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APPROXIMATE SOLUTIONS OF WEAKLY NONLINEAR DIFFERENTIAL EQUATIONS

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APPROXIMATE SOLUTIONS OF WEAKLY NONLINEAR DIFFERENTIAL EQUATIONS

Abstract

In this study, we analyzed the most useful approximation methods for obtaining approximate solutions to a weak second-order nonlinear differential equation using a power series with small parameters. We established the third-order periodic approximate solution and the best third-order approximation of the weakly nonlinear differential equation.

We focused our study on the approximate solutions of the van der Pol equation in its general form. First, we demonstrated the approximate analytical solutions to this equation using different perturbation methods, including the Simple Perturbation Method (SPM), the Lindstedt-Poincaré Method (LPM), and the Averaging Method (AM). Then, we compared these approximations with each other and with the exact solution.

Résumé

Dans cette étude, nous avons analysé les méthodes d'approximation les plus utiles pour démontrer des solutions approximatives à une équation différentielle non linéaire du second ordre en utilisant une série de puissances avec de petits paramètres. Nous avons établi la solution d'approximation périodique du troisième ordre ainsi que la meilleure approximation du troisième ordre pour cette équation différentielle faiblement non linéaire.

Nous avons concentré notre étude sur les solutions approximatives de l'équation de van der Pol sous sa forme générale. Tout d'abord, nous avons démontré les solutions analytiques approximatives de cette équation en utilisant différentes méthodes de perturbation, notamment la méthode de perturbation simple (SPM), la méthode de Lindstedt-Poincaré (LPM) et la méthode de moyennage (AM). Ensuite, nous avons comparé ces approximations entre elles et avec la solution exacte.

ملخص

في هذه الدراسة، قمنا بتحليل أكثر الطرق المفيدة للتقريب من أجل إثبات الحلول التقريبية لمعادلة تفاضلية غير خطية من الدرجة الثانية بإستخدام متسلسلة القوى ذات المعلمات الصغيرة.

قمنا بإثبات الحل التقريبي الدوري من الدرجة الثالثة وأفضل تقريب من الدرجة الثالثة لهذه المعادلة غير الخطية الضعيفة.

ركزنا دراستنا على الحلول التقريبية لمعادلة فان دير بول بصيغتها العامة. وأثبتنا الحلول التحليلية التقريبية لهذه المعادلة باستخدام عدة طرق اضطرابية، بما في ذلك طريقة الاضطراب البسيطة

وطريقة المعدّل المتوسط(LPM) ، وطريقة ليندشت-بوانكاريه(SPM) ، وقمنا بمقارنة هذه التقريبات فيما بينها ومع الحل الدقيق للمسألة (AM)

Acknowledgement

I thank ≪ Allah ≫ who gave me the will, the strength and the patience for the realization of this modest memoir.

Having reached the end of this work, I would like to express my gratitude, my deepest gratitude and my unwavering attachment to my memoir supervisors, Professor ***MEFTAH Safia***

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Dedication

Thank ALLAH Almighty first and foremost for the great grace that he has bestowed upon me.
I dedicate this humble research project as a sign of respect, appreciation and thanks
To my mum ***Memouna Ghouri*** *and dad* ***Bachir*** *for their great support even when things were really tough because they kept encouraging me to work even harder.*
To my dear sisters: ***Aicha*** *and* ***Zahra****.*
To my dear brothers: ***Lakhder****,* ***Aymen*** *and* ***Nadhir****.*
To all my aunts, uncles, their children and grandchildren.
I also dedicate this dissertation to my many Your friendship makes my life a wonderful experience. I cannot list all the names here, but you are always on my mind. Thank you, Lord, for always being there for me. This memoir is only the beginning of my journey.

General notations

$\sigma_n(\delta)$ order functions.

$E(z)$ the approximation error.

$F(z, \dot{z})$ a numerical function with respect to z et $\dot{z}$.

$\dot{z} = \frac{dz}{d\theta}$ the prime derivative of z with respect to θ.

$\ddot{z} = \frac{d^2z}{d\theta^2}$ the second derivative of z with respect to θ.

$F_z = \frac{\partial F}{\partial z}$ the first derivative of F with respect to z.

$F_{zz} = \frac{\partial^2 F}{\partial z^2}$ the second derivative of F with respect to z.

$F_{\dot{z}} = \frac{\partial F}{\partial \dot{z}}$ the first derivative of F with respect to $\dot{z}$.

$\frac{\partial h}{\partial t}$ the first derivative of h with respect to t.

$\frac{\partial^2 h}{\partial t^2}$ the second derivative of h with respect to t.

$\frac{\partial^2 z}{\partial t^2}$ the second derivative of z with respect to t.

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General Introduction

Nonlinear differential equations form the foundation of mathematical modeling for both natural and engineered systems. They are crucial for capturing real-world complexity across various fields, including electrical engineering, mechanical vibrations, fluid dynamics, biology, chemical reactions, control theory, and economics. These equations extend beyond the simplicity of linear relationships, describing intricate phenomena such as oscillations, bifurcations, limit cycles, and chaos.

Unlike linear differential equations whose solutions are well understood within an established theoretical framework nonlinear equations often resist exact analytical treatment. The absence of general closed-form solutions compels researchers to rely on approximation techniques. While these methods are not exact, they effectively describe both qualitative and quantitative behaviors, especially in systems influenced by small parameters or periodic effects.

Among the many nonlinear systems that have garnered significant attention, the Van Der Pol oscillator stands out as a widely studied and influential model. Introduced by Balthasar Van Der Pol in the 1920s to describe nonlinear electrical circuits, this oscillator exhibits a broad spectrum of dynamic behaviors and serves as a benchmark for testing new analytical techniques. Its relevance extends beyond electronics to fields such as biology (e.g., modeling the heartbeat), seismology, and economic cycles.

This thesis aims to investigate approximate solution methods for nonlinear differential equations by applying and comparing three widely used perturbation techniques:

Simple Perturbation Method (SPM)

LindstedtPoincaré Method (LPM)

Averaging Method (AM)

Each of these methods is based on expanding the solution in terms of a small parameter. Their primary objective is to derive periodic solutions while eliminating secular terms undesirable terms that cause solutions to grow unbounded over time, leading to a loss of accuracy. Although each method has its advantages, their effectiveness varies depending on the equation's structure and order.

The thesis is structured into three main chapters:

Chapter 1 establishes the theoretical foundation, introducing essential concepts in differential equations and perturbation theory.

Chapter 2 focuses on deriving and analyzing the second approximate solutions for the general second-order equation and also the Van Der Pol equation.

Chapter 3 expands on the calculation of the more complex third-order approximation of the equation.

Through comparative analysis and discussion, this study evaluates the strengths and limitations of each method, highlighting the contexts in which they are most effective.

Chapter 1

Preliminaries

This chapter serves as the theoretical foundation of the memo. It begins with a general review of differential equations, emphasizing the distinction between linear and nonlinear types and the challenges posed by the latter. The chapter then introduces perturbation theory, explaining its significance and the principles behind its application to nonlinear problems. Fundamental definitions and methods are presented alongside illustrative examples, followed by a detailed discussion of specific techniques, including simple expansion, the LindstedtPoincaré method, and the averaging method. Special emphasis is placed on the issue of secular terms and techniques for their elimination. The chapter concludes with a review of the Fundamental Theorem of Existence and Uniqueness, which ensures the mathematical validity and well-posedness of the systems under consideration. Additionally, we introduce fundamental concepts related to dynamical systems, differential equations, and perturbation theory, which will be utilized in Chapters 2 and 3, see [2, 3, 8].

1.1 Differential equations

Differential equations have been used since Newton's time to understand physical, engineering, and biological sciences. In addition to their contributions to mathematical analysis, their applications have extended to economics and social sciences, gaining importance across all areas of science and its applications.

Definition 1.1.1. *[11, 4, 5] A differential equations are a relation between an independent variable h, and a dependent variable $z\left(t\right)$ and one or more differential derivatives $z, z^{'}, z^{''}, ...$.*

This means that its general formula is of the form

$$F\left(t, z, z^{'}, z^{''}, ...\right) = 0, \tag{1.1}$$

this equation is called ordinary differential equation. But if number of variables independent is greater than an independent t, z and $h(t, z)$ is a dependent variable that is partially differentiable for each of t, z, the equation that includes the independent variables, the dependent variable and its partial derivatives is called a partial differential equation, in the form of

$$G\left(t, z, h, \tfrac{\partial h}{\partial t}, \tfrac{\partial^2 h}{\partial t^2}, ...\right) = 0.$$

Example 1.1.1. *Differential equations*

$$z^{''2} + 2z^{'3} - 5z = \sin t, \tag{1.2}$$

$$z^{'} + tz = t^2, \tag{1.3}$$

$$\frac{\partial^2 h}{\partial t^2} + 2tz\frac{\partial^2 h}{\partial tz^2} + \frac{\partial h}{\partial t} = t. \tag{1.4}$$

note that the equations (1.2) and (1.3) are both ordinary differential equations, while equations (1.4) is a partial differential equations.

There are linear differential equations, which are linear equations in the dependent variable and all its derivatives.

Example 1.1.2.

$$t^2 z^{''} + tz^{'} + t^2 z = exp\left(t\right) sint. \tag{1.5}$$

It is a second order linear equations where the dependent variable z and its derivatives $z^{'}, z^{''}$ are linear. that is, each of them is raised to the power to the power of one, and there are no common products them and it does not matter whether their coefficients are constants or functions in t.

If a differential equations is not linear, then it is a nonlinear differential equations. For example

$$zz^{''} + z^{'} = t. \tag{1.6}$$

Note: Nonlinearity does not affect the order of differential equations, equation (1.6) is second-order nonlinear.

1.2 Perturbation Theory

The development of the theory concerning the influence of small perturbations on the solutions of differential equations began in the 18th century. Since Poincaré's perturbation theory emerged around 1900, it has flourished with new ideas and significant fundamental applications.

The principle of perturbation theory is to study dynamical systems that are small perturbations of "simple" systems. An important example is the practice of measurement, where quantities such as distance and volume are considered.

This theory, while historically established, is currently receiving great attention among scientists and mathematicians, particularly in fields such as mechanics.

Multiscale analysis in mathematics and physics involves techniques used to construct regular approximations for solving perturbation problems. It does so by introducing fast and slow media within a moving medium, treating them as independent, and thereby obtaining regular and periodic solutions, see [3].

Example 1.2.1. *A simple example of a perturbation arising in a natural way is the following problem. Consider a harmonic oscillation, described by the equation*

$$\ddot{z} + z = 0. \tag{1.7}$$

In deriving this equation the effect of friction has been neglected; in practice however, friction will always be present. If the oscillator is such that the friction is small, an improved model for the oscillations is given by the equation

$$\ddot{z} + \delta\dot{z} + z = 0. \tag{1.8}$$

The term $\delta\dot{z}$ is called "friction term" or "damping term" and this particular simple form of the friction term has been based on certain assumptions concerning the mechanics of friction.

The parameter δ is small

$$0 \leq \delta << 1,$$

as will always be the case, in this chapter and in subsequent chapters. If we put $\delta = 0$ in equation (1.8) one recovers the original equation(1.8), we call equation (1.8) the "unperturbed problem".

One of the interesting conclusions which we shall draw is, that in a great many problems, the introduction of small perturbations triggers off qualitatively and quantitatively behaviour of the solutions which diverges very much from the behaviour of the solutions of the unperturbed problem. We can observe this already in the simple example of the damped oscillator described by equation (1.8).

Example 1.2.2. *Consider the initial value problem*

$$\dot{z} = -z + \delta,\ z(0) = 1. \tag{1.9}$$

The solution is $z_\delta(t) = \delta + (1-\delta)e^{-t}$. *The unperturbed problem is*

$$\dot{z} = -z,\ z(0) = 1.$$

The solution is $z\,(t) = e^{-t}$. *It is clear that*

$$|z_\delta(t) - z\,(t)\,| = \delta - \delta e^{-t} \leq \delta,\ t \geq 0.$$

The error, arising in approximating $z\,(t)$ *by* $z_\delta(t)$ *is never bigger then* δ.

Example 1.2.3. *The situation is very different in the problem*

$$\dot{z} = 2z + \delta,\ z(0) = 1.$$

The solution is $z_\delta(t) = -\delta + (1+\delta)e^{2t}$. *The unperturbed problem is*

$$\dot{z} = +z,\ z(0) = 1,$$

with solution $z\,(t) = e^{2t}$. *We find*

$$|z_\delta(t) - z\,(t)\,| = \delta(1 - e^{2t}).$$

On the interval $0 \leq t \leq 1$, *the error caused by approximating* $z\,(t)$ *by* $z_\delta(t)$ *is of the order of magnitude* δ. *This is not true anymore for* $t \geq 0$, *where the difference increases without bounds.*

In the example 1.2.3 the solutions are not bounded for $t \geq 0$.
We shall show now that also in the case of bounded solutions the difference between the solutions of the perturbed and the unperturbed problem can be considerable.
Consider the initial value problem in the following example.

Example 1.2.4.

$$\ddot{z} + (1+\delta)^2 z = 0,\ z(0) = 1,\ \dot{z}(0) = 0. \tag{1.10}$$

The solution is

$$z_\delta(t) = \frac{1}{1+\delta}\cos((1+\delta)t).$$

The unperturbed problem is

$$\ddot{z} + z = 0,\ z(0) = 1,\ \dot{z}(0) = 0, \tag{1.11}$$

with the solution $z\,(t) = \cos t$*. We should analyse the difference* $z\,(t) - z_\delta(t)$*. The solutions are close for a long time, but if we wait long enough, take for example* $t = \pi/(2\delta)$*, the difference becomes considerable.*

These examples illustrate that, when constructing approximations of solutions to initial value problems, one must specify the time interval over which the approximation is sought. In many cases, we aim for this time interval to be as large as possible.

We now introduce a number of concepts that allow us to estimate vector functions in terms of a small parameter δ.

1.3 Basic material

Consider the vector function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, the functions continuous in the variables $t \in \mathbb{R}$ and $z \in D \subset \mathbb{R}^n$, δ is a small parameter.
The function $f(t,\, z,\, \delta)$ has to be expanded with respect to the small parameter δ. In the simple case that f has a Taylor expansion with respect to δ near $\delta = 0$, we have

$$f(t,\, z,\, \delta) = f(t,\, z,\, 0) + \delta f_1(t, z) + \delta^2 f_2(t,\, z) + \cdots \delta^n f_n(t,\, z) + \cdots, \tag{1.12}$$

with coefficients f_1, f_2, $\cdots$ which depend on t and z. The expressions δ, δ^2, $\cdots$, δ^n, $\cdots$ are called order functions. We shall look for an expansion in the form of

$$f(t,\, z,\, \delta) = \sum_{n=0}^{N} \sigma_n(\delta) f_n(t,\, z) + \cdots \tag{1.13}$$

in which $\sigma_n(\delta)$, $n = 0, 1, 2, \cdots$ are order functions, see [2, 6].

Definition 1.3.1. *The function* $\sigma(\delta)$ *is still positive continuous in* $[0, \delta_0]$ *and monotonically such that exists when tends to zero* $\sigma(\delta)$ *is called an order function.*

In the case that f has a Taylor expansion, the order functions which have been used are

$$\{\delta^n\}_{n=0}^{\infty}\,.$$

Other examples of order functions on $[0, 1]$

$$\delta|\ln\delta|,\ \sin\delta,\ e^{-1/e}.$$

Very often we shall compare the magnitude of order functions with a characterisation like δ^{4n} goes faster to zero that δ^{2n}. What we mean is that we compare the behaviour of these two order functions as δ tends to zero, for this comparison we shall use the Landau O-symbols.

Definition 1.3.2. *1.* $\sigma_1(\delta) = O\left(\sigma_2(\delta)\right)$ *as* $\delta \to 0$ *if there exists a constant* k *such that*

$$\sigma_1(\delta) \leq k\sigma_2(\delta),\ as\, \delta \to 0$$

2. $\sigma_1(\delta) = o\left(\sigma_2(\delta)\right)$ *as* $\delta \to 0$ *if*

$$\lim_{\delta \to 0} \frac{\sigma_1(\delta)}{\delta_2(\delta)} = 0.$$

Remark 1.3.1. *In a number of cases the following limit exists*

$$\lim_{\delta \to 0} \frac{\sigma_1(\delta)}{\sigma_2(\delta)} = k.$$

In this case we find with the definition $\sigma_1(\delta) = O\left(\sigma_2(\delta)\right)$. So if this limit exists, we have in a simple way the O-estimate for order functions.

Example 1.3.1. *In the example given above, we have*

$$\sigma_1(\delta) = \delta,\ \sigma_2(\delta) = 2\delta;$$
$$\sigma_1(\delta) = \delta^2,\ g(\delta) = \delta^{\frac{1}{2}}.$$

Example 1.3.2.

$$\delta t \sin z, \quad t \geq 0,\ z \in \mathbb{R};$$
$$\delta^2 t \sin x, \quad t \geq 0,\ z \in \mathbb{R}.$$

In perturbation theory we usually omit ^{n} *as* $\delta \to 0^n$ *as this is always the case to be considered. We are now able to compare functions of* δ*, but this does not apply to vector functions* $f\,(t,\ z,\ \delta)$*. Consider for instance the function.*

$$\delta t \sin z,\ 0 \leq t \leq 1,\ z \in \mathbb{R}.$$

Intuitively we would estimate $\delta t \sin z = O(\delta)$ *on the domain* $[0, 1] \times \mathbb{R}$*; to do this, we have to extend our definitions.*

Definition 1.3.3. *Consider the function* $f\,(t,\ z,\ \delta)\,,\ t \in I \subset \mathbb{R},\ z \in D \subset \mathbb{R}^n,\ 0 \leq \delta \leq \delta_0,$
a. $f\,(t,\ z,\ \delta)$ *is* $O(\sigma(\delta))$ *if there exists a constant* k *such that* $\|f\| \leq k\sigma(\delta)$ *as* $\delta \to 0$ *with* $\sigma(\delta)$ *an order function* ($\|.\|$ *is the sup norm on* $I \times D$*).*
b. $f(t,\ z,\ \delta)$ *is* $o(\sigma(\delta))$ *as* $\delta \to 0$ *if* $\lim\limits_{\delta \to 0} \dfrac{\|f\|}{\sigma(\delta)} = 0.$

In the example 1.3.2 given above we have

$$\delta t \sin z = O(\delta),\ 0 \leq t \leq 1,\ z \in \mathbb{R},$$

Note however that the $O(\delta)$ estimate does not hold for the functions:

$$\delta t \sin z, \quad t \geq 0,\ z \in \mathbb{R};$$
$$\delta^2 t \sin z, \quad t \geq 0,\ z \in \mathbb{R}.$$

As we are especially interested in initial value problems for differential equations, the variable t plays a special part. Often we shall study a solution or its approximation on an interval which we would like to take as large as possible. So in these cases we shall not fix the interval of time apriori. It turns out to be useful to characterise the size of the interval of time in terms of the small parameter δ.

1.4 Simple expansion

Consider the initial value problem

$$\dot{z} = f\left(t,\, z,\, \delta\right),\, z(0) = z_0, \tag{1.14}$$

with $t \geq 0$, $z \in D \subset \mathbb{R}^n$.

If we can expand $f\left(t,\, z,\, \delta\right)$ in a Taylor series with respect to δ,

$$f\left(t,\, z,\, \delta\right) = f_0(t,z) + \delta f_1(t,z) + \delta^2 f_2(t,z)\cdots, \tag{1.15}$$

then we could assume that there is a similar expansion for the solution

$$z\left(t,\, \delta\right) = z_0(t) + \delta z_1(t) + \delta^2 z_2(t)\cdots. \tag{1.16}$$

It seems natural to expect that the formal expansion represents an asymptotic approximation to the solution. We will show that this is correct, but only in a very restricted sense. It seems natural to expect that the formal expansion represents an asymptotic approximation to the solution. We will show that this is correct, but only in a very restricted sense. The general procedure for simple approximation is to substitute into the equation, expand in powers of δ, and set all coefficients of powers of δ equal to zero. This gives a system of nor homogeneous linear differential equations that we can solve recursively, see[2].

Definition 1.4.1. 1. *The solution*

$$z\left(t,\, \delta\right) = z_0(t) + \delta z_1(t) + \delta^2 z_2(t). \tag{1.17}$$

is called the second order approximate solution of the equation (1.14)*. Or, the approximate solution to order* δ^2*.*

2. *The solution*

$$z\left(t,\, \delta\right) = z_0(t) + \delta z_1(t) + \delta^2 z_2(t) + \delta^3 z_3(t). \tag{1.18}$$

is called the third order approximate solution of the equation (1.14)*. Or, the approximate solution to order* δ^3*.*

We will give a simple example to illustrate the technique.

Example 1.4.1. *[9] Let the following nonlinear differential equation be*

$$\frac{d^2z}{dt^2} + z + 2\delta z^3 = 0. \tag{1.19}$$

Let us assume that the solution of this equation is of the following form

$$z(t) = z_0(t) + \delta z_1(t) + \dots. \tag{1.20}$$

We can show that all solutions of equation (2.3) are periodic. Substituting (1.20) into equation (2.3) gives

$$\left(\frac{d^2z_0}{dt^2} + z_0\right) + \delta\left(\frac{d^2z_1}{dt^2} + z_1 + 2{z_0}^3 + 6z_0\right) + \delta^2\left(\frac{d^2z_2}{dt^2} + z_2 + 6{z_0}^2 z_1 + 6yz_1\right) + ... = 0. \tag{1.21}$$

$$\frac{d^2z_0}{dt^2} + z_0 = 0. \tag{1.22}$$

$$\frac{d^2z_2}{dt^2} + z_2 = -6z_1\left(z_0^2 - 1\right). \tag{1.23}$$

Where F_n is a cubic polynomial function in its arguments. It should be noted that this system of equations can be solved recursively, that is, determining z_k only requires knowledge of the functions z_m for $0 \leq m \leq k-1$.

