People's Democratic Republic Of Algeria

Ministry Of Higher Education and Scientific Research

Echahid Hamma Lakhdar University of El-Oued

Faculty Of Exact Sciences

Department of Mathematics and informatics

Dissertation Submitted

For Obtaining The Master's Degree Of Mathematics

Domain: Mathematics and informatics

Section: Mathématiques

Specialty: Fundamental mathematics

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Fixed Point Theorems of Multi-Maps in a Metric Spaces

In Front Of The Jury Composed Of

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University year : 2022-2023

Remerciement

Merci avant tout au bon Dieu **ALLAH**, le clément,
le miséricordieux, le plus puissant.

En premier lieu nos remerciements s'adressent
à nos encadrant **Dr. Djedidi Mostefa**
pour l'attention et la disponibilité dont ils ont su faire preuve au cours de leurs encadrement.

Nous exprimons nos profonde reconnaissance à
Dr.Meneceur Bekkar
pour l'honneur qu'il me fait en acceptant de présider ce jury.

Nos remerciements vont ensuite à
Dr.Ghendir Aoun Abdellatif
qui a accepté d'examiner ce travail.

Merci à tous nos professeurs et le directeur de l'université Echahid Hamma Lakhdar, d'Eloued.

Enfin, nous remercions toutes les personnes, famille, amis, qui
directement ou indirectement ont contribué
à la réalisation de ce travaille.

Dédicace

Je dédier ce modeste travail :

Á mon père Abdelaziz, ma mère Besseriani Sabah,
mes frères Tayeb,Mohammed sedik et Massoud ,
mes sœurs Hanane,Cherouk , Chahinaze Merieme et Safia et ma grande familles .

Á tous mes amis. Aussi bien à tous ceux qui m'ont aidé.

Merci

Maamri Sana.

Dedicace

«وآخر دعواهم أن الحمد لله ربّ العالمين»

To the one who was the first supporter of achieving ambition... To the one who saw the path of my life... To the great mother: Abla Assami .

To the one who supported me and helped me in times of adversity throughout my life... To the perfect man, may God prolong his life, so that he continues to help me... To my kind father: Abdel-Baki Assami .

To the high-ranking woman with a fragrant biography... My dear aunt: Masouda Assami .

To those whom God strengthened my arm, and they were the best supporter, to my sisters and brothers: Sarah, Basma, Shorouk, Saba, Nasreddin, Nour al-Islam, Ahmed, may God protect them and provide them with His bounty.

I dedicate to you the fruit of my effort represented in this humble note, hoping that it will be an ongoing charity for you and me.

Ikram Assami.

Abstract

In this work, we study and presented some fixed point theorems for multi-function in metric space, such as Nadler's theorem and Caristi's theorem and generalized them to partial and cone metric space.

We concluded this work by proving the existence of solutions to some integral inclusions and the issue of differential inclusions by applying a set of theorems and results.

Keywords: metric space, fixed point, multi-valued applications, partial metric space, cone metric space, integral equation, integral inclusions, differential inclusions.

ملخص

في هذا العمل ، درسنا وقدمنا بعض نظريات النقطة الثابتة متعددة القيم في الفضاء المتري ، مثل نظرية نادلر ونظرية كاريستي وعممناها. في الفضاء المتري جزئي ومخروطي.
اختتمنا هذا العمل بإثبات وجود حلول لبعض الاحتواءات التكاملية ومسألة الاحتواءات التفاضلية من خلال تطبيق مجموعة من النظريات والنتائج.

الكلمات المفتاحية : الفضاء المتري ، النقطة الثابتة ، التطبيقات متعددة القيم ، الفضاء المتري الجزئي, الفضاء المتري المخروطي, المعادلة التكاملية, الاحتواءات التكاملية, الاحتواءات التفاضلية.

Résumé

Dans ce travail, nous avons étudié et présenté quelques théorème du point fixe multivoque dans l'espace métrique, telles que, la théorème de Nadler et la théorème de Caristi et les avons généralisées aux espaces métriques partiels et cone.
Nous avons conclu ce travail en prouvant l'existence de solutions à certaines inclusions intégrales et la question des inclusions différentielles en appliquant un ensemble de théorème et de résultats.

Mots clés : espace métrique, point fixe, applications multivoque, espace métrique partiel, espace métrique cone, équation intégrale, inclusions intégrales, inclusions différentielles. .

Contents

Introduction

Why study the fixed point? And what is its purpose?

We have always had this question, and we often ask ourselves, what is the meaning of a fixed point? The fixed point of the function is the point antecedent is equal to the image of the function $f(x)=x$, if we express this property graphically, we find that it is the point of intersection between the curve of the function f and the first bisector at the point (x,x).

We note that the fixed point was explained by Banach in his classical theory of 1922, which presents the existence and unity of the fixed point in the integral equations, so that the world Nadler generalized it in his famous multi-valued fixed point theorem. Caristi also showed the existence of the fixed point only without proving its unity to complete his research and generalize his theory Multi-valued fixed point. We have studied these theories of the fixed point according to some types of metric space and the form of its existence, unity or multi-values, as the researchers were not satisfied with these researches and theorems reached in the metric space, to re-study them in the partial metric space and prove their theorems and beliefs about the fixed point for multi-value applications, so we noticed that Both Nadler and Caristi did not stop there in their research, to find that they expanded their research and theorems to a different dimension and space than before, so we see that they took their theorems and research to the cone metric space. To verify and prove the validity of these theorems in various metric spaces.

In this work, we introduced a set of theorems, including the fixed point theory of Banach and Caristi in metric space, as preliminary information that helps us understand the multi-valued fixed point theorem.

In Chapter 1 we discussed some intuitive notions of metric space, as well as the theorems of Banach and Caristi. As for the second chapter, we have dealt with some fixed point theorems for multi-functions in metric space, as well as their generalization in partial metric space and cone metric space. We conclude this work with the third chapter, which we devoted to studying the existence of solutions for some integrated inclusions applications and nonlinear differential inclusions. using the previously studied theorems and results.

Notations

The most frequently used notations, symbols and abbreviations are listed below:

$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{R}$	:	The set of real numbers.
$\mathbb{R}_+$	:	The set of positive real numbers.
$H(A,B)$	:	The Hausdorff distance between A and B.
X,Y	:	Are metric spaces if no other indication is given.
$CB(X)$	:	The set of all closed and bounded subsets of X.
$CL(X)$	:	The set of closed subsets in X.
$K(X)$	:	The set of all compact subsets of X.
$T:X\to Y$	:	T is a multivalued map from $XtoP(Y)$.
$Dom(T)$	:	The domain of T.
$Graph(T)$	:	The graph of T.
$Im(T)$	:	The image of T.
$d(x,A)$	:	The distance between a point x and $a$set A.
$p(x,y)$	:	The distance between a point x and a set A in partiel metric space.
$e(A,B)$	:	Denotes the excess of a set $A\in A$.
$L^1(I)$	:	The space of integrable maps defined on I.

CHAPTER 1

Preliminary Notions

In this chapter we will provide some definitions, basic mathematical concepts and some preliminary properties that we will use in this work.

1.1 Metric Space

Definition 1.1.1. [38] Let X be a non-empty set. We call distance on X any map $d : X \times X \longrightarrow \mathbb{R}_+$ satisfying the following axioms :

$(d1)$ Separation: $\forall\ x,y \in X\ :\ d(x,y) = 0 \Longleftrightarrow x = y$.

$(d2)$ Symmetry : $\forall\ x,y \in X\ :\ d(x,y) = d(y,x)$.

$(d3)$ Triangle inequality : $\forall\ x,y,z \in X\ :\ d(x,y) \leq d(x,z)+d(z,y)$.

Definition 1.1.2. [38] A metric space is a pair (X,d) where X is a set and d is a distance.

Example 1.1.1. [27] $d(x,y) = |\ x-y\ |$ provided also in $\mathbb{Z}$ and $\mathbb{Q}$.

Example 1.1.2. [10] We can define a metric on any set X, assuming for $x,y \in X$:

$$d(x,y) = \begin{cases} 0 & if\ \ x = y \\ 1 & if\ \ x \neq y, \end{cases}$$

we call it discrete metric.

Definition 1.1.3. [10] Let (X,d) be a metric space, $x_0 \in X$ and $r > 0$.

- We call closed ball with center x_0 and radius r the set

$$B_f(x_0,r) = \{x \in X : d(x_0,x) \leq r\}.$$

- We call open ball with center x_0 and radius r the set

$$B(x_0,r) = \{x \in X : d(x_0,x) < r\}.$$

- We call sphere with center x_0 and radius r the set

$$S(x_0,r) = \{x \in X : d(x_0,x) = r\}.$$

Remark 1.1.

$$B_f(x_0,r) = B(x_0,r) \bigcup S(x_0,r)$$

Example 1.1.3. [10] Let $\mathbb{R}$ be the set of real numbers endowed with the usual distance $d(x,y)=|x-y|$, $(x,y) \in \mathbb{R}$ therefore

1.$B(x_0,r)=]x_0-r,x_0+r[$.

2.$B_f(x_0,r)=[x_0-r,x_0+r]$.

3.$S(x_0,r)=\{x_0-r,x_0+r\}$.

1.1.1 Cauchy Sequence and Convergent Sequence

Definition 1.1.4. [10] A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is said to be Cauchy if:

$$\forall\ \varepsilon > 0, \exists\ \delta(\varepsilon) \in \mathbb{N}, \forall\ n,m \geq \delta(\varepsilon) : d(x_n,x_m) < \varepsilon$$

i.e:

$$\lim_{n,m\to+\infty} d(x_n,x_m)=0$$

Remark 1.2. We can write $\forall\ \varepsilon > 0, \exists\ N_\varepsilon \in \mathbb{N}, \forall\ n \geq N_\varepsilon,\ \forall\ m \in \mathbb{N} : d(x_{n+m},x_n) \leq \varepsilon$.

Example 1.1.4. Let $(\mathbb{R}, |.|)$ be a sequence $(x_n)_{n\in\mathbb{N}}$ defined by:

$$x_n = \frac{1}{n}$$

Let $n, m \in \mathbb{N}, n > m$:

$$\begin{aligned} d(x_n, x_m) &= \left|\tfrac{1}{n} - \tfrac{1}{m}\right| \\ &= \left|\tfrac{n-m}{nm}\right| \\ &\leq \left|\tfrac{n}{nm}\right| \\ &\leq \tfrac{n}{nm} \\ &\leq \tfrac{1}{m} \end{aligned}$$

then $\lim\limits_{m\to+\infty} \dfrac{1}{m} = 0$, so x_n is a Cauchy sequence.

Definition 1.1.5. [10] Let (X, d) be a metric space then $(x_n)_{n\in\mathbb{N}}$ a sequence in X is called convergent sequence if and only if:

$$\forall\ \varepsilon > 0, \exists\ \delta(\varepsilon) \in \mathbb{N}, \forall\ n \geq \delta(\varepsilon) \Rightarrow d(x_n, x) \leq \varepsilon,$$

we then denote

$$\lim_{n\to+\infty} x_n = x.$$

Properties 1.1.1.

- Any convergent sequence is a Cauchy sequence.
- Any convergent sequence is Cauchy, the converse is generally false.

Example 1.1.5. [10] Let $\{x_n\}$ be the sequence in the metric space $(\mathbb{R} - \{1\}, |.|)$, such that $x_n = \frac{n}{n+1}$.

We have $\lim\limits_{n\to\infty} x_n = 1 \notin \mathbb{R}-\{1\}$, so the sequence x_n diverges. Let $n,m \in \mathbb{N}, n > m$:

$$\begin{aligned} d(x_n, x_m) &= \left| \tfrac{n}{n+1} - \tfrac{m}{m+1} \right| \\ &= \left| \tfrac{n-m}{(n+1)(m+1)} \right| \\ &\le \tfrac{n}{n(m+1)} \\ &\le \tfrac{1}{m+1}, \end{aligned}$$

then $\lim\limits_{m\to+\infty} \dfrac{1}{m+1} = 0$. Hence $\lim\limits_{n,m\to+\infty} d(x_n, x_m) = 0$, so the sequence x_n is of Cauchy.

1.1.2 Continuous Map

Definition 1.1.6. [10] Let (X, d_X) and (Y, d_Y) be two metric spaces, $f : X \longrightarrow Y$ a map and $a \in X$, we say that f is continuous at the point a if:

$$\lim_{x\to a} f(x) = f(a),$$

that's to say :

$$\forall\ \varepsilon > 0. \exists\ \delta > 0. \forall\ x \in X, d_X(x, a) \le \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon.$$

1.1.3 Contraction

Definition 1.1.7. [10] A mapping f from a metric space (X, d) to itself is said to be k-contractive if $0 \le k < 1$ and if, for any pair of points x and y of X,

$$d(f(x), f(y)) \le k d(x, y).$$

It is said to be contracting if it is k-contracting for some constant k.

