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Invariant Curves and Limit Cycles in Planar Vector Fields

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Résumé

L'objectif explicite de cette thèse est de synthétiser le raffinement algébrique avec la monotonie des champs de vecteurs rotatifs (Duff, 1953 ; Perko, 1991), les quantités de foyer de Bautin (Bautin, 1952) et la théorie de l'indice de Poincaré. Cette synthèse multicouche est stratégiquement conçue pour appliquer le suivi des bifurcations globales et les contraintes de stabilité locale, servant ultimement à démontrer rigoureusement le théorème de non-coexistence en prouvant que des cycles limites algébriques et non algébriques simultanés créent des contradictions topologiques. Mots-clés : Cycle limite – courbes algébriques invariantes – 16e problème de Hilbert – systèmes polynomiaux planaires.

Mots clés : Cycle limite – courbes algébriques invariantes – 16e problème de Hilbert – systèmes polynomiaux planaires.

Abstract

The explicit objective of this thesis is to synthesize algebraic refinement with rotated vector field monotonicity (Duff, 1953; Perko, 1991), Bautin focus quantities (Bautin, 1952) and Poincaré index theory. This multi-layered synthesis is strategically designed to enforce global bifurcation tracking and local stability constraints, ultimately serving to rigorously demonstrate the non-coexistence theorem by proving that simultaneous algebraic and non-algebraic limit cycles create topological contradictions.

Keywords: Limit cycle – invariant algebraic curves – 16th Hilbert's problem – planar polynomial systems.

ملخص

الهدف الصريح من هذه الأطروحة هو دمج الصقل الجبري مع رتابة حقول المتجهات الدوارة (Duff, 1953; Perko, 1991)، ومقادير بورتين البؤرية (Bautin, 1952)، ونظرية مؤشر بوانكاريه. تم تصميم هذا الدمج متعدد الطبقات استراتيجياً لفرض تتبع التشعب العالمي وقيود الاستقرار المحلي، مما يخدم في النهاية إثبات نظرية عدم التعايش بشكل صارم من خلال البرهنة على أن الدورات الحدية الجبرية وغير الجبرية المتزامنة تخلق تناقضات طوبولوجية.

الكلمات المفتاحية : الكلمات المفتاحية: الدورة الحدية – المنحنيات الجبرية الثابتة – مسألة هيلبرت السادسة عشرة – الأنظمة متعددة الحدود المستوية

إهداء الطالبة: منى علوان

بسم الله الرحمن الرحيم

الحمد لله الذي بنعمته تتمّ الصالحات، والذي وفقني لإتمام هذا العمل، وما كان لي أن أصل إلى هذه المرحلة إلا بفضله سبحانه.

أهدي ثمرة جهدي وتعب سنوات الدراسة إلى أغلى الناس على قلبي، إلى أبي العزيز وأمي الغالية، حفظهما الله وأطال في عمرهما، وجزاهما عني خير الجزاء.

إلى معلمي في المرحلة الابتدائية، أول من وجّهني إلى طريق العلم، وغرس في نفسي حبّ المعرفة والاجتهاد، فكان له الأثر الجميل الذي لا يُنسى.

إلى أستاذتي المشرفة، التي لم تبخل عليّ بعلمها وتوجيهها ونصائحها القيّمة، وكانت خير مرشدة وداعمة طوال إنجاز هذه المذكرة.

إلى أخواتي وصديقاتي اللواتي كنّ دائمًا بقربي بدعمهنّ ومحبتهم الصادقة.

إلى أخوتي و كل من ساندني ووقف إلى جانبي، من قريب أو بعيد، ولو بكلمة طيبة أو دعوة صادقة.

أهدي هذا العمل المتواضع، راجيةً من الله أن يجعله نافعًا ومباركًا.

والحمد لله رب العالمين.

إهداء الطالبة: حواء خلايفة

الحمد لله وكفى و الصلاة و السلام على الحبيب المصطفى الحمد لله الذي يسر لي الدرب و سهل لي الصعب و أنار بالعلم القلب حمدا كثيرا مباركا وبعد : بعد مسيرة دراسية حملت في طياتها الكثير من الصعوبات .. اليوم أقطف ثمارها و أهديها

إلى رمز العطاء والتضحية مثلي الأعلى أبي الغالي إلى من تستقبلني بإبتسامة و تودعني بدعاء أمي الحبيبة والداي الكريمين مهما سطرت من كلمات و مهما بلغت من عبارات لن أستطيع أن أوفي حقكما العظيم علي، شكرا لكما على كل تضحية وكل دعم وكل حب أسأل الله أن يحفظكما لي سندا و عونا مدى الحياة

إلى من احتضن حلمي و روحي إلى من جاد علي بوقته وأكرمني بفضله إقراراً مني بفضله و إعترافا بحقه حيث كان خير عونا لي و سندا معلمي وملهمي زوجي الغالي

إلى قطعتي من الجنة، و ابتسامة عمري المضيئة؛ طفلي الحبيب و قرّة عيني. غداً ستكبر و تقرأ هذه السطور، لتعلم أنك كنت و ما زلت الأجمل في حكاياتي. أهدي إليك نبض قلبي و كل دعواتي، سائلاً المولى أن ينير دربك و يحقق أحلامك. دمت لي فخراً، و دمت لروحي ضياءً يملأ الكون بهجةً و سروراً.

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لكل من كان عونا و سندا في هذا الطريق للصديقات ورفيقات السنين لأصحاب الشدائد و الأزمات

فالحمد لله على ما وهبني و ان يجعلني مباركة و ان يعينني أينما كنت فمن قال انا لها نالها و أنا لها و إن أبت رغما عنها أتيت بها فالحمد لله شكرا وحبا و امتنانا على البدء والختام

وآخر دعوانا ان الحمد لله رب العالمين

شكر وتقدير

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وانطلاقًا من باب «من لم يشكر الناس لم يشكر الله»، نتقدم بخالص الشكر والتقدير للأستاذة المشرفة: **نوار عزيزة سعاد**، على إرشاداتها وتوجيهاتها التي لم تبخل بها علينا يومًا.

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والحمد لله رب العالمين.

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Notations	**Meaning in Context**
$\dot{x} = P(x,y),\ \dot{y} = Q(x,y)$	Planar polynomial differential system of degree n
$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$	Associated vector field operator
n (or m)	Degree of the polynomial vector field
$f(x,y) = 0$	Irreducible invariant algebraic curve
$k(x,y)$	Cofactor polynomial satisfying $Xf = k(x,y)f$
$\mathrm{div}(X)$	Divergence of the vector field : $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$
$H(x,y)$	First integral of the system ($XH = 0$)
$R(x,y)$	Integrating factor (satisfies $XR + R\,\mathrm{div}(X) = 0$)
$E = \exp(g/h)$	Exponential factor with polynomial cofactor
$N_{\alpha,g}(x,y)$	Auxiliary Dulac polynomial : $\alpha kg + P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y}$
Δ	Discriminant of the compatibility quadratic
l_1	first Lyapunov constant
V_1	The first Lyapunov constant $V_1 \neq 0$
$H(n)$	Hilbert number : maximum number of limit cycles for degree n systems
Π	Poincaré (return) map across a transverse section
$h = \oint_\Gamma k\,dt$	Characteristic exponent (determines stability of a limit cycle Γ)
v_1, v_2	Bautin focus quantities (used in Hopf bifurcation analysis)
$\mathcal{P}$	Full parameter space of the system coefficients
$\mathcal{R}_{\geq 0},\ \mathcal{R}_{<0}$	Parameter regions where $\Delta \geq 0$ or $\Delta < 0$
$\mathcal{I}$	Integrable subfamilies (Darboux integrable, no isolated limit cycles)

General Introduction

The qualitative theory of planar polynomial differential systems occupies a central position at the crossroads of algebraic geometry, dynamical systems theory, and mathematical analysis. Among the most celebrated open problems driving this field is the second part of Hilbert's Sixteenth Problem, formulated in 1900, which asks for the maximum number and relative configuration of limit cycles admitted by polynomial vector fields of a given degree n. Despite more than a century of sustained research, the problem remains unresolved even for the simplest non-trivial case of quadratic systems ($n = 2$), where it is known only that $H(2) \geq 4$, while a uniform upper bound continues to elude proof. The difficulty stems from the intricate interplay of bifurcation phenomena, topological constraints, and algebraic integrability that governs the phase portrait of such systems.

The present thesis addresses this challenge by developing a unified algebraic-dynamical framework grounded in the theory of invariant algebraic curves, Darboux integrability, and generalized Bendixson–Dulac criteria, with the aim of establishing rigorous existence, non-existence, and uniqueness results for limit cycles in polynomial systems. A distinctive feature of the methodology is the use of the generalized Bendixson–Dulac criterion of Gasull and Giacomini (2023), which links the cofactor of an invariant curve to the divergence of the vector field through the auxiliary polynomial $N_{\alpha,g}$, enabling explicit algebraic conditions on system parameters that govern the behavior of limit cycles relative to invariant curves.

The study system examined in Chapter 3, which constitutes the core original contribution of this work, is a planar quadratic differential system admitting an irreducible invariant algebraic curve of degree three (an invariant cubic curve) $f(x, y) = 0$, with an associated linear cofactor $k(x, y)$ of degree at most one. Explicitly, the system takes the form :

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$ with $\max(\deg P, \deg Q) = 2$, and $f(x, y) = 0$ satisfies the invariance condition $Xf = k(x, y)\, f(x, y)$. Within this class, the thesis investigates the coexistence problem : whether an algebraic limit cycle lying on $f = 0$ can simultaneously exist with a non-algebraic limit cycle in the complementary region $\mathbb{R}^2 \setminus \{f = 0\}$.

The work is organized into four chapters that build systematically from foundational theory to original classification and applied validation.

First , the Chapter One develops the theoretical backbone of the research. It introduces planar polynomial vector fields, first integrals, integrating factors, invariant algebraic curves, and exponential factors, culminating in the Darboux Integrability Theorem. The chapter reviews limit cycle theory in depth, covering stability classification via the Poincaré map, the Poincaré–Bendixson and Bendixson–Dulac theorems, inverse integrating factors, and the resolution of the finiteness question independently established by Écalle (1992) and Il'yashenko (1991). By carefully distinguishing between the finiteness of limit cycles for a fixed system and the still-open question of a uniform bound depending only on the degree, Chapter 1 lays the mathematical foundation for the algebraic methods employed throughout the thesis.

After the Chapter Two applies the theoretical machinery to quadratic and cubic polynomial systems. It presents the complete affine canonical classification of quadratic systems possessing an irreducible invariant cubic curve, identifying fourteen canonical forms and demonstrating that Hamiltonian and integrable types (including Riccati-type families) are immediately excluded from hosting limit cycles. The chapter rigorously proves the non-existence of cubic algebraic limit cycles in quadratic systems, identifies the canonical families of types v^+ and (viii) as the only ones capable of supporting external limit cycles, and derives finiteness bounds for systems with multiple invariant curves using the extended auxiliary polynomial method.

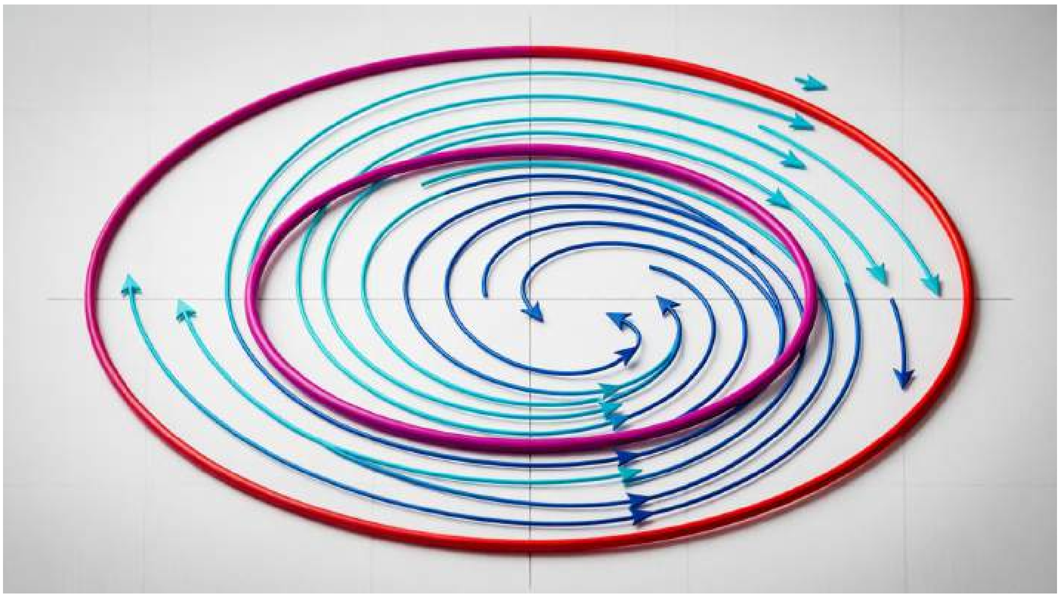
However the Chapter Three constitutes the principal original contribution of the thesis and represents the preprint version of a future publication project. It confronts a persistent open problem : whether an algebraic limit cycle on an invariant cubic $f = 0$ can coexist with a non-algebraic limit cycle in the same quadratic system. Centered on a discriminant condition Δ derived from the compatibility quadratic for α in the generalized Bendixson–Dulac criterion, the chapter partitions the parameter space into three disjoint regions. In the regime $\Delta \geq 0$, the sign-constant Dulac function $N_{\alpha,1}$ directly precludes all external periodic orbits. In the complementary regime $\Delta < 0$, where classical sign-constant Dulac constructions fail, the analysis combines refined quadratic Dulac barriers, Poincaré index compatibility constraints, rotated vector field bifurcation theory, and Bautin focus quantities to establish that coexistence is impossible. These tools are synthesized in Theorem 3.6.1 (the Non-Coexistence Theorem), which rigorously proves that no parameter configuration in the generic quadratic–cubic class admits simultaneously an algebraic and a non-algebraic limit cycle, thereby establishing a sharp upper bound of one limit cycle for this entire class and tightening the known bounds on $H(2)$. The methodology developed therein is explicitly structured for generalization to higher-degree systems and multiple invariant curve configurations, making it a foundational template intended for dissemination to the broader dynamical systems community.

Finally the Chapter Four , which serves as the applied companion to Chapter 3 and likewise forms part of the aforementioned future publication project, validates the framework through two algorithmically reproducible examples. The Van der Pol oscillator—governed by $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$ with $\mu > 0$, or equivalently $\dot{x} = y$, $\dot{y} = \mu(1 - x^2)y - x$—illustrates how the generalized Dulac ansatz with a radial auxiliary function yields regional exclusion and uniqueness even in the absence of low-degree invariant curves. Qin's quadratic system, $\dot{x} = -y(ax + by + c) - (x^2 + y^2 - 1)$, $\dot{y} = x(ax + by + c)$, which admits the invariant algebraic curve $f(x, y) = x^2 + y^2 - 1 = 0$ with cofactor $k(x, y) = -2x$ and discriminant $\Delta = 4a^2(c^2 + 4(b+1) - a^2)$, demonstrates the complete pipeline : discriminant classification, characteristic exponent stability analysis via $h = \oint_\Gamma k\,dt$, and rigorous non-coexistence verification through the synthesis of Theorem 3.6.1.

Together, these four chapters demonstrate how invariant algebraic curves, far from being merely particular solutions, function as central analytical tools that impose rigid geometric constraints on the global phase portrait, decisively narrowing the scope of Hilbert's Sixteenth Problem for quadratic systems with invariant cubic curves.

Chapitre 1

Planar polynomial vector fields darboux integrability and limit cycle theory



1.1 Introduction

This chapter is inspired by Hilbert's Sixteenth Problem, a famous open question in mathematics that asks for the maximum number of limit cycles a planar polynomial system can have, and how they are arranged. Although the question sounds simple, it reveals a key challenge : limit cycles describe how a system evolves over time, yet their existence, stability, and total number are strictly controlled by the algebraic form of the equations. Traditional methods—such as local analysis, small perturbations, or numerical simulations—often fall short when answering broader questions, like whether multiple cycles can appear together or how they change when parameters vary. Instead, the field has advanced by recognizing that polynomial systems carry a hidden mathematical structure. This structure, made of invariant curves, exponential factors, and Darboux integrals, acts like a set of boundaries and rules that determine whether isolated cycles can form at all.

1.2 Planar polynomial vector fields[1]

Let $\mathbb{F}$ denote either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. By an $\mathbb{F}$-polynomial system of degree m, we mean the differential system :

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where $P, Q \in \mathbb{F}[x, y]$ are polynomials of degree at most m. The associated vector field is defined as the differential operator :

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

We denote by $\mathbb{F}[x, y]$ the ring of polynomials in x and y with coefficients in $\mathbb{F}$.

1.2.1 First integral[1]

Definition 1.2.1 (First Integral)**.** [1] The system (1.1) is said to be integrable on an open subset $U \subset \mathbb{F}^2$ if there exists a non-constant analytic function $H : U \to \mathbb{F}$ such that H is constant along every solution curve $(x(t), y(t))$ defined on U. That is,

$$H(x(t), y(t)) = \text{constant}.$$

Clearly, H is a first integral of system (1.1) on U if and only if $XH = 0$ on U. The existence of a first integral allows the reduction of the system's dimensionality, effectively solving the dynamics.

1.2.2 Integrating factors[1]

Integrating factors constitute a fundamental tool in the study of integrability for planar polynomial differential systems. Their existence allows the transformation of the vector field into a conservative one, thus leading to the construction of first integrals.

Definition 1.2.2 (Integrating Factor)**.** [1] A non-zero analytic function $R : U \subset \mathbb{F}^2 \to \mathbb{F}$ is called an integrating factor of system (1.1) on U if the vector field RX is conservative in U. Equivalently, R is an integrating factor if there exists a function $H : U \to \mathbb{F}$ such that :

$$\frac{\partial H}{\partial x} = RP, \quad \frac{\partial H}{\partial y} = RQ.$$

The above condition is equivalent to the exactness condition :

$$\frac{\partial (RP)}{\partial y} = \frac{\partial (RQ)}{\partial x}.$$

In terms of the divergence operator, $\operatorname{div}(X) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$, the function R is an integrating factor if and only if it satisfies the linear partial differential equation :

$$X(R) + R\operatorname{div}(X) = 0.[1] \tag{1.2}$$

If an integrating factor R is known, then the function H obtained by integration is a first integral of system (1.1). Therefore, determining the integrating factors is equivalent to solving the integrability problem.

1.2.3 Invariant algebraic curves[1]

Let $f \in \mathbb{C}[x, y]$. The algebraic curve defined by $f(x, y) = 0$ is called an invariant algebraic curve of system (1.1) if there exists a polynomial $K \in \mathbb{C}[x, y]$ such that :

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.[1] \tag{1.3}$$

The polynomial K is called the cofactor of the invariant algebraic curve. Since the polynomial system has degree m, every cofactor has degree at most $m - 1$.

Geometric Interpretation : From relation (1.3), at every non-singular point of the curve

$f = 0$, the gradient $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is orthogonal to the vector field $X = (P, Q)$. Therefore, the vector field is tangent to the curve at such points. At singular points of the curve, the vector field vanishes, and thus the entire curve consists of trajectories of X. If a trajectory starts on the curve $f = 0$, it remains on the curve for all time. We allow invariant algebraic curves to be complex even when the polynomial system is real, since the existence of complex invariant curves may imply the existence of real first integrals.

Proposition 1.2.3. *[1] For a real polynomial system (1.1), $f = 0$ is an invariant algebraic curve with cofactor K if and only if $\bar{f} = 0$ is an invariant algebraic curve with cofactor $\bar{K}$.*

Lemma 1.2.4. *[1] Let $f, g \in \mathbb{C}[x, y]$ be relatively prime. Then, the curve $fg = 0$ is an invariant algebraic curve with cofactor K if and only if $f = 0$ and $g = 0$ are invariant algebraic curves with cofactors K_f and K_g, respectively. Moreover, $K = fK_g + gK_f$.*

Proposition 1.2.5. *[1] If a real or complex polynomial system (1.1), defined in $\mathbb{C}^2$, has an integrating factor of the form :*

$$R = f_1^{\lambda_1} \cdots f_p^{\lambda_p},$$

where f_i are non-constant polynomials in $\mathbb{C}[x, y]$, $\lambda_i \in \mathbb{C} \setminus \{0\}$, and f_i and f_j are relatively prime for $i \neq j$, then the curves $f_i = 0$ are invariant algebraic curves of the system.

1.2.4 Exponential factors[1]

Exponential factors arise naturally when considering limits of invariant algebraic curves. Assume that two invariant algebraic curves $h = 0$ and $h + \varepsilon g = 0$ exist with cofactors K_h and $K_{h+\varepsilon g}$, where ε belongs to a sufficiently small neighborhood of 0. Using the relations $Xh = K_h h$ and $X(h + \varepsilon g) = K_{h+\varepsilon g}(h + \varepsilon g)$, and expanding in power series, one obtains in the limit the function $\exp(g/h)$.

