w} = (w_i) : w_i \in L^2(\Omega), 1 \le i \le d\},$$

$$Q = L^2(\Omega)_s^{d\times d} = \{\varsigma = (\varsigma_{ji}) : \varsigma_{ji} = \varsigma_{ij} \in L^2(\Omega), 1 \le i \le d\}.$$

The spaces H and Q are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{w}, \mathbf{v})_{L^2(\Omega)^d} = \int_\Omega w_i v_i dx = \int_\Omega \mathbf{w}.\mathbf{v} dx,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\varsigma})_Q = \int_\Omega \sigma_{ji}\varsigma_{ji} dx = \int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varsigma} dx,$$

and the associated norms are denoted by $|.|_H$ and $|.|_Q$, respectively.
We need specific spaces for the stress and displacement field.

3.1.2 Spaces for the displacement field

In the contact problems presented in this thesis, displacements are sought in space

$$H^1(\Omega)^d = \{\boldsymbol{w} = (w_i) : w_i \in H^1(\Omega), 1 \le i \le d\},$$

or its subsets or subspaces, depending on the given boundary conditions.
We note that $H^1(\Omega)^d$ is a real Hilbert space endowed with the inner product

$$(\mathbf{w}, \mathbf{v})_{H^1(\Omega)^d} = \int_\Omega (w_i v_i + w_{i,j} v_{i,j}) dx,$$

and the associated norm

$$|w|_{H^1(\Omega)^d} = (\int_\Omega (w_i w_i + w_{i,j} w_{i,j}) dx)^{\frac{1}{2}}.$$

Here $\varepsilon : H^1(\Omega)^d \to Q$ is the deformation operator defined by

$$\varepsilon(\mathbf{w}) = (\varepsilon_{ij}(\mathbf{w})), \varepsilon_{ij}(\mathbf{w}) = \frac{1}{2}(w_{i,j} + w_{j,i}).$$

The tensor ε is called the linear deformation tensor.

$$((\mathbf{w}, \mathbf{v}))_{H^1(\Omega)^d} = (\mathbf{w}, \mathbf{v})_{L^2(\Omega)^d} + (\varepsilon(\mathbf{w}), \varepsilon(\mathbf{v}))_Q \ \forall \mathbf{w}, \mathbf{v} \in H^1(\Omega)^d, \tag{3.1}$$

the associated norm is defined by

$$|\mathbf{w}|^2_{H^1(\Omega)^d} = |\mathbf{w}|^2_{L^2(\Omega)^d} + |\varepsilon(\mathbf{w})|^2_Q. \tag{3.2}$$

Moreover, by the Sobolev's trace theorem, we can define the trace$\gamma\mathbf{w}$ of a function $\mathbf{w} \in H^1(\Omega)^d$ on the boundary Γ, if $w \in H^1(\Omega)^d \cap C(\overline{\Omega})^d$ we have that $\gamma\mathbf{w} = w|\Gamma$. For an element $\mathbf{w} \in H^1(\Omega)^d$ we use the notation $\mathbf{w}$ to designate the trace $\gamma\mathbf{w}$ of $\mathbf{w}$ on Γ. The trace operator $\gamma : H^1(\Omega)^d \to L^2(\Gamma)^d$ is a linear continuous operator, so we can see that

$$|\mathbf{w}|_{L^2(\Gamma)^d} \leq C|\mathbf{w}|_{H^1(\Omega)^d} \quad \forall \mathbf{w} \in H^1(\Omega)^d, \tag{3.3}$$

such that C is a positive constant depending only on Ω.
For any vector field $\mathbf{w} \in H^1(\Omega)^d$,we denote by w_ν and $\mathbf{w}_\tau$ the normal and the tangential components of $\mathbf{w}$ on Γ given by

$$w_\nu = \mathbf{w}.\nu \quad \mathbf{w}_\tau = \mathbf{w} - w_\nu.\nu. \tag{3.4}$$

In the study of contact problems, we consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\mathbf{w} \in H^1(\Omega)^d : \mathbf{w} = 0 \text{ on } \Gamma_e\}. \tag{3.5}$$

Since $meas(\Gamma_e) > 0$, the following Korn's inequality holds:

$$|\varepsilon(\mathbf{w})|_Q \geq C_0|\mathbf{w}|_{H^1(\Omega)^d} \ \forall \mathbf{w} \in V, \tag{3.6}$$

where $C_0 > 0$ is a constant which depends only on Ω and Γ_e.
Now, we define the inner product $(,)_V$ on V by

$$(\mathbf{w}, \mathbf{v})_V = (\varepsilon(\mathbf{w}), \varepsilon(\mathbf{v}))_Q, \tag{3.7}$$

and let $|.|_V$ be the associated norm, i.e.

$$|\mathbf{w}|_V = |\varepsilon(\mathbf{w})|_Q. \tag{3.8}$$

It follows from (3.2), (3.6) and (3.8) that $|.|_V$ and $|.|_{H^1(\Omega)^d}$ are equivalent norms on V and therefore $(V, |.|_V)$ is a real Hilbert space. From (3.3), (3.6) and (3.8), we find that

$$|\mathbf{w}|_{L^2(\Gamma_c)^d} \leq C_1 |\mathbf{w}|_V \quad \forall \mathbf{w} \in V, \tag{3.9}$$

where C_1 is a positive constant depending only on the domain Ω, Γ_e and Γ_c.

3.1.3 Spaces for the stress field

For the divergence of $\boldsymbol{\sigma}$, we use the notation $\operatorname{div} \boldsymbol{\sigma}$ or $\sigma_{ij,j}$ and we define the space

$$Q_1 = \{\boldsymbol{\sigma} \in Q : \operatorname{div} \boldsymbol{\sigma} \in L^2(\Omega)^d\}. \tag{3.10}$$

The space Q_1 is a Hilbert space with a inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\varsigma})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\varsigma})_Q + (\operatorname{div} \boldsymbol{\sigma}, \operatorname{div} \boldsymbol{\varsigma})_{L^2(\Omega)^d},$$

and the associated norm $|.|_{Q_1}$.
We denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\sigma \in_{Q_1}$ such that

$$\sigma_\nu = (\sigma\nu).\nu \quad \boldsymbol{\sigma}_\tau = \sigma\nu - \sigma_\nu \nu. \tag{3.11}$$

Furthermore, the Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\mathrm{w}) dx + \int_\Omega \operatorname{div} \boldsymbol{\sigma}.\mathbf{w} dx = \int_\Gamma \boldsymbol{\sigma\nu}.\mathbf{w} dx \;\; \forall \mathbf{w} \in H^1(\Omega)^d. \tag{3.12}$$

3.1.4 Spaces for temperators contact problems

Let F be a closed subspace of $H^1(\Omega)$ defined by

$$F = \{y \in E : y = 0 \text{ sur } \Gamma_e \cup \Gamma_d\}, \;\; E = H^1(\Omega).$$

By the poincaré inequality, there exists a constant c such that

$$|\nabla y|_H \geq c |y|_{H^1(\Omega)}, \tag{3.13}$$

Here, c is a positive constant depending on the problem data but is independent of the solutions, it's value can be change from one line to another. We define the inner product on F by

$$(z, y)_E = (\nabla z, \nabla y)_H \forall y, z \in F, \tag{3.14}$$

it follows from (3.13) and (3.14) that $|.|_{H^1(\Omega)}$ and $|.|_E$ are equivalent norms on F. So $(F, |.|_F)$ is a Hilbert space. Moreover, by Sobolev's trce theorem, there exists a constant c_Γ which depends only on $\Omega, \Gamma_e, \Gamma_d$ and Γ_c such that

$$|\theta|_{L^2(\Gamma_c)} \leq c_\Gamma |\theta|_F \tag{3.15}$$

$F^{'}$ is the dual space of F. Identifier $L^2(\Omega)$ with it's own dual we can write

$$F \subset L^2(\Omega) \subset F^{'}.$$

We denote by $\langle ., . \rangle$ the hook duality between F and $F^{'}$, $|.|_{F^{'}}$ denotes the norm on $F^{'}$.
Also,

$$\langle \theta, \vartheta \rangle_{E' \times E} = (\theta, \vartheta)_{L^2(\Omega)}, \tag{3.16}$$

for $\theta \in L^2(\Omega)$ and $\vartheta \in F$.

3.2 Modelling of Thermo-Viscoelastic and Viscoplastic contact

In this section, we present a general physical framework for the various contact problems discussed in this thesis. We then introduce the equation of motion, the constitutive laws of different materials, and the boundary conditions used in this study.

3.2.1 Physical Setting

We consider the general physical setting shown in Figure 3.2.1 that we describe in what follows. A deformable body which occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$, with a regular boundary surface Γ, composed of three sets Γ_e, Γ_c and Γ_d, relatively closed with mutually disjoint relatively open interiors. On Γ_e the body is held fixed, time-dependent surface tractions of density $\boldsymbol{f}_0$ act on Γ_d and body forces of density $\boldsymbol{f}_1$ act in Ω. In the reference configuration the body may come in contact over Γ_c with a conductive obstacle, the so called foundation. We assume that in the reference configuration there exists a gap $h = h(\boldsymbol{x})$ between the foundation and Γ_c, as measured in the direction normal $\boldsymbol{\nu}$ to Γ_c.
We are interested in mathematical models that describe the equilibrium of the mechanical state of the body during the time interval $[0, T]$, $T > 0$. We denote by $\mathbf{u} : \Omega \times [0, T] \longrightarrow \mathbb{R}^d$ the field of displacements, $\boldsymbol{\sigma} : \Omega \times [0, T] \longrightarrow \mathbb{S}^d$ the stress field. The state of the system is completely determined by $(\mathbf{u}, \boldsymbol{\sigma})$.

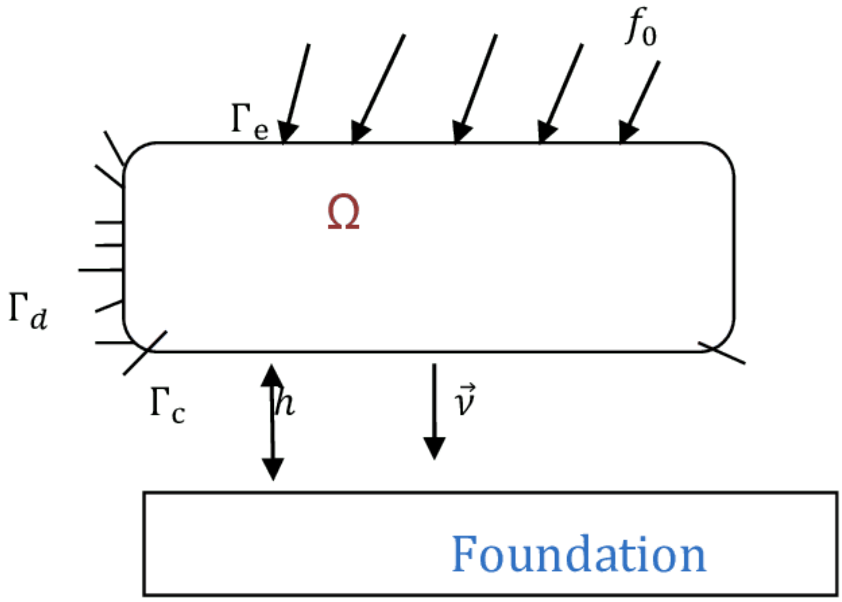

Figure 3.1: The physical setting

3.2.2 Equations of motion and of equilibrium

The equation of motion that governs the evolution of the mechanical state of the body is

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f}_1 = \varrho \ddot{\mathbf{u}} \tag{3.17}$$

where $\varrho : \Omega \to \mathbb{R}_+$ is the mass density and $\boldsymbol{f}_1$ is the density of applied forces, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ represent first and second time derivative, respectively. In certain situations, equation (3.17) be further simplified. For example when the velocity field $\dot{\mathbf{u}}$ varies very slowly with respect to time and as a result the term $\varrho\ddot{\mathbf{u}}$ debecomes negligible: it is then a quasistatic processes. In this case equation (4.66) becomes

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f}_1 = 0. \tag{3.18}$$

The description of our model is not yet finished, since we have unknown more than Equations. The equation is therefore insufficient, on its own, to describe the balance of material bodies, it must be completed by other relationships which characterize the behavior of each type of material and which we refer to as the general term laws of behavior, this is the subject of the following paragraph.

3.2.3 Constitutive equations

Viscoplastic constitutive laws
The viscoplastic constitutive law can be written in the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}),$$

where $\mathcal{A}$ represents the viscosity operator and $\mathcal{G}$ is the plasticity operator.
Thermo-viscelastic laws with damage A general Thermo-viscelastic laws with damage can be written in the form

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) + \int_0^t \mathcal{B}(t-s, \boldsymbol{\varepsilon}(\mathbf{u}(s)), \varsigma(s), \theta(s))ds.$$

Where $\mathcal{A}$ and $\mathcal{C}$ denote the linear viscosity operator and the elastic operator, respectively and $\mathcal{B}$ is the relaxation tensor depending on the damage ς and the temperature θ. Such that the evolution of damage function ς is described by the following differential inclusion

$$\dot{\varsigma} - \Delta\varsigma + \partial\Psi_k(\varsigma) \ni \mathcal{S}(\varepsilon(\mathbf{u}), \varsigma, \theta),$$

where $\partial\Psi_k$ is the sub-differential of the indicator function on K.
And the evolution of the tomperature field θ is governed by the heat equation , obtained fromthe conservation of energy and defined by the following parabolic equation

$$\dot{\theta} - \mathcal{K}\Delta\theta = \Phi(\varepsilon(\mathbf{u}), \varsigma, \theta) + q,$$

where $\mathcal{K} > 0$, θ is a non linear constitutive function that represents the the heat generated by internal forces and q is the amount of heat in the volume during the time interval.

3.2.4 Contact boundary conditions

The boundary conditions on the contact surface are described in both the normal direction and in the tangent plane, the latter being called friction conditions. In the normal direction, we can distinguish unilateral contact (when the obstacle is rigid), bilateral contact (when there is no separation between the body and the obstacle), normal compliance (when the obstacle is deformable) or instantaneous normal response (when the contact surface is reactive). Apart from the limit case when the tangential constraint is zero (the case without friction), friction can be at a threshold (slippage occurs when the friction force reaches a critical value) or without a threshold (when slippage occurs for any friction force). Among the threshold friction laws, the most commonly used in literature are those of Coulomb and Tresca; they model dry friction, while laws of non-threshold friction model lubricated friction.

Boundary conditions are also influenced by taking into account the different underlying phenomena such as frictional contact, adhesion, wear, and thermal effects. Furthermore, even if we neglect these phenomena and limit ourselves to sliding contact with a threshold, the dependence of the friction threshold on slip or slip velocity can be considered, thus influencing the boundary conditions of the mathematical model under consideration.

We now turn to the boundary conditions for the system. To this end, we assume that the surface Γ is Lipschitz.

The displacement and traction boundary condition We assume that the body is held fixed on Γ_e and, therefore

$$\mathbf{u} = 0 \text{ on } \Gamma_e. \tag{3.19}$$

Known tractions of density f act on Γ_d, so we use the condition

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_0 \text{ on } \Gamma_d. \tag{3.20}$$

This condition is called the traction boundary condition.
To complete the mathematical model which describes the evolution of the body, it is necessary specify the boundary conditions on Γ_c, this is the subject of the contact conditions and frictionless or frictional laws that we will describe in what follows.

Conditions with normal compliance

In this condition we assume that the body is in contact with a deformable foundation such that the reaction of the foundation involves the normal compliance law

$$-\sigma_\nu = p_\nu(u_\nu - h),$$

where $p_\nu(.)$ is a nonnegative prescribed function that vanishes for negative argument. Indeed, when there is no contact (i.e., when $u_\nu < h$) then $\sigma_\nu = 0$ and, therefore, the normal pressure vanishes. When there is contact then $\sigma_\nu \leq 0$ and, therefore, the reaction of the foundation is towards the body.
As example of a normal compliance function p_ν we can consider

$$p_\nu(s) = c_\nu s_+,$$

or, more generally

$$p_\nu(s) = c_\nu (s_+)^m,$$

here the constant $c_\nu > 0$ is the surface stiffness coefficient and $m \geq 1$ is the normal compliance exponent, we recall that $s_+ = \max\{0, s\}$ is the positive part of s.
A second example is given by

$$p_\nu(s) = \begin{cases} c_\nu s_+ & \text{if } s \leq \alpha, \\ c_\nu \alpha & \text{if } s > \alpha, \end{cases}$$

where α is a positive coefficient related to the wear and hardness of the surface. In this case, the normal compliance contact condition means that if the penetration is too large the obstacle offers no additional resistance to penetration.

Normal damped response contact

A contact condition with normal damped response has been primarily used to describe a lubricated contact. This condition assumes that the normal stress on the contact surface is related to the normal velocity, that is to say

$$-\sigma_\nu = p_\nu(\dot{u}_\nu).$$

Where, p_ν is a non negative prescribed function which vanishes for negative argument.
We now move on to the conditions in the tangential directions, called friction conditions or friction laws.

Contact condition

It means that the contact between the body and the foundation is maintained at all time and is modelled by the equation

$$u_\nu = 0, \ \textit{on } \Gamma_c.$$

This condition can be found in many machines and in moving parts and components of mechanical equipment.

Signorini contact condition

They say that the contact between the body and a rigid foundation is frictionless if the tangential movements are free, which results in

$$\boldsymbol{\sigma}_\tau = 0. \tag{3.21}$$

Since the base is rigid, it will not undergo any deformations. Therefore, the body will not be able to penetrate it. This property is mathematically expressed by the following inequality.

$$u_\nu \leq 0, \quad \text{on } \Gamma_c \times [0, T]. \tag{3.22}$$

At the points of Γ_c such as $u_\nu < 0$, the deformable body leaves the rigid base. The normal stresses are then zero. Consequently, we have

$$u_\nu < 0 \Longrightarrow \sigma_\nu = 0, \quad \text{on } \Gamma_c \times [0, T]. \tag{3.23}$$

At the points of Γ_c such that $u_\nu = 0$, it is assumed that the rigid base exerts an unknown reaction in the direction of the normal and oriented towards Ω. We have

$$u_\nu = 0 \Longrightarrow \sigma_\nu \leq 0, \quad \text{on } \Gamma_c \times [0, T]. \tag{3.24}$$

To summarize, the contact conditions (3.21)- (3.24) are written in a combined manner as follows.

$$u_\nu \leq 0, \ \sigma_\nu \leq 0, \ \boldsymbol{\sigma}_\tau = 0, \ \sigma_\nu u_\nu = 0, \quad \text{on } \Gamma_c \times [0, T]. \tag{3.25}$$

The boundary conditions of contact of the form (3.25) are also called Signorini contact conditions.

Type contact condition with adhesion

The signorini contact condition with adhesion is given by the following relations

$$\begin{aligned} &u_\nu \leq 0, \ \sigma_\nu - \gamma_\nu R_\nu(u_\nu)\beta^2 \leq 0, \\ &(\sigma_\nu - \gamma_\nu R_\nu(u_\nu)\beta^2)u_\nu = 0, \quad \text{on } \Gamma_c \times [0, T]. \end{aligned} \tag{3.26}$$

Where β represent the adhesion field. The evolution of the adhesion field is described by a differential equation of the form

$$\dot{\beta} = -(\gamma_\nu \beta (R_\nu(u_\nu))^2 - \epsilon_a)_+,$$

$$\beta(0) = \beta_0.$$

γ_ν is the adhesion coefficient, R_ν is a truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if} \quad s < -L, \\ -s & \text{if} \quad -L \leq s \leq 0, \\ 0 & \text{if} \quad s > 0, \end{cases} \tag{3.27}$$

where $L > 0$ is the characteristic lenght of the link.
Clearly, the function R satisfies

$$R(s)s \leq 0,\ |R(s)| \leq |s|,\ |R(s)| \leq L,\quad \forall s \in \mathbb{R}, \tag{3.28}$$
$$|R(s_1) - R(s_2)| \leq |s_1 - s_2|,\quad \forall s_1, s_2 \in \mathbb{R}. \tag{3.29}$$

The Signorini contact conditions model the contact between a deformable body and a rigid base. Therefore, it is possible to consider contact conditions between a deformable body and a deformable base.

Coulomb's friction laws

This is one of the most common friction laws in mathematical literature.