Example 1.1.6. the case on $\mathbb{R}$ of the application $x \longmapsto (x + \sin x)/3$, avec $k = 2/3$.

Definition 1.1.8. Let (X,d) be a metric space. A mapping $f:X\to X$ is said to be an F-contraction on X, if there exists $\tau>0$ such that for all $x,y\in X$,

$$d(f(x),f(y))>0\Rightarrow \tau+F(d(f(x),f(y)))\leq F(d(x,y)).$$

and $F:(0,+\infty)\to\mathbb{R}$ a mapping satisfying the following:

$F1$: F is strictly increasing, that is for all $x,y\in\mathbb{R}_+$ such that $x<y\Rightarrow F(x)<F(y)$.

$F2$: For each sequence $\{x_n\}_{n\geq 1}$ of positive numbers, $\lim\limits_{n\to\infty} x_n=0$ if and only if $\lim\limits_{n\to\infty} F(x_n)=-\infty$.

$F3$: There exists $k\in(0,1)$ such that $\lim\limits_{n\to\infty} x^k F(x)=0$.

The set of all functions satisfying the conditions $(F1)-(F3)$is denoted by ΔF

1.1.4 Lipschitzienne

Definition 1.1.9. [10] be (X,d_X) and (Y,d_Y) metric spaces, $f:X\to Y$ an application and k a positive real. We say that f is k-Lipschitz si

$$\forall(x,y)\in X^2, d_Y(f(x),f(y))\leq k d_X(x,y).$$

Example 1.1.7. Consider the metric space $\mathbb{R}^2$ equipped with the Manhattan distance. The app $(x,y)\longrightarrow(x+y)$ is then 1-lipschitzian. More generally, any continuous linear map between two $e.v.n.$ (normed vector spaces) is Lipschitz (in particular, any linear map from a finite-dimensional $e.v.n.$ to an $e.v.n.$)

1.1.5 Non Expansive

Definition 1.1.10. [19] Let $f:X\longrightarrow X$ be a mapping of the metric space X into itself. We call f non-expansive if

$$d(f(x),f(y))\leq d(x,y) \quad (x,y\ \in\ X).$$

Remark 1.3. [30] that contraction $\Rightarrow$ non expansive $\Rightarrow$ Lipchitz, and that all these functions are uniformly continuous.

1.1.6 Complete Space

Definition 1.1.11. [38] A metric space (X,d) is said to be complete if any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges in X.

Properties 1.1.2. [38] Let (X,d) be a metric space and $A\subset X$

- If (A,d) is complete, then A is a closed set of X.
- If (X,d) is complete and A is a closed set of X, then (A,d) is complete.

Proof.

- If $A\subset X$ is complete and $(x_n)_{n\in\mathbb{N}}\subset A$ is a sequence which converges to $a\in X$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence (since in X it converges), then it must converge in A to a point $b\in A$. According to is the uniqueness of the limit (in X) where $a=b\in A$ so A is closed.
- If X is complete, and A closed in X, any Cauchy sequence $(x_n)_{n\in\mathbb{N}}\subset A$ must converge in X to a point $a\in X$, since this point is complete and A closed, this limit must belong to A, which proves that A is complete. □

1.2 Norm Vector Space

Definition 1.2.1. [7] A norm on X is a map from X to $\mathbb{R}_+$ denoted $\|.\|$ satisfying for all $x,y\in X$ and all $\lambda\in\mathbb{K}$:

$(N1)$ $\|x\|=0\Leftrightarrow x=0$,

$(N2)$ $\|\lambda x\|=|\lambda|\,\|x\|$ (homogeneity).

$(N3)$ $\|x+y\|\leq\|x\|++\|y\|$ (triangular inequality).

Definition 1.2.2. [7] A normed vector space is a pair $(X,\|.\|)$ where X is a vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $\|.\|$ is a norm over X.

The following proposition specifies the meaning of normed vector spaces as metric spaces . It follows that all the properties of the distances given above have a translation into norm in the normed vector spaces (In particular, as for the distances if it is not necessary to assume the positive or null norm, it is a consequence of the axioms $(N1)$, $(N2)$ and $(N3)$).

Properties 1.2.1. [7] If $(X,\|.\|)$ is a normed vector space then $d(x,y)=\|x-y\|$ defines a distance on X.

Proof. It is clear that the hypotheses $(N1)$ with $\lambda = 0$ and $(N2)$ for the norm $\|.\|$ entail the property $(d1)$ of the distance d. The property $(N1)$ with λ = -1 for the norm leads to the property $(d2)$ for the distance. The triangle inequality immediately follows. □

Remark 1.4. [7] Balls or spheres with center 0 and radius 1 are called unit balls, unit sphere.

Example 1.2.1. The n-dimensional vector space $\mathbb{K}^n$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is endowed with three important norms (the; standard norms) $\|.\|_1, \|.\|_2, \|.\|_\infty$, defined by $x = (x_1, \ldots, x_n)$ and $y = (y_1,, y_n)$:

$$\|x\|_1 = \sum_{i=1}^{n} | x_i |, \ \|x\|_2 = \left(\sum_{i=1}^{n} (x_i^2)^{\frac{1}{2}} \right), \text{ and } \|x\|_\infty = \max_{1 \leq i \leq n} \|x_i\|.$$

The three norms satisfy the inequalities $\|.\|_\infty \leq \|x\|_1 \leq \sqrt{n}\|.\|_2 \leq n\|x\|_\infty$.

Definition 1.2.3. [18] Let $(X, \|.\|)$ be a normed vector space, and $\{x_n\}_{n\in\mathbb{N}^*}$ a sequence in $(X, \|.\|)$

- $\{x_n\}_{n\in\mathbb{N}^*}$ is said to converge to $x \in X$ if

$$\varepsilon > 0, \exists n_0 \in \mathbb{N}, \ \forall n \geq n_0, \|x_n - x\| \leq \varepsilon.$$

- $\{x_n\}_{n\in\mathbb{N}^*}$ is called a Cauchy sequence if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \ \forall n, m \geq n_0, \|x_n - x_m\| \leq \varepsilon.$$

1.3 Banach spaces

Definition 1.3.1. A Banach space is a normed vector space $(X, \|.\|)$ which is complete for the distance associated with the norm.

Example 1.3.1. For $1 \leq p < \infty$, we define the p-norm on $\mathbb{R}^n$ (or $\mathbb{C}^n$) by

$$\|(x_1, x_2, \ldots, x_n)\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{\frac{1}{p}}.$$

For $p = \infty$, we define the ∞, or maximum, norm by

$$\|(x_1, x_2, \ldots, x_n)\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Then $\mathbb{R}^n$ equipped with the p-norm is a finite-dimensional Banach space for $1 \leq p \leq \infty$.

1.4 Weak-Weak* Topology

1.4.1 Weak Topology

Let Y be a Banach space and $f \in X'$, X' is the dual of x. We denote by $\phi_f : X \longrightarrow \mathbb{R}$ the map defined by $\phi_f(x) = < f, x >$.

When f describes X', we obtain a family of maps $(\phi_f)_{f \in X'}$ from X to $\mathbb{R}$.

Definition 1.4.1. The weak topology $\sigma(X, X')$ is the least fine topology on X making all the maps $(\phi_f)_{f \in X'}$ continuous.

Definition 1.4.2. . Let $(x_n)_n$ be a sequence of $\bar{\mathbb{R}}$. We define the lower and upper bounds of $(x_n)_n$ as follows

$$\liminf_{n \to \infty} x_n = \sup_{n \in \mathbb{N}} (\inf_{k \geq n} x_k)$$

$$\limsup_{n \to \infty} x_n = \inf_{n \in \mathbb{N}} (\sup_{k \geq n} x_k)$$

Properties 1.4.1. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of Y. We have:

1. $x_n \rightharpoonup^* x$ for $\sigma(X, X') \Leftrightarrow < f, x_n > \to < f, x >$, $\forall\ x \in X'$.

2. If $x_n \longrightarrow x$ then $x_n \rightharpoonup x$ for $\sigma(X, X')$.

3. If $x_n \rightharpoonup x$ for $\sigma(X, X')$, then $\|x_n\|$ is bounded and $\|x\| \leq \liminf_{n \to +\infty} \|x_n\|$.

4. If $x_n \rightharpoonup x$ for $\sigma(X, X')$ and if $f_n \longrightarrow f$ in X', then $< f_n, x_n > \longrightarrow < f, x >$.

1.4.2 Weak* Topology

We will now define the weak* topology denoted by $\sigma(X', X)$. For each $x \in X$ we consider the map $\varphi_x : X' \longrightarrow \mathbb{R}$ defined by $f \mapsto \varphi_x(f) = < f, x >$. When x runs through X we obtain a family of maps $(\varphi_x)_{x \in X}$ from X' to $\mathbb{R}$.

Definition 1.4.3. The weak* topology also denoted $\sigma(X', X)$ is the least fine topology on X' rendering the maps $(\phi_x)_{x \in X}$.

Properties 1.4.2. Let $(f_n)_n \in \mathbb{N}$ be a sequence of X'. We have

1. $f_n \rightharpoonup^* f$ for $\sigma(X', X) \Leftrightarrow < f_n, x > \to < f, x >$, $\forall\ x \in X$.

2. If $f_n \rightharpoonup^* f$ for $\sigma(X', X)$, then $\|f_n\|$ is bounded and $\|f\| \leq \liminf_{n\to+\infty} \|f_n\|$.

3. If $f_n \rightharpoonup^* f$ for $\sigma(X', X)$ and if $x_n \longrightarrow x$ strongly in X, then $< f_n, x_n > \longrightarrow < f, x >$.

Definition 1.4.4. A topological space is metrizable if there exists a metric which induces its topology.

Theorem 1.4.1. Let X be a separable Banach space. Then the closed unit ball of X' is metrizable for the topology $\sigma(X', X)$ i.e. we can construct on this ball a metric which induces the same topology as the weak* topology ($\sigma(X', X)$).

1.5 Fixed Point Theorem for Functions in Metric Spaces

Definition 1.5.1. Let (X, d) be a metric space, and $f : X \longrightarrow X$ a map. We call fixed point of f any point $x \in X$ such that $f(x) = x$. (We also use the notation $fx = x$).

1.5.1 Banach's Fixed Point Theorem

Theorem 1.5.1. [37] **(Banach 1922)**. Let (X, d) be a complete metric space (or a Banach space if X has a norm) and $f : X \longrightarrow X$ a contraction. Then f admits a unique fixed point in X, i.e. there exists $x \in X$ such that $fx = x$. Moreover, this point can be obtained as the limit of any sequence generated by alliteration

Proof.

(*1*) **Existence:**

Let x_0 be any initial point and $(x_n)_{n\in\mathbb{N}}$ the sequence defined by:

$$\begin{cases} x_0 \in X \\ x_{n+1} = f(x_n)\, n \in \mathbb{N}. \end{cases}$$

We will establish that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. For all $n \in \mathbb{N}$, since f is contractive, we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \lambda d(x_{n-1}, x_n) \leq \cdots \leq \lambda^n d(x_0, x_1)$$

Thus, for $m > n$, where $n \geq 0$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_1, x_0) + \lambda^{n+1} d(x_1, x_0) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &\leq \lambda^n [1 + \lambda + \lambda^2 + \ldots + \lambda^{m-n-1}] d(x_0, x_1) \\ &\leq \tfrac{\lambda^n}{1-\lambda} d(x_0, f(x_0)) \longrightarrow 0 \ \textit{when } n \longrightarrow \infty \ \textit{because } \lambda \in [0,1[. \end{aligned}$$

This shows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and since X is a complete space, then there exists $x \in X$ such that $x_n \longrightarrow x$. Moreover, since f is continuous, we have:

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x).$$

so x is a fixed point of f.

(*2*) **Uniqueness:**

Suppose there are two points $x, y \in X$ such that $x \neq y$, $x = f(x)$ and $y = f(y)$. Then, we have

$$d(f(x), f(y)) = d(x, y),$$

SO

$$\frac{d(f(x), f(y))}{d(x, y)} = 1,$$

on the other hand, contracting f therefore

$$\frac{d(f(x), f(y))}{d(x, y)} \leq \lambda \frac{d(x, y)}{d(x, y)} < 1,$$

which is a contradiction, hence the uniqueness. □

Example 1.5.1. [37] Let $X = \mathbb{R}$ and the map $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by:

- $f(x) = \frac{1}{2}x + 1$

Then f is contractive and has a unique fixed point $x = 2$.

- $f(x) = \frac{1}{3}x^2$

Then f is contractive and has a unique fixed point $x = 3$.

Theorem 1.5.2. [26] **(Cantor's intersection)**. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of closed non-empty subsets of X such that $A_1 \supset A_2 \supset A_3 \supset \cdots$ and such that $\lim\limits_{n\to\infty} diam A_n = 0$. Then, the set $\bigcap\limits_{n\in\mathbb{N}} A_n$ consists of a single point.