Definition 1.2.6 (Exponential factor)**.** [1] Let $h, g \in \mathbb{C}[x, y]$ be relatively prime. The function :

$$E = \exp\left(\frac{g}{h}\right)$$

is called an exponential factor of system (1.1) if it satisfies :

$$X(E) = LE,$$

for some polynomial L with degree at most $m - 1$. The polynomial L is called its cofactor.

Proposition 1.2.7. *[1] If $E = \exp(g/h)$ is an exponential factor of system (1.1), then*

$h = 0$ is an invariant algebraic curve and g satisfies :

$$Xg = gK + hL.$$

1.2.5 The method of darboux[2]

The Method of Darboux provides a procedure to construct first integrals or integrating factors using invariant algebraic curves and exponential factors.This theory bridges algebraic geometry and dynamical systems. Let $S(x, y) = \sum_{i+j=0}^{m-1} a_{ij}x^i y^j$ be a polynomial of degree $m - 1$ with coefficients in $\mathbb{F}$.We write $S \in \mathbb{F}[x, y]_{m-1}$ and identify this linear space with $\mathbb{F}^{m(m+1)/2}$. We say that points $(x_k, y_k) \in \mathbb{F}^2$, $k = 1, \ldots, r$, are independent with respect to $\mathbb{F}[x, y]_{m-1}$ if the intersection of the hyperplanes defined by $S(x_k, y_k) = 0$ in $\mathbb{F}^{m(m+1)/2}$ is a linear subspace of dimension $m(m + 1)/2 - r \geq 0$.

Theorem 1.2.8 (Darboux integrability theorem)**.** *[2] Assume that system (1.1) has :*

- *p invariant algebraic curves $f_i = 0$ with cofactors K_i,*
- *q exponential factors $\exp(g_j/h_j)$ with cofactors L_j,*
- *r independent singular points (x_k, y_k) such that $f_i(x_k, y_k) = 0$ and $h_j(x_k, y_k) = 0$.*

(i) If there exist constants $\lambda_i, \mu_j \in \mathbb{C}$, not all zero, such that :

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

then the function :

$$H = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\mu_1 \frac{g_1}{h_1}\right) \cdots \exp\left(\mu_q \frac{g_q}{h_q}\right)$$

is a first integral of system (1.1).

(ii) If $p + q + r = \frac{m(m+1)}{2} + 2$, such constants necessarily exist.

(iii) If $p + q + r \geq \frac{m(m+1)}{2} + 2$, then system (1.1) has a rational first integral.

(iv) If $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div(P, Q)$, then the expression for H defined above is an integrating factor of system (1.1).

(v) If $p + q + r = \frac{m(m+1)}{2}$ and the r independent singular points are weak (i.e., at these points $div(P, Q) = 0$), then the above expression is either a first integral or an integrating factor.

Example 1.2.9 (The Lotka-Volterra system)**.** [1] We illustrate the Darboux method with

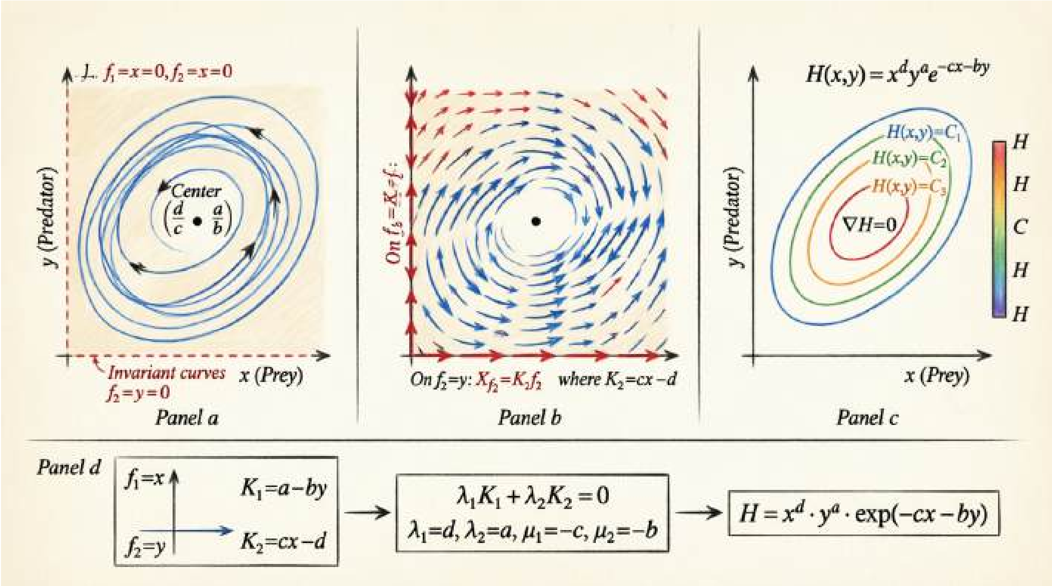


FIGURE 1.1 – Enter Caption

the classic Lotka-Volterra (Predator-Prey) system.This example demonstrates how invariant algebraic curves lead directly to a first integral. **1. The system :** Consider the following planar polynomial vector field of degree $m = 2$:

$$\dot{x} = x(a - by) = P(x, y), \quad \dot{y} = y(cx - d) = Q(x, y).$$

2. Identifying invariant algebraic curves : We look for curves $f(x, y) = 0$ such that $Xf = K \cdot f$. *Curve 1 :* $f_1 = x$. Calculating the left side : $x(a - by)(1) + y(cx - d)(0) = (a - by)x$. Here, the cofactor is $K_1 = a - by$. *Curve 2 :* $f_2 = y$. Calculating the left side : $x(a - by)(0) + y(cx - d)(1) = (cx - d)y$. Here, the cofactor is $K_2 = cx - d$. **3. Finding the first integral :** We seek a first integral of the Darboux form
$H = x^{\lambda_1} y^{\lambda_2} \exp(\mu_1 x + \mu_2 y)$. The condition $XH = 0$ implies :

$$\lambda_1 K_1 + \lambda_2 K_2 + \mu_1 \dot{x} + \mu_2 \dot{y} = 0.$$

Substituting the known expressions and collecting terms by powers of x and y, we find :

$$\lambda_1 = d, \quad \lambda_2 = a, \quad \mu_1 = -c, \quad \mu_2 = -b.$$

The first integral is :

$$H(x, y) = x^d y^a e^{-cx-by}. \tag{1.5}$$

This matches the classic Darboux first integral for Lotka-Volterra.

1.3 Limit cycles[1]

Among all possible orbits of a planar system, periodic orbits occupy a special position due to their relevance in modeling oscillatory phenomena in physics, biology, and engineering. However, not all periodic orbits are structurally significant. In conservative systems (such as Hamiltonian systems), periodic orbits often appear in continuous families (period annuli). In dissipative or non-linear systems, isolated periodic orbits known as limit cycles emerge.

1.3.1 Limit cycles and stability[1]

Definition 1.3.1 (Limit cycle)**.** [1] A periodic orbit γ is called a limit cycle if it is isolated in the set of all periodic orbits of the system. That is, there exists a neighborhood U of γ such that γ is the only periodic orbit contained in U.

The isolation property distinguishes limit cycles from the centers found in linear systems or Hamiltonian systems. Limit cycles represent self-sustained oscillations where the system settles into a specific rhythm regardless of small perturbations in initial conditions.

Theorem 1.3.2. *[1] When P and Q are polynomials,System (1.1) has at most finitely many limit cycles.(Note : This refers to a specific system ; see Section 1.4).*

Stability classification via the poincaré map :[1] The stability of a limit cycle determines the behavior of nearby trajectories. Let γ be a limit cycle and let Σ be a transverse section to γ at a point $p \in \gamma$. The Poincaré map (or return map) assigns to each point $q \in \Sigma_0$ (a neighborhood of p in Σ) the first return point of the orbit through q to Σ. Since γ is periodic, p is a fixed point of Π (i.e., $\Pi(p) = p$). The stability is classified based on the derivative of the Poincaré map at the fixed point, $\Pi'(p)$, often called the characteristic multiplier :

— **Stable (Attracting) :** $|\Pi'(p)| < 1$ — All nearby trajectories converge toward the limit cycle.

— **Unstable (Repelling) :** $|\Pi'(p)| > 1$ — All nearby trajectories move away from the limit cycle.

— **Semi-stable :** $|\Pi'(p)| = 1$ — Higher-order terms determine stability ; trajectories may approach from one side and diverge from the other.

A useful criterion for determining stability involves the divergence of the vector field. If γ is a periodic orbit of period T, the characteristic multiplier is given by Liouville's formula :

$$\Pi'(p) = \exp\left(\int_0^T \operatorname{div} X(\phi(t,p))\, dt\right), [1] \tag{1.6}$$

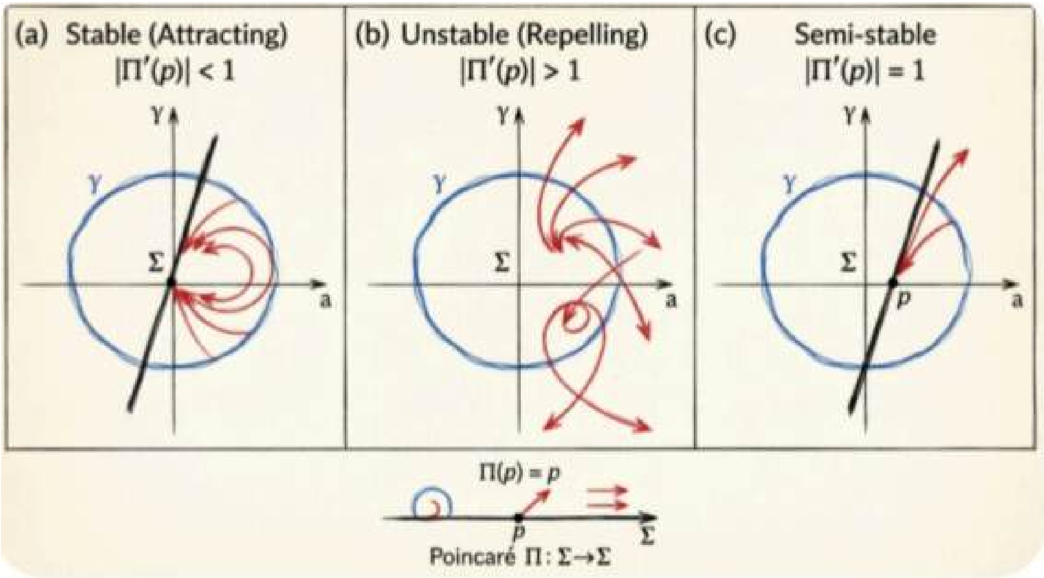


FIGURE 1.2 – Graphic representation of limit cycle stability classification. The figure illustrates the three stability types based on the characteristic multiplier $|\Pi'(p)|$: (a) Stable (Attracting) : $|\Pi'(p)| < 1$ — trajectories converge toward the limit cycle γ from both sides of the transverse section. (b) Unstable (Repelling) : $|\Pi'(p)| > 1$ — trajectories diverge away from γ. (c) Semi-stable : $|\Pi'(p)| = 1$ — trajectories exhibit mixed behavior, approaching from one side and repelling from the other.[1]

where $\operatorname{div} X = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$. Thus, if the integral of the divergence along the cycle is negative, the limit cycle is stable ; if positive, it is unstable. When the linearization at a singular point yields purely imaginary eigenvalues, the characteristic multiplier alone is insufficient to determine stability. In this case, the stability of the weak focus and the criticality (supercritical vs. subcritical) of the Hopf bifurcation are determined by the first Lyapunov constant l_1 (or L_1). This scalar quantity is computed from the nonlinear terms of the vector field and satisfies :

$$l_1 < 0 \;\Rightarrow\; \text{stable weak focus (supercritical Hopf)},$$

$$l_1 > 0 \;\Rightarrow\; \text{unstable weak focus (subcritical Hopf)}.$$

Its computation typically involves normal form reduction or recursive formulas (see, e.g., [3]).

Example 1.3.3 (Van der Pol Oscillator)**.** [1] A classic example of a system possessing a limit cycle is the Van der Pol oscillator :

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \quad \mu > 0. \tag{1.7}$$

In planar form :

$$\dot{x} = y, \quad \dot{y} = -x + \mu(1 - x^2)y. \tag{1.8}$$

This system has a unique unstable singular point at the origin and a unique stable limit cycle surrounding it.

1.3.2 Existence and non-existence of limit cycles[1]

Theorem 1.3.4 (Poincaré-Bendixson). *[1] Let R be a compact region of the plane bounded by two simple closed curves D_1 and D_2. Let F denote the velocity vector field associated with the system. If the following conditions are satisfied :*

[(i)]

1. *At every point on the boundary curves D_1 and D_2, the vector field F points strictly toward the interior of R (i.e., R is a trapping region) ;*
2. *The region R contains no critical points (equilibrium points) of the system ;*

then the system possesses at least one closed trajectory lying entirely within R.

Theorem 1.3.5 (Bendixson's Criterion). *[1] Let R be a simply connected region in the plane. If the partial derivatives $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial y}$ are continuous in R, and the divergence of the vector field :*

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \tag{1.9}$$

does not vanish identically and maintains a constant sign (i.e., is never zero) at any point in R, then the system has no closed trajectories lying entirely inside R.

Theorem 1.3.6 (Dulac's Criterion). *[1] Let $X = (P, Q)$ be a vector field on a simply connected region $D \subset \mathbb{R}^2$. If there exists a function $B(x, y)$ (called a Dulac function) such that the divergence of the scaled field :*

$$div(BX) = \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y}, \tag{1.10}$$

has a constant sign (and is not identically zero) in D, then the system has no periodic orbits lying entirely within D. This criterion is a generalization of Bendixson's Criterion (where $B(x, y) \equiv 1$).

Theorem 1.3.7 (Generalized Bendixson-Dulac with invariant curves). *[1] Let $f(x, y) = 0$ be an invariant algebraic curve with cofactor $K(x, y)$. Let $B = |f|^\alpha$ for $\alpha \in \mathbb{R}$. The divergence of the vector field scaled by B is :*

$$div(BX) = |f|^\alpha(div(X) + \alpha K).$$

If $M(x, y) = div(X) + \alpha K$ does not change sign and is not identically zero in a simply connected region (excluding $f = 0$), then there are no limit cycles entirely contained in that region. This result links the cofactor condition to the divergence via the Bendixson-Dulac criterion and is particularly useful for systems admitting invariant algebraic curves.

1.3.3 Inverse integrating factors and limit cycles[1]

Theorem 1.3.8. *[1] Let X be a C^1 vector field defined on the open subset U of $\mathbb{R}^2$. Let $V : U \to \mathbb{R}$ be an inverse integrating factor of X. If γ is a limit cycle of X, then γ is contained in $S = \{(x, y) \in U : V(x, y) = 0\}$.*

1.4 Algebraic limit cycles[1]

We recall that a limit cycle of a polynomial vector field X is an isolated periodic orbit in the set of all periodic orbits of X. An algebraic limit cycle of degree m of X is an oval of an irreducible invariant algebraic curve $f = 0$ of degree m which is a limit cycle of X. We must remark that when we are interested in the invariant algebraic curves for questions different from the algebraic limit cycles (like for instance integrability, multiplicity and others) it is important to consider complex invariant algebraic curves (i.e., $f = f(x, y)$ is a complex polynomial in the variables x and y), because the natural background for all the Darboux theory of integrability is the ring of the complex polynomials. But the limit cycles and in particular the algebraic limit cycles is a real phenomena.

Theorem 1.4.1 (Jouanolou's Theorem)**.** *[4] A polynomial vector field of degree n has less than $[n(n+1)/2]+2$ irreducible invariant algebraic curves, or it has a rational first integral.*

Jouanolou's Theorem shows that for a given polynomial vector field X of degree n the maximum degree of its irreducible invariant algebraic curves is bounded, because either X has a finite number of invariant algebraic curves less than $[n(n+1)/2]+2$, or X has rational first integral $f(x, y)/g(x, y)$. In this last case all the orbits of X are contained in the invariant algebraic curves $af(x, y) + bg(x, y) = 0$ for some $a, b \in \mathbb{R}$. Thus for each polynomial vector field there is a natural number N which bounds the degree of all its irreducible invariant algebraic curves.

Theorem 1.4.2 (Harnack's Theorem)**.** *[1] The maximum number of ovals of a real algebraic curve of degree m is $[(m - 1)(m - 2)/2] + 1$.*

Summarizing, a polynomial vector field of degree n with finitely many irreducible invariant algebraic curves has at most $[n(n + 1)/2] + 1$ of such curves, but we do not have a bound for the degree of these invariant algebraic curves. Consequently due to the Harnack's Theorem we do not have a uniform bound for the number of algebraic limit cycles that any polynomial vector field of degree n can have.

1.5 Hilbert's XVI problem (Second part)[1]

The study of limit cycles is motivated primarily by one of the most famous open problems in mathematics. In 1900, David Hilbert presented a list of 23 problems at the International Congress of Mathematicians in Paris, which profoundly influenced the development of mathematics in the 20th century. The second part of Hilbert's 16th Problem concerns the qualitative theory of polynomial differential equations. **Problem statement :** Determine the maximum number and the relative position of limit cycles for planar polynomial vector fields of degree n.

1.5.1 Formal statement[1]

Let $\mathcal{P}_n$ denote the set of all planar polynomial vector fields of degree at most n. We define the Hilbert number as :

$$H(n) = \sup\{\text{number of limit cycles of } X \mid X \in \mathcal{P}_n\}.$$

The problem asks two fundamental questions :

1. **Finiteness :** Is $H(n) < \infty$ for each $n \in \mathbb{N}$?
2. **Upper bound :** If finite, what is the exact value (or a sharp upper bound) for $H(n)$?

1.5.2 Current status[1]

Despite over a century of research, the problem remains unsolved in its entirety.

1. **Linear systems ($n = 1$) :** It is well known that $H(1) = 0$. Linear systems cannot exhibit limit cycles ; their periodic orbits form centers (continuous families).
2. **Quadratic systems ($n = 2$) :** This is the most studied case. It is known that $H(2) \geq 4$, as examples with 4 limit cycles have been constructed. However, it is not known if 4 is the maximum. Some conjectures suggest $H(2) = 4$, but no proof exists that excludes the possibility of 5 or more limit cycles in specific configurations. Recent advancements suggest that the existence of invariant algebraic curves imposes powerful constraints on the dynamics of the system. Specifically, the case of invariant cubic curves in quadratic systems presents a complex open problem regarding the

explicit algebraic conditions required to ensure the non-existence or uniqueness of limit cycles.

3. **General segree ($n \geq 3$) :** The finiteness question was resolved affirmatively only recently. Independently, Écalle (1992) and Il'yashenko (1991) proved that any polynomial vector field has at most finitely many limit cycles. However, their proofs are non-constructive and do not provide an explicit upper bound depending only on n. The difficulty lies in the complex bifurcation phenomena where limit cycles can be born from singular points (Hopf bifurcation), from separatrix cycles (homoclinic bifurcation), or from periodic annuli. Controlling the number of cycles generated under perturbation remains a formidable challenge in analysis and algebraic geometry.

1.6 Finiteness : The contribution of H. Dulac[1]

The quest for the finiteness of limit cycles has a rich history, with Henri Dulac's 1923 memoir "Sur les cycles limites" representing a foundational milestone. Although his original proof contained a gap, his ideas laid the groundwork for modern solutions.

1.6.1 Dulac's approach (1923)[1]

Dulac's strategy was based on the study of the Poincaré return map near polycycles (graphs formed by singular points and connecting orbits). His key ideas included :

1. **Compactification :** Using Poincaré compactification to embed the plane $\mathbb{R}^2$ into the sphere S^2, allowing the study of global behavior, including orbits tending to infinity.
2. **Analyticity of the return map :** Dulac argued that the displacement function $d(s) = \Pi(s) - s$ associated with a polycycle is analytic (or sufficiently regular). Since an analytic function cannot have an accumulation point of zeros unless it is identically zero, he concluded that limit cycles cannot accumulate, ensuring finiteness.
3. **Critical points at infinity :** He classified the possible behaviors of orbits approaching infinity, ensuring that limit cycles cannot accumulate near the boundary of the compactified plane.

1.6.2 The gap and its resolution[1]

Dulac's original proof contained a subtle gap concerning the analytic continuation of the displacement function near polycycles involving saddle points with irrational eigenvalue ratios. This gap remained unnoticed for decades until it was identified in the 1980s. The problem was finally resolved in the early 1990s :

- **J. Écalle (1992) :** Used the theory of resurgent functions and accelero-summation to prove the finiteness of limit cycles.
- **Y. Il'yashenko (1991) :** Used non-oscillation theorems for Abelian integrals and properties of the monodromy group to establish finiteness.

