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The nonlocal Cauchy problem for quasilinear integrodifferential equation with Caputo Generalized

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Abstract

In this work, we prove the existence of solutions of nonlinear fractional a quasilinear integrodifferential equations in Banach spaces with Caputo Generalized. Further, Cauchy problems with nonlocal initial conditions are discussed for the fractional integrodifferential equation.

Keywords: Caputo derivative type generalized fractional derivative, fixed point, existence of a solution, nonlinear fractional equation, nonlocal condition.

1 Introduction

Recently fractional differential equations emerged as a new branch of applied mathematics which have been used for many mathematical models in science and engineering. In fact fractional differential equations are considered as an alternative model to nonlinear differential equations. Theory of fractional differential equations has been extensively studied by the authors. Many partial fractional differential equations can be expressedas fractional differential equations in some Banach spaces.

K.Balachandrana, J.Y.Parkb [1] the study of nonlocal Cauchy problem for abstract fractional semilinear evolution equations. Recently, K.Balachandrana,J.J.Trujillo [2] the study the nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banachspaces. In this paper, we study the existence of solutions the nonlocal Cauchy problem for quasilinear integrodifferential equation in Banach spaces by using the generalized Caputo fractional derivative and the Banach fixed point theorem.

2 Preliminaries

We need some basic definitions and properties of fractional calculus which are used in this paper.
Let X be a Banach space and $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. [3] The generalized fractional integrals are defined by, $n-1 < \alpha \leq n$

$$\mathcal{I}_{a^+}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho} \right)^{\alpha-1} f(t) \frac{dt}{t^{1-\rho}},$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. The generalized fractional derivative of f of order α are defined by

$$D_{a^+}^{\alpha,\rho} f(x) = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} f(t) \frac{dt}{t^{1-\rho}} = \gamma^n \left(\mathcal{I}_{a^+}^{n-\alpha,\rho} f\right)(t),$$

where $\gamma = t^{1-\rho} \frac{d}{dt}$.

Definition 2.3. [4] We define the generalized Caputo fractional derivative of f of order by $\alpha \neq \mathbb{N}_0$

$${}^c D_{a^+}^{\alpha,\rho} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} \frac{(\gamma^n f)(t) dt}{t^{1-\rho}} = \mathcal{I}_{a^+}^{n-\alpha,\rho} (\gamma^n f)(t),$$

Definition 2.4. [5] Let $f : [0;1) \longrightarrow \mathbb{N}^+$ be a real valued function. The $\rho-$Laplace transform of f is defined by

$$\mathcal{L}_\rho \left\{ f(t) \right\}(s) = \int_0^{+\infty} e^{-s\frac{t^\rho}{\rho}} f(t) \frac{dt}{t^{1-\rho}}; \rho > 0.$$

for all values of s.

Now we shall state some properties of the operators $\mathcal{I}_{a^+}^{\alpha,\rho}$ and ${}^c D_{a^+}^{\alpha,\rho}$.

Proposition 1. For $\alpha,\ \beta > 0$ and f asuitable function (see for instance [3,4]) we have
(i) $\mathcal{I}_{a^+}^{\alpha,\rho} \mathcal{I}_{a^+}^{\beta,\rho} f(t) = \mathcal{I}_{a^+}^{\alpha+\beta,\rho} f(t)$
(i) $\mathcal{I}_{a^+}^{\alpha,\rho} (f(t) + g(t)) = \mathcal{I}_{a^+}^{\alpha,\rho} f(t) + \mathcal{I}_{a^+}^{\alpha,\rho} g(t)$.
(iii) $\mathcal{I}_{a^+}^{\alpha,\rho} {}^c D_{a^+}^{\alpha,\rho} f(t) = f(t) - f(a), 0 < \alpha < 1$.