It is characterized by the involvement of normal stress in the friction threshold and can be stated as follows

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \leq \mu|\sigma_\nu|, \\ |\boldsymbol{\sigma}_\tau| < \mu|\sigma_\nu| \Longrightarrow \dot{\mathbf{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = \mu|\sigma_\nu| \Longrightarrow \exists\lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda\dot{\mathbf{u}}_\tau, \end{cases} \tag{3.30}$$

where $\mu > 0$ is the coefficient of friction.
A quasi-static version of Coulomb's laws can be stated in the form

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \leq F_b, \\ \\ -\boldsymbol{\sigma}_\tau = F_b \frac{\dot{\mathbf{u}}_\tau}{|\dot{\mathbf{u}}_\tau} \text{ if } \dot{\mathbf{u}}_\tau \neq 0. \end{cases} \quad \text{on} \Gamma_c \times (0, T) \tag{3.31}$$

Here, $\dot{\mathbf{u}}_\tau$ is the tangential velocity or slip ratio and F_b represents the friction threshold (this threshold is not fixed and depends on the coefficient of friction and the solution of the problem).
We can take the following choice for F_b

$$F_b = F_b(\sigma_\nu) = \mu|\sigma_\nu|,$$

where $\mu > 0$ is the friction coefficient. This choice in (3.27) leads to the classic version of Coulomb's law. The choices

$$F_b = F_b(\sigma_\nu) = p_\tau(u_\nu - h)| \text{ and } F_b = F_b(\sigma_\nu) = p_\tau(\dot{u}_\nu)$$

are compatible with the normal compliance and normal damped response condition, respectively. Here, p_τ is a positive function that becomes zero for

negative arguments, i.e. when there is no contact.
Coulomb's law is often used for rigid or elastic bodies. It should also be noted that this is a threshold law. As long as the threshold is not reached, there is no slippage. This threshold is variable and depends on the normal stress, which represents a major difficulty for the mathematical study of this law of friction.

3.2.5 Wear contact process

Since the bilateral contact between the base and the body, the surface is eroded on the part Γ_c. We introduce the wear function $\omega : (0,T)\times\Gamma_c \longrightarrow \mathbb{R}_+$ which measures the wear of the surface. To describe the evolution of wear we use the modified version of Archard's law. The rate form of Archard's law is

$$\dot{\omega} = -k_1\sigma_\nu|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|, \tag{3.32}$$

where $\mathbf{v}^*$ is the tangential velocity of the foundation, k_1 is the wear coefficient and $|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|$ represents the slip speed between the contact surface and the foundation. If we assume that v^* is large so

$$\dot{\omega} = -k_1\sigma_\nu|\dot{\mathbf{u}}_\tau|.$$

For simplicity we assume that $\omega(0) = 0$, i.e. there is no wear initially.
Friction condition with wear
Now we describe the process of frictional contact on the surface Γ_c. We use a version of the normal compliance condition to model the contact. Since the process involves mating surface wear, we take consider changing the geometry by replacing h with $h+\omega$. Therefore

$$-\sigma_\nu = p_\nu(u_\nu - \omega - h) \text{ on } \Gamma_c \times (0,T),$$

The friction law is chosen as(3.30), so

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \le \mu p_\nu, \\ |\boldsymbol{\sigma}_\tau| < \mu|\sigma_\nu| \Longrightarrow \dot{\mathbf{u}}_\tau = \mathbf{v}^*, \\ |\boldsymbol{\sigma}_\tau| = \mu|\sigma_\nu| \Longrightarrow \exists\lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = \mathbf{v}^* - \lambda\dot{\mathbf{u}}_\tau, \end{cases}$$

where p_ν and μ is the coefficient of friction that depends on the density of the wear particles, the temperature and the rate of slippage.

3.2.6 Thermal conditions

The boundary condition associated with the derivative of the Fourier condition temperature

$$-\mathcal{K}_{ij}\frac{\partial\theta}{\partial\nu}\nu_j = \mathcal{K}_e(\theta-\theta_f) - h_\tau(|\dot{\boldsymbol{u}}_\tau|),\ \text{on}\Gamma_c\times[0,T], \tag{3.33}$$

$$\theta = 0,\ \text{in }\Gamma_d\cup\Gamma_e\times[0,T], \tag{3.34}$$

where θ_f is the known temperature of the foundation, h_τ the tengential function and $\mathcal{K}_e$ is the temperature exchange between the foundation and the deformable body. (3.34) is the temperature boundary condition on $\Gamma_d\cup\Gamma_e$. We also consider the conditions.

$$\mathcal{K}_0\frac{\partial\theta}{\partial\nu} + B\theta = 0,\ \text{on}\Gamma\times[0,T], \tag{3.35}$$

$$\frac{\partial\varsigma}{\partial\nu} = 0,\ \text{on}\Gamma\times[0,T], \tag{3.36}$$

which represent, respectively on Γ, a Fourier boundary condition for the temperature and a homogeneous Neumann boundary condition for the damage field on Γ, where $\frac{\partial\varsigma}{\partial\nu}$ represents the normal derivative of ς.

Chapter 4

Frictional contact problem with wear for thermo-viscoelastic materials with damage

In this chapter, we consider a mathematical model describing the contact with friction between a thermo-viscoelastic body and a base.The contact is described by a normal compliance condition with friction. We establish a result of existence and uniqueness of the solution using fixed point techniques and first-order evolution equations with monotone operatorsn which we can find also in reference [19].

4.1 Problem statement and variational formulation

Under these considerations, the considered mechanical problem is formulated as follows.

Problem$P$

Find a displacement field $\mathbf{u} : \Omega \times [0, T] \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \longrightarrow \mathbb{S}^d$, a damage field $\varsigma : \Omega \times [0, T] \longrightarrow \mathbb{R}$, a temperature field $\theta : \Omega \times [0, T] \longrightarrow \mathbb{R}$

and a wear field $w : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}$ such that

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{C}\varepsilon(\mathbf{u}) + \int_0^t \mathcal{B}(t-s, \boldsymbol{\varepsilon}(\mathbf{u}(s)), \varsigma(s), \theta(s))ds \quad \text{in } \Omega \times (0,T), \tag{4.1}$$

$$\dot{\varsigma} - \Delta\varsigma + \partial\Psi_k(\varsigma) \ni \mathcal{S}(\varepsilon(\mathbf{u}), \varsigma, \theta) \quad \text{in } \Omega \times (0,T), \tag{4.2}$$

$$\dot{\theta} - \mathcal{K}\Delta\theta = \Phi(\varepsilon(\mathbf{u}), \varsigma, \theta) + q \quad \text{in } \Omega \times (0,T), \tag{4.3}$$

$$\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f}_1 = \varrho\ddot{\mathbf{u}} \quad \text{in } \Omega \times (0,T), \tag{4.4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_e \times (0,T), \tag{4.5}$$

$$\sigma\nu = \boldsymbol{f}_0 \quad \text{on } \Gamma_d \times (0,T), \tag{4.6}$$

$$\left\{\begin{array}{l} -\sigma_\nu = p_\nu(u_\nu - w), |\sigma_\tau| \le p_\tau(u_\nu - w), \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu - w) \Longrightarrow \dot{\mathbf{u}}_\tau = \mathbf{v}^*, \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu - w) \Longrightarrow \dot{\mathbf{u}}_\tau = \mathbf{v}^* - \lambda\boldsymbol{\sigma}_\tau, \ \lambda > 0, \\ \dot{w} = k_\omega|\mathbf{v}^*|p_\nu(u_\nu - w), \end{array}\right. \quad \text{on } \Gamma_c \times (0,T), \tag{4.7}$$

$$\mathcal{K}\frac{\partial\theta}{\partial\nu} + \mu_1\theta = 0 \quad \text{on } \Gamma \times (0,T), \tag{4.8}$$

$$\frac{\partial\varsigma}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{4.9}$$

$$w(0) = 0 \quad \text{on } \Gamma_c, \tag{4.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \ \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0, \ \theta(0) = \theta_0, \ \varsigma(0) = \varsigma_0 \quad \text{on } \Omega. \tag{4.11}$$

We will now describe issue (4.1) – (4.11). The thermo-viscoelastic constitutive with long memory and damage is represented by equation (4.1), where $\mathcal{B}$ is the relaxation tensor that depends on the temperature and damage. $\mathcal{A}$ and $\mathcal{C}$ stand for the linear viscosity operator and the elastic operator, respectively. The damage field's evolution, which is determined by the source damage function $\mathcal{S}$, is shown in equation (4.2). The subdifferential of the indicator function within the permissible damage function set is denoted by $\partial\Psi_K$. The temperature field evolution is represented by equation (4.3), where μ_0, q is the density of volume heat sources, and Φ is a nonlinear constitutive function that reflects the heat generated by the work of internal forces. The boundary conditions for displacement and traction are represented by equations (4.5) and (4.6), respectively. Equation (4.4) represents the equilibrium equation for the stress displacement fields. The state with wear, Coulomb's friction law, and normal compliance is described by equation (4.7). The wear function w calculates the total amount of surface wear. The differential form of Archard's law, where $k_w > 0$ is a wear coefficient and $\mathbf{v}^*$ is a constant vector that reflects the foundation's displacement, controls the evolution of the wear of the contacting surface, Prescribed functions of the friction bound

and normal compliance, respectively, are denoted by p_ν and p_τ. (4.8) denotes the conduction coefficient of Γ and is a Fourier boundary condition for the temperature θ when $\mu_1 > 0$. For the damage ς, the homogeneous Niemann border condition is (4.9), where $\frac{\partial \varsigma}{\partial \nu}$ is the normal derivative of Γ. The wear function's initial state, shown by (4.10), indicates that the foundation is new at this point. Ultimately, the initial data in (4.11) are the functions $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, θ_0 and ς_0.
In examining the mechanical problem (4.1)-(4.11), we take into account the following presumptions.

$$\begin{cases} (1) & \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d; \\ (2) & \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2, \\ & \ \text{for any } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ a.e } \boldsymbol{x} \in \Omega; \\ (3) & \text{There exists } M_1, M_2 > 0 \text{ such that} \\ & |(\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})| \leq M_1 |\boldsymbol{\varepsilon}| + M_2, \ \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ a.e } x \in \Omega; \\ (4) & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is continuous on } \mathbb{S}^d, \ \ \text{a.e. } \boldsymbol{x} \in \Omega; \\ (5) & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{is Lebesgue measurable on } \Omega \\ & \ \text{for any } \varepsilon \in \mathbb{S}^d. \end{cases} \tag{4.12}$$

$$\begin{cases} (1) & \mathcal{C} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d; \\ (2) & \text{There exists } m_{\mathcal{C}} > 0 \text{ such that} \\ & \mathcal{C}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon} \geq m_{\mathcal{C}} |\boldsymbol{\varepsilon}|^2, \ \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ a.e } \boldsymbol{x} \in \Omega; \\ (3) & \mathcal{C}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) \cdot \boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_1 \cdot \mathcal{C}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2) \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega; \\ (4) & \mathcal{C}_{ijkl} \in L^\infty(\Omega), \forall i, j, k, l = 1, .., d \end{cases} \tag{4.13}$$

$$\begin{cases} (1) & \mathcal{B} : \Omega \times [0, T] \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d; \\ (2) & \text{There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ & |\mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1, \zeta_1, \theta_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \zeta_2, \theta_2)| \leq L_{\mathcal{B}}(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\zeta_1 - \zeta_2| + |\theta_1 - \theta_2|) \\ & \text{for all } t \in [0, T], \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \zeta_1, \zeta_2, \theta_1, \theta_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Omega; \\ (3) & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}, \zeta, \theta) \text{ is Lebesgue measurable on } \Omega \\ & \ \text{for any } t \in [0, T], \boldsymbol{\varepsilon} \in \mathbb{S}^d, \zeta, \theta \in \mathbb{R}; \\ (4) & \text{The mapping } t \mapsto \mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}, \zeta, \theta) \text{ is continuous measurable on } [0, T] \\ & \ \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \zeta \in, \theta \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Omega; \\ (5) & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, t, \mathbf{0}, 0, 0) \in Q. \end{cases} \tag{4.14}$$

$$\begin{cases} (1) & \mathcal{S} : \Omega \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}; \\ (2) & \text{There exists } L_S > 0 \text{ such that} \\ & |\mathcal{S}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \zeta_1, \theta_1) - \mathcal{S}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \zeta_2, \theta_2)| \leq L_S(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\zeta_1 - \zeta_2| + |\theta_1 - \theta_2|) \\ & \forall \boldsymbol{x} \in \Omega, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \forall\, \zeta_1, \zeta_2, \theta_1, \theta_2 \ \in \ \mathbb{R}; \\ (3) & \text{The function } \boldsymbol{x} \longrightarrow \mathcal{S}(\boldsymbol{x}, \boldsymbol{\varepsilon}, \zeta, \theta) \text{ is Lebesgue measurable on } \Omega \\ & \quad \forall\, \boldsymbol{\varepsilon} \ \in \mathbb{S}^d, \forall \zeta \in \mathbb{R} \\ (4) & \text{The function } x \longrightarrow \mathcal{S}(x, \mathbf{0}, 0, 0) \ \in \ L^2(\Omega). \end{cases} \tag{4.15}$$

$$\begin{cases} (1) & \Phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}; \\ (2) & \text{There exists } L_\Phi > 0 \text{ such that} \\ & |\Phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \varsigma_1, \theta_1) - \Phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \varsigma_2, \theta_2)| \leq L_\Phi(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\varsigma_1 - \varsigma_2| + |\theta_1 - \theta_2|) \\ & \forall \boldsymbol{x} \in \Omega, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \ \forall\, \varsigma_1, \varsigma_2, \theta_1, \theta_2 \ \in \ \mathbb{R}; \\ (3) & \text{The mapping } \boldsymbol{x} \longrightarrow \Phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \varsigma, \theta) \text{ is Lebesgue mesurable on } \Omega \\ & \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \varsigma, \theta \ \in \mathbb{R}; \\ (4) & \text{The function } \boldsymbol{x} \longrightarrow \Phi(\boldsymbol{x}, \mathbf{0}, 0, 0) \in L^2(\Omega). \end{cases} \tag{4.16}$$

$$\begin{cases} (1) & p_r : \Gamma_c \times \mathbb{R} \longrightarrow \mathbb{R}_+, (r = \nu, \tau); \\ (2) & \text{There exists } L_r > 0 \text{ such that } |p_r(\boldsymbol{x}, a_1) - p_r(\boldsymbol{x}, a_2)| \leq L_r |a_1 - a_2| \\ & \forall \boldsymbol{x} \in \Omega, \forall a_1, a_2 \in \mathbb{R}, a.e.\ \boldsymbol{x} \in \Gamma_c; \\ (3) & p_r(\boldsymbol{x}, a) = 0 \quad \forall a \leq 0, \ a.e.\ \boldsymbol{x} \in \Gamma_c; \\ (4) & \text{The mapping } \boldsymbol{x} \longrightarrow p_r(\boldsymbol{x}, a) \text{ is Lebesgue measurable on } \Gamma_c \quad \forall a \in \mathbb{R}. \end{cases} \tag{4.17}$$

For $\varrho^* > 0$, we assume that the mass density fulfills

$$\varrho \in L^\infty(\Omega), \ \varrho(\boldsymbol{x}) \geq \varrho^*, \quad \text{a.e.} \boldsymbol{x} \in \Omega. \tag{4.18}$$

We assume that the mechanical and heat forces satisfy

$$q \in L^2(0, T; L^2(\Omega)), \ \boldsymbol{f}_1 \in W^{1,1}(0, T; H), \ \boldsymbol{f}_0 \in W^{1,1}(0, T; L^2(\Gamma_d)^d). \tag{4.19}$$

We indicate the following element by $\boldsymbol{f}_2(t) \in V$

$$(\boldsymbol{f}_2(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_1.\boldsymbol{v} dx + \int_{\Gamma_d} \boldsymbol{f}_0.\boldsymbol{v} da, \quad \forall \boldsymbol{v} \in V. \tag{4.20}$$

Let us define $j : V \times V \times L^2(\Gamma_c) \to \mathbb{R}$ be the functional

$$j(\boldsymbol{u}, \boldsymbol{v}, w) = \int_{\Gamma_c} p_\nu(u_\nu - w) v_\nu da + \int_{\Gamma_c} p_\tau(u_\nu - w) \|\boldsymbol{v}_\tau\| da, \tag{4.21}$$

the functional j satisfies

$$\begin{aligned} &\text{for all } \boldsymbol{u} \in V \text{and } w \in L^2(\Gamma_c) \\ &j : \boldsymbol{v} \to j(\boldsymbol{u}, \boldsymbol{v}, w) \text{ is lower semicontinuous, convex, and proper on } V. \end{aligned} \tag{4.22}$$

When we combine assumptions (4.17) and (3.9), we get

$$\begin{aligned} & j(\boldsymbol{u}_1, \boldsymbol{v}_2, w) - j(\boldsymbol{u}_1, \boldsymbol{v}_1, w) + j(\boldsymbol{u}_2, \boldsymbol{v}_1, w) - j(\boldsymbol{u}_2, \boldsymbol{v}_2, w) \\ & \leq C_1^2(L_\nu + L_\tau)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V, \\ & \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V, \ \forall w \in L^2(\Gamma_c). \end{aligned} \tag{4.23}$$

As we can see, condition (4.19) suggests

$$\boldsymbol{f}_2 \in W^{1,1}(0, T; V) \tag{4.24}$$

We also suppose that the initial data satisfy

$$\theta_0 \in L^2(\Omega), \ \varsigma_0 \in K, \ w_0 \in L^2(\Gamma_c), \ \boldsymbol{u}_0 \in V, \ \dot{\boldsymbol{u}}_0 \in D(\partial_2 j), \tag{4.25}$$

where $\partial_2 j$ indicates the partial subdifferential with respect to the second argument of the operator j and $D(\partial_2 j)$ symbolize its domain.
There exists $\mathbf{g} \in H$ such that

$$\begin{aligned} & \big(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}_0) + \mathcal{B}\varepsilon(\boldsymbol{u}_0), \varepsilon(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}_0)\big)_Q + j(\boldsymbol{u}_0, \boldsymbol{v}, 0) - j(\boldsymbol{u}_0, \dot{\boldsymbol{u}}_0, 0) \\ & \geq (\mathbf{g}, \boldsymbol{v} - \dot{\boldsymbol{u}}_0), \quad \forall \boldsymbol{v} \in V. \end{aligned} \tag{4.26}$$

The following continuous functionals $A : V \to V'$ and $B : V \to V'$ are presented by $\forall \boldsymbol{u} \in V$, $\forall \boldsymbol{v} \in V$

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V' \times V} = (\mathcal{C}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q, \tag{4.27}$$

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle_{V' \times V} = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q. \tag{4.28}$$

We define the bilinear forms $a : E \times E \to \mathbb{R}$ and $b : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a(\xi, \kappa) = \mu_0 \int_\Omega \nabla\xi \cdot \nabla\kappa dx + \mu_1 \int_\Gamma \xi\kappa da, \ \ \forall \, \xi, \ \kappa \in E, \tag{4.29}$$

$$b(\varsigma, \zeta) = \int_\Omega \nabla\varsigma \cdot \nabla\zeta dx, \ \ \forall \, \varsigma, \ \zeta \in H^1(\Omega). \tag{4.30}$$

Applied to the Hilbert space H, we shall employ a modified inner product given by

$$((\boldsymbol{u}, \boldsymbol{v}))_H = (\varrho\boldsymbol{u}, \boldsymbol{v})_H, \ \ \forall \boldsymbol{u}, \boldsymbol{v} \in H, \tag{4.31}$$

that is, we consider $||\cdot||_H$ to be the associated norm, meaning that it is weighted with ϱ

$$||\boldsymbol{v}||_H = (\varrho\boldsymbol{v}, \boldsymbol{v})_H^{\frac{1}{2}}, \ \ \forall \boldsymbol{v} \in H. \tag{4.32}$$

It follows from assumptions (4.18) that the inclusion mapping of $(V, |\cdot|_H)$ into $(H, ||\cdot||_H)$ is dense and continuous, and that $||\cdot||_H$ and $|\cdot|_H$ are comparable norms on H. We identify H with its own dual by writing V', which stands for the dual space of V.

$$V \subset H \subset V'.$$

We recall that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} = ((\boldsymbol{u}, \boldsymbol{v}))_H, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H. \tag{4.33}$$

From (4.4) and use Green's formula (3.12), we get

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_Q + ((\ddot{\boldsymbol{u}}, \boldsymbol{v}))_H = (\boldsymbol{f}_1(t), \boldsymbol{v})_V + \int_\Gamma \boldsymbol{\sigma}\nu.\boldsymbol{v}da, \quad \forall \boldsymbol{v} \in V.$$

Then, using conditions (4.5), (4.6), (4.7) and (4.11) combined with definitions (4.21) and (4.20) to find

$$\begin{aligned}&(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \int_0^t \mathcal{B}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \varsigma(s), \theta(s)\big)ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_Q \\ &+ ((\ddot{\boldsymbol{u}}, \boldsymbol{v} - \dot{\boldsymbol{u}}))_H + j(\boldsymbol{u}, \boldsymbol{v}, w) - j(\boldsymbol{u}, \dot{\boldsymbol{u}}, w) \geq (\boldsymbol{f}_2(t), \boldsymbol{v} - \dot{\boldsymbol{u}})_V, \ \ \forall\ \boldsymbol{v}\ \in\ V.\end{aligned} \tag{4.34}$$

Now, multiplying Eq (4.3) by $\alpha \in E$

$$\int_\Omega \dot{\theta}\alpha dx - \int_\Omega \mathcal{K}\Delta\theta\alpha dx = \int_\Omega \Phi(\varepsilon(\boldsymbol{u}), \varsigma, \theta)\alpha dx + \int_\Omega q\alpha dx.$$

According to the Green's formula, we get

$$\int_\Omega \dot{\theta}\alpha dx - \int_\Gamma (\mathcal{K}_{ij}\theta, i)\nu_j.\alpha da + \int_\Omega (\mathcal{K}_{ij}\theta_{,j})\alpha_{,j}dx = \int_\Omega \Phi(\varepsilon(\boldsymbol{u}), \varsigma, \theta)\alpha dx + \int_\Omega q\alpha dx.$$

using the boundary condition (4.8), we obtain

$$\int_\Omega \dot{\theta}\alpha dx + \mu_1 \int_\Gamma (\theta\alpha da + \int_\Omega (\mathcal{K}_{ij}\theta_{,j})\alpha_{,j}dx = \int_\Omega \Phi(\varepsilon(\boldsymbol{u}), \varsigma, \theta)\alpha dx + \int_\Omega q\alpha dx. \tag{4.35}$$

Next, taking in to account inclusion (4.2) and using the definition of the subdifferential of the indicator function Ψ_k and the integration by parts with the boundary condition (4.9), we have

$$\int_\Omega \dot{\varsigma}(\beta - \varsigma(t))dx + \int_\Omega \varsigma(t).(\beta - \varsigma(t))dx \geq \int_\Omega \mathcal{S}(\varepsilon(\boldsymbol{u}(t)), \varsigma(t), \theta(t))(\beta - \varsigma(t))dx, \tag{4.36}$$

$$\forall\ \ \beta \in K.$$

We gather now (4.34), (4.35) and (4.36) and obtain the variational formulation of the Problem$P$.