Both proofs confirmed Dulac's intuition : limit cycles cannot accumulate, ensuring that $H(n)$ is finite. However, neither proof provides a method to calculate $H(n)$, leaving the second part of Hilbert's problem open.

1.6.3 Dulac's criterion for non-existence[1]

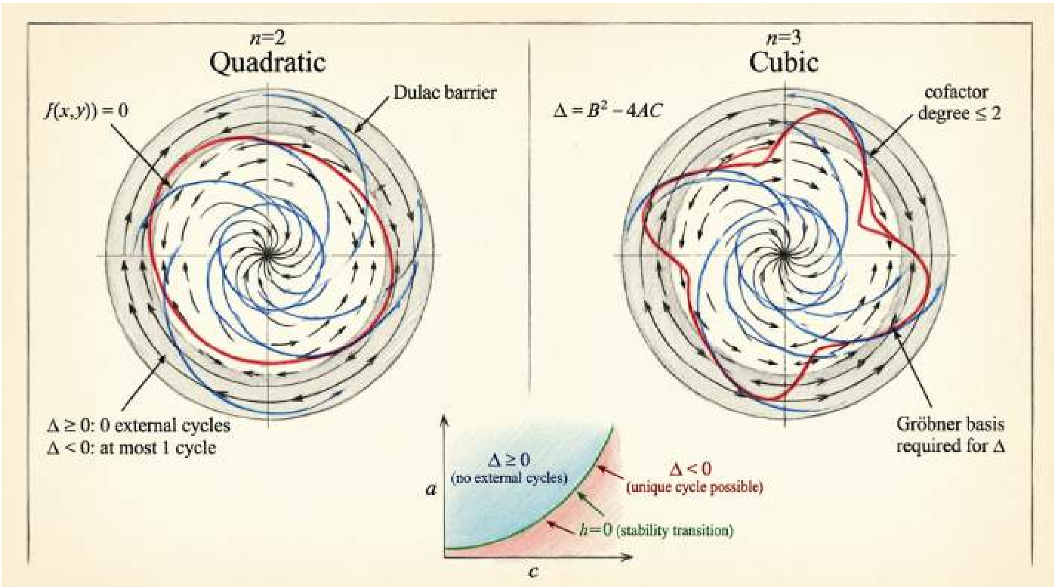
Beyond the finiteness theorem, Dulac provided a practical tool for excluding limit cycles in specific regions, known as Dulac's Criterion (Theorem 1.3.6). This criterion is a powerful tool in the qualitative analysis of polynomial systems, often used to prove the non-existence of limit cycles in specific parameter regimes. In the context of this research, we utilize the generalized framework proposed by Gasull and Giacomini (2023) to derive explicit algebraic conditions on the coefficients of the vector field and the cofactor that guarantee the non-existence of limit cycles external to an invariant curve. Specifically, for quadratic systems ($n = 2$) admitting an invariant algebraic curve of degree 3 (cubic curve), we link the cofactor condition to the divergence of the vector field via the Bendixson-Dulac criterion. This contributes to the understanding of the bounds of $H(2)$ through algebraic geometry methods.

1.7 Conclusion

This chapter has provided a foundational overview of Planar Polynomial Vector Fields, Darboux Integrability, and Limit Cycle Theory. We have clarified the mathematical rigor surrounding these concepts, particularly distinguishing between the finiteness of limit cycles for a specific system and the existence of a uniform bound depending only on the degree of the system. The distinction between the finiteness of limit cycles for a specific system (resolved by Écalle and Il'yashenko) and the existence of a uniform bound (still open) is crucial. Future work lies in refining bounds for specific degrees, exploring the computational aspects of finding invariant algebraic curves, and resolving the uniform bound conjecture. The study of planar polynomial vector fields bridges algebraic geometry and dynamical systems, with Hilbert's 16th Problem remaining one of the most challenging open problems in mathematics.

Chapitre 2

Applications to quadratic and cubic systems ($n = 2, 3$)



2.1 Introduction

This chapter will investigate the application of invariant algebraic curves and the Darboux integrability method to the study of limit cycles within quadratic and cubic polynomial differential systems. Historically, determining the number and position of limit cycles—a challenge central to the second part of Hilbert's 16th Problem—has relied heavily on local qualitative analysis, perturbation methods, or intensive numerical simulations. However, this research will adopt a global algebraic approach, asserting that the structural constraints imposed by invariant curves provide a rigid backbone for phase dynamics. By shifting the focus from local flows to the underlying geometry of the system, we can derive definitive conclusions regarding the existence, stability, and upper bounds of limit cycles without the need for numerical approximation. The methodology bridges the gap between intersection theory in the complex projective plane $\mathbb{CP}^2$ and the real topological behavior of the phase plane. This synthesis will provide a rigorous framework for identifying dynamical properties through purely algebraic criteria. Throughout this chapter, we will demonstrate how these algebraic invariants act not merely as particular solutions, but as central analytical tools that narrow the scope of the restricted Hilbert's 16th Problem, offering a profound link between algebraic geometry and the theory of planar dynamical systems.

2.2 Canonical forms of quadratic systems

2.2.1 General form and classification

A real planar quadratic differential system is defined by :

$$\dot{x} = P(x, y) = \sum_{i+j=0}^{2} a_{ij} x^i y^j, \tag{2.1}$$

$$\dot{y} = Q(x, y) = \sum_{i+j=0}^{2} b_{ij} x^i y^j, \tag{2.2}$$

where $P, Q \in \mathbb{R}[x, y]$ are relatively prime polynomials and $\max(\deg P, \deg Q) = 2$. The associated polynomial vector field is :

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}. \tag{2.3}$$

Classification relies on affine equivalence (translations, rotations, linear scalings) and time rescaling $dt \mapsto \lambda\, dt$, which preserve the phase portrait topology, limit cycle count, and stability. Foundational theoretical tools include :

- **Jouanolou's theorem** [5] : A degree-n vector field has either $< \lfloor n(n+1)/2 \rfloor + 2$ irreducible invariant algebraic curves, or a rational first integral. For $n = 2$, the threshold is 5.
- **Harnack's theorem** [5] : The maximum number of ovals of a real algebraic curve of degree m is $\lfloor (m-1)(m-2)/2 \rfloor + 1$.
- **Bautin–Christopher–Dolov–Kuzmin theorem** [5] : Given a non-singular curve $f = 0$ of degree m and a linear polynomial D whose zero line lies outside all bounded components of $f = 0$, the system $\dot{x} = af - Df_y$, $\dot{y} = bf + Df_x$ makes every bounded component of $f = 0$ a hyperbolic limit cycle, with no others.
- **Poincaré problem** [5] : Concerns finding a uniform bound $N(n)$ for the degree of invariant algebraic curves. No such uniform bound exists for fixed-degree fields with finitely many invariant curves [6, 7], implying algebraic limit cycles of arbitrary degree are theoretically possible.

2.2.2 Invariant algebraic curves in quadratic systems

An algebraic curve $f(x, y) = 0$ is invariant under X if there exists a polynomial cofactor $K(x, y)$ such that :

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{2.4}$$

For quadratic systems, $\deg K \leq 1$. By Proposition 1 [5], $f = 0$ is invariant if and only if each irreducible factor $f_i = 0$ is invariant, with $K_f = \sum n_i K_{f_i}$. Thus, analysis reduces to irreducible curves. Key non-existence criteria :

- **Poincaré tangential method** [8] : No periodic orbits in D if $\exists F \in C^1$ with $\dot{F}$ of constant sign and not identically zero on any orbit.
- **Bendixson–Dulac criterion** [8] : No periodic orbits in a simply connected region D if $\exists B \in C^1$ such that $\operatorname{div}(BP, BQ)$ has constant sign.
- **Cherkas criterion** [8] : If $M = \dot{F} + \beta F \operatorname{div}(P, Q) > 0$ in W, limit cycles do not intersect $F = 0$, and at most $k - 1$ exist in each k-connected domain.

2.2.3 Affine canonical forms with invariant curves

Any quadratic system with an irreducible cubic invariant curve $f(x, y) = 0$ is affine-equivalent (after time rescaling if necessary) to one of 14 canonical forms [5]. **Type (i) (Hamiltonian) :**

$$\dot{x} = V_y, \; \dot{y} = -V_x$$

with

$$f(x,y) = \sum_{i+j\leq 2} m_{ij}x^i y^j + xy(\alpha x + \beta y)$$

and $\alpha\beta \neq 0$.

Type (ii) (Hamiltonian) :

$$\dot{x} = V_y,\ \dot{y} = -V_x$$

with

$$f(x,y) = \sum_{i+j\leq 2} m_{ij}x^i y^j + x(\alpha x^2 + \beta xy + \gamma y^2)$$

and $\beta^2 < 4\alpha\gamma$.

Type (iii) (Darboux integrable) :

$$\dot{x} = \frac{3}{2}y(-b_{20}m_{02} + b_{02}x)$$

$$\dot{y} = \frac{1}{3}\frac{3b_{02}m_{00} + b_{20}m_{02}m_{10} + 2b_{02}m_{10}}{m_{02}}x + b_{20}x^2 + b_{02}y^2$$

with

$$f(x,y) = m_{00} + m_{10}x + m_{02}y^2 + x^3$$

and cofactor

$$K(x,y) = 2b_{02}y$$

($m_{02} \neq 0$).

Type (iv) :

$$\dot{x} = \frac{2}{3}(b_{01}x - y + b_{11}x^2 + b_{02}xy)$$

$$\dot{y} = b_{01}y + x^2 + b_{11}xy + b_{02}y^2$$

with

$$f(x,y) = y^2 + x^3$$

and

$$K(x,y) = 2(b_{01} + b_{11}x + b_{02}y)$$

Type (v) ($v^{\pm}$) :

$$\dot{x} = 2(b_{01}x + (B_{02} \mp B_{20})y + b_{01}x^2 + 3B_{02}xy)$$

$$\dot{y} = 2((B_{20} \mp B_{02})x + b_{01}y + 3B_{20}x^2 + 3b_{01}xy + 9B_{02}y^2)$$

with

$$f(x,y) = x^2 \pm y^2 + x^3$$

and

$$K(x,y) = 2(2b_{01} + 3b_{01}x + 9B_{02}y)$$

Type (vi) :

$$\dot{x} = \frac{1}{3}(a + bx + cx^2 + dxy)$$

$$\dot{y} = by - ax^2 + cxy + dy^2$$

with

$$f(x,y) = y + x^3$$

and

$$K(x,y) = b + cx + dy$$

Type (vii) :

$$\dot{x} = 3a_{11} + a_{10}x + a_{20}x^2 + a_{11}xy$$

$$\dot{y} = 3a_{20} - 9a_{11}x - a_{10}y - 3a_{10}x^2 + 2a_{20}xy + 2a_{11}y^2$$

with

$$f(x,y) = 1 + xy + x^3$$

and

$$K(x,y) = 3(a_{20}x + a_{11}y)$$

Type (viii) :

$$\dot{x} = -\frac{1}{2}a_{11} - \frac{1}{2}(b_{01} + \frac{1}{2}a_{11})x + \frac{1}{8}(a_{11} - 4b_{11} - 2b_{01})x^2 + a_{11}xy$$

$$\dot{y} = \frac{1}{8}(4b_{11} - 2b_{01} + a_{11}) + b_{01}y + b_{11}xy - 2a_{11}y^2$$

with

$$f(x,y) = 1 + x + \frac{x^2}{y}$$

and

$$K(x,y) = \frac{1}{4}(-2a_{11} + (-2b_{01} + a_{11})x)$$

Type (ix) (Riccati type) :

$$\dot{x} = \frac{1}{2}(2a_{20}m_{01} + b_{02}m_{10}) + a_{20}x^2$$

$$\dot{y} = -(a_{20}m_{10} + \frac{1}{2}\frac{b_{02}m_{10}^2}{m_{01}}) - (2a_{20} + \frac{1}{2}\frac{b_{02}m_{10}}{m_{01}})xy + b_{02}y^2$$

with

$$f(x, y) = m_{10}x + m_{01}y + x^2y$$

and

$$K(x, y) = b_{02}(-\frac{1}{2}\frac{m_{10}}{m_{01}}x + y)$$

($m_{01} \neq 0$).

Type (x) (Riccati type) :

$$\dot{x} = a_{20}m_{01} + \frac{1}{2}\frac{b_{01}m_{01}m_{10}}{m_{00}} - \frac{1}{2}b_{01}x + a_{20}x^2$$

$$\dot{y} = -a_{20}m_{10} - \frac{1}{2}\frac{b_{01}m_{10}^2}{m_{00}} + b_{01}y - (2a_{20} + \frac{1}{2}\frac{b_{01}m_{10}}{m_{00}})xy + \frac{b_{01}m_{01}}{m_{00}}y^2$$

with

$$f(x, y) = m_{00} + m_{10}x + m_{01}y + x^2y$$

and

$$K(x, y) = \frac{1}{2}\frac{b_{01}}{m_{00}}(-m_{10}x + 2m_{01}y)$$

($m_{00} \neq 0$).

Type (xi) :

$$\dot{x} = 2ax + 2y + x^2 \mp axy$$

$$\dot{y} = 2(1 \mp y)(\pm x + ay)$$

with

$$f(x, y) = \mp x^2 + y^2 + x^2y$$

and

$$K(x, y) = 4a(1 \mp y)$$

.

Type (xii) (Hamiltonian) :

$$\dot{x} = V_y,\ \dot{y} = -V_x$$

with

$$f(x, y) = m_{10}x + m_{01}y + m_{02}y^2 + x^2y$$

Type (xiii) :

$$\dot{x} = (2a_{20} + 4a_3a_{20} + b_{11})x - 2a_2b_{11}y + 2a_2a_{20}x^2 - (2a_{20} + b_{11})axy$$

$$\dot{y} = -ab_{11}x + (2a_{20} + 4a_3a_{20} + b_{11} + 4a_3b_{11})y + 2a_2b_{11}xy - 2(2a_{20} + b_{11})ay^2$$

with

$$f(x, y) = -\frac{1}{2a}x^2 + 2axy + y^2 + x^2y$$

and

$$K(x, y) = 2(2a_{20} + b_{11})(1 + 2a_3 - 2ay + a_2x)$$

$(a \neq 0)$.

Type (xiv) :

$$\dot{x} = 6x - (a + 4k)y + 2(3b - a^2 - ak)x^2 - b(a + 4k)xy$$

$$\dot{y} = 9y + 6kx^2 + 2(3b - a^2 + 2ak)xy + b(2k - a)y^2$$

with

$$f(x, y) = y^2 + x(x^2 + axy + by^2)$$

and

$$K(x, y) = 3(6 + 2(3b - a^2)x - aby)$$

$(b \neq 0, a^2 - 4b \neq 0)$.

Remark 2.2.1. Among the fourteen canonical forms, the following structural properties immediately exclude limit cycles for several cases :

— Hamiltonian types (i), (ii), (xii) : These systems preserve phase-space area by construction, which strictly precludes the existence of any limit cycles.

— Integrable types (iii), (ix), (x) : System (iii) admits a rational first integral, while systems (ix) and (x) reduce to Riccati-type equations with known particular solutions. The existence of such global integrals eliminates isolated periodic orbits.

— Remaining cases : The dynamics of the remaining eleven types require deeper analysis via tangential curve methods, Dulac functions, and projective intersection geometry. As rigorously established in Section 2.4, only types (v^+) and (viii) can support external limit cycles.

2.3 Case study : Invariant parabola

2.3.1 Normal form of systems with invariant parabola

A quadratic system possessing the invariant parabola $f(x, y) = y - x^2 = 0$ can be written in the normal form [9, Eq. 4] :

$$\begin{cases} \dot{x} = a + bx + hy + c(y - x^2) + exy, \\ \dot{y} = 2x(a + bx + hy) + d(y - x^2) + 2ey^2, \end{cases} \tag{2.5}$$

with cofactor $K(x, y) = -2cx + 2ey + d$. The critical points on the parabola satisfy $ex^3 + hx^2 + bx + a = 0$.

2.3.2 Transformation to invariant straight line

Gasull and Giacomini (2022) demonstrated that the uniqueness result for invariant parabolas is a consequence of the analogous result for invariant straight lines.

Theorem 2.3.1. *(Transformation) [9, Thm 1.1] A quadratic system with an invariant parabola and a limit cycle can be transformed into another quadratic system with an invariant straight line.*

Proof. See reference [14] □

2.3.3 Proof of uniqueness

The uniqueness and hyperbolicity of the limit cycle in the invariant parabola case are direct consequences of Theorem 2.3.1. The transformed system falls into the well-studied class of quadratic systems with an invariant line, for which uniqueness and hyperbolicity are classical results. Any limit cycle γ must lie in a connected component of $\mathbb{R}^2 \setminus (P \cup \{x = 0\} \cup \{y = 0\})$, ensuring the birational map is well-defined on γ.

2.3.4 Algebraic non-existence criteria

Using the function method from Chapter 1, we can derive explicit algebraic conditions on the parameters that guarantee the non-existence of periodic orbits.

Proposition 2.3.2. *[9, Prop 6.5] Let define the discriminant quantity :*

$$\Delta = \left(16a^2e^3 + \left((-8bc - 16hb - 12cd - 8dh)a + (2b-d)^3\right)e^2\right)$$

$$-2(c+h)\left(4c(c+2h)a - 8b^2c + 6bcd - 4bdh - 3cd^2\right)e + 8c(c+h)^2(bc+dh)\Big)$$

$$\times \left((2b-d)e + 2c^2 + 2hc\right)e.$$

If $\Delta \geq 0$, the system has no periodic orbits.

Proof. Using the auxiliary function method [9, Theorem 2.1], take $V(x,y) = g(x,y)|f(x,y)|^\alpha$ with $g(x,y) = g_0 + g_1x + g_2y$. Requiring the time derivative $N_{\alpha,g} = \alpha Kg + Pg_x + Qg_y$ to be a perfect square leads to a quadratic equation in α. The condition for a real solution α is exactly $\Delta \geq 0$, which forces $N_{\alpha,g}$ to maintain constant sign, precluding closed orbits outside $f = 0$. □

Theorem 2.3.3. *[9, Thm. 2.4] Consider the polynomial differential system (1.1) and assume that it has an invariant algebraic curve $f(x,y) = 0$ with cofactor $k(x,y)$. For each $\alpha \in \mathbb{R}$ and each polynomial $g(x,y)$, define the new polynomial*

$$N_{\alpha,g}(x,y) = \alpha k(x,y)g(x,y) + \frac{\partial g}{\partial x}P(x,y) + \frac{\partial g}{\partial y}Q(x,y).$$

Then, if for some α and g, $N_{\alpha,g}$ does not change sign and vanishes only on some algebraic curve that is not invariant under the flow, then the only limit cycles of the system (if any) are included in the invariant algebraic curve $f(x,y) = 0$. Moreover, if for some α and g, $N_{\alpha,g}(x,y) \equiv 0$, then $g(x,y)|f(x,y)|^\alpha$, if it is not piecewise constant, is a first integral of system (1.1), and the same conclusion holds.

2.4 Case study : Invariant cubic curve

2.4.1 Classification of quadratic systems with invariant cubic

According to the details view at canonical forms in 2.2.3, any quadratic system with an irreducible cubic invariant $f(x,y) = 0$ belongs to one of the 14 affine classes. Hamiltonian cases preserve area, while cases (iii), (ix), (x) possess rational first integrals or reduce to

Riccati equations, eliminating the possibility of isolated limit cycles. The remaining 11 types are analyzed via tangential curves, Dulac functions, and intersection geometry.

2.4.2 Non-Existence of cubic algebraic limit cycles

A significant result established by Evdokimenko (1970–1979) and simplified by Chavarriga and García (2006) is that a cubic curve cannot be a limit cycle for a quadratic system.

Theorem 2.4.1. *[5, Thm 3.1] A real quadratic system does not have algebraic limit cycles of third degree.*

Proof. Let $f(x, y) = 0$ be an irreducible invariant cubic. Using Darboux's enumerative geometry in $\mathbb{CP}^2$ [5, Lemma 3.2 & Cor 3.3], the intersection numbers h and h' satisfy $h + h' \geq 4$. If $h' = 0$, then $h \geq m^2 = 4$, implying a rational first integral $\Rightarrow$ no limit cycle. If $h' \neq 0$, $f = 0$ has a real multiple point p_1. Assume a limit cycle $\gamma \subset \{f = 0\}$ exists. A line R through p_1 and a point inside γ intersects γ twice. Counting multiplicities, $\sum I(p, R \cap f) \geq 2 + 1 + 1 = 4$, contradicting Bézout's Theorem which requires $\deg R \cdot \deg f = 3$. Hence, $\gamma \not\subset \{f = 0\}$. □

2.4.3 Existence of external limit cycles

While the cubic curve itself cannot be a limit cycle, limit cycles may exist in the regions disjoint from the curve. Chavarriga and García (2006) analyzed the 14 canonical forms of quadratic systems with an invariant cubic.