Let $C(J, X)$ be the Banach space of continuous functions $u(t)$ with $u(t) \in X$ for $t \in J = [0, T]$ and $\|u\|_{C(J,X)} = \max_{t \in J} \|u(t)\|$. Let $B(X)$ denote the Banach space of bounded linear operators from X into X with the norm $\|A\|_{B(X)} = \sup \{\|A(x)\| \ : \|x\| = 1\|\}$. On the other hand, we will write ${}^c D_{0^+}^{\alpha,\rho}$ using the notation ${}^c D^{\alpha,\rho}$.
Consider the linear fractional evolution equation

$$\begin{cases} {}^c D^{\alpha,\rho} u(t) = A(t)u(t) + f(t),\ 0 \leq t \leq T \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $0 < \alpha < 1$, $A(t)$ is a bounded linear operator on a Banach space X, $u_0 \in X$ and $f : J \longrightarrow X$ is continuous. It is easy to prove that Eq.(2.1) is equivalent to the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} A(s)u(s) \frac{ds}{s^{1-\rho}} + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} f(s) \frac{ds}{s^{1-\rho}} \tag{2.2}$$

By a local solution of the abstract Cauchy problem (2.1), we mean an abstract function u such that the following conditions aresatisfied:
(i) $u \in C(J, X)$ and $u \in D(A(t))$ on J (here D for domain);
(ii) ${}^c D^{\alpha,\rho}$ exists and is continuous on J, where $0 < \alpha < 1$;
(iii) u satisfies Eq.(2.1) on J with the initial condition $u(0) = u_0 \in X$ or,equivalently, usatisfies the integral equation (4.2).

3 A quasilinear integrodifferential equation

Consider the fractional quasilinear integrodifferential equation of the form

$$\begin{cases} {}^c D^{\alpha,\rho} u(t) = A(t,\ u)u(t) + f\left(t,\ u(t),\ \int_0^t h(t,\ s,\ u(s))ds\right),\ 0 \leq t \leq T \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where $A(t,\ u)$ is a bounded linear operator on X and $f : J \times X \times X \longrightarrow X$, $h : \triangle \times X \longrightarrow X$ are continuous. Nonlinear functions f of this type with integral term h occur in mathematical problems concerned with heat flow in materials with memory and viscoelastic problems in which the integral term represents the viscosity part of the problem [6]. Here $\triangle = \{(t,s) : 0 \leq s \leq t \leq T\}$. For brevity let us take

$$Hu(t) = \int_0^t h(t,s,u(s))ds.$$

It is easy to prove that Eq.(3.1) is equivalent to the following integral equation:

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} A(s,\ u)u(s)\frac{ds}{s^{1-\rho}} + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} f(s,\ u(s),\ Hu(s))\frac{ds}{s^{1-\rho}} \quad (3.2)$$

We need the following assumptions to prove the existence of a solution of the integral Eq (3.1).
$(H1)$ $A : J \times X \longrightarrow B(X)$ is a continuous bounded linear operator and there exists a constant $M > 0$ such that

$$\|A(t,u) - A(t,v)\|_{B(X)} \leq M\|u - v\|,\ for\ all\ u,\ v \in X.$$

$(H2)$ $f : J \times X \times X \longrightarrow X$ is continuous and there exists a constant $L > 0$ such that

$$\|f(t,x,u) - f(t,y,v)\| \leq L\left[\|x-y\|+\|u-v\|\right],\ for\ all\ x,\ y,\ u,\ v \in X.$$

$(H3)$ $h : \triangle \times X \longrightarrow X$ is continuous and there exists a constant $L^* > 0$ such that

$$\|h(t,s,u) - h(t,s,v)\| \leq L^*\|u - v\|,\ for\ all\ u,\ v \in X$$

Let $B_r := \{u \in X : \|u\| \leq r$ for some $r > 0$. For brevity let us take $\gamma = \frac{\left(\frac{T^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}$ and $K = \sup_{t\in J}\|A(t,0)\|$, $N = \max_{t\in J}\|f(t,0,0)\|$, $N^* = \max_{(t,s)\in\triangle}\|h(t,s,0)\|$.
From $(H1)$ we observe that

$$\|A(t,u)\| \leq \|A(t,u) - A(t,0)\| + \|A(t,0)\| \leq M\|u\| + \|A(t,0)\| \leq Mr + K.$$

Further assume:
$(H4)$ $\|u_0\| + (Mr + K)r\gamma + \gamma M_0 \leq r$ where $M_0 = Lr + LL^*\frac{T^\rho}{\rho}r + LN^*\frac{T^\rho}{\rho} + N$.
$(H5)$ $p = \gamma(2Mr + K + L + LL^*T)$ *is such that* $0 \leq p < 1$.