Problem PV

Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \longrightarrow V$, a damage field $\varsigma : \Omega \times [0,T] \longrightarrow L^2(\Omega)$, a temperature field $\theta : \Omega \times [0,T] \longrightarrow E$ and a wear field $w : \Gamma_3 \times [0,T] \longrightarrow L^2(\Gamma_c)$ such that

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \int_0^t \mathcal{B}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \varsigma(s), \theta(s)\big)ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_Q \\
&+ ((\ddot{\boldsymbol{u}}, \boldsymbol{v} - \dot{\boldsymbol{u}}))_H + j(\boldsymbol{u}, \boldsymbol{v}, w) - j(\boldsymbol{u}, \dot{\boldsymbol{u}}, w) \geq (\boldsymbol{f}_2(t), \boldsymbol{v} - \dot{\boldsymbol{u}})_V, \quad \forall\, \boldsymbol{v} \in V,
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
&(\dot{\varsigma}, \beta - \varsigma(t))_{L^2(\Omega)} + b(\varsigma(t), \beta - \varsigma(t)) \\
&\geq (\mathcal{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \varsigma(t), \theta(t)), \beta - \varsigma(t))_{L^2(\Omega)}, \quad \forall\, \beta \in K,
\end{aligned} \tag{4.38}
$$

$$
\begin{aligned}
&\langle \dot{\theta}(t), \alpha \rangle_{E' \times E} + a(\theta(t), \alpha) = \langle \Phi(\varsigma(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \theta(t)), \alpha \rangle_{E' \times E} \\
&+ (q(t), \alpha)_{L^2(\Omega)}, \quad \forall\, \alpha \in E, \quad \text{a.e } \ t \in (0,T),
\end{aligned} \tag{4.39}
$$

$$
\dot{w} = k_\omega \|\mathbf{v}^*\| p_\nu(u_\nu - w), \qquad \text{a.e.} \ \ t \in (0,T). \tag{4.40}
$$

To study Problem (4.37)-(4.40), we need the following smallness assumption

$$
L_\nu + L_\tau < \frac{2m_{\mathcal{A}}}{C_1^2 + C_1}, \tag{4.41}
$$

where $m_{\mathcal{A}}$, C_1 and $L_r, (r = \nu, \tau)$ are given in (4.12), (3.9) and (4.17), respectively.
Our main existence and uniqueness result is stated and proved in the next section.

4.2 Existence and uniqueness result

Theorem 4.1. *Let the assumptions (4.12)-(4.26) and (4.41). Then there exists an unique solution $(\boldsymbol{u}, \varsigma, \theta, w)$ to problem PV. Moreover, the solution satisfies*

$$
\boldsymbol{u} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H), \tag{4.42}
$$

$$
\varsigma \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{4.43}
$$

$$
\theta \in C(0,T;L^2(\Omega)) \cap L^2(0,T;E), \ \ \dot{\theta} \in L^1(0,T;E') \tag{4.44}
$$

$$
w \in C^1(0,T;L^2(\Gamma_c)). \tag{4.45}
$$

The functions $\boldsymbol{u}, \boldsymbol{\sigma}, \varsigma, \theta$ and w which satisfy (4.1) and (4.37)-(4.40) referred to as a feeble resolution of the contact issue P.
We conclude that, under the assumptions (4.12)-(4.26) and (4.41), the mechanical problem (4.1)-(4.11) has a unique weak solution satisfying (4.42)-(4.45). The regularity of the weak solution is given by (4.42)-(4.45) and, in

term of stress,

$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1). \tag{4.46}$$

Indeed, it follows from (4.1), (4.12)(3), (4.13)(3), (4.13)(4), (4.14)(2) and the regularity (4.42)-(4.44) implies $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q)$. Let $t \in [0,T]$ and we choose as a test function $\boldsymbol{v} = \dot{\boldsymbol{u}}(t) \pm \mathbf{z}$ where $\mathbf{z} \in \mathcal{D}(\Omega)^d$ in (4.37) and using (3.12), (4.31) and (4.33) to obtain

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}_1 = \varrho \ddot{\boldsymbol{u}} \quad \text{in } H.$$

It now follows from (4.19) and (4.42) that $\operatorname{div} \boldsymbol{\sigma} \in W^{1,\infty}(0,T;H)$ which show (4.46).
The following is a proof of each step involved in the proof of Theorem 4.1, everywhere in this section assuming the validity of Theorem 4.1's assumptions, we examine C as a generic positive constant whose value varies depending on the problem's facts but remains independent over time.
Let $w \in C(0,T;L^2(\Gamma_c))$ and $\eta \in L^2(0,T;Q)$. In the first step we consider the following variational problem

Problem $PV_{w\eta}$

Find $\boldsymbol{u}_{w\eta} : [0,T] \to V$ such that

$$\begin{aligned}
&((\ddot{\boldsymbol{u}}_{w\eta}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{w\eta}(t)))_H + \langle A\boldsymbol{u}_{w\eta}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{w\eta}(t)\rangle_{V'\times V} \\
&+ \langle B\dot{\boldsymbol{u}}_{w\eta}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{w\eta}(t)\rangle_{V'\times V} + j(\boldsymbol{u}_{w\eta}(t), \boldsymbol{v}, w(t)) \\
&- j(\boldsymbol{u}_{w\eta}(t), \dot{\boldsymbol{u}}_{w\eta}(t), w(t)) \geq (\mathbf{F}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{w\eta}(t))_V, \quad \forall\, \boldsymbol{v} \in V, \\
&\boldsymbol{u}_{w\eta}(0) = \boldsymbol{u}_0,\ \dot{\boldsymbol{u}}_{w\eta}(0) = \dot{\boldsymbol{u}}_0,
\end{aligned} \tag{4.47}$$

where $(\boldsymbol{F}(t), \boldsymbol{v})_V = (\boldsymbol{f}_2(t), \boldsymbol{v})_V - (\eta(t), \varepsilon(\boldsymbol{v}))_Q, \quad \forall \boldsymbol{v} \in V.$
In the study of Problem $PV_{w\eta}$ we have the following result.

Lemma 4.1. *The problem $PV_{w\eta}$ has a unique solution which satisfies $\boldsymbol{u}_{w\eta} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H)$*

Proof. We may observe that the operator A based on assumptions (4.27), (3.8) and (4.13)(3), and satisfies the condition (2.11) with $\lambda = 0$ and $\omega = m_{\mathcal{G}}$. After, we define the set-valued operator $\psi : V \to V'$ by

$$\psi = B + \partial_2 j. \tag{4.48}$$

From (4.12)(2), we deduce that the operator B defined by (4.28), is monotone. Using (4.28) and (3.8), we have

$$|B\boldsymbol{u} - B\boldsymbol{v}|_{V'} \leq |\mathcal{A}\varepsilon(\boldsymbol{u}) - \mathcal{A}\varepsilon(\boldsymbol{v})|_Q,\ \forall \boldsymbol{u}, \boldsymbol{v} \in V,$$

keeping in mind (4.12)(2), (4.12)(3), (4.12)(4) and Krasnoselski's theorem (see [25, p.60]), we find that $B : V \to V'$ is a continuous operator. Using again (4.28) and (4.12)(2), we find that B is bounded.
From (4.23) and (4.22), thus, j must be maximally monotone. Consequently, since B is monotone, bounded and hemicontinuous from V to V', we conclude (see [5, p.39]) that $\psi = B + \partial_2 j$ is maximal monotone.
Moreover, the initial data $\boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}_0$ satisfy (2.12) due to (4.25) and (4.26). Thus, all the requirements of Theorem 2.8, with A defined by (4.27), $M = \psi$ given in (4.48) and $\boldsymbol{f} = \mathbf{F}$, are satisfied, it follows that there exists a unique solution $\boldsymbol{u}_{w\eta}$ to Problem $PV_{w\eta}$ satisfying the regularity expressed in (4.42). □

Let $\eta \in L^2(0,T;\mathcal{H})$. In the second step, we consider the operator $\chi : C(0,T;L^2(\Gamma_c)) \to C(0,T;L^2(\Gamma_c))$ defined by

$$\chi w(t) = k_w|\mathbf{v}^*| \int_0^t p_\nu(u_{w\eta\nu}(s) - w(s))ds \qquad \forall t \in [0,T]. \tag{4.49}$$

Lemma 4.2. *The operator χ has a unique fixed point $w^* \in C(0,T;L^2(\Gamma_c))$.*

Proof. Let $w_1, w_2 \in C(0,T;L^2(\Gamma_c))$ and denote by $\boldsymbol{u}_i$, $i = 1,2$ the solutions to the problem $PV_{w\eta}$, for $w = w_i$ i.e. $\boldsymbol{u}_i = \boldsymbol{u}_{w_i\eta}$ and $\boldsymbol{v}_i = \dot{\boldsymbol{u}}_i = \dot{\boldsymbol{u}}_{w_i\eta}$. From the definition (4.49) of χ, we can write

$$|\chi w_1(t) - \chi w_2(t)|_{L^2(\Gamma_c)} \le k_w|\mathbf{v}^*| \int_0^t |(p_\nu(u_{1\nu}(t) - w_1(t)) - p_\nu(u_{2\nu}(t) - w_2(t)))|ds$$

Using (4.17)(2) and (3.9), we get

$$|\chi w_1(t) - \chi w_2(t)|^2_{L^2(\Gamma_c)} \le C\Big(\int_0^t |w_1(s) - w_2(s)|^2_{L^2(\Gamma_c)}ds + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_V\Big)ds \tag{4.50}$$

Using the relation (4.47), we find

$$\begin{aligned}
&((\dot{\boldsymbol{v}}_1(t) - \dot{\boldsymbol{v}}_2(t), \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)))_H + \langle A\boldsymbol{u}_1(t) - A\boldsymbol{u}_2(t), \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\rangle_{V'\times V} \\
&+ \langle B\boldsymbol{v}_1(t) - B\boldsymbol{v}_2(t), \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\rangle_{V'\times V} \\
&\le j(\boldsymbol{u}_1(t), \boldsymbol{v}_2(t), w_1(t)) - j(\boldsymbol{u}_2(t), \boldsymbol{v}_2(t), w_2(t)) \\
&+ j(\boldsymbol{u}_2(t), \boldsymbol{v}_1(t), w_2(t)) - j(\boldsymbol{u}_1(t), \boldsymbol{v}_1, w_1(t)).
\end{aligned}$$

By virtue of (4.27), (4.28), (4.12)(2), (4.13)(2), (4.17) and (3.8), this inequality becomes

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)\|_H^2+\frac{m_{\mathcal{C}}}{2}\frac{d}{dt}|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V^2+m_{\mathcal{A}}|\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)|_V^2$$
$$\leq L_\nu\int_{\Gamma_c}(|u_{1\nu}-u_{2\nu}|+|w_1-w_2|)|v_{2\nu}|da-L_\nu\int_{\Gamma_c}(|u_{1\nu}-u_{2\nu}|+|w_1-w_2|)|v_{1\nu}|da$$
$$+L_\tau\int_{\Gamma_c}(|u_{1\nu}-u_{2\nu}|+|w_1-w_2|)\|\boldsymbol{v}_{2\tau}\|da-L_\tau\int_{\Gamma_c}(|u_{1\nu}-u_{2\nu}|+|w_1-w_2|)\|\boldsymbol{v}_{1\tau}\|da.$$

Integrating this inequality over the interval time variable $(0,t)$, using (3.9) and the inequality $2ab\leq a^2+b^2$ leads to

$$\|\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)\|_H^2+\frac{m_{\mathcal{C}}}{2}|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V^2+m_{\mathcal{A}}\int_0^t|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)|_V^2ds$$
$$\leq(L_\nu+L_\tau)\frac{C_1}{2}\int_0^t\big(C_1|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V^2+(C_1+1)|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)|_V^2$$
$$\|w_1(s)-w_2(s)|_{L^2(\Gamma_c)}^2\big)ds,$$

and keeping in mind (4.41), we obtain

$$\int_0^t|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)|_V^2ds\leq C\int_0^t\big(|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V^2+|w_1(s)-w_2(s)|_{L^2(\Gamma_c)}^2\big)ds. \tag{4.51}$$

On the other hand, since $\boldsymbol{u}_i(t)=\boldsymbol{u}_0+\int_0^t\mathbf{v}_i(s)\,ds$, we have

$$|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V^2\leq\int_0^t|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)|_V^2\,ds, \tag{4.52}$$

and using this inequality in (4.51) yields

$$|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V^2\leq C\int_0^t\Big(|w_1(t)-w_2(t)|_V^2+|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V^2\Big)ds.$$

It follows now from a Gronwall-type argument that

$$|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V^2\,ds\leq C\int_0^t|w_1(s)-w_2(s)|_V^2\,ds.$$

which implies for $s\leq t\leq T$

$$\begin{aligned}\int_0^t|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V^2ds\leq&C\int_0^t\int_0^s|w_1(r)-w_2(r)|_{L^2(\Gamma_c)}^2drds\\ \leq&C\int_0^t\int_0^t|w_1(r)-w_2(r)|_{L^2(\Gamma_c)}^2drds\\ \leq&C\int_0^t|w_1(r)-w_2(r)|_{L^2(\Gamma_c)}^2dr\int_0^T ds.\end{aligned}$$

Then

$$\int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|_V^2 ds \leq CT \int_0^t |w_1(s) - w_2(s)|_{L^2(\Gamma_c)}^2 ds. \tag{4.53}$$

From (4.49) and (4.53), we deduce that

$$|\chi w_1(s) - \chi w_2(s)|_{L^2(\Gamma_c)}^2 \leq CT \int_0^t |w_1(s) - w_2(s)|_{L^2(\Gamma_c)}^2 ds. \tag{4.54}$$

Reiterating the last inequality n times, we infer that

$$|\chi^n w_1 - \chi^n w_2|_{C(0,T;L^2(\Gamma_c))} \leq \left(\frac{C^n T^{n+1}}{n!} \right)^{\frac{1}{2}} |w_1 - w_2|_{C(0,T;L^2(\Gamma_c))}.$$

Thus, for n sufficiently large, a power χ^n of χ is a contraction in the Banach space $C(0,T;L^2(\Gamma_c))$. Which implies that the operator χ has a unique fixed point $w^* \in C(0,T;L^2(\Gamma_c))$. □

In the third step, let $\gamma \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem PV_γ

Find a damage field $\varsigma_\gamma : [0,T] \to H^1(\Omega)$ such that

$$\begin{aligned} \varsigma_\gamma \in K, \quad & (\dot{\varsigma}_\gamma, \beta - \varsigma_\gamma(t))_{L^2(\Omega)} + b(\varsigma_\gamma(t), \beta - \varsigma_\gamma(t)) \\ & \geq (\gamma(t), \beta - \varsigma_\gamma(t))_{L^2(\Omega)} \quad \forall\, \beta \in K, \\ & \varsigma_\gamma(0) = \varsigma_0. \end{aligned} \tag{4.55}$$

Lemma 4.3. *The problem PV_γ has a unique solution ς_γ satisfying*

$$\varsigma_\gamma \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \tag{4.56}$$

Proof. Using (4.30) and (4.25), after some algebraic computations and from a classical existence and uniqueness result of parabolic equations (see for example the reference [38, p.60]), we find that the problem PV_γ has a unique solution $\varsigma_\gamma \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. □

In the forth step, let $\varphi \in L^2(0,T;E')$, we consider the following variational problem

Problem PV_φ

Find a temperature $\theta_\varphi : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned}
&\langle \dot{\theta}_\varphi(t), \alpha\rangle_{E' \times E} + a(\theta_\varphi(t), \alpha) = \langle \varphi(t) + q(t), \alpha\rangle_{E' \times E} \\
&\forall\, \alpha \in E \; a.e\; t \in (0, T), \\
&\theta_\varphi(0) = \theta_0.
\end{aligned} \tag{4.57}$$

Lemma 4.4. *The problem PV_φ has a unique solution θ_φ satisfies the regularity (4.44).*

Proof. From the Friedrichs-poincaré inequality, we can find that there exists a constant $\mu_2 > 0$ such that

$$\int_\Omega |\nabla(\xi)|^2 dx + \frac{\mu_1}{\mu_0} \int_\Gamma |\xi|^2 da \geq \mu_2 \int_\Omega |\xi|^2 dx.$$

Thus, we obtain

$$a(\xi, \xi) \geq \mu_3 |\xi|_E^2, \tag{4.58}$$

where $\mu_3 = \frac{\mu_0 \min(1, \mu_2)}{2}$, which implies that a is elliptic on E. Consequently, based on a classical arguments of functional analysis concerning parabolic equations (see [5, p.140]), we conclude that the problem PV_φ has a unique solution θ_φ which satisfies the regularity (4.44). □

Let us now consider the operator $\Lambda : L^2(0, T; Q \times L^2(\Omega) \times E) \to L^2(0, T; Q \times L^2(\Omega) \times E)$

$$\Lambda(\eta, \gamma, \varphi)(t) = (\Lambda_1(\eta, \gamma, \varphi)(t), \Lambda_2(\eta, \gamma, \varphi)(t), \Lambda_3(\eta, \gamma, \varphi)(t)), \tag{4.59}$$

defined by

$$\Lambda_1(\eta, \gamma, \varphi)(t) = \int_0^t \mathcal{B}\big(\varepsilon(\boldsymbol{u}_{w^*\eta})(s), \varsigma_\gamma(s), \theta_\varphi(s)\big) ds, \tag{4.60}$$

$$\Lambda_2(\eta, \gamma, \varphi)(t) = \mathcal{S}(\varepsilon(\boldsymbol{u}_{w^*\eta}(t)), \varsigma_\gamma(t), \theta_\varphi(t)), \tag{4.61}$$

$$\Lambda_3(\eta, \gamma, \varphi)(t) = \Psi(\varepsilon(\boldsymbol{u}_{w^*\eta}(t)), \varsigma_\gamma(t), \theta_\varphi(t)). \tag{4.62}$$

Here, w^* be the fixed point of the operator χ. For every $(\eta, \gamma, \varphi) \in L^2(0, T; Q \times L^2(\Omega) \times E')$, $\boldsymbol{u}_{w^*\eta}, \varsigma_\gamma$ and θ_φ represent the displacement, the damage and the temperature obtained in Lemma 4.1, Lemma 4.3 and Lemma 4.4 respectively. The last step in the proof of Theorem 4.1 is the next result.