Lemma 1.5.1. (Bishop-Phelps). Let $\phi : X \longrightarrow \mathbb{R}$ be a lower bounded semi-continuous function and λ a positive constant. Then, for all $x_0 \in X$, there exists $x^* \in X$ such that

$$\lambda d(x_0, x^*) \le \phi(x_0) - \phi(x^*)$$

and

$$\lambda d(x, x^*) > \phi(x^*) - \phi(x) \text{ for all } x \in X \text{ such that } x \neq x^*.$$

Proof. For all $z \in X$, let $f(z) = \{y \in X : \phi(y) + \lambda\ d(z, y) \le \phi(z)\}$.

Since the function ϕ is lower semi-continuous, $f(z)$ is closed for all $z \in X$.

Let $x_0 \in X$. First choose $x_1 \in f(x_0)$ such that

$$\phi(x_1) \le 1 + \inf[\phi(f(x_0))]$$

. Then choose inductively $x_n \in f(x_{n-1})$ such that

$$\phi(x_n) \le \frac{1}{n} + \inf[\phi(f(x_{n-1}))].$$

It is clear that $f(x_0) \supset f(x_1) \supset f(x_2) \supset \cdots$. Let us estimate the diameter of $f(x_n)$ for a given n. Let $w \in f(x_n) \subset f(x_{n-1})$. We have that

$$\phi(w) \ge \inf[\phi(f(x_{n-1}))] \ge \phi(x_n) - \frac{1}{n}.$$

Since $w \in f(x_n)$, we have that

$$\lambda d(x_n, w) \leq \phi(x_n) - \phi(w) \leq \frac{1}{n}.$$

So, $diam f(x_n) \leq \frac{2}{\lambda n}$ for all $n \in \mathbb{N}$

By Cantor's intersection theorem, there exists a unique $x^* \in \bigcap_{n=0}^{\infty} f(x_n)$. Since $x^* \in f(x_0)$

$$\lambda d(x_0, x^*) \leq \phi(x_0) - \phi(x^*).$$

Moreover, if $\lambda d(x, x^*) \leq \phi(x^*) - \phi(x)$, then $x \in \bigcap_{n=0}^{\infty} f(x_n)$, since for all $n \in \mathbb{N}$

$$\begin{aligned} \lambda d(x_n, x) &\leq \lambda d(x_n, x^*) + \lambda d(x^*, x) \\ &\leq \phi(x_n) - \phi(x^*) + \phi(x^*) - \phi(x) \\ &= \phi(x_n) - \phi(x), \end{aligned}$$

□

and by uniqueness $x = x^*$. Here is a proof of Caristi's theorem using the Bishop-Phelps lemma

1.5.2 Caristi's Fixed Point Theorem:

In 1976, using transfinite induction, Caristi obtained a fixed point theorem.

Definition 1.5.2. A function $\phi : Y \longrightarrow \mathbb{R}$ is said to be lower semi-continuous if, for all $x \in Y$,

$$\liminf_{y \to x} \phi(y) \geq \phi(x)$$

or, equivalently, if the set $\{x \in Y : \phi(x) \leq a\}$ is closed for all $a \in \mathbb{R}$.It is upper semi-continuous if $-\phi$ is lower semi-continuous.

Theorem 1.5.3. [15] [13] Let $\phi : X \longrightarrow \mathbb{R}$ be a lower bounded semi-continuous function. Let $f : X \longrightarrow X$ be any function such that

$$d(x, f(x)) \leq \phi(x) - \phi(f(x))$$

for all $x \in X$. Then f has a fixed point.

A simpler proof using a result of Bishop and Phelps was later made. Before stating this lemma, here is Cantor's intersection theorem which we will need to prove the Bishop-Phelps lemma

Proof. Let $\lambda = 1$ and choose $x_0 \in X$ arbitrarily. By Lemma (1.5.1), there exists $x^* \in X$ such that, if

$$x \neq x^*, d(x, x^*) > \phi(x^*) - \phi(x).$$

Now, by hypothesis,

$$d(x^*, f(x^*)) \leq \phi(x^*) - \phi(f(x^*)).$$

From these two inequalities, we deduce that $x^* = f(x^*)$.

□

CHAPTER 2

Multi-Valued Analysis

In this chapter we introduce some concepts and characteristics of multi-applications We will also present some fixed point theorem in each of the spaces, metric space, Partial metric and Cone metric.

2.1 Pompieu-Hausdorff Distance

Definition 2.1.1. [2] Let (X,d) be a metric space, $x \in X$ and A a subset of X. The distance from point x to A denoted $d(x,A)$ is defined by:

$$d(x,A) = \inf\{d(x,y) : y \in A\}.$$

By convention, $d(x,\lambda) = +\infty$. If on the contrary, A is non-empty, then for all $\varepsilon > 0$, there exists an element $y \in A$ such that $d(x,y) \leq d(x,A) + \varepsilon$.

Lemma 2.1.1. [14] Let (X,d) be a metric space. For $A \in CL(X)$ and $x \in X, d(x,A) = 0$ if and only if $x \in A$.

Theorem 2.1.1. [35] Let $x \in X$ and $A \in K(X)$, then there exists $x_0 \in A$ such that:

$$d(x,A) = d(x,x_0)$$

.

Definition 2.1.2. [35] Let (X,d) be a metric space and $A,B \in CB(X)$. We define the excess from A to B as follows:

$$e(A,B)=\sup\{d(a,B): a \in A\}$$

Remark 2.1. Obviously, the excess is not symmetric, so it is not a metric. Let us recall in the following proposition some properties of the excess.

Properties 2.1.1. [2] Let (X,d) be a metric space and $A,B,C \in CB(X)$. Then we have:

(*1*) $e(A,B) = 0 \Leftrightarrow A \subset B$.

(*2*) $B \subset C \Rightarrow e(A,C) \leq e(A,B)$.

(*3*) $e(A\bigcup B,C)=\max\{e(A,C),e(B,C)\}$.

(*4*) $e(A,B) \leq e(A,C)+e(C,B)$.

Proof.
(*1*) According to definition 2.1.2, we have :

$$\begin{aligned} e(A,B) &= 0 \Leftrightarrow \sup_{a\in A} d(a,B)=0 \\ &\Leftrightarrow \quad d(a,B)=0, \textit{for all } a\in A. \end{aligned}$$

Since B is closed in X, we deduce that

$$d(a,B)=0 \Leftrightarrow a\in B.$$

SO,

$$e(A,B)=0 \Leftrightarrow A\subset B.$$

(*2*) We have

$$B\subset C \Rightarrow d(a,C)\leq d(a,B), \text{ for all } a \in X$$

Then taking the upper bound on the a in A, we get

$$B \subset C \Rightarrow \sup_{a\in A} d(a,C) \leq \sup_{a\in A} d(a,B).$$

SO,

$$B \subset C \Rightarrow e(A,C) \leq e(A,B).$$

(3) We have:

$$e(A \bigcup B, C) = \sup_{a\in A\cup B} d(a,C) = max\{\sup_{a\in A} d(a,C), \sup_{a\in B} d(a,C)\}.$$

SO,

$$e(A \bigcup B, C) = max\{e(A,C), e(B,C)\}.$$

(4) We first show the following inequality:

$$d(a,B) \leq d(a,c) + d(c,B), \text{ for all } a, c \in X. \tag{2.1}$$

Indeed, let $a, c \in X$, by the triangular inequality, it follows that

$$\begin{aligned} d(a,B) \leq d(a,b) &\leq d(a,c) + d(c,b), \text{ for all } b \in B \\ &\leq d(a,c) + \inf_{b\in B} d(c,b). \end{aligned}$$

Then, the inequality 2.1 holds, so for all $a \in A$ and $c \in C$, we have

$$d(a,B) \leq d(a,c) + d(c,B) \leq d(a,c) + e(C,B).$$

This implies

$$d(a,B) \leq d(a,c) + e(C,B).$$

Taking then the lower bound on the c in C, we get (for any a)

$$d(a,B) \leq d(a,C) + e(C,B).$$

Whene (4) the result by taking the upper bound on the a in A.

The other notion of distance that we will need is that of Hausdorff. The Hausdorff distance between two sets A and B corresponds to the maximum between $e(A,B)$ and $e(B,A)$. □

Definition 2.1.3. [22] Let (X,d) be a metric space. The Pompieu-Hausdorff distance between two sets $A,B \in CB(X)$ is defined by:

$$H(A,B) = max\{e(A,B), e(B,A)\} = max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

Remark 2.2. [32] Let $A,B \in CB(X)$ and $a \in A$. Then, as a consequence of the definition above, we conclude that for each $\varepsilon > 0$ there exists $b \in B$ such that

$$d(a,b) \leq H(A,B) + \varepsilon.$$

Example 2.1.1.

- Let $X = R$, $A = [1,2]$ and $B = [2,3]$ with the usual distance. SO,

$$e(A,B) = \sup\{d(a,B):\ a \in A\ \} = 1 \text{ and } e(B,A) = \sup\{d(b,A):\ b \in B\ \} = 1.$$

Therefore, $H(A,B) = max\{e(A,B), e(B,A)\} = 1$.

Example 2.1.2.

- Let $X = R$, $A = [1,3]$ and $B = [3,4]$ with the usual distance. SO,

$$e(A,B) = \sup\{d(a,B):\ a \in A\ \} = 2 \text{ and } e(B,A) = \sup\{d(b,A):\ b \in B\ \} = 1.$$

Therefore, $H(A,B) = max\{e(A,B), e(B,A)\} = 2$.

Remark 2.3. If $A = \{a\}$ and $B = \{b\}$, then $H(A,B) = d(a,b)$.

Properties 2.1.2. [2] Let (X,d) be a metric space, then H is a metric on $CB(X)$.

Proof.

(*1*) By the definition of H, we have $H(A,B) \geq 0$.

Note,

$$\begin{aligned} H(A,B)=0 &\iff max\{e(A,B),e(B,A)\}=0 \\ &\iff e(A,B)=0 \text{ and } e(B,A)=0 \\ &\iff A\subset B \text{ and } B\subset A \\ &\iff A=B. \end{aligned}$$

(2) It is clear that $max\{e(A,B),e(B,A)\}=max\{e(B,A),e(A,B)\}$.

So,

$$H(A,B)=H(B,A).$$

(3) Using Proposition 2.1.1, we get

$$\begin{aligned} H(A,B) &= max\{e(A,B),e(B,A)\} \\ &\leq max\{e(A,C)+e(C,B),e(B,C)+e(C,A)\} \\ &\leq max\{e(A,C),(C,A)\}+max\{e(B,C),e(C,B)\} \\ &= H(A,C)+H(C,B). \end{aligned}$$

We will now give definitions concerning the sequences in $(CB(X),H)$ □

Definition 2.1.4. [36] Let $(CB(X),H)$ be a metric space.

(1) We say that a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $CB(X)$ converges to $A\in CB(X)$, if for all $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that

$$H(A_n,A)<\varepsilon,\ \textit{for all } n\ \geq N.$$

We write $\lim\limits_{n\to+\infty} A_n=A$.

(2) A sequence $\{A_n\}_{n\in\mathbb{N}}$ in $CB(X)$ is said to be Cauchy if

$$\forall\varepsilon>0,\exists N\in\mathbb{N},\forall n,m\in\mathbb{N}: n>m\geq N\Rightarrow H(A_n,A_m)<\varepsilon,$$

and we write

$$H(A_n, A_m) \longrightarrow 0.$$

$$n, m \longrightarrow +\infty$$

Theorem 2.1.2. [36] If (X,d) is a complete metric space, then $(CB(X),H)$ and $(K(X),H)$ are also.

Our next results are based on the following lemma:

Lemma 2.1.2. [9] Let (X,d) be a metric space and $A,B \in CB(X)$. Then, for each $h > 1$ and for all $a \in A$, there exists $b \in B$, such that

$$d(a,b) \leq hH(A,B). \tag{2.2}$$

Proof. Note that if $H(A,B) = 0$, then $A = B$ and $a \in B$, we conclude that inequality 2.2 holds for $b = a$.

On the other hand, if $H(A,B) > 0$ let

$$\varepsilon = (h-1)H(A,B) > 0. \tag{2.3}$$

Using the definition of $d(a,B)$ and $H(A,B)$, it becomes that for each $\varepsilon > 0$ there exists $b(a) = B$ such that

$$d(a,b) \leq d(a,B) + \varepsilon \leq H(A,B) + \varepsilon. \tag{2.4}$$

Then, it suffices to replace 2.3 in 2.4 to obtain the inequality 2.2 □

Lemma 2.1.3. [32] Let (X,d) be a metric space $A,B \in CB(X)$ and $a \in A$. Then $\forall\ \varepsilon > 0$, $\exists\ b \in B$ such that

$$d(a,b) \leq H(A,B) + \varepsilon.$$

2.2 Some Notions of Multi-Map Analysis

2.2.1 Mutlti-Applications

Definition 2.2.1. [17] A multi-function $T\colon X \longrightarrow Y$ is a map which to each $x \in X$ associates Tx. Tx is a non-empty set of Y.