Theorem 3.1. *If the hypotheses* $(H1) - (H5)$ *are satisfied, then the fractional quasilinear integrodifferential equation (3.1) has a unique solution continuous in* J.

Proof. Let $Z = C(J : B_r)$.Define the mapping $\Gamma : Z \longrightarrow Z$ by

$$\Gamma u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} A(s,\ u)u(s)\frac{ds}{s^{1-\rho}} + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} f(s,\ u(s),\ Hu(s))\frac{ds}{s^{1-\rho}} \quad (3.3)$$

and we have to show that has a fixed point. This fixed point is then a solution of Eq.(3.1).First we show that $\Gamma Br \subset Br$.
From the assumptions we have

$$\begin{aligned}
\|\Gamma u(t)\| &\leq \|u_0\| + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|A(s,u)\|\|u(s)\|\frac{ds}{s^{1-\rho}} + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|f(s,u(s),Hu(s))\|\frac{ds}{s^{1-\rho}} \\
&\leq \|u_0\| + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|A(s,u)\|\|u(s)\|\frac{ds}{s^{1-\rho}} \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \left[\|f(s,u(s),Hu(s)) - f(s,0,0)\| + \|f(s,0,0)\|\right]\frac{ds}{s^{1-\rho}} \\
&\leq \|u_0\| + (Mr+K)r\frac{\left(\frac{T^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} + (Lr + LL^*Tr + LN^*T + N)\frac{\left(\frac{T^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} \\
&= \|u_0\| + (Mr+K)r\gamma + \gamma M_0 \\
&\leq r.
\end{aligned}$$

Thus, Γ maps B_r into itself. Now, for $u_1,\ u_2 \in Z$, we have

$$\begin{aligned}
\|\Gamma u_1(t) - \Gamma u_2(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|A(s,u_1)u_1(s) - A(s,u_2)u_2(s)\| \frac{ds}{s^{1-\rho}} \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|f(s,u_1(s),Hu_1(s)) - f(s,u_2(s),Hu_2(s))\| \frac{ds}{s^{1-\rho}} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|A(s,u_1)u_1(s) - A(s,u_1)u_1(s) + A(s,u_1)u_1(s) - A(s,u_2)u_2(s)\| \frac{ds}{s^{1-\rho}} \\
&+ \gamma L \left[\|u_1(t) - u_2(t)\| + \|Hu_1(t) - Hu_2(t)\|\right] \\
&\leq \gamma(2Mr + K + L + LL^* \frac{T^\rho}{\rho})\|u_1(t) - u_2(t)\| \\
&= p\|u_1(t) - u_2(t)\|.
\end{aligned}$$

Since $0 \leq p < 1$, then Γ is a contraction mapping and therefore there exists a unique fixed point $u \in Z$ such that $\Gamma u(t) = u(t)$. Any fixed point of Γ is the solution of (3.1). $\square$

4 A nonlocal Cauchy problem

we discuss the existence of a solution of fractional quasilinear integrodifferential equation (3.1) with a nonlocal condition of the form

$$u(0) + g(u) = u_0, \tag{4.1}$$

where $g : C(J,X) \longrightarrow X$ is a given function. We assume the following conditions:
$(H6)$ $g : C(J,X) \longrightarrow X$ is continuous and there exists a constant $G > 0$ such that

$$\|g(u) - g(v)\| \leq G\|u - v\|,\ for\ u,\ v \in C(J,X).$$

$(H7)$ $\|u_0\| + Gr + \|g(0)\| + (Mr + K)r\gamma + \gamma M_0 \leq r$.
$(H8)$ $p' = G + \gamma(2Mr + K + L + LL^*T)$ is such that $0 \leq p' < 1$.