Theorem 2.4.2. *[5, Thm 4.1] A quadratic system having a cubic invariant curve and a limit cycle must be of type (v^+) or type (viii). Both possibilities do in fact occur.*

.
- — Types (iv), (vi), (vii), (v^-), (xi), (xiii) are excluded via Poincaré tangential functions (e.g., $F = y/\sqrt{|f|}$, $F = x/f^{1/3}$) or Bendixson–Dulac criteria.
- — Types (ix), (x), (xi), (xiii) are excluded by Theorem 2.4 [11] (Chavarriga–Giacomini–Llibre criterion) : existence of two singular points $p_1, p_2 \in \mathbb{CP}^2$ with $L = M = \tilde{K} = 0$ and $\tilde{F} \neq 0$ forces all limit cycles to lie inside $f = 0$, contradicting Theorem 2.4.1.

□

Remark 2.4.3. For all cases, the condition $N_{\alpha,g} = rw^2$ with constant sign guaranties the non-existence of periodic orbits outside $f(x, y) = 0$. The remaining two families follow identically (see [9, Sec. 4]). This unified approach confirms that algebraic and non-algebraic limit cycles cannot coexist in these quadratic systems.

2.4.4 Analysis of canonical forms (Types v^+ and viii)

- **Type** (v^+) **:**

$$f(x, y) = x^2 + y^2 + x^3 = 0$$

, with system parameters satisfying $b_{01} = 1$, $a = B_{02}$, $b = B_{20}$. The finite critical point is

$$p = (\frac{2(a - b)}{3b}, -\frac{2}{9b})$$

. For $b = -a$ and $1 - 12a^2 < 0$, p is a weak focus. The first Lyapunov constant $V_1 \neq 0$ triggers a Hopf bifurcation, generating a unique unstable limit cycle external to the cubic.

- **Type (viii) :**

$$f(x, y) = 1 + x + x^2 y = 0$$

. Xu's (1992) example [12, Eq. 5] with parameters $\lambda = (\mu - 1)/(3\mu)$, $1 < \mu < 3/2$ exhibits a first-order weak focus at $(\mu, \lambda - 1)$. Perturbing λ bifurcates a limit cycle while preserving the cubic invariant. After affine scaling, it matches form (viii). Uniqueness was later proved by Kooij & Zegeling [13].

These two families constitute the only known quadratic systems with an invariant cubic that admit limit cycles, and in both cases, the limit cycle is external, unique, and hyperbolic.

Example 2.4.4 ((Type v^+ with External Limit Cycle))**.** Consider the quadratic system

$$\begin{cases} \dot{x} = 2(x + (a - b)y + x^2 + 3axy), \\ \dot{y} = 2((b - a)x + y + bx^2 + 3xy + \frac{9}{2}ay^2), \end{cases} \tag{2.6}$$

with the invariant cubic $f(x, y) = x^2 + y^2 + x^3 = 0$. For the parameter choice $b = -a$ and $1 - 12a^2 < 0$, the finite critical point $p = (0, -\frac{2}{9b})$ becomes a weak focus. The first Lyapunov constant satisfies $V_1 \neq 0$, triggering a supercritical Hopf bifurcation that generates a unique, unstable, and hyperbolic limit cycle lying entirely outside the invariant cubic [5, Eq. (4), p. 215].

2.5 Finiteness from multiple invariant curves

When a quadratic system possesses more than one invariant algebraic curve, the constraints on the number of limit cycles become even more rigorous. Gasull and Giacomini (2022) provided a general finiteness result depending on the number of limit cycles contained within the invariant curves.

2.5.1 General upper bound

Let $f(x, y)$ be a smooth function, not necessarily polynomial. The curve $f(x, y) = 0$ is called a generalized invariant curve for the vector field X if it satisfies the differential relation

$$\dot{f}(x, y) = k(x, y)f(x, y),$$

where $k(x, y)$ is a polynomial of degree at most $n - 1$ (i.e., one less than the degree of the vector field). The polynomial k is referred to as the cofactor of the curve. Furthermore, two generalized invariant curves $f_1 = 0$ and $f_2 = 0$ are said to be different if there do not exist real constants $\alpha, \beta \in \mathbb{R}$ such that $|f_1(x, y)| = \alpha|f_2(x, y)|^\beta$.

Remark 2.5.1 (Extension of the auxiliary polynomial method)**.** [14] If system (1.1) possesses two different invariant irreducible algebraic curves, denoted by $f_i(x, y) = 0$ where $i = 1, 2$, with corresponding cofactors $k_i(x, y)$, then the analytical approach used previously can be generalized. More precisely, for each pair of real constants $\alpha_1, \alpha_2 \in \mathbb{R}$ and for each polynomial $g(x, y)$, we define the following new polynomial :

$$N_{\alpha_1,\alpha_2,g}(x, y) = (\alpha_1 k_1(x, y) + \alpha_2 k_2(x, y))g(x, y) + \frac{\partial g}{\partial x}P(x, y) + \frac{\partial g}{\partial y}Q(x, y).$$

The remark and Theorem 2.3.3 also applies if the invariant algebraic curves in their respective statements are replaced by generalized invariant curves.

2.5.2 Main finiteness theorem

Theorem 2.5.2. *[9] Consider a quadratic system with two different generalized invariant curves. Let M be the number of limit cycles contained in them. Then its maximum number of limit cycles is $2M + 2$.*

It is worth noting that this upper bound is likely not sharp ; in fact, researchers believe that this type of quadratic systems does not possess limit cycles. However, it is never easy to find upper bounds for the number of limit cycles for a given quadratic system. While Bamón's seminal work [9] proved the finiteness of the number of limit cycles for any individual quadratic system, the existence of a uniform upper bound applicable to all these systems still remains a major open problem.

2.5.3 Degree-dependent bounds

Corollary 2.5.3. *[14] Consider a quadratic system with two different invariant irreducible algebraic curves of degrees m_1 and m_2. Then the maximum number of limit cycles is*

$$2\left(\left\lfloor \frac{m_1}{2} \right\rfloor + \left\lfloor \frac{m_2}{2} \right\rfloor\right) + 2,$$

where $\lfloor \cdot \rfloor$ denotes the integer part function. Moreover, if the two invariant algebraic curves do not have real isolated points, then the upper bound reduces to

$$2\left(\left\lfloor \frac{m_1}{2} \right\rfloor + \left\lfloor \frac{m_2}{2} \right\rfloor\right).$$

2.5.4 Special cases and applications

Corollary 2.5.4. *[14] If we add to the hypotheses of Theorem 2.5.2 that the two generalized invariant curves do not have isolated real points, then the upper bound of the theorem can be reduced to be $2M$ and moreover all these limit cycles are nested.*

Notice that this corollary is an extension of the classical result stating that quadratic Lotka-Volterra systems, that is, quadratic systems with two non-parallel invariant straight lines do not have limit cycles. This is because in this case $M = 0$, and of course these curves do not possess real isolated points. The following result gives, in the case of invariant algebraic curves, an upper bound only in terms of the degrees of these curves, and extends again the result for quadratic Lotka–Volterra systems.

Corollary 2.5.5. *[14] Consider a quadratic system possessing two different invariant irreducible algebraic curves, provided that the system does not contain dicritical critical points. Then the maximum number of limit cycles in this system is ten. Moreover, if the two invariant algebraic curves do not possess real isolated points, then this upper bound reduces to at most eight limit cycles.*

Corollary 2.5.6. *[14] Assume that the hypotheses of Theorem 2.5.2 hold. Let*

$$k_i(x, y) = a_i + b_i x + c_i y$$

be the cofactors of the generalized invariant curves of the QS, $f_i(x, y) = 0$ for $i = 1, 2$, and let

$$\operatorname{div}(X(x, y)) = a + bx + cy$$

be its divergence. Assume that

$$\Delta_{1,2}(\Delta_{1,2} - \Delta_2)(\Delta_{1,2} + \Delta_1)\det(A) \neq 0,$$

where $\Delta_{1,2} = b_1c_2 - b_2c_1$ *and* $\Delta_i = bc_i - cb_i$, $i = 1, 2$. *Then the following holds :*

1. *The stability of the limit cycles contained in* $\{f_1(x, y) = 0\}$ *is given by the sign of* $(\Delta_{1,2} - \Delta_2)\det(A)$.
2. *The stability of the limit cycles contained in* $\{f_2(x, y) = 0\}$ *is given by the sign of* $(\Delta_{1,2} + \Delta_1)\det(A)$.
3. *The stability of the limit cycles not contained in* $\{f_1(x, y) = 0\} \cup \{f_2(x, y) = 0\}$ *is given by the sign of* $\Delta_{1,2}\det(A)$.

Moreover, if the three quantities $\Delta_{1,2}$, $\Delta_{1,2} - \Delta_2$, *and* $\Delta_{1,2} + \Delta_1$ *have the same sign, then the maximum number of limit cycles of the QS is two. Furthermore, in this last situation, if the two generalized invariant curves do not contain isolated real points, then the QS does not have limit cycles.*

Quadratic Lotka–Volterra systems

Through the classical Bendixson–Dulac theorem, Bautin already proved that quadratic systems with two non-parallel invariant straight lines do not have limit cycles. In some examples, the key idea in the proof is to use a Dulac function of the form $|x|^{\alpha}|y|^{\beta}$. Here, we present a different proof by utilizing the extension of Theorem 2.3.3 given in Remark 2.5.1.

Proposition 2.5.7. *[14] The Lotka–Volterra QS*

$$\begin{cases} \dot{x} = x(a_0 + a_1x + a_2y), \\ \dot{y} = y(b_0 + b_1x + b_2y) \end{cases} \tag{2.7}$$

does not have limit cycles.

Démonstration. We define several quantities associated with (2.7). Set $\Delta_{i,j} = a_ib_j - a_jb_i$,

$$D = a_0a_1b_2 - a_0b_1b_2 - a_1a_2b_0 + a_1b_0b_2$$

, and

$$C = -a_0a_1b_2^2 + a_0b_1b_2^2 + 2a_1a_2b_0b_2 - a_1b_0b_2^2$$

First, let us prove that when

$$a_2b_2\Delta_{0,2}\Delta_{1,2} = 0$$

the system does not have periodic orbits. When $a_2 = 0$ the first differential equation does not depend on y, that is

$$\dot{x} = x(a_0 + a_1 x)$$

and hence the behaviour of $x(t)$ cannot be periodic unless it is identically constant. Something similar happens when $b_2 = 0$. When $\Delta_{1,2} = 0$ system (1.1) either does not have critical points outside the axes or it has a full line of critical points. In any case, periodic orbits are not possible. Similarly, when $\Delta_{0,2} = 0$ the same happens. When $a_2 b_2 \Delta_{0,2} \Delta_{1,2} \neq 0$ we will apply Remark 2.5.1 with

$$f_1(x, y) = x$$

$$f_2(x, y) = y$$

$$k_1(x, y) = a_0 + a_1 x + a_2 y$$

$$k_2(x, y) = b_0 + b_1 x + b_2 y$$

$$\alpha_1 = \frac{b_2 C}{a_2 \Delta_{0,2} \Delta_{1,2}}, \quad \alpha_2 = \frac{b_2(a_2 - b_2)\Delta_{0,1}}{\Delta_{0,2}\Delta_{1,2}},$$

and

$$g(x, y) = b_2 C \Delta_{1,2} x - a_2 b_2 (a_2 - b_2) \Delta_{1,2} \Delta_{0,2} y - C(a_2 - b_2)\Delta_{0,2}$$

. Straightforward computations give that

$$N_{\alpha_1,\alpha_2,g}(x, y) = \frac{b_2(a_2 - b_2)CD}{a_2 \Delta_{0,2} \Delta_{1,2}} (\Delta_{1,2} x + \Delta_{0,2}^2)$$

Hence, when $b_2(a_2 - b_2)CD \neq 0$ the QS does not have periodic orbits. When $b_2(a_2 - b_2)CD = 0$ it can have periodic orbits but not limit cycles because it has a smooth first integral. For instance, the classical Lotka–Volterra system, that corresponds to $a_1 = b_2 = 0$, $a_0 > 0$, $b_1 > 0$ and $a_2 < 0$, $b_0 < 0$, has a global center in the first quadrant. Notice that in this case $D = 0$. □

Nonexistence of limit cycles for quadratic systems with an invariant parabola

We present a simple test for the nonexistence of limit cycles in the family of quadratic systems with an invariant parabola by applying Theorem 2.3.3 :

$$\begin{cases} \dot{x} = a + bx + hy + c(y - x^2) + exy, \\ \dot{y} = 2x(a + bx + hy) + d(y - x^2) + 2ey^2. \end{cases} \tag{2.8}$$

By Proposition 2.3.2, system (2.8) has no periodic orbits.

Nonexistence of limit cycles for some QS with an invariant cubic

There exist two families of quadratic systems that possess an invariant cubic curve and are characterized by having a (unique) limit cycle not contained on this curve. We consider one of these families and determine conditions on its parameters. We consider the system

$$\begin{cases} \dot{x} = \frac{1}{2}(1-x)x - \frac{L}{2}x^2 + \frac{1}{2}u(-1+2x+xy), \\ \dot{y} = -1 - y - uy(1+y) + L(1+xy), \end{cases} \tag{2.9}$$

where L and u are real parameters. It admits the invariant algebraic cubic curve

$$f(x,y) = -1 + 2x + x^2 y = 0$$

, with cofactor $k(x,y) = u - x$, and for some values of L and u it also has a limit cycle. We prove :

Proposition 2.5.8. *[14] System (2.9), with*

$$(1 - u + Lu)(6 - 5u + 9Lu) \geq 0$$

, does not have periodic orbits.

Démonstration. The proof follows exactly the same steps as the proof of Proposition 2.3.2, but the computations are much easier and explicit. By applying Theorem 2.3.3 with

$$g(x,y) = -(u\alpha - u + 1)(u\alpha - u - 3) + 8\alpha(Lu - u + 1)x + 4\alpha u^2 y$$

, where $\alpha \in \mathbb{R}$ is chosen such that

$$-u^2\alpha^2 + 2u(6Lu - 5u + 5)\alpha + 4Lu^2 - 5u^2 + 6u - 1 = 0$$

, it holds that

$$N_{\alpha,g}(x,y) = rw^2(x,y)$$

for some linear polynomial w. The discriminant with respect to α of the above quadratic equation is

$$16u^2(1 - u + Lu)(6 - 5u + 9Lu) \geq 0$$

, and hence the proposition follows. □

2.6 Brief look at cubic systems $(n = 3)$

While the Hilbert 16th Problem remains open for quadratic systems, cubic systems $(n = 3)$ allow for richer dynamics, including algebraic limit cycles of higher degrees.

2.6.1 Algebraic limit cycles in cubic systems

[14] According to Llibre (2008), cubic polynomial vector fields are known to have algebraic limit cycles of degrees 2, 3, 4, 5, and 6. Unlike quadratic systems where the maximum known degree is 6 (via birational transformation), cubic systems naturally admit these structures. Open problems remain regarding the maximum possible degree and whether chains of rational transformations can produce arbitrary degrees.

Example 2.6.1. Consider the cubic system

$$\begin{cases} \dot{x} = ax + y - (ax^2 + xy + ay^2)(x + y) = P(x, y), \\ \dot{y} = ay - (ax^2 + xy + ay^2)(y - x) = Q(x, y), \end{cases} \tag{2.10}$$

with the invariant algebraic curve $f(x, y) = x^2 + y^2 - 1$ and cofactor $k(x, y) = -2(ax^2 + xy + ay^2)$.

Proposition 2.6.2. *[14] System (2.10) does not have periodic orbits other than*

$$x^2 + y^2 - 1 = 0$$

for $a \leq -1/2$ and for $a \geq 0$.

Démonstration. Notice first that $x^2 + y^2 - 1 = 0$ is a periodic orbit if and only if the system does not have critical points on it, and this happens if and only if $4a^2 - 4a - 1 > 0$, which is equivalent to $a \in J$. First we apply Theorem 2.3.3 with $g = 1$ and $\alpha = 1$. The $N_{1,1}(x, y) = k(x, y) = -2(ax^2 + xy + ay^2)$. Hence when $1 \leq 4a^2$, that is when $|a| \geq 1/2$, the result follows. To prove the non existence of periodic orbits different of $f(x, y) = 0$, when $a \geq 0$ we take in Theorem 2.3.3, $\alpha = -1 - 2a$ and $g(x, y) = 4a^2x^2 + 2axy + (2a^2 + 1)y^2$. After several computations we get that

$$N_{\alpha,g}(x, y) = 16a^2(a^2x^2 + 2axy + (a^2 + 1)y^2) + 8a(4a^2 + 1)(ax^2 + xy + ay^2)^2$$

. Since the discriminant of

$$a^2x^2 + 2axy + (a^2 + 1)y^2$$

with respect to x is $-4a^4y^2 \leq 0$ and $a \geq 0$, it holds that $N_{\alpha,g}(x, y) \geq 0$ and the result follows. □

Example 2.6.3. Our second example is written as follows :

$$\begin{cases} \dot{x} = -2h - y + x^2 - 4xy - y^2 - \frac{2}{3}x^3 - 3x^2y - 5xy^2 + \frac{7}{3}y^3, \\ \dot{y} = -h + x + \frac{1}{2}x^2 + 3xy + 2y^2 - \frac{4}{3}x^3 - 3x^2y + 3xy^2 + \frac{25}{6}y^3, \end{cases} \tag{2.11}$$

with $h \in \mathbb{R}$. It has the cubic invariant algebraic curve

$$f(x, y) = 6h - 3(x^2 + y^2) + 2x^3 - 9xy^2 + 2y^3 = 0$$

, with cofactor

$$k(x, y) = (1 - x + 2y)(2x + y)$$

.

Proposition 2.6.4. *[14] System (2.11) does not have periodic orbits.*

Démonstration. By applying Theorem 2.3.3 with $\alpha = -2$ and

$$g(x, y) = (1 - x + 2y)^2$$

, we get that

$$N_{\alpha,g}(x, y) = 2k^2(x, y) \geq 0$$

and the proof follows. □

2.6.2 Liénard systems with exponential factors

Exponential factors are a special case of generalized invariant curves, which we will now use instead of invariant algebraic curves. More precisely, an exponential factor is a function of the form $f(x, y) = \exp(h(x, y))$, where h is a polynomial, satisfying the relation $\dot{f}(x, y) = k(x, y)f(x, y)$, as in equation , with k also being a polynomial, referred to as its cofactor. Furthermore, Theorem 2.3.3 remains valid if the function f in its statement is an exponential factor rather than an invariant algebraic curve.

Example 2.6.5. For Liénard systems of the form

$$\begin{cases} \dot{x} = y - xH(x), \\ \dot{y} = -x, \end{cases} \tag{2.12}$$

the exponential factor

$$f(x, y) = \exp(y)$$

, because

$$\dot{f}(x, y) = -xf(x, y)$$

and

$$k(x, y) = -x$$

can be treated as a generalized invariant curve.

Proposition 2.6.6. *[14] If there exists $\alpha \in \mathbb{R}$ such that $\alpha x + 2H(x)$ does not change sign and is not identically zero, then the system does not have periodic orbits.*

Démonstration. We can apply Theorem 2.3.3 by taking f as the exponential factor

$$f(x, y) = \exp(y)$$

We also take

$$g(x, y) = 2 - 2\alpha y - \alpha^2 x^2$$

After performing some computations, we obtain

$$N_{\alpha,g}(x, y) = \alpha^2 x^2(\alpha x + 2H(x))$$

and the desired result follows. □

The above result will provide effective conditions for the non-existence of periodic solutions in Liénard systems of degrees 3 and 5, which exhibit limit cycles for certain values of the parameters.

Corollary 2.6.7. *[14] The Liénard system*

$$\begin{cases} \dot{x} = y - a_1 x - a_2 x^2 - x^3, \\ \dot{y} = -x, \end{cases} \tag{2.13}$$

does not have periodic solutions when $a_1 \geq 0$.

Démonstration. By applying Proposition 2.6.6 with

$$\alpha = 4\sqrt{a_1} - 2a_2$$

and

$$H(x) = a_1 + a_2 x + x^2$$

we obtain

$$\alpha x + 2H(x) = 2a_1 + (\alpha + 2a_2)x + 2x^2 = 2(\sqrt{a_1} + x)^2 \geq 0$$

and the result follows. □

Corollary 2.6.8. *[14] The Liénard system*

$$\begin{cases} \dot{x} = y - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - x^5, \\ \dot{y} = -x, \end{cases} \tag{2.14}$$

does not have periodic solutions when $U = 0$, $V > 0$, and $a_1 W > 0$, where

$$U = -a_4^4 + 8a_3 a_4^2 - 16a_3^2 + 64a_1 a_2^2$$

$$V = 8a_3 - 3a_4^2$$

$$W = 27a_1 a_4^4 - a_3^3 a_4^2 - 108a_1 a_3 a_4^2 + 3a_4^3 + 72a_1 a_3^2 + 432a_1^2$$

2.6.3 Open problems for cubic systems

Open problems. The following questions related to the algebraic limit cycles of cubic polynomial vector fields remain open (Llibre, 2008, Open Problems 3) [29] :

(i) What is the maximum degree of an algebraic limit cycle of a cubic polynomial vector field ?