These equations can be solved using the initial conditions.

$$z\,(0) = A, \qquad \frac{dz\,(0)}{dt} = 0, \tag{1.24}$$

Thus,

$$\begin{cases} z_0\,(0) = A, \qquad z_i\,(0) = 0 \qquad pour \qquad i \geq 1, \\ \dfrac{dz_k\,(0)}{dt} = 0, \qquad\qquad pour \qquad k \geq 0. \end{cases} \tag{1.25}$$

The solution of (2.5) with the conditions (1.25) is

$$z_0\,(t) = Acos\,t. \tag{1.26}$$

By substituting this result into the right-hand side of equation (2.11), we obtain

$$\frac{d^2z_1}{dt^2} + z_1 = -2{z_0}^3 - 6z_0 = -2A^3\cos^3 t - 6A\cos t = -\frac{A^3}{2}\left(3\cos t + \cos 3t\right) - 6A\cos t. \tag{1.27}$$

The particular solution of this equation is

$$z_1p\,(t) = \left(\frac{A^3}{16}\right) cos\,3t - \left(\frac{3A^3}{4}\right) sin\,t - 3At\sin t. \tag{1.28}$$

Therefore, the general solution is

$$z_1\left(t\right) = C_1 cos\, t + C_2 sin\, t + \left(\frac{A^3}{16}\right) cos\, 3t - \left(\frac{3A^3}{4}\right) sin\, t - 3At \sin t. \tag{1.29}$$

The initial conditions, specified by equations (1.25), allow us to determine the arbitrary constants C_1 and C_2, such that $C_1 = \frac{-A^3}{32}$ and $C_2 = 0$, thus $z_1(t)$ becomes

$$z_1\left(t\right) = \left(\frac{A^3}{16}\right)\left(cos\, 3t - cos\, t\right) - 3At \sin t. \tag{1.30}$$

$$z\left(t,\delta\right) = A cos\, t + \delta\left[\left(\frac{A^3}{16}\right)\left(cos\, 3t - cos\, t\right) - 3At\, sin\, t\right]. \tag{1.31}$$

However, the solution (2.8) shows that $z_1(t)$, which is the correction term to the periodic function $z_0(t)$ (assumed to be small), is not only non-periodic but also unbounded as $t \to \infty$. Therefore, directly applying (2.8) leads to difficulties when trying to compute analytical and periodic approximations of nonlinear differential equations of the form given by (2.3).

Example 1.4.2. *[1]*

We consider the nonlinear differential equation

$$\frac{d^2z}{dt^2} + \delta \sin(z) = 0, \tag{1.32}$$

where $\delta \ll 1$ is a small parameter. Using the Taylor series

$$\sin(z) = z - \frac{z^3}{6} + \cdots,$$

we approximate

$$\sin(z) \approx z - \frac{z^3}{6}.$$

Substituting this into the equation gives

$$\frac{d^2z}{dt^2} + \delta\left(z - \frac{z^3}{6}\right) = 0. \tag{1.33}$$

We have

$$z(t) = z_0(t) + \delta z_1(t) + \delta^2 z_2(t) + \cdots.$$

$$\frac{d^2z}{dt^2} = z_0'' + \delta z_1'' + \delta^2 z_2'' + \cdots,$$

and expand

$$z^3 = (z_0 + \delta z_1)^3 = z_0^3 + 3\delta z_0^2 z_1 + \cdots.$$

$$z_0'' + 0 = 0 \quad \Rightarrow \quad z_0(t) = A\cos(t).$$

$$z_1'' + z_1 = \frac{1}{6} z_0^3.$$

We compute

$$z_0^3 = A^3\cos^3(t) = A^3\left(\frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t)\right),$$

so

$$z_1'' + z_1 = \frac{A^3}{8}\cos(t) + \frac{A^3}{24}\cos(3t).$$

The solution is

$$z_1(t) = -\frac{A^3}{16} t\sin(t) + \frac{A^3}{192}\cos(3t).$$

$$z(t) \approx A\cos(t) + \delta\left(-\frac{A^3}{16} t\sin(t) + \frac{A^3}{192}\cos(3t)\right).$$

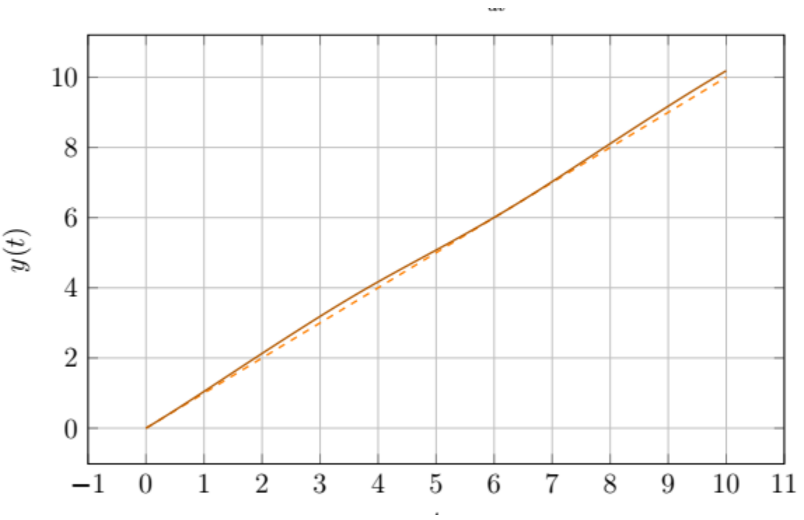

Figure 1.1: Approximate Solution of $\frac{d^2z}{dt^2} + \delta\sin(z) = 0$, with Simple Method.

1.5 Lindstedt-Poincaré Method (LPM)

We present an approximation method, based on the expansion of a solution of a differential equation in a series in a small parameter. It is used to construct uniformly valid periodic solutions to second-order nonlinear differential equations in the form, see [6, 7].

$$\frac{d^2z}{dt^2}(t,\delta) + z(t,\delta) = \delta F\left(z(t,\delta), \frac{dz}{dt}(t,\delta)\right), \tag{1.34}$$
$$z(0,\delta) = A, \qquad (dz/dt)(0,\delta)(0) = 0, \qquad 0 < \delta << 1.$$

If $\delta = 0$ we obtain the following unperturbed problem

$$\frac{d^2z}{dt^2}(t,0) + z(t,0) = 0. \tag{1.35}$$

The starting point of the perturbation method is the assumption that a periodic solution of equation (1.34) can be written in the form

$$z\left(t,\delta\right) = \sum_{m=0}^{n} \delta^m z_m\left(t\right) + O\left(\delta^{n+1}\right). \tag{1.36}$$

1.5.1 Secular Terms

Te conservation of a finite numbers of terms on the right-side of expansion (1.36) determines a function that is not only nonperiodic, but also unbounded as $t \to +\infty$.

Definition 1.5.1. *Terms such as* $t^m \cos(pt)$ *or* $t^m \sin(nt)$ *where* $m, n \in \mathbb{N}^*, p \in \mathbb{N}$ *are called secular terms.*

These expressions appear because expansion (1.36) is not uniformly valid. The existence of such expressions destroys the periodicity of expansion (1.36) when only a finite number of terms is conserved.

Therefore, to obtain a uniformly valid solution, we must look for an approximation that eliminates secular terms.

A technique to avoid the presence of secular terms and allows for an approximation that is valid for all time has been developed by Lindstedt-Poincare.

The essence of the method is to introduce a transformation of the independent variable. This transformation will allow us to avoid the occurrence of secular terms in the solutions of the perturbation series equations, see [8].

The fundamental idea originates from the astronomer Lindstedt-Poincaré and is based on the observation that one of the effects of the nonlinear term in equation (2.12) is to change the system's frequency from the linear value $\nu_0 = 1$. To $\nu(\delta)$ account for this change in frequency, a new variable $\theta = \nu t$ is introduced. When δ the resulting periodic solution has unit frequency. If δ and is small, then the frequency depends on δ and remains close to unity. If $\nu(t)$ has a period of $2\pi/\nu(\delta)$, then by setting $\theta = \nu t$, we obtain a new function $\nu(\theta)$ with period 2π both ν and are then expanded in powers of δ as follows

$$z(\theta, \delta) = z_0(\theta) + \delta z_1(\theta) + \cdots + \delta^n z_n(\theta) + \ldots, \tag{1.37}$$

$$\theta = \nu.t = 1 + \delta\nu_1 + \cdots + \delta^n \nu_n + \ldots. \tag{1.38}$$

At this point, the ν_j are unknown constants. Now, we introduce the following notations

$$\dot{z} \equiv \frac{dz}{d\theta}, \qquad \ddot{z} \equiv \frac{d^2 z}{d\theta^2}, \tag{2.3a}$$

$$F_z(z, \dot{z}) \equiv \frac{\partial F(z, \dot{z})}{\partial z}, \qquad F_{\dot{z}}(z, \dot{z}) \equiv \frac{\partial F(z, \dot{z})}{\partial \dot{z}}. \tag{2.3b}$$

we have $\theta = \nu t$

$$\frac{dz}{d\theta} = \frac{dz}{d(\nu t)} = \frac{1}{\nu}\frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \nu \frac{dz}{d\theta} \tag{1.39}$$

$$\Rightarrow \frac{d^2 z}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dz}{d\theta}\right) = \frac{1}{\nu^2}\frac{d^2 z}{dt^2}, \tag{1.40}$$

then the equation (2.12) becomes

$$\nu^2 \frac{d^2 z}{d\theta^2}(\theta, \delta) + z(\theta, \delta) = \delta F\left(z, \nu\frac{dz}{d\theta}\right). \tag{1.41}$$

If the equations (1.39) and (1.40) are substituted in the equation (1.41) and the coefficients of the different powers of δ are equal to zero, we get

$$\ddot{z}_0 + z_0 = 0, \tag{1.42}$$

$$\ddot{z}_1 + z_1 = -2\nu_1 \ddot{z}_0 + F(z_0, \dot{z}_0), \tag{1.43}$$

$$\ddot{z}_2 + z_2 = -2\nu_1 \ddot{z}_1 - \left(\nu_1^2 + 2\nu_2\right)\ddot{z}_0 + F_z(z_0, \ddot{z}_0)\, z_1 + F_{\dot{z}}(z_0, \dot{z}_0)\left(\nu_1 \dot{z}_0 + \dot{z}_1\right), \tag{1.44}$$

$$\ddot{z}_3 + z_3 = G_3(\dot{z}_0, \dot{z}_1, \ldots, \dot{z}_2; \dot{\nu}_0, \dot{z}_1, \ldots, \dot{z}_2). \tag{1.45}$$

$$\ddot{z}_n + z_n = G_n(\dot{z}_0, \dot{z}_1, \ldots, \dot{z}_{n-1}; \dot{\nu}_0, \dot{z}_1, \ldots, \dot{z}_{n-1}). \tag{1.46}$$

If F is a polynomial function in z and dz/dt, then G is also a polynomial function with respect to its arguments.

To calculate an approximate periodic solutions of (1.44), we must solve (1.42), (1.43), (1.44) and (1.46).

Example 1.5.1. *[1] Our aim is to solve the nonlinear differential equation*

$$\frac{d^2z}{dt^2} + \delta \sin(z) = 0,$$

using the ***Lindstedt-Poincaré method****, which eliminates secular terms that appear in regular perturbation expansions, into steps*

Introduce a rescaled time variable

$$\theta = \nu t, \quad \nu = 1 + \delta\nu_1 + \delta^2\nu_2 + \cdots$$

Then, derivatives transform as $\frac{d}{dt} = \nu\frac{d}{d\theta}, \quad \frac{d^2}{dt^2} = \nu^2\frac{d^2}{d\theta^2}$*. Assume the solution has a regular expansion*

$$z\left(t\right) = z_0(\theta) + \delta z_1(\theta) + \delta^2 z_2(\theta) + \cdots .$$

$$\sin(z) = \sin(z_0) + \delta z_1 \cos(z_0) + \cdots .$$

Collecting Orders of δ*;*

Order δ^0

$\ddot{z}_0 + z_0 = 0 \quad \Rightarrow \quad z_0(\theta) = A\cos(\theta).$

Order δ^1

$\ddot{z}_1 + \ddot{z}_0 \cdot 2\nu_1 + \sin(z_0) = 0.$

Substitute: $z_0 = A\cos(\theta), \quad \ddot{z}_0 = -A\cos(\theta), \quad \sin(z_0) \approx A\cos(\theta) - \frac{A^3}{6}\cos^3(\theta).$

Then $\ddot{z}_1 - 2\nu_1 A\cos(\theta) + A\cos(\theta) = 0.$ *To eliminate the secular term* $\cos(\theta)$*, we choose* $\nu_1 = \frac{1}{2}$*,*

so $\nu = 1 + \frac{\delta}{2}$*.*

The approximate solution is: $z\left(t\right) \approx A\cos\left(\left(1 + \frac{\delta}{2}\right)t\right).$

Proposition 1.5.1. *Let the equation*

$$\ddot{z} + z = G\left(\theta\right), \; z\left(0\right) = 0, \; \dot{z}\left(0\right) = 0. \tag{1.47}$$

Where $G\left(\theta\right) = -2\nu_1\ddot{z}_0 + F(z_0, \dot{z}_0)$*. The solution to this problem is*

$$z\left(\theta\right) = \int_0^\theta \sin\left(\theta - \tau\right) G\left(\tau\right) d\tau. \tag{1.48}$$

Moreover, the equation (1.47) has a periodic solution $z_1\left(\theta\right)$ *if and only if*

$$\begin{cases} \int_0^{2\pi} F\left(A\,\cos\theta, -A\,\sin\theta\right)\sin\theta\, d\theta = 0, \\ 2\pi\nu_1 A + \int_0^{2\pi} F\left(A\,\cos\theta, -A\,\sin\theta\right)\cos\theta\, d\theta = 0. \end{cases}$$

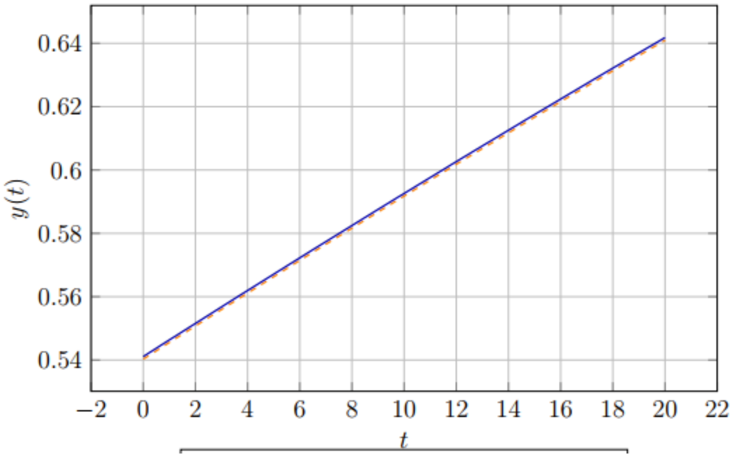

Figure 1.2: Approximate solution of $\ddot{z} + \delta \sin(z) = 0$, with Lindstedt-Poincaré Method.

proof See [10]

We know that the solution of (1.47) is $z(\theta) = C_1 \cos(\theta) + C_2 \sin(\theta) + z_p(\theta)$, such that $z_p(\theta) = C_1(\theta)\cos\theta + C_2(\theta)\sin\theta$. By variation of constants we find

$$\begin{cases} C_1'(\theta)\cos\theta + C_2'(\theta)\sin\theta = 0, \\ -C_1'(\theta)\sin\theta + C_2'(\theta)\cos\theta = G(\theta). \end{cases}$$

$$\Rightarrow \begin{cases} -C_1'(\theta) = -\sin\theta\, G(\theta) \Rightarrow C_1(\theta) = -\int_0^\theta \sin\tau\, G(\tau)\, d\tau, \quad C_1(0) = 0, \\ -C_2'(\theta) = \cos\theta\, G(\theta) \Rightarrow C_1(\theta) = \int_0^\theta \cos\tau\, G(\tau)\, d\tau, \quad C_2(0) = 0, \end{cases}$$

$$z_p(\theta) = (-\int_0^\theta \sin\tau\, G(\tau)\, d\tau)\cos\theta + (\int_0^\theta \cos\tau\, G(\tau)\, d\tau)\sin\theta = \int_0^\theta (-\sin\tau\cos\theta + \cos\tau\sin\theta) G(\tau)\, d\tau,$$

$\Rightarrow z_p(\theta) = \int_0^\theta \sin(\theta - \tau)\, G(\tau)\, d\tau \Rightarrow z(\theta) = C_1\cos\theta + C_2\sin\theta + \int_0^\theta \sin(\theta-\tau)\, G(\tau)\, d\tau,$ with the initial values $z_0(0) = \dot{z}(0) = 0$ owe have $C_1 = C_2 = 0$. so we deduce that problem (1.47) admits (1.48)as a solution.

Moreover, (1.47) gives

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = -z_1 + G(\tau). \end{cases}$$

On the other hand, the condition of periodicity for the new variable θ can be expressed as

$$z(\theta) = z(\theta + 2\pi). \tag{1.49}$$

so the corresponding conditions for $z_n(\theta)$ are

$$z_n(\theta) = z_n(\theta + 2\pi), \quad n = 1, 2, ... \tag{1.50}$$

$$\Rightarrow \begin{cases} z_1(2\pi) = z_1(0) = 0, \\ z_2(2\pi) = z_2(0) = 0, \end{cases}$$

which yields to the periodicity condition

$$\int_{\theta}^{\theta+2\pi} \sin(\theta - \tau) G(\tau) d\tau = 0, \tag{1.51}$$

$$\Rightarrow \begin{cases} \int_0^{2\pi} \cos\theta G(\theta) d\theta = 0, \\ \int_0^{2\pi} \sin\theta\, G(\theta)\ d\theta = 0. \end{cases} \tag{1.52}$$

According to (1.43) we have $G_1(\theta) = -2\nu_1 \ddot{z}_0 + F(z_0, \dot{z}_0)$, $z_0 = A\ \cos\theta$.
we rewrite (1.52) as

$$\Rightarrow \begin{cases} \int_0^{2\pi} \cos\theta \left[2\nu_1 A\ \cos\theta + F(A\ \cos\theta, -A\ \sin\theta)\right] d\theta = 0, \\ \int_0^{2\pi} \sin\theta \left[2\nu_1 A\ \cos\theta + F(A\ \cos\theta, -A\ \sin\theta)\right] d\theta = 0, \end{cases} \tag{1.53}$$

$$\Rightarrow \begin{cases} 2\nu_1 A \int_0^{2\pi} \cos^2\theta d\theta + \int_0^{2\pi} \cos\theta F(A\ \cos\theta, -A\ \sin\theta) d\theta = 0, \\ 2\nu_1 A\ \int_0^{2\pi} \sin\theta \cos\theta d\theta + \int_0^{2\pi} \sin\theta F(A\ \cos\theta, -A\ \sin\theta) d\theta = 0, \end{cases}$$

which is required.

1.6 Averaging method (AM)

Here we present a third method for determining the perturbation solutions of the differential equation for a nonlinear oscillator.
The main advantage of the method is that it not only allows to determine the periodic motions at steady state, but also allows to determine the transient behavior of the motion at a periodic solution, see[2].
This method applies to equations of the form

$$\ddot{z} + \nu^2 z + \delta F(z, \dot{z}) = 0. \tag{1.54}$$

For $\delta = 0$, the general solution is

$$z\left(t\right) = A\,\sin\left(\nu t + \Phi\right),\ \text{ ou } A \text{ and } \Phi \text{ are any constants.} \tag{1.55}$$

For $\delta \neq 0$ small, Krylov and Boyolinbov posed the solution

$$z\left(t\right) = A\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right), \tag{1.56}$$

$$\dot{z}\left(t\right) = A\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right). \tag{1.57}$$

Let $z = \dot{z}$ in (1.54), we find

$$\begin{cases} \dot{z} = k, \\ \dot{k} = -\nu^2 z - \delta F\left(z, w\right). \end{cases} \tag{1.58}$$

$$\Rightarrow \dot{z} = z\dot{z} = A\left(t\right)\nu\,cos\left(\nu t + \Phi\left(t\right)\right) \tag{1.59}$$

$$\Rightarrow A\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right) = \dot{A}\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right) + A\left(t\right)\left(\nu + \dot{\Phi}\left(t\right)\right)\,\cos\left(\nu t + \Phi\left(t\right)\right)$$

$$\Rightarrow \dot{A}\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right) + A\left(t\right)\dot{\Phi}\left(t\right)\,\cos\left(\nu t + \Phi\left(t\right)\right) = 0. \tag{1.60}$$

Similarly we have

$$\dot{A}\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right) - A\left(t\right)\nu\left(\nu + \dot{\Phi}\left(t\right)\right)\,\sin\left(\nu t + \Phi\left(t\right)\right) =$$

$$-A\left(t\right)\nu^2\,sin\left(\nu t + \Phi\left(t\right)\right) - \delta f\left(A\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right), A\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right)\right),$$

$$\Rightarrow \dot{A}\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right) - A\left(t\right)\nu\dot{\Phi}\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right) =$$

$$\delta f\left(A\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right), A\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right)\right). \tag{1.61}$$

By solving (2.13) and (1.56) with respect to $\dot{A}$ and $\dot{\Phi}$, we obtain (by the method of cramer).

$$\begin{cases} \dot{A}\left(t\right) = -\dfrac{\delta}{\nu}\,\cos\left(\nu t + \Phi\left(t\right)\right) f\left(A\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right), A\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right)\right), \\ \dot{\Phi}\left(t\right) = -\dfrac{\delta}{A\left(t\right)\nu}\,\sin\left(\nu t + \Phi\left(t\right)\right) f\left(A\left(t\right)\,\sin\left(\nu t + \Phi\left(t\right)\right), A\left(t\right)\nu\,\cos\left(\nu t + \Phi\left(t\right)\right)\right). \end{cases} \tag{1.62}$$

The Krylov and Boyolinbov approximation is to replace $A\left(t\right)$ and $\Phi\left(t\right)$ in (1.62) by their average values over a period $T = \frac{2\pi}{\nu}$, (i.e $\frac{1}{T}\int_0^T f\left(t\right)dt$).

$A(t)$ and $\Phi(t)$ are considered constants by taking the average. This process is known as the averaging method.

$$\begin{cases} \dot{A} = -\dfrac{\delta}{2\pi} \int_0^{\frac{2\pi}{\nu}} \cos(\nu t + \Phi(t)) f(A(t) \sin(\nu t + \Phi(t)), A(t)\nu \cos(\nu t + \Phi(t))) dt, \\ \dot{\Phi} = \dfrac{\delta}{2\pi A} \int_0^{\frac{2\pi}{\nu}} sin(\nu t + \Phi(t)) f(A(t) \sin(\nu t + \Phi(t)), A(t)\nu \cos(\nu t + \Phi(t))) dt. \end{cases}$$

Let $\theta = \nu t + \Phi$, we find

$$\begin{cases} \dot{A} = -\dfrac{1}{2\pi} \int_0^{2\pi} \cos(\theta) f(A \sin\theta, A\nu \cos\theta) d\theta, \\ \dot{\Phi} = \dfrac{1}{2\pi A\nu} \int_0^{2\pi} \sin(\theta) f(A\, sin\theta, A\nu \cos\theta) d\theta. \end{cases} \tag{1.63}$$

Once these integrals have been found, we will have to solve differential equations for $A(t)$ and $\Phi(t)$.

Remark 1.6.1. *We recall that*

$$I_{m,n} = \int_0^{2\pi} \sin^m \cos^n x dx = 0, \text{ if} \quad m, n \text{ sont impaires},$$

and further

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}, \quad I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}.$$

We arrive at $I_{0,0} = 2\pi$.