Lemma 4.5. *The operator Λ has a unique fixed point $(\eta^*, \gamma^*, \varphi^*) \in L^2(0, T; Q \times L^2(\Omega) \times E')$.*

Proof. Let $(\eta_1, \gamma_1, \varphi_1), (\eta_2, \gamma_2, \varphi_2) \in L^2(0, T; Q \times L^2(\Omega) \times E^{'})$.
We use the notation $\boldsymbol{u}_{w^*\eta_i} = \boldsymbol{u}_i, \dot{\boldsymbol{u}}_{w^*\eta_i} = \boldsymbol{v}_i, \varsigma_{\gamma_i} = \varsigma_i$ and $\theta_{\varphi_i} = \theta_i$ for $i = 1, 2$.
Using (4.60)-(4.62) and from (4.14)-(4.16) and (3.8), we get

$$\begin{aligned}&|\Lambda(\eta_1, \gamma_1, \varphi_1)(t) - \Lambda(\eta_2, \gamma_2, \varphi_2)|_{Q\times L^2(\Omega)\times E'}\\ &\leq L_{\mathcal{B}} \int_0^t \big(|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|_V + |\varsigma_1(s) - \varsigma_2(s)|_{L^2(\Omega)} + |\theta_1(s) - \theta_2(s)|_E\big)ds\\ &+ (L_S + L_\Phi)\big(|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V + |\varsigma_1(t) - \varsigma_2(t)|_{L^2(\Omega)} + |\theta_1(t) - \theta_2(t)|_{L^2(\Omega)}\big)\end{aligned}$$

Employing Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}&|\Lambda(\eta_1, \gamma_1, \varphi_1)(t) - \Lambda(\eta_2, \gamma_2, \varphi_2)|^2_{Q\times L^2(\Omega)\times E'}\\ &\leq C \int_0^t \big(|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V + \|\varsigma_1(s) - \varsigma_2(s)|^2_{L^2(\Omega)}\\ &+ |\theta_1(s) - \theta_2(s)|^2_E\big)ds + C\big(|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_V\\ &+ |\varsigma_1(t) - \varsigma_2(t)|^2_{L^2(\Omega)} + |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega)}\big)\end{aligned} \tag{4.63}$$

Using the relation (4.47), we obtain

$$\begin{aligned}&(\mathcal{A}\varepsilon(\boldsymbol{v}_1) - \mathcal{A}\varepsilon(\boldsymbol{v}_2), \varepsilon(\boldsymbol{v}_1 - \boldsymbol{v}_2)_V + ((\dot{\boldsymbol{v}}_1 - \dot{\boldsymbol{v}}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2))_H\\ &\leq (\mathcal{C}\varepsilon(\boldsymbol{u}_1) - \mathcal{C}\varepsilon(\boldsymbol{u}_2), \varepsilon(\boldsymbol{v}_1 - \boldsymbol{v}_2))_Q + j(\boldsymbol{u}_1, \boldsymbol{v}_2, w) - j(\boldsymbol{u}_1, \boldsymbol{v}_1, w) + j(\boldsymbol{u}_2, \boldsymbol{v}_1, w)\\ &\quad - j(\boldsymbol{u}_2, \boldsymbol{v}_2, w) - (\eta_1 - \eta_2, , \varepsilon(\boldsymbol{v}_1) - \varepsilon(\boldsymbol{v}_2))_Q\end{aligned}$$

We use similar arguments that those used in the proof of the relation (4.53) to obtain that

$$\int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_V ds \leq C \int_0^t |\eta_1(s) - \eta_2(s)|^2_Q ds. \tag{4.64}$$

From (4.55), we get

$$(\dot{\varsigma}_1 - \dot{\varsigma}_1, \varsigma_1 - \varsigma_2)_{L^2(\Omega)} + b(\varsigma_1 - \varsigma_2, \varsigma_1 - \varsigma_2) \leq (\varsigma_1 - \varsigma_2, \varsigma_1 - \varsigma_2)_{L^2(\Omega)} \quad \text{a.e. } t \in (0, T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\varsigma_1(0) = \varsigma_2(0) = \varsigma_0$ and the inequality $b(\varsigma_1 - \varsigma_2, \varsigma_1 - \varsigma_2) \geq 0$, we find

$$\frac{1}{2}|\varsigma_1(t) - \varsigma_2(t)|^2_{L^2(\Omega)} \leq \int_0^t (\varsigma_1(s) - \varsigma_2(s), \varsigma_1(s) - \varsigma_2(s))_{L^2(\Omega)} ds,$$

which implies that

$$|\varsigma_1(t) - \varsigma_2(t)|^2_{L^2(\Omega)} \leq \int_0^t \big(|\gamma_1(s) - \gamma_2(s)|^2_{L^2(\Omega)} + |\varsigma_1(s) - \varsigma_2(s)|^2_{L^2(\Omega)}\big)ds,$$

this inequality combined with Gronwall's inequality leads to

$$|\varsigma_1(t)-\varsigma_2(t)|^2_{L^2(\Omega)} \leq C\int_0^t |\gamma_1(s)-\gamma_2(s)|^2_{L^2(\Omega)}ds, \quad \forall t\in[0,T]. \tag{4.65}$$

In order words from (4.57), it follows

$$(\dot{\theta}_1-\dot{\theta}_1,\theta_1-\theta_2)_{E'\times E}+a(\theta_1-\theta_2,\theta_1-\theta_2)=(\varphi_1-\varphi_2,\theta_1-\theta_2)_{E'\times E} \quad \text{a.e. } t\in(0,T).$$

We integrate the previous equality, using (3.16), he initial conditions $\theta_1(0)=\theta_2(0)=\theta_0$ and as a is E-elliptic, we get

$$\begin{aligned}&|\theta_1(t)-\theta_2(t)|^2_{L^2(\Omega)}+\mu_3\int_0^t|\theta_1(t)-\theta_2(t)|^2_E\\ &\leq \int_0^t|\varphi_1(s)-\varphi_2(s)|_{E'}|\theta_1(s)-\theta_2(s)|_E \quad \text{a.e. } t\in(0,T),\end{aligned}$$

employing Young's and Hölder's inequalities, we get

$$\begin{aligned}&|\theta_1(t)-\theta_2(t)|^2_{L^2(\Omega)}+\int_0^t|\theta_1(s)-\theta_2(s)|^2_E ds\\ &\leq C\int_0^t|\varphi_1(s)-\varphi_2(s)|^2_{E'}ds \quad \text{a.e. } t\in(0,T).\end{aligned} \tag{4.66}$$

Now, we combine (4.64), (4.65) and (4.66), we find that

$$|\Lambda(\eta_1,\gamma_1,\varphi_1)(t)-\Lambda(\eta_2,\gamma_2,\varphi_2)|^2_{Q\times L^2(\Omega)\times E'} \leq C\int_0^T |(\eta_1,\gamma_1,\varphi_1)(t)-(\eta_2,\gamma_2,\varphi_2)|^2_{Q\times L^2(\Omega)\times E'}.$$

Reiterating this inequality n times we are led to

$$\begin{aligned}&|\Lambda(\eta_1,\gamma_1,\varphi_1)(t)-\Lambda(\eta_2,\gamma_2,\varphi_2)(t)|^2_{Q\times L^2(\Omega)\times E'}\\ &\leq\ C^n\underbrace{\int_0^t\int_0^s...\int_0^m}_{n\text{ integrals}}\int_0^T|(\eta_1,\gamma_1,\varphi_1)(r)-(\eta_2,\gamma_2,\varphi_2)(r)|^2_{Q\times L^2(\Omega)\times E'}\ dr...ds,\end{aligned}$$

which implies that

$$\begin{aligned}&|\Lambda^n(\eta_1,\gamma_1,\varphi_1)-\Lambda^n(\eta_2,\gamma_2,\varphi_2)|^2_{L^2(Q\times L^2(\Omega)\times E')}\\ &\leq\frac{C^n\ T^n}{n!}|(\eta_1,\gamma_1,\varphi_1)-(\eta_2,\gamma_2,\varphi_2)|^2_{L^2(Q\times L^2(\Omega)\times E')}\end{aligned} \tag{4.67}$$

Since $\lim\limits_{n\to\infty}\frac{C^n\ T^n}{n!}=0$, it follows that there exists a positive integer n such that $\frac{C^n\ T^n}{n!}<1$ and, therefore, (4.67) shows that the operator Λ^n is a contraction on the Banach space $L^2(Q\times L^2(\Omega)\times E^{'})$ and, so, there exists a unique fixed point $(\eta^*,\gamma^*,\varphi^*)\in L^2(0,T;Q\times L^2(\Omega)\times E^{'})$ such that $\Lambda(\eta^*,\gamma^*,\varphi^*)=(\eta^*,\gamma^*,\varphi^*)$. □

We have now all the ingredient to prove Theorem 4.1 which we complete now.

Existence

Let w^* be the fixed point of the operator χ given by (4.49) and $(\eta^*, \gamma^*, \varphi^*)$ be the fixed point of the operator Λ given by (4.59)-(4.62) and denote

$$\boldsymbol{u}_* = \boldsymbol{u}_{w^*\eta^*}, \quad \varsigma_* = \varsigma_{\gamma^*}, \quad \theta_* = \theta_{\varphi^*}. \tag{4.68}$$

It follows from (4.60)-(4.62) that

$$\begin{aligned}
\eta^*(t) &= \int_0^t \mathcal{B}\big(\varepsilon(\boldsymbol{u}_*(s)), \varsigma_*(s), \theta_*(s)\big)ds,\\
\gamma^*(t) &= \mathcal{S}(\varepsilon(\boldsymbol{u}_*(t)), \varsigma_*(t), \theta_*(t)),\\
\varphi^*(t) &= \Psi(\varepsilon(\boldsymbol{u}_*(t)), \varsigma_*(t), \theta_*(t)).
\end{aligned}$$

and, therefore, (4.47), (4.55), (4.57) and (4.49) imply that $(\boldsymbol{u}_*, \varsigma_*, \theta_*, w^*)$ is a solution of problem PV.

Uniqueness

The uniqueness of the solution follows from the uniqueness of the fixed point of the operators χ and Λ defined by (4.49) and (4.59)-(4.62) respectively.

Chapter 5

Analysis of mechanical contact problem with unilateral constraint and adhesion

5.1 Problem statement and variational formulation

This chapter is dedicated to the mathematical analysis of a frictionless contact problem with adhesion between a deformable solid and a rigid obstacle in the case of small deformations.

Problem P

Find a displacement $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$ and an adhesion field: $\beta : \Gamma_3 \times (0,T) \longrightarrow [0,1]$ such that

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u})) \quad \text{in} \;\; \Omega \times (0,T), \tag{5.1}$$

$$\text{div}\, \boldsymbol{\sigma} + \boldsymbol{f}_1 = \varrho \ddot{\boldsymbol{u}} \quad \text{in } \Omega \times (0,T), \tag{5.2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_e \times (0,T), \tag{5.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_0 \quad \text{on } \Gamma_d \times (0,T), \tag{5.4}$$

$$\begin{cases} u_\nu \leq g, \quad \sigma_\nu + p(u_\nu) - C_\nu R(u_\nu)\beta^2 \leq 0, \\ (\sigma_\nu + p(u_\nu) - C_\nu R(u_\nu)\beta^2)(u_\nu - g) = 0, \end{cases} \quad \text{on } \Gamma_c \times (0,T), \tag{5.5}$$

$$\boldsymbol{\sigma}_\tau = 0 \quad \text{on } \Gamma_c \times (0,T), \tag{5.6}$$

$$\dot{\beta} = -(C_\nu \beta (R(u_\nu))^2 - \epsilon_a)_+ \quad \text{on } \Gamma_c \times (0,T), \tag{5.7}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \;\; \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \;\; \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega, \tag{5.8}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_c. \tag{5.9}$$

let us denote by

$$U = \{\boldsymbol{v} \in V : \quad v_\nu \leq g \text{ on } \Gamma_c\}. \tag{5.10}$$

the convex subset of admissible displacements and we introduce the set of admissible bonding fields

$$\mathcal{Z} = \{\theta \in L^2(\Gamma_c) : \ 0 \leq \theta \leq 1 \ \text{ a.e. on } \Gamma_c\}. \tag{5.11}$$

Let's now specify the assumption about the data. we assume that the operators $\mathcal{A}, \mathcal{G}$ and the functions p satisfy the following conditions

$$\left\{\begin{array}{l} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d. \\ \text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that } \mathcal{A}\varepsilon \cdot \varepsilon \geq m_{\mathcal{A}}|\varepsilon|^2. \\ \text{(c) } \mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{klij}; \mathcal{A}_{ijkl} \in L^\infty(\Omega) \ \forall i,j,k,l = 1,..,d. \end{array}\right. \tag{5.12}$$

$$\left\{\begin{array}{l} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \longrightarrow \mathbb{S}^d. \\ \text{(b) There exists } m_{\mathcal{G}} > 0 \text{ such that} \\ \quad |\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)| \leq m_{\mathcal{G}}(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|) \\ \quad \forall \boldsymbol{x} \in \Omega \ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d. \\ \text{(c) The operator } \boldsymbol{x} \longrightarrow \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is Lebesgue mesurable on } \Omega \ \forall \ \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The operator } \boldsymbol{x} \longrightarrow \mathcal{G}(\boldsymbol{x}, 0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}) \in Q. \end{array}\right. \tag{5.13}$$

$$\left\{\begin{array}{l} \text{(a) } p : \Gamma_c \times \mathbb{R} \to \mathbb{R}^+. \\ \text{(b) There exists } L_\nu > 0 \text{ such that} \\ p(\boldsymbol{x}, z_1) - p(\boldsymbol{x}, z_2) \leq L_\nu |z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{R}, \text{a.e.} \boldsymbol{x} \in \Gamma_c. \\ \text{(c) } (p(\boldsymbol{x}, z_1) - p(\boldsymbol{x}, z_2))(z_1 - z_2) \geq 0 \ \ \forall r_1, r_2 \in \mathbb{R}, \ \text{a.e.} \boldsymbol{x} \in \Gamma_c. \\ \text{(d) } p(\boldsymbol{x}, z) = 0, \quad \forall z \leq 0, \text{a.e.} \boldsymbol{x} \in \Gamma_c. \\ \text{(e) the mapping } \boldsymbol{x} \to p(\boldsymbol{x}, z) \text{ is Lebesgue measurable on } \Gamma_c, \forall z \in \mathbb{R}. \end{array}\right. \tag{5.14}$$

We also suppose that the mass density satisfies

$$\varrho \in L^\infty(\Omega) \text{ and there exist } \ \varrho^* > 0 \text{ such that } \ \varrho(x) \geq \varrho^*, \ \text{a.e. } x \in \Omega. \tag{5.15}$$

In addition, we assume that the densities of body forces, surface tractions, the adhesion coefficient C_ν, the Dupré energy ϵ_a, and initial bonding field β_0, have regularity

$$(i) \ \boldsymbol{f}_1 \in W^{2,\infty}(0,T;H), \qquad (ii) \ \boldsymbol{f}_0 \in W^{2,\infty}(0,T;L^2(\Gamma_d)^d). \tag{5.16}$$

$$C_\nu \in L^\infty(\Gamma_c), \quad C_\nu \geq 0, \quad \epsilon_a \in L^\infty(\Gamma_c), \quad \epsilon_a \geq 0, \tag{5.17}$$

$$g \in L^2(\Gamma_c), \beta_0 \in L^2(\Gamma_c), \quad 0 \leq \beta_0 \leq 1 \ \text{ a.e. on } \Gamma_c. \tag{5.18}$$

Finally, we assume that the initial data $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ and $\boldsymbol{\sigma}_0$ satisfy

$$\boldsymbol{u}_0 \in U, \quad \boldsymbol{v}_0 \in H, \quad \boldsymbol{\sigma}_0 \in Q_1. \tag{5.19}$$

We are now defining the applications $\boldsymbol{f}(t) \in V'$ for a.e. $t \in (0,T)$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V'\times V} = \int_\Omega \boldsymbol{f}_1(t).\boldsymbol{v}dx + \int_{\Gamma_d} \boldsymbol{f}_0(t).\boldsymbol{v}da \quad \forall v \in V, \tag{5.20}$$

$\phi : V \times V \to \mathbb{R}$, $\psi : V \times V \times L^\infty(\Gamma_c) \to \mathbb{R}$ and $J : V \times V \times L^\infty(\Gamma_c) \to \mathbb{R}$, defined by

$$\phi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_c} P(u_\nu)v_\nu da, \tag{5.21}$$

$$\psi(\boldsymbol{u}, \boldsymbol{v}, \beta) = -\int_{\Gamma_c} C_\nu R(u_\nu)\beta^2 v_\nu da, \tag{5.22}$$

$$J(\boldsymbol{u}, \boldsymbol{v}, \beta) = \phi(\boldsymbol{u}, \boldsymbol{v}) + \psi(\boldsymbol{u}, \boldsymbol{v}, \beta), \tag{5.23}$$

and note that

$$\boldsymbol{f} \in W^{2,\infty}(0,T;V'). \tag{5.24}$$

If we use (3.9), (3.28) and (3.29), we conclude that the functional ψ satisfies

$$\begin{aligned} |\psi(w,v,\beta_1) - \psi(w,v,\beta_2)| \leq & C_\nu C_1^2 |w|_V |v|_V \big(|\beta_1|_{L^\infty(\Gamma_c)} \\ & + |\beta_2|_{L^\infty(\Gamma_c)}\big)|\beta_1 - \beta_2|_{L^2(\Gamma_c)}, \end{aligned} \tag{5.25}$$

$$|\psi(\boldsymbol{z}, \boldsymbol{v}, \beta) - \psi(\boldsymbol{z}, \boldsymbol{w}, \beta)| \leq C_\nu C_1^2 |\boldsymbol{z}|_V |\beta|^2_{L^\infty(\Gamma_c)} |\boldsymbol{v} - \boldsymbol{w}|_V, \tag{5.26}$$

$$|\psi(\boldsymbol{w}, \boldsymbol{z}, \beta) - \psi(\boldsymbol{v}, \boldsymbol{z}, \beta)| \leq C_\nu C_1^2 |\boldsymbol{w} - \boldsymbol{v}|_V |\beta|^2_{L^\infty(\Gamma_c)} |\boldsymbol{z}|_V, \tag{5.27}$$

$$|\psi(\boldsymbol{w}, \boldsymbol{v}, \beta)| \leq C_\nu |\boldsymbol{w}|_{L^2(\Gamma_c)} |\beta|_{L^\infty(\Gamma_c)} |\boldsymbol{v}|_{L^2(\Gamma_c)}, \tag{5.28}$$

$$\psi(\boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}, \beta) - \psi(\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{v}, \beta) \geq 0, \tag{5.29}$$

$$\psi(\boldsymbol{w}, \boldsymbol{w}, \beta) \geq 0. \tag{5.30}$$

Next, we check that under hypotheses (3.9) and (5.14) the functional ϕ satisfies

$$\phi(\boldsymbol{y}, \boldsymbol{v}) - \phi(\boldsymbol{y}, \boldsymbol{w}) + \phi(\boldsymbol{z}, \boldsymbol{w}) - \phi(\boldsymbol{z}, \boldsymbol{v}) \leq C_1^2 L_\nu |\boldsymbol{y} - \boldsymbol{z}|_V |\boldsymbol{v} - \boldsymbol{w}|_V, \tag{5.31}$$

$$\phi(\boldsymbol{y}, -\boldsymbol{v}) - \phi(\boldsymbol{y}, \boldsymbol{w} - \boldsymbol{v}) \leq C_1^2 L_\nu |\boldsymbol{y}|_V |\boldsymbol{w}|_V, \tag{5.32}$$

$$\phi(\boldsymbol{v}, \boldsymbol{w}) - \phi(\boldsymbol{z}, \boldsymbol{w}) \leq C_1^2 L_\nu |\boldsymbol{v} - \boldsymbol{z}|_V |\boldsymbol{w}|_V, \tag{5.33}$$

$$\phi(\boldsymbol{y}, \boldsymbol{v}) - \phi(\boldsymbol{y}, \boldsymbol{w}) = \phi(\boldsymbol{y}, \boldsymbol{v} - \boldsymbol{w}), \tag{5.34}$$

$$|\phi(\boldsymbol{y}, \boldsymbol{v}) - \phi(\boldsymbol{y}, \boldsymbol{w})| \leq C_1^2 L_\nu |\boldsymbol{y}|_V |\boldsymbol{v} - \boldsymbol{w}|_V, \tag{5.35}$$

$$\phi(\boldsymbol{w}, \boldsymbol{z}) \leq C_1^2 L_\nu |\boldsymbol{w}|_V |\boldsymbol{z}|_V. \tag{5.36}$$