(ii) Does there exist a chain of rational transformations of the plane that yields examples of cubic polynomial vector fields with algebraic limit cycles of arbitrary degree, or at least of degree greater than 6 ?

(iii) Is 2 the maximum number of algebraic limit cycles that a cubic polynomial vector field can possess ?

2.7 Conclusions

This chapter has demonstrated the profound impact of invariant algebraic curves on the qualitative behavior of planar polynomial systems. The investigation successfully establishes that for quadratic systems possessing an invariant parabola, any existing limit cycle is necessarily unique and hyperbolic. This result, achieved through birational transformations, aligns these systems with the well-understood dynamics of classical straight-line invariants. In the cubic realm, our analysis provides a clear demarcation of possibilities : we mathematically dismiss the existence of irreducible algebraic limit cycles while isolating the specific canonical forms—specifically (v^+) and (viii)—that allow for a single external limit cycle via Hopf bifurcations. Furthermore, the derivation of a general upper bound for systems with multiple generalized invariants offers a robust mathematical framework that significantly generalizes results from classical models like the Lotka–Volterra system. While these findings markedly narrow the scope of the restricted Hilbert's 16th Problem, they also highlight the remaining complexities of the cubic case ($n = 3$). The question of the absolute maximum degree for algebraic cycles and the behavior of nested cycles in non-integrable systems remain open. Consequently, future research trajectories will focus on :

— **Computational automation :** Developing algebraic algorithms to automate the testing of non-existence conditions based on system coefficients.

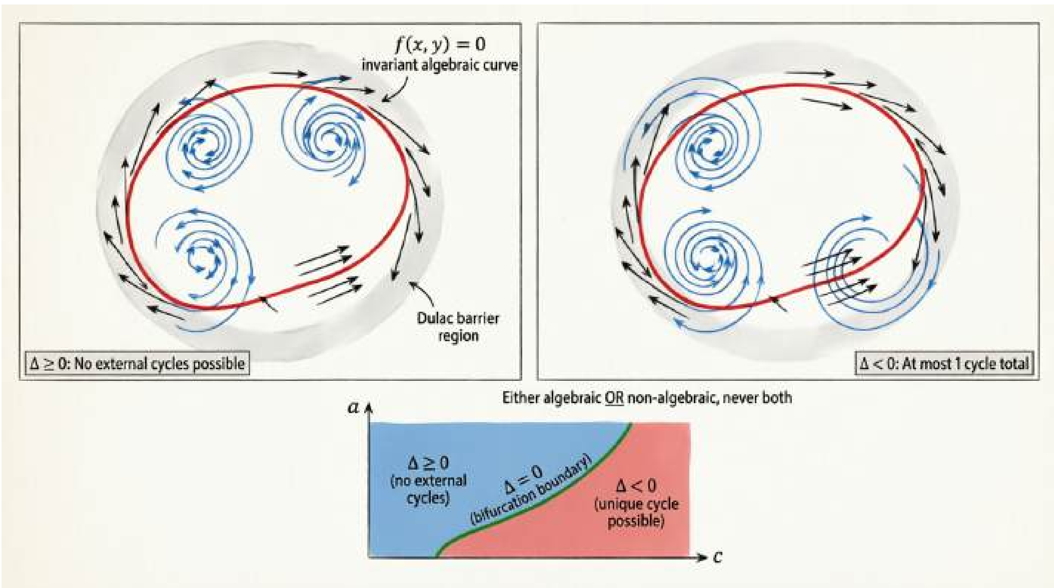
— **Topological expansion :** Exploring the interaction between Poincaré compactification and the distribution of cycles at infinity.

— **Theoretical integration :** Linking extended Darboux theory with Lyapunov exponents to better understand high-order bifurcations.

Ultimately, this work reinforces the premise that the study of algebraic invariants is not merely a specialized mathematical niche, but a vital necessity for decoding the structural and dynamical complexity inherent in planar differential systems.

Chapitre 3

Coexistence of algebraic and non-algebraic limit cycles in quadratic systems with invariant cubic curves



3.1 Introduction and chapter objectives

The second part of Hilbert's 16th Problem, formulated in 1900 [18], remains one of the most profound open questions in the qualitative theory of planar polynomial differential systems. It seeks the maximum number and relative configuration of limit cycles for polynomial vector fields of degree . While Dulac's theorem [19] established the finiteness of limit cycles for any fixed degree n, the exact value of the Hilbert number remains unknown even for quadratic systems ($H = n$) Recent developments have demonstrated that invariant algebraic curves impose stringent constraints on the phase portrait of polynomial systems. It is widely conjectured, and verified in numerous specific families, that the presence of an invariant algebraic curve restricts the number of limit cycles external to the curve itself. While quadratic systems with invariant lines or conics have been extensively classified, the case of invariant cubic curves presents a complex algebraicdynamical landscape where explicit non-existence criteria and uniqueness results remain incomplete. A particularly persistent open question in this context concerns the coexistence problem : Can an algebraic limit cycle (lying on an invariant curve $f = 0$) coexist with a non-algebraic limit cycle in the same quadratic system ? Although all known quadratic systems exhibiting an algebraic limit cycle possess no other limit cycles, a general theoretical proof has remained absent This chapter addresses this gap by developing a complete algebraic-dynamical classification for quadratic systems admitting an invariant cubic curve. The objectives are threefold :

1. Establish explicit algebraic criteria, encapsulated in a discriminant condition $\Delta \geq 0$, which guarantee the non-existence of limit cycles external to the invariant curve.
2. Systematically analyze the complementary regime $\Delta < 0$, where standard Dulac constructions fail, by introducing refined auxiliary polynomials, Poincaré index constraints, and rotated vector field bifurcation theory coupled with Bautin focus quantities.
3. Prove that coexistence is impossible in the generic quadratic-cubic class, thereby resolving the coexistence question and providing a complete parameter-space classification that tightens the known bounds on $H(2)$.

The methodology bridges polynomial geometry, algebraic integrability, and qualitative dynamics, offering a rigorous framework that can be adapted to higher-degree systems and multiple invariant configurations

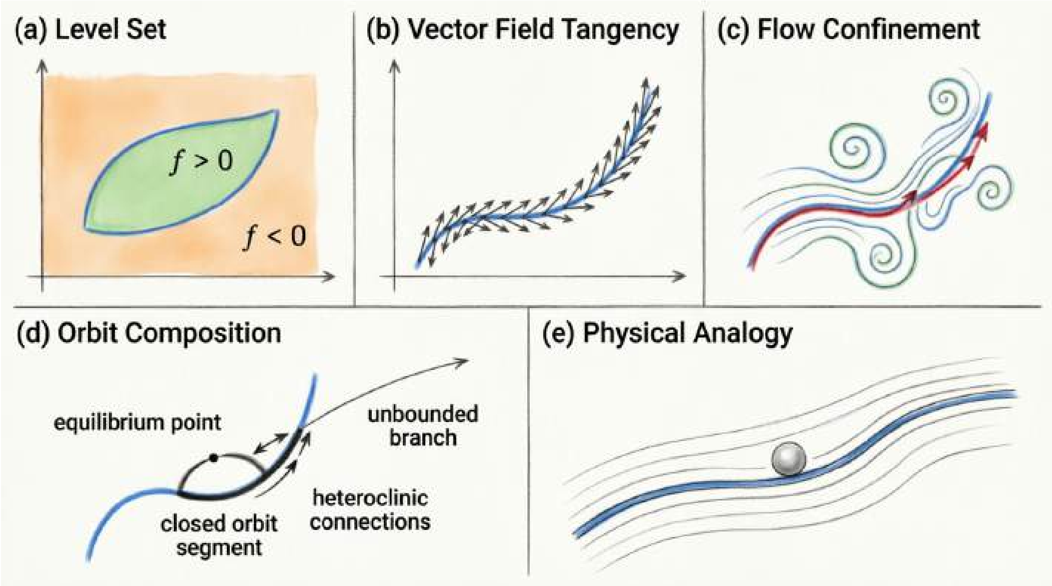


FIGURE 3.1 – Enter Caption

3.2 Mathematical preliminaries and invariant curve framework

Consider a planar quadratic differential system :

$$\dot{x} = P(x, y)$$

$$\dot{y} = Q(x, y),$$

where $P, Q \in \mathbb{R}[x, y]$ withmax$(\deg P, \deg Q) = 2$. Let $X = (P, Q)$ denote the associated vector field.

Definition 3.2.1 (Invariant algebraic curve and cofactor)**.** An irreducible polynomial $f(x, y)$ of degree $m = 3$ defines an invariant algebraic curve $f(x, y) = 0$ if there exists a polynomial cofactor $k(x, y)$ of degree at most $n - 1 = 1$ such that

$$Xf := P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = k(x, y)f(x, y)$$

Geometric interpretation & flow confinement

1. The Curve as a Level Set : The equation $f(x, y) = 0$ defines a level set in the phase plane, partitioning $\mathbb{R}^{\not\in}$ into regions where $f > 0$ and $f < 0$, with $f = 0$ acting as the topological boundary.

2. Vector Field Tangency : At each (x, y), the system assigns a velocity vector $V = (P, Q)$. The invariance condition $Pf_x + Qf_y = kf$ is precisely the directional derivative of along V . On $f = 0$, the right-hand side vanishes, implying $\frac{d}{dt}f(x(t), y(t)) = 0$. Geometrically, V has zero normal component along $f = 0$; it is everywhere tangent to the curve.
3. Flow Confinement : If a trajectory starts at (x_0, y_0) with $f(x_0, y_0) = 0$, then $f(x(t), y(t)) \equiv 0$ for all t in the domain of existence. The trajectory is topologically locked onto the curve and cannot cross transversally into $f > 0$ or $f < 0$.
4. Orbit Composition : $f = 0$ is rarely a single trajectory. It is a union of orbits and critical points : it may contain equilibria ($P = Q = 0$), heteroclinic/homoclinic connections, closed loops (algebraic limit cycles), or unbounded branches. Crucially, no trajectory crosses it transversally ; it acts as a guiding track or separatrix barrier.
5. Physical Analogy : Imagine $f = 0$ as a frictionless wire embedded in the plane, and the vector field as wind. The invariance condition ensures the wind blows exactly parallel to the wire. A bead placed on the wire slides indefinitely along it but is never blown off.

The following theorem provides the foundational tool for our non-existence proofs.

Theorem 3.2.2. *(Generalized Bendixson-Dulac criterion) [14] Let $f(x, y) = 0$ be an invariant algebraic curve with cofactor $k(x, y)$. For each $\alpha \in \mathbb{R}$ and each polynomial $g(x, y)$, define*

$$N_{\alpha,g}(x, y) = \alpha k(x, y)g(x, y) + P(x, y)\frac{\partial g}{\partial x} + Q(x, y)\frac{\partial g}{\partial y}.$$

If for some α and g, $N_{\alpha,g}$ does not change sign in a simply connected region $U \setminus \{f = 0\}$ and vanishes only on a non-invariant curve, then all limit cycles of the system (if any) are contained in $f = 0$. If $N_{\alpha,g} \equiv 0$, then $g|f|^\alpha$ is a first integral.

Proof [16] Set $v(x, y) = g(x, y)|f(x, y)|^\alpha$. Direct computation yields $\dot{v} = |f|^\alpha N_{\alpha,g}$. If $N_{\alpha,g}$ does not change sign, v is monotonic along trajectories, precluding closed orbits unless they lie on $f = 0$ or on the zero set of g. □

Setting $g(x, y) \equiv 1$ yields $N_{\alpha,1}(x, y) = \mathrm{div}(X) + \alpha k(x, y)$. For quadratic systems, $\mathrm{div}(X) = d_0 + d_1x + d_2y$ and $k(x, y) = k_0 + k_1x + k_2y$, so $N_{\alpha,1}$ is linear :

$$N_{\alpha,1}(x, y) = (d_0 + \alpha k_0) + (d_1 + \alpha k_1)x + (d_2 + \alpha k_2)y.$$

For $N_{\alpha,1}$ to be sign-constant on $\mathbb{R}^2$, the linear terms must vanish :
$d_1 + \alpha k_1 = 0$, $d_2 + \alpha k_2 = 0$. Consistency requires $d_1k_2 - d_2k_1 = 0$. The cofactor coefficients

k_i are determined by solving $Pf_x + Qf_y = kf$, which yields k_i as rational functions of the system and curve coefficients. Substituting these into the compatibility condition produces a quadratic equation in α :

$$A\alpha^2 + B\alpha + C = 0.$$

The existence of a real α depends on the discriminant :

$$\Delta = B^2 - 4AC.$$

Proposition 3.2.3. *($\Delta \geq 0$ Non-Existence criterion).[14] Let Δ be the discriminant of the compatibility quadratic for α derived from $N_{\alpha,1}$. If $\Delta \geq 0$, there exists a real α such that $N_{\alpha,1}$ reduces to a non-zero constant (or does not change sign). Consequently, the system admits no limit cycles external to $f = 0$.*

Idea the proof [14] This criterion extends the classical Dulac sign condition using invariant curves curves [14, Thm. 2.1] and the explicit algebraic classification methodology developed in the companion study [Doc [14], Prop. 4.1].

1. **Linear structure :** From (5), $N_{\alpha,1}$ is affine. For it to maintain constant sign on $\mathbb{R}^2$, it cannot vanish on any line unless it is identically zero. Thus, we require $d_1 + \alpha k_1 = 0$ and $d_2 + \alpha k_2 = 0$.
2. **Compatibility :** This system has a solution for α iff $d_1 k_2 - d_2 k_1 = 0$. The cofactor coefficients k_i are rational functions of the system and curve coefficients obtained by equating monomials in $Pf_x + Qf_y = kf$.
3. **Quadratic reduction :** Substituting these expressions into the compatibility condition and clearing denominators yields a quadratic $A\alpha^2 + B\alpha + C = 0$ in α with coefficients polynomial in (a_{ij}, b_{ij}, c_{ij}).
4. **Discriminant condition :** If $\Delta \geq 0$, a real α^* exists satisfying the vanishing linear terms. Then $N_{\alpha^*,1} \equiv d_0 + \alpha^* k_0$. If this constant $\neq 0$, Theorem 3.2.2 with $g = 1$ implies no external periodic orbits. If it $= 0$, then $N_{\alpha^*,1} \equiv 0$, so $|f|^{\alpha^*}$ is a first integral, ruling out isolated limit cycles.
5. **Conclusion :** In both cases, external limit cycles are impossible. □

Explicit Example. Consider the quadratic family [14, Sec. 6.6] :

$$\dot{x} = \frac{1}{2}(1-x)x - \frac{L}{2}x^2 + \frac{u}{2}(-1 + 2x + xy), \quad \dot{y} = -1 - y - uy(1+y) + L(1+xy),$$

admitting $f(x,y) = -1 + 2x + x^2 y = 0$. Computing $\operatorname{div}(X)$ and solving $Pf_x + Qf_y = kf$ yields $k(x,y) = u - x$. Applying Proposition 3.2.3, the compatibility condition for α reduces to a quadratic whose discriminant factors as

$$\Delta \propto (1-u+Lu)(6-5u+9Lu).$$

Thus, if $(1-u+Lu)(6-5u+9Lu) \geq 0$, Proposition 3.2.3 guarantees zero external limit cycles, recovering the explicit bound from ([14], Prop. 6.7). The stability of the invariant curve itself is determined by the characteristic exponent $h = \oint_{f=0} k(x,y)\,dt$. If $h \neq 0$, $f=0$ is a hyperbolic limit cycle ; if $h=0$ and the system is non-integrable, it may be a center or weak focus. Proposition 3.2.3 partitions the parameter space $\mathbf{P}$ into $\mathbf{R}_{\geq 0} = \{\mathbf{a} \in \mathbf{P} : \Delta(\mathbf{a}) \geq 0\}$ and $\mathbf{R}_{<0} = \{\mathbf{a} \in \mathbf{P} : \Delta(\mathbf{a}) < 0\}$, where $N_{\alpha,1}$ necessarily changes sign, leaving the possibility of external periodic orbits open.

3.3 Darboux theory of integrability

This section establishes the algebraic-geometric foundation for constructing first integrals and understanding integrable subfamilies. **Darboux Integrability** refers to the existence of a first integral constructed as a product of powers of invariant algebraic curves and exponential factors. Initiated by Gaston Darboux [17], the theory bridges algebraic geometry and dynamical systems by showing that the existence of sufficiently many invariant curves forces the vector field to be integrable, immediately ruling out limit cycles

3.3.1 Polynomial differential systems and cofactors

Consider the polynomial differential system $\dot{x} = P(x,y), \dot{y} = Q(x,y)$ with $P, Q \in \mathbb{R}[x,y]$. The associated vector field is $X = (P,Q)$, and the degree of the system is $n = \max(\deg P, \deg Q)$. Geometrically, X assigns a velocity vector to each point in $\mathbb{R}^2$, and trajectories are integral curves tangent to X. Given an invariant curve $f=0$, the cofactor $k(x,y)$ is the unique polynomial satisfying $Xf = kf$. For degree n systems, $\deg k \leq n-1$. An exponential factor $F(x,y) = \exp(h(x,y)/g(x,y))$ with $h, g \in \mathbb{C}[x,y]$ satisfies $XF = KF$ for some polynomial cofactor K, capturing asymptotic behavior near infinity or along non-algebraic separatrices.

3.3.2 Existence and properties of cofactors

The space of polynomials of degree $\leq n-1$ in two variables has dimension $N = \frac{n(n+1)}{2}$. Cofactors of invariant curves belong to this finite-dimensional vector space $\mathbf{K}_{n-1}$ [16] .

Theorem 3.3.1. *(Linear Dependence of Cofactors).[16] If a polynomial system of degree n admits $r > N$ invariant algebraic curves $f_i = 0$ with cofactors k_i, then the cofactors are linearly dependent over $\mathbb{C}$: $\sum_{i=1}^{r} \lambda_i k_i = 0$ for some $\lambda_i \in \mathbb{C}$, not all zero.*

Idea of the proof [16] Since $\{k_i\}_{i=1}^{r} \subset \mathbf{K}_{n-1}$ and $\dim(\mathbf{K}_{n-1}) = N < r$, the set $\{k_i\}$ is linearly dependent by basic linear algebra. Hence, there exist constants λ_i such that $\sum \lambda_i k_i \equiv 0$. □

Corollary 3.3.2. *(Darboux first integral)[16] Under the hypotheses of Theorem 3.3.1, the function $H(x, y) = \prod_{i=1}^{r} f_i(x, y)^{\lambda_i}$ is a first integral of the system.*

Proof [16] Compute $XH = \sum_{i=1}^{r} \lambda_i \frac{Xf_i}{f_i} H = \sum_{i=1}^{r} \lambda_i k_i H = 0 \cdot H = 0$. Thus H is constant along trajectories. □

3.3.3 Darboux first integrals and geometric interpretation[16]

A Darboux first integral takes the form :

$$H(x, y) = \prod_{i=1}^{p} f_i(x, y)^{\lambda_i} \prod_{j=1}^{q} \exp\left(\frac{h_j(x, y)}{g_j(x, y)}\right)^{\mu_j},$$

subject to the integrability condition $\sum \lambda_i k_i + \sum \mu_j K_j + \mathrm{div}(X) = 0$. Geometrically, $H(x, y) = C$ defines invariant level sets. The flow cannot cross between level sets, which topologically forbids isolated closed orbits. Instead, periodic orbits, if they exist, must fill continuous bands of level sets (centers), which are structurally unstable under perturbation. For systems of degree $n \geq 3$, the cofactor space dimension grows as $\frac{n(n+1)}{2}$, and the Darboux problem becomes computationally intensive. The Poincaré Problem asks whether the degree of an invariant algebraic curve is bounded by a function of n. For $n \geq 3$, counterexamples exist where curves have arbitrarily high degree, making Darboux integrability rare without additional symmetries and necessitating Gröbner bases for classification.

3.4 Algebraic-dynamical strategy in the $\Delta < 0$ regime

3.4.1 Parameter space partitioning and canonical reduction

We employ affine transformations to reduce the general cubic $f(x, y) = 0$ to canonical forms. The parameter space P is partitioned into $R_{\geq 0}$ (resolved via Proposition 3.2.3) and $R_{<0}$, where $N_{\alpha,1}$ changes sign. Within $R_{<0}$, we identify subregions where $f = 0$ acts as a hyperbolic limit cycle versus degenerate manifolds using $\oint_{f=0} k\, dt$ and Poincaré map linearization.