Example 1.6.1. *see[1] We solving the following equation by the averaging method.*

$$\ddot{z} + \delta \sin(z) = 0, \quad \text{with } 0 < \delta \ll 1.$$

With Taylor Expansion, by assuming small z, *we approximate* $\sin(z) \approx z - \frac{z^3}{6}$. *Thus, the equation becomes*
$\ddot{z} + \delta z - \frac{\delta}{6} z^3 = 0.$
Rewriting $\ddot{z} + z = -\delta\left(\frac{1}{6}z^3 - z\right).$
Assume the form of the solution. We assume the solution has the form

$$z(t) = A(t) \sin(t + \Phi(t)).$$

Therefore
$\frac{1}{6}z^3 = \frac{A^3}{8} \sin(t + \Phi) - \frac{A^3}{24} \sin(3(t + \Phi)).$

By averaging and neglecting the higher harmonic $\sin(3(t+\Phi))$*, we obtain the averaged equations* $A'(t)=0, \qquad \Phi'(t)=\frac{3\delta}{8}A^2$.
The amplitude is constant $A(t)=A_0$*, and the phase evolves is* $\Phi(t)=\frac{3\delta A_0^2}{8}t$.
Thus, the approximate averaging solution is

$$z\left(t\right)=A_0\sin\left(t+\frac{3\delta A_0^2}{8}t\right)=A_0\sin\left[\left(1+\frac{3\delta A_0^2}{8}\right)t\right].$$

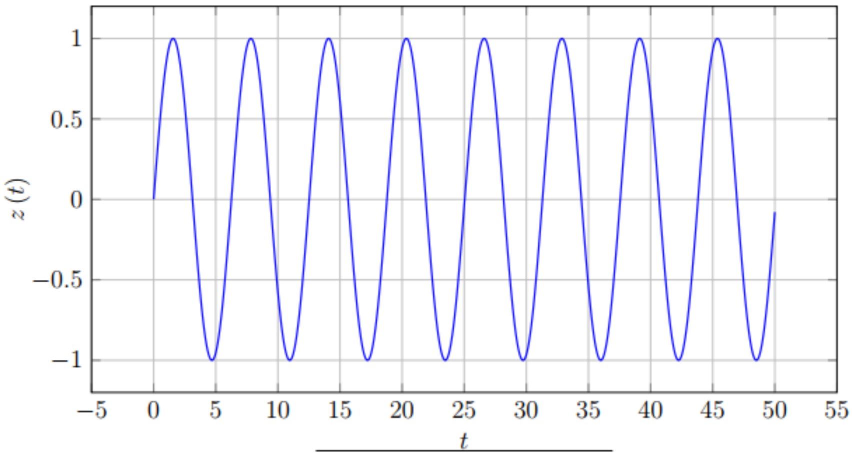

Figure 1.3: Approximate solution of $\ddot{z}+\delta\sin(z)=0$, with Averaging Method.

1.7 Fundamental Theorem of Existence and Uniqueness

Definition 1.7.1. *see[**?**]*
Suppose that $f\in C(E)$*, where* E *is an open subset of* $\mathbb{R}^n$*.*
Then, a function $z\left(t\right)$ *is a solution of the differential equation* $\dot{z}=f(z)$ *on an interval* I *if*

- $z\left(t\right)$ *is differentiable on* I*,*
- *For every* $t\in I$*,* $z\left(t\right)\in E$*,*
- *And* $z^{'}\left(t\right)=f(z\left(t\right))$*.*

Given $z_0 \in E$, $z(t)$ *is a solution to the initial value problem* $\dot{z} = f(z), \quad z(t_0) = z_0$. *on an interval* I *if* $t_0 \in I$, $z(t_0) = z_0$, *and* $z(t)$ *solves the differential equation on* I.

To apply the method of successive approximations to establish existence of a solution, we define the **Lipschitz condition** and show that C^1 functions are locally Lipschitz.

Let $E \subset \mathbb{R}^n$ be open. A function $f : E \to \mathbb{R}^n$ satisfies a Lipschitz condition on E if there exists a constant $K > 0$ such that for all $z, y \in E$, $|f(z) - f(y)| \leq K|z - y|$.

Definition 1.7.2. f *is said to be* ***locally Lipschitz*** *on* E *if for every* $z_0 \in E$, *there exists* $\delta > 0$ *and a constant* $K_0 > 0$ *such that for all* $z, y \in N_\delta(z_0)$, $|f(z) - f(y)| \leq K_0|z - y|$, *where the* δ*-neighborhood of* z_0 *is defined as* $N_\delta(z_0) = \{z \in \mathbb{R}^n \mid |z - z_0| < \delta\}$.

Lemma 1.7.1. *Let* $E \subset \mathbb{R}^n$ *be open and let* $f : E \to \mathbb{R}^n$. *If* $f \in C^1(E)$, *then* f *is locally Lipschitz on* E.

proof : Since E is open, for any $z_0 \in E$, there exists $\delta > 0$ such that $N_\delta(z_0) \subset E$. We define

$$K = \max_{|z - z_0| \leq \frac{\delta}{2}} \|Df(z)\|.$$

This maximum exists because the domain is compact and Df is continuous. Let $N_0 = N_{\delta/2}(z_0)$. For z, $y \in N_0$, set $u = y - z$. Then $z + su \in N_0$ for $s \in [0, 1]$.
Let $F(s) = f(z + su)$., by the chain rule,we have

$$F'(s) = Df(z + su) \cdot u,$$

thus $f(y) - f(z) = F(1) - F(0) = \int_0^1 Df(z + su) \cdot u \, ds.$
Then

$$|f(y) - f(z)| \leq \int_0^1 \|Df(z + su)\| \cdot |u| \, ds \leq K|y - z|.$$

This proves the lemma.

The **Picard successive approximation method** is based on the fact that $z(t)$ is a solution to the initial value problem

$$\dot{z} = f(z), \quad z(0) = z_0,$$

if and only if $z\left(t\right)$ is a continuous function satisfying the integral equation

$$z\left(t\right) = z_0 + \int_0^t f(z(s))\, ds.$$

The successive approximations are defined as $u_0(t) = z_0, \quad u_{k+1}(t) = z_0 + \int_0^t f(u_k(s))\, ds.$

Example 1.7.1. *Let the initial value problem: $\dot{z} = az$, $z(0) = z_0$ using successive approximations with*

$u_0(t) = z_0$, $u_1(t) = z_0 + \int_0^t az_0\, ds = z_0(1 + at)$,

$u_2(t) = z_0 + \int_0^t az_0(1 + as)\, ds = z_0\left(1 + at + \frac{a^2t^2}{2}\right)$,

$u_3(t) = z_0 + \int_0^t az_0\left(1 + as + \frac{a^2s^2}{2}\right) ds = z_0\left(1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!}\right)$.

By induction

$u_k(t) = z_0\left(1 + at + \cdots + \frac{(at)^k}{k!}\right)$, *and* $\lim\limits_{k\to\infty} u_k(t) = z_0e^{at}$.

Thus, the successive approximations converge to the exact solution $z\left(t\right) = z_0e^{at}$.

Chapter 2

Second order approximate solutions of weakly nonlinear differential equations

This chapter focuses on the practical application of the methods introduced in Chapter 1. We study a second order weakly nonlinear differential equation in its general form and select the second-order Van der Pol equation as a case study due to its historical and theoretical significance in the study of nonlinear oscillations, see [9, 13].

2.1 Second order approximate solution by the Simple-Pertubation (SPM) method of the general weakly nonlinear differential equation

In this section, we present the second order approximate solution by the simple-pertubation (SPM) method of the general weakly nonlinear differential equation, for more explanation, see [9]. We consider the following nonlinear differential equation

$$\frac{d^2z}{dt^2} + z = \delta F\left(z, \frac{dz}{dt}\right). \tag{2.1}$$

We assume the solution can be expanded as a power series in δ

$$z(t) = z_0(t) + \delta z_1(t) + \delta^2 z_2(t) + \mathcal{O}(\delta^3). \tag{2.2}$$

To calculate the second order approximate solution, we applied the equation (2.11). In the following example, we calculate $z_2(t)$.

Example 2.1.1. *Let the nonlinear differential equation be*

$$\frac{d^2z}{dt^2} + z + 2\delta z^3 = 0. \tag{2.3}$$

$$\begin{aligned}\left(\frac{d^2z_0}{dt^2} + z_0\right) + \delta\left(\frac{d^2z_1}{dt^2} + z_1 + 2{z_0}^3 + 6z_0\right) + \delta^2\left(\frac{d^2z_2}{dt^2} + z_2 + 6{z_0}^2 z_1 + 6yz_1\right)\\ + \delta^3\left(\frac{d^2z_3}{dt^2} + z_3 - z_0^3 - 3z_0^2 z_1 - 3z_0^2 z_2 - 3z_0 z_1^2\right) = 0.\end{aligned}$$

$$\frac{d^2z_0}{dt^2} + z_0 = 0. \tag{2.4}$$

$$\frac{d^2z_1}{dt^2} + z_1 = -2{y_0}^3 - 6z_0. \tag{2.5}$$

$$\frac{d^2z_2}{dt^2} + z_2 = -6z_1\left(z_0^2 - 1\right). \tag{2.6}$$

$$\frac{d^2z_3}{dt^2} + z_3 = z_0^3 + 3z_0^2 z_1 + 3z_0^2 z_2 + 3z_0 z_1^2, \tag{2.7}$$

We have

$$z\left(t,\delta\right) = Acos\,t + \delta\left[\left(\frac{A^3}{16}\right)\left(cos\,3t - cos\,t\right) - 3At\,sin\,t\right]. \tag{2.8}$$

So

$$\begin{aligned}\frac{d^2z_2}{dt^2} + z_2 = -6\Bigg[&\frac{A^2}{2}\left(4\cos^3(t)\cos(2t)\right) + \left(1 - \frac{A^2}{2}\right)\cos(t)\\ &- \frac{A^2}{2}\cos(t)\cos(2t) + \left(-\frac{3A^3}{2} + 3A\right)t\sin(t) - \frac{3A^3 t\sin(t)\cos(2t)}{2}\Bigg].\end{aligned} \tag{2.9}$$

Then

$$\begin{aligned}z_2(t) = C_1\cos(t) + C_2\sin(t) - 6\Bigg[&\left(1 + \frac{A^2}{4}\right)\cos(t) + \frac{A^2}{2}\cos(3t) + \frac{A^2}{4}\cos(5t)\\ &+ \left(-\frac{9A^3}{4} + 3A\right)t\sin(t) - \frac{3A^3}{4}t\sin(3t)\Bigg].\end{aligned}$$

$$z_2(0) = C_1 + (1 + \frac{3A^2}{4}) = 0 \Longrightarrow C_1 = (-1 - \frac{3A^2}{4}). \tag{2.10}$$

$$\dot{z}_2(0) = 0 + C_2 + 0 + 0 \Longrightarrow C_2 = 0. \tag{2.11}$$

So

$$\begin{aligned} z_2(t) = (-1 - \frac{3A^2}{4})\cos(t) - 6\Bigg[&\left(1 + \frac{A^2}{4}\right)\cos(t) + \frac{A^2}{2}\cos(3t) + \frac{A^2}{4}\cos(5t) \\ &+ \left(-\frac{9A^3}{4} + 3A\right) t\sin(t) - \frac{3A^3}{4} t\sin(3t)\Bigg], \end{aligned}$$

2.2 Second order approximate solution by the Lindstedt-Poincaré (LPM) method of the general weakly nonlinear differential equation

In this section, we present the second order approximate solution by the Lindstedt-Poincaré (LPM) method of the general weakly nonlinear differential equation, for more explanation, see [9]. We consider the following nonlinear differential equation

$$\frac{d^2z}{dt^2} + z = \delta F(z, \frac{dz}{dt}). \tag{2.12}$$

$$z(0,\delta) = A, \qquad (dz/dt)(0,\delta)(0) = 0, \qquad 0 \leq \delta << 1$$

If $\delta = 0$ we obtain the following unperturbed problem

$$\frac{d^2z}{dt^2}(t,0) + z(t,0) = 0. \tag{2.13}$$

Proposition 2.2.1. *[9] Let $z_2(\theta)$ be a solution of the equation (1.44). Then $z_2(\theta)$ is periodic if and only if*

$$\begin{cases} \nu_2 = -\dfrac{\nu_1^2}{2} + \dfrac{1}{2A\pi}\int_0^{2\pi} [2z_1\cos\theta\,\ddot{z}_1 - \cos\theta\,F_z(A\cos\theta, -A\sin\theta)\,z_1 \\ \quad -(\nu_1 A\sin\theta\cos\theta + \dot{z}_1\cos\theta)\,F_{\dot{z}}(A\cos\theta, -A\sin\theta)]\,d\theta, \\ \\ \int_0^{2\pi}\sin\theta\,[F_z(A\cos\theta, -A\sin\theta)\,z_1 + F_{\dot{z}}(A\cos\theta, -A\sin\theta)\,\dot{z}_1]\,d\theta \\ -\nu_2\int_0^{2\pi}\left[2\sin\theta\,\ddot{z}_1 + AF_{\dot{z}}(A\cos\theta, -A\sin\theta)\sin^2\theta\right]d\theta = 0. \end{cases} \tag{2.14}$$

proof : The periodic condition requires that

$$\int_{\theta}^{\theta+2\pi} \sin(\theta-\tau)\,G(\tau)\,d\tau = 0 \quad \Longrightarrow \quad \begin{cases} \int_0^{2\pi} \cos\theta\,G(\theta)\,d\theta = 0, \\ \int_0^{2\pi} \sin\theta\,G(\theta)\,d\theta = 0. \end{cases} \tag{2.15}$$

From the first chapter, we find that its approximate solutions, according to equation (1.44) and in [9], we have

$$G_2(\theta) = -\,2\nu_1\ddot{z}_1 - \left(\nu_1^2 + 2\nu_2\right)\ddot{z}_0 + F_z\left(z_0,\ddot{z}_0\right)z_1 + F_{\dot{z}}\left(z_0,\dot{z}_0\right)\left(\nu_1\dot{z}_0 + \dot{z}_1\right),$$

with $z_0(\theta) = A\cos\theta, \quad z_1(\theta) = \int_0^{\theta} \sin(\theta-\tau)\left(2\nu_1 A\cos\tau + F(A\cos\tau, -A\sin\tau)\right)d\tau.$
We rewrite (2.15) as

$$\begin{cases} \int_0^{2\pi} \cos(\theta)\left[-2\nu_1\ddot{z}_1 - (\nu_1^2+2\nu_2)(-A\cos(\theta)) + F_z(A\cos(\theta), -A\cos(\theta))z_1\right]d\theta \\ + \int_0^{2\pi} \cos(\theta)F_{\dot{z}}(A\cos(\theta), -A\sin(\theta))(-\nu_1 A\sin(\theta) + \dot{z}_1(\theta))d\theta = 0, \\ \\ \int_0^{2\pi} \sin(\theta)\left[-2\nu_1\ddot{z}_1 - (\nu_1^2+2\nu_2)(-A\cos(\theta)) + F_z(A\cos(\theta), -A\cos(\theta))z_1\right]d\theta \\ - \int_0^{2\pi} \cos(\theta)F_{\dot{z}}(A\cos(\theta), -A\sin(\theta))(-\nu_1 A\sin(\theta) + \dot{z}_1(\theta))d\theta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \nu_2 = -\dfrac{\nu_1^2}{2} + \dfrac{1}{2A\pi}\int_0^{2\pi}\left[2\nu_1\cos\theta\,\ddot{z}_1 - \cos\theta\,F_z(A\cos\theta, -A\cos\theta)z_1\right. \\ \quad \left.+\nu_1 A F_{\dot{z}}(A\cos\theta, -A\sin\theta)\sin\theta\cos\theta - F_{\dot{z}}(A\cos\theta, -A\sin\theta)\dot{z}_1\cos\theta\right]d\theta, \\ \quad -2\nu_1\int_0^{2\pi}\sin\theta\,\ddot{z}_1\,d\theta + \int_0^{2\pi}\sin\theta\,F_z(A\cos\theta, -A\cos\theta)z_1\,d\theta \\ +\nu_1 A\int_0^{2\pi} F_{\dot{z}}(A\cos\theta, -A\sin\theta)\sin^2\theta\,d\theta + \int_0^{2\pi} F_{\dot{z}}(A\cos\theta, -A\sin\theta)\dot{z}_1\sin\theta\,d\theta = 0. \end{cases}$$

so

$$\begin{cases} \nu_2 = -\dfrac{\nu_1^2}{2} + \dfrac{1}{2A\pi}\int_0^{2\pi}\left[2\nu_1\cos\theta\,\ddot{z}_1 - \cos\theta\,F_z(A\cos\theta, -A\cos\theta)z_1\right. \\ \left. - \left(\nu_1 A\sin\theta\cos\theta + \dot{z}_1\cos\theta\right)F_{\dot{z}}(A\cos\theta, -A\sin\theta)\right]d\theta, \\ \\ + \int_0^{2\pi}\sin\theta\left[F_z(A\cos\theta, -A\cos\theta)z_1 + F_{\dot{z}}(A\cos\theta, -A\sin\theta)\dot{z}_1\right]d\theta \\ \quad -\nu_2\int_0^{2\pi}\left[2\sin\theta\,\ddot{z}_1 + AF_{\dot{z}}(A\cos\theta, -A\sin\theta)\sin^2\theta\right]d\theta = 0. \end{cases}$$

Example 2.2.1. *[9] Let the differential equation*

$$\frac{d^2z}{dt^2}+z+\delta\left(2z^3\right)=0. \tag{2.16}$$

With the initial conditions $z(0)=A$ *and* $(dz/dt)(0)=0$, *(in order to obtain periodic solutions, the amplitude of the motion must remain sufficiently small).*
Suppose that the second order approximate solution is:

$$z(\theta,\delta)=z_0(\theta)+\delta z_1(\theta)+\delta^2 z_2(\theta). \tag{2.17}$$

We have $\theta=\nu t$ *such that*

$$\nu=1+\delta\nu_1+\delta^2\nu_2,\ \nu(\delta)\neq 1.$$

So, by using (1.39) *and* (1.40) *the equation* (2.16) *becomes*

$$\nu^2\ddot{z}+z=-2\nu z^3. \tag{2.18}$$

With compensation, we find

$$(1+\delta\nu_1+\delta^2\nu_2)^2)(\ddot{z}_0+\delta\ddot{z}_1+\delta^2\ddot{z}_2)+(z_0+\delta z_1+\delta^2 z_2)=-2\delta(z_0+\delta z_1+\delta^2 z_2)^3.$$

$$\ddot{z}_0+z_0=0,\ z_0(0)=A,\dot{z}_0=0. \tag{2.19}$$

$$\ddot{z}_1+z_1=-2\nu_1\ddot{z}_0-2z_0^3,\ z_1(0)=\dot{z}_1(0)=0. \tag{2.20}$$

$$\ddot{z}_2+z_2=-2\nu_1\ddot{z}_1-(\nu_1^2+2\nu_1)\ddot{z}_0-6z_0^2z_1,\ z_2(0)=\dot{z}_2(0)=0. \tag{2.21}$$

The following equations (2.19)(2.20)(2.21) were solved by[9] and their solution is given by

$$z(\theta,\delta)=Acos\theta+\delta\left(\frac{A^3}{16}\right)[cos3\theta-cos\theta]+\delta^2\left(\frac{A^5}{512}\right)[cos\theta-cos5\theta]+o(\delta^2),$$

with $\theta=\nu t$, *such that*

$$\nu=1+\delta\left(3\frac{A^2}{4}\right)+\delta^2\left(15\frac{A^4}{128}\right)+...$$

2.3 Second order Approximate solutions for the "Van Der Pol" equation

2.3.1 Simple perturbation method (SPM)

In this section, we consider solving the second-order "Van Der Pol" equation of different perturbation methods: the simple perturbation method (SPM), the Lindstedt-Poincaré method (LPM) and the simple method (AM). Then we prove the approximate analytical solutions of the generalized Van Der Pol oscillator by these methods, see [13].
Consider the initial value problem

$$\ddot{z}+\delta\left(az^2+b\dot{z}^2-1\right)\dot{z}+z=0,\quad z(0)=A \text{ and } \dot{z}(0)=0.\, a,b\,\text{and}A\in\mathbb{R}. \tag{2.22}$$

and

$$z\left(t\right)=z_0(t)+\delta z_1(t)+\delta^2 z_2(t)+o(\delta^2).$$

The solutions of the equation 2.22 in the second order is

$$\begin{aligned}
z_P(t,\delta) &= A\cos(t)+\delta\left[(\frac{A^3}{32}[7a+9b]-\frac{A}{2})\sin(t)+(\frac{A}{2}-\frac{A^3(a+3b)}{8})t\cos(t)-\frac{A^3(a-b)}{32}\sin(3t)\right]\\
&+\delta^2((-\frac{A^5}{3072}[9(a+3b)(a-b)+9(a-b)(a-3b)+24a(a+3b)+18a(a-b)-(5a-9b)(a-b)]\\
&+\frac{2A^3}{256}[7a-3b])\cos(t)+(\frac{A^5}{256}[(a-9b)(a-b)-9(a-b)^2+2(a-9b)(a+3b)+6(a-b)]\\
&+\frac{3A^3}{64}[7b+a]-\frac{A}{8})t\sin(t)+(\frac{3A^5}{128}(a+3b)^2+\frac{A}{8}-\frac{A^3}{8}(a+3b))t^2\cos(t)\\
&+(\frac{A^5}{1024}[3(a+3b)(a-b)+3(a-b)(a-3b)+8a(a+3b)+6a(a-b)]\\
&-\frac{2A^3}{256}[7a-3b])\cos(3t)-\frac{A^5}{3072}(5a-9b)(a-b)\cos(5t)\\
&+(\frac{3A^5}{256}(a-b)(a+3b)-\frac{3A^3}{64}(a-b))t\sin(3t))+o(\delta^2).
\end{aligned} \tag{2.23}$$

2.3.2 Lindstedt-Poincaré Method (LPM)

Let be the following general formula of Van Der Pol differential equation

$$\ddot{z}+\delta\left(az^2+b\dot{z}^2-1\right)\dot{z}+z=0.\quad z(0)=A \text{ and } \dot{z}(0)=0. \tag{2.24}$$

In order to apply the Lindstedt method, we put $\theta = \nu t$, $z\,(t) = z(\theta)$, $\ddot{z} = \nu^2 \ddot{z}$ and $\dot{y} = z\dot{z}$, so (2.22) become

$$\nu^2 \ddot{z} - \delta(1 - az^2 - b\dot{z}^2)\nu \dot{z} + z = 0 \quad z(0) = 1 \text{ and } \dot{z}(0) = 0. \tag{2.25}$$

Proposition 2.3.1. *[13] The equation* $\ddot{z}_1 + z_1 = 2A\nu_1 \cos\theta - A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin\theta$, *has a periodic solution* $z_1\,(\theta)$, *if and only if*

$$\begin{cases} \int_0^{2\pi} -A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin^2\theta\, d\theta = 0, \\ 2\pi\nu_1 A + \int_0^{2\pi} -A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin\theta cos\theta\, d\theta = 0. \end{cases} \tag{2.26}$$

$z_2\,(\theta)$ *has a periodic solution if and only if*

$$\int_\theta^{\theta+2\pi} \sin(\theta - \tau)\, G_2(\tau)\, d\tau = 0, \quad \Longrightarrow \quad \begin{cases} \int_0^{2\pi} \cos\theta\, G_2(\theta)\, d\theta = 0, \\ \int_0^{2\pi} \sin\theta\, G_2(\theta)\, d\theta = 0. \end{cases} \tag{2.27}$$

proof:

$$(2.26) \Rightarrow \begin{cases} A = \frac{2}{\sqrt{a+3b}}, \\ \nu_1 = 0. \end{cases}$$

$$(2.27) \Longrightarrow \nu_2 = -[\frac{a(a-b)}{8(a+3b)^2} - \frac{3(a-b)}{16(a+3b)} + \frac{9b(a-b)}{8(a+3b)^2}].$$

The solution of (2.25) in [13] is

$$\begin{aligned} z(\theta, \delta) &= \frac{2}{\sqrt{a+3b}}\cos(\theta) + \delta(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(\theta) - \sin(3\theta))) \\ &+ \delta^2((\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3a}} + \frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(\theta) \\ &+ (\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)2\sqrt{a+3b}})\cos(3\theta) \\ &+ (-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(5\theta)) + o(\delta^2), \end{aligned}$$

$$\nu = 1 - [\frac{a(a-b)}{8(a+3b)^2} - \frac{3(a-b)}{16(a+3b)} + \frac{9b(a-b)}{8(a+3b)^2}]\delta^2 + o(\delta^2),$$

with $\theta = \nu t$.

