Mathematical analysis of a generic reaction diffusion model applied in medical imaging

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Abstract

This paper provides the existence of a global solution to a generic reaction diffusion system. We show the existence of a global weak solution to the considered system in the case of quasi-positivity and a triangular structure condition on the nonlinearities. Application examples of our results are presented on new models of biology.

Keywords : reaction diffusion systems, mathematical biology model, parabolic system, global existence, Schauder fixed point.

MSC 2010 : 35J55 ; 35J60 ; 35J70.

1 Introduction

The modeling and the mathematical analysis of reaction-diffusion models play an important role in information processing, and has been the subject of in-depth studies of several mathematicians in recent years, as they appear in the modeling of a large variety of phenomena in biology and in chemistry such as gas dynamics, fusion processes, cellular processes, disease propagation, industrial processes , catalytic transport of contaminants in the environment, population dynamics, flame spread and others.

In this work, we tackle the problem of the global existence of solutions of a reaction-diffusion system applied to a medical imaging model. The model considered is the following

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\ \left(A\left|\nabla u_\sigma\right|\nabla u\right) = f\left(t,x,u,v\right), & \text{in } Q_T \\ \dfrac{\partial v}{\partial t} - \text{div}\ \left(B\left|\nabla v_\sigma\right|\nabla v\right) = g\left(t,x,u,v\right) & \text{, in } Q_T \\ \dfrac{\partial u}{\partial v} = \dfrac{\partial v}{\partial v} = 0 & \text{, on } \Sigma_T \\ u\left(0,x\right) = u_0\left(x\right) & \text{, in } \Omega \\ v\left(0,x\right) = v_0\left(x\right) & \text{, in } \Omega \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $T \in (0,\infty[$, $Q_T =]0,T[\times \Omega$ and $\Sigma_T =]0,T[\times \partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω. v is the outward normal to the domain and ∂_v is the normal derivative.

Let $\sigma > 0$, ∇u_σ and ∇v_σ are the regularizations by convolution of ∇u and ∇v respectively. They are defined as

$$\nabla u_\sigma = \nabla(G_\sigma * u) \quad \text{and} \quad \nabla v_\sigma = \nabla(G_\sigma * v)$$

where G_σ is the gaussian function. The anisotropic diffusivities A and B are smooth nonincreasing functions, such that $A(0) = B\left(0\right) = 1$ and $\lim\limits_{s\to\infty} A(s) = \lim\limits_{s\to\infty} B(s) = 0$.

The nonlinear functions $f,\ g : Q_T \times \mathbb{R}^2 \to \mathbb{R}$ are measurable and $f(t,x,.),\ g(t,x,.) : \mathbb{R}^2 \to \mathbb{R}$ are continuous. In addition the nonlinearities satisfy the positivity property

$$f(t,x,0,s) \geq 0 \quad \forall s \geq 0 \quad \text{and} \quad g(t,x,r,0) \geq 0 \quad \forall r \geq 0 \tag{2}$$

and a triangular structure

$$(f+g)(t,x,r,s) \leq L_1(r+s+1) \quad \text{and} \quad g(t,x,r,s) \leq L_2(r+s+1) \tag{3}$$

where L_1 and L_2 are positive constant. Furthermore,

$$\sup_{|r|+|s|\leq R} \left(|f(t,x,r,s)| + |g(t,x,r,s)|\right) \in L^1(Q_T) \tag{4}$$

for $R > 0$. However there is no further assumption on their growth. The initial conditions u_0, v_0 are only assumed to be square integrable.

Before tackling the main problem, we clearly state our definition of weak solution to the reaction-diffusion system.

Definition 1 *We call* (u,v) *a weak solution of the system* (1)*, if*

$u,\ v \in L^2\left(0,T;H^1\left(\Omega\right)\right) \cap C\left(\left[0,T\right];L^2\left(\Omega\right)\right)$

$\forall \phi, \psi \in C^1(Q_T)$ such that $\phi(.,T) = 0$ and $\psi(.,T) = 0$, we have

$$\int_{Q_T} -u\frac{\partial \phi}{\partial t} + A(|\nabla u_\sigma|)\nabla u \nabla \phi = \int_{Q_T} f(t,x,u,v)\phi + \int_\Omega u_0\phi(0,.)$$
$$\int_{Q_T} -v\frac{\partial \psi}{\partial t} + B(|\nabla v_\sigma|)\nabla v \nabla \psi = \int_{Q_T} g(t,x,u,v)\psi + \int_\Omega v_0\psi(0,.)$$

where $f(t,x,u,v)$, $g(t,x,u,v) \in L^1(Q_T)$.

Now, we enunciate the main result of the paper.

Theorem 2 *Under the assumptions $(2)-(4)$ and for a continuous functions A and B described above. The reaction diffusion system (1) admits a global weak solution (u,v), within the meaning of definition 1, for all u_0, $v_0 \in L^2(\Omega)$ such that u_0, v_0 are positive.*

To prove our main result, we will proceed by steps. We truncate the problem and show that the approximate problem admits weak solutions using a Schauder fixed point. Afterward, we will provide some essential compactness and equi-integrablity results in order to pass to the limit and rigourously demonstrate the existence of global weak solution to the considered model.

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