Definition 2.2.2. A point $x_0 \in X$ is said to be a fixed point of T if: $x_0 \in Tx_0$
In the literature, these maps are also called multi-maps, multi-valued functions, multi-functions or correspondences

Remark 2.4. Note that a mapping is also a multi-mapping since for $x \in X$ the singleton

$$f(x) = \{y\}$$

is a subset of Y.

Example 2.2.1. Any application is an image of a multi-application. Let $f : X \longrightarrow Y$ a univocal map, we define $T : X \longrightarrow 2^Y$ by $Tx = \{f(x))\}$.

Definition 2.2.3. [17] Let X and Y be two sets and $T\colon X \longrightarrow 2^Y$ a multi-application.
(1) We call domain of T the set:

$$domT := \{x \in X / Tx \neq \varnothing\}$$

(2) We call image of T, the set:

$$ImT := \{y \in Y / \exists x \in X; y = Tx\}$$

(3) We call image of a part A of X by T the set:

$$TA := \bigcup_{x \in A} Tx$$

(4) The graph of T is the subset of $X \times Y$ defined by:

$$gphT := \{(x, y) \in X \times Y / y \in Tx\}$$

2.2.2 Continuity of Multi-Applications

- **Upper semi-continuity** ($s.c.s.$)

Definition 2.2.4. Let X and Y be two topological spaces and let $T : X \rightrightarrows Y$ be a multi-function.
▹ We say that T is upper ($s.c.s.$) semi-continuous at the point $x_0 \in X$, if for any open set V of Y such that $Tx_0 \subset V$, there exists a neighborhood Ω of x_0, such that

$$Tx \subset V, \forall\, x \in \Omega.$$

▹ We say that T is s.c.s. on X if it is $s.c.s.$ at any point of X.

Properties 2.2.1. Let X and Y be two topological spaces and let $T : X \rightrightarrows Y$ be a closed-valued multi-function, (Tx a closed set of Y,$\forall\, x \in X$).
▹ If T is $s.c.s.$ on X then the graph of T is closed.

Properties 2.2.2. Let X and Y be two topological spaces, such that Y is compact and let $T : X \rightrightarrows Y$ be a multi-map.
▹ If the graph of T is closed then T is $s.c.s.$ on X.

Theorem 2.2.1. Let X and Y be two topological spaces, $T : X \rightrightarrows Y$ a multi-map, then the following assertions are equivalent
i) T is upper semi-continuous on X,
ii) The set $T_+^{-1}V = \{x \in X : Tx \subset V\}$ is an open set of X for any open set V of Y,
iii) the set $T^{-1}M = \{x \in X : Tx \cap M \neq \varnothing\}$ is a closed set of X for every closed set M of Y.

Properties 2.2.3. [5] Let $T : X \rightrightarrows Y$ be an upper semicontinuous compact-valued multi-map on X. Then for each compact set $K \subset X$, the image $F(K) = \bigcup_{x \in K} Tx$ is compact in Y.

Properties 2.2.4. [26] Let X and Y be two metric spaces, $T : X \rightrightarrows Y$ a compact-valued multi-function, then T is upper semi-continuous on X if and only if for each $x \in X$ and each sequence $(x_n)_n$ of X such that $x_n \longrightarrow x$, and $(y_n)_n$ of Y with $y_n \in Tx_n$, there exists a subsequence (y_m) of (y_n) such that

$$\lim_{m \to +\infty} y_m \in Tx.$$

- **Lower Semi-Continuity** $(s.c.i.)$

Definition 2.2.5. Let X and Y be two topological spaces, $T : X \rightrightarrows Y$ a multi-map.
▹ We say that T is lower semi-continuous $(s.c.i.)$ at the point $x_0 \in X$, if for any open set V of X such that
$T x_0 \cap V \neq \varnothing$, there exists a neighborhood Ω of x_0, such that

$$T x \cap V \neq \varnothing, \forall\, x \in \Omega.$$

▹ We say that T is $s.c.i.$ on X if it is $s.c.i.$ at any point of X.

Remark 2.5. ▹ A multi-map is continuous at a point if it is $s.c.s.$ and $s.c.i.$ at this point.
▹ A multi-map is continuous on a set A if it is continuous at every point of A.

- **Upper Hemi-Continuity** $(h.c.s.)$

Definition 2.2.6. Let $\mathscr{T}_1$,$\mathscr{T}_2$ be two topological spaces equipped with the weak topology and F:$\mathscr{T}_1 \rightrightarrows \mathscr{T}_2$ a multi-map.
We say that F is upper semi-continuous (or scalar upper semi-continuous) at x_0 if for all (t, t') $\in$ $(\mathscr{T}_1$,$\mathscr{T}'_2)$,($\mathscr{T}'_2$ is the dual of $\mathscr{T}_2$) the function $I^*_{Tt}(t')$ is upper semi-continuous at x_0.

Properties 2.2.5. Any upper semi-continuous multi-map defined from $\mathscr{T}_1$ to values in $\mathscr{T}_2$ which is endowed with the weak topology is upper hemi-continuous.

2.2.3 Measurability of Multi-Applications

Definition 2.2.7. Let (Y, Σ) be a measurable space, X a metric space, $T\,:\, Y \rightrightarrows X$ a multi-function, we say that T is Σ-measurable if for any open V of X,

$$T^{-1} V \in \Sigma.\; T^{-1} V = \{t \in Y : T t \cap V \neq \varnothing\}$$

Theorem 2.2.2. (Kuratowski and Ryll-Nardzewski theorem) [29]
Let (J, Σ) be a measurable space, X a separable complete metric space and $T\,:\, J \rightrightarrows X$ a closed-valued Σ-measurable multi-function, then T admits at least one measurable selection.

Properties 2.2.6. Let X be a topological space, and Y a metric space, F an upper semi-continuous multi-map defined as $T : X \rightrightarrows Y$. Then the inverse image of the open sets are Borelians

2.2.4 Convergence Theorem

Theorem 2.2.3. [5] Let F be a multi-function $h.c.s.$ defined on a locally convex Hausdorff space X with convex closed values in a Banach space Y.

Let I be an interval of $\mathbb{R}$, $x_k(.)$ and $y_k(.)$ measurable functions from I to X (respecti-tively Y) satisfying

for almost any $t \in I$, and any neighborhood V of 0 in $X \times Y$, there exists $k_0 = k_0(t, V)$ such that

$$\forall\, k \geq k_0, (x_k(t), y_k(t)) \in gph(T) + V,$$

if we have

i) $x_k(.)$ converges p.p. to a function $x(.)$,

ii) $y_k(.) \in L^1(I, Y)$ and converges weakly to $y(.)$ in $L^1(I, Y)$,

then

$$(x(t), y(t)) \in gph(T),$$

i.e.

$$y(t) \in Tx(t)\, for \quad almost \quad all \;\; t \in I.$$

Theorem 2.2.4. [16] Let I be a compact interval of $\mathbb{R}$, E a finite dimensional Banach space and let (x_n) be a sequence of absolutely continuous functions defined on I with values in E satisfying the following conditions

1. $\forall\; t \in I, (x_n(t))_n$ is a relatively compact subset in E,
2. there exists a positive real-valued function $c \in L^1(I, \mathbb{R}_+)$ such that

$$\|x_0(t)\| \leq c(t), \;\; p.p. \;\; on \;\; I.$$

Then there exists a subsequence (again denoted $(x_n)_n$) and an absolutely continuous function $x : I \longrightarrow E$ in the following sense:

$i)$ $(x_n)_n$ converges uniform ement to x on a compact set of I,

$ii)$ $(x'_n)_n$ converges weakly to x' in $L^1(I,E)$.

2.3 Partial and Cone Metric Spaces

2.3.1 Partial Metric Spaces

Definition 2.3.1. [11] Let X be a non-empty set. We say that the map $p : X\times X \longrightarrow \mathbb{R}_+$ is a partial metric on X if for all $x,y,z \in X$, the following conditions are satisfied:

$(P1)$ $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

$(P2)$ $p(x,x) \le p(x,y)$,

$(P3)$ $p(x,y) = p(y,x)$,

$(P4)$ $p(x,z) \le p(x,y) + p(y,z)$ - $p(y,y)$.

Then (X,p) is said to be a partial metric space such that X is a non-empty set and p is a partial metric on X.

Example 2.3.1. the function p: $X\times X \longrightarrow \mathbb{R}_+$ defined by:

• $p(x,y) = p(x,y) + max\{w(x), w(y)\}$.

is a partial metric in X, $w : X \longrightarrow \mathbb{R}_+$ is an arbitrary function

Let $x,y,z \in X$.

• **For the partial metric** p

For $x = y$, In effect :

$$\begin{aligned} p(x,x) &= p(x,x)+max\{w(x),w(x)\} \\ &= p(x,y)+max\{w(x),w(y)\} \\ &= p(y,y)+max\{w(y),w(y)\}. \end{aligned}$$

So

$$x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y).$$

And more

$$p(x,y) = p(x,y)+max\{w(x),w(y)\} = p(y,x)+max\{w(y),w(x)\} = p(y,x).$$

In effect also :

$$\begin{aligned} p(x,y) &= p(x,y)+max\{w(x),w(y)\} \\ &\leq p(x,y)+p(y,z)-p(y,y)+max\{w(x),w(y)\}+max\{w(y),w(z)\}-max\{w(y),w(y)\} \\ &\leq p(x,y)+p(y,z)-p(y,y) \end{aligned}$$

Hence (X,p) is a partial metric space

Example 2.3.2. We consider the function p: $\mathbb{R}_- \times \mathbb{R}_- \longrightarrow \mathbb{R}_+$ de ned for all $x,y \in \mathbb{R}_-$, by:

$$p(x,y) = -min\{x,y\},$$

the even $(\mathbb{R}_-,p)$ is a partial metric space.

1.In effect:

$$x = y \Leftrightarrow p(x,x) = -min\{x,x\} = -min\{x,y\} = -min\{y,y\}$$

So

$$x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y).$$

2.Let $x,y \in \mathbb{R}_-$,In effect

$$min\{x,y\} \leq x \Leftrightarrow -min\{x,y\} \geq -x.$$

So

$$p(x,y) = -min\{x,y\} \geq -x = p(x,x)$$

3..Let $x,y,z \in \mathbb{R}_-$, We ensure that:

$$p(x,z) \leq p(x,y)+p(y,z)-p(y,y). \tag{2.5}$$

We consider the following three cases: $y \leq x \leq z . x \leq y \leq z$ et $x \leq z \leq y$.

- First case $y \leq x \leq z$. In effect :

$$p(x,z) = -min\{x,z\} = -x,$$
$$p(x,y) = -min\{x,y\} = -y,$$
$$p(y,z) = -min\{y,z\} = -y,$$
$$p(y,y) = -min\{y,y\} = -y.$$

By transferring the values of $p(x,z), p(x,y), p(y,z)$ in (2.5), it comes that :

$$-x \leq -y - y + y \Leftrightarrow x \geq y.$$

So

$$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$$

- Second case $x \leq y \leq z$.In effect :

$$p(x,z) = -min\{x,z\} = -x,$$
$$p(x,y) = -min\{x,y\} = -x,$$
$$p(y,z) = -min\{y,z\} = -y,$$
$$p(y,y) = -min\{y,y\} = -y.$$

By transferring the values of $p(x,z), p(x,y), p(y,z)$ and $p(y,y)$ in (2.5), it comes that :

$$\begin{aligned} -x \leq -x - x + y &\Leftrightarrow -x \leq -2x + y \\ &\Leftrightarrow -x + 2x \leq y \\ &\Leftrightarrow x \leq y. \end{aligned}$$

So

$$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$$

- Third case $x \leq z \leq y$. In effect :

$$p(x,z) = -min\{x,z\} = -x,$$
$$p(x,y) = -min\{x,y\} = -x,$$
$$p(y,z) = -min\{y,z\} = -z,$$
$$p(y,y) = -min\{y,y\} = -y.$$

By transferring the values of $p(x,z), p(x,y), p(y,z)$ and $p(y,y)$ in (2.5), it comes that :

$$-x \leq -x - z + y \Leftrightarrow 0 \leq -z + y \Leftrightarrow z \leq y.$$

So

$$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$$

It turns out that the hypothesis *4* is satisfied. Hence (X,p), is a partial metric space

Lemma 2.3.1. [21] Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $h > 1$. For all $a \in A$, there exists $b = b(a) \in B$ such that

$$p(a,b) \leq hH_p(A,B). \tag{2.6}$$

Definition 2.3.2. [25]
(i) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if

$$p(x,x) = \lim_{n\to+\infty} p(x,x_n).$$

(ii) a sequence $\{x_n\}$ in a partial metric space (X,p) is said to be Cauchy if and only if

$$\lim_{n,m\to+\infty} p(x_n,x_m) \textit{ exists (and finite).}$$

(iii) a partial metric space (X,p) is complete if any Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$$

.