2.3.3 Averaging method (AM)

Example 2.3.1. *We solving the equation* (2.24) *by averaging method.*
We have $\nu = 1$, $F(z, \dot{z}) = \left(az^2 + b\dot{z}^2 - 1\right)\dot{z}$.
The system (1.63) *is written*

$$\begin{cases} \dot{A} = \dfrac{\delta}{2\pi} \int_0^{2\pi} \cos\theta[A\cos\theta\left(1 - aA^2\sin^2\theta - bA^2\cos^2(\theta)\right)]d\theta, \\ \dot{\Phi} = -\dfrac{\delta}{2\pi A} \int_0^{2\pi} \sin\theta[A\cos\theta\left(1 - aA^2\sin^2\theta - bA^2\cos^2(\theta)\right)]d\theta. \end{cases} \tag{2.28}$$

$$\begin{aligned} \dot{A} &= \frac{dA}{dt} \\ &= \frac{\delta A}{2\pi}\left(A^n I_{0,n+1} - aA^n + 2I_{2,n+1} - bA^n + 2I_{0,n+1}\right) \\ &= \frac{\delta A}{2\pi}\left(\frac{1}{2}2\pi - aA^2\frac{1}{4}\frac{1}{2}2\pi - 3bA^2\frac{1}{4}\frac{1}{2}2\pi\right) \\ &= \frac{\delta A}{8}\left(4 - A^2(a+3b)\right). \end{aligned}$$

Then

$$\begin{aligned} &\int \frac{dA}{A\left(4 - A^2(a+3b)\right)} = \frac{\delta}{8}\int dt \\ &\Rightarrow 8\int\left(\frac{1}{4A} + \frac{1}{8\left(2 - A\sqrt{(a+3b)}\right)} - \frac{1}{8\left(2 + A\sqrt{(a+3b)}\right)}\right)dA = \delta t + \ln(c) \\ &\Rightarrow 2\ln A - \ln\left(2 - A\sqrt{(a+3b)}\right) - \ln\left(2 + A\sqrt{(a+3b)}\right) = \delta t + \ln(c) \\ &\Rightarrow \ln\left(\frac{A^2}{c\left(4 - A^2(a+3b)\right)}\right) = \delta t. \textit{ Let's pose } A(0) = A_0 \\ &\Rightarrow c = \frac{A_0^2}{4 - A_0^2(a+3b)}, \textit{ On the other hand } \ln\left(\frac{A^2}{c\left(4 - A^2(a+3b)\right)}\right) = \delta t \\ &\Rightarrow \frac{A^2}{c\left(4 - A^2(a+3b)\right)} = e^{\delta t} \\ &\Rightarrow A^2 = c\left(4 - A^2(a+3b)\right)e^{\delta t} \\ &\Rightarrow A^2(t) = \frac{\frac{4A_0^2}{4 - A_0^2(a+3b)}e^{\delta t}}{1 + \frac{A_0^2}{4 - A_0^2(a+3b)}e^{\delta t}} \\ &\Rightarrow A(t) = \frac{2}{\left[\left(\frac{4}{A_0^2} - (a+3b)\right)e^{-\delta t} + 1\right]^{\frac{1}{2}}}. \end{aligned}$$

On the other hand

$$\begin{aligned}
\dot{\Phi}\left(t\right) &= -\frac{\delta}{2\pi A}\int_0^{2\pi}\sin\theta[A\cos\theta\left(1-A^2\sin^2\theta\right)]d\theta\\
&= -\frac{\delta}{2\pi}\int_0^{2\pi}[\sin\theta\cos\theta - A^2\cos\theta\sin^3\theta]d\theta\\
&= -\frac{\delta}{2\pi}I_{1,1}+\frac{A^2\delta}{2\pi}I_{3,1}\\
&= 0+0\\
&= 0,
\end{aligned}$$

$$\Rightarrow\ \dot{\Phi}=\frac{d\Phi}{dt}=0,$$

$$\Phi=\Phi_0.$$

The averaging approximate solution is

$$z\left(t,\delta\right)=\frac{2}{\left[\left(\frac{4}{A_0^2}-(a+3b)\right)e^{-\delta t}+1\right]^{\frac{1}{2}}}\sin\left(t+\Phi_0\right),$$

with $z_L(t,\delta)=z\left(t,\delta\right)$.

2.4 Comparison of second approximate solutions

In this section we analyze the approximate solutions of (2.22) are obtained by the three numerical methods (SPM, LPM, AM), [12].
We compare $z_E(t,0)$ the exact solution, $z_P(t,\delta)$ the approximate solution with simple perturbation method, $z_L(\theta,\delta)$ the approximate solution with Lindestedt method and $z_A(t,\delta)$ the approximate solution with Averaging method of Ven Der Pol equation.
The figures from 2.1 to 2.4 give the comparison of the approximate solutions in order δ, obtained by the three methods (SPM, LM and AM). In order δ^2, we have the figures from 2.5 to 2.8 obtained by the three methods (SPM, LM, AM).

2.4.1 Comparison of approximate solutions to order δ^2

In this part, we compare approximate solutions to order δ^2. where $g(t)$ is the Taylor series expansion of $z_L((1+[\frac{a(a-b)}{8(a+3b)^2}-\frac{3(a-b)}{16(a+3b)}+\frac{9b(a-b)}{8(a+3b)^2}]\delta^2)t,\delta)$ to order δ^2, in the neighbourhood of $\delta = 0$. We have

$$\begin{aligned}
g(t) = {} & \frac{2}{\sqrt{a+3b}}\cos(t) + \delta\Big(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(t)-\sin(3t))\Big) \\
& + \delta^2\Big(\Big(\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} \\
& + \frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}}\Big)\cos(t) + \Big(\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} \\
& + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)^2\sqrt{a+3b}}\Big)\cos(3t) \\
& + \Big(-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}}\Big)\cos(5t).
\end{aligned}$$

And $h(t)$ is the Taylor series expansion of $z_A(t,\delta)$, to order δ^2, in the neighbourhood of $\delta = 0$.

$$h(t) = (A + (\frac{A}{2} - \frac{A^3(a+3b)}{8})\delta t + (\frac{3A^5}{128}(a+3b)^2 + \frac{A}{8} - \frac{A^3}{8}(a+3b)\,(\delta t)^2)\cos(t).$$

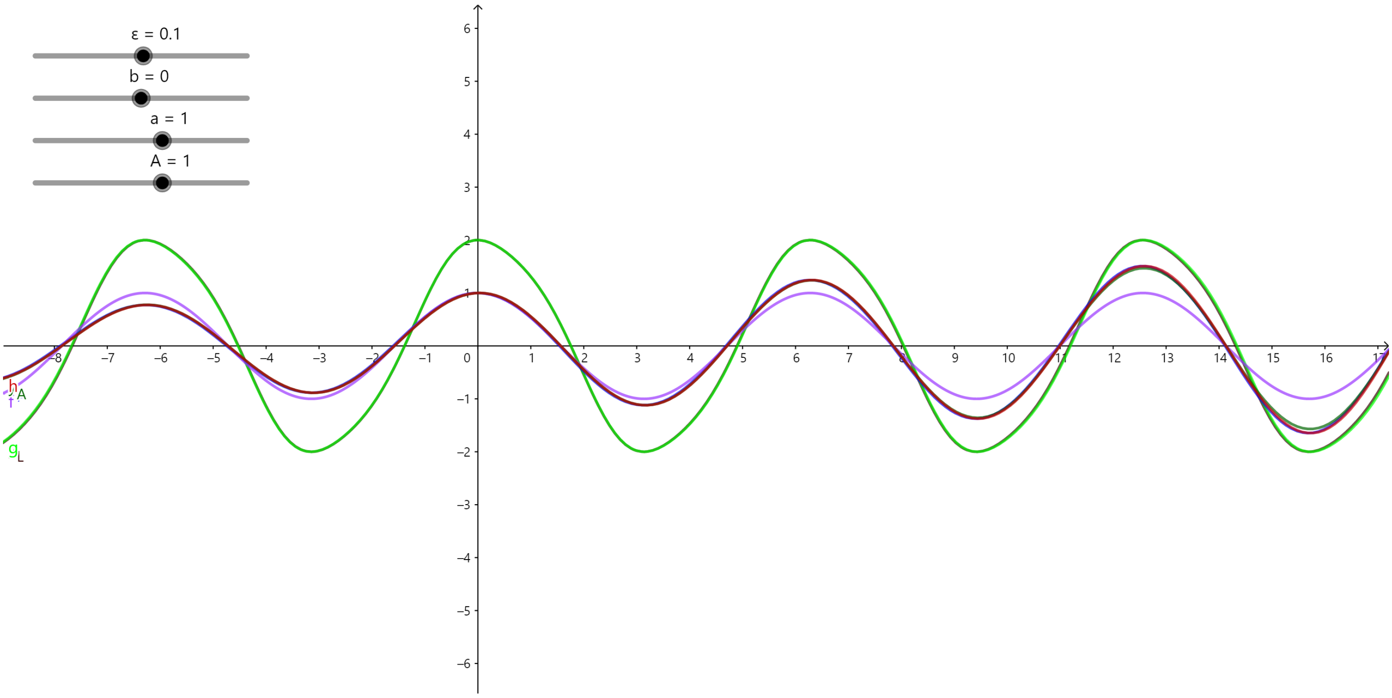

Figure 2.1: Comparison of the SPM solution to order δ^2, LPM solution and AM solution for $\delta = 0.1$ and $A = 1$, to order δ^2.

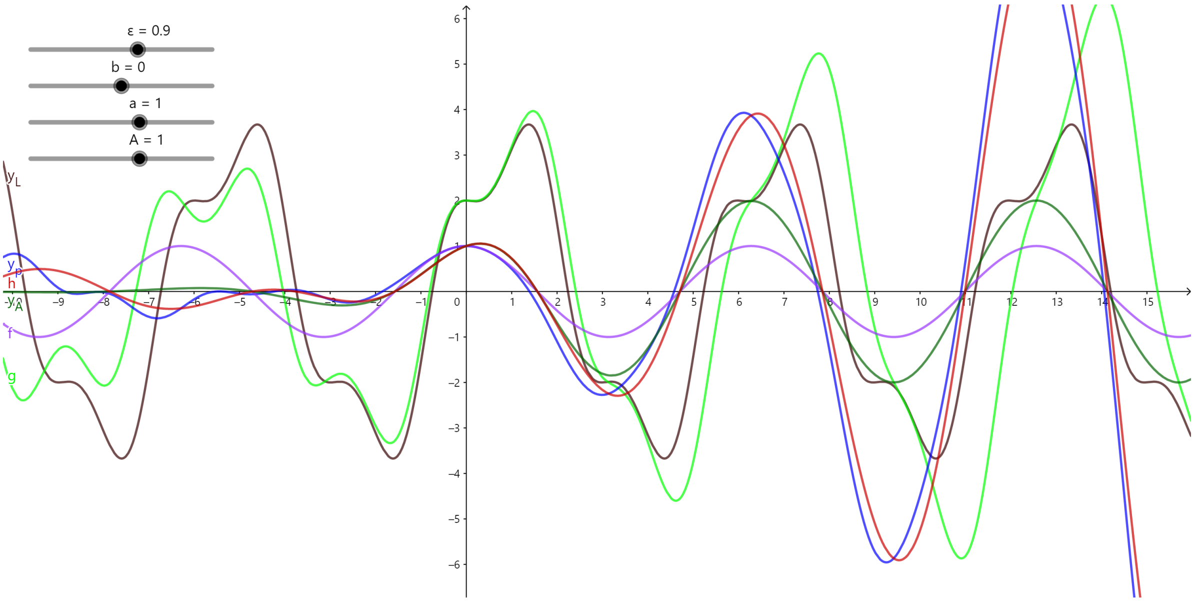

Figure 2.2: Comparison of the SPM solution, LPM solution and AM solution for $\delta = 0.9$ and $A = 1$, to order δ^2.

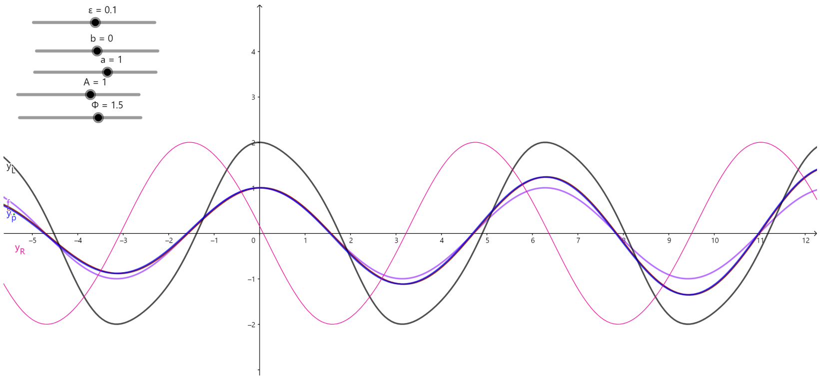

Figure 2.3: Comparison of the SPM solution in order δ^2, LPM solution and AM solution for $\epsilon = 0.9$ and $A = 2$, to order δ^2.

2.4.2 Discussion of results

When $A = 1$ and $\delta = 0.1$, in Fig.2.1, shows analytic approximate solutions. Based on the obtained results, we found that the solutions z_S, z_A and $h(t)$ are indetical to the exact solution z_E, but the solutions z_L, are convergent to the exact solution z_E.
When $A = 1$ and $\delta = 0.9$, in Fig.2.2, shows analytic approximate solutions. Based on the obtained results, we found that z_A closest to the exact solution z_E , but z_L and z_S spaced on the exact solution z_E. Also, we note that the solutions $h(t)$ are indetical to the exact solution z_E.
When $A = 2$ and $\delta = 0.9$, in Fig.2.3, shows analytic approximate solutions. Based on the obtained results, we found that in the domain $t \in [-5, 5]$, the solutions z_S are congruent to the exact solution z_E, but when $t \longrightarrow \infty$ the solutions z_S, spaced on the exact solution z_E, we note that the solutions z_L, are indetical to the exact solution z_E.

2.5 Conclusion

In Chapter 2, we presented an approximate study of a second-order nonlinear differential equation in its general form, and we also devoted a case study to the van der Poel differential equation, a classical second-order nonlinear model representing oscillatory systems. We applied three analytical approximation techniques: the simple perturbation method (SPM), the Lindstedt-Poincaré method (LPM), and the average method (AM) to compute the second-order approximation. These methods aim to construct periodic solutions under weak nonlinearity assumptions. The results reveal that while the SPM is suitable for non-very small nonlinearities, the LPM effectively removes secular terms, and the AM provides a time-stable approximation. These approximation methods are useful for very small oscillatory variables and also for small domains .

Chapter 3

Third order approximate solutions of weakly nonlinear differential equations

Second approximate Solutions of Nonlinear Differential Equation in Their General Form n the final chapter, the methodology is extended to the third-order Van Der Pol equation, which introduces additional mathematical complexity and analytical challenges. This higher-order system allows for a more rigorous test of the robustness and scalability of the perturbation methods.

Here, the SPM and LPM are re-applied and adapted to handle the increased complexity. The performance of each method is evaluated based on solution behavior, convergence, and practicality. The findings from this chapter are then 0compared with those of Chapter 2, offering a broader perspective on the application of perturbation techniques across varying system complexities.

We dedicate our study to the approximate solutions of Van Der Pol equation in their general form.

First, we prove the approximate analytic solutions to this equation by different perturbation methods. Then we compare these approximations with each other and with the exact solution.

3.1 Third-order approximate solution by the Simple-Pertubation (SPM) method of the general weakly nonlinear differential equation

In the following example, we prove the third-order approximate $z_3(t)$. We mention here that $z_1(t)$ is calculated in example 1.4.1, and we proved $z_2(t)$ in example 2.1.1.

Example 3.1.1. *We will poved the third-order approximate $z_3(t)$ of the equation (2.3), by solving the equation (2.11), we have*

$$\begin{aligned}
\ddot{z}_3 + z_3 =& A^3\cos^3(t) + 3A^2\delta\left(\frac{A^3}{16}\right)\cos^2(t)\left(\cos(3t)-\cos(t)\right) - 3A^2\delta\left(\frac{A^3}{16}\right)\cos^2(t)\cos(t)\\
&- 9A^3t\cos^2(t)\sin(t) + 3A^2\left(-1-\frac{3A^2}{4}\right)\cos(t) - 18A^2\left(1+\frac{A^2}{4}\right)\cos^2(t)\cos(t)\\
&- 9A^2\cos^2(t)\cos(3t)\\
&- 18A^2\cos^2(t)\cos(5t) + 6A^2\left(-\frac{9A^3}{4}+3A\right)t\sin(t) + 6A^2\frac{3A^3}{4}t\sin(3t)\\
&+ 3A\cos(t)\left(\delta\left(\frac{A^3}{16}\right)\left(\cos(3t)-\cos(t)\right) - 3At\sin(t)\right)^2.
\end{aligned}$$

$$\begin{aligned}
\ddot{z}_3 + z_3 =& \left(\frac{3A^3\cos(t)}{4} + \frac{A^3\cos(3t)}{4}\right) + 3A^2\delta\left(\frac{A^3}{16}\right)\cdot\frac{\cos(3t)-\cos(t)}{2} - 3A^2\delta\left(\frac{A^3}{16}\right)\cdot\frac{\cos(3t)+\cos(t)}{2}\\
&- 9A^3t\cos^2(t)\sin(t) + 3A^2\left(-1-\frac{3A^2}{4}\right)\cos(t) - 18A^2\left(1+\frac{A^2}{4}\right)\cdot\frac{\cos(3t)+\cos(5t)}{2}\\
&- 9A^2\cdot\frac{\cos(3t)+\cos(5t)}{2} - 18A^2\cdot\frac{\cos(5t)+\cos(7t)}{2} + 6A^2\left(-\frac{9A^3}{4}+3A\right)t\sin(t)\\
&+ 6A^2\cdot\frac{3A^3}{4}t\sin(3t) + 3A\cos(t)\left(\delta\left(\frac{A^3}{16}\right)\left(\cos(3t)-\cos(t)\right) - 3At\sin(t)\right)^2.
\end{aligned}$$

So

$$\begin{aligned}
z_3(t) =& C_1\cos(t) + C_2\sin(t) + t\left(A_1\cos(t) + B_1\sin(t)\right) + A_3\cos(3t) + B_3\sin(3t) + A_5\cos(5t)\\
&+ B_5\sin(5t) + A_7\cos(7t) + B_7\sin(7t) + t\left(C_3\sin(3t) + D_3\cos(3t)\right),
\end{aligned}$$

and we have

$$z_3(0) = C_1 + tA_1 + A_3 + A_5 + A_7 + D_3 \rightarrow C_1 = \left(-tA_1 - A_3 - A_5 - A_7 - D_3\right),$$

and

$$\dot{z}_3(0) = 0 + C_2 + 0 + 0 \Longrightarrow C_2 = 0. \tag{3.1}$$

The general solution in order δ^3*is*

$$\begin{aligned}
z(t,\delta) = {} & A\cos t \\
& + \delta\left[\left(\frac{A^3}{16}\right)(\cos 3t - \cos t) - 3At\sin t\right] \\
& + \delta^2\left[\left(-1 - \frac{3A^2}{4}\right)\cos t - 6\left(1 + \frac{A^2}{4}\right)\cos t + \frac{A^2}{2}\cos(3t) + \frac{A^2}{4}\cos(5t)\right] \\
& + \delta^2\left[\left(-\frac{9A^3}{4} + 3A\right)t\sin t - \frac{3A^3}{4}t\sin(3t)\right] \\
& + \delta^3\Big[(-tA_1 - A_3 - A_5 - A_7 - D_3)\cos t + t\,(A_1\cos t + B_1\sin t) \\
& \qquad + A_3\cos(3t) + B_3\sin(3t) \\
& + A_5\cos(5t) + B_5\sin(5t) + A_7\cos(7t) + B_7\sin(7t) \\
& + t\,(C_3\sin(3t) + D_3\cos(3t))\Big].
\end{aligned}$$

3.2 Third-order approximate solution by the Lindstedt-Poincaré (LPM) method of the general weakly nonlinear differential equation

First, we give a periodic approximate solution for the nonlinear differential equation In the following proposition, we prove the periodic conditions of tsolution $z_3(\theta)$. We mention here that the proof of the periodicity of the solution $z_1(\theta)$, $z_2(\theta)$ in [9, 6].