Next we turn to derive the variational formulation for Problem. From the equation (5.2) and the Green's formula, we have

$$\int_{\Omega} -\operatorname{div} \boldsymbol{\sigma}.(\boldsymbol{v}-\boldsymbol{u})dx + \int_{\Omega} \rho\ddot{\mathbf{u}}.(\boldsymbol{v}-\boldsymbol{u})dx = \int_{\Omega} \boldsymbol{f}.(\boldsymbol{v}-\boldsymbol{u})dx$$

Using an integration by parts and equalities (5.3), (5.4), it is straightforward to see that

$$\begin{aligned} \textstyle\int_{\Omega} -\operatorname{div} \boldsymbol{\sigma}.(\boldsymbol{v}-\boldsymbol{u})dx &= \textstyle\int_{\Omega} \boldsymbol{\sigma}.\varepsilon(\boldsymbol{v}-\boldsymbol{u})dx - \int_{\Gamma} \boldsymbol{\sigma\nu}(\boldsymbol{v}-\boldsymbol{u})da \\ &= \textstyle\int_{\Omega} \boldsymbol{\sigma}\varepsilon(\boldsymbol{v}-\boldsymbol{u})dx - \int_{\Gamma_d} \boldsymbol{\sigma\nu}(\boldsymbol{v}-\boldsymbol{u})da \\ &\quad \textstyle- \int_{\Gamma_e} \boldsymbol{\sigma}.\boldsymbol{\nu}(\boldsymbol{v}-\boldsymbol{u})da - \int_{\Gamma_c} \boldsymbol{\sigma}.\boldsymbol{\nu}(\boldsymbol{v}-\boldsymbol{u})da \\ &= \textstyle\int_{\Omega} \boldsymbol{\sigma}.\varepsilon(\boldsymbol{v}-\boldsymbol{u})dx - \int_{\Gamma_e} \boldsymbol{f}_0.(\boldsymbol{v}-\boldsymbol{u})da \\ &\quad \textstyle- \int_{\Gamma_c} \boldsymbol{\sigma}.\boldsymbol{\nu}.(\boldsymbol{v}-\boldsymbol{u})da \end{aligned} \tag{5.37}$$

which imply that

$$\int_{\Omega} \boldsymbol{\sigma}.\varepsilon(\boldsymbol{v}-\boldsymbol{u})dx + \int_{\Omega} \rho\ddot{\mathbf{u}}.(\boldsymbol{v}-\boldsymbol{u})dx - \int_{\Gamma_c} \boldsymbol{\sigma}.\boldsymbol{\nu}.(\boldsymbol{v}-\boldsymbol{u})da$$

$$= \int_{\Omega} \boldsymbol{f}_1.(\boldsymbol{v}-\boldsymbol{u})dx + \int_{\Gamma_e} \boldsymbol{f}_0.(\boldsymbol{v}-\boldsymbol{u})da$$

and, by (5.5) we find

$$\begin{aligned} &\textstyle\int_{\Omega} \boldsymbol{\sigma}\varepsilon(\boldsymbol{v}-\boldsymbol{u})dx + \int_{\Omega} \rho\ddot{\boldsymbol{u}}(\boldsymbol{v}-\boldsymbol{u})dx + \int_{\Gamma_c} (P(u_\nu) - C_\nu R(u_\nu)\beta^2)(v_\nu - u_\nu)da \\ &\geq \textstyle\int_{\Omega} \boldsymbol{f}_1(\boldsymbol{v}-\boldsymbol{u})dx + \int_{\Gamma_e} \boldsymbol{f}_0(\boldsymbol{v}-\boldsymbol{u})da \end{aligned} \tag{5.38}$$

Using the Riesz representation theorem, we can introduce the operator $A : V \to V'$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q + (\boldsymbol{\sigma}_0 - \mathcal{A}\varepsilon(\boldsymbol{u}_0), \varepsilon(\boldsymbol{v}))_Q \quad \forall u, v \in V.$$

Using assumption (5.12), we conclude that

$$\text{There exists } L_{\mathcal{A}} > 0 \text{ such that } |A\boldsymbol{u} - A\boldsymbol{v}|_{V'} \leq L_{\mathcal{A}}|\boldsymbol{v}-\boldsymbol{u}|_V \quad \forall \boldsymbol{v}, \boldsymbol{u} \in V, \tag{5.39}$$

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{v}-\boldsymbol{u}\rangle_{V'\times V} \geq m_{\mathcal{A}}|\boldsymbol{v}-\boldsymbol{u}|_V^2 \quad \forall \boldsymbol{v}, \boldsymbol{u} \in V. \tag{5.40}$$

Finally, we recall that the unilateral constraints imposed to the displacement and bonding field, combined with the definitions (5.10) and (5.11), yield

$$\boldsymbol{u}(t) \in U, \quad \beta(t) \in \mathcal{Z}. \tag{5.41}$$

We will use a modified inner product on the Hilbert space H given by

$$((\boldsymbol{u},\boldsymbol{v}))_H = (\varrho\boldsymbol{u},\boldsymbol{v})_H \quad \forall \boldsymbol{v},\boldsymbol{u}\in H,$$

that is, it is weighted with ϱ, and we let $\|.\|_H$ be the associated norm, i.e.,

$$\|\boldsymbol{u}\|_H = (\varrho\boldsymbol{u},\boldsymbol{u})_H^{\frac{1}{2}} \quad \forall \boldsymbol{u}\in H.$$

It follows from assumptions (5.15) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V,|.|_V)$ into $(H,\|.\|_H)$ is continuous and dense. We denote by V' the dual space of V . Identifying H with its own dual, we can write the Gelfand triple

$$V\subset H\subset V'.$$

We recall that

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle_{V'\times V} = ((\boldsymbol{u},\boldsymbol{v}))_H \quad \forall\boldsymbol{u}\in H,\quad \forall\boldsymbol{v}\in V.$$

Then we see that there exixts a constant $C_2>0$ such that

$$\|\boldsymbol{v}\|_{V'} \le C_2\|\boldsymbol{v}\|_V. \tag{5.42}$$

By a standard procedure based on Green's formula (3.12) we derive the following variational formulation of problem P, in terms of stress, displacement and adhesion fields.

Problem P_V

Find a displacement $\boldsymbol{u}:[0,T]\to V$, a stress field $\boldsymbol{\sigma}:[0,T]\to\mathcal{S}$ and an adhesion field $\beta:[0,T]\longrightarrow L^\infty(\Gamma_3)$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s),\varepsilon(\boldsymbol{u}(s)))ds - \mathcal{A}(\varepsilon(\boldsymbol{u}_0)) + \boldsymbol{\sigma}_0, \tag{5.43}$$

$$\begin{aligned}&(\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{v})-\varepsilon(\boldsymbol{u}))_Q + \langle\ddot{\boldsymbol{u}},\boldsymbol{v}-\boldsymbol{u}\rangle_{V'\times V} + J(\boldsymbol{u},\boldsymbol{v},\beta) - J(\boldsymbol{u},\boldsymbol{u},\beta)\\ &\ge \langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}\rangle_{V'\times V} \quad \forall\boldsymbol{v}\in U,\end{aligned} \tag{5.44}$$

$$\beta(t) = -\int_0^t C_\nu R(u_\nu)\beta(s)^2 ds - \beta_0. \tag{5.45}$$

To study Problem P_V, we assume the following condition of compatibility

$$(\boldsymbol{\sigma}_0,\varepsilon(\boldsymbol{v})-\varepsilon(\boldsymbol{u}_0))_Q + J(\boldsymbol{u}_0,\boldsymbol{v},\beta_0) - J(\boldsymbol{u}_0,\boldsymbol{u}_0,\beta_0) \ge \langle\boldsymbol{f}(0),\boldsymbol{v}-\boldsymbol{u}_0\rangle_{V'\times V} \quad \forall\boldsymbol{v}\in U, \tag{5.46}$$

and we make the following smallness assumption

$$m_{\mathcal{A}} > 2C_1^2 L_\nu + L_{\mathcal{A}}, \tag{5.47}$$

where, C_1, L_ν, $m_{\mathcal{A}}$ and $L_{\mathcal{A}}$ are given in (3.9), (5.14), (5.39) and (5.40) respectively.

5.2 Existence and uniqueness result

Our main result which states the existence of a unique weak solution to Problem P_V, is the following

Theorem 5.1. *Assume that* (5.12)-(5.17), (5.19), (5.46) *and* (5.47) *hold. Then, there exists an unique solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ *to problem* P_V. *Moreover, the solution has the regularity*

$$\boldsymbol{u} \in W^{1,\infty}(0,T;V), \quad \ddot{u} \in L^{\infty}(0,T;V^{'}), \tag{5.48}$$

$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1), \tag{5.49}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)). \tag{5.50}$$

We start with the proof of Theorem 5.1 which will be carried out in several steps. We assume in the rest of this section that (5.12)-(5.17) and (5.19) hold and c will denote a generic positive constant which may depend on Ω, Γ_e, Γ_d, Γ_c, $\mathcal{A}$, $\mathcal{G}$, p and T , but does not depend on t, nor on the rest of the input data, and whose value may change from place to place.

We will employ the time discritization method, for each $N \in \mathbb{N}^*$, every where in this section, we use the notation h for the time step size, i.e $h = \frac{T}{N}$, $t_i = ih$, $i = \overline{0,N}$ for a sequence $(\boldsymbol{v}^i)_{i=0}^N$, we denote by $\delta \boldsymbol{v}^{i+1} = \frac{\boldsymbol{v}^{i+1} - \boldsymbol{v}^i}{h}$, $\delta^2 \boldsymbol{v}^{i+1} = \frac{\delta \boldsymbol{v}^{i+1} - \delta \boldsymbol{v}^i}{h}$, $\delta^3 \boldsymbol{v}^{i+1} = \frac{\delta^2 \boldsymbol{v}^{i+1} - \delta^2 \boldsymbol{v}^i}{h}$ and we use the notation $g^i = g(t_i)$ for all continuous function g with values in a normed space X (i.e $g \in C([0,T];X)$). Then we consider the following incremental Problems P_V^{i+1}, $i \in \{0, \dots, N-1\}$.

Problem P_V^{i+1}. Find $\boldsymbol{u}^{i+1} \in U$, such that

$$\begin{aligned}
&\langle A\boldsymbol{u}^{i+1}, \boldsymbol{v} - \boldsymbol{u}^{i+1}\rangle_{V' \times V} + \frac{1}{h^2}\langle \boldsymbol{u}^{i+1}, \boldsymbol{v} - \boldsymbol{u}^{i+1}\rangle_{V' \times V} \\
&+ \big(h\sum_{j=0}^{i} \mathcal{G}(\boldsymbol{\sigma}^j, \varepsilon(\boldsymbol{u}^j)), \varepsilon(\boldsymbol{v} - \boldsymbol{u}^{i+1})\big)_Q + \psi(\boldsymbol{u}^{i+1}, \boldsymbol{v} - \boldsymbol{u}^{i+1}, \beta^{i+1}) \\
&+ \frac{1}{h^2}\langle -2\boldsymbol{u}^i + \boldsymbol{u}^{i-1}, \boldsymbol{v} - \boldsymbol{u}^{i+1}\rangle_{V' \times V} + \phi(\boldsymbol{u}^{i+1}, \boldsymbol{v}) - \phi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}) \\
&\geq \langle \boldsymbol{f}^{i+1}, \boldsymbol{v} - \boldsymbol{u}^{i+1}\rangle_{V' \times V} \quad \forall \boldsymbol{v} \in\ U,
\end{aligned} \tag{5.51}$$

$$\boldsymbol{\sigma}^{i+1} = \mathcal{A}\varepsilon(\boldsymbol{u}^{i+1}) + h\sum_{j=0}^{i} \mathcal{G}(\boldsymbol{\sigma}^j, \varepsilon(\boldsymbol{u}^j)) + \boldsymbol{\sigma}^0 - \mathcal{A}\varepsilon(\boldsymbol{u}^0), \tag{5.52}$$

$$\beta^{i+1} = -h\sum_{j=0}^{i} (C_\nu \beta^j (R(u_\nu^j))^2 - \epsilon_a)_+ + \beta^0, \tag{5.53}$$

$$\boldsymbol{u}^0 = \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0,\ \beta^0 = \beta(0) = \beta_0,\ \boldsymbol{f}^0 = \boldsymbol{f}(0), \tag{5.54}$$

where $\boldsymbol{u}^j$ is the unique solution of Problems P_V^j, $j \in \{0, ..., i\}$.

Theorem 5.2. *The Problem P_V^{i+1} has an unique solution, which satisfies $\boldsymbol{u}^{i+1} \in U$.*

Proof. The Problem P_V^{i+1} can be transformed as follows.
Find $\boldsymbol{u}^{i+1} \in U$, such that

$$\langle \eta \boldsymbol{u}^{i+1}, \boldsymbol{v}-\boldsymbol{u}^{i+1}\rangle_{V'\times V}+h^2\phi(\boldsymbol{u}^{i+1},\boldsymbol{v})-h^2\phi(\boldsymbol{u}^{i+1},\boldsymbol{u}^{i+1}) \geq \ h^2\langle \boldsymbol{f}^{i+1}, \boldsymbol{v}-\boldsymbol{u}^{i+1}\rangle_{V',V} \ \forall v \in \ U,$$

where the operator $\eta : U \to V'$ for $i = \overline{1, N-1}$ define by

$$\begin{aligned}\langle \eta \boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} =& h^2\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} + h^2(h\sum_{j=0}^{i} G(\boldsymbol{\sigma}^j, \varepsilon(\boldsymbol{u}^j)), \varepsilon(\boldsymbol{v}))_Q + h^2\psi(\boldsymbol{v}, \boldsymbol{u}, \beta^{i+1})\\ &+ \langle -2\boldsymbol{u}^i + \boldsymbol{u}^{i-1}, v\rangle_{V'\times V} + \langle \boldsymbol{v}, \boldsymbol{u}\rangle_{V'\times V} \qquad \forall \boldsymbol{v}, \boldsymbol{u} \in U.\end{aligned} \tag{5.55}$$

Let $\boldsymbol{v}, \boldsymbol{u} \in U$. From the definition (5.55), we get

$$\begin{aligned}\langle \eta\boldsymbol{v} - \eta\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V'\times V} =& h^2\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V'\times V} + \langle \boldsymbol{v} - \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V'\times V}\\ &+ h^2\psi(\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}, \beta^{i+1}) - h^2\psi(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}, \beta^{i+1}).\end{aligned}$$

Using (5.12) and (5.29), the previous equality implies that

$$\langle \eta\boldsymbol{v} - \eta\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V'\times V} \geq \rho^*|\boldsymbol{v} - \boldsymbol{u}|_V^2.$$

We also note that (5.39), (5.42) and (5.27) imply that

$$|\eta\boldsymbol{v} - \eta\boldsymbol{u}|_{V'} \leq (T^2L_{\mathcal{A}} + C_2 + T^2C_\nu C_1^2|\beta^{i+1}|^2_{L^\infty(\Gamma_c)})|\boldsymbol{v} - \boldsymbol{u}|_V.$$

If we use (5.53), (5.18) and applying Lemma 2.2, we deduce that

$$|\beta^{i+1}|_{L^\infty(\Gamma_c)} \leq c \ \ \forall\, i \in \{0, .., N-1\}. \tag{5.56}$$

Since η is a strongly monotone operator and Lipschitz continuous operator on V and since ϕ is a proper convex lower semicontinuous functional and checked the relation (5.31), it follows from an abstract result for elliptic quasi-variational inequalities that the problem P_V^{i+1} has an unique solution $\boldsymbol{u}^{i+1} \in U$. □

In the following, c is a variable constant, that it does not have a single value.

Lemma 5.1. *There are $c > 0$, where for every $n \in \mathbb{N}$ we find*

$$|\boldsymbol{\sigma}^{i+1}|_Q + |\boldsymbol{u}^{i+1}|_V + |\beta^{i+1}|_{L^\infty(\Gamma_c)} \leq c \quad \forall\, i \in \{0, ..., N-1\}. \tag{5.57}$$

$$|\delta^2 \boldsymbol{u}^{i+1}|_{V'} \leq c \quad \forall\, i \in \{1, ..., N-1\}. \tag{5.58}$$

$$|\delta \boldsymbol{u}^{i+1}|_V \leq c \quad \forall\, i \in \{0, ..., N-1\}. \tag{5.59}$$

$$|\delta^3 \boldsymbol{u}_{i+1}|_{V'} \leq c \quad \forall\, i \in \{0, ..., N-1\}. \tag{5.60}$$

$$\Big|\frac{\boldsymbol{\sigma}^{i+1} - \boldsymbol{\sigma}^i}{h}\Big|_Q \leq c \quad \forall i \in \{0, \dots, N-1\}. \tag{5.61}$$

$$|\delta \beta_{i+1}|_{L^\infty(\Gamma_c)} \quad \forall\, i \in \{0, ..., N-1\}. \tag{5.62}$$

Proof. With the assumptions (5.18), (5.19) and (5.54), we see that

$$|\boldsymbol{\sigma}^0|_Q + |\boldsymbol{u}^0|_V + |\beta^0|_{L^\infty(\Gamma_c)} \leq c. \tag{5.63}$$

For $i \geq 0$, we take $v = 0_V$ in (5.51), we have that

$$\begin{aligned} \langle \delta^2 \boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} \rangle_{V' \times V} \leq &-(A\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1})_V - (h \sum_{j=0}^{i} (\mathcal{G}(\boldsymbol{\sigma}^j, \varepsilon(\boldsymbol{u}^j)), \varepsilon(\boldsymbol{u}^{i+1}))_Q \\ &- \psi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}, \beta^{i+1}) - \phi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}) + \langle \boldsymbol{f}^{i+1}, \boldsymbol{u}^{i+1} \rangle_{V' \times V}. \end{aligned} \tag{5.64}$$

It follows from (5.13), (5.14), (5.39), (5.30), (3.9) and (5.24) that

$$\begin{aligned} |\delta^2 \boldsymbol{u}^{i+1}|_{V'} \leq & h m_{\mathcal{G}} \sum_{l=0}^{i} (|\boldsymbol{\sigma}^l|_Q + |\boldsymbol{u}^l|_V) + L_{\mathcal{A}} |\boldsymbol{u}^{i+1}|_V + h \sum_{l=0}^{i} |\mathcal{G}(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d})|_Q \\ &+ |A(0_V)|_V + C_1^2 L_\nu |\boldsymbol{u}^{i+1}|_V + |\boldsymbol{f}^{i+1}|_{V'}. \end{aligned} \tag{5.65}$$

Using similar arguments that those used in the proof of the relation (5.64), to find that
for $i \geq 0$

$$\begin{aligned} \langle A\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} \rangle_{V' \times V} \leq &-\psi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}, \beta^{i+1}) - \langle \delta^2 \boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} \rangle_{V' \times V} - \phi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}) \\ &- h(\sum_{j=0}^{i} \mathcal{G}(\boldsymbol{\sigma}^j, \varepsilon(\boldsymbol{u}^j)), \varepsilon(\boldsymbol{u}^{i+1}))_Q + \langle \boldsymbol{f}^{i+1}, u^{i+1} \rangle_{V' \times V}. \end{aligned}$$