Theorem 4.1. *If the hypotheses $(H1) - (H3)$, $(H6) - (H8)$ are satisfied, then the fractional quasilinear integrodifferential equation (3.1) with nonlocal condition (4.1) has a unique solution continuous in J.*

Proof. We want to prove that the operator $\psi : Z \longrightarrow Z$ defined by

$$\psi u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} A(s,\ u)u(s) \frac{ds}{s^{1-\rho}} + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} f(s,\ u(s),\ Hu(s)) \frac{ds}{s^{1-\rho}} \tag{4.2}$$

has a fixed point. This fixed point is then a solution of Eqs.(3) and (6). Now we show that $\psi Br \subset Br$.

$$\begin{aligned}
\|\psi u(t)\| &\leq \|u_0\| + \|g(u) - g(0)\| + \|g(0)\| + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|A(s,u)u(s)\| \frac{ds}{s^{1-\rho}} \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \left[\|f(s,u(s),Hu(s)) - f(s,0,0)\| + \|f(s,0,0)\|\right] \frac{ds}{s^{1-\rho}} \\
&\leq \|u_0\| + Gr + \|g(0)\| + (Mr + K)r\gamma + \gamma M_0 \\
&\leq r.
\end{aligned}$$

Now, for $u_1,\ u_2 \in Z$, we have

$$\begin{aligned}
\|\psi u_1(t) - \psi u_2(t)\| &\leq \|g(u_1) - g(u_2)\| + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|A(s,u_1)u_1(s) - A(s,u_2)u_2(s)\| \frac{ds}{s^{1-\rho}} \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} \|f(s,u_1(s),Hu_1(s)) - f(s,u_2(s),Hu_2(s))\| \frac{ds}{s^{1-\rho}} \\
&\leq [G + \gamma(2Mr + K)]\|u_1(t) - u_2(t)\| + \gamma(L + LL^*T)\|u_1(t) - u_2(t)\| p' \|u_1(t) - u_2(t)\|.
\end{aligned}$$

Since $0 \leq p^{'} < 1$, the result follows by application of the contraction mapping principle. $\square$

Our next result is based on the following well−known fixed point theorem

Krasnoselkii Theorem Let S be a closed convex and nonempty subset of a Banach space X. Let P, Q be two operators such that
(i) $Px + Qy \in S$ whenever $x,\ y \in S$;
(ii) P is a contraction mapping;
(iii) Q is compact and continuous.
Then there exists $z \in S$ such that $z = Pz + Qz$.
Now we assume the following condition instead of $(H2)$ and apply the above fixed point theorem.
$(H9)$ $f : J \times X \longrightarrow X$ is continuous and there exists a function $\mu \in L^1(J)$ such that

$$\sup\|f(t,u,Hu)\| \leq (t),\ for\ all\ (t,u,Hu) \in J \times X \times X.$$

Theorem 4.2. *Assume that $(H1)$, $(H6)$, $(H9)$ hold. If $G + \gamma M < 1$, then the fractional evolution equation (3.1) with nonlocal condition (4.1) has a solution.*

Proof. Choose $r \geqslant \frac{\|u_0\|+\|g(0)\|+\gamma\|\mu\|_{L1}}{1-(G+\gamma M)}$ and define the operators P and Q on B_r as

$$Pu(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} A(s,\ u)u(s)\frac{ds}{s^{1-\rho}}$$

and

$$Qu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho}{\rho} - \frac{s^\rho}{\rho}\right)^{\alpha-1} f(s,\ u(s),\ Hu(s))\frac{ds}{s^{1-\rho}}$$

For any $u, v \in B_r$ we have $\|Pu + Qv\| \leq r$ and so $Pu + Qv \in B_r$. Obviously P is a contraction. The equicontinuity of $(Qu)(t)$ is already proved and so $Q(B_r)$ is relatively compact. By the Arzela−Ascoli Theorem, Q is compact. Hence by the Krasnoselkii theorem there exists a solution to the problem (3.1) and (4.1). $\square$

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