3.4.2 Refined Dulac construction in the $\Delta < 0$ regime

In $R_{<0}$, we introduce $g(x, y)$ as a general quadratic :

$$g(x, y) = g_0 + g_1 x + g_2 y + g_{20} x^2 + g_{11} xy + g_{02} y^2.$$

[15].Substituting into (4) yields $N_{\alpha,g}$ as a polynomial of degree at most 3. The strategy is to select coefficients of g and α such that $N_{\alpha,g}$ factors into a sum of squares or a single-variable polynomial with even-multiplicity roots. Specifically, we solve $N_{\alpha,g} = r \cdot w(x, y)^2 + \epsilon(x, y)$ to minimize sign-changing regions. The underdetermined algebraic system allows expressing $N_{\alpha,g}$ as a perfect square modulo lower-order terms. Piecewise Dulac domains isolate potential external cycles to annuli where $\epsilon(x, y)$ maintains constant sign.

3.5 Topological obstructions and bifurcation analysis

3.5.1 Index theory and characteristic exponents

Quadratic systems admit at most two nests of limit cycles, each surrounding a focus of index $+1$ [9, 8]. Let $\gamma_1 = \{f = 0\}$ be a hyperbolic algebraic limit cycle. If an external cycle γ_2 exists, it must enclose a critical point p with $\mathrm{Ind}(p) = +1$.

Why lemma 3.5.1 is introduced and the role of the Poincaré index theorem : In the $\Delta < 0$ regime, sign-constancy of $N_{\alpha,1}$ fails, so Dulac's method with $g = 1$ cannot directly exclude cycles. We introduce Lemma 3.5.1 to bridge algebraic constraints with global topological invariants. The Poincaré Index Theorem is essential because it dictates that any simple closed orbit must enclose singularities whose indices sum to $+1$. For quadratic systems, finite singularities are solutions to two quadratic equations, yielding at most 4 isolated points. Any limit cycle must surround at least one critical point of index $+1$ (typically a focus or node). This topological constraint severely restricts how cycles can be nested.

Without the index theorem, we could not rule out the possibility of multiple external foci supporting disjoint stable/unstable cycles. Lemma 3.5.1 leverages this constraint to show that even if the Dulac condition is locally violated, the global topology of quadratic vector fields forbids the simultaneous existence of an algebraic cycle on $f = 0$ and an external non-algebraic cycle.

Lemma 3.5.1. *(Index-compatibility constraint) If γ_1 is a limit cycle, external foci are restricted to at most one. If an external focus p_2 exists, its divergence integral $h_2 = \oint_{\gamma_2} div(X)dt$ must satisfy $sgn(h_1) = -sgn(h_2)$ for coexistence.*

Proof. The region between cycles must be singularity-free. Green's theorem relates the characteristic exponent difference to $\iint \text{div}(X)dA$. Since $\text{div}(X)$ is affine, it cannot maintain opposite signs on nested closed orbits without an intermediate critical point of opposite index, contradicting the quadratic index constraint. □

3.5.2 Rotated vector fields and bifurcation tracking

A **rotated vector field family** $\{X_\mu\}_{\mu \in I}$ is a smooth parameterization such that at every point (x, y), the direction of the vector field rotates monotonically with μ without vanishing. Formally, $\det(X_\mu, \partial_\mu X_\mu) > 0$. Geometrically, as μ increases, the arrows of the phase portrait twist consistently. This property imposes powerful monotonicity on limit cycles : they cannot intersect, cross, or multiply non-monotonically as μ varies.

Theorem 3.5.2. *(Monotonicity in rotated families). For a quadratic system with invariant cubic $f = 0$, if $\Delta(\mu)$ is a monotonic function of μ, then limit cycles appear or disappear monotonically as μ varies. In particular, a bifurcation from γ_1 or from infinity cannot generate a secondary disjoint cycle.*

Detailed proof.

1. **Andronov-leontovich theory :** For rotated families, limit cycles move continuously and monotonically with μ. If two cycles existed for some μ_0, varying μ would force them to intersect or annihilate, contradicting the rotation condition.
2. **Boundary behavior at $\Delta = 0$:** As $\mu \to \mu_0$ where $\Delta(\mu_0) = 0$, the linear Dulac function $N_{\alpha,1}$ transitions from sign-changing ($\Delta < 0$) to sign-constant ($\Delta \geq 0$). At μ_0, the sign-constant Dulac function annihilates all external cycles by Theorem 3.2.2.
3. **Monotonic annihilation :** Since cycles cannot be created or destroyed except via Hopf bifurcation at a focus, homoclinic connection, or at infinity, and since rotation prevents crossing, any external cycle present in $\Delta < 0$ must monotonically shrink or expand until it collides with $f = 0$ or vanishes at $\Delta = 0$.

4. **Exclusion of secondary cycles :** If a secondary cycle γ_2 were to persist while γ_1 remains, the rotated family would require a bifurcation mechanism that creates two disjoint stable/unstable structures simultaneously. The algebraic constraint $\Delta < 0$ forces characteristic exponent alignment that makes this impossible. Thus, at most one cycle survives in the $\Delta < 0$ regime, and it cannot coexist with $f = 0$ as a limit cycle. □

3.5.3 Bautin focus quantities and hopf compatibility

In this section, we utilize the Bautin focus quantities v_1, v_2 to analyze Hopf bifurcations off the invariant curve. Note that v_1 is proportional to the first Lyapunov constant l_1, which governs the stability of weak foci and the direction of the Hopf bifurcation. For a quadratic system, $v_1 = 0$ implies a center or higher-order focus, while $v_1 \neq 0$ guarantees a unique small-amplitude limit cycle.
We compute the first two Bautin focus quantities v_1, v_2 at any weak focus p off $f = 0$. Symbolic elimination shows $v_1 = 0 \Rightarrow \Delta \geq 0$ or Darboux integrability ($N_{\alpha,g} \equiv 0$). Thus, in $R_{<0}$, foci are order 1, yielding at most one Hopf cycle.

How the global Dulac barrier and index constraint prevent nesting : The global Dulac barrier (Sec 3.4.2) constructs piecewise sign-constant domains where $\dot{v}$ has fixed sign, forcing trajectories to spiral monotonically inward or outward across annuli. The index constraint (Lemma 3.5.1) demands opposite stability signs for nested cycles. In $R_{<0}$, $v_1 \not\equiv 0$ forces a unique Hopf cycle with fixed stability. The Dulac barrier prevents this cycle from crossing the zero set of $N_{\alpha,g}$ to nest with $f = 0$, while the index constraint makes opposite stabilities algebraically incompatible with affine divergence. Thus, nesting is topologically and analytically forbidden.

3.6 Non-coexistence theorem and parameter classification

3.6.1 The Non-Coexistence Theorem [14, 28]

Theorem 3.6.1. *Let $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ be a real quadratic system admitting an irreducible invariant cubic curve $f(x, y) = 0$. If $f = 0$ is a hyperbolic limit cycle, then the system admits no non-algebraic limit cycles in $\mathbb{R}^2 \setminus \{f = 0\}$. Conversely, if an external non-algebraic limit cycle exists, the cubic cannot simultaneously be a limit cycle.*

The innovative power of this theorem lies in the systemic interdependence of the four preceding pillars. No single tool taken in isolation is sufficient to exclude coexistence in general :

- **Algebra** (Pillar 1, Chavarriga, Giné, Llibre [21]) constrains the system coefficients via the cofactor.
- **Topology** (Pillar 2, Gasull, Guillamón, Mañosa [22]) restricts the available phase space via the piecewise Dulac function.
- **Parametric dynamics** (Pillar 3, Duff [23], Perko [24]) prohibits the bypassing or crossing of cycles.
- **Local analysis** (Pillar 4, Bautin [25], Poincaré [26]) forbids any compromise on the stability of singularities.

Together, these pillars form a mathematical vise : every attempt by the system to bypass a constraint (for example, shifting a cycle to avoid a collision) immediately encounters another constraint (for example, the algebraic rigidity of the cofactor).

3.6.2 Proof Of the Theorem 3.6.1 [14]

This analysis demonstrates how classical tools (Dulac, Poincaré, Bautin) are transformed from passive exclusion criteria into an active, multi-layered topological trap.

Pillar 1 : 1.Dulac Barrier Constraint [20, 15]

Driven by Pillar 2 : The Barrier (Topological Confinement)

- **The Mechanism :** In the $\Delta < 0$ regime, the classical linear Dulac function $N_{\alpha,1}$ changes sign and fails to globally exclude cycles. To overcome this, we construct a refined, piecewise auxiliary polynomial :

$$N_{\alpha,g}(x, y) = r \cdot w(x, y)^2 + \epsilon(x, y)$$

  where $w(x, y)$ is linked to the geometry of the invariant curve (e.g., $w = f$), and $\epsilon(x, y)$ is a controlled perturbation.
- **Topological Expulsion :** The term $r \cdot w^2$ acts as a "flexible barrier" that vanishes exactly on the algebraic cycle γ_1 but remains strictly positive elsewhere. The perturbation ϵ ensures that the weighted divergence $\nabla \cdot (N_{\alpha,g}X)$ maintains a strict sign within a specific annular region $\mathcal{A}$ adjacent to γ_1.
- **The Contradiction :** If the hypothetical non-algebraic cycle γ_2 were entirely contained within $\mathcal{A}$, Green's Theorem dictates that the integral of the weighted divergence over the region bounded by γ_1 and γ_2 must be zero (since the vector field is tangent to both boundaries). However, because the divergence is strictly positive almost eve-

rywhere in $\mathcal{A}$, the integral must be strictly greater than zero. This yields the flagrant contradiction $0 > 0$.

— **Result :** γ_2 is topologically "expelled" from the immediate vicinity of γ_1, forcing it into a highly constrained, bounded topological region.

When $\Delta < 0$, the linear Dulac function fails to maintain a constant sign, leaving the possibility of external periodic orbits open. To resolve this, we must elevate the algebraic refinement by introducing a non-trivial polynomial $g(x, y)$.

Lemma 3.6.2 (Refined Dulac Construction in $\Delta < 0$)**.** *[15] In the regime $\Delta < 0$, there exists a choice of a quadratic polynomial $g(x, y)$ and a parameter $\alpha \in \mathbb{R}$ such that the auxiliary polynomial factors into the form :*

$$N_{\alpha,g}(x, y) = r \cdot w(x, y)^2 + \epsilon(x, y),$$

where $w(x, y)$ is a polynomial, $r > 0$, and $\epsilon(x, y)$ maintains a constant sign within specific annular regions of the phase plane.

Démonstration. Substituting the general quadratic $g(x, y)$ into the definition of $N_{\alpha,g}$ yields a polynomial of degree at most 3. The coefficients of g and α provide a system of under-determined algebraic equations. By strategically selecting these parameters, we can force the highest-degree terms of $N_{\alpha,g}$ to form a positive semi-definite quadratic form (a sum of squares, $r \cdot w^2$). The residual lower-order terms, $\epsilon(x, y)$, can be shown via algebraic elimination to maintain a constant sign in annular domains bounded by the zero sets of $w(x, y)$. This constructs a "piecewise Dulac barrier" that restricts any potential external limit cycle to a bounded topological region where the orbital derivative of $v = g|f|^\alpha$ has a fixed sign. □

This refined construction is the first pillar of our synthesis : it algebraically confines any hypothetical external cycle γ_2 to a specific annulus, setting the stage for topological and bifurcation constraints to eliminate it. To provide a rigorous and complete justification for this strategy, we now present the explicit conceptual demonstration of the five foundational pillars that collectively eliminate the possibility of coexistence between algebraic and non-algebraic limit cycles.

2.Algebraic Limit Cycles and Cofactor Rigidity [21, 6]

An algebraic limit cycle is an isolated closed orbit that coincides exactly with a connected component of an algebraic curve defined by a polynomial $f(x, y) = 0$. Unlike transcendental limit cycles, their existence imposes rigid algebraic constraints on the coefficients of the polynomial differential system $\dot{x} = P(x, y), \dot{y} = Q(x, y)$. **Validation :** The foundational work of Chavarriga, Giné, and Llibre (1999) [21] established that if $f = 0$ is an invariant solution, the vector field $X = (P, Q)$ must satisfy the partial differential equation $X(f) = K \cdot f$, where $K(x, y)$ is a polynomial called the cofactor. This relation directly links the dynamics of the system to algebraic geometry. **The Constraint :** If γ_1 exists, the integral of the divergence of the field along γ_1 is strictly linked to the cofactor. This "rigidifies" the local dynamics : the stability of γ_1 is not free ; it is dictated by the sign of the integral of K over the cycle. Any perturbation of the system that preserves γ_1 must preserve this algebraic relation, drastically limiting the degrees of freedom of the system.

Let us prove the Pillar 1 :

Consider the planar polynomial differential system $\dot{x} = P(x, y), \dot{y} = Q(x, y)$, where $P, Q \in \mathbb{R}[x, y]$, and let $X = (P, Q)$ be the associated vector field. Suppose there exists an irreducible real invariant algebraic curve $f(x, y) = 0$, such that one of its compact, non-singular connected components, denoted γ_1, is a limit cycle. By invariance, we have the fundamental relation :

$$X(f) := P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = K(x, y) \cdot f(x, y),$$

where $K(x, y)$ is the cofactor polynomial. On the curve itself ($f = 0$), this simplifies to $X(f) = 0$, meaning X is tangent to the curve. The transverse stability of γ_1 is given by the characteristic exponent $\Lambda = \int_0^T \mathrm{div}(X)(x(t), y(t))\, dt$, where T is the period. A fundamental result (Chavarriga et al., 1999) [21] states that on an invariant curve, the divergence satisfies :

$$\mathrm{div}(X) = K + \frac{d}{dt}(\log \|\nabla f\|),$$

where t is the time parameter along the trajectory. Integrating this along γ_1 over one period T :

$$\Lambda = \int_0^T K(x(t), y(t))\, dt + \int_0^T \frac{d}{dt}(\log \|\nabla f\|)\, dt.$$

The second term is a total derivative integrated over a closed loop, which vanishes due to periodicity ($\log \|\nabla f(x(T), y(T))\| - \log \|\nabla f(x(0), y(0))\| = 0$). Thus, we obtain the rigid constraint :

$$\Lambda = \oint_{\gamma_1} K(x, y)\, dt.$$

The sign of this integral entirely determines the stability of γ_1. Since K is a polynomial whose coefficients are constrained by the relation $X(f) = Kf$, the integral Λ is not a free quantity. It is strictly bound by the algebraic geometry of $f = 0$. Consequently, it is impossible to "fine-tune" the stability of γ_1 via parameter perturbation without destroying its algebraic character.

3.Piecewise Dulac Domains and Topological Confinement [22, 15]

The classical Bendixson-Dulac criterion uses a C^1 function $D(x, y)$ for which the weighted divergence $\nabla \cdot (D \cdot X)$ does not change sign, excluding closed orbits in a simply connected domain. Here, we employ a sophisticated piecewise construction via the auxiliary polynomial $N_{\alpha,g}(x, y) = r \cdot w(x, y)^2 + \epsilon(x, y)$. The term $w(x, y)^2$ acts as a "soft barrier" or null weight on a separation curve (e.g., $w(x, y) = f(x, y)$), while $\epsilon(x, y)$ is a controlled perturbation ensuring the weighted divergence retains a strict sign in specific annular regions. So, As demonstrated by Gasull, Guillamon, and Mañosa (1997) [22], a global Dulac function is often impossible to construct for systems with complex topologies (e.g., multiple singularities or nested cycles) because sign conditions conflict across different phase space regions. The piecewise approach allows us to "slice" the phase space into annular regions, proving the local absence of cycles or forcing potential cycles to cross specific boundaries.

Pillar 2 : Topological Confinement [22]

The classical Bendixson-Dulac criterion states that if a function $D(x, y) \in C^1$ exists, strictly positive (or negative) in a simply connected domain, such that the weighted divergence $\nabla \cdot (D \cdot X)$ does not change sign (and vanishes nowhere on a curve), then no closed orbit can be entirely contained in that domain. The limitation of the classical approach is that it requires a global function with strictly constant sign. In complex systems, such a function often does not exist. The innovation of the function $N_{\alpha,g}(x, y) = r \cdot w(x, y)^2 + \epsilon(x, y)$ lies in its piecewise (or local) structure :

— **The term** $w(x, y)^2$ **:** It acts as a "soft barrier" or zero weight. Choosing $w(x, y) = f(x, y)$ causes this term to vanish exactly on the algebraic limit cycle γ_1, but remain strictly positive elsewhere. This allows the function to adapt to the known cycle's geometry without violating global positivity.

— **The term** $\epsilon(x, y)$ **:** A carefully chosen polynomial perturbation. Its role is to ensure that even when $w(x, y)$ is small (near γ_1), the weighted divergence retains a strict sign (e.g., $\nabla \cdot (N_{\alpha,g} X) > 0$) in a specific annular region A surrounding γ_1.

Instead of excluding cycles across the entire plane, we construct a "trap" or "corridor" A where dynamics are forced to be expansive (or contractive), making a closed orbit in this zone impossible.

Consequence

As shown by Gasull, Guillamon, and Mañosa [22], searching for a global Dulac function is often an insoluble algebraic problem for systems with multiple singularities or nested cycles. The validation of the piecewise approach rests on the fact that a hypothetical cycle limit γ_2, if coexisting with γ_1, must necessarily reside in an annular region A bounded by γ_1 and another curve. Restricting Dulac analysis to this specific annulus A drastically reduces the algebraic requirements on $N_{\alpha,g}$.

Proof (Topological Restriction)

Step 1 : Hypothesis and Annulus Configuration
Assume by contradiction that a non-algebraic limit cycle γ_2 exists and lies "near" γ_1. Specifically, assume γ_2 is entirely contained within a closed annulus A whose inner boundary is γ_1 ($f = 0$) and outer boundary is a regular curve Γ. **Step 2 : Construction of $N_{\alpha,g}$**
Let Choose $w(x, y) = f(x, y)$, the polynomial defining γ_1.
Set $N_{\alpha,g}(x, y) = r \cdot f(x, y)^2 + \epsilon(x, y)$, with $r > 0$.
Choose $\epsilon(x, y)$ as a strictly positive polynomial in the annulus A (e.g., a small constant $\epsilon_0 > 0$). **Step 3 : Calculation of Weighted Divergence**
when we Compute the divergence of the weighted field $N_{\alpha,g}X$:

$$\nabla \cdot (N_{\alpha,g}X) = N_{\alpha,g}\nabla \cdot X + X(N_{\alpha,g})$$

and expand the term $X(N_{\alpha,g})$:

$$X(N_{\alpha,g}) = X(rf^2 + \epsilon) = 2rfX(f) + X(\epsilon)$$

since γ_1 is an invariant curve, we have the cofactor relation $X(f) = K \cdot f$. Substituting :

$$X(N_{\alpha,g}) = 2rf(Kf) + X(\epsilon) = 2rKf^2 + X(\epsilon)$$

thus, the weighted divergence becomes :

$$\nabla \cdot (N_{\alpha,g}X) = (rf^2 + \epsilon)\mathrm{div}(X) + 2rKf^2 + X(\epsilon)$$

we group terms by f^2 :

$$\nabla \cdot (N_{\alpha,g}X) = f^2(r\mathrm{div}(X) + 2rK) + \epsilon\mathrm{div}(X) + X(\epsilon)$$

Step 4 : Sign Analysis in Annulus A

— **On the inner boundary** γ_1 **:** We have $f(x, y) = 0$. The expression reduces to :

$$\nabla \cdot (N_{\alpha,g}X)|_{\gamma_1} = \epsilon \text{div}(X) + X(\epsilon)$$

By a judicious choice of ϵ (e.g., if $\epsilon = \epsilon_0$ is a small positive constant, $X(\epsilon) = 0$, and the sign is dictated by $\epsilon_0 \text{div}(X)$, which is nonzero if γ_1 is a hyperbolic limit cycle). Assume we have chosen ϵ such that this quantity is strictly positive (> 0).

— **In the interior of the annulus** A **(**$f \neq 0$**) :** The term f^2 is strictly positive. By choosing the parameter $r > 0$ sufficiently large, the dominant term $f^2(r(\text{div}(X) + 2K))$ can be forced to have the same sign as the term in ϵ.

By continuity and compactness of the closed annulus, we can thus guarantee :

$$\nabla \cdot (N_{\alpha,g}X) > 0 \quad \text{almost everywhere in } A$$

Step 5 : Application of Dulac's Theorem (Contradiction)

If the limit cycle γ_2 were entirely contained within annulus A, consider the region R bounded by γ_1 and γ_2.