We use (5.12), (5.13), (5.36), (5.30), (5.19), (5.24), (5.65) and (3.9) to get

$$\begin{aligned} m_{\mathcal{A}} |\boldsymbol{u}^{i+1}|_V^2 \;\leq\; &+ L_{\mathcal{A}} |\boldsymbol{u}^{i+1}|_V^2 + ch \sum_{j=0}^{i} (|\boldsymbol{\sigma}^j|_Q + |\boldsymbol{u}^j|_V) |\boldsymbol{u}^{i+1}|_V \\ &+ 2C_1^2 L_\nu |\boldsymbol{u}^{i+1}|_V^2 + c|\boldsymbol{u}^{i+1}|_V. \end{aligned} \tag{5.66}$$

Hence, taking into account (5.47), to deduce that

$$|\boldsymbol{u}^{i+1}|_V \leq ch\sum_{j=0}^{i}(|\boldsymbol{\sigma}^j|_Q + |\boldsymbol{u}^j|_V) + c. \tag{5.67}$$

On the other hand, it follows from (5.12), (5.13) and (5.52) that

$$|\boldsymbol{\sigma}^{i+1}|_Q \leq hm_{\mathcal{G}}\sum_{j=0}^{i}(|\boldsymbol{\sigma}^j|_Q + |\boldsymbol{u}^j|_V) + h\sum_{j=0}^{i}\big|\mathcal{G}(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d})\big|_Q + c|\boldsymbol{u}^{i+1}|_V + c,$$

this inequality combined with (5.67), gives

$$|\boldsymbol{\sigma}^{i+1}|_Q + |\boldsymbol{u}^{i+1}|_V \leq ch\sum_{j=0}^{i}(|\boldsymbol{\sigma}^j|_Q + |\boldsymbol{u}^j|_V) + c.$$

Applying Lemma 2.2, it follows that

$$|\boldsymbol{\sigma}^{i+1}|_Q + |\boldsymbol{u}^{i+1}|_V \leq c \ \ \forall i \in \{0, \dots, N-1\}. \tag{5.68}$$

We combine (5.56) and (5.68), we get (5.57).
Now, from (5.53) we find that

$$|\beta^{i+1} - \beta^i|_{L^\infty(\Gamma_c)} = |h(C_n\beta^i(R(u_\nu^j))^2 - \epsilon_a)_+|_{L^\infty(\Gamma_c)},$$

which, together with (5.1) leads to

$$|\delta\beta^{i+1}|_{L^2(\Gamma_c)} \leq c.$$

The relations (5.68), (5.65) and assumption (5.24) show that

$$|\delta^2\boldsymbol{u}^{i+1}|_{V'} \leq c \qquad \forall i \in \{0, \dots, N-1\}. \tag{5.69}$$

For $i \geq 0$, we take $\boldsymbol{v} = \boldsymbol{u}_{i+1}$ in the inequality realized by $\boldsymbol{u}_i$ and $\boldsymbol{v} = \boldsymbol{u}_i$ in inequality realized by $\boldsymbol{u}_{i+1}$ and adding the above two inequalities, we have

$$\begin{aligned}
&\langle\delta^2\boldsymbol{u}^{i+1} - \delta^2\boldsymbol{u}^i, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i\rangle_{V'\times V} \leq -\psi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i, \beta^{i+1}) \\
&+ \psi(\boldsymbol{u}^i, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i, \beta^i) + \psi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i, \beta^i) \\
&- \psi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i, \beta^i) - \Phi(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\
&+ \Phi(\boldsymbol{u}^i, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) - \langle A\boldsymbol{u}^{i+1} - A\boldsymbol{u}^i, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i\rangle_{V'\times V} \\
&+ \langle\boldsymbol{f}^{i+1} - \boldsymbol{f}^i, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i\rangle_{V'\times V} - h(\mathcal{G}(\boldsymbol{\sigma}^i, \varepsilon(\boldsymbol{u}^i)), \varepsilon(\boldsymbol{u}^{i+1} - \boldsymbol{u}^i))_Q.
\end{aligned} \tag{5.70}$$

Hence, using (5.13), (5.24), (5.56), (5.29), (5.33), (5.25) and (5.39) we get the following result

$$\begin{aligned}|\delta^2\boldsymbol{u}^{i+1}-\delta^2\boldsymbol{u}^i|_{V'} \leq & C_1^2 L_\nu |\boldsymbol{u}^{i+1}-\boldsymbol{u}^i|_V + c|\beta^{i+1}-\beta^i|_{L^2(\Gamma_c)} \\ & + ch\sum_{j=0}^{i}(|\boldsymbol{\sigma}^j|_Q+|\boldsymbol{u}^j|_V) + h\sum_{j=0}^{i}\big|\mathcal{G}(0_{\mathbb{S}^d},0_{\mathbb{S}^d})\big|_Q \\ & + |\boldsymbol{f}^{i+1}-\boldsymbol{f}^i|_{V'} + L_{\mathcal{A}}|\boldsymbol{u}^{i+1}-\boldsymbol{u}^i|_V.\end{aligned} \tag{5.71}$$

We take $\boldsymbol{v}=\boldsymbol{u}^1$ in the inequality (5.46), and $\boldsymbol{v}=\boldsymbol{u}^0$ in inequality (5.51) realized by $\boldsymbol{u}_1$, and add the resulting inequalities to obtain

$$\begin{aligned}\langle A\boldsymbol{u}^1-A\boldsymbol{u}^0,\boldsymbol{u}^1-\boldsymbol{u}^0\rangle_{V'\times V} \leq & -\langle\delta^2\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^0\rangle_{V'\times V} - \psi(u^1,\boldsymbol{u}^1-\boldsymbol{u}^0,\beta^1) \\ & + \psi(\boldsymbol{u}^0,\boldsymbol{u}^1-\boldsymbol{u}^0,\beta^0) - \phi(\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^0) \\ & + \phi(\boldsymbol{u}^0,\boldsymbol{u}^1-\boldsymbol{u}^0) - h(\mathcal{G}(\boldsymbol{\sigma}^0,\varepsilon(\boldsymbol{u}^0)),\varepsilon(\boldsymbol{u}^1-\boldsymbol{u}^0))_Q \\ & + \langle\boldsymbol{f}^1-\boldsymbol{f}^0,\boldsymbol{u}^1-\boldsymbol{u}^0\rangle_{V'\times V}.\end{aligned} \tag{5.72}$$

We use (5.12), (5.13), (5.30), (5.33), (3.9), (5.25) and (5.71) we get

$$\begin{aligned}m_{\mathcal{A}}|\delta\boldsymbol{u}^1|_V^2 \leq \quad & c(|\boldsymbol{\sigma}^0|_Q+|\boldsymbol{u}^0|_V)|\delta\boldsymbol{u}^1|_V + (2C_1^2L_\nu+L_{\mathcal{A}})|\delta\boldsymbol{u}^1|_V^2 \\ & + c|\delta\boldsymbol{u}^1|_V|\delta\beta^1|^2_{L^2(\Gamma_c)} + |\delta\boldsymbol{f}^1|_{V'}|\delta\boldsymbol{u}^1|_V,\end{aligned}$$

Thus, from (5.24),(5.56),(5.47) and (5.68), we have that

$$|\delta\boldsymbol{u}^1|_V \leq c, \tag{5.73}$$

for $i \geq 1$. Thanks to (5.70) we get

$$\begin{aligned}&\langle A\boldsymbol{u}^{i+1}-A\boldsymbol{u}^i,\boldsymbol{u}^{i+1}-\boldsymbol{u}^i\rangle_{V'\times V} \leq -\psi(\boldsymbol{u}^{i+1},\boldsymbol{u}^{i+1}-\boldsymbol{u}^i,\beta^{i+1}) \\ &+\psi(\boldsymbol{u}^i,\boldsymbol{u}^{i+1}-\boldsymbol{u}^i,\beta^i) - \Phi(\boldsymbol{u}^{i+1},\boldsymbol{u}^{i+1}-\boldsymbol{u}^i) + \Phi(\boldsymbol{u}^i,\boldsymbol{u}^{i+1}-\boldsymbol{u}^i) \\ &+\psi(\boldsymbol{u}^{i+1},\boldsymbol{u}^{i+1}-\boldsymbol{u}^i,\beta^i) - \psi(\boldsymbol{u}^{i+1},\boldsymbol{u}^{i+1}-\boldsymbol{u}^i,\beta^i) \\ &-\langle\delta^2\boldsymbol{u}^{i+1}-\delta^2\boldsymbol{u}^i,\boldsymbol{u}^{i+1}-\boldsymbol{u}^i\rangle_{V'\times V} - h(\mathcal{G}(\boldsymbol{\sigma}^i,\varepsilon(\boldsymbol{u}^i)),\varepsilon(\boldsymbol{u}^{i+1}-\boldsymbol{u}^i))_Q \\ &+\langle\boldsymbol{f}^{i+1}-\boldsymbol{f}^i,\boldsymbol{u}^{i+1}-\boldsymbol{u}^i\rangle_{V'\times V},\end{aligned} \tag{5.74}$$

the relations (5.25) (5.29), (5.33),(5.71) (5.56), (5.68), (5.47) with the assumptions (5.12), (5.13),(5.24) and (3.9) show that

$$|\delta\boldsymbol{u}^{i+1}|_V \leq c \;\; \forall i \in \{0,\dots,N-1\}. \tag{5.75}$$

We use (5.52), (5.12) and (5.13), to have

$$\big|\boldsymbol{\sigma}^{i+1}-\boldsymbol{\sigma}^i|_Q \leq c|\boldsymbol{u}^{i+1}-\boldsymbol{u}^i|_V + ch(|\boldsymbol{\sigma}^i|_Q+|\boldsymbol{u}^i|_V+c),$$

and, by (5.52), (5.68) and (5.75) we find

$$\left|\frac{\boldsymbol{\sigma}^{i+1}-\boldsymbol{\sigma}^{i}}{h}\right|_{Q} \le c \ \ \forall i \in \{0,\ldots,N-1\}.$$

Thus, the following inequality can be obtained from (5.71), (5.56), (5.24), (5.75) and (5.68)

$$|\delta^3 \boldsymbol{u}^{i+1}|_V \le c \ \ \forall i \in \{0,\ldots,N-1\}. \tag{5.76}$$

From (5.53) we find that

$$|\beta_{i+1}-\beta_i|_{L^\infty(\Gamma_c)} = |h(C_n\beta_i(R(u_\nu^j))^2-\epsilon_a)_+|_{L^\infty(\Gamma_c)},$$

which, together with (5.17) leads to

$$|\delta\beta_{i+1}|_{L^2(\Gamma_c)} \le c$$

□

For each $N \in \mathbb{N}$, let $\boldsymbol{u}^{i+1}$ be the unique solution of the Problem $P_V^{i+1}, i \in \{0,\ldots,N-1\}$. We introduce the following functions $\boldsymbol{u}_N : [0,T] \to V$, $\tilde{\boldsymbol{u}}_N : [0,T] \to V$, $\tilde{\boldsymbol{f}}_N : [0,T] \to V$, $\tilde{\boldsymbol{\sigma}}_N : [0,T] \to Q$, $\tilde{\beta}_N : [0,T] \to L^2(\Gamma_c)$ and $G_N : [0,T] \to Q$ defined, respectively, by

$$\boldsymbol{u}_N(t) = \boldsymbol{u}^i + (t-t_i)\delta\boldsymbol{u}^{i+1} \qquad \forall t \in (t_i,t_{i+1}], \tag{5.77}$$

$$\boldsymbol{u}_N(0) = \boldsymbol{u}_0, \tilde{\boldsymbol{u}}_N(t) = \boldsymbol{u}^{i+1} \qquad \forall t \in (t_i,t_{i+1}], \tag{5.78}$$

$$\tilde{\boldsymbol{f}}_N(t) = \boldsymbol{f}^{i+1}, \ \dot{\boldsymbol{f}}_N(t) = \delta\boldsymbol{f}^{i+1} \qquad \forall t \in (t_i,t_{i+1}], \tag{5.79}$$

$$\tilde{\boldsymbol{\sigma}}_N(t) = \boldsymbol{\sigma}^{i+1} \qquad \forall t \in (t_i,t_{i+1}], \tag{5.80}$$

$$\tilde{\beta}_N(t) = \beta^{i+1} = -h\sum_{j=0}^{i} C_n\beta^j(\tilde{R}(u_\nu^j))^2 - \beta_0 \qquad \forall t \in (t_i,t_{i+1}], \tag{5.81}$$

$$G_N(t) = h\sum_{l=0}^{i} \mathcal{G}(\boldsymbol{\sigma}^l, \varepsilon(\boldsymbol{u}^l)) \qquad \forall t \in (t_i,t_{i+1}], \tag{5.82}$$

for all, $i = 0, ...N-1$. Here $(\boldsymbol{\sigma}^j)_0^N$, $(\beta^j)_0^N$ and $\boldsymbol{u}^0$ are given in (5.52) -(5.54). We use the following notations

$$\tilde{\boldsymbol{v}}_N(t) = \delta\boldsymbol{u}^{i+1}, \quad \boldsymbol{v}_N(t) = \delta\boldsymbol{u}^i + (t-t_i)\delta^2\boldsymbol{u}^{i+1}, \ \ \forall t \in (t_i,t_{i+1}], \tag{5.83}$$

$$\tilde{\boldsymbol{\omega}}_N(t) = \delta^2\boldsymbol{u}^{i+1}, \ \ \boldsymbol{\omega}_N(t) = \delta^2\boldsymbol{u}^{i+1} + (t-t_i)\delta^3\boldsymbol{u}^{i+1}, \ \ \forall t \in (t_i,t_{i+1}]. \tag{5.84}$$

Lemma 5.2.

$$\left|\tilde{\boldsymbol{\sigma}}_N(t)|_Q + \left|\tilde{\boldsymbol{u}}_N(t)|_V + |\tilde{\beta}_N(t)|_{L^\infty(\Gamma_c)} \leq c \qquad \forall t \in [0,T], \tag{5.85}$$

$$|\boldsymbol{u}_N(t)|_V \leq c \qquad \forall t \in [0,T], \tag{5.86}$$

$$\left|\boldsymbol{v}_N(t)\right|_V \leq c \qquad \forall t \in [0,T], \tag{5.87}$$

$$|\tilde{\boldsymbol{\omega}}_N(t)|_{V'} \leq c \qquad \forall t \in [0,T], \tag{5.88}$$

$$\left|\tilde{\boldsymbol{f}}_N(t) - \boldsymbol{f}(t)\right|_V \leq ch \qquad \forall t \in [0,T], \tag{5.89}$$

$$\left|\tilde{\boldsymbol{u}}_N(t) - \boldsymbol{u}_N(t)\right|_V \leq ch \qquad \forall t \in [0,T], \tag{5.90}$$

$$\left|\tilde{\boldsymbol{v}}_N(t) - \boldsymbol{v}_N(t)\right|_V \leq ch \qquad \forall t \in [0,T], \tag{5.91}$$

$$\left|\tilde{\boldsymbol{\omega}}_N(t) - \boldsymbol{\omega}_N(t)\right|_{V'} \leq ch \qquad \forall t \in [0,T], \tag{5.92}$$

$$\left|\dot{\boldsymbol{f}}_N(t) - \dot{\boldsymbol{f}}(t)\right|_V \leq ch \qquad \forall t \in [0,T], \tag{5.93}$$

$$|\boldsymbol{u}_N(t) - \boldsymbol{u}_N(s)|_V \leq c \mid t-s \mid \qquad \forall t,s \in [0,T], \tag{5.94}$$

$$|\boldsymbol{u}_N(t) - \boldsymbol{u}_N(s)|_{L^2(\Gamma_c;\mathbb{R}^d)} \leq c \mid t-s \mid \qquad \forall t,s \in [0,T], \tag{5.95}$$

$$\left|\boldsymbol{v}_N(t) - \boldsymbol{v}_N(s)\right|_V \leq c \mid t-s \mid \qquad \forall t,s \in [0,T], \tag{5.96}$$

$$\left|\boldsymbol{v}_N(t) - \boldsymbol{v}_N(s)\right|_{L^2(\Gamma_c;\mathbb{R}^d)} \leq c \mid t-s \mid \qquad \forall t,s \in [0,T]. \tag{5.97}$$

Proof. (5.85)-(5.88) are consequences of Lemma 5.1. For proof of (5.90) and (5.89) we see that

$$\tilde{\boldsymbol{f}}_N(t) = \boldsymbol{f}_0 + \int_0^{t_{i+1}} \dot{\boldsymbol{f}}(s)ds \; \forall t \in (t_i, t_{i+1}], i = 0, ...N-1,$$

so

$$\begin{aligned}
\left|\tilde{\boldsymbol{f}}_N(t) - \boldsymbol{f}(t)\right|_V &= \Big| \int_0^{t_{i+1}} \dot{\boldsymbol{f}}(s)ds - \int_0^t \dot{\boldsymbol{f}}(s)ds\Big|_V \\
&= \Big| \int_t^{t_{i+1}} \dot{\boldsymbol{f}}(s)ds\Big|_V \\
&\leq \Big| \int_t^{t_{i+1}} |\dot{\boldsymbol{f}}(s)|_V ds\Big| \\
&\leq ch,
\end{aligned}$$

we have that

$$\begin{aligned}
\boldsymbol{u}^0 + \int_0^t \tilde{\boldsymbol{v}}_N(r)dr &= \boldsymbol{u}^0 + \sum_{l=1}^{i} \int_{t_{l-1}}^{t_l} \tilde{\boldsymbol{v}}_N(r)dr + \int_{t_i}^{t} \tilde{\boldsymbol{v}}_N(r)dr \quad \forall t \in (t_i, t_{i+1}] \\
&= \boldsymbol{u}^0 + \sum_{l=1}^{i} h\delta\boldsymbol{u}^l + \delta\boldsymbol{u}^{i+1} \int_{t_i}^{t} dr \\
&= \boldsymbol{u}^i + (t - t_i)\delta\boldsymbol{u}^{i+1} \\
&= \boldsymbol{u}_N(t);
\end{aligned}$$

thus we find

$$\begin{aligned}
\Big|\tilde{\boldsymbol{u}}_N(t) - \boldsymbol{u}_N(t)\Big|_V &= \Big|\int_0^{t_{i+1}} \tilde{\boldsymbol{v}}_N(s)ds - \int_0^t \tilde{\boldsymbol{v}}_N(s)ds\Big|_V \\
&= \Big|\int_t^{t_{i+1}} \tilde{\boldsymbol{v}}_N(s)ds\Big|_V \\
&\le \Big|\int_t^{t_{i+1}} \Big|\tilde{\boldsymbol{v}}_N(s)\Big|_V ds\Big| \\
&\le ch.
\end{aligned}$$

In the same way we prove (5.92), (5.93) and (5.91).
If we use (5.87) we get

$$|\boldsymbol{u}_N(t) - \boldsymbol{u}_N(s)|_V \le \Big|\int_s^t \Big|\tilde{\boldsymbol{v}}_N(r)\Big|_V dr\Big| \le c \mid t - s \mid \qquad \forall t, s \in [0, T];$$

(5.95) is a result of (5.94) and (3.9),
also we have that

$$\begin{aligned}
\boldsymbol{v}^0 + \int_0^t \tilde{\boldsymbol{\omega}}_N(r)dr &= \boldsymbol{v}^0 + \sum_{l=1}^{i} \int_{t_{l-1}}^{t_l} \tilde{\boldsymbol{\omega}}_N(r)dr + \int_{t_i}^{t} \tilde{\boldsymbol{\omega}}_N(r)dr \quad \forall t \in (t_i, t_{i+1}] \\
&= \boldsymbol{v}^0 + \sum_{l=1}^{i} h\delta^2\boldsymbol{u}^i + \delta^2\boldsymbol{u}^{i+1} \int_{t_i}^{t} dr \\
&= \delta\boldsymbol{u}^i + (t - t_i)\delta^2\boldsymbol{u}^{i+1} \\
&= \boldsymbol{v}_N(t),
\end{aligned}$$

so we get

$$|\boldsymbol{v}_N(t) - \boldsymbol{v}_N(s)|_V = \Big| \int_0^t \tilde{\boldsymbol{\omega}}_N(r)dr - \int_0^s \tilde{\boldsymbol{\omega}}_N(r)dr \Big|_V$$

$$\leq \Big| \int_s^t \Big|\tilde{\boldsymbol{\omega}}_N(r)\Big|_V dr \Big|$$
$$\leq c|t-s|.$$