Lemma 2.3.2. [12] Let (X,p) be a partial metric space. Then the mapping $p^s : X \times X \to [0,\infty)$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y).$$

for all $x,y \in X$, defines a metric on X. Thus, associated with any partial metric space (X,p) and

(X,d) is a metric space, where $d=p^s$, induced by p.

Lemma 2.3.3. [25]

(*1*) A sequence $\{x_n\}$ is Cauchy in a partial metric space (X,p) if and only if $\{x_n\}$ is Cauchy in the metric space (X,p^s).

(*2*) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.

$$\lim_{n\to\infty} p^s(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m) \tag{2.7}$$

Definition 2.3.3. Let (X_p) , be a partial metric space and $T:X\longrightarrow CB^p(X)$ be a mapping. Then,T is said to be a multi-valued F-contraction if $F\in\Delta_F$ and there exists $\tau>0$ such that

$$H_p(Tx,Ty)>0 \Rightarrow \tau+F(H_p(Tx,Ty))\leq F(p(x,y)). \tag{2.8}$$

The following lemma is very useful in our results:

Lemma 2.3.4. Let X be a partial metric space and $K(X)$ a compact subset of X. Let $A\subseteq K(X)$, and define a function $f:A\longrightarrow K(X)$, then the following statements are equivalent:

(i) f is continuous.

(ii) Since $K(X)$ is compact, then for any convergence subsequence $x_k\to c, f(x_k)\to f(c)$ for any point $c\in A$.

Proof.

$i\Rightarrow ii$

For any $\varepsilon>0$, there exists $\delta>0$ such that $p(x,c)-p(c,c)<\delta$ and $p(x,c)-p(x,x)<\delta$ imply $p(f(x),f(c))-p(f(c),f(c))<\varepsilon$ and $p(f(x),f(c))-p(f(x),f(x))<\varepsilon$. Using the fact that sequence $x_{n_k}\to c$ to show that $f(x_{n_k})\to f(c)$.Now $p(x_{n_k},c)-p(c,c)<\delta$ and $p(x_{n_k},c)-p(x_{n_k},x_{n_k})<\delta$ imply $p(f(x_{n_k}),f(c))-p(f(c),f(c))<\varepsilon$ and $p(f(x_{n_k}),f(c))-p(f(x_{n_k}),f(x_{n_k}))<\varepsilon$ which shows that $f(x_{n_k})\to f(c)$.

$ii\Rightarrow i$

We know that $f(x_{n_k})\to f(c)$, then we will show that f is continuous by contradiction. Suppose f is not continuous, then there exists $\varepsilon>0$ such that $\delta>0$, $p(x,c)-p(c,c)<\delta$ and $p(x,c)-p(x,x)<\delta$

imply $p(f(x), f(c)) - p(f(c), f(c)) \geq \varepsilon$ and $p(f(x), f(c)) - p(f(x), f(x)) \geq \varepsilon$. Choosing any $\delta = \frac{1}{x_{n_k}}$ for any $n \in \mathbb{N}$ such that $p(x_{n_k}, c) - p(c, c) < \frac{1}{x_{n_k}}$ and $p(x_{n_k}, c) - p(x_{n_k}, x_{n_k}) < \frac{1}{x_{n_k}}$ imply $p(x_{n_k}, c) - p(c, c) \geq \varepsilon$ and $p(x_{n_k}, c) - p(x_{n_k}, x_{n_k}) \geq \varepsilon$, which shows that $x_{n_k} \to c$ while $f(x_{n_k}) \nrightarrow f(c)$, and this contradicts the fact that $f(x_{n_k}) \to f(c)$. Hence,f is continuous. □

Theorem 2.3.1. Let (x_p) , be a complete partial metric space and let $T : X \to K(X)$ be a multivalued $F-$contraction, then T has a fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since Tx is non-empty for all $x \in X$, we can choose $x_1 \in Tx_0$.

If $x_1 \in Tx_1$, this makes x_1 a fixed point of T and this completes the proof. Now, suppose $x_1 \notin Tx_1$. Since Tx_1 is closed $p(x_1, Tx_1) > 0$.

On the other hand, from $p(x_1, Tx_1) \leq H_p(Tx_0, Tx_1)$ and $(F1)$ of Definition 1.1.8 we can have, $F(p(x_1, Tx_1) \leq F(H_p(Tx_0, Tx_1)$.

From contractive condition (2.8) we have,

$$F(p(x_1, Tx_1)) \leq F(H_p(Tx_0, Tx_1) \leq F(p(x_0, x_1) - \tau.$$

Since Tx_1 is compact, then there exists $x_2 \in Tx_1$ such that $p(x_1, x_2) = p(x_1, Tx_2)$, with

$$F(p(x_1, x_2)) \leq F(H_p(Tx_0, Tx_1)) \leq F(p(x_1, x_0)) - \tau.$$

Continuing with the same process recursively, we obtain $\{x_n\} \in X$ such that $x_{n+1} \in Tx_n$, with

$$F(p(x_n, x_{n+1})) \leq F(p(x_n, x_{n-1})) - \tau, \quad \textit{for all} \quad n \in \mathbb{N}. \tag{2.9}$$

If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of f and this completes the proof. Now suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote by $d_n = p(x_n, x_{n+1})$ for all $n = (0, 1, 2, \ldots$. Then $d_n > 0$, for all $n \in \mathbb{N}$ and using (2.9) the following is true:

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \cdots \leq F(d_0) - n\tau. \tag{2.10}$$

From 2.10 we obtain $F(d_n) = -\infty$ then by $(F2)$ of Definition 1.1.8 we have $\lim\limits_{n\to\infty} d_n = 0$. Then by $(F3)$ of Definition 1.1.8 there exists $k \in (0,1)$ such that, $\lim_{n\to\infty} d_n^k F(d_n) = 0$. From (2.10) the following holds for all $n \in \mathbb{N}$:

$$d_n^k(F(d_n) - F(d_0)) \leq -d_n^k n\tau \leq 0. \tag{2.11}$$

Letting $n \to \infty$ in (2.11), we obtain

$$\lim_{n\to\infty} n d_n^k = 0. \tag{2.12}$$

From (2.12), there exists $n_1 \in \mathbb{N}$ such that, $n d_n^k \leq 1$, for all $n \geq n_1$ then we have,

$$d_n \leq \frac{1}{n^{\frac{1}{k}}} \quad \textit{for all} \quad n \geq n_1. \tag{2.13}$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence.

Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$, then from (2.13) and by $(P3)$ of Definition 2.3.1 we have,

$$\begin{aligned}
p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) - \sum_{j=n+1}^{m-1} p(x_j, x_j) \\
&\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\
&= d_n + d_{n+1} + \cdots + d_{m-1} = \sum_{j=n}^{m-1} d_j \leq \sum_{j=n}^{\infty} d_j \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}.
\end{aligned}$$

The convergence of the series $\sum_{j=1}^{\infty} \frac{1}{\frac{1}{jk}}$ implies that $\lim_{n\to\infty} p(x_n, x_m) = 0$ .By Lemma 2.3.2,we have for any $n, m \in \mathbb{N}, p^s(x_n, x_m) \leq 2p(x_n, x_m) \to 0$ as $n \to \infty$. This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p^s and hence converges by Lemma 2.3.3. Thus, there exists $v \in X$ such that $\lim_{n\to\infty} p^s(x_n, v) = 0$.

From contractive condition (2.8), for all $x, y \in X$, with $H(Tx, Ty) > 0$, we obtain $H(Tx, Ty) < p(x, y)$, and so $H(Tx, Ty) \leq p(x, y)$ for all $x, y \in X$. Thus,

$$p(x_{n+1}, Tv) \leq H_p(Tx_n, Tv) \leq p(x_n, v). \tag{2.14}$$

From (2.14) taking limit $n \to \infty$ we obtain $p(v, Tv) = 0$. This gives $v \in \bar{Tv} = Tv$. Hence, v is a fixed point of T. □

2.3.2 Cone Metric Space

Definition 2.3.4. [18] (Cone). P is called a cone if

- P is non-empty, closed and $P \neq \{0\}$.
- $\forall\ x, y \in P\ \forall\ \alpha, \beta \in \mathbb{R}_+ \Rightarrow \alpha x + \beta y \in P$.
- $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset X$, we define a partial order $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will represent $y - x \in int(P)$ ($int(P)$ denotes the interior of P).

Definition 2.3.5. [1] A cone P in $(X, \|.\|)$ is called:

(N)Normal: If there is a constant $k > 0$ such that:

$$0 \leq x \leq y \ \ \textit{implies that} \ \ \| x \| \leq k \| y \|.$$

The least positive integer k is called the normal constant of P, we will see that there is no cone with constant $k < 1$.

(R) Regular: If every increasing sequence that is upper bounded is convergent. That is, if $\{x\}_{n \geq 1}$ is a sequence such that $x_1 \leq x_2 \leq . \leq y$ for some $y \in X$ then there exists $x \in X$ such that: $\lim_{n \to \infty} \|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if each decreasing sequence which is lower bounded converges.

(M) Minihedral : If $\sup\{x, y\}$ exists $\forall\ x, y \in X$, and strongly Minihedral if any subset of X that is bounded above has an upper.

(S) Solid : If $P^\circ \neq \emptyset$.

Lemma 2.3.5. [1]

(*1*) Any regular cone is normal.

(*2*) For all $k > 1$, there exists a normal cone with a normal constant $K > k$.

Properties 2.3.1. [1] There are normal cones which are not regular. The following example illustrates

Example 2.3.3. Let $E = C[0,1]$ with the upper norm and $P = \{f \in X : f \geq 0\}$ then P is a normal cone with $k = 1$ which is not regular. Since the sequence $\{x_n\}$ is decreasing, but is not uniformly convergent to 0. Thus, P is not strongly minihedral.

Definition 2.3.6. [23] Let X be a non-empty set. Suppose that the map $d : X \times X \longrightarrow Y$ satisfies:
$(c1)$ $0 < d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$,
$(c2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$,
$(c3)$ $d(x,y) = d(x,z) + d(y,z)$ for all $x, y, z \in X$. Then d is called a cone metric on X and (X,d) is called a cone metric space. It is obvious that cone metric spaces generalize metric spaces.

Example 2.3.4. Let $Y = \mathbb{R}^2, P = \{(x,y) \in Y,\ x, y \geq 0\ \} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d : X \times X \longrightarrow Y$ such that $d(x,y) = (\mid x - y \mid, \alpha \mid x - y \mid)$, where $\alpha \geq 0$ constant. Then (X,d) is a cone metric space.

Definition 2.3.7. [33] Let (X,d) be a cone metric space, $x \in X$ and (x_n) a sequence in X. Then
(i) (x_n) is said to converge to x if for all $c \in E$ with $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$, and we denote by $\lim\limits_{n \to +\infty} x_n = x$ or $x_n \xrightarrow[n \to +\infty]{} x$.
(ii) (x_n) is said to be a Cauchy sequence if for all $c \in E$ with $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$.
(iii) We say that (X,d) is a complete cone metric space if all the Cauchy sequences are convergent.

2.4 Fixed Point Theorems in Metric Space

2.4.1 Nadler's Fixed Point Theorem

Nadler's theorem is a generalization of Banach's contraction principle for multi-functions

Theorem 2.4.1. [32] **(Nadler 1969)**. Let $(X;d)$ be a complete metric space, if $T\colon X \longrightarrow CB(X)$ is a multi-function contracting application. Then T has a fixed point.