Proposition 3.2.1. *The general formula of the equation* (1.45) *is*

$$\ddot{z}_3 + z_3 = -2\nu_1\ddot{z}_2 - \left(\nu_1^2 + 2\nu_2\right)\ddot{z}_1 - (2\nu_3 + 2\nu_1\nu_2)\ddot{z}_0 + z_2F_z(z_0,\dot{z}_0) + \frac{{z_1}^2}{2}F_{zz}(z_0,\dot{z}_0) +$$

$$+z_1(\nu_1\dot{z}_0 + \dot{z}_1)F_{z\dot{z}}(z_0,\dot{z}_0) + (\nu_2\dot{z}_0 + \nu_1\dot{z}_1 + \dot{z}_2)F_{\dot{z}}(\nu_0,\dot{z}_0) + \frac{1}{2}(\nu_1\dot{z}_0 + \dot{z}_1)^2F_{\dot{z}\dot{z}}(z_0,\dot{z}_0). \tag{3.2}$$

proof: First, by substituting z_0, z_1 and z_2 in (2.12), we obtain in the third order

$$\delta^3\left(\ddot{z}_3 + z_3 + 2\nu_1\ddot{z}_2 + \left({\nu_1}^2 + 2\nu_2\right)\ddot{z}_1 + (2\nu_3 + 2\nu_1\nu_2)\ddot{z}_0\right) = \frac{\delta^3}{2}\frac{\partial^2F(z_0,\dot{z}_0)}{\partial\delta^2}$$

$$\Rightarrow \ddot{z}_3 + z_3 = -2\nu_1\ddot{z}_2 - \left({\nu_1}^2 + 2\nu_2\right)\ddot{z}_1 - \left(2\nu_3 + 2\nu_1\nu_2\right)\ddot{z}_0 + \frac{1}{2}\frac{\partial^2 F\left(z_0, \dot{z}_0\right)}{\partial\delta^2} \tag{3.3}$$

such that

$$\frac{\partial^2 F\left(z, \nu\dot{z}\right)}{\partial\delta^2} = \frac{\partial}{\partial\delta}\left(\frac{\partial F}{\partial z}\frac{\partial z}{\partial\delta} + \frac{\partial F}{\partial\dot{z}}\frac{\partial\nu\dot{z}}{\partial\delta}\right)$$

$$= \frac{\partial z}{\partial\delta}\left(\frac{\partial}{\partial\delta}\frac{\partial F}{\partial z}\right) + \frac{\partial^2 z}{\partial\delta^2}\frac{\partial F}{\partial z} + \frac{\partial\nu\dot{z}}{\partial\delta}\left(\frac{\partial}{\partial\delta}\frac{\partial F}{\partial\dot{z}}\right) + \frac{\partial^2\nu\dot{z}}{\partial\delta^2}\frac{\partial F}{\partial\dot{z}}$$

$$= \frac{\partial z}{\partial\delta}\left(\frac{\partial^2 F}{\partial z^2}\frac{\partial w}{\partial\delta} + \frac{\partial^2 F}{\partial w\partial\dot{z}}\frac{\partial\nu\dot{z}}{\partial\delta}\right) + \frac{\partial^2 w}{\partial\delta^2}F_z + \frac{\partial^2\nu\dot{z}}{\partial\delta^2}F_{\dot{z}} + \frac{\partial\nu\dot{z}}{\partial\delta}\left(\frac{\partial^2 F}{\partial z\partial\dot{z}}\frac{\partial z}{\partial\delta} + \frac{\partial^2 F}{\partial\dot{z}^2}\frac{\partial\nu\dot{z}}{\partial\delta}\right)$$

$$= \left(\frac{\partial z}{\partial\delta}\right)^2 F_{zz} + 2\frac{\partial z}{\partial\delta}\frac{\partial\nu\dot{z}}{\partial\delta}F_{z\dot{z}} + \frac{\partial^2 z}{\partial\delta^2}F_z + \frac{\partial^2\nu\dot{z}}{\partial\delta^2}F_{\dot{z}} + \left(\frac{\partial\nu\dot{z}}{\partial\delta}\right)^2 F_{\dot{z}\dot{z}},$$

or

$$F_{zz} \equiv \frac{\partial^2 F}{\partial z^2}, \quad F_{z\dot{z}} \equiv \frac{\partial^2 F}{\partial z\partial\dot{z}}, \quad F_{\dot{z}\dot{z}} \equiv \frac{\partial^2 F}{\partial\dot{z}\partial\dot{z}}.$$

On the other hand, at third order, we have

$$z = z_0 + \delta z_1 + \delta^2 z_2 + \delta^3 z_3 \Rightarrow \frac{\partial z}{\partial\delta} = z_1 + 2\delta z_2 + 3\delta^2 z_3 \Rightarrow \frac{\partial^2 z}{\partial\delta^2} = 2z_2 + 6\delta z_3,$$

and

$$\frac{\partial\nu\dot{z}}{\partial\delta} = \frac{\partial\nu}{\partial\delta}\dot{z} + \frac{\partial\dot{z}}{\partial\delta}\nu = \left(\nu_1 + 2\delta\nu_2 + 3\delta^2\nu_3\right)\left(\dot{z}_0 + \delta\dot{z}_1 + \delta^2\dot{z}_2 + \delta^3\dot{z}_3\right)$$
$$+ \left(\dot{z}_1 + 2\delta\dot{z}_2 + 3\delta^2\dot{z}_3\right)\left(1 + \delta\nu_1 + \delta^2\nu_2 + \delta^3\nu_3\right),$$

and

$$\frac{\partial^2\nu\dot{z}}{\partial\delta^2} = \left(2\nu_2 + 6\delta\nu_3\right)\left(\dot{z}_0 + \delta\dot{z}_1 + \delta^2\dot{z}_2 + \delta^3\dot{z}_3\right) + 2\left(\dot{z}_1 + 2\delta\dot{z}_2 + 3\delta^2\dot{z}_3\right)\left(\nu_1 + 2\delta\nu_2 + 3\delta^2\nu_3\right)$$

$$+ \left(2\dot{z}_2 + 6\delta\dot{z}_3\right)\left(1 + \delta\nu_1 + \delta^2\nu_2 + \delta^3\nu_3\right),$$

$$\Rightarrow \frac{1}{2}\frac{\partial^2 F\left(z_0, \dot{z}_0\right)}{\partial\delta^2} = \frac{{z_1}^2}{2}F_{zz}\left(z_0, \dot{z}_0\right) + \left(\nu_1\dot{z}_0 + \dot{z}_1\right)z_1 F_{z\dot{z}}\left(z_0, \dot{z}_0\right) + z_2 F_z\left(z_0, \dot{z}_0\right)$$
$$+ \left(\nu_2\dot{z}_0 + \nu_1\dot{z}_1 + \dot{z}_2\right)F_{\dot{z}}\left(z_0, \dot{z}_0\right) + \frac{1}{2}\left(\nu_1\dot{z}_0 + \dot{z}_1\right)^2 F_{\dot{z}\dot{z}}\left(z_0, \dot{z}_0\right). \tag{3.4}$$

When we substitute (3.4) into equation (3.3) we obtain (3.5).

Thus

$$\ddot{z}_3 + z_3 = G_3\left(\theta\right) = -2\nu_1\ddot{z}_2 - \left(\nu_1^2 + 2\nu_2\right)\ddot{z}_1 - \left(2\nu_3 + 2\nu_1\nu_2\right)\ddot{z}_0 + z_2 F_z\left(z_0, \dot{z}_0\right) + \frac{{z_1}^2}{2}F_{zz}\left(z_0, \dot{z}_0\right) +$$

$$+z_1\left(\nu_1\dot{z}_0+\dot{z}_1\right)F_{z\dot{z}}\left(z_0,\dot{z}_0\right)+\left(\nu_2\dot{z}_0+\nu_1\dot{z}_1+\dot{z}_2\right)F_{\dot{z}}\left(z_0,\dot{z}_0\right)+\frac{1}{2}\left(\nu_1\dot{z}_0+\dot{z}_1\right)^2F_{\dot{z}\dot{z}}\left(z_0,\dot{z}_0\right). \quad (3.5)$$

We prove that $z_3(\theta)$ is a periodic solution

Proposition 3.2.2. *$z_3(\theta)$ is a periodic solution of the equation (3.2) if and only if*

$$\Rightarrow\begin{cases}\nu_3=\nu_1\nu_2+\frac{1}{2\pi}\nu_1\int_0^{2\pi}\Big[2\nu_1^2A\pi F_{z\dot{z}z}+A\frac{3}{4}\pi F_{z\dot{z}\dot{z}}-(\nu_1^2+\nu_2)A\pi+2\nu_1A\pi\\ -\pi F(A\cos\theta,-A\sin\theta)+A\pi F(A\cos\theta,-A\sin\theta)F_{zz}\\ +2\nu_1\frac{3}{4}\pi F_{\dot{z}zz}+\frac{3}{4}\pi F_{\dot{z}zz}-\pi\frac{3}{4}F_{z\dot{z}z}+\pi\frac{3}{4}F_{z\dot{z}\dot{z}}\\ -\pi\frac{3}{4}F_{\dot{z}\dot{z}\dot{z}}F_{\dot{z}}+\pi\frac{3}{4}F_{\dot{z}z}F_{z\dot{z}}-\frac{3}{4}\pi\frac{3}{2}\nu_1+\frac{3}{4}\pi F_{\dot{z}\dot{z}\dot{z}}+\frac{3}{4}\pi F_{\dot{z}}F_{z\dot{z}\dot{z}\dot{z}}\Big]d\theta.\\ +\int_0^{2\pi}\Big[+\nu_1\pi\frac{3}{4}F_z\left(\frac{3}{4}\pi+4\nu_1^2\pi+(\nu_1^2-2\nu_2)^2F_z+2\nu_1\pi F_z\right)\\ +\pi F(A\cos\theta,-A\sin\theta)F_{\dot{z}}F_{z\dot{z}}+\pi\frac{3}{4}F_{\dot{z}}F_{z\dot{z}z}d\theta\\ \int_0^{2\pi}\sin\theta\Big[-2\nu_1\ddot{z}_2-\left(\nu_1^2+2\nu_2\right)\ddot{z}_1-\left(2\nu_3+2\nu_1\nu_2\right)\ddot{z}_0+z_2F_z\left(z_0,\dot{z}_0\right)+\frac{{z_1}^2}{2}F_{z\dot{z}}\left(z_0,\dot{z}_0\right)+\Big]d\theta\\ +\int_0^{2\pi}\cos\theta\left[+\nu_1\left(\nu_1\dot{z}_0+\dot{z}_1\right)F_{z\dot{z}}\left(\nu_0,\dot{z}_0\right)+\left(\nu_2\dot{z}_0+\nu_1\dot{z}_1+\dot{z}_2\right)F_{\dot{z}}\left(z_0,\dot{z}_0\right)\right]\\ \left[+\frac{1}{2}\left(\nu_1\dot{z}_0+\dot{z}_1\right)^2F_{\dot{z}\dot{z}}\left(\nu_0,\dot{z}_0\right)\right]d\theta=0.\end{cases} \quad (3.6)$$

proof The periodic condition gives

$$\int_\theta^{\theta+2\pi}\sin(\theta-\tau)G_3(\tau)d\tau=0, \quad (3.7)$$

$$\Rightarrow\begin{cases}\int_0^{2\pi}\cos\theta G_3(\theta)=0.\\ \int_0^{2\pi}\sin\theta G_3(\theta)=0.\end{cases} \quad (3.8)$$

According to the equation:(3.2), we have

$$\begin{cases}G_3(\theta)=-2\nu_1\ddot{z}_2-\left(\nu_1^2+2\nu_2\right)\ddot{z}_1-\left(2\nu_3+2\nu_1\nu_2\right)\ddot{z}_0+z_2F_z\left(z_0,\dot{z}_0\right)\\ +\frac{{z_1}^2}{2}F_{zz}\left(z_0,\dot{z}_0\right)++z_1\left(\nu_1\dot{z}_0+\dot{z}_1\right)F_{z\dot{z}}\left(z_0,\dot{z}_0\right)\\ +\left(\nu_2\dot{z}_0+\nu_1\dot{z}_1+\dot{z}_2\right)F_{\dot{z}}\left(z_0,\dot{z}_0\right)+\frac{1}{2}\left(\nu_1\dot{z}_0+\dot{z}_1\right)^2F_{\dot{z}\dot{z}}\left(z_0,\dot{z}_0\right).\end{cases} \quad (3.9)$$

We have $z_0(\theta)$, $z_1(\theta)$and$z_2(\theta)$in [9], we rewrite(3.8) as

$$\Rightarrow \begin{cases} \int_0^{2\pi} \cos\theta \left[-2\nu_1 \ddot{z}_2 - \left(\nu_1^2 + 2\nu_2\right) \ddot{z}_1 - \left(2\nu_3 + 2\nu_1\nu_2\right) \ddot{z}_0 + z_2 F_z\left(z_0, \dot{z}_0\right) + \frac{{z_1}^2}{2} F_{zz}\left(z_0, \dot{z}_0\right)\right] d\theta \\ + \int_0^{2\pi} \cos\theta \left[+z_1\left(\nu_1 \dot{z}_0 + \dot{z}_1\right) F_{z\dot{z}}\left(z_0, \dot{z}_0\right) + \left(\nu_2 \dot{z}_0 + \nu_1 \dot{z}_1 + \dot{z}_2\right) F_{\dot{z}}\left(z_0, \dot{z}_0\right)\right] \\ \left[+\frac{1}{2}\left(\nu_1 \dot{z}_0 + \dot{z}_1\right)^2 F_{\dot{z}\dot{z}}\left(\nu_0, \dot{z}_0\right)\right] d\theta = 0, \\ \\ \int_0^{2\pi} \sin\theta \left[-2\nu_1 \ddot{z}_2 - \left(\nu_1^2 + 2\nu_2\right) \ddot{z}_1 - \left(2\nu_3 + 2\nu_1\nu_2\right) \ddot{z}_0 + z_2 F_z\left(z_0, \dot{z}_0\right) + \frac{{z_1}^2}{2} F_{z\dot{z}}\left(z_0, \dot{z}_0\right) + \right] d\theta \\ + \int_0^{2\pi} \cos\theta \left[+\nu_1\left(\nu_1 \dot{z}_0 + \dot{z}_1\right) F_{z\dot{z}}\left(\nu_0, \dot{z}_0\right) + \left(\nu_2 \dot{z}_0 + \nu_1 \dot{z}_1 + \dot{z}_2\right) F_{\dot{z}}\left(z_0, \dot{z}_0\right)\right] \\ \left[+\frac{1}{2}\left(\nu_1 \dot{z}_0 + \dot{z}_1\right)^2 F_{\dot{z}\dot{z}}\left(\nu_0, \dot{z}_0\right)\right] d\theta = 0. \end{cases} \tag{3.10}$$

$$\Rightarrow \begin{cases} -2\nu_1 \int_0^{2\pi} \Big[\cos\theta \cdot 2\nu_1 \ddot{z}_0^2 \dot{z}_0 F_{z\dot{z}z} + \dddot{z}_0 \dot{z}_0 z_0 \cos\theta F_{z\dot{z}\dot{z}} - \left(\nu_1^2 + \nu_2\right) \ddddot{z}_0 \cos\theta + 2\nu_1 A \cos^2\theta \\ - \cos^2\theta F(A\cos\theta, -A\sin\theta) + A\cos^2\theta F(A\cos\theta, -A\sin\theta) F_{zz} \\ +2\nu_1 \dot{z}_0^2 z_0 \cos\theta F_{\dot{z}zz} + \dot{z}_0^2 z_0 \cos\theta F_z F_{\dot{z}zz} - \dot{z}_0^2 \cos\theta F_{z\dot{z}z} + \dot{z}_0^2 \cos\theta F_{z\dot{z}\dot{z}} \\ + \dddot{z}_0 \dot{z}_0 z_0^2 \cos\theta F_{\dot{z}\dot{z}\dot{z}} F_{\dot{z}} + \dot{z}_0^2 z_0^2 F_{\dot{z}z} F_{z\dot{z}} - 2\nu_1 \cos^2\theta + \ddot{z}_0 \cos\theta F_z \Big] d\theta \\ + \int_0^{2\pi} \Big[-\left(2\nu_3 + 2\nu_1\nu_2\right)\cos^2\theta + F_z\left(4\nu_1^2 \cos^2\theta + \left(\nu_1^2 - 2\nu_2\right)\cos^2\theta F_z + 2\nu_1 \cos^2\theta F_z\right) \\ +\ddot{z}_0 \cos\theta F(A\cos\theta, -A\sin\theta) F_{\dot{z}} F_{z\dot{z}} + \ddot{z}_0^2 z_0^2 \cos\theta F_{\dot{z}} F_{z\dot{z}z} \Big] d\theta = 0, \\ \\ -2\nu_1 \int_0^{2\pi} \Big[\sin\theta \cdot 2\nu_1 \ddot{z}_0^2 \dot{z}_0 F_{z\dot{z}z} + \dddot{z}_0 \dot{z}_0 z_0 \sin\theta F_{z\dot{z}\dot{z}} - \left(\nu_1^2 + \nu_2\right) \ddddot{z}_0 \sin\theta + 2\nu_1 A \cos\theta \sin\theta \\ - \cos\theta \sin\theta F(A\cos\theta, -A\sin\theta) + A\cos\theta \sin\theta F(A\cos\theta, -A\sin\theta) F_{zz} \\ +2\nu_1 \dot{z}_0^2 z_0 \sin\theta F_{\dot{z}zz} + \dot{z}_0^2 z_0 \sin\theta F_z F_{\dot{z}zz} - \dot{z}_0^2 \sin\theta F_{z\dot{z}z} + \dot{z}_0^2 \sin\theta F_{z\dot{z}\dot{z}} \\ + \dddot{z}_0 \dot{z}_0 z_0^2 \sin\theta F_{\dot{z}\dot{z}\dot{z}} F_{\dot{z}} + \dot{z}_0^2 z_0^2 F_{\dot{z}z} F_{z\dot{z}} - 2\nu_1 \cos\theta \sin\theta + \ddot{z}_0 \sin\theta F_z \Big] d\theta \\ + \int_0^{2\pi} \Big[-\left(2\nu_3 + 2\nu_1\nu_2\right)\cos\theta \sin\theta + F_z\left(4\nu_1^2 \cos\theta \sin\theta + \left(\nu_1^2 - 2\nu_2\right)\cos\theta \sin\theta F_z + 2\nu_1 \cos^2\theta F_z\right) \\ +\ddot{z}_0 \sin\theta F(A\cos\theta, -A\sin\theta) F_{\dot{z}} F_{z\dot{z}} + \ddot{z}_0^2 z_0^2 \sin\theta F_{\dot{z}} F_{z\dot{z}z} \Big] d\theta = 0. \end{cases} \tag{3.11}$$

$$\Rightarrow \left\{ \begin{array}{l} \nu_3 = \nu_1\nu_2 + \frac{1}{2\pi}\nu_1 \int_0^{2\pi} \Big[2\nu_1^2 A\pi F_{z\dot{z}z} + A\frac{3}{4}\pi F_{z\dot{z}\dot{z}} - (\nu_1^2+\nu_2)A\pi + 2\nu_1 A\pi \\ -\pi F(A\cos\theta, -A\sin\theta) + A\pi F(A\cos\theta, -A\sin\theta)F_{zz} \\ +2\nu_1\frac{3}{4}\pi F_{\dot{z}zz} + \frac{3}{4}\pi F_{\dot{z}zz} - \pi\frac{3}{4}F_{z\dot{z}z} + \pi\frac{3}{4}F_{z\dot{z}\dot{z}} \\ -\pi\frac{3}{4}F_{\dot{z}\dot{z}\dot{z}}F_{\dot{z}} + \pi\frac{3}{4}F_{\dot{z}z}F_{z\dot{z}} - \frac{3}{4}\pi\frac{3}{2}\nu_1 + \frac{3}{4}\pi F_{\dot{z}\dot{z}\dot{z}} + \frac{3}{4}\pi F_{\dot{z}}F_{z\dot{z}\dot{z}\dot{z}} \Big] \, d\theta \\ + \int_0^{2\pi} \Big[+ \nu_1\pi\frac{3}{4}F_z \left(\frac{3}{4}\pi + 4\nu_1^2\pi + (\nu_1^2-2\nu_2)^2F_z + 2\nu_1\pi F_z\right) \\ +\pi F(A\cos\theta, -A\sin\theta)F_{\dot{z}}F_{z\dot{z}} + \pi\frac{3}{4}F_{\dot{z}}F_{z\dot{z}z} \Big] \, d\theta = 0, \\ \\ -2\nu_1 \int_0^{2\pi} \Big[\sin\theta \cdot 2\nu_1 \ddot{z}_0^2 \dot{z}_0 F_{z\dot{z}z} + \dddot{z}_0 \dot{z}_0 z_0 \sin\theta F_{z\dot{z}\dot{z}} - (\nu_1^2+\nu_2) \ddddot{z}_0 \sin\theta + 2\nu_1 A\cos\theta\sin\theta \\ -\cos\theta\sin\theta F(A\cos\theta, -A\sin\theta) + A\cos\theta\sin\theta F(A\cos\theta, -A\sin\theta)F_{zz} \\ +2\nu_1\dot{z}_0^2 z_0\sin\theta F_{\dot{z}zz} + \dot{z}_0^2 z_0\sin\theta F_z F_{\dot{z}zz} - \dot{z}_0^2\sin\theta F_{z\dot{z}z} + \dot{z}_0^2\sin\theta F_{z\dot{z}\dot{z}} \\ + \dddot{z}_0\dot{z}_0 z_0^2\sin\theta F_{\dot{z}\dot{z}\dot{z}}F_{\dot{z}} + \dot{z}_0^2 z_0^2 F_{\dot{z}z}F_{z\dot{z}} - 2\nu_1\cos\theta\sin\theta + \ddot{z}_0\sin\theta F_z \Big] \, d\theta \\ + \int_0^{2\pi} \Big[- (2\nu_3+2\nu_1\nu_2)\cos\theta\sin\theta + F_z\left(4\nu_1^2\cos\theta\sin\theta + (\nu_1^2-2\nu_2)\cos\theta\sin\theta F_z + 2\nu_1\cos^2\theta F_z\right) \\ +\ddot{z}_0\sin\theta F(A\cos\theta, -A\sin\theta)F_{\dot{z}}F_{z\dot{z}} + \ddot{z}_0^2 z_0^2\sin\theta F_{\dot{z}}F_{z\dot{z}z} \Big] \, d\theta = 0. \end{array} \right. \tag{3.12}$$

$$\Rightarrow \left\{ \begin{array}{l} \nu_3 = \nu_1\nu_2 + \frac{1}{2\pi}\nu_1 \int_0^{2\pi} \Big[2\nu_1^2 A\pi F_{z\dot{z}z} + A\frac{3}{4}\pi F_{z\dot{z}\dot{z}} - (\nu_1^2+\nu_2)A\pi + 2\nu_1 A\pi \\ -\pi F(A\cos\theta, -A\sin\theta) + A\pi F(A\cos\theta, -A\sin\theta)F_{zz} \\ +2\nu_1\frac{3}{4}\pi F_{\dot{z}zz} + \frac{3}{4}\pi F_{\dot{z}zz} - \pi\frac{3}{4}F_{z\dot{z}z} + \pi\frac{3}{4}F_{z\dot{z}\dot{z}} \\ -\pi\frac{3}{4}F_{\dot{z}\dot{z}\dot{z}}F_{\dot{z}} + \pi\frac{3}{4}F_{\dot{z}z}F_{z\dot{z}} - \frac{3}{4}\pi\frac{3}{2}\nu_1 + \frac{3}{4}\pi F_{\dot{z}\dot{z}\dot{z}} + \frac{3}{4}\pi F_{\dot{z}}F_{z\dot{z}\dot{z}\dot{z}} \Big] \, d\theta \\ + \int_0^{2\pi} \Big[+ \nu_1\pi\frac{3}{4}F_z \left(\frac{3}{4}\pi + 4\nu_1^2\pi + (\nu_1^2-2\nu_2)^2F_z + 2\nu_1\pi F_z\right) \\ +\pi F(A\cos\theta, -A\sin\theta)F_{\dot{z}}F_{z\dot{z}} + \pi\frac{3}{4}F_{\dot{z}}F_{z\dot{z}z} d\theta \\ \int_0^{2\pi} \sin\theta \left[-2\nu_1\ddot{z}_2 - \left(\nu_1^2+2\nu_2\right)\ddot{z}_1 - \left(2\nu_3+2\nu_1\nu_2\right)\ddot{z}_0 + z_2F_z\left(z_0,\dot{z}_0\right) + \frac{{z_1}^2}{2}F_{z\dot{z}}\left(z_0,\dot{z}_0\right) + \right] d\theta \\ + \int_0^{2\pi} \cos\theta \left[+\nu_1\left(\nu_1\dot{z}_0+\dot{z}_1\right)F_{z\dot{z}}\left(\nu_0,\dot{z}_0\right) + \left(\nu_2\dot{z}_0+\nu_1\dot{z}_1+\dot{z}_2\right)F_{\dot{z}}\left(z_0,\dot{z}_0\right)\right] \\ \left[+\frac{1}{2}\left(\nu_1\dot{z}_0+\dot{z}_1\right)^2 F_{\dot{z}\dot{z}}\left(\nu_0,\dot{z}_0\right)\right] d\theta = 0. \end{array} \right. \tag{3.13}$$

In the following example, we will calculate the third approximation solution of the example 2.2.1.