(5.97) is a consequence of (5.96) and (3.9). □

Lemma 5.3. *There is $c > 0$ where for each $M, N \in \mathbb{N}^*$ with $N > M$, we take $h' = \frac{T}{M}$*

$$|\mathcal{G}_N(t) - \mathcal{G}_M(t)|_Q \leq c \int_0^t |\tilde{\boldsymbol{u}}_N(s) - \tilde{\boldsymbol{u}}_M(s)|_V ds + ch'$$

Proof. The last inequality is self-evident for $t = 0$.
So for all $t \in (0, T]$ there is $l \in \{0, ..., N-1\}$, $q \in \{0, ..., M-1\}$ where

$$t \in (t_l, t_{l+1}] \cap (t_q, t_{q+1},] \tag{5.98}$$

using (5.52), (5.54), (5.78) and (5.80), to get

$$\tilde{\boldsymbol{\sigma}}_N(t) = \mathcal{A}\varepsilon(\tilde{\boldsymbol{u}}_N(t)) + h\mathcal{G}(\boldsymbol{\sigma}^0, \varepsilon(\boldsymbol{u}^0)) + \boldsymbol{\sigma}^0 - \mathcal{A}\varepsilon(\boldsymbol{u}^0) \quad \forall\, t \in (0, t_1],$$

and therefore $t \in (t_l, t_{l+1}]$

$$\tilde{\boldsymbol{\sigma}}_N(t) = \mathcal{A}\varepsilon(\tilde{\boldsymbol{u}}_N(t)) + \sum_{j=1}^{l} \int_{t_{j-1}}^{t_j} \mathcal{G}(\tilde{\boldsymbol{\sigma}}_N(s), \varepsilon(\tilde{\boldsymbol{u}}_N(s)))ds + h\mathcal{G}(\boldsymbol{\sigma}^0, \varepsilon(\boldsymbol{u}^0)) + \boldsymbol{\sigma}^0 - \mathcal{A}\varepsilon(\boldsymbol{u}^0).$$

So

$$\begin{aligned} \tilde{\boldsymbol{\sigma}}_N(t) &= \mathcal{A}\varepsilon(\tilde{\boldsymbol{u}}_N(t)) + h\mathcal{G}(\boldsymbol{\sigma}^0, \varepsilon(\boldsymbol{u}^0)) + \int_0^t \mathcal{G}(\tilde{\boldsymbol{\sigma}}_N(s), \varepsilon(\tilde{\boldsymbol{u}}_N(s)))ds \\ &+ \int_t^{t_l} \mathcal{G}(\tilde{\boldsymbol{\sigma}}_N(s), \varepsilon(\tilde{\boldsymbol{u}}_N(s)))ds + \boldsymbol{\sigma}^0 - \mathcal{A}\varepsilon(\boldsymbol{u}^0) \quad \forall\, t \in (t_l, t_{l+1}]. \end{aligned} \tag{5.99}$$

We conclude from the last inequality, (5.12), (5.13) and (5.98) that $\forall t \in [0, T]$,

$$\begin{aligned} |\tilde{\boldsymbol{\sigma}}_N(t) - \tilde{\boldsymbol{\sigma}}_M(t)|_Q &\leq c|\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)|_V + c\int_0^t |\tilde{\boldsymbol{u}}_N(s) - \tilde{\boldsymbol{u}}_M(s)|_V ds \\ &+ c\int_0^t |\tilde{\boldsymbol{\sigma}}_N(s) - \tilde{\boldsymbol{\sigma}}_M(s)|_Q ds + ch + ch', \end{aligned}$$

using the Gronwall lemma we get

$$\left|\tilde{\boldsymbol{\sigma}}_N(t) - \tilde{\boldsymbol{\sigma}}_M(t)\right|_Q \leq c\left|\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\right|_V + c\int_0^t \left|\tilde{\boldsymbol{u}}_N(s) - \tilde{\boldsymbol{u}}_M(s)\right|_V ds + ch', \tag{5.100}$$

we use (5.78),(5.80),(5.82) and (5.13), we find that for each $t \in (t_l, t_{l+1}] \cap (t_q, t_{q+1}]$

$$\begin{aligned}|\mathcal{G}_N(t) - \mathcal{G}_M(t)\Big|_Q \leq & c\int_0^t \left|\tilde{\boldsymbol{\sigma}}_N(s) - \tilde{\boldsymbol{\sigma}}_M(s)\right|_Q ds + c\int_0^t \left|\tilde{\boldsymbol{u}}_N(s) - \tilde{\boldsymbol{u}}_M(s)\right|_V ds \\ & + ch + ch' + \int_{t_l}^t \left|\mathcal{G}(\tilde{\boldsymbol{\sigma}}_N(s), \varepsilon(\tilde{\boldsymbol{u}}_N(s))\right|_Q ds \\ & + \int_{t_q}^t \left|\mathcal{G}(\tilde{\boldsymbol{\sigma}}_M(s), \varepsilon(\tilde{\boldsymbol{u}}_M(s))\right|_Q ds,\end{aligned}$$

by (5.85) we find

$$\left|\mathcal{G}_N(t) - \mathcal{G}_M(t)\right|_Q \leq c\int_0^t \left|\tilde{\boldsymbol{u}}_n(s) - \tilde{\boldsymbol{u}}_m(s)\right|_V ds + ch'.$$

□

Lemma 5.4. *There exists two subsequences of $\{u_N\}$ and $\{\tilde{u}_N\}$ also denoted by $\{u_N\}$, $\{\tilde{u}_N\}$ respectively and an element $u \in W^{1,2}(0,T;V)$ such that*

$$\boldsymbol{u}_N \rightharpoonup \boldsymbol{u} \ \textit{weakly in } L^2(0,T;V). \tag{5.101}$$
$$\tilde{\boldsymbol{v}}_N \rightharpoonup \dot{\mathbf{u}} \ \textit{weakly in } L^2(0,T;V). \tag{5.102}$$
$$\boldsymbol{v}_N \rightarrow \dot{\boldsymbol{u}} \ \textit{strongly in } C([0,T];V). \tag{5.103}$$
$$\tilde{\boldsymbol{v}}_N \rightarrow \dot{\boldsymbol{u}} \ \textit{strongly in } L^2(0,T;V). \tag{5.104}$$
$$\tilde{\boldsymbol{\omega}}_N \rightharpoonup \ddot{\mathbf{u}} \ \textit{weakly in} L^2(0,T;H). \tag{5.105}$$
$$\tilde{\boldsymbol{\omega}}_N \rightarrow \ddot{\mathbf{u}} \ \textit{strongly in } L^2(0,T;V'). \tag{5.106}$$
$$\boldsymbol{u}_N \rightarrow \boldsymbol{u} \ \textit{strongly in } C([0,T];L^2(\Gamma_3;\mathbb{R}^d)). \tag{5.107}$$
$$\boldsymbol{u}_N \rightarrow \boldsymbol{u} \ \textit{strongly in } C([0,T];V). \tag{5.108}$$
$$\tilde{\boldsymbol{u}}_N \rightarrow \boldsymbol{u} \ \textit{strongly in } L^2(0,T;V). \tag{5.109}$$

Proof. We know that $L^2(0,T;V)$ is a Hilbert space endowed to the inner product

$$(\boldsymbol{u}, \boldsymbol{z})_{L^2(0,T;V)} = \int_0^t (\boldsymbol{u}(s), \boldsymbol{z}(s))_V ds \quad \forall \boldsymbol{u}, \boldsymbol{z} \in L^2(0,T;V).$$

From (5.86) and (5.87), we find that we have two functions $\boldsymbol{u}, \boldsymbol{w}$ in $L^2(0,T;V)$ and a subsequence of $\{\boldsymbol{u}_N\}$ denoted by $\{\boldsymbol{u}_N\}$ such that

$$\boldsymbol{u}_N \rightharpoonup \boldsymbol{u} \textit{ weakly in } L^2(0,T;V),$$

$$\widetilde{\boldsymbol{v}}_N \rightharpoonup \boldsymbol{w} \textit{ weakly in } L^2(0,T;V),$$

the functionals $\boldsymbol{z} \to (\boldsymbol{v}, \int_0^T \boldsymbol{z}(s)\varsigma(s)ds)_V$ and $\boldsymbol{z} \to (\boldsymbol{v}, \int_0^T \boldsymbol{z}(s)\dot{\varsigma}(s)ds)_V$ are linear continuous functionals for all $\boldsymbol{v} \in K$ and $\varsigma \in D(0,T)$ than we find that

$$\begin{aligned}\Big(\boldsymbol{v}, \int_0^T \boldsymbol{u}(s)\dot{\varsigma}(s)ds\Big)_V &= \lim_{N\to\infty}\Big(\boldsymbol{v}, \int_0^T \boldsymbol{u}_N(s)\dot{\varsigma}(s)ds\Big)_V\\ &= \lim_{N\to\infty}\Big(\boldsymbol{v}, -\int_0^T \tilde{\boldsymbol{v}}_N(s)\varsigma(s)ds\Big)_V\\ &= \Big(\boldsymbol{v}, -\int_0^T \boldsymbol{w}(s)\varsigma(s)ds\Big)_V,\end{aligned}$$

and from it we deduce that $\boldsymbol{w} = \dot{\boldsymbol{u}}$.
For the proof of (5.107), we deduce frome (5.95) that the set $W = \{\boldsymbol{u}_N : [0,T] \to L^2(\Gamma_3), \mathbb{R}^d), N \in \mathbb{N}^*\}$ is equicontinous, and since the trace map is compact operator and from (5.86), we conclude that $W(t) = \{\boldsymbol{u}_N(t) \ / \ \boldsymbol{u}_N \in W\}$ is relatively compact, in the last from the Ascoli-Arzela theorem, we get (5.107).
It follows from (5.51), (5.78), (5.77), (5.81), (5.79) and (5.82) that

$$\begin{aligned}&\Big\langle A\tilde{\boldsymbol{u}}_N(t), v - \tilde{\boldsymbol{u}}_N(t)\Big\rangle_{V\times V'} + \Big(\mathcal{G}_N(t), \varepsilon(\boldsymbol{v} - \tilde{\boldsymbol{u}}_N(t)\Big)_Q + \psi\Big(\tilde{\boldsymbol{u}}_N(t), v - \tilde{\boldsymbol{u}}_N(t), \tilde{\beta}_N(t)\Big)\\ &+ \Phi\Big(\tilde{\boldsymbol{u}}_N(t), \boldsymbol{v}\Big) - \Phi\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t)\Big) + ((\tilde{\boldsymbol{\omega}}_N, \boldsymbol{v} - \tilde{\boldsymbol{u}}_N(t)))_H\\ &\geq \Big\langle \boldsymbol{f}_N(t), \boldsymbol{v} - \tilde{\boldsymbol{u}}_N(t)\Big\rangle_{V\times V'} \qquad \forall \boldsymbol{v} \in U \quad \forall t \in [0,T].\end{aligned}$$

For each $M, N \in \mathbb{N}^*$, let $N > M$. We take $v = -h\tilde{\boldsymbol{v}}_M + \boldsymbol{u}^{i+1}$ in inequality (5.51) which satisfaite with $\boldsymbol{u}_i$, $v = h\tilde{\boldsymbol{v}}_M + \boldsymbol{u}^i$ in inequality which satisfaite with $\boldsymbol{u}_{i+1}$,$\boldsymbol{v} = -h'\tilde{\boldsymbol{v}}_N + \boldsymbol{u}^{j+1}$ in inequality which satisfaite with $\boldsymbol{u}_j$ and $\boldsymbol{v} = h\widetilde{\boldsymbol{v}}_N + \boldsymbol{u}^j$ in inequality which satisfaite with $\boldsymbol{u}_{j+1}$ and we put $\tilde{\dot{\boldsymbol{v}}}_N(t) = \delta^2\boldsymbol{u}_i$, $\ddot{\boldsymbol{v}}_N(t) = \delta^2\boldsymbol{u}^{i-1} + (t - t_{i-1})\delta^3\boldsymbol{u}^i$, we get

$$\Big\langle A\tilde{\boldsymbol{v}}_N(t) - A\tilde{\boldsymbol{v}}_M(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big\rangle_{V'\times V}$$
$$\leq \quad -\Big(h\mathcal{G}(\boldsymbol{\sigma}^i, \varepsilon(\boldsymbol{u}^i)) - h^{'}\mathcal{G}(\boldsymbol{\sigma}^j, \varepsilon(\boldsymbol{u}^j)), \varepsilon(\tilde{\boldsymbol{v}}_N - \tilde{\boldsymbol{v}}_M)\Big)_Q + \Phi\Big(\boldsymbol{u}^i, \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big)$$
$$- \Phi\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big) + \Phi\Big(\tilde{\boldsymbol{u}}_M(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big)$$
$$- \Phi\Big(\boldsymbol{u}^j, \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big) + \Big\langle \dot{\boldsymbol{f}}_N(t) - \dot{\boldsymbol{f}}_M(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big\rangle_{V'\times V}$$
$$+ \Big\langle \tilde{\ddot{\boldsymbol{v}}}_N(t) - \tilde{\boldsymbol{\omega}}_N(t) + \tilde{\boldsymbol{\omega}}_M(t) - \tilde{\ddot{\boldsymbol{v}}}_M(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big\rangle_{V'\times V}$$
$$- \psi\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t), \tilde{\beta}_N(t)\Big) + \psi\Big(\boldsymbol{u}^i, \tilde{\boldsymbol{v}}(t) - \tilde{\boldsymbol{v}}_M(t), \beta^i\Big)$$
$$+ \psi\Big(\tilde{\boldsymbol{u}}_M(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t), \tilde{\beta}_M(t)\Big) - \psi\Big(\boldsymbol{u}^j, \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t), \beta^j\Big)$$
$$+ \psi\Big(\boldsymbol{u}^i(t)\tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t), \tilde{\beta}_N(t)\Big) - \psi\Big(\boldsymbol{u}^i(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t), \tilde{\beta}_N(t)\Big)$$
$$\psi\Big(\boldsymbol{u}^j(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t), \tilde{\beta}_M(t)\Big) - \psi\Big(\boldsymbol{u}^j(t), \tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t), \tilde{\beta}_M(t)\Big),$$

which combined with (5.12), (5.13),(5.25),(5.33),(5.62),(5.60) and (5.83)

$$\begin{aligned} & m_{\mathcal{A}}\Big|\tilde{v}_N(t) - \tilde{v}_M(t)\Big|_V^2 \\ & \leq +ch^{'}\Big|\tilde{v}_N(t) - \tilde{v}_M(t)\Big|_V + ch\Big|\tilde{v}_N(t) - \tilde{v}_M(t)\Big|_V + ch^{'} + ch \\ & \quad + |\dot{f}_N(t) - \dot{f}_M(t)|_V\Big|\tilde{v}_N(t) - \tilde{v}_M(t)\Big|_V, \end{aligned}$$

applying the following inequality

$$ab \leq \frac{2}{m_{\mathcal{A}}}a^2 + \frac{m_{\mathcal{A}}}{8}b^2,$$

we find

$$\Big|\tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big|_V^2 \leq c\Big|\dot{\boldsymbol{f}}_N(t) - \dot{\boldsymbol{f}}(t)\Big|_V^2 + c\Big|\dot{\boldsymbol{f}}(t) - \dot{\boldsymbol{f}}_M(t)\Big|_V^2 + ch + ch^{'} + ch^2 + ch^{'2},$$

by (5.93), we have

$$\Big|\tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big|_V^2 \leq ch + ch^{'} + ch^2 + ch^{'2}.$$

In the oder said from (5.91) we have that

$$\begin{aligned} |\boldsymbol{v}_N(t) - \boldsymbol{v}_M(t)|_V^2 & \leq \Big|\tilde{\boldsymbol{v}}_N(t) - \tilde{\boldsymbol{v}}_M(t)\Big|_V^2 + ch^2 + ch^{'2} \\ & \leq ch + ch^{'} + ch^2 + ch^{'2}. \end{aligned} \tag{5.110}$$

Then we conclude that $\{\tilde{\boldsymbol{v}}_N\}$ is the Cauchy sequence in $L^2([0,T];V)$ and $\{\boldsymbol{v}_N\}$ is the Cauchy sequence in $C([0,T];V)$, if we use the convergence (5.102), we obtain (5.104). The convergence (5.103) is a result of (5.87) and theorem 2.39 in [29].
About (5.105) we have that for all $v \in U$ and $\boldsymbol{\varsigma} \in D(0,T)$ than we find that

$$\begin{aligned}\Big(\Big(\int_0^T \boldsymbol{v}(s)\dot{\boldsymbol{\varsigma}}(s)ds, v\Big)\Big)_H &= \lim_{N\to\infty}\Big(\Big(\int_0^T \boldsymbol{v}_N(s)\dot{\boldsymbol{\varsigma}}(s)ds, v\Big)\Big)_H \\ &= \lim_{N\to\infty}\Big(\Big(-\int_0^T \tilde{\boldsymbol{\omega}}_N(s)\boldsymbol{\varsigma}(s)ds, v\Big)\Big)_H \\ &= \Big(\Big(-\int_0^T g(s)\boldsymbol{\varsigma}(s)ds, v\Big)\Big)_H,\end{aligned}$$

so we conclude that $\tilde{\boldsymbol{\omega}}_N(t) \rightharpoonup g(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{u}}(t)$ in H.
Since H is compactly embedded in V' and from (5.105), we have $\tilde{\boldsymbol{\omega}}_N \longrightarrow \ddot{\boldsymbol{u}}$ strongly in $C(0,T;V')$.
On the other hand, we have

$$\begin{aligned}&\Big(\mathcal{A}\varepsilon(\tilde{\boldsymbol{u}}_N(t)) - \mathcal{A}\varepsilon(\tilde{\boldsymbol{u}}_M(t)), \varepsilon(\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t))\Big)_Q \\ &\leq \Big(\mathcal{G}_N(t) - \mathcal{G}_M(t), \varepsilon(\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t))\Big)_Q - \Big\langle \tilde{\boldsymbol{\omega}}_N(t) - \tilde{\boldsymbol{\omega}}_M(t), \tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\Big\rangle_{V'\times V} \\ &\quad + \psi\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t), \tilde{\beta}_N(t)\Big) - \psi\Big(\tilde{\boldsymbol{u}}_M(t), \tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t), \tilde{\beta}_M(t)\Big) \\ &\quad + \Phi\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\Big) - \Phi\Big(\tilde{\boldsymbol{u}}_M(t), \tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\Big) \\ &\quad + \Big\langle \tilde{\boldsymbol{f}}_N(t) - \tilde{\boldsymbol{f}}_M(t), \tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\Big\rangle_{V'\times V} \qquad \forall \boldsymbol{v} \in U,\ \forall t \in [0,T],\end{aligned}$$

from (5.12),(5.28), (5.36), (5.85) and applying the inequality

$$ab \leq \frac{2}{m_{\mathcal{A}}}a^2 + \frac{m_{\mathcal{A}}}{8}b^2$$

we obtain

$$\begin{aligned}&\Big|\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\Big|_V^2 \\ &\leq c\Big|\mathcal{G}_N(t) - \mathcal{G}_M(t)\Big|_Q^2 + c\Big|\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\Big|_{L^2(\Gamma_c;\mathbb{R}^d)} + c\Big|\tilde{\boldsymbol{f}}_N(t) - \boldsymbol{f}(t)\Big|_V^2 \\ &\quad + c\Big|\tilde{\boldsymbol{f}}_M(t) - \boldsymbol{f}(t)\Big|_V^2 + c|\tilde{\boldsymbol{\omega}}_N(t) - \tilde{\boldsymbol{\omega}}_M(t)|_{V'}^2 \qquad \forall t \in [0,T],\end{aligned}$$

and from (5.90) and (3.9) we have that

$$\left|\widetilde{\boldsymbol{u}}_N(t) - \widetilde{\boldsymbol{u}}_M(t)|_V \leq |\boldsymbol{u}_N(t) - \boldsymbol{u}_M(t)\right|_V + ch + ch^{'} \tag{5.111}$$

$$\left|\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\right|_{L^2(\Gamma_c;\mathbb{R}^d)} \leq |\boldsymbol{u}_N(t) - \boldsymbol{u}_M(t)|_{L^2(\Gamma_c;\mathbb{R}^d)} + ch + ch^{'}, \tag{5.112}$$

next, by (5.112), (5.111) ,(5.89) and lemma (5.3) we find that

$$\begin{aligned}\left|\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\right|_V^2 \leq\ & c|\boldsymbol{u}_N(t) - \boldsymbol{u}_M(t)|_{L^2(\Gamma_c;\mathbb{R}^d)} + ch + ch^{'} + ch^2 + ch^{'2} \\ & + c\int_0^t |\boldsymbol{u}_N(s) - \boldsymbol{u}_M(s)|_V^2 ds + c\|\tilde{\boldsymbol{\omega}}_N(t) - \tilde{\boldsymbol{\omega}}_M(t)\|_{V'}^2.\end{aligned} \tag{5.113}$$