Proof. Let $\alpha < 1$ be a Lipshitz constant for T,(we can assume that $\alpha > 0$) and let $p_0 \in X$. We choose $p_1 \in Tp_0$. As $Tp_0; Tp_1 \in CB(X)$ and $p_1 \in Tp_0$, then there exists a point $p_2 \in Tp_1$ such that

$$d(p_1, p_2) \leq H(Tp_0, Tp_1) + \alpha$$

(See the remark which follows). Now having

$$Tp_1; Tp_2 \in CB(X) \, \text{and} \, p_2 \in Tp_1$$

there exists a point $p_3 \in Tp_2$ such that $d(p_2, p_3) \leq H(Tp_1, Tp_2) + \alpha^2$. Continuing under this same line, construct a sequence $\{p_i\}_{i=1}^{\infty}$ of point of X such that $p_{i+1} \in Tp_i$ and $d(p_i, p_{i+1}) \leq H(Tp_{i-1}, Tp_i) + \alpha^i$ for all $i \geq 1$. Note that:

$$\begin{aligned}
d(p_i, p_{i+1}) &\leq H(Tp_{i-1}, Tp_i) + \alpha^i \leq \alpha d(p_{i-1}, p_i) + \alpha^i \\
&\leq \alpha[H(Tp_{i-2}, Tp_{i-1}) + \alpha^{i-1}] + \alpha^i \\
&\leq \alpha^2 d(p_{i-2}, p_{i-1}) + 2\alpha^i \leq \ldots \leq \alpha^i d(p_0, p_1) + i.\alpha^i
\end{aligned}$$

for all $i \geq 1$. Hence

$$\begin{aligned}
d(p_i, p_{i+j}) &\leq d(p_i, p_{i+1}) + d(p_{i+1}, p_{i+2}) + \ldots + d(p_{i+j-1}, p_{i+j}) \\
&\leq \alpha^i d(p_0, p_1) + i.\alpha^i + \alpha^{i+1} d(p_0, p_1) + (i+1).\alpha^{i+1} + \cdots \\
&\quad + \alpha^{i+j-1} d(p_0, p_1) + (i+j-1).\alpha^{i+j-1} \\
&= \left(\sum_{n=i}^{i+j-1} \alpha^n \right) d(p_0, p_1) + \sum_{n=i}^{i+j-1} n\alpha^n
\end{aligned}$$

For all $i, j \geq 1$. It follows that the sequence $\{p_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since $(X; d)$ is complete, the sequence$\{p_i\}_{i=1}^{\infty}$ converges to a point $x_0 \in X$. Therefore, the sequence $\{Tp_i\}_{i=1}^{\infty}$ converges to Tx_0 and, since $p_i \in Tp_{i-1}$ for all i, it follows that $x_0 \in Tx_0$. This completes the proof of Theorem □

Remark 2.6. Let $A;B \in CB(X)$ and let $a \in A$. If $\eta > 0$, then it is a simple consequence of the definition of $H(A, B)$ that there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \eta$ (in the proof of the previous theorem the Lipschitz constant a and subsequently $\underline{c}$ $\underline{\alpha^i}$ plays the role of such η) . However, there may not be a point $b \in B$ such that $d(a, b) \leq H(A, B)$ (if B is compact, then such a point b exists).

2.4.2 Generalization of Nadler's Fixed Point Theorem

Theorem 2.4.2. [20] Let (X,d) be a complete metric space and T a multi-application of X in $CB(X)$ satisfying the following condition:

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta[d(x,Tx)+d(y,Ty)] + \gamma[d(x,Ty)+d(y,Tx)]$$

for all x ,$y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+2\gamma < 1$. Then T has a fixed point.

Proof. Let x_0 be arbitrary in $X, x_1 \in Tx_0$. We define a sequence (x_n) in X by $x_{n+1} \in Tx_n$. Let

$r = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} > 0$. By Lemma 2.1.3 we have

$$\left\{\begin{array}{ll} \exists x_2 \in Tx_1: & d(x_1,x_2) \leq H(Tx_0,Tx_1)+r \\ \exists x_3 \in Tx_2: & d(x_2,x_3) \leq H(Tx_1,Tx_2)+r^2 \\ & \dots \\ \exists x_{n+1} \in F(x_n): & d(x_n,x_{n+1}) \leq H(Tx_{n-1},Tx_n)+r^n \end{array}\right\}$$

So

$$\begin{aligned} d(x_n,x_{n+1}) &\leq H(Tx_{n-1},Tx_n)+r^n \\ &\leq \alpha d(x_{n-1},x_n)+\beta[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)]+\gamma[d(x_{n-1},Tx_n)+ \\ &\quad d(x_n,Tx_{n-1})]+r^n \\ &\leq \alpha d(x_{n-1},x_n)+\beta[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+\gamma[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+r^n, \end{aligned}$$

$$d(x_n,x_{n+1}) \leq (\alpha+\beta+\gamma)d(x_{n-1},x_n)+(\beta+\gamma)d(x_n,x_{n+1})+r^n,$$

for all $n \in \mathbb{N}$.

From where

$$d(x_n, x_{n+1}) \le r d(x_{n-1}, x_n) + \frac{r^n}{\beta+\gamma},$$

for all $n \in \mathbb{N}$. It can be concluded that

$$d(x_n, x_{n+1}) \le r^n d(x_0, x_1) + \frac{n r^n}{\beta+\gamma},$$

for all $n \in \mathbb{N}$.

$$\begin{aligned}
d(x_n, x_{n+m}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\
&\le r^n d(x_0, x_1) + \tfrac{n r^n}{1-(\beta+\gamma)} + r^{n+1} d(x_0, x_1) + \tfrac{(n+1) r^{n+1}}{1-(\beta+\gamma)} + \cdots + \\
&\quad (n+m-1) d(x_0, x_1) + \tfrac{(n+m-1) r^{n+m-1}}{1-(\beta+\gamma)} \\
&= d(x_0, x_1) \sum_{i=n}^{n+m-1} r^i + \frac{1}{1-(\beta+\gamma)} \sum_{i=n}^{n+m-1} i r^i \quad \textit{pour tout}\ n, m \in \mathbb{N}.
\end{aligned}$$

In the same way as the previous proof, we find

$$\lim_{n,m \to +\infty} d(x_n, x_{n+m}) = 0$$

(x_n) is a Cauchy sequence in X which is complete, hence there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.

We will show that x^* is a fixed point of T. We have

$$\begin{aligned}
d(x^*, T x^*) &\le d(x^*, x_{n+1}) + d(x_{n+1}, T x^*) \\
&\le d(x^*, x_{n+1}) + H(T x_n, T x^*) \\
&\le d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta[d(x_n, T x_n) + d(x^*, T x^*)] + \gamma[d(x_n, T x^*) \\
&\quad + d(x^*, f(x_n))],
\end{aligned}$$

for all $n \in \mathbb{N}$.So
$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta[d(x_n, x_{n+1}) + d(x^*, Tx^*)] + \gamma[d(x_n, Tx^*) + d(x^*, x_{n+1})]$,
for all $n \in \mathbb{N}$. By passing to limit $n \to \infty$, we obtain

$$d(x^*, Tx^*) \leq (\beta + \gamma) d(x^*, Tx^*),$$

as $\beta + \gamma < 1$, then $d(x^*, Tx^*) = 0$, which implies that $x^* \in Tx^*$. □

Remark 2.7. If $\beta = \gamma = 0$ in Theorem 2.4.2, we get Nadler's theorem.

2.4.3 Caristi's Fixed Point Theorem

There is a generalization of Caristi's theorem to multi-functions

Theorem 2.4.3. [31] Let ϕ: $X \longrightarrow \mathbb{R}$ a lower bounded semi-continuous eigenfunction and T : $X \longrightarrow X$ a multi-function such that for all $x \in X$ there exists $y \in Tx$ such that:

$$d(x, y) \leq \phi(x) - \phi(y)$$

Then T has a fixed point.

Proof. For each $x \in X$ let $f(x) = y$ where y is an element of X such that $y \in Tx$ and $d(x, y) \leq \phi(x) - \phi(y)$. Then, f is a function from X to X satisfying :

$$d(x, f(x)) \leq \phi(x) - \phi(f(x))$$

for all $x \in X$.
Since is a proper function, there exists $u \in X$ such that $\phi(u) < +\infty$. Let:

$$X' = \{x \in X : \phi(x) \leq \phi(u) - d(u, x)\}$$

X' is not empty since $u \in X'$. Moreover, by the lower semi-continuity of ϕ, X' is closed and therefore a complete metric space

Note that the function ϕ restricted to the domain $X^{'}$ has real values and that it is also lower semi-continuous and lower bounded.
Let us now show that $X^{'}$ is invariant under f. For all $x \in X^{'}$, we have

$$\begin{aligned} \phi(f(x)) &\leq \phi(x)-d(x,f(x)) \\ &\leq \phi(u)-(d(u,x)+d(x,f(x))) \\ &\leq \phi(u)-d(u,f(x)) \end{aligned}$$

Therefore $f(x) \in X^{'}$ for all $x \in X^{'}$.
Caristi's theorem 1 allows us to conclude that f has a fixed point $z \in X$. Therefore, $z = f(z) \in Tz$. □

2.5 Fixed Point Theorems in The Partial Metric Space

2.5.1 Nadler's Fixed Point Theorem

Theorem 2.5.1. [6] Let (X,p) be a complete partial metric space. If $T : X \longrightarrow CB^p(X)$ is a multi-function such that for all $x, y \in X$, we have

$$H_p(Tx,Ty) \leq kp(x,y).$$

Where $k \in [0,1[$. Then F admits a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. By Lemma 2.3.1 with $h = \frac{1}{\sqrt{k}}$, there exists $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0,Tx_1).$$

As,$H_p(Tx_0,Tx_1) \leq kp(x_0,x_1)$, then $p(x_1,x_2) \leq \sqrt{k}p(x_0,x_1)$. For $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that

$$p(x_2,x_3) \leq \frac{1}{\sqrt{k}} H_p(Tx_1,Tx_2) \leq \sqrt{k}p(x_1,x_2)$$

Hence, we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} \in T x_n \text{ and } p(x_{n+1}, x_n) \leq \sqrt{k} p(x_n, x_{n-1}) \textit{ for all } n \geq 1. \tag{2.15}$$

Now by 2.15, we get

$$p(x_{n+1}, x_n) \leq (\sqrt{k})^n p(x_0, x_1) \textit{ for all } n \in \mathbb{N}. \tag{2.16}$$

According to 2.16 and the property $(p4)$ of the partial metric, for all $m \in \mathbb{N}^*$, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m}) \\ &\leq (\sqrt{k})^n p(x_0, x_1) + (\sqrt{k})^{n+1} p(x_0, x_1) + \cdots + (\sqrt{k})^{n+m-1} p(x_0, x_1) \\ &= ((\sqrt{k})^n + (\sqrt{k})^{n+1} + \cdots + (\sqrt{k})^{n+m-1}) p(x_0, x_1) \\ &\leq (\sqrt{k})^n 1 - \sqrt{k} p(x_0, x_1) \xrightarrow[n \to +\infty]{} 0 \end{aligned}$$

since $0 < k < 1$. By definition of p^s , we obtain for all $m \in \mathbb{N}^*$,

$$p^s(x_n, x_{n+m}) \leq 2p(x_n, x_{n+m}) \xrightarrow[n \to +\infty]{} 0.$$

This gives $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete, then by Lemma 2.3.1 (X, p^s) is a complete metric space. Hence, the sequence $\{x_n\}$ converges to $x^* \in X$ with respect to the metric p^s, i.e.,

$$\lim_{n \to +\infty} p^s(x_n, x_*) = 0.$$

Moreover, by 2.3.3, we have

$$p(x^*, x^*) = \lim_{n \to +\infty} p^s(x_n, x^*) = \lim_{n \to +\infty} p^s(x_n, x_n) \tag{2.17}$$

Since $H_p(T x_n, T x_*) \leq k p(x_n, x^*)$, then

$$\lim_{n\to+\infty} H_p(Tx_n, Tx_*) \tag{2.18}$$

Now $x_{n+1} \in Tx_n$ gives that

$$p(Tx_{n+1}, Tx_*) \leq \delta p(Tx_n, Tx_*) \leq H_p(Tx_n, Tx_*)$$

From 2.18, we get

$$\lim_{n\to+\infty} p(Tx_{n+1}, Tx_*) = 0 \tag{2.19}$$

On the other hand, we have

$$p(x^*, Tx_*) \leq p(x^*, Tx_{n+1}) + p(Tx_{n+1}, Tx_*)$$

We take $n \to +\infty$ and we use 2.17 and 2.19, we get $p(x^*, Tx_*) = 0$. So, from 2.1

$$p(x^*, x^*) = 0$$

, we get

$$p(x^*, x^*) = p(x^*, Tx_*).$$

By $a \in \bar{A} \Leftrightarrow p(a, A) = p(a, a)$ we find that $x^* \in \bar{Tx_*} = Tx_*$. Then T admits a fixed point. □

2.5.2 Caristi's Fixed Point Theorem

Theorem 2.5.2. [18] Let(X, d) be a complete partial metric space and T a map from X to X (not necessarily continuous). For T to admit a fixed point, it suffices that there exists a lower semi-continuous map $f : X \longrightarrow [0, +\infty)$ such that for any point x of X

$$p(x, Tx) \leq f(x) - f(Tx)$$

Proof. look at [18] □

Example 2.5.1. Let $X = \mathbb{R}_+$ and let $p(x, y) = max\{x, y\}$, then (X, p) is a complete partial metric space.

suppose that $T : X \longrightarrow X$ such that $Tx = \frac{x}{8}$ for all $x \in X$ and $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $f(t) = 2t$, then

$$p(x,Tx)=max\{x,\frac{x}{8}\}=x$$

and

$$f(x)-f(Tx)=\frac{7x}{4}.$$

Thus, it satisfies all the conditions of Theorem 2.5.2. it guarantees that T has a fixed point, indeed $x=0$ is the obtained point.

