# Existence and uniqueness of positive solutions for first-order nonlinear Liouville–Caputo fractional differential equations

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**Abstract**

We study the existence and uniqueness of positive solutions of the first-order nonlinear Liouville–Caputo fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}\left(x\left(t\right)-g\left(t,x\left(t\right)\right)\right)=f\left(t,x\left(t\right)\right),\ 0<t\leq 1, \\ x\left(0\right)=x_0>g\left(0,x_0\right)>0, \end{cases}$$

where $0<\alpha\leq 1$. In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct appropriate mappings and employ the Krasnoselskii fixed point theorem and the method of upper and lower solutions to show the existence of a positive solution of this equation. We also use the Banach fixed point theorem to show the existence of a unique positive solution.

**Keywords**
Fractional differential equation, Positive solutions, Fixed point theorem, Existence and uniqueness.

## 1 Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional differential equations. We concentrate on the positivity of solutions for the first-order nonlinear Liouville–Caputo fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}\left(x\left(t\right)-g\left(t,x\left(t\right)\right)\right)=f\left(t,x\left(t\right)\right),\ 0<t\leq 1, \\ x\left(0\right)=x_0>g\left(0,x_0\right)>0, \end{cases} \tag{1}$$

where $0<\alpha\leq 1$. $g,\ f:[0,1]\times[0,\infty)\to[0,\infty)$ are continuous. To show the existence and uniqueness of the positive solution, we transform (1) into an integral equation and then by the method of upper and lower solutions and use the Krasnoselskii and Banach fixed point theorems.

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