Example 3.2.1. *Let the differential equation*

$$\frac{d^2z}{dt^2} + z + \delta\left(2z^3\right) = 0. \tag{3.14}$$

With the initial conditions $z(0) = A$ *and* $(dz/dt)(0) = 0$, *(in order to obtain periodic solutions, the amplitude of the motion must remain sufficiently small).*
Suppose that the third-order approximate solution is

$$z(\theta,\delta) = z_0(\theta) + \delta z_1(\theta) + \delta^2 z_2(\theta) + \delta^3 z_3(\theta). \tag{3.15}$$

We have $\theta = \nu t$ *such that*

$$\nu = 1 + \delta\nu_1 + \delta^2\nu_2 + \delta^3\nu_3 +,\ \nu(\delta) \neq 1.$$

So, by using (1.39) *and* (1.40) *the equation* (3.14) *becomes*

$$\nu^2\ddot{z} + z = -2\nu z^3. \tag{3.16}$$

With compensation, we find

$$(1+\delta\nu_1+\delta^2\nu_2+\delta^3\nu_3)^2(\ddot{z}_0+\delta\ddot{z}_1+\delta^2\ddot{z}_2+\delta^3\ddot{z}_3)+(z_0+\delta z_1+\delta^2 z_2+\delta^3 z_3) = -2\delta(z_0+\delta z_1+\delta^2 z_2+\delta^3 z_3)^3.$$

we find

$$\ddot{z}_3 + z_3 = G_3(\theta) = -2\nu_1\ddot{z}_2 - \left(\nu_1^2 + 2\nu_2\right)\ddot{z}_1 - (2\nu_3 + 2\nu_1\nu_2)\ddot{z}_0 + z_2 F_z(z_0,\dot{z}_0) + \frac{{z_1}^2}{2}F_{zz}(z_0,\dot{z}_0). \tag{3.17}$$

$$+z_1(\nu_1\dot{z}_0 + \dot{z}_1)F_{z\dot{z}}(z_0,\dot{z}_0) + (\nu_2\dot{z}_0 + \nu_1\dot{z}_1 + \dot{z}_2)F_{\dot{z}}(z_0,\dot{z}_0) + \frac{1}{2}(\nu_1\dot{z}_0 + \dot{z}_1)^2 F_{\dot{z}\dot{z}}(z_0,\dot{z}_0),$$

with $z_3(0) = \dot{z}_0 = 0$.
The following equations (2.19)(2.20)(2.21) were solved by[9] and their solution is given by

$$z(\theta,\delta) = Acos\theta + \delta\left(\frac{A^3}{16}\right)[cos3\theta - cos\theta] + \delta^2\left(\frac{A^5}{512}\right)[cos\theta - cos5\theta] + o(\delta^2),$$

with $\theta = \nu t$ *such that*

$$\nu = 1 + \delta\left(3\frac{A^2}{4}\right) + \delta^2\left(15\frac{A^4}{128}\right) + ...$$

the solution of the following (3.17) *is*

$$\ddot{z}_3 + z_3 = -\frac{6A^7}{512}\cos^2 t(\cos t - \cos 5t) - \frac{6A^7}{256}\cos t(\cos 3t - \cos t)^2$$

$$+\left(\frac{90A^6}{512} + A\nu_3^2 - \frac{3A^7}{64}\right)\cos t + \frac{459A^7}{1024}\cos(3t) - \frac{75A^7}{1024}\cos(5t) + \frac{6A^7}{512}\cos^2 t\cos(5t).$$

so

$$\begin{aligned}
\ddot{z}_3 + z_3 = &-\left(\frac{6A^7}{512}\right)\left(\frac{1}{4}\cos(t) + \frac{3}{4}\cos(3t) - \frac{1}{2}\cos(4t) - \frac{1}{2}\cos(6t)\right) \\
&-\left(\frac{6A^7}{256}\right)\left(-\frac{3}{4}\cos(3t) + \frac{3}{4}\cos(t) - \frac{1}{4}\cos(5t) + \frac{1}{4}\cos(7t)\right) \\
&+\left(\frac{90A^6}{512} + A\nu_3^2 - \frac{3A^7}{64}\right)\cos t \\
&+\frac{459A^7}{1024}\cos(3t) \\
&-\frac{75A^7}{1024}\cos(5t) \\
&+\left(\frac{6A^7}{512}\right)\left(\frac{1}{4}\cos(3t) + \frac{1}{2}\cos(5t) + \frac{1}{4}\cos(7t)\right), \text{ with } z_3(0) = \dot{z}_3(0) = 0.
\end{aligned}$$

The requirement of no secular terms gives the following results

$$\nu_3 = \sqrt{\left(\frac{90A^5}{512} - \frac{57A^6}{1024}\right)}.$$

$$\begin{aligned}
\ddot{z}_3 + z_3 = &-\left(\frac{6A^7}{512}\right)\left(\frac{3}{4}\cos(3t) - \frac{1}{2}\cos(4t) - \frac{1}{2}\cos(6t)\right) \\
&-\left(\frac{6A^7}{256}\right)\left(-\frac{3}{4}\cos(3t) + \frac{1}{4}\cos(7t)\right) \\
&+\frac{459A^7}{1024}\cos(3t) \\
&+\left(\frac{6A^7}{512}\right)\left(\frac{1}{4}\cos(3t) + \frac{1}{4}\cos(7t)\right).
\end{aligned}$$

$$\ddot{z}_3 + z_3 = \left(\frac{18A^7}{2028} + \frac{477A^7}{1024} + \frac{6A^7}{2048}\right)\cos(3t) + \left(\frac{3A^7}{512}\right)\cos(4t) + \left(\frac{3A^7}{512}\right)\cos(6t) + \left(\frac{6A^7}{1024} - \frac{6A^7}{2048}\right)\cos(7t).$$

$$\ddot{z}_3 + z_3 = \underbrace{A\cos(3t)}_{f1} + \underbrace{B\cos(4t)}_{f2} + \underbrace{D\cos(6t)}_{f3} + \underbrace{E\cos(7t)}_{f4}.$$

$$z_{3_{pf_1}} = \left(\frac{18A^7}{2028} + \frac{477A^7}{1024} + \frac{6A^7}{2048}\right)\cos(3t).$$

$$z_{3_{pf_2}} = \left(\frac{3A^7}{512}\right)\cos(4t).$$

$$/z_{3_{pf_3}} = (\frac{3A^7}{512})\cos(6t).$$

$$z_{3_{pf_4}} = (\frac{6A^7}{1024} - \frac{6A^7}{2048})\cos(7t).$$

$$\begin{aligned} z_3 &= C_1\cos(t) + C_2\sin(t) \\ &+ \left(\frac{A^7}{113} + \frac{480A^7}{1024}\right)\cos(3t) \\ &+ \frac{3A^7}{512}\cos(4t) \\ &+ \frac{3A^7}{512}\cos(6t) \end{aligned}$$

$$+\frac{3A^7}{1024}\cos(7t). \tag{3.18}$$

we have

$$\begin{aligned} z_3(0) &= 0, \;\Rightarrow \\ c_1 &= (\frac{18A^7}{2028} + \frac{477A^7}{1024} + \frac{6A^7}{2048}) + (\frac{6A^7}{512}) + (\frac{6A^7}{1024} - \frac{6A^7}{2048}), \end{aligned}$$

and

$$\dot{z}_3(0) = 0 \;\Rightarrow\; c_2 = 0.$$

The solution to (3.18) *is*

$$\begin{aligned} z_3 = {} & (\frac{18A^7}{2028} + \frac{477A^7}{1024} + \frac{6A^7}{2048}) + (\frac{6A^7}{512}) + (\frac{6A^7}{1024} - \frac{6A^7}{2048})\cos(t) \\ & + \left(\frac{A^7}{113} + \frac{480A^7}{1024}\right)\cos(3t) + \frac{3A^7}{512}\cos(4t) + \frac{3A^7}{512}\cos(6t) \\ & + \frac{3A^7}{1024}\cos(7t). \end{aligned}$$

The solution to (3.17) *is*

$$
\begin{aligned}
z(\theta,\delta) = {} & A\cos\theta + \delta\left(\frac{A^3}{16}\right)[\cos(3\theta)-\cos\theta] + \delta^2\left(\frac{A^5}{512}\right)[\cos\theta-\cos(5\theta)] \\
& + \delta^3\Bigg[\left(\frac{18A^7}{2028}+\frac{477A^7}{1024}+\frac{6A^7}{2048}\right)+\frac{6A^7}{512}+\left(\frac{6A^7}{1024}-\frac{6A^7}{2048}\right)\cos(\theta) \\
& \quad + \left(\frac{A^7}{113}+\frac{480A^7}{1024}\right)\cos(3\theta)+\frac{3A^7}{512}\cos(4\theta)+\frac{3A^7}{512}\cos(6\theta)+\frac{3A^7}{1024}\cos(7\theta)\Bigg] \\
& + o(\delta^3)
\end{aligned}
$$

with

$$\nu(\theta) = 1 + \delta\left(3\frac{A^2}{4}\right) + \delta^2\left(15\frac{A^4}{128}\right) + \delta^3\sqrt{\left(\frac{90A^5}{512}-\frac{57A^6}{1024}\right)} + ...,\ A < \frac{60}{19}.$$

3.3 Third order Approximate solutions for the "Van Der Pol" equation

3.3.1 Simple perturbation Method (SPM)

In the following proposition, we solve $z_3(\theta)$. We mention here the solution of the equations $z_1(\theta)$, $z_2(\theta)$ in [13].

$$\ddot{z} + \delta\left(az^2 + b\dot{z}^2 - 1\right)\dot{z} + z = 0, \quad z(0) = A, \quad \dot{z}(0) = 0, \quad a, b, \text{ and } A \in \mathbb{R}. \tag{3.19}$$

Suppose that the approximate solution is

$$z_P(t,\delta) = z\left(t,\delta\right) = z_0\left(t\right) + \delta z_1\left(t\right) + \delta^2 z_2(t) + \delta^3 z_3(t) + o(\delta^3). \tag{3.20}$$

To determine $z_0(t)$, $z_1(t)$ and $z_2(t)$, substituting (3.20) into (3.19), and calculating, we find To determine $z_0(t)$, $z_1(t)$ and $z_2(t)$, by substituting (3.20) into (3.19), and calculating, we find

$$
\begin{aligned}
& (\ddot{z}_0 + z_0) + \delta\left(\ddot{z}_1 + z_1 - \dot{z}_0 + a\dot{z}_0 z_0^2 + b z_0 \dot{z}_0{}^2\right) \\
& + \delta^2\left(\ddot{z}_2 + z_2 - \dot{z}_1 + a\dot{z}_1 z_0^2 + 2b\dot{z}_0{}^2\dot{z}_1 + b\dot{z}_0{}^2\dot{z}_1\right) \\
& + \delta^3\left(\ddot{z}_3 + z_3 + b\dot{z}_0\dot{z}_1{}^2 + 2b\dot{z}_0{}^2\dot{z}_2 + 2b\dot{z}_0\dot{z}_1{}^2 + b\dot{z}_0{}^2\dot{z}_2 + a\dot{z}_0\dot{z}_1{}^2 + 2a\dot{z}_0 z_0 z_2 + a\dot{z}_2\dot{z}_0{}^2 - \dot{z}_2\right) \\
& + o(\delta^3) = 0.
\end{aligned}
$$

$$\ddot{z}_3 + z_3 = -\ddot{z}_3 - z_3 - b\dot{z}_0\dot{z}_1{}^2 - 2b\dot{z}_0{}^2\dot{z}_2 - 2b\dot{z}_0\dot{z}_1{}^2 - b\dot{z}_0{}^2\dot{z}_2 - a\dot{z}_0\dot{z}_1{}^2$$

$$-2a\dot{z}_0 z_0 z_2 - a\dot{z}_2\dot{z}_0{}^2 + \dot{z}_2, \tag{3.21}$$

we have z_0, z_1 and z_2 in (2.23), and here we calculate z_3.

$$\begin{aligned}
\ddot{z}_3 + z_3 = &\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[144a^2(a-b) + a(5a-9b)(a-b)\right]\right)\\
&+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[-72a(a-b) - \frac{9}{8}a(a-b) - 3a^2\right]\right)\\
&+\left(\frac{1}{(a+3b)\sqrt{a+3b}}\left[6b + 7a + 3a^2t^2 - 4at^2 + \frac{3b}{a+3b} - \frac{24b}{(a+3b)^2}\right]\cos^2 t\sin t\right)\\
&+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[\frac{-9}{12}a^2(a-3b)\right]\right)\\
&+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[\frac{-9}{12}a(a-b) - (2ba + 5a - 9b)(a-b) + b(7a-3b)\right]\right)\\
&+\left(\frac{1}{(a+3b)\sqrt{a+3b}}\left[2bt^2\right]\sin^3 t\right)\\
&+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[2(a-9b)(a-b)^2 - 18(a-b)^2 - 48bt\right]\right)\\
&+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[7b(7a+9b) - 2b(7a+9b)t + 4(a-9b)(a-b) + 12(a-b)^2\right]\right)\\
&+\left(\frac{1}{(a+3b)\sqrt{a+3b}}\left[242bt + 120b\right]\cos t\sin^2 t\right)\\
&+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\cdot 18b(a-b)\right)\cos^2(3t)\sin t\\
&+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[\frac{2304}{1024}b(57a-9b)(a-b) + 9(a-3b)(a-b) + \frac{3}{4}a(a-b)\right]\right)\\
&+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[9b(a-b) + a + 9(a-b) - 4b(7a-3b)\right]\right)\cos t\cos(3t)\sin t\\
&+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[-9b(a-b) + -36b(a-b)\right]\right)\cos(3t)\sin^2 t
\end{aligned}$$

$$+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[\frac{3}{4}b(a-b)(a-3b)+\frac{3}{2}ab(a-b)\right]\right)$$

$$+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[-\frac{5}{4}b(a-b)+2ab\right]\right)\sin(3t)\sin^2 t$$

$$+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[\frac{5}{12}b(5a-9b)(a-b)\right]\right)\sin(5t)\sin^2 t$$

$$+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[\frac{1}{12}(5a-9b)(a-b)\right]\right)\cos(t)\cos(5t)\sin t$$

$$+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[3at(a-b)\right]\right)\sin(t)\sin(3t)\cos t$$

$$+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[-2a(a-9b)(a-b)+18(a-b)^2-12b(a-b)\right]\right)$$

$$+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[-4a(a-9b)-at\right]\right)$$

$$+\left(\frac{1}{(a+3b)\sqrt{a+3b}}\left[-2at\right]\right)\cos^3 t$$

$$+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[\frac{3}{4}a(a-3b)(a-b)+\frac{3}{4}(a-b)a^2+\frac{3}{2}a(a-b)+\frac{9}{2}(a-b)\right]\right)$$

$$+\left(\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[-\frac{3}{4}a(a-b)+a^2-\frac{9}{2}(a-b)\right]\right)\sin(3t)\cos^2 t$$

$$+\left(\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[\frac{5}{24}a(a-b)(a-9b)\right]\right)\sin(5t)\sin^2 t.$$

$$\ddot{z}_3+z_3=\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[144a^2(a-b)+a(5a-9b)(a-b)\right]$$

$$+\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[-72a(a-b)-\frac{9}{8}a(a-b)-3a^2\right]$$

$$+\frac{1}{(a+3b)\sqrt{a+3b}}\left(6b+7a+3a^2t^2-4at^2+\frac{3b}{a+3b}-\frac{24b}{(a+3b)^2}\right)\cdot\left(\frac{1}{4}\sin t+\frac{1}{4}\sin 3t\right)$$

$$+\frac{-9}{12}\cdot\frac{a^2(a-3b)}{(a+3b)^3\sqrt{a+3b}}$$

$$+\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[-\frac{9}{12}a(a-b)-(2ab+5a-9b)(a-b)+b(7a-3b)\right]$$

$$+\frac{2bt^2}{(a+3b)\sqrt{a+3b}}\cdot\sin^3 t\cdot\left(\frac{3}{4}\sin t-\frac{1}{4}\sin 3t\right)$$

$$+\frac{1}{(a+3b)^3\sqrt{a+3b}}\left[2(a-9b)(a-b)^2-18(a-b)^2-48bt\right]$$

$$+\frac{1}{(a+3b)^2\sqrt{a+3b}}\left[7b(7a+9b)-2b(7a+9b)t+4(a-9b)(a-b)+12(a-b)^2\right]$$

$$+\frac{1}{(a+3b)\sqrt{a+3b}}(242bt+120b)\cdot\left(\frac{1}{4}\cos t-\frac{1}{4}\cos 3t\right)$$

$$+\frac{18b(a-b)}{(a+3b)^3\sqrt{a+3b}}\cdot\left(\frac{1}{2}\sin t+\frac{1}{4}\sin 7t-\frac{1}{4}\sin 5t\right)$$

$$
\begin{aligned}
&+ \frac{1}{(a+3b)^3\sqrt{a+3b}} \left[\frac{2304}{1024} b(57a-9b)(a-b) + 9(a-3b)(a-b) + \frac{3}{4}a(a-b)\right] \\
&+ \frac{1}{(a+3b)^2\sqrt{a+3b}} \left[9b(a-b) + a + 9(a-b) - 4b(7a-3b)\right] \\
&+ \frac{1}{4}\sin 5t + \frac{1}{4}\sin t \frac{-45b(a-b)}{(a+3b)^2\sqrt{a+3b}} \cdot \frac{1}{2}\cos(3t) - \frac{1}{4}\cos(t) - \frac{1}{4}\cos(5t) \\
&+ \frac{\frac{3}{4}b(a-b)(a-3b) + \frac{3}{2}ab(a-b)}{(a+3b)^3\sqrt{a+3b}} \\
&+ \frac{-\frac{5}{4}b(a-b) + 2ab}{(a+3b)^2\sqrt{a+3b}} \cdot \left(\frac{1}{2}\sin(3t) - \frac{1}{4}\sin(5t) - \frac{1}{4}\sin(t)\right) \\
&+ \frac{\frac{5}{12}b(5a-9b)(a-b)}{(a+3b)^3\sqrt{a+3b}} \cdot \left(\frac{1}{2}\sin(5t) - \frac{1}{4}\sin(7t) - \frac{1}{4}\sin(3t)\right) \\
&+ \frac{\frac{1}{12}(5a-9b)(a-b)}{(a+3b)^3\sqrt{a+3b}} \cdot \left(\frac{1}{4}\sin(7t) - \frac{1}{4}\sin(3t)\right) \\
&+ \frac{3at(a-b)}{(a+3b)^2\sqrt{a+3b}} \cdot \left(\frac{1}{4}\cos(t) - \frac{1}{4}\cos(5t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{-2a(a-9b)(a-b) + 18(a-b)^2 - 12b(a-b)}{(a+3b)^3\sqrt{a+3b}} \\
&+ \frac{-4a(a-9b) - at}{(a+3b)^2\sqrt{a+3b}} \\
&+ \frac{-2at}{(a+3b)\sqrt{a+3b}} \cdot \left(\frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t)\right) \\
&+ \frac{\frac{3}{4}a(a-3b)(a-b) + \frac{3}{4}a^2(a-b) + \frac{3}{2}a(a-b) + \frac{9}{2}(a-b)}{(a+3b)^3\sqrt{a+3b}} \\
&+ \frac{-\frac{3}{4}a(a-b) + a^2 - \frac{9}{2}(a-b)}{(a+3b)^2\sqrt{a+3b}} \cdot \left(\frac{1}{2}\sin(3t) + \frac{1}{4}\sin(5t) + \frac{1}{4}\sin(t)\right) \\
&+ \frac{\frac{5}{24}a(a-b)(a-9b)}{(a+3b)^3\sqrt{a+3b}} \cdot \left(\frac{1}{2}\sin(5t) + \frac{1}{4}\sin(7t) + \frac{1}{4}\sin(3t)\right).
\end{aligned}
$$

$$
\begin{aligned}
\ddot{z}_3 + z_3 = {} & \frac{sint}{(a+3b)^3\sqrt{a+3b}}[-9a^2(a-b) + \frac{a}{2}(5a-9b)(a-b) + (a-b)(a-3b) \\
& + 36(a-b)b + 9(7a+9b)(a-b)b + 36(a-b)(a-3b) + 3a(a-b) - 3(a-b)(a-3b)b \\
& + 6ab(a-b)(a-3b) + 3(a-b)(a-3b)a + 3a^2(a-b) \\
& + 6(a-b)a + 18(a-b)]
\end{aligned}
$$

$$+ \frac{sint}{(a+3b)^2\sqrt{a+3b}}[-\frac{9}{2}a(a-b) - 24b(7a+9b) - 2a(a-b) - 12a^2 - b(a-b) - \frac{8}{3}ab$$