2.6 Fixed Point Theorems in Cone Metric Space

2.6.1 Nadler's Fixed Point Theorem

Theorem 2.6.1. [4] Let (X,d) be a complete cone metric space. Let $\mathscr{M}$ be a nonempty family of nonempty closed subsets of X and let $H:\mathscr{M}\times\mathscr{M}\longrightarrow E$ be a cone H-metric induced by d. If for the map $T:X\longrightarrow\mathscr{M}$ there exists $\lambda\in[0,1[$ such that for all $x,y\in X$

$$H(Tx,Ty)\leq\lambda d(x,y),$$

Then $Fix\ T\neq\varnothing$.

Proof. Let $x_0\in X$ and $x_1\in Tx_0$. If $x_0=x_1$, then $x_0\in Fix\ T$ and if $x_0\neq x_1$, we have

$$H(Tx_0,Tx_1)\leq\lambda d(x_0,x_1)<\sqrt{\vartheta}d(x_0,x_1),$$

we can choose $x_2\in X$ such that $x_2\in Tx_1$ and

$$d(x_1,x_2)<\sqrt{\lambda}d(x_0,x_1),$$

in the same way, in the case $x_1\neq x_2$ we can choose $x_3\in X$ such that $x_3\in Tx_2$ and

$$d(x_2,x_3)<\sqrt{\lambda}d(x_1,x_2)<(\sqrt{\lambda})^2d(x_0,x_1),$$

then we build a sequence $\{x_n\}$ of X such that $x_{n-1} \in Tx_n,\ n = 0,1,2,\dots$

$$d(x_n, x_{n+1}) < \sqrt{\lambda} d(x_{n-1}, x_n) < (\sqrt{\lambda})^2 d(x_{n-2}, x_{n-1}) < \cdots < (\sqrt{\lambda})^n d(x_0, x_1)$$

Now for $m > n$

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq ((\sqrt{\lambda})^n + (\sqrt{\lambda})^{n+1} + \cdots + (\sqrt{\lambda})^{m-1}) \\
&\leq (\sqrt{\lambda})^n (1 + \sqrt{\lambda} + \cdots + (\sqrt{\lambda})^{m-n-1}) d(x_0, x_1) \\
&\leq (\sqrt{\lambda})^n \left(\tfrac{1-(\sqrt{\lambda})^{m-n}}{1-\sqrt{\lambda}}\right) d(x_0, x_1) \\
&\leq \left(\tfrac{(\sqrt{\lambda})\sqrt{n-1}}{1-\sqrt{\lambda}}\right) d(x_0, x_1).
\end{aligned}$$

For a given $0 \ll c$, we choose a symmetric neighborhood V of 0 such that $c + V \subseteq intP$. Also, We choose $N_1 \in \mathbb{N}$ such that $\left(\frac{(\sqrt{\lambda})^n}{1-\sqrt{\lambda}}\right) d(x_0, x_1) \in V$, for all $n \geq N_1$. Then, $\left(\frac{(\sqrt{\lambda})^n}{1-\sqrt{\lambda}}\right) d(x_0, x_1) \ll c$, for all $n \geq N_1$. Thus,

$$d(x_m, x_n) \leq \left(\frac{(\sqrt{\lambda})^n}{1-\sqrt{\lambda}}\right) d(x0, x1) \ll c,$$

for all $m > n$. So $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \longrightarrow u$. Since

$$H(Tx_n, Tu) \leq \lambda d(x_n, u) < \sqrt{\lambda} d(x_n, u).$$

For all n, $x_{n+1} \in Tx_n)$, we have $y_n \in Tu$ such that $d(x_{n+1}, y_n) < \sqrt{\lambda}(x_n, u)$.
Now, we choose $N_2 \in \mathbb{N}$ such that

$$d(x_n, u) \ll \frac{c}{2}, \forall n \geq N_2.$$

Then for all $n \geq N_2$,

$$\begin{aligned} d(u, y_n) &\leq d(u, x_{n+1}) + d(x_{n+1}, y_n) \\ &\leq d(u, x_{n+1}) + \sqrt{\lambda} d(x_n, u) \\ &\leq d(u, x_{n+1}) + d(x_n, u) \ll \tfrac{c}{2} + \tfrac{c}{2} = c. \end{aligned}$$

Then $y_n \longrightarrow u$, which gives $u \in Tu$. □

2.6.2 Caristi's Fixed Point Theorem

Theorem 2.6.2. [18] Let (X, d) be a complete cone metric space, $P \subset E$ a strongly minihedral normal cone, and T a map from X to X (not necessarily continuous). For T to admit a fixed point, it suffices that there exists an inferior semi-continuous $f : X \longrightarrow P$ such that for any point x of X

$$d(x, Tx) \leq f(x) - f(Tx).$$

Proof. look at [18] □

CHAPTER 3

Applications

In this chapter we apply some theorem in order to study the existence a fixed point of multi-functions and the solutions to integral equation, differential Inclusion and integral Inclusion.

3.1 Application 1

As an application of the fixed point Theorem 2.3.1, we provide its application for the existence of solution of an integral equation. Consider the following integral equation:

$$g(t) = f(t) + \lambda \int_{t_0}^{t} K(t,s,g(s))ds, \quad t \in [0, I], \tag{3.1}$$

where $\lambda > 0$, $t, s \in I$, and $K : [a,b] \times [a,b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $f : [a,b] \longrightarrow \mathbb{R}^2$.
Let $X = C([a,b],\mathbb{R}^n)$ with usual sup-norm.Define $\|g\|^{\tau} = \sup_{t\in[a,b]} \{|g(t)|^{e^{-\tau t}}\}$,where $\tau > 0.\|.\|_{\tau}$ is a Banach space norm equivalent to maximum norm and $(X, \|.\|_{\tau})$ endowed with a metric d_{τ} defined by

$$d_{\tau}(g,h) = \sup_{t\in[a,b]} \{|g(t) - h(t)|^{e^{-\tau t}}\}$$

for all $g, h \in X = C([a,b],\mathbb{R}^n)$. One can see O'Regan and Petrusel [34] and Al-Rawashdeh et al. [3] for more details on it. For any increasing sequences $\{x\} \in X$ converging to $x \in X$, we have $\{x_n(t)\} \ll \{x(t)\}$ for all $t \in [a,b]$.
A standard approach in finding the solution of the integral equation is the use of the iteration procedure. The main reference [28].

Theorem 3.1.1. Suppose that the following conditions hold:

(i) $g(t,s)$, is continuous on the compact subset of X of $\mathbb{R}^2$;

(ii) there exists $x \in X$ such that $x_n \in Tx_{n-1}$;

(iii) there exists a continuous function $q:[a,b]\times[a,b]\longrightarrow[a,b]$ such that

$$|K(t,s,g(s))-K(t,s,h(s))| \leq q(g(s),h(s))|g(s)-h(s)|,$$

for each $t,s\in[a,b]$ and $g,h\in\mathbb{R}^2$. If $\alpha>0$ such that $q(g(s),h(s))\leq\alpha$ for all $s\in[a,b]$. Then, the integral equation 3.1 has a solution.

Proof. Define a self-map $T:X\longrightarrow X$ by

$$Tg(t)=f(t)+\lambda\int_{t_0}^{t}K(t,s,g(s))ds,\ \ t\in[O,I].$$

Now, by using conditions $(i-iii)$, we obtain

$$\begin{aligned}
|Tg(t)-Th(t)| &= |\lambda\int_{t_0}^{t}K(t,s,g(s))ds-\lambda\int_{t_0}^{t}K(t,s,h(s))ds| \\
&\leq \lambda\int_{t_0}^{t}|K(t,s,g(s))-K(t,s,h(s))|ds \\
&\leq \int_{t_0}^{t}|q(g(s),h(s))|g(s)-h(s)|ds \leq \alpha\|g-h\|.
\end{aligned}$$

Let $x=g$ and $y=h$ in $\lim\limits_{n\to+\infty}p(T_{n+1},Ty)=0$, we obtain

$$\begin{aligned}
\tau+F(p(Tg,Th)) &\leq F(p(g,h)), \\
F(p(Tg,Th)) &\leq F(p(g,h))-\tau.
\end{aligned} \tag{3.2}$$

Passing natural logarithms in 3.2, we obtain

$$p(Tg, Th) \leq e^{-\tau} p(g, h).$$

If we choose $\alpha = e^{-\tau}$ in the above equation, we obtain

$$p(Tg, Th) \leq \alpha p(g, h).$$

Therefore, T is a contraction mapping and by virtue of Theorem 2.3.1, T has a fixed point. Hence, the integral equation 3.1 has a solution □

3.2 Application 2

In this section we will study the Friedholm integral inclusions of the second kind, and show the analytical side by showing the existence of the solution, under assumptions that are more or less applicable in practice. The main reference [24].

Theorem 3.2.1. [24] Let (X, d) be a complete metric space endowed with a graph G and $F : X \longrightarrow CB(X)$ (resp. $K(X)$) a multi-map. Suppose the following conditions are satisfied:

(1) For all $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$.

(2) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$.

(3) T is G-semi-continuous from below, i.e. for $x \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $\lim\limits_{n\to+\infty} \inf d(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$, implies

$$\lim_{n\to+\infty} \inf d(x_n, Tx_n) \geq d(x, Tx),$$

or for any sequence $\{x_n\}_{n\epsilon\mathbb{N}}$ in X such that $x_n \longrightarrow x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x)\, in\, E(G)$ for all $n \in \mathbb{N}$.

(4) There exist $F \in \mathscr{F}_*$, $\beta \in \Omega$ and $\tau > 0$ and positive real numbers a_1, a_2, a_3, a_4, a_5 with $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$ such that

$$\tau + F(H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)), \tag{3.3}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $H(Tx, Ty) > 0$ where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Then, T has a fixed point.

▹ Existence Of a Solution For Integral Fredholm-Type Inclusion

This section is devoted to the nonlinear integral inclusion of Fredholm type of the second kind. We will study this kind of inclusion in depth, showing the analytical side by demonstrating the existence of the solution, under assumptions that are quite applicable in practice.

Properties 3.2.1. Let $X = C([a, b], \mathbb{R})$ be the space of all continuous functions on $[a, b]$ with $d(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|$, for all $x, y \in X$. Then, (X, d) is a complete metric space.

Now, we discuss the application of the fixed point technique to the existence of a solution for the following integral inclusion of Fredholm type:

$$x(t) \in f(t) + \int_a^b K(t, s, x(s))ds, \quad t \in J = [a, b], \tag{3.4}$$

where $f \in X$ and $K\colon J \times J \times \mathbb{R} \longrightarrow CB(R)$.

Let $X = C([a, b], \mathbb{R})$ be the space of all continuous functions on $[a, b]$. By Proposition 3.2.1, X endowed with the metric $d(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|$, for all $x, y \in X$ is a complete metric space

problem stance

Let $T : X \longrightarrow CB(X)$ be a multi-map defined by:

$$Tx(t) = \{y \in X : y \in f(t) + \int_a^b K(t, s, x(s))ds \quad t \in J\}.$$

We say that x is a solution of the integral inclusion (3.4) if and only if x is a fixed point of T.

Indeed, if $x \in Tx$, then

$$x \in f(t) + \int_a^b K(t, s, x(s))ds,$$

which implies x is a solution of the integral inclusion (3.4). Conversely, if x is a solution of (3.4)

, then

$$x \in f(t) + \int_a^b K(t,s,x(s))ds,$$

i.e. $x \in F(x)$.

Theorem 3.2.2. Let $T : X \longrightarrow CB(X)$ be a multi-map defined by:

$$Tx(t) = y \in X : y \in f(t) + \int_a^b K(t,s,x(s))ds \quad t \in J.$$

Suppose the following conditions are satisfied:

(A) For all $x \in X$, the multi-map $K_x(t,s) := K(t,s,x(s))$, $(t,s) \in J \times J$ is lower semi-continuous.

(B) There exists a continuous map $\rho : J \times J \longrightarrow [0,+\infty[$ such that

$$|K(t,s,u(s)) - K(t,s,v(s))| \leq \rho(t,s).ln(|u(s)-v(s)|+1),$$

for all $u,v \in X$ with $(u,v) \in E(G)$ and $u \neq v$ and for all $(t,s) \in J \times J$.

(C) There exists $\tau > 0$ such that

$$\sup_{t \in J} \int_a^b \rho(t,s)ds \leq \exp^{-\tau}$$

(D) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$.

(E) For all $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$ for all $z \in Ty$.

(F) For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \longrightarrow x \in X$ and $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n,x) \in E(G)$ for all $n \in \mathbb{N}$. Then, the integral inclusion (3.4) admits a solution in X.

Proof.

Evidence. From condition (A), the multi-map $K_x(t,s) : J \times J \longrightarrow CB(\mathbb{R})$ is lower semi-continuous and has non-empty closed convex values. Then, by Michael's selection theorem for all $x \in X$, there exists a continuous selection $k_x : J \times J \longrightarrow \mathbb{R}$ such that

$$k_x(t,s) \in K_x(t,s),$$

for all $(t,s) \in J \times J$.