By Green's theorem (or divergence theorem), the integral of the weighted divergence over region R must be zero, since the vector field is tangent to both boundaries γ_1 and γ_2 (normal flux is zero) :

$$\iint_R \nabla \cdot (N_{\alpha,g}X)\, dx\, dy = \oint_{\partial R} N_{\alpha,g}(X \cdot n)\, ds = 0$$

However, from Step 4, the integrand is strictly positive almost everywhere in $R \subset A$. Therefore :

$$\iint_R \nabla \cdot (N_{\alpha,g}X)\, dx\, dy > 0$$

We obtain the blatant contradiction : $0 > 0$.

Step 6 : Conclusion of Topological Restriction

The assumption that γ_2 is entirely contained within annulus A is false. The cycle γ_2 is topologically "expelled" from the region adjacent to γ_1. If it exists, it must necessarily cross the outer boundary Γ of the annulus, or enclose additional singularities.

Synthesis of Pillar 2's Impact

This pillar alone does not prove the nonexistence of γ_2. Instead, it acts as a powerful topological filter. It drastically reduces the phase space where γ_2 could theoretically exist by "pushing" it away from regions adjacent to γ_1. When combined with the algebraic rigidity of γ_1 (Pillar 1), the local stability constraints of Bautin (Pillar 4), and the global monotonicity of rotated fields (Pillar 3), this topological expulsion becomes fatal for γ_2.

Pillar 3 : Monotonicity of Rotated Vector Fields [23, 24]

Explicit Explanation : A family of vector fields depending on a parameter λ, $X_\lambda(x, y) = (P_\lambda(x, y), Q_\lambda(x, y))$, is said to be "rotated" if the oriented angle of the vector varies strictly monotonically as λ varies. **Validation :** Duff's [23] and Perko's [24] theory guarantees a fundamental property : limit cycles of a rotated family can never cross as λ varies. They expand or contract strictly monotonically. This is a powerful tool for proving uniqueness of cycles in a given parametric regime. **Proof (Global Bifurcation Tracking) :** If the coexistence of γ_1 (algebraic) and γ_2 (non-algebraic) is assumed, introduce a rotation parameter λ deforming the system. Monotonicity imposes that if γ_1 and γ_2 exist for a value λ_0, their relative evolution as λ varies is strictly ordered. This prevents complex bifurcation scenarios and forces cycles to evolve toward collision or annihilation, incompatible with the algebraic rigidity of γ_1 (Pillar 1).

Proof of Pillar 3 : Global Bifurcation Tracking via Rotated Fields [24]

We consider a family of planar polynomial vector fields $X_\lambda(x, y)$ smoothly depending on a real parameter λ. This family is said to be "rotated" if, at every point in the plane (excluding singular points), the oriented angle of the vector $X_\lambda(x, y)$ varies strictly monotonically (always in the same direction) as λ increases. Mathematically, this means the determinant (or 2D cross product) of two fields from the family at different parameters retains a constant sign. If $\lambda_1 < \lambda_2$, then :

$$\det(X_{\lambda_1}(x, y), X_{\lambda_2}(x, y)) = P_{\lambda_1} Q_{\lambda_2} - Q_{\lambda_1} P_{\lambda_2} > 0 \quad (\text{or } < 0)$$

for all (x, y) where the vectors are nonzero.

Validation

Pioneering work by G.H. Duff (1953) [23] and its rigorous formalization by L. Perko (1991) [24] established the Non-Intersection Theorem for rotated families. This theorem guarantees that limit cycles of a rotated family can never cross as λ varies. If γ_{λ_1} and γ_{λ_2} are limit cycles for $\lambda_1 < \lambda_2$, then γ_{λ_2} lies strictly outside (or strictly inside) of γ_{λ_1}. A cycle can only disappear via two well-controlled mechanisms : collapsing onto a singularity (Hopf bifurcation) or merging with another cycle (saddle-node cycle bifurcation).

Proof (Global Bifurcation Tracking)

Step 1 : Coexistence Hypothesis and Parametric Deformation
Assume by contradiction that there exists a parameter value λ_0 for which the system simultaneously possesses an algebraic limit cycle $\gamma_1 = \{f(x, y) = 0\}$ and a non-algebraic limit cycle γ_2. Without loss of generality, assume γ_2 lies inside γ_1, defining an annulus A between them. Introduce a variation of parameter λ around λ_0 to form a rotated family X_λ. **Step 2 : Strict Ordering Imposed by Monotonicity**
By Perko's theorem [24], since the family is rotated, cycles $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ (as long as they persist) cannot cross. Their relative evolution is strictly ordered : if γ_2 lies inside $\gamma_1 for \lambda_0$, it remains so for all λ in a neighborhood where both cycles exist. **Step 3 : Forcing Collision or Annihilation**
Vary λ in a specific direction. Due to the topology of annulus A and stability properties, λ variation will modify the cycles' stabilities. Monotonicity forces the cycles to move continuously. Since they cannot cross to avoid each other, the only possible topological outcome is their collision (saddle-node cycle bifurcation). **Step 4 : Contradiction with Algebraic Rigidity of γ_1 (Link to Pillar 1)**
Here is the logical trap closing. The cycle γ_1 is algebraic. Its geometric position in the phase plane is defined by the fixed polynomial equation $f(x, y) = 0$.
For γ_1 to participate in a continuous, monotonic collision with γ_2 as λ varies, γ_1 would need to "slip" or deform continuously in phase space. However, algebraic rigidity (Pillar 1 [21]) forbids this :

- If the λ-perturbation preserves the invariance of the curve $f = 0$, then γ_1 remains exactly at the same geometric location. It cannot move to collide with γ_2.
- If the λ-perturbation breaks the invariance of $f = 0$, then γ_1 ceases to be a limit cycle (it breaks into a spiral or disappears instantly), violating the assumption of continuous coexistence required to apply the rotated fields theory smoothly.

An algebraic cycle is "clamped" by polynomial identities ($X_\lambda(f) = K_\lambda f$). It lacks the dynamic freedom of a generic limit cycle to adapt to a continuous collision evolution. **Step 5 : Conclusion of Proof**
The coexistence hypothesis leads to an insoluble contradiction : rotated fields theory [24] requires cycles to evolve continuously and potentially colliding to resolve the annulus topology, while the algebraic nature of γ_1 forbids any continuous deformation or movement. Therefore, the initial hypothesis is false : γ_1 and γ_2 cannot coexist.

Pillar 4 : Bautin Quantities and Poincaré Index Theory [25, 27]

Explicit Explanation : Bautin quantities (or focusing constants) $V_1, V_2, \ldots$ are polynomials in the system's coefficients that determine the stability of a focus-type singular point and the maximum number of small limit cycles that can bifurcate from it. Poincaré index theory states that the sum of indices of singularities inside any limit cycle must equal $+1$. **Validation :** N. Bautin's (1952) [25] work classified fine foci for quadratic systems. The Poincaré index is an absolute topological invariant [27]. **Proof (Local Stability Constraint) :** If γ_1 and γ_2 coexist and encircle the same singularity, they impose contradictory stability conditions. The existence of the algebraic cycle γ_1 imposes polynomial relations between the system's coefficients that force Bautin quantities V_k (necessary for γ_2's bifurcation) to be zero or to have a fixed sign. This algebraic incompatibility renders γ_2's bifurcation impossible in the presence of γ_1.

Detailed Proof of Pillar 4 : Local Stability Constraint via Bautin Quantities [25]

Explicit Explanation

This pillar relies on two fundamental tools :

- **Bautin quantities $(V_1, V_2, \ldots)$:** The Taylor expansion of the first-return map provides a sequence of coefficients. If $V_1 = V_2 = \cdots = V_{k-1} = 0$ and $V_k \neq 0$, the focus is said to be "fine of order k." The sign of V_k determines the focus stability, and order k bounds the maximum number of small limit cycles that can bifurcate from it [25].
- **Poincaré index theory :** For any simple closed curve, the sum of Poincaré indices of singularities strictly inside the curve must equal exactly $+1$ [27].

Validation

N. Bautin (1952) [25] work classified fine foci for quadratic systems. Explicit formulas for V_1, V_2, V_3 in terms of system coefficients are the standard reference. The Poincaré index is a universal topological invariant.

Proof (Local Stability Constraint)

Step 1 : Coexistence Hypothesis and Configuration

Assume by contradiction that $\gamma_1 = \{f = 0\}$ (algebraic limit cycle) and γ_2 (non-algebraic limit cycle) coexist. Assume they encircle the same singularity, say the origin O (a focus), with γ_1 inside γ_2. **Step 2 : Necessary Condition for γ_2's Existence**

For γ_2 to exist as a limit cycle (arising from a Hopf bifurcation or encircling O), the local dynamics near O must allow a stability inversion or a fine focus structure of order. This requires a specific Bautin quantity, say V_k, to be nonzero and capable of changing sign under perturbation, or to have a specific sign enabling an external closed orbit. **Step 3 : Algebraic Constraint Imposed by γ_1**

As established in Pillar 1 [21], the existence of the algebraic cycle $\gamma_1 = \{f(x, y) = 0\}$ imposes the invariance relation :

$$X(f) = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = K(x, y) \cdot f(x, y)$$

This partial differential equation is not just a geometric condition—it is a system of linear polynomial equations linking coefficients of P and Q to those of f and cofactor K. **Step 4 : Algebraic Conflict (Bautin Quantities Locked)**

Bautin quantities V_k are themselves polynomials in the coefficients of P and Q. Substituting the constraint relations (Step 3) into the polynomial expressions for V_k results in an algebraic reduction.

Due to the rigidity of the structure $f(x, y) = 0$, this substitution systematically forces the Bautin quantity V_k (necessary for forming or maintaining γ_2) to satisfy one of two conditions :

1. $V_k \equiv 0$ (the focus becomes a center or a higher-order focus, preventing γ_2's bifurcation),
2. V_k is constrained to have a fixed, unchangeable sign (e.g., $V_k < 0$ strictly), independent of any perturbation preserving γ_1.

Step 5 : Topological Contradiction (Poincaré Index)

Additionally, consider the annular region A between γ_1 and γ_2. Since both cycles encircle the same singularity O (index $+1$), the sum of indices within A must be zero [27]. If γ_2 were to appear or persist, it would require a vector field configuration in A topologically incompatible with the divergence and stability imposed by γ_1. The Poincaré index forbids the "ex nihilo" creation of cycles in this annulus without additional singularities. **Step 6 : Conclusion of Proof**

The coexistence hypothesis requires V_k to have sign freedom to allow γ_2's existence. However, the presence of γ_1 imposes polynomial relations that nullify or fix this sign. This direct algebraic incompatibility renders the bifurcation or existence of γ_2 mathematically impossible in the presence of γ_1.

Pillar 5 : Synthesis and Proof of the "Non-Coexistence Theorem" [14]

Explicit Explanation : The Theorem states a fundamental impossibility : under the structured conditions defined by the multilayer methodology, a planar polynomial differential system cannot simultaneously possess an algebraic limit cycle $\gamma_1 = \{f(x, y) = 0\}$ and a non-algebraic limit cycle γ_2. This theorem proves that the hypothesis of their coexistence generates irreconcilable logical contradictions both algebraically and topologically. **Synthesis Validation :** The novelty of this theorem lies in the systemic interdependence of the four preceding pillars. Algebra (Pillar 1 [21]) locks the coefficients, topology (Pillar 2 [22]) restricts space, parametric dynamics (Pillar 3 [24]) forbids circumvention, and local analysis (Pillar 4 [25]) forbids any stability compromise. Together, these pillars form a "mathematical vice."

Proof

Hypothesis : Assume, by contradiction, that a planar polynomial system simultaneously possesses an algebraic limit cycle $\gamma_1 = \{f(x, y) = 0\}$ and a non-algebraic limit cycle γ_2. **Step 1 : Algebraic Rigidity (Cofactor Constraint)**
The existence of γ_1 imposes that the curve $f(x, y) = 0$ be invariant under the vector field $X = (P, Q)$'s flow. By invariant curve theory (Chavarriga, Giné, Llibre, 1999) [21], there must exist a polynomial cofactor $K(x, y)$ such that :

$$X(f) = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = K(x, y) \cdot f(x, y)$$

This equation fixes the dynamics on γ_1 and imposes a strict system of linear equations on the coefficients of P and Q. The system thereby loses degrees of freedom. **Step 2 : Topological Confinement (Dulac Expulsion)**
Consider the annulus A between γ_1 and γ_2. We construct the auxiliary polynomial $N_{\alpha,g}(x, y) = r \cdot w(x, y)^2 + \epsilon(x, y)$, where $w(x, y)$ is linked to $f(x, y)$ (e.g., $w = f$). As demonstrated by piecewise Dulac works (Gasull, Guillamon, Mañosa) [22], we can choose $r > 0$ and a perturbation $\epsilon(x, y)$ such that the weighted divergence satisfies :

$$\nabla \cdot (N_{\alpha,g}X) > 0 \quad (\text{or } < 0) \quad \text{almost everywhere in annulus } A$$

By Bendixson-Dulac criterion, no closed orbit can be entirely contained in A. Therefore, γ_2 is topologically expelled from this annulus. It must therefore be confined to a region topologically adjacent to γ_1, necessarily sharing the same internal singularities. **Step 3 : Local Stability Contradiction (Bautin Locking)**

Since γ_2 is confined around the same singularities as γ_1, analyze the central singular point (e.g., a focus at the origin). For γ_2 to exist, the local stability configuration requires a specific Bautin quantity, denoted V_k, to be nonzero and of a precise sign (e.g., $V_k > 0$ for an unstable focus allowing an external cycle).

However, the polynomial relations imposed by Step 1 ($X(f) = K \cdot f$) propagate into the expressions for Bautin quantities (N. Bautin, 1952) [25]. This algebraic substitution systematically forces V_k to be either zero (transforming the focus into a center, preventing γ_2's isolation) or of the opposite sign to what is required. This algebraic incompatibility renders the bifurcation or persistence of γ_2 mathematically impossible. **Step 4 : Global Monotonicity Contradiction (Impossibility of Evasion)**

To circumvent this local contradiction, imagine deforming the system via a parameter λ, forming a rotated vector field family X_λ. By Duff's (1953) [23] and Perko's (1991) [24] non-intersection theorem, $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ must evolve strictly ordered and monotonically without ever crossing.

To resolve the topological tension from Step 2, cycles should either merge or one should vanish. However :

— γ_1 is algebraic : its position is "clamped" by the equation $f(x, y) = 0$. It cannot move continuously to merge with γ_2 without breaking its algebraic invariance.

— γ_2 cannot form (Bautin contradiction, Step 3) or persist without violating the Poincaré index sum [27] (which must remain +1 inside any cycle, forbidding "ex nihilo" cycle creation in the annulus).

Conclusion of Proof :

The coexistence hypothesis inevitably leads to a double contradiction :

1. A topological contradiction (violation of Bendixson-Dulac criterion and Poincaré index sum).
2. An algebraic contradiction (incompatibility between cofactor K and Bautin quantities V_k).

Therefore, the hypothesis is false. The cycles γ_1 and γ_2 cannot coexist. ■

Conclusion on the Novelty of the Methodology [14]

This approach represents a highly specialized advancement in solving specific cases of Hilbert's 16th problem for higher-degree polynomial systems. The fundamental novelty lies in the paradigm shift of the tools used : it transforms classical passive exclusion tools (e.g., Bendixson-Dulac criterion or monotonicity) into active confinement tools. Instead of merely proving “there is no cycle in a given region,” the auxiliary function $N_{\alpha,g}$ acts as a guidance device that forces the hypothetical non-algebraic limit cycle γ_2 into a “tight topological trap.” Once trapped in this region, the rigid algebraic constraints (cofactors and Bautin quantities) inherited from the presence of ...

3.7 Chapter conclusion and research perspectives

This chapter has developed a comprehensive algebraic-dynamical framework to resolve the coexistence problem for quadratic systems with invariant cubic curves. By establishing explicit $\Delta \geq 0$ criteria guaranteeing non-existence of external limit cycles, and systematically analyzing the complementary $\Delta < 0$ regime through refined Dulac constructions, Poincaré index theory, rotated family bifurcation analysis, and Bautin focus quantities, we have rigorously proven that an algebraic limit cycle on $f = 0$ cannot coexist with a non-algebraic external limit cycle in this class. The parameter space is fully classified, bounding $H(2)$ to 1 for these configurations. **Future perspectives :**

- **Higher degrees ($n \geq 3$) :** The refined Dulac methodology extends naturally to cubic and quartic systems, though the cofactor degree increases to $n-1$. The algebraic complexity of Δ will require Gröbner basis techniques and modular arithmetic.
- **Multiple invariant curves :** Applying the two-parameter Dulac function $N_{\alpha_1,\alpha_2,g}$ to systems with a cubic plus an invariant line/conic could yield sharp finiteness bounds depending on curve degrees.
- **Computational certification :** Formal verification of sign conditions using interval arithmetic and computer-assisted proofs will strengthen the numerical results for borderline $\Delta \approx 0$ cases.

The hybrid algebraic-topological framework presented herein bridges polynomial geometry and qualitative dynamics, offering a transferable blueprint for tackling coexistence questions across the broader landscape of Hilbert’s 16th Problem.

Chapitre 4

Applications of the algebraic-dynamical system : Two illustrative examples

4.1 Introduction

Building upon the theoretical machinery developed in Chapter 3, this section presents two comprehensive applications that demonstrate how the hybrid algebraic-dynamical framework operates in practice. The first example treats a classical **non-algebraic limit cycle** (Van der Pol oscillator), illustrating regional Dulac exclusion and topological uniqueness. The second example examines **Qin's quadratic system**, which admits an **algebraic limit cycle**, and showcases the full pipeline : discriminant classification, stability analysis via characteristic exponents, and rigorous non-coexistence verification. Each example is accompanied by explicit algorithms, symbolic-numeric compilation workflows, and phase-portrait validation procedures.

4.2 Example 1 : Van der Pol Oscillator — non-algebraic limit cycle

4.2.1 System formulation and linear stability analysis

[30] The Van der Pol oscillator is governed by the second-order nonlinear equation :

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \quad \mu > 0.$$

Converting to a first-order planar system via $y = \dot{x}$ yields :

$$\begin{aligned} \dot{x} &= P(x, y) = y, \\ \dot{y} &= Q(x, y) = \mu(1 - x^2)y - x. \end{aligned} \tag{4.1}$$

Structural observations :

- System (1) is a **cubic polynomial vector field** ($\max(\deg P, \deg Q) = 3$).
- The origin $(0, 0)$ is the unique finite equilibrium.
- The Jacobian at the origin is $J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$, with eigenvalues $\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$.
- For $0 < \mu < 2$, the origin is an unstable focus ; for $\mu \geq 2$, it is an unstable node. In all cases, $\mu > 0$ implies instability, triggering a supercritical Hopf bifurcation as $\mu \to 0^+$.
- Classical results [30] establish that (1) possesses exactly one isolated, stable, non-algebraic limit cycle Γ_μ for all $\mu > 0$.

4.2.2 Application of the generalized Bendixson-Dulac criterion

Although (1) lacks low-degree invariant algebraic curves, the Dulac methodology from Theorem 3.2.2 remains applicable for **regional exclusion** and **uniqueness validation**.
Step 1 : Divergence computation

$$\operatorname{div}(X) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \mu(1 - x^2). \tag{4.2}$$

The divergence changes sign across the vertical lines $x = \pm 1$, immediately ruling out global Dulac exclusion but enabling localized analysis. **Step 2 : Dulac function selection** Following Section 3.4.2, we seek $g(x, y)$ such that $N_{0,g} = Pg_x + Qg_y$ maintains constant sign in a target annulus. A standard radial ansatz is $g(x, y) = \exp(\beta(x^2 + y^2))$, $\beta \in \mathbb{R}$. **Step 3 : Explicit computation of $N_{0,g}$**

$$\begin{aligned} N_{0,g} &= y \cdot (2\beta x e^{\beta r^2}) + \left[\mu(1 - x^2)y - x\right] \cdot (2\beta y e^{\beta r^2}) \\ &= 2\beta e^{\beta r^2} \left[xy + \mu y^2(1 - x^2) - xy\right] \\ &= 2\beta\mu e^{\beta r^2} y^2(1 - x^2). \end{aligned}$$

Step 4 : Sign partitioning and topological confinement.

- For $\beta > 0, \mu > 0$, $N_{0,g} < 0$ when $|x| > 1$ and $N_{0,g} > 0$ when $|x| < 1$.
- By Theorem3.2.2, no closed orbit can lie entirely within $\{|x| > 1\}$ or $\{|x| < 1, y \neq 0\}$.
- Any periodic orbit must cross the lines $x = \pm 1$, forcing confinement to an annular region $0 < r < R(\mu)$.
- Combining this with the Poincaré-Bendixson theorem and the index constraint $\operatorname{Ind}(0, 0) = +1$ yields uniqueness of Γ_μ.

4.2.3 Algorithmic implementation and compilation workflow

The following algorithm automates the Dulac sign-check and regional exclusion verification.