$$+ \frac{4}{3}(5a-9b)(a-b)$$

$$+ \frac{4}{3}b(7a-3b) - 36(a-b)b - 36(a-b) - 4a + 16(7a-3b) + (a-b)b$$

$$- 8ab - 3a(a-b) + 4a^2 - 18(a-b) - \frac{9}{48}(b)(a-b) + (5a-9b) - \frac{1}{16}(7a-3b)]$$

$$+ \frac{sint}{(a+3b)\sqrt{a+3b}}[24b + 28a + 12a^2t^2 + 12at^2 + 8bt^2 + 36(a-b) - \frac{9}{96}(a-b) - \frac{1}{4}a - \frac{1}{2}t^2]$$

$$+ \frac{sin3t}{(a+3b)^3\sqrt{a+3b}}[-9a^2(a-b) + \frac{a}{2}(5a-9b)(a-b) - 3(a-3b)(a-b) + 3(a-b)(a-3b)b$$

$$+ 6ab(a-b) - \frac{5}{3}b(5a-9b)(a-b) - \frac{1}{3}a(5a-9b)(a-b) + \frac{3}{2}a(a-b)(a-3b) + \frac{3}{2}a^2(a-b)$$

$$+ 3a(a-b) + 9(a-b) + \frac{5}{6}a(a-b)(a-9b)$$

$$+ \frac{sin3t}{(a+3b)^2\sqrt{a+3b}}[-24b(7a+9b) - \frac{9}{2}a(a-b) - 12a^2 + \frac{9}{3}b(a-b) + 8ab - 4(5a-9b)(a-b)$$

$$- 4b(7a-3b) - \frac{5}{2}b(a-b) + 4ab - \frac{3}{2}a(a-b) + 2a^2 - 9(a-b) - \frac{3}{32}(a-b)(a-3b) + \frac{3}{16}a(a-b)]$$

$$+ \frac{sin3t}{(a+3b)\sqrt{a+3b}}[24b + 28a + 12a^2t^2 - 8bt^2] - \sin(-\frac{3}{4}(a-b) - \frac{1}{4}a)$$

$$+ \frac{\sin 5t}{(a+3b)^3\sqrt{a+3b}}\Big[- 72(a-b)b + 9(7a+9b)(a-b)b + 36(a-b)(a-3b)$$

$$+ 3a(a-b) - 3(a-b)(a-3b)b - 6ab(a-b) + \frac{5}{6}(5a-9b)(a-b)$$

$$+ 3a(a-b)(a-3b) + 3a^2(a-b) + 6(a-b)a + 18(a-b) + \frac{15}{12}a(a-9b)(a-b)]$$

$$+ \frac{sin5t}{(a+3b)^2\sqrt{a+3b}}[41(a-b)b + 2(a-b) + 4a - 16(7a-3b) + -8ab - 3a(a-b)4a^2$$

$$- \frac{5}{96}(5a-9b)(a-b)]$$

$$+ \frac{sin7t}{(a+3b)^3\sqrt{a+3b}}[72(a-b)b - \frac{5}{3}(5a-9b)(a-b) + \frac{1}{3}(5a-9b)(a-b) + \frac{5}{6}a(a-9b)(a-b)]$$

$$+ \frac{cost}{(a+3b)^3\sqrt{a+3b}}[8(a-9b)(a-b)^2 - 72(a-b)^3 - 192bt - \frac{8}{3}a(a-9b)(a-b)$$

$$+ 24a(a-b)^2 - 12b(a-b)]$$

$$+\frac{cost}{(a+3b)^2\sqrt{a+3b}}[-4b(7a+9b)+32b(7a+9b)+8bt(7a+9b)+16(a-9b)(a-b)$$

$$+48(a-b)^2+8(a+7b)(a-b)+36(a-b)+144(a-b)b+12(a-b)at-\frac{16}{3}a(a-9b)$$

$$-\frac{4}{3}at+\frac{1}{8}(a-9b)(a-b)-\frac{9}{8}(a-b)^2+b(a-b)]$$

$$+\frac{cost}{(a+3b)\sqrt{a+3b}}[[480b+968bt-\frac{8}{3}at]+2(a-9b)+\frac{3}{8}(7b-a)]$$

$$+\frac{cos3t}{(a+3b)^3\sqrt{a+3b}}[-8(a-9b)(a-b)^2+72(a-b)^3++192bt-8a(a-9b)(a-b)$$

$$+72a(a-b)^2-48(a-b)b]$$

$$+\frac{cos3t}{(a+3b)^2\sqrt{a+3b}}[-20b(7a+9b)-8b(7a+9b)t$$

$$-16(a-9b)(a-b)-48(a-b)^2-8(a+7b)(a-b)-18(a-b)-72(a-b)b-16a(a-9b)-5at]$$

$$+\frac{cos3t}{(a+3b)\sqrt{a+3b}}[-480b-968bt-8at]\frac{3}{4}(a-b)-\frac{3}{8}(a-b)$$

$$+\frac{cos5t}{(a+3b)^2\sqrt{a+3b}}[36(a-b)+144(a-b)b-12(a-b)at].$$

So we have

$$\ddot{z}_3+z_3=\underbrace{A\cos t+B\sin t}_{f_1}+\underbrace{C\cos 3t+D\sin 3t}_{f_2}+\underbrace{E\cos 5t+F\sin t}_{f_3}+\underbrace{G\cos 7t+H\sin 7t}_{f_4}$$
$$+\underbrace{tk\cos t}_{f_5}+\underbrace{Yt^2\sin t}_{f_6}+\underbrace{Nt\cos 3t}_{f_7}+\underbrace{Wt^2\sin 3t}_{f_8}+\underbrace{tV\cos 5t}_{f_9},$$

and

$$z_{3_{p_{f1}}}=tA\cos t+tB\sin t,\, z_{3_{p_{f2}}}=C\cos 3t+D\sin 3t.$$

$$z_{3_{p_{f3}}}=E\cos 5t+F\sin 5t,\, z_{3_{p_{f4}}}=G\cos 7t+H\sin 7t.$$

$$z_{3_{p_{f5}}}=Kt^2\sin t+tK\cos t,\, z_{3_{p_{f6}}}=\frac{1}{3}t^2\sin t-\frac{3}{50}t^2\cos t+N\sin 3t+tN\cos 3t.$$

$$z_{3_{p_{f7}}}=N\sin 3t+tN\cos 3t,\, z_{3_{p_{f8}}}=Y\frac{2}{25}t^2\sin 3t.$$

$$z_{3_{p_{f9}}}=V\sin 5t+tV\cos 5t.$$

We get

$$\begin{aligned}
z_3(t) = {} & c_1 \cos t + c_2 \sin t \\
& + tA \cos t + tB \sin t + C \cos 3t + D \sin 3t + E \cos 5t + F \sin 5t \\
& + G \cos 7t + H \sin 7t \\
& + Kt^2 \sin t + tK \cos t + \frac{1}{3} t^2 \sin t - \frac{3}{50} t^2 \cos t \\
& + N \sin 3t + tN \cos 3t + \frac{2}{25} Y t^2 \sin 3t \\
& + V \sin 5t + tV \cos 5t.
\end{aligned}$$

Then

$$z_2(0) = c_1 + 0 + 0 + 0 + C + E + G,$$

$$c_1 = C + E + G,$$

$$\dot{z}(0) = 0 + c_2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0, \Rightarrow \ c_2 = 0.$$

So

$$\begin{aligned}
z_3(t) = {} & C + E + G \cos t \\
& + tA \cos t + tB \sin t + C \cos 3t + D \sin 3t + E \cos 5t + F \sin 5t \\
& + G \cos 7t + H \sin 7t \\
& + Kt^2 \sin t + tK \cos t + \frac{1}{3} t^2 \sin t - \frac{3}{50} t^2 \cos t \\
& + N \sin 3t + tN \cos 3t + \frac{2}{25} Y t^2 \sin 3t \\
& + V \sin 5t + tV \cos 5t.
\end{aligned}$$

The final solution to (3.21) is

$$
\begin{aligned}
z_P(t,\delta) = {} & A\cos(t) \\
& + \delta\Bigg[\left(\frac{A^3}{32}[7a+9b]-\frac{A}{2}\right)\sin(t)+\left(\frac{A}{2}-\frac{A^3(a+3b)}{8}\right)t\cos(t)-\frac{A^3(a-b)}{32}\sin(3t)\Bigg] \\
& + \delta^2\Bigg[\Bigg(-\frac{A^5}{3072}\left[9(a+3b)(a-b)+9(a-b)(a-3b)+24a(a+3b)+18a(a-b)\right] \\
& -(5a-9b)(a-b)+\frac{2A^3}{256}[7a-3b]\Bigg)\cos(t) \\
& + \left(\frac{A^5}{256}\left[(a-9b)-9(a-b)^2+2(a-9b)+6(a-b)\right]+\frac{3A^3}{64}[7b+a]-\frac{A}{8}\right)t\sin(t). \\
& + \left(\frac{3A^5}{128}(a+3b)^2+\frac{A}{8}-\frac{A^3}{8}(a+3b)\right)t^2\cos(t) \\
& + \left(\frac{A^5}{1024}\left[3(a+3b)(a-b)+3(a-b)a+8a(a+3b)+6a(a-b)\right]-\frac{2A^3}{256}[7a-3b]\right)\cos(3t) \\
& - \frac{A^5}{3072}(5a-9b)(a-b)\cos(5t) \\
& + \left(\frac{3A^5}{256}(a-b)(a+3b)-\frac{3A^3}{64}(a-b)\right)t\sin(3t)\Bigg]+ \\
& \delta^3(+C+E+G\cos t \\
& + tA\cos t+tB\sin t+C\cos 3t+D\sin 3t+E\cos 5t+F\sin 5t \\
& + G\cos 7t+H\sin 7t \\
& + Kt^2\sin t+tK\cos t+\frac{1}{3}t^2\sin t-\frac{3}{50}t^2\cos t \\
& + N\sin 3t+tN\cos 3t+\frac{2}{25}Yt^2\sin 3t \\
& + V\sin 5t+tV\cos 5t) \\
& + o(\delta^3).
\end{aligned}
$$

With $z_L(t,\delta)=z(\theta,\delta)$.

3.3.2 Lindstedt-Poincaré Method (LPM)

In the following proposition, we solve $z_3(\theta)$. We mention here that $z_1(\theta)$, $z_2(\theta)$ in [13].

$$\nu^2\ddot{z}-\delta(1-az^2-b\dot{w}^2)\nu\dot{z}+z=0\,\text{and}\,z(0)=1,\ \dot{z}(0)=0. \tag{3.22}$$

$$z(\theta) = z_0(\theta) + \delta z_1(\theta) + \delta^2 z_2(\theta) + \delta^3 z_3(\theta) + o(\delta^3),$$

with

$$\nu = 1 + \delta\nu_1 + \delta^2\nu_2 + \delta^3\nu_3 + (\delta^3),$$

we get

$$\begin{aligned} z(\theta) = {} & (1 + \delta\nu_1 + \delta^2\nu_2 + \delta^3\nu_3 + o(\delta^3))^2(\ddot{z}_0 + \delta\ddot{z}_1 + \delta^2\ddot{z}_2 + \delta^3\ddot{z}_3 + o(\delta^3)) \\ & + (z_0 + \delta z_1 + \delta^2 z_2 + \delta^3\ddot{z}_3 + o(\delta^3)) \\ & - \delta(1 - a(z_0 + \delta z_1 + \delta^2 z_2 + \delta^3 z_3 + o(\delta^3))^2 - b(\dot{z}_0 + \delta\dot{z}_1 + \delta^2\dot{z}_2 + \delta^3\dot{z}_3 + o(\delta^3))^2) \\ & (1 + \delta\nu_1 + \delta^2\nu_2 + \delta^3\nu_3 + o(\delta^3))(\dot{z}_0 + \delta\dot{z}_1 + \delta^2\dot{z}_2 + \delta^3\nu_3 + O(\delta^3)) = 0, \end{aligned}$$

and

$$\begin{aligned} & = \ddot{z}_0 + z_0 + \delta(\ddot{z}_1 + z_1 + 2\nu_1\ddot{z}_0 - \dot{z}_0 + az_0^2\dot{z}_0 + b\dot{z}_0^3) \\ & + \delta^2(\ddot{z}_2 + z_2 + 2\nu_1\ddot{z}_1 + (\nu_1^2 + 2\nu_2)\ddot{z}_0 + 2az_0az_1\dot{z}_0 \\ & - \dot{z}_1 - \nu_1\dot{z}_0 + az_0^2\dot{z}_1 + a\nu_1z_0^2\dot{z}_0 + 3b\dot{z}_0^2\dot{z}_1 + 3b\nu_1\dot{z}_0^3 \\ & + \delta^3(-\ddot{z}_1\nu_1^2 - 2\ddot{z}_1\nu_2 - 2\nu_1\nu_2\ddot{z}_0 - 3b\dot{z}_1^2z_0 - 3b\dot{z}_0^2\dot{z}_2 - b\nu_1^2\dot{z}_0^2 - 2b\nu2\dot{z}_0^3 - 6b\nu1\dot{z}_0^2\dot{z}_1 \\ -2b\dot{z}_0^3\nu_2 & - az_1^2\dot{z}_0 - 2az_0z_2\dot{z}_0 - -2az_0z_1\dot{z}_1 - a\dot{z}_2z_0^2 + \dot{z}_2 + \nu_1\dot{z}_1 + \nu2\dot{z}_0) = 0. \end{aligned}$$

So

$$\ddot{z}3 + z3 = -\ddot{z}1\nu1^2 - 2\nu_2\ddot{z}1^2 - 2\nu1\nu_2\ddot{z}0 - 3b\dot{z}1^2\dot{z}0 - 3bz0^2\dot{z}2 - b\nu1^2\dot{z}_0^2 \tag{3.23}$$

$$- 4b\nu_2\dot{z}0^3 - 6b\nu1\dot{z}0^2\dot{z}1 - az1^2\dot{z}0 - 2a\dot{z}0\dot{z}2\dot{z}_0,$$

we have z_0, z_1 and z_2 in (2.24), and here we calculate z_3.

$$\begin{aligned} \ddot{z}_3 + z_3 = {} & -2\nu_2\ddot{z}_1^2 - 3b\dot{z}_1^2\dot{z}_0 \\ & - 3bz_0^2\dot{z}_2 - -4b\nu_2\dot{z}_0^3 \\ & - az_1^2\dot{z}_0 - 2a\dot{z}_0\dot{z}_2\dot{z}_0 \end{aligned}$$

$$-2az_0z_1\dot{z}_1 - a\dot{z}_2z_0^2 + \dot{z}_2 + \nu_2\dot{z}_0 - 2\nu_3\ddot{z}_0. \tag{3.24}$$

Substituting Eq.(3.23) into Eq.(3.24) and simplifying the resulting expression gives

$$
\begin{aligned}
\ddot{z}_3 + z_3 = {} & [(\frac{1}{4(a+3b)^3\sqrt{a+3b}}(\frac{137}{48}a^2(a-b) + \frac{26}{7}b^2(a-b) + \frac{1069}{32}ab(a-b) + \frac{57}{8}a^3 \\
& + \frac{5}{8}ab^2 - 2a^2b + 4ab + \frac{1}{4}b^2) + \frac{(a-b)}{4(a+3b)^2\sqrt{a+3b}}(\frac{2}{6}a^2 + \frac{9}{2}ab + \frac{17}{32}a - \frac{35}{4}b) \\
& + \frac{3}{8}\frac{(a-b)}{(a+3b)\sqrt{a+3b}}]\sin\theta \\
& + [\frac{1}{16(a+3b)^3\sqrt{a+3b}}(\frac{19}{12}a^2(a-b) + \frac{7}{2}ba(a-b) + 36ab^2(a-b) + \frac{51}{4}a(a-b) \\
/ & + \frac{171}{8}b(a-b) + \frac{45}{3}ab - \frac{25}{6}a^2 - \frac{187}{8}a^3 - \frac{33}{2}ab^2 + 23ba^2 + \frac{1}{2}b^2) \\
& + (\frac{1}{16(a+3b)^2\sqrt{a+3b}}(-8ab + 4a^2 + \frac{135}{4}a + 3ab + 45b - 27b^2)]\sin 3\theta \\
& [\frac{1}{16(a+3b)^3\sqrt{a+3b}}(\frac{179}{12}a^2(a-b) + \frac{421}{8}8ab(a-b) + 399a(a-b) + \frac{50}{6}a^2 - 12a^3
\end{aligned}
$$

$$
- \frac{7}{2}ab^2 - 23a^2b + \frac{1}{2}b^2) + (\frac{(a-b)}{16(a+3b)^2\sqrt{a+3b}}\frac{85}{8}a + \frac{147}{16}b + \frac{135}{3}b^2 + 2a^2)]\sin 5\theta \tag{3.25}
$$

$$
+ [\frac{1}{16(a+3b)^3\sqrt{a+3b}}(\frac{15}{64}a^2(a-b) + \frac{15}{125}b(a-b) + \frac{65}{16}a(a-b) - \frac{75}{96}ab(a-b) \tag{3.26}
$$

$$
- \frac{13}{2}a^3 - 7ab^2 + \frac{25}{16}a^2 - 7a^2b + \frac{1}{2}b^2) + \frac{135}{48}\frac{(a-b)}{16(a+3b)^2\sqrt{a+3b}}]\sin 7\theta \tag{3.27}
$$

$$
+ [(\frac{-54a^2b - 54b^3 + 108ab^2}{16(a+3b)^2\sqrt{a+3b}})(\frac{7}{4}cos\theta + \frac{5}{4}cos3\theta + \frac{3}{4}cos5\theta + \frac{1}{4}cos7\theta) + \frac{2}{\sqrt{a+3b}}\nu_3 cos\theta. \tag{3.28}
$$

We proved the periodic of $z_3\left(\theta\right)$.

Proposition 3.3.1. *$z_3\left(\theta\right)$ has a periodic solution of the equation (3.25) if and only if*

$$
\begin{cases}
\nu_3(\theta) = \dfrac{378a^2b}{32(a+3b)^3} - \dfrac{378b^3}{32(a+3b)^3} + \dfrac{756ab^2}{32(a+3b)^3}, \\
[(75(a-b)(a+3b)^6(-5ab + 26a + 800b))/32 + \\
(1/384)(a+3b)^(7/2)(-96a^2 + 120ab - 810a + 648b^2 - 1080b - 3(a-b)(32a^2 + 170a + 720b^2 + 147b) \\
+(a-b)(32a^2 + 432ab + 51a - 840b)) \\
+(1/896)(a+3b)^(5/2)(4249a^3 + 840a^2b + 1120a^2(a-b) - 700a^2 - 2016ab^2(a-b) \\
+1456ab^2 - 39865ab(a-b) + 56ab - 45402a(a-b) + 832b^2(a-b) - 28b^2 - 1197b(a-b))]\pi = 0.
\end{cases}
\tag{3.29}
$$

proof: The periodicity condition gives

$$\int_{\theta}^{\theta+2\pi} \sin(\theta-\tau)G_3(\tau)d\tau = 0, \tag{3.30}$$

$$\Rightarrow \begin{cases} \int_0^{2\pi} \cos\theta G_3(\theta) = 0. \\ \int_0^{2\pi} \sin\theta G_3(\theta) = 0. \end{cases} \tag{3.31}$$

we rewrite(3.31) as

$$\begin{cases} \int_0^{2\pi} \cos\theta[((\frac{1}{4(a+3b)^3\sqrt{a+3b}}(\frac{137}{48}a^2(a-b)+\frac{26}{7}b^2(a-b)+\frac{1069}{32}ab(a-b)+\frac{57}{8}a^3 \\ +\frac{5}{8}ab^2-2a^2b+4ab+\frac{1}{4}b^2)+\frac{(a-b)}{4(a+3b)^2\sqrt{a+3b}}(\frac{2}{6}a^2+\frac{9}{2}ab \\ +\frac{17}{32}a-\frac{35}{4}b)+\frac{3}{8}\frac{(a-b)}{(a+3b)\sqrt{a+3b}})\sin\theta+(+\frac{2}{\sqrt{a+3b}}\nu_3+(\frac{-54a^2b-54b^3+108ab^2}{16(a+3b)^2\sqrt{a+3b}}))cos\theta]d\theta = 0, \\ \\ \int_0^{2\pi} \sin\theta[((\frac{1}{4(a+3b)^3\sqrt{a+3b}}(\frac{137}{48}a^2(a-b)+\frac{26}{7}b^2(a-b)+\frac{1069}{32}ab(a-b)+\frac{57}{8}a^3 \\ +\frac{5}{8}ab^2-2a^2b+4ab+\frac{1}{4}b^2)+\frac{(a-b)}{4(a+3b)^2\sqrt{a+3b}}(\frac{2}{6}a^2+\frac{9}{2}ab \\ +\frac{17}{32}a-\frac{35}{4}b)+\frac{3}{8}\frac{(a-b)}{(a+3b)\sqrt{a+3b}})\sin\theta+(+\frac{2}{\sqrt{a+3b}}\nu_3+(\frac{-54a^2b-54b^3+108ab^2}{16(a+3b)^2\sqrt{a+3b}}))cos\theta]d\theta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \int_0^{2\pi}[((\frac{1}{4(a+3b)^3\sqrt{a+3b}}(\frac{137}{48}a^2(a-b)+\frac{26}{7}b^2(a-b)+\frac{1069}{32}ab(a-b)+\frac{57}{8}a^3 \\ +\frac{5}{8}ab^2-2a^2b+4ab+\frac{1}{4}b^2)+\frac{(a-b)}{4(a+3b)^2\sqrt{a+3b}}(\frac{2}{6}a^2+\frac{9}{2}ab \\ +\frac{17}{32}a-\frac{35}{4}b)+\frac{3}{8}\frac{(a-b)}{(a+3b)\sqrt{a+3b}})\sin\theta\cos\theta d\theta+\int_0^{2\pi}+(+\frac{2}{\sqrt{a+3b}}\nu_3+(\frac{-54a^2b-54b^3+108ab^2}{16(a+3b)^2\sqrt{a+3b}}))cos^2\theta]d\theta = 0, \\ \\ \int_0^{2\pi}[((\frac{1}{4(a+3b)^3\sqrt{a+3b}}(\frac{137}{48}a^2(a-b)+\frac{26}{7}b^2(a-b)+\frac{1069}{32}ab(a-b)+\frac{57}{8}a^3 \\ +\frac{5}{8}ab^2-2a^2b+4ab+\frac{1}{4}b^2)+\frac{(a-b)}{4(a+3b)^2\sqrt{a+3b}}(\frac{2}{6}a^2+\frac{9}{2}ab \\ +\frac{17}{32}a-\frac{35}{4}b)+\frac{3}{8}\frac{(a-b)}{(a+3b)\sqrt{a+3b}})\sin^2\theta d\theta+\int_0^{2\pi}(+\frac{2}{\sqrt{a+3b}}\nu_3+(\frac{-54a^2b-54b^3+108ab^2}{16(a+3b)^2\sqrt{a+3b}}))\sin\theta\cos\theta]d\theta = 0. \end{cases}$$

$$\Longrightarrow \begin{cases} +\frac{2}{\sqrt{a+3b}}\nu_3+(\frac{-54a^2b-54b^3+108ab^2}{16(a+3b)^2\sqrt{a+3b}}) = 0, \end{cases} \tag{3.32}$$

we get

$$\Longrightarrow \begin{cases} \nu_3 = \frac{378a^2b}{32(a+3b)^3}-\frac{378b^3}{32(a+3b)^3}+\frac{756ab^2}{32(a+3b)^3}. \end{cases} \tag{3.33}$$

so

$$\sqrt{a+3b} \neq 0, \text{ and } A = 0,$$

$$\begin{cases} \nu_3(\theta) = \frac{378a^2b}{32(a+3b)^3} - \frac{378b^3}{32(a+3b)^3} + \frac{756ab^2}{32(a+3b)^3}. \end{cases} \tag{3.34}$$

