Modeling and mathematical analysis of a reaction diffusion model applied in engineering

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Abstract

In this paper, we show the existence of continuous positive solutions of a class of nonlinear parabolic reaction diffusion systems with initial conditions using techniques of functional analysis and potential analysis.

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1 Introduction

The modeling and the mathematical analysis of parabolic systems, in particular, reaction diffusion systems, has been the subject of in-depth studies of several mathematicians in recent years, as they appear in the modeling of a large variety of phenomena, not only in biology and chemistry, but also in engineering, economics and ecology, such as gas dynamics, fusion processes, cellular processes, disease propagation, industrial processes , catalytic transport of contaminants in the environment, population dynamics, flame spread and others.

The subject of this paper is in this context, we will take care to study the existence of positive solutions of the following nonlinear parabolic reaction diffusion system

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \Delta u = \lambda p(x,t) f(v) \\ -\dfrac{\partial v}{\partial t} + \Delta v = \mu q(x,t) g(w) \\ -\dfrac{\partial w}{\partial t} + \Delta w = \eta r(x,t) h(z) \\ -\dfrac{\partial z}{\partial t} + \Delta z = \varrho e(x,t) k(u) \end{cases} \tag{1.1}$$

with $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and the initial conditions

$$\begin{cases} u(x,0) = \varphi(x) \quad , \quad v(x,0) = \psi(x) \\ w(x,0) = \gamma(x) \quad , \quad z(x,0) = \zeta(x) \end{cases} , \ \forall x \in \mathbb{R}^n \tag{1.2}$$

where $n \geq 3$, φ, ψ, γ and $\zeta : \mathbb{R}^n \rightarrow [0,\infty)$ are continuous, the constants λ, μ, η and ϱ are nonnegative, f, g, h and $k : (0,\infty) \rightarrow [0,\infty)$ are nondecreasing and continuous. p, q, r and $e : \mathbb{R}^n \times (0,\infty) \rightarrow [0,\infty)$ are measurable functions and satisfy an appropriate hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n)$.

To study problem $(1.1) - (1.2)$, we consider the following definition and hypotheses :

Definition 1 *We say that a nonnegative superharmonic function ω satisfies condition (H_0) if ω is locally bounded in $\mathbb{R}^n$ $(n \geq 3)$ and the map $(x,t) \mapsto P\omega(x,t)$ is continuous in $\mathbb{R}^n \times (0,\infty)$.*

We fix four nonnegative superharmonic functions ω, θ, δ and ϕ satisfying condition (H_0). Let us introduce the required hypotheses on the initial values φ, ψ, γ and ζ the nonlinear terms :

(H_1) There exist four constants $c_i > 1$, $1 \leq i \leq 4$, such that

$$\frac{1}{c_1}\omega(x) \leq \psi(x) \leq c_1\omega(x) \quad , \quad \frac{1}{c_2}\theta(x) \leq \varphi(x) \leq c_2\theta(x)$$
$$\frac{1}{c_3}\delta(x) \leq \gamma(x) \leq c_3\delta(x) \quad , \quad \frac{1}{c_4}\phi(x) \leq \zeta(x) \leq c_4\phi(x)$$

and

$$\lim_{t\rightarrow 0} P_t\psi(x) = \psi(x) \quad , \quad \lim_{t\rightarrow 0} P_t\varphi(x) = \varphi(x)$$
$$\lim_{t\rightarrow 0} P_t\gamma(x) = \gamma(x) \quad , \quad \lim_{t\rightarrow 0} P_t\zeta(x) = \zeta(x)$$

(H_2) $f,\ g,\ h,\ k : (0,\infty) \rightarrow [0,\infty)$ are nondecreasing and continuous.

(H_3) The functions p, q, r and e are measurable nonnegative and for each $c > 0$, the functions

$$\tilde{p}_c = \frac{pf(cP\omega)}{P\theta} \quad , \quad \tilde{q}_c = \frac{qg(cP\delta)}{P\omega} \quad , \quad \tilde{r}_c = \frac{rh(cP\phi)}{P\delta} \quad , \quad \tilde{e}_c = \frac{ek(cP\theta)}{P\phi}$$

belong to the parabolic Kato class $P^\infty(\mathbb{R}^n)$.

$(P_t)_{t>0}$ on $\mathbb{R}^n$ denotes the Gauss semigroup.

To study $(1.1) - (1.2)$, a basic assumptions on p, q, r and e requires to fix four superharmonic functions ω, θ, δ and ϕ on $\mathbb{R}^n$ satisfying condition (H_0).

2 The main result

Now, we can state the main result of this work :

Theorem 2 *Assume* $(H_1)-(H_3)$. *Then there exist four constants* λ_0,μ_0, η_0 *and* ϱ_0 *such that for each* $\lambda \in [0, \lambda_0)$, $\mu \in [0, \mu_0)$, $\eta \in [0, \eta_0)$ *and* $\varrho \in [0, \varrho_0)$ *the problem* $(1.1) - (1.2)$ *has a positive continuous solution* (u, v, w, z) *in* $(\mathbb{R}^n \times (0, \infty))^4$ *satisfying for each* $t > 0$ *and* $x \in \mathbb{R}^n$

$$\begin{cases} (1 - \frac{\lambda}{\lambda_0})P\varphi(x,t) \leq u(x,t) \leq P\varphi(x,t) \\ (1 - \frac{\mu}{\mu_0})P\psi(x,t) \leq v(x,t) \leq P\psi(x,t) \\ (1 - \frac{\eta}{\eta_0})P\gamma(x,t) \leq w(x,t) \leq P\gamma(x,t) \\ (1 - \frac{\varrho}{\varrho_0})P\zeta(x,t) \leq z(x,t) \leq P\zeta(x,t) \end{cases}$$

3 References

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