First, we will show that the map T is well defined. Indeed, we have

$$f(t)+\int_a^b k_x(t,s)ds \in Tx.$$

Then, Tx is not empty for all $x \in X$.

In what follows, we will assert that Tx is bounded and closed.

First, the lower semi-continuity of the map K_x on $[a,b] \times [a,b]$ implies the existence of a continuous function k_x: $[a,b]\times[a,b] \longrightarrow \mathbb{R}$, this after Michael's selection theorem.

As the image of a closed interval in $\mathbb{R}$ by a continuous function is a closed and bounded interval.

Let $y \in Tx$, then there exists k_x such that

$$y(t,s)=f(t)+\int_a^b k_x(t,s)ds.$$

Since f and k_x are continuous on $[a,b]$ and $[a,b]\times[a,b]$ respectively, then y is a continuous function on $[a,b]\times[a,b]$.

Therefore, the image of y is a closed and bounded interval. Therefore, Tx is closed and bounded, that is,

$$T : X \longrightarrow CB(X).$$

Now, we will check that T is $(\alpha-F)$-Geraghty contraction of Hardy-Rogers type.

Let $x_1, x_2 \in X$ with $(x_1,x_2) \in E(G)$, $x_1 \neq x_2$ and $v_1 \in Tx_1$.

SO,

$$v_1(t) \in f(t)+\int_a^b K(t,s,x_1(s))ds \;\; t \in J.$$

Which implies that there is

$$k_{x1}(t,s) \in K_{x1}(t,s),$$

such as

$$v_1(t)=f(t)+\int_a^b k_{x1}(t,s)ds \;\; (t,s) \in J \times J.$$

From (B), we have

$$|K(t,s,x_1(s))-K(t,s,x_2(s))| \leq \rho(t,s).ln(||x_1(s)-x_2(s)|+1),$$

for all $(t,s) \in J \times J$.

This gives

$$\begin{aligned} |K(t,s,x_1(s)) - K(t,s,x_2(s))| &= \sup |k_{x1} - k_{x2}|, k_{x1} \in K_{x1}, k_{x2} \in K_{x2} \\ &\leq \rho(t,s).ln(|x_1(s) - x_2(s)| + 1). \end{aligned}$$

So there exists

$$w(t,s) \in K_{x2}(t,s),$$

such that

$$|k_{x1}(t,s) - w(t,s)| \leq \rho(t,s).ln(|x_1(s) - x_2(s)| + 1),$$

for all $(t,s) \in J \times J$.

Let us now consider multi-map L which is defined by:

$$L(t,s) = \{w(t,s) \in K_{x2}(t,s) : |k_{x1}(t,s) - w(t,s)| \leq \rho(t,s)ln(\alpha |x_1(s) - x_2(s)| + 1)\},$$

for all $(t,s) \in J \times J$.

By condition A, we find that the multi-map L is lower semi-continuous, then there exists a continuous function

$$k_{x2}(t,s) \in L(t,s)$$

for $(t,s) \in J$.

So we have

$$v_2(t) = f(t) + \int_a^b k_{x2}(t,s)ds \in f(t) + \int_a^b K(t,s,x_2(s))ds \quad t \in J$$

And

$$\begin{aligned}
|v_1(t,s)-v_2(t,s)| &\leq \int_a^b |k_{x1}(t,s)-k_{x2}(t,s)|\,ds \\
&\leq \int_a^b \rho(t,s)ln(|x_1(s)-x_2(s)|+1)ds \\
&\leq ln(d(x_1,x_2)+1)\int_a^b \rho(t,s)ds \\
&\leq e^{-\tau} ln(d(x_1,x2)+1) \\
&= e^{-\tau}\tfrac{ln(d(x_1,x_2)+1)}{d(x_1,x_2)}.d(x_1,x_2),
\end{aligned}$$

for all $t\in J$.

Therefore, we get

$$dv_1,v_2)\leq e^{-\tau}\frac{ln(d(x_1,x_2)+1)}{d(x_1,x_2)}.d(x_1,x_2),$$

By exchanging the role of x_1 and x_2, we deduce

$$H(Tx_1,Tx_1)\leq e^{-\tau}\frac{ln(d(x_1,x_2)+1)}{d(x_1,x_2)}.d(x_1,x_2).$$

Taking the logarithm of two sides in the above inequality, we get

$$\tau+lnH(Tx_1,Tx_1)\leq ln\left(\frac{ln(d(x_1,x_2)+1)}{d(x_1,x_2)}.d(x_1,x_2)\right),$$

for all x_1,$x_2\in X$ with $(x_1,x_2)\in E(G)$ and $x_1\neq x_2$.

So, we observe that the map T satisfying the condition (3.3) with

$$Tt=ln\ t,$$

$$\beta(t)=\frac{ln(t+1)}{t},$$

$$a_1=1\ and\ a_2=a_3=a_4=a_5=0.$$

Therefore, all the conditions of Theorem 3.2.1 are satisfied. Then, T has a fixed point, i.e. the integral inclusion of Fredholm type (3.4) admits a solution in X. □

Remark 3.1. We can apply the results obtained to study the existence problem of the solution of differential inclusions.

3.3 Application 3

In this section we will examine the existence of solutions to a nonlinear differential equation, under assumptions that are more or less applicable in practice.The main reference [8].

Theorem 3.3.1. [15] Let Ω be an open set of $\mathbb{R}\times\mathbb{R}^n$ and $F:\Omega\rightrightarrows\mathbb{R}^n$ a semi-continuous upper multi-map with compact convex non-empty values. Therefore, there exists $T>0$, and an absolutely continuous function $x:[t_0-T,t_0+T]\longrightarrow\mathbb{R}^n$ solution of the differential inclusion

$$\begin{cases} x'(t) & \in F(t,x(t)) \;\; p.p. \;\; sur \;\; [t_0-T,t_0+T], \\ \\ x(t_0) & = x_0. \end{cases} \tag{3.5}$$

Proof. step 1:

1. There exists $I=[t_0-T,t_0+T]$ and M such that

$$\begin{cases} i)I\times B_f(x_0,r) \text{ is contained in } \Omega, \\ ii)\|F(t,x)\|\leq M \text{ on } I\times B_f(x_0,r). \end{cases}$$

Indeed:

as Ω is an open set, there exists $r>0$ and $T>0$ such that the compact set $k=[t_0-T,t_0+T]\times B_f(x_0,r)$ is included in Ω. Moreover as F is $s.c.s.$ with compact convex non-empty values, by Proposition 2.2.3 $F(K)=\bigcup_{x\in K}F(x)$ is a compact, and therefore there exists M such that:

$$\sup\{\|u\|, u\in F(t,x),(t,x)\in k\}\leq M.$$

2. Let $H=\{x\in C(I,\mathbb{R}^n), x \text{ is Lipschitz with constant } M \text{ and } x(t_0)=x_0\}$, for all x in $\mathscr{H}$, define the integral operator $\mathscr{L}$ as follows

$$\mathscr{L}(x)=\{z\in\mathscr{H}, z'(t)\in F(t,x(t))\; p.p.\; in\; I\}.$$

Let us show that for all $x \in \mathscr{H}$, $\mathscr{L}(x)$ is nonempty.

Indeed:

Since the multi-map $t \longrightarrow F(t, x(t))$ is upper semi-continuous by Proposition 2.2.6 the hypothesis of the measurable selection theorem 2.2.2 is satisfied, i.e. there exists a measurable selection $v : I \longrightarrow \mathbb{R}^n$, such that $v(t) \in F(t, x(t))$.

We put $z(t) = x_0 + \int_{t_0}^{t} v(s)ds$.

• Check that $z \in \mathscr{L}(x)$; indeed

for t', $t \in I$ we have

$$\begin{aligned}
\|z(t') - z(t)\| &= \|x_0 + \int_{t_0}^{t'} v(s)ds - x_0 - \int_{t_0}^{t} v(s)ds\| \\
&= \|\int_{t}^{t'} v(s)ds\| \\
&\leq \int_{t}^{t'} \| v(s) \| ds \\
&\leq M \mid t' - t \mid,
\end{aligned}$$

whence z is M-Lipschitz.

It's clear that $z(t_0) = x_0$, so $z \in \mathscr{H}$.

Moreover we have

$$z'(t) = v(t) \; p.p.$$

from where

$$z'(t) \in F(t, x(t)),$$

which shows that $z \in \mathscr{L}(x)$.

Step 2: Let us show that

$$\begin{cases} 1)\, \textit{for } x \in \mathscr{H}, \mathscr{L}(x) \textit{ is convex,} \\ \\ 2)\, \textit{the multi-map } x \longrightarrow \mathscr{L}(x) \textit{ is s.c.s.,} \\ \\ 3)\ \textit{ there exists a fixed point } x^* \in \mathscr{L}\,(x^*). \end{cases}$$

• $\mathscr{L}$ is convex:

Let $z_1, z_2 \in \mathscr{L}(x)$. Then $z_1, z_2 \in \mathscr{H}$, and $z_1'(t), z_2'(t) \in F(t, x(t))$. Indeed for $t', t \in I$ and $\alpha \in [0,1]$ we have

$$\|z_1(t')-z_1(t)\| \leq M \mid t'-t \mid \ and\ \|z_2(t')-z_2(t)\| \leq M \mid t'-t \mid$$

we multiply the 1^{st} by α and the 2^{nd} by $(1-\alpha)$, by the addition we find:

$$\begin{aligned} \|(\alpha z_1(t')+(1-\alpha)z_2(t'))-(\alpha z_1(t)+(1-\alpha)z_2(t))\| &\leq \alpha M \mid t'-t \mid +(1-\alpha)M \mid t'-t \mid \\ \\ &= M \mid t'-t \mid. \end{aligned}$$

Hence $\alpha z_1 + (1-\alpha)z_2$ is M-Lipschitz.

And $\alpha z_1(t_0)+(1-\alpha)z_2(t_0)-z_0$. So $\alpha z_1 + (1-\alpha)z_2 \in \mathscr{H}$.

Moreover, we have $z_1'(t), z_2'(t) \in F(t, x(t))$, by the convexity of the values of F

$$\alpha z_1'(t)+(1-\alpha)z_2'(t) \in F(t, x(t)), \forall\, \alpha \in [0,1].$$

which shows the convexity of $\mathscr{L}$.

• $\mathscr{L}$ is upper semi-continuous: we first show that the graph of $\mathscr{L}$ is closed, indeed:

let $x_n \in \mathscr{H}$, $z_n \in \mathscr{L}(x_n)$, and we assume that $x_n \longrightarrow \bar{x}, z_n \longrightarrow \bar{z}$, and we call y_n the multi-map $y_n : t \longrightarrow z_n'(t)$ such that $\|y_n(t)\| \leq M$ of $L^{\infty}(I)$, hence equent by Alaoglu's theorem, and Theorem 1.4.1 we can extract a subsequence $(y_n)_n$ such that

$$y_n \rightharpoonup^* \bar{y}\ in\, \sigma(L^{\infty}, L^1),$$

in particular for all φ in $L^\infty(I)$,

$$< y_n, \varphi > \to < \bar{y}, \varphi >,$$

that is to say we have, on the one hand (y_n) converges in $\sigma(L^1, L^\infty)$, and on the other hand

$$z^n(t) = \int_{t_0}^{t} y_n(t) \to \int_{t_0}^{t} y\bar{(t)},$$

i.e. $\bar{y} = \bar{z}'$.

By the convergence theorem (Theorem 2.2.3) we deduce that $\bar{z}'(t) \in F(t, \bar{x}(t))$, whence the graph is closed.

We show that $\mathscr{H}$ is compact:

Let $(x_n)_n$ be a sequence of $\mathscr{H}$, so $(x_n)_n$ is an absolutely continuous function sequence on the compact I. By the corollary of Ascoli's theorem Arzela (theorem 2.2.4) there exists a subsequence also denoted by $(x_n)_n$ which uniformly converges to an absolutely continuous function x, whence the compactness of $\mathscr{H}$.

For values in the compact $\mathscr{H}$, Proposition 2.2.2 ensures that $\mathscr{L}$ is upper semi-continuous.

By application of kakutani's theorem there exists a fixed point $x^* \in L(x^*)$ i.e. $x^{*'}(t) \in F(t, x^*(t))$.

Therefore the problem 3.5 admits at least one solution. □

Conclusion

We studied some theorems of the fixed point that contribute to solving many non linear problems, which are based on sufficient conditions in each of the metric space, partial metric space, and cone metric space.

As some applications, we have studied the existence of solutions for each of the nonlinear integral inclusions and the issue of differential inclusions, by employing the studied theorems and some results in the metric space and the partial metric space.

Perspectives

In the future, we will look at the following issues as examples

(1) We will try to apply the results obtained to study the problem of the existence of the solution for a fractional differential inclusion.

(2) We will look for other combinations of the various theorems to improve our results.

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