We calculate $z_3(\theta)$

$$\ddot{z}_3 + z_3 = \underbrace{[(\frac{1}{4(a+3b)^3\sqrt{a+3b}}(\frac{137}{48}a^2(a-b) + \frac{26}{7}b^2(a-b) + \frac{1069}{32}ab(a-b) + \frac{57}{8}a^3}_{A}$$

$$+ \underbrace{\frac{5}{8}ab^2 - 2a^2b + 4ab + \frac{1}{4}b^2) + \frac{(a-b)}{4(a+3b)^2\sqrt{a+3b}}(\frac{2}{6}a^2 + \frac{9}{2}ab + \frac{17}{32}a - \frac{35}{4}b)}_{A}$$

$$+ \underbrace{\frac{3}{8}\frac{(a-b)}{(a+3b)\sqrt{a+3b}}]}_{A} \sin\theta$$

$$+ \underbrace{[\frac{1}{16(a+3b)^3\sqrt{a+3b}}(\frac{19}{12}a^2(a-b) + \frac{7}{2}ba(a-b) + 36ab^2(a-b) + \frac{51}{4}a(a-b)}_{B}$$

$$+ \underbrace{\frac{171}{8}b(a-b) + \frac{45}{3}ab - \frac{25}{6}a^2 - \frac{187}{8}a^3 - \frac{33}{2}ab^2 + 23ba^2 + \frac{1}{2}b^2)}_{B}$$

$$+ \underbrace{(\frac{1}{16(a+3b)^2\sqrt{a+3b}}(-8ab + 4a^2 + \frac{135}{4}a + 3ab + 45b - 27b^2)]}_{B} \sin 3\theta$$

$$\underbrace{[\frac{1}{16(a+3b)^3\sqrt{a+3b}}(\frac{179}{12}a^2(a-b) + \frac{421}{8}8ab(a-b) + 399a(a-b) + \frac{50}{6}a^2 - 12a^3}_{C}$$

$$- \underbrace{\frac{7}{2}ab^2 - 23a^2b + \frac{1}{2}b^2) + (\frac{(a-b)}{16(a+3b)^2\sqrt{a+3b}}\frac{85}{8}a + \frac{147}{16}b + \frac{135}{3}b^2 + 2a^2)]}_{C} \sin 5\theta$$

$$+ \underbrace{[\frac{1}{16(a+3b)^3\sqrt{a+3b}}(\frac{15}{64}a^2(a-b) + \frac{15}{125}b(a-b) + \frac{65}{16}a(a-b) - \frac{75}{96}ab(a-b)}_{D}$$

$$- \underbrace{\frac{13}{2}a^3 - 7ab^2 + \frac{25}{16}a^2 - 7a^2b + \frac{1}{2}b^2) + \frac{135}{48}\frac{(a-b)}{16(a+3b)^2\sqrt{a+3b}}]}_{D} \sin 7\theta$$

$$+ \underbrace{[(\frac{-54a^2b - 54b^3 + 108ab^2}{16(a+3b)^2\sqrt{a+3b}}}_{E})(+\frac{5}{4}cos3\theta + \frac{3}{4}cos5\theta + \frac{1}{4}cos7\theta).$$

We have

$$\ddot{z}_3 + z_3 = \underbrace{Ecos3\theta + Bsin3\theta}_{f1} + \underbrace{Ecos5\theta + Csin5\theta}_{f2} + \underbrace{Ecos7\theta + Dsin7\theta}_{f3},$$

so

$$z_{3_{pf_1}} = -(\frac{-54a^2b - 54b^3 + 108ab^2}{128(a+3b)^2\sqrt{a+3b}}cos3\theta - \frac{B}{8}sin3\theta.$$

$$z_{3_{pf_2}} = -(\frac{-54a^2b - 54b^3 + 108ab^2}{384(a+3b)^2\sqrt{a+3b}}cos5\theta - \frac{C}{24}sin5\theta.$$

$$z_{3_{pf_3}} = -(\frac{-54a^2b - 54b^3 + 108ab^2}{752(a+3b)^2\sqrt{a+3b}}cos7\theta - \frac{D}{47}sin7\theta.$$

So we have

$$Z_3 = c_1\cos(\theta) + c_2\sin(\theta) - \frac{E}{8}\cos 3\theta - \frac{B}{8}\sin 3\theta - (\frac{-54a^2b - 54b^3 + 108ab^2}{384(a+3b)^2\sqrt{a+3b}})\cos 5\theta$$
$$- \frac{C}{24}sin5\theta - (\frac{-54a^2b - 54b^3 + 108ab^2}{752(a+3b)^2\sqrt{a+3b}})\cos 7\theta - \frac{D}{47}sin7\theta,$$

and

$$\begin{aligned} Z_2(0) &= 0, \Rightarrow \\ c_1 &= 3E = 3(\frac{-54a^2b - 54b^3 + 108ab^2}{16(a+3b)^2\sqrt{a+3b}}) \\ &= (\frac{-162a^2b - 162b^3 + 324ab^2}{16(a+3b)^2\sqrt{a+3b}}) \\ \dot{z}_2(0) &= 0, \Rightarrow c_2 = B + C + D, \end{aligned}$$

$$\Rightarrow c_2 = B + C + D.$$

So

$$Z_3 = (\frac{-162a^2b - 162b^3 + 324ab^2}{16(a+3b)^2\sqrt{a+3b}})\cos(\theta) + (B+C+D)\sin(\theta) - \frac{E}{8}cos3\theta$$
$$- \frac{B}{8}sin3\theta - \frac{E}{24}cos5\theta - \frac{C}{24}sin5\theta - \frac{E}{47}cos7\theta - \frac{D}{47}sin7\theta.$$

The solution to (3.23) is

$$
\begin{aligned}
z(\theta,\delta) &= \frac{2}{\sqrt{a+3b}}\cos(\theta)+\delta(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(\theta)-\sin(3\theta))) \\
&+\delta^2((\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}}-\frac{3(a-b)}{32(a+3b)\sqrt{a+3a}}+\frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(\theta) \\
&+(\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}}+\frac{3(a-b)}{32(a+3b)\sqrt{a+3b}}-\frac{9b(a-b)}{8(a+3b)2\sqrt{a+3b}})\cos(3\theta) \\
&+(-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}}+\frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(5\theta))+\delta^3(\frac{-162a^2b-162b^3+324ab^2}{16(a+3b)^2\sqrt{a+3b}})\cos(\theta) \\
&+(B+C+D)\sin(\theta)-\frac{E}{8}cos3\theta-\frac{B}{8}sin3\theta-\frac{E}{24}cos5\theta-\frac{C}{24}sin5\theta-\frac{E}{47}cos7\theta \\
&-\frac{D}{47}sin7\theta)+o(\delta^3).
\end{aligned}
$$

3.4 Comparison of third approximate solutions

In this section we analyze the approximate solutions of (2.22) are obtained by the three numerical methods (SPM, LPM, AM) [12].

We compare $z_E(t,0)$ the exact solution, $z_P(t,\delta)$ the approximate solution with simple perturbation method, $z_L(\theta,\delta)$ the approximate solution with Lindestedt method and $z_A(t,\delta)$ the approximate solution with Averaging method of Ven Der Pol equation.

3.4.1 Comparison of approximate solutions to order δ^3

In this part, we compare approximate solutions to order δ^3. where $g(t)$ is the Taylor series expansion of $z_L((1+[\frac{a(a-b)}{8(a+3b)^2}-\frac{3(a-b)}{16(a+3b)}+\frac{9b(a-b)}{8(a+3b)^2}]\delta^2+[\frac{378a^2b}{32(a+3b)^3}-\frac{378b^3}{32(a+3b)^3}+\frac{756ab^2}{32(a+3b)^3}]\delta^3)t,\delta)$.In the order δ^3 in the neighbourhood of $\delta = 0$. We have

$$\begin{aligned}
g(t) = {} & \frac{2}{\sqrt{a+3b}}\cos(t) + \delta\Big(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(t)-\sin(3t))\Big) \\
& + \delta^2\Big(\Big(\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3a}} + \frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}}\Big)\cos(t) \\
& + \Big(\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)2\sqrt{a+3b}}\Big)\cos(3t) \\
& + \Big(-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}}\Big)\cos(5t)\Big) + \delta^3\Big(\frac{-162a^2b-162b^3+324ab^2}{16(a+3b)^2\sqrt{a+3b}}\Big)\cos(t) \\
& + (B+C+D)\sin(t) - \frac{E}{8}cos3t - \frac{B}{8}sin3t - \frac{E}{24}cos5t - \frac{C}{24}sin5t - \frac{E}{47}cos7t \\
& - \frac{D}{47}sin7t\Big).
\end{aligned}$$

And $h(t)$ is the Taylor series expansion of $z_A(t,\delta)$, in the order δ^3 in the neighbourhood of $\delta = 0$.

$h(t) = \left(A + (\frac{A}{2} - \frac{A^3(a+3b)}{8})\delta t + \delta^2 t^2\left((\frac{3A^5}{128}(a+3b)^2 + \frac{A}{8} - \frac{A^3}{8}(a+3b)\right) + \delta^3(\frac{A^3}{32} - \frac{15A^7}{1024}t^3)\right)\cos(t).$

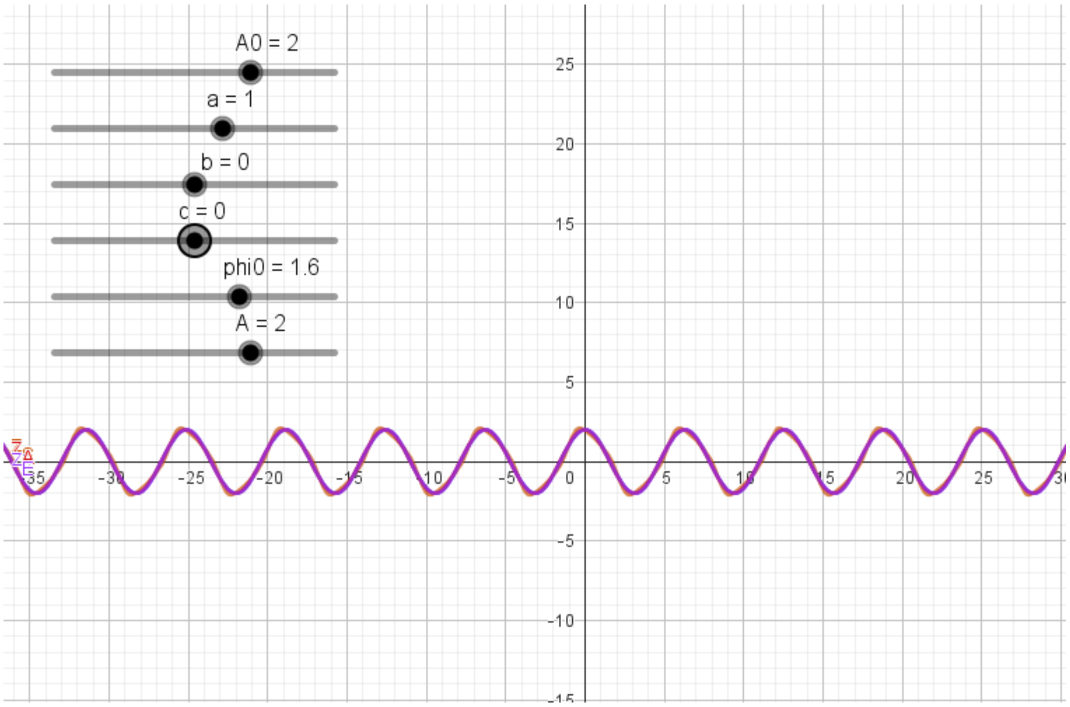

Figure 3.1: Comparison of the SPM solution, LPM solution and AM solution for $\delta = 0$ and $A = 2$, to order δ^3.

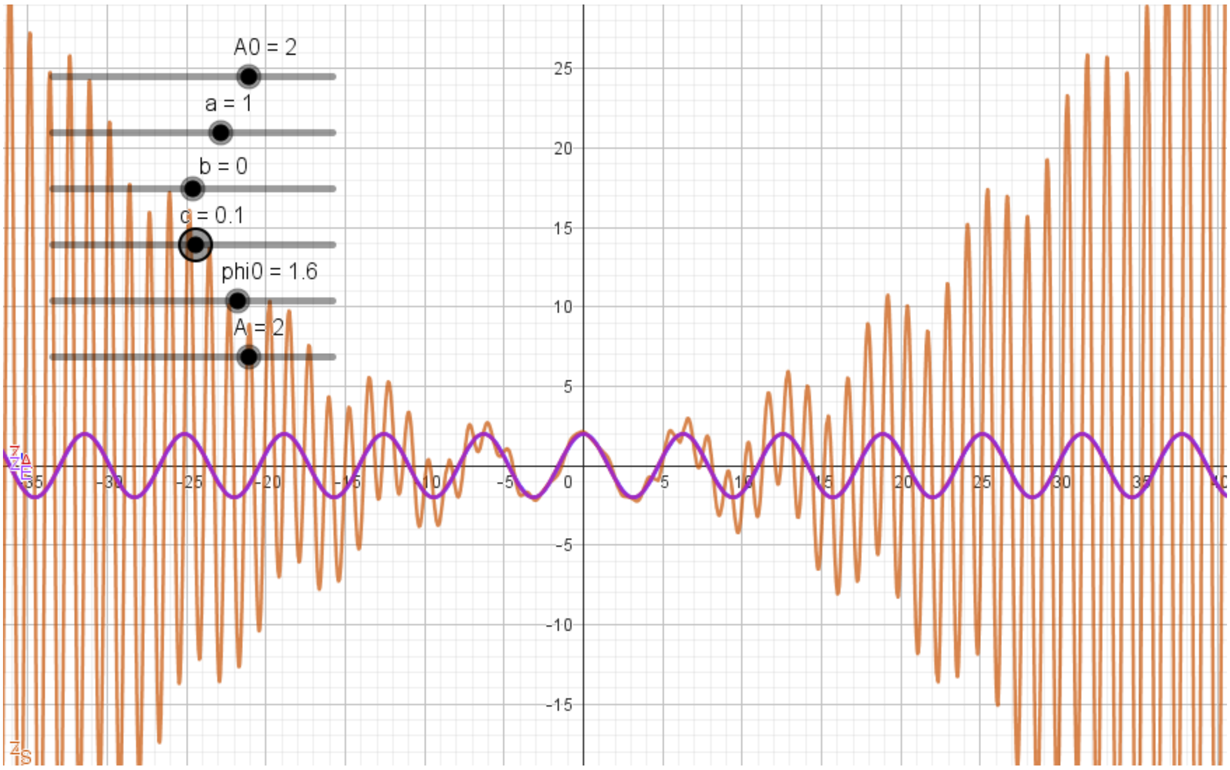

Figure 3.2: Comparison of the SPM solution, LPM solution and AM solution for $\delta = 0.1$ and $A = 2$, to order δ^3.

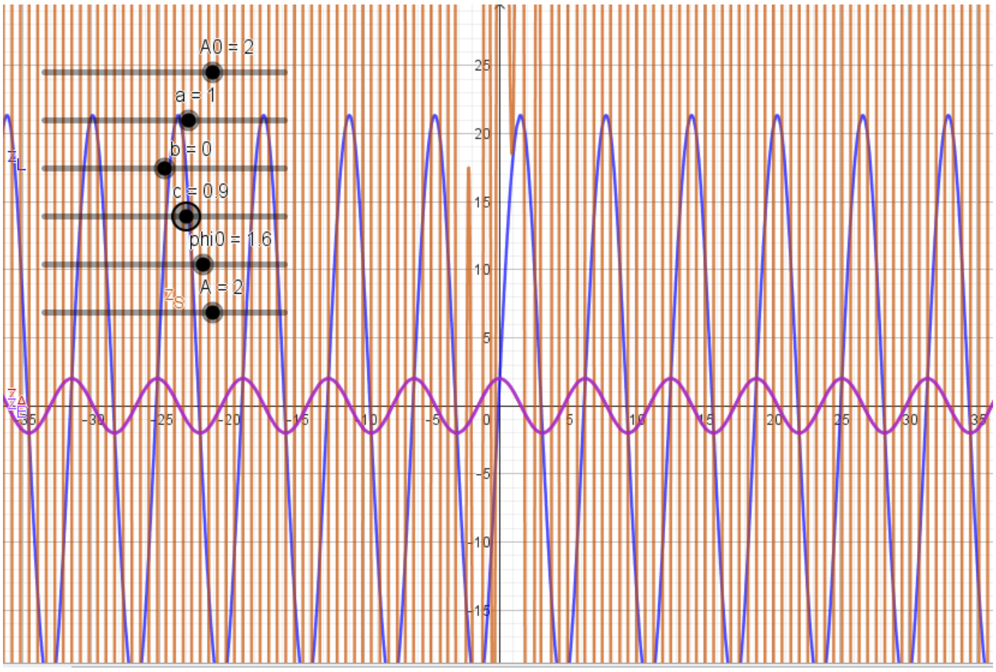

Figure 3.3: Comparison of the SPM solution, LPM solution, AM and RGM solution for $\delta = 0.9$, and $A = 2$, to order δ^3.

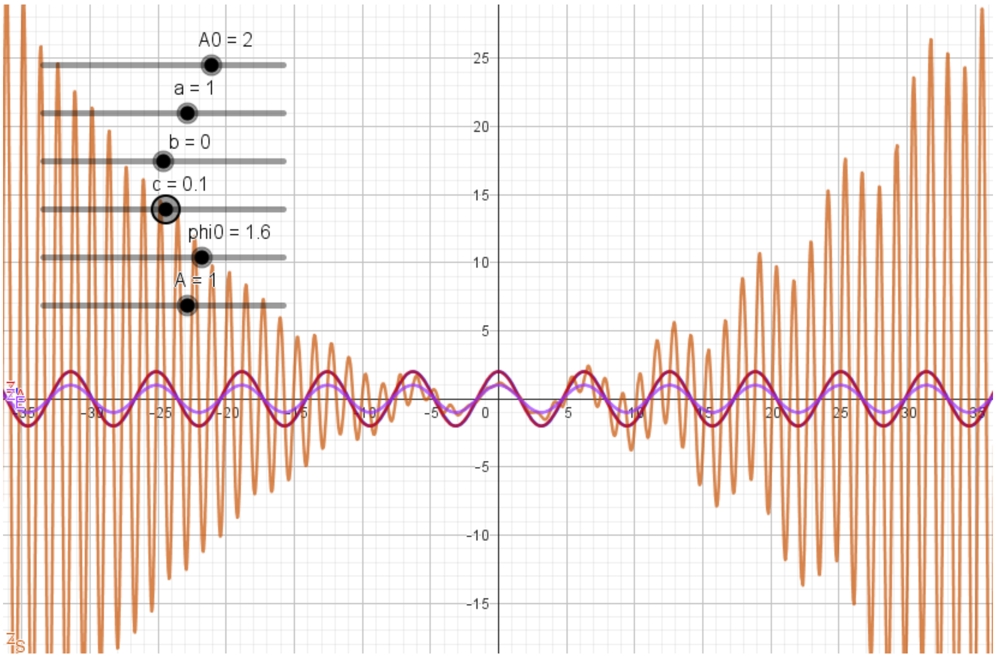

Figure 3.4: Comparison of the SPM solution, LPM solution, AM and solution RGM solution for $\delta = 0.1$, and $A = 1$, to order δ^3.

3.4.2 Discussion of results

When $A = 2$ and $\delta = 0$, in Fig.3.1, shows analytic approximate solutions. Based on the obtained results, we found that the solutions z_L, z_A and z_S, are indetical to the exact solution z_E.
When $A = 2$ and $\delta = 0.1$, in Fig.3.2, shows analytic approximate solutions. Based on the obtained results, we found that in the domain $t \in [-5, 5]$ the solutions z_L, z_A and z_S and $h(t)$, are indetical to the exact solution z_E, but when $t \longrightarrow \infty$ the solutions z_S, are spaced on the exact solution z_E. Also, we note that the solutions $z_L$and $h(t)$ it will be indetical to the exact solution z_E.
When $A = 2$ and $\delta = 0.9$, in Fig.3.3, shows analytic approximate solutions. Based on the obtained results, we found that the solutions z_A, is indetical to the exact solution z_E, but z_L and z_S, are spaced on the exact solution z_E, we note that the solutions $h(t)$ are indetical to the exact solution z_E.
When $A = 1$ and $\delta = 0.1$, in Fig.3.4, shows analytic approximate solutions. Based on the obtained results, we found that in the domain $t \in [-5, 5]$, the solutions z_S, are convergent to the exact solution z_E, and z_L, z_A, are indetical to the exact solution z_E, but when $t \longrightarrow \infty$ the solutions z_S spaced on the exact solution z_E, we note that the solutions $h(t)$ convergent to the exact solution z_E and z_L, are indetical the exact solution z_E.

3.5 Conclusion

Chapter Three presents a comparative analysis of the three previously applied approximation methods for studying third-order approximate solutions, where accuracy increases. The chapter evaluates the accuracy, stability, and mathematical complexity of each method. The average method (AM) proves to be the most stable for long-term behavior, while the Lindstedt-Poincaré method (LPM) achieves high accuracy at the expense of more complex calculations. The simple perturbation method (SPM), while easy to implement, is limited in scope. The comparison highlights that the choice of method depends on the specific characteristics of the problem and its objective.

General Conclusion

This note explores approximate analytical methods for solving second-order weakly nonlinear differential equations, focusing on the van der Poel equation as a case study. It demonstrates the application and effectiveness of SPM, LPM, and AM algorithms in extracting approximate periodic solutions. The results emphasize the importance of approximation techniques in studying complex systems where exact solutions are difficult to achieve. This work encourages further exploration of advanced methods and their application to broader nonlinear phenomena in science and engineering.

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