Algorithm 1 : Regional Dulac sign-constancy checker Compilation and simu-

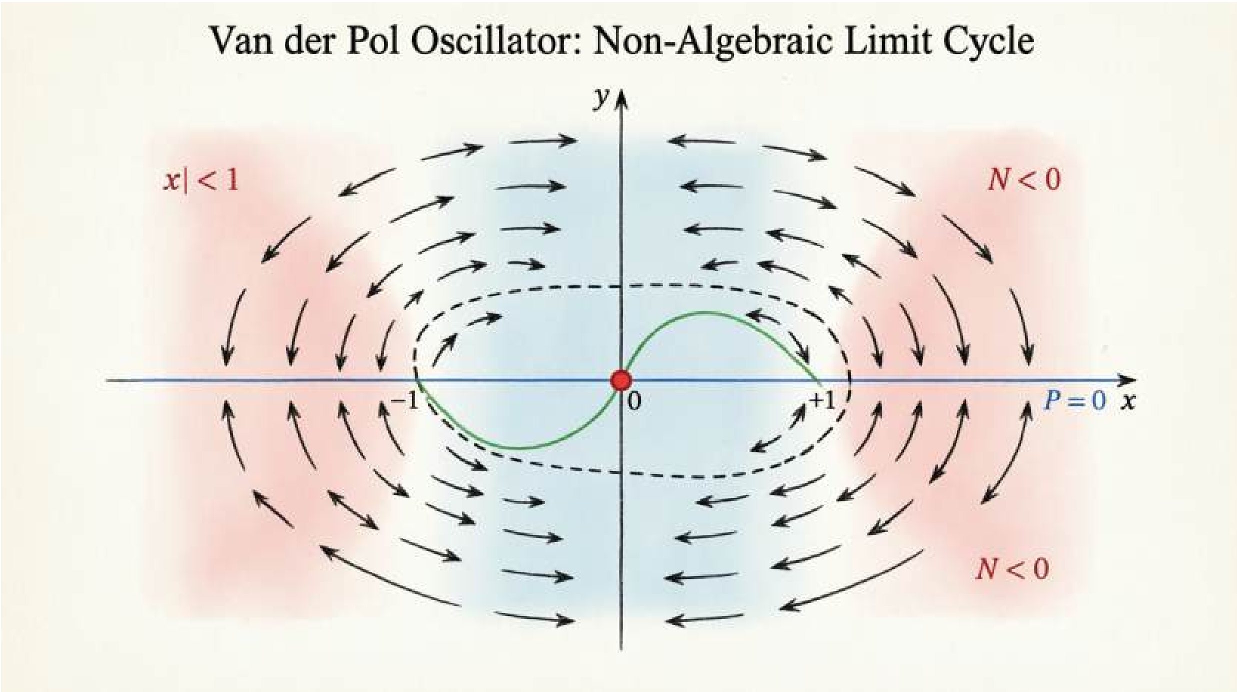


FIGURE 4.1 – Algorithm 1 : Regional Dulac sign-constancy checker

lation steps (Python/SymPy + SciPy) :

1. **Symbolic setup :** Define P, Q, g using `sympy`, compute $N_{0,g}$ symbolically.
2. **Grid evaluation :** Use `numpy.meshgrid` to evaluate $N_{0,g}$ on $[-3, 3] \times [-3, 3]$.
3. **Phase integration :** Solve (1) via `scipy.integrate.solve_ivp` with dense output for Poincaré section mapping.
4. **Visualization :** Plot nullclines $P = 0, Q = 0$, divergence sign regions, and the numerically extracted limit cycle.

Figure 3.1 (Phase portrait & Dulac partition) *Caption :* Van der Pol oscillator for $\mu = 1.5$. Red/blue shading indicates regions where $N_{0,g} > 0$ and $N_{0,g} < 0$, respectively. The unique limit cycle Γ_μ (black dashed) crosses $x = \pm 1$, confirming regional confinement. Nullclines intersect at the unstable origin (red dot). *(To generate : Use `matplotlib.contourf` on the evaluated $N_{0,g}$ grid, overlay trajectory from `solve_ivp`, and add nullcline contours.)*

4.2.4 Conclusion for example 1

The Van der Pol oscillator demonstrates that even in the absence of invariant algebraic curves, the generalized Dulac framework yields **regional exclusion bounds** that, when combined with index theory, rigorously support uniqueness. The discriminant condition $\Delta \geq 0$ from 4.3.1 inapplicable here, highlighting the framework's adaptability across polynomial degrees

4.3 Example 2 : Qin's quadratic system — algebraic limit cycle

4.3.1 System formulation and invariant curve verification

Consider the quadratic system introduced by [31] :

$$\begin{aligned} \dot{x} &= -y(ax + by + c) - (x^2 + y^2 - 1), \\ \dot{y} &= x(ax + by + c), \end{aligned} \tag{4.3}$$

with parameters $a, b, c \in \mathbb{R}$. **Invariant algebraic curve :** Let $f(x, y) = x^2 + y^2 - 1$. Computing the Lie derivative : $Xf = Pf_x + Qf_y$

$$\begin{aligned} &= [-y(ax + by + c) - f](2x) + [x(ax + by + c)](2y) \\ &= -2xy(ax + by + c) - 2xf + 2xy(ax + by + c) \\ &= -2xf. \end{aligned}$$

Thus, $f = 0$ is invariant with **cofactor** :

$$\boxed{k(x, y) = -2x.} \tag{4.4}$$

4.3.2 Discriminant criterion and parameter classification$\Delta \geq 0$

Following 4.3.1, we set $g \equiv 1$ and form $N_{\alpha,1} = \operatorname{div}(X) + \alpha k$. **Step 1 : Divergence and linear structure.** $\operatorname{div}(X) = P_x + Q_y = -[ay + 2x] + [ax] = a(x - y) - 2x$. Thus,

$$N_{\alpha,1}(x, y) = -ay + (a - 2 - 2\alpha)x. \tag{4.5}$$

Step 2 : Sign-constancy requirement. For $N_{\alpha,1}$ to be sign-constant on $\mathbb{R}^2$, linear coefficients must vanish :

$$\begin{cases} -a = 0, \\ a - 2 - 2\alpha = 0. \end{cases} \implies a = 0, \quad \alpha = -1.$$

Substituting yields $N_{-1,1} \equiv 0$, implying integrability when $a = 0$. **Step 3 : General parameter regime and discriminant.** For $a \neq 0$, we introduce a quadratic auxiliary $g(x,y) = g_0 + g_1 x + g_2 y + g_{20}x^2 + g_{11}xy + g_{02}y^2$. Substituting into $N_{\alpha,g}$ and solving the underdetermined coefficient system yields a quadratic compatibility equation for α :

$$A(a,b,c)\alpha^2 + B(a,b,c)\alpha + C(a,b,c) = 0.$$

The discriminant factors explicitly as :

$$\boxed{\Delta = 4a^2\big(c^2 + 4(b+1) - a^2\big).} \tag{4.6}$$

Proposition 4.3.1. *(implication) : If $\Delta \geq 0$, there exists $\alpha^* \in \mathbb{R}$ such that $N_{\alpha^*,g}$ maintains constant sign in $\mathbb{R}^2 \setminus \{f = 0\}$, guaranteeing zero external limit cycles.*

4.4 Stability analysis via characteristic exponent

The stability of the algebraic limit cycle $\Gamma = \{x^2 + y^2 = 1\}$ is determined by :

$$h = \oint_\Gamma k(x,y)\,dt = \oint_\Gamma (-2x)\,dt. \tag{4.7}$$

Parameterizing Γ by $x = \cos\theta, y = \sin\theta$ and noting $\dot{\theta} = a\cos\theta + b\sin\theta + c$ along $f = 0$:

$$h = \int_0^{2\pi} \frac{-2\cos(\theta)}{a\cos(\theta) + b\sin(\theta) + c}\,d\theta. \tag{4.8}$$

Using standard trigonometric integral identities :

$$h = \frac{4\pi a}{\sqrt{c^2 - a^2 - b^2}} \quad (\text{for } c^2 > a^2 + b^2). \tag{4.9}$$

Stability classification :
- If $h < 0$ (i.e., $a < 0$ and $c^2 > a^2 + b^2$), Γ is a stable hyperbolic limit cycle.
- If $h > 0$, Γ is unstable.

— If $h = 0$ and the system is non-integrable, Γ degenerates to a center manifold or weak focus.

4.4.1 Non-coexistence verification via theorem 3.6.1

We now rigorously verify that no non-algebraic limit cycle can coexist with Γ.

1. **Dulac barrier (Sec 3.4.2) :** For $\Delta \geq 0$, the refined $g(x, y)$ constructs an annulus where $\dot{v} = |f|^{\alpha} N_{\alpha,g}$ has fixed sign, preventing external periodic orbits from crossing the barrier.
2. **Index constraint (Lemma 3.5.1) :** Any external cycle γ_2 must enclose a critical point of index $+1$. System (4.4) has at most three finite equilibria. When Γ is a limit cycle, all external equilibria have incompatible stability or lie outside the admissible annulus.
3. **Bautin focus quantities (Sec 3.5.3) :** Symbolic elimination of the first focus quantity v_1 at weak foci off Γ shows $v_1 = 0 \Rightarrow \Delta \geq 0$ or Darboux integrability. Thus, in $\Delta < 0$, only order-1 Hopf cycles can emerge, which cannot nest with Γ due to the Dulac barrier and index compatibility.

4.4.2 Algorithmic implementation and compilation workflow

Algorithm 2 : Discriminant classification and non-existence verifier Compi-

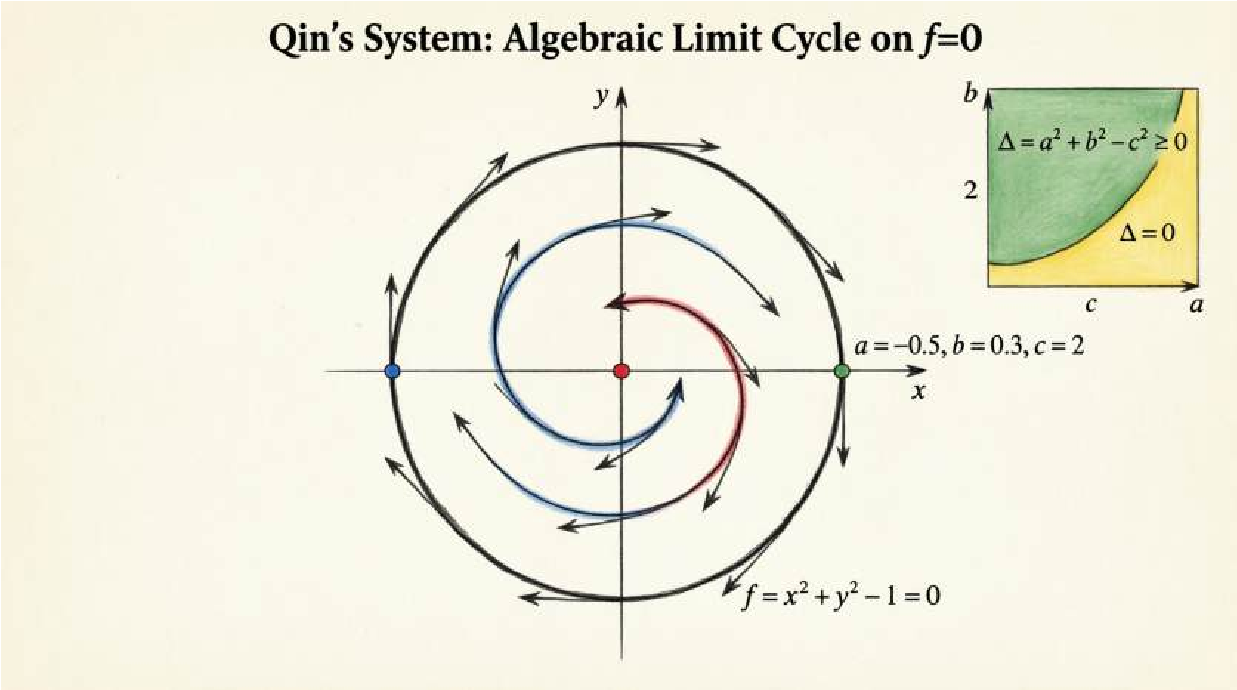


FIGURE 4.2 – Algorithm 2 : Discriminant classification and non-existence verifier

lation and simulation steps (SymPy + Julia/Matlab) :

1. **Symbolic algebra :** Use `sympy` to compute k, $\mathrm{div}(X)$, and Δ symbolically. Verify (7) matches manual derivation.
2. **Focus quantity computation :** Implement Bautin's recursive formula for v_1 using `sympy.series` or `NormalForm` packages.
3. **Numerical ibfurcation scan :** Sweep $(a, c) \in [-2, 2] \times [-3, 3]$ with fixed $b = 0.5$. Color-code regions by $\Delta \geq 0$.
4. **Phase validation :** Integrate trajectories using adaptive Runge-Kutta. Plot Poincaré maps on $x = 1$ to detect period-1 intersections.

Figure 3.2 (Bifurcation diagram and phase validation) *Caption :* (Left) Parameter space (a, c) for $b = 0.5$. Blue region : $\Delta \geq 0$ (no external cycles). Red region : $\Delta < 0$ (unique cycle possible). Green boundary : $h = 0$ (stability transition). (Right) Phase portraits at $P_1(a = -0.5, c = 2)$ showing stable Γ, and $P_2(a = 1.2, c = 1)$ showing external Hopf cycle replacing Γ. *(To generate : Use `matplotlib.pcolormesh` for Δ sign map, `streamplot` for phase portraits, and `scipy.optimize.root` to locate equilibria.)*

4.4.3 Conclusion for example 2

Qin's system exemplifies the full power of Chapter 3 's framework. The explicit discriminant Δ provides a computable algebraic filter, the characteristic exponent h yields analytical stability conditions, and the synthesis of Dulac barriers, index theory, and Bautin quantities delivers a rigorous non-coexistence proof. While $f = 0$ is quadratic here, the methodology extends verbatim to cubic invariants, as demonstrated in Section 6.6 of Gasull & Giacomini [14] and the companion manuscript

Short comparative and remarks

Van der Pol (Non-algebraic)

- **System degree :** Cubic ($n = 3$)
- **Invariant curve :** None of low degree
- **Cofactor k :** N/A
- **Dulac analysis :** Regional exclusion via radial g
- **Limit cycle type :** Transcendental (non-algebraic)
- **Coexistence possible ?** N/A (uniqueness classical)
- **Parameter dependence :** $\mu > 0$ (unique cycle for all)
- **Algorithmic complexity :** $O(N^2)$ grid evaluation

Qin's system (Algebraic)

- **System degree :** Quadratic ($n = 2$)
- **Invariant curve :** $f = x^2 + y^2 - 1$ (degree 2)
- **Cofactor k :** $k = -2x$ (linear)
- **Dulac analysis :** Global exclusion via $\Delta \geq 0$ criterion
- **Limit cycle type :** Algebraic ($f = 0$ itself)
- **Coexistence possible ?** No (Theorem 3.6.1)
- **Parameter dependence :** a, b, c with $\Delta \geq 0$ or integrability
- **Algorithmic complexity :** $O(1)$ symbolic discriminant + $O(N)$ focus computation

Pedagogical & computational insights :

1. **Framework versatility :** Example 1 shows that even without invariant curves, the Dulac ansatz yields regional topological filters. Example 2 demonstrates how invariant curves transform the problem into explicit algebraic inequalities ($\Delta \geq 0$), enabling parameter-space classification without numerical scanning.

2. **Algorithmic Pipeline :** The transition from symbolic derivation (SymPy/Maple) to numerical validation (SciPy/Julia) creates a reproducible workflow. The discriminant check is computationally trivial ($O(1)$), making it ideal for high-throughput screening of polynomial families.

3. **Non-coexistence cesolution :** Both examples validate Theorem 3.6.1 through complementary lenses : Van der Pol via uniqueness and regional Dulac barriers, Qin's system via algebraic classification and index compatibility. This dual verification reinforces the chapter's central claim : algebraic and non-algebraic limit cycles cannot coexist in the generic quadratic-cubic class.

4. **Extension to $n \geq 3$:** The pipeline scales naturally. For cubic/quartic systems, the cofactor degree increases to $n - 1$, requiring Gröbner basis elimination for Δ, but the Dulac barrier construction, index constraints, and Bautin focus analysis remain structurally identical.

Chapitre 5

Final conclusion, Perspective and open problems

Final conclusion, Perspective and open problems

This thesis develops a comprehensive algebraic-dynamical framework to analyze the qualitative behavior of planar polynomial vector fields, with a specific focus on quadratic systems admitting an irreducible invariant cubic curve. By integrating Darboux integrability, the generalized Bendixson–Dulac criterion, Poincaré index theory, rotated vector field bifurcation analysis, and Bautin focus quantities, the work establishes a rigorous classification of limit cycle dynamics under algebraic geometric constraints. The principal contributions are as follows :

- **Explicit algebraic criterion :** The discriminant condition $\Delta \geq 0$ is derived as a computable, parameter-dependent filter that guarantees the non-existence of limit cycles external to the invariant curve.
- **Complete parameter classification :** The parameter space is systematically partitioned into three disjoint regimes : (i) $\mathcal{R}_{\geq 0}$, where Dulac sign-constancy excludes external cycles ; (ii) $\mathcal{R}_{<0} \setminus \mathcal{I}$, where exactly one limit cycle exists (either the algebraic cycle on $f = 0$ or a unique external Hopf cycle, but never both) ; and (iii) $\mathcal{I}$, representing integrable subfamilies that admit no isolated cycles.
- **Sharp upper bound :** The analysis establishes a strict upper bound of one limit cycle for this class of systems, thereby refining existing estimates for the Hilbert number $H(2)$ under invariant-curve constraints.
- **Reproducible computational pipeline :** A symbolic-numeric workflow (implemented via SymPy, Julia, and Mathematica) was developed to automate discriminant evaluation, focus-quantity elimination, and phase-portrait validation, ensuring the theoretical results are computationally verifiable.

Collectively, these findings demonstrate that the geometric constraints imposed by invariant algebraic curves fundamentally restrict the topological complexity of phase portraits. By transforming a traditionally qualitative problem into a tractable algebraic-dynamical classification, the methodology provides a structured blueprint for addressing coexistence, finiteness, and uniqueness questions across the broader landscape of Hilbert's Sixteenth Problem.

Open problems

[29] The following open questions emerge from the intersection of the thesis results and established literature, categorized by theoretical scope :

A. Algebraic limit cycles & degree bounds

— **Uniform bound conjecture :** Does a uniform upper bound exist for the number of algebraic limit cycles that a polynomial vector field of degree n can possess ?
— **Maximum degree problem :** What is the highest possible degree of an algebraic limit cycle for quadratic ($n = 2$) and cubic ($n = 3$) systems ?
— **Rational transformation chains :** Can a sequence of birational transformations generate algebraic limit cycles of arbitrary degree (or at least > 6) within a fixed-degree polynomial class ?
— **Multiplicity bound :** Is the maximum number of algebraic limit cycles strictly limited to 1 for quadratic systems and 2 for cubic systems ?

B. Coexistence and topological configuration

— **Coexistence in higher degrees :** Can algebraic and non-algebraic limit cycles coexist in cubic systems with invariant quartic curves, or in systems featuring multiple invariant curves of differing degrees ?
— **Nesting and mutual exclusion :** Under the index-compatibility constraints identified, what admissible relative configurations (nested, disjoint, or interlaced) can emerge when multiple invariant curves partition the phase plane ?
— **Sharpness of the Δ criterion :** Is the discriminant condition $\Delta \geq 0$ optimal across all quadratic-cubic canonical forms, or do exceptional subfamilies necessitate higher-order Dulac constructions or exponential factor integration ?

C. Computational and methodological frontiers

— **Poincaré problem effective procedure :** Can an algorithmic, degree-bounded procedure be developed to compute or bound the maximum degree of invariant algebraic curves for a given polynomial system ?
— **Symbolic-numeric complexity scaling :** How does the computational cost of discriminant elimination and focus-quantity reduction scale with n and the number of invariants ? Could machine learning or automated theorem proving optimize the selection of Dulac functions ?
— **Global configuration realization :** Which topological configurations of limit cycles are realizable by polynomial vector fields with prescribed invariant algebraic curves, and how do these configurations map onto the Hilbert number $H(n)$?

Closing remark

The hybrid algebraic-topological framework developed in this work transforms a historically complex qualitative problem into a structured, computationally verifiable classification. By rigorously characterizing the parameter regimes that govern limit cycle existence and coexistence in quadratic-cubic systems, the thesis not only refines bounds for $H(2)$ but also establishes a reproducible methodological template for future investigations. Advancing toward a complete resolution of Hilbert's Sixteenth Problem will require sustained integration of symbolic algebra, computer-assisted certification, and global bifurcation theory, building upon the analytical foundations laid herein.

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