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Field : Mathematics

SUBJECT OF THE THESIS:

Boundary control and asymptotic stability of an axially moving string

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Defended on January 25, 2024, in front of the jury composed of:

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ملخص

في هذه الأطروحة، قمنا بدراسة استقرار حركة محورية مثل الخيوط والأحزمة والأسلاك والكابلات والأشرطة المغناطيسية والسلاسل، يتعلق العمل الأول بسلسلة متحركة محوريًا تخضع لاضطراب الحدود غير المحدود. يتم استخدام طريقة ليابونوف لإظهار فعالية التحكم في الحدود لضمان تقليل الاهتزاز. النتائج التي تم الحصول عليها تحسن بعض النتائج السابقة، المسألة الثانية هي تحريك شعاع اللزوجة المطاطية محوريًا. في المسألة الثالثة درسنا تثبيت سلسلة كيرشوف المطاطية اللزجة المتحركة محوريًا. أخيرًا، درسنا في المسألة الرابعة تثبيت شعاع لزج مطاطي متحرك محوريًا بشروط مصدر لوغاريتمي. لإظهار استقرار هذه الأنظمة، استخدمنا تقنية المضاعفات.

الكلمات المفتاحية: الاضمحلال العام، تقنية المضاعفات، شعاع أويلر- برنولي، هيكل متحرك، استقرار تكيفي عالي الكسب، تقدير الاضطراب، شبه المجموعة غير الخطية.

Abstract

In this thesis, we are interested in studying the stabilization of an axially moving, like threads, belts, wires, cables, magnetic tapes and chains. The first work is concerned with an axially moving string subject to unbounded boundary disturbance. The Lyapunov method is employed to show the effectiveness of the boundary control for ensuring the vibration reduction. The obtained results improves certain previous results. The second problem is axially moving viscoelastic beam. In third problem we studied the stabilization of an axially moving viscoelastic Kirchhoff string. Finally forth problem we studied the stabilization of an axially moving viscoelastic beam with logarithmic source terms. To show the stabilization of these system, we use a multipliers technique.

Key words : General decay, multiplier technique, Disturbance estimate, EulerBernoulli beam, Moving structure, High-gain adaptive stabilization, Nonlinear semigroup.

Résumé

Dans cette thèse, nous nous intéressons à l'étude de la stabilisation d'un mouvement axial, comme les fils, les courroies, les câbles, les bandes magnétiques et les chaînes. Le premier travail concerne une corde se déplaçant axialement soumis à des perturbations aux limites illimitées. La méthode de Lyapunov est utilisée pour montrer l'efficacité du contrôle des limites pour assurer la réduction des vibrations. Les résultats obtenus améliorent certains résultats antérieurs. Le deuxième problème est le déplacement axial de la poutre viscoélastique. Dans le troisième problème, nous avons étudié la stabilisation d'une corde de Kirchhoff viscoélastique se déplaçant axialement. Enfin, quatrième problème, nous avons étudié la stabilisation d'une poutre viscoélastique à déplacement axial avec des termes sources logarithmiques. Pour montrer la stabilisation de ces systèmes, nous utilisons une technique de multiplicateurs.

Mots clés : Décomposition général, technique du multiplicateur, Euler-Bernoulli en mouvement axial, Structure mobile, Stabilisation adaptative à gain élevé, Estimation des perturbations, Semi-groupe non linéaire.

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Finally, Thanks to my parents for their unwavering support, love, and advice over the years and for encouraging me to pursue my education. I am so grateful for my family and friends who have constantly surrounded me with love, joy and encouragement.

Dedication

I dedicate this thesis to my father and mother, to my family, to all my friends, and to everyone who contributed to this work.

Tikialine Belgacem

Contents

Publications

The following results were published or submitted:

1. **B. Tikialine**, A. Kelleche and H. A. Tedjani, High-gain adaptive boundary stabilization for an axially moving string subject to unbounded boundary disturbance, Annals of the University of Craiova, Mathematics and Computer Science Series. 48(1), 112-126, 2021.

2. **B. Tikialine**, A. Kelleche and H. A. Tedjani, General decay of the solution energy of an axially moving viscoelastic beam with Logarithmic Source Terms, Advances in Mathematics: Scientific Journal 11 (2022), no.1, 59-82.

3. **B. Tikialine**, A. Kelleche and H. A. Tedjani, General decay for an axially moving viscoelastic beam.

4. **B. Tikialine**, A. Kelleche and H. A. Tedjani, General decay for an axially moving viscoelastic Kirchhoff string.

Conferences attended

1. 5^{th} international conference of mathematical sciences ICMS 2021 23-27 june 2021 Istanbul, Turkey.

2. International conference on PDE and applications, modeling and simulation, jun 2-3, 2021 Morocco.

3. International seminar on mathematics and computer science, Oran, Algeria, 06/02/2021.

4. National conference applied mathematics and mathematical modeling, march 30-31, 2021, Eloued (Algeria).

5. National conference on applied mathematics and didactics (NCAMD2021), école normale supérieure assia djebar de Constantine - Constantine (Algeria), 19 jun 2021.

6. National conference of mathematics and applications (CNMA2021), 11- december Mila (Algeria).

7. 7^{th} international mardin artuklu scientific researches conference, december 10-12, 2021, Mardin, Turkey.