Note that thanks to (5.110), There is a constant c where

$$|\tilde{\boldsymbol{\omega}}_N(t) - \tilde{\boldsymbol{\omega}}_M(t)|_{V'}^2 \leq ch,$$

so, we get

$$\begin{aligned}\left|\tilde{\boldsymbol{u}}_N(t) - \tilde{\boldsymbol{u}}_M(t)\right|_V^2 \leq\ & c|\boldsymbol{u}_N(t) - \boldsymbol{u}_M(t)|_{L^2(\Gamma_3;\mathbb{R}^d)} + ch + ch^{'} \\ & + c\int_0^t |\boldsymbol{u}_N(s) - \boldsymbol{u}_M(s)|_V^2 ds + ch^2 + ch^{'2},\end{aligned} \tag{5.114}$$

which together with (5.90), leads us to

$$\begin{aligned}\left|\boldsymbol{u}_N(t) - \boldsymbol{u}_M(t)\right|_V^2 \leq\ & c|\boldsymbol{u}_N(t) - \boldsymbol{u}_M(t)|_{L^2(\Gamma_c;\mathbb{R}^d)} + ch + ch^{'} \\ & + c\int_0^t |\boldsymbol{u}_N(s) - \boldsymbol{u}_M(s)|_V^2 ds + ch^2 + ch^{'2},\end{aligned} \tag{5.115}$$

we apply Lemma 4.19

$$\begin{aligned}|\boldsymbol{u}_N(t) - \boldsymbol{u}_M(t)|_V^2 \leq & c|\boldsymbol{u}_{N(t)-\boldsymbol{u}_M}(t)|_{L^2}(\Gamma_c;\mathbb{R}^d) \\ & + c\int_0^t |\boldsymbol{u}_N(s) - \boldsymbol{u}_M(s)|_{L^2(\Gamma_c;\mathbb{R}^d)} ds + ch^{'} \qquad \forall t \in [0,T].\end{aligned} \tag{5.116}$$

So we obtain

$$|\boldsymbol{u}_N - \boldsymbol{u}_M|_{C([0,T];V)}^2 \leq\ c|\boldsymbol{u}_N - \boldsymbol{u}_M|_{C([0,T];L^2(\Gamma_c))} + ch^{'}. \tag{5.117}$$

Now, using (5.107) we conclude that $\{\boldsymbol{u}_N\}$ is the Cauchy sequence in $C([0,T];V)$ and if we use (5.101) we deduce the convergence (5.108), and in the last (5.109) is the result of (5.90) and (5.108). □

We define the operator $F : C([0,T];Q) \to C([0,T];Q)$ as follows:

$$F\boldsymbol{\vartheta}(t) = \mathcal{A}\varepsilon(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\vartheta}(s), \varepsilon(\boldsymbol{u}(s)))ds + \boldsymbol{\sigma}^0 - \mathcal{A}\varepsilon(\boldsymbol{u}^0), \qquad \forall t \in [0,T] \tag{5.118}$$

Lemma 5.5. *The operator F has a unique fixed point $\sigma \in C([0,T];Q)$*

Proof. Let $\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in C([0,T];Q)$, using 5.13 we get

$$|F\boldsymbol{\vartheta}_1(t) - F\boldsymbol{\vartheta}_2(t)|_Q^2 \leq c\int_0^t |\boldsymbol{\vartheta}_1(s) - \boldsymbol{\vartheta}_2(s)|_Q^2, ds \tag{5.119}$$

reiterating (5.119) n times, we find that

$$|F^n\boldsymbol{\vartheta}_1 - F^n\boldsymbol{\vartheta}_2|^2_{C([0,T];Q)} \leq \frac{(cT)^n}{n!}|\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2|^2_{C([0,T];Q)},$$

for a great value for n the operator F^n is a contraction in $C([0,T];Q)$, hence from the Banach fixed point theorem the operator F has a unique fixed point $\boldsymbol{\sigma} \in C([0,T];Q)$ □

Problem P_V^β

Find $\beta : [0,T] \to L^\infty(\Gamma_c)$ such that:

$$\beta(0) = \beta_0 \tag{5.120}$$

$$\dot{\beta}(t) = -(C_\nu\beta(t)(\tilde{R}(u_\nu(t)))^2 - \epsilon_a)_+ \qquad a.e\ t\ \in (0,T). \tag{5.121}$$

Lemma 5.6. *The problem P_V^β has a unique solution β wich satisfies*

$$\beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_c)). \tag{5.122}$$

$$\tag{5.123}$$

Proof. Fix $u \in L^2(0,T;V)$ We define $B : [0,T]\times L^\infty(\Gamma_c) \to L^\infty(\Gamma_c)$ as follows

$$B(t,\beta) = -(C_\nu\beta(t)(\tilde{R}(u_\nu(t)))^2 - \epsilon_a)_+,$$

we see that

$$\begin{aligned}
&|B(t,\beta_1) - B(t,\beta_2)|_{L^\infty(\Gamma_c)} \\
&= \Big| -(C_\nu(\tilde{R}(u_\nu(s)))^2\beta_1(s) - \epsilon_a)_+ + (C_\nu(\tilde{R}(u_\nu(s)))^2\beta_2(s) - \epsilon_a)_+\Big|_{L^\infty(\Gamma_c)} \\
&\leq C_\nu(\tilde{R}(u_\nu))^2|\beta_1(s) - \beta_2(s)|_{L^\infty(\Gamma_c)}.
\end{aligned}$$

In addition to $t \to B(t,\beta) \in L^\infty(0,T;L^\infty(\Gamma_c))$, $\forall\beta\ \in L^\infty(\Gamma_c)$. Then according to Theorem 2.7 there is only one unique $\beta \in L^\infty(0,T;L^\infty(\Gamma_c))$ such that

$$\dot{\beta}(t) = B(t,\beta(t)),$$
$$\beta(0) = \beta_0.$$

(5.7) show that the function $t \to \beta(t)$ is decreasing and its derivative vanishes when $C_\nu\beta(t)(\tilde{R}(u_\nu(t)))^2 \leq \epsilon_a$. Moreover, from (5.18) we conclude that $0 \leq \beta \leq 1$ for all $t \in [0,T]$. □

$$\mathcal{Q}(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\boldsymbol{u}(s)))ds \quad \forall t \in [0, T]. \tag{5.124}$$

Lemma 5.7. $\forall \boldsymbol{v} \in L^2(0, T; V)$

$$\begin{aligned}
&\lim_{N\to+\infty} \Big\langle A\tilde{\boldsymbol{u}}_N(t), \boldsymbol{v}(t) - \tilde{\boldsymbol{u}}_N(t)\Big\rangle_{V'\times V} = \langle A\boldsymbol{u}(t), \boldsymbol{v}(t) - u(t)\rangle_{V'\times V},\\
&\lim_{N\to+\infty} \Big\langle \tilde{\boldsymbol{f}}_N(t), \boldsymbol{v}(t) - \tilde{\boldsymbol{u}}_N(t)\Big\rangle_{V'\times V} = \langle \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle_{V'\times V},\\
&\lim_{N\to+\infty} \Big(\mathcal{G}_N(t), \boldsymbol{v}(t) - \varepsilon(\tilde{\boldsymbol{u}}_N(t))\Big)_Q = (\mathcal{Q}(t), \varepsilon(\boldsymbol{v}(t) - \boldsymbol{u}(t)))_Q,\\
&\lim_{N\to+\infty} \Big(\Big(\tilde{\boldsymbol{\omega}}_N(t), \boldsymbol{v}(t) - \tilde{\boldsymbol{u}}_N(t)\Big)\Big)_H ds = ((\ddot{\boldsymbol{u}}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)))_H,\\
&\lim_{N\to+\infty} J\Big(\tilde{\boldsymbol{u}}_N(t), \boldsymbol{v}(t), \tilde{\beta}_N(t)\Big)ds = J(\boldsymbol{u}(t), \boldsymbol{v}(t), \beta(t)),\\
&\lim_{N\to+\infty} \Big(J\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t), \tilde{\beta}_N(t)\Big) - J(\boldsymbol{u}(t), \boldsymbol{u}(t), \beta(t))\Big) = 0,\\
&\lim_{N\to+\infty} \mathit{inf}\, J\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t), \tilde{\beta}_N(t)\Big) \geq J(\boldsymbol{u}(t), \boldsymbol{u}(t), \beta(t)).
\end{aligned}$$

Proof. We note that from (5.39) and (5.109)

$$A\tilde{\boldsymbol{u}}_N \to A\boldsymbol{u} \text{ strongly in } L^2(0, T; V),$$

in addition to the result (5.109) we get the first equation in Lemma 5.7. Using the result (5.89) we find that

$$\tilde{\boldsymbol{f}} \to \boldsymbol{f} \text{ strongly in } L^2(0, T; V),$$

and with (5.109) we obtained the second equation of Lemma (5.97).
Let $\boldsymbol{\sigma}$ be the fixed point of (5.118), it follows from (5.12), (5.13) and (5.99) that

$$\begin{aligned}
\Big|\tilde{\boldsymbol{\sigma}}_N(t) - \boldsymbol{\sigma}(t)\Big|_Q \leq c\Big|\tilde{\boldsymbol{u}}_N(t) - \boldsymbol{u}(t)\Big|_V + c\int_0^t \Big|\tilde{\boldsymbol{\sigma}}_N(s) - \boldsymbol{\sigma}(s)\Big|_Q ds\\
+ c\int_0^t \Big|\tilde{\boldsymbol{u}}_N(s) - \boldsymbol{u}(s)\Big|_Q ds + ch \quad \forall t \in [0, T],
\end{aligned}$$

we use now Lemma 4.19 to obtain

$$\Big|\tilde{\boldsymbol{\sigma}}_N(t) - \boldsymbol{\sigma}(t)\Big|_Q \leq c\Big|\tilde{\boldsymbol{u}}_N(t) - \boldsymbol{u}(t)\Big|_V + c\int_0^t \Big|\tilde{\boldsymbol{u}}_N(s) - \boldsymbol{u}(s)\Big|_V ds + ch \quad \forall t \in [0, T],$$

it follows from (5.124), (5.78),(5.80) and (5.82) that

$$\begin{aligned}\Big|\mathcal{G}_N(t)-\mathcal{Q}(t)\Big|_Q &\leq c\int_0^t \Big|\tilde{\boldsymbol{u}}_N(s)-\boldsymbol{u}(s)\Big|_V ds + c\int_0^t \Big|\tilde{\boldsymbol{\sigma}}_N(s)-\boldsymbol{\sigma}(s)\Big|_Q ds + ch\\ &\leq \quad c\int_0^t \Big|\tilde{\boldsymbol{u}}_N(s)-\boldsymbol{u}(s)\Big|_V ds + ch \quad \forall t\in[0,T],\end{aligned}$$

using (5.109), we deduce that

$$\mathcal{G}_N \to \mathcal{Q} \ \ \textit{strongly in } L^2(0,T;S)$$

and from it the third inequality have been verified. By (5.109) we get the fourth equation of Lemma (5.7). Using the definition of the functional J and the properties of functions ψ and ϕ, the trace theorem, (5.85), (5.122) we obtain

$$\begin{aligned}&\Big|[J(\tilde{\boldsymbol{u}}_N(t),\boldsymbol{v}(t),\tilde{\beta}_N(t))-J(\boldsymbol{u}(t),\boldsymbol{v}(t),\beta(t))]\Big|\\ &\leq c|v|_V(\Big|\tilde{\boldsymbol{u}}_N-\boldsymbol{u}\Big|_V+\Big|\tilde{\beta}_N-\beta\Big|_{L^2(\Gamma_c)} \quad \forall v\in U.\end{aligned} \tag{5.125}$$

Now, using relations (5.81), (5.85), (5.78) and (5.53) it follows that

$$\begin{aligned}&\Big|\tilde{\beta}_N(t)-\beta(t)\Big|_{L^2(\Gamma_c)}\\ &=\Big|-\int_0^t (C_\nu\tilde{\beta}(s)(\tilde{R}(\tilde{u}_\nu(s))^2-\epsilon_a)_+ds+\int_0^t (C_\nu\beta(s)(\tilde{R}(u_\nu(s))^2-\epsilon_a)_+ds\\ &\qquad -\int_t^{t_i}(C_\nu\tilde{\beta}(s)(\tilde{R}(\tilde{u}_\nu(s))^2-\epsilon_a)_+ - h(C_\nu\beta_0(\tilde{R}(u_\nu^0))^2-\epsilon_a)_+\Big|_{L^2(\Gamma_c)}\\ &\leq\int_0^t \Big|C_\nu\beta(s)(\tilde{R}(u_\nu(s))^2-C_\nu\tilde{\beta}(s)(\tilde{R}(\tilde{u}_\nu(s))^2\Big|_{L^2(\Gamma_c)}ds\\ &\qquad+\int_t^{t_i}\Big|C_\nu\tilde{\beta}(s)(\tilde{R}(\tilde{u}_\nu(s))^2\Big|_{L^2(\Gamma_c)}ds+ch\\ &\leq c|\tilde{\boldsymbol{u}}_N(s)-\boldsymbol{u}(s)|_V ds + c|\tilde{\beta}_N(s)-\beta(s)|_{L^2(\Gamma_c)}ds+ch,\end{aligned}$$

applying the Gronwall lemma, we get

$$\Big|\tilde{\beta}_N(t)-\beta(t)\Big|_{L^2(\Gamma_c)}\leq c\int_0^t |\tilde{\boldsymbol{u}}_N(s)-\boldsymbol{u}(s)|_V ds+ch.$$

Using the last inequality in (5.125) and from (5.109), (5.105) we get the convergences of the fifth and sixth equations of the previous Lemma.
Let $\varphi : L^2(0,T;V)\to\mathbb{R}$ be the functional defined by

$$\varphi(v)=J(u(t),v(t),\beta(t)).$$

By (5.23), (5.26) and (5.35), we find that the functional $\varphi(v)$ is continuous and convex. Thus, we conclude that J is weakly lower semicontinuous function, which with the convergence (5.109) one can establish the following estimate

$$\lim_{N\to+\infty} \inf \varphi(\tilde{\boldsymbol{u}}_N) \geq \varphi(\boldsymbol{u}).$$

On the other side, we have that

$$J\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t), \tilde{\beta}_N(t)\Big) = \varphi(\tilde{\boldsymbol{u}}_N) + J\Big(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t), \tilde{\beta}_N(t)\Big) - J\Big(\boldsymbol{u}(t), \tilde{\boldsymbol{u}}_N(t), \beta(t)\Big),$$

passing to the lim inf as $N \to \infty$, we get the last inequality of the previous lemma. □

Now we are ready to prove Theorem 5.1

Proof. We induce that from (5.51), (5.55), (5.78), (5.81) and (5.82) that

$$\begin{cases} \langle A\tilde{\boldsymbol{u}}_N(t), \boldsymbol{v} - \tilde{\boldsymbol{u}}_N(t)\rangle_{V'\times V} + (G_N(t), \varepsilon(\boldsymbol{v} - \tilde{\boldsymbol{u}}_N(t)))_Q \\ +\Big(\Big(\tilde{\boldsymbol{\omega}}_N(s), \boldsymbol{v}(s) - \tilde{\boldsymbol{u}}_N(s)\Big)\Big)_H + J(\tilde{\boldsymbol{u}}_N(t), \boldsymbol{v}(t), \tilde{\beta}_N(t)) \\ -J(\tilde{\boldsymbol{u}}_N(t), \tilde{\boldsymbol{u}}_N(t), \tilde{\beta}_N(t)) \geq \ \langle \tilde{\boldsymbol{f}}_N(t), \boldsymbol{v} - \tilde{\boldsymbol{u}}_N(t)\rangle_{V'\times V} \quad \forall\, v \in\ U, \end{cases}$$

in order to pass to the lim supl when N tends to infinity and applying Lemma 5.7, we not that the last one is equivalent to

$$\begin{cases} \langle A\boldsymbol{u}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle_{V'\times V} + (\mathcal{Q}(t), \varepsilon(\boldsymbol{v}(t) - \boldsymbol{u}(t)))_Q + \big((\ddot{\boldsymbol{u}}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t))\big)_H \\ +J(\boldsymbol{u}(t), \boldsymbol{v}(t), \beta(t))J(\boldsymbol{u}(t), \boldsymbol{u}(t), \beta(t)) \geq \ \langle \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle_{V'\times V}\ \forall\, \boldsymbol{v} \in\ U, \end{cases}$$

We deduce that $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ is a solution of Problem P^1V.
In the other side from (5.94) and (5.96) we have that

$$\begin{aligned} |\boldsymbol{u}(t) - \boldsymbol{u}(s)|_V &\leq |\boldsymbol{u}(t) - \boldsymbol{u}_N(t)|_V + |\boldsymbol{u}_N(t) - \boldsymbol{u}_N(s)|_V + |\boldsymbol{u}_N(s) - \boldsymbol{u}(s)|_V \\ &\leq |\boldsymbol{u}(t) - \boldsymbol{u}_N(t)|_V + c|t-s| + |\boldsymbol{u}_N(s) - \boldsymbol{u}(s)|_V \quad \forall t, s\ \in [0, T], \end{aligned}$$

when N tends to infinity, we conclude that

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)|_V \leq c|t-s| \quad \forall t, s\ \in [0, T].$$

On the other hand, we have the following inequality

$$\begin{aligned} |\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}(s)|_V \leq&\ \ \dot{\boldsymbol{u}}(t) - \boldsymbol{v}_N(t)|_V + |\boldsymbol{v}_N(t) - \boldsymbol{v}_N(s)|_V + |\boldsymbol{v}_N(s) - \dot{\boldsymbol{u}}(s)|_V \\ &\leq |\dot{\boldsymbol{u}}(t) - \boldsymbol{v}(t)|_V + c|t-s| + |\boldsymbol{v}_N(s) - \dot{\boldsymbol{u}}(s)|_V \quad \forall t, s\ \in [0, T] \end{aligned}$$

so when N tends to infinity

$$|\dot{\boldsymbol{u}}(t)-\dot{\boldsymbol{u}}(s)|_V \leq c|t-s| \quad \forall t,s \ \in [0,T],$$

which implies that

$$|\dot{\boldsymbol{u}}(t)-\dot{\boldsymbol{u}}(s)|_{V'} \leq c|t-s| \quad \forall t,s \ \in [0,T],$$

Thus, we have

$$(i)\ u \in W^{1,\infty}(0,T;V) \quad (ii)\ \ddot{\boldsymbol{u}} \in L^\infty(0,T;V). \tag{5.126}$$

And we also have that

$$\begin{aligned}|\boldsymbol{\sigma}(t)-\boldsymbol{\sigma}(s)|_Q &\leq c|\boldsymbol{u}(t)-\boldsymbol{u}(s)|_V + c\Big|\int_s^t (|\boldsymbol{u}(r)|_V + |\boldsymbol{\sigma}(r)|_Q + 1)dr\Big| \\ &\leq c|t-s| \quad \forall t,s \ \in [0,T].\end{aligned} \tag{5.127}$$

We take $\boldsymbol{v}(t)=\boldsymbol{u}(t)\pm\boldsymbol{w}(t)$ where $\boldsymbol{w}$ in $[D(\Omega)]^d$ in the inequality of problem P^2V, we conclude that

$$\mathrm{Div}\,\boldsymbol{\sigma}(t)+\boldsymbol{f}_0(t) \ = \ \varrho\ddot{\boldsymbol{u}}(t) \ \ in\ \Omega,\ \forall t \ \in \ [0,T], \tag{5.128}$$

Now, (5.127), (5.16)(i), (5.126)(ii) and (5.128) imply that

$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1). \tag{5.129}$$

□

Conclusion

This thesis proposes a contribution to the study of boundary problems in contact mechanics. Two viscoplastic and thermo-viscoelastic behavior laws were considered for this purpose. Contact problems with and without friction were studied in a dynamic process with boundary conditions, where thermal effects and adhesion are coupled. For each of these problems, we provide the variational formulation, followed by the existence and uniqueness of the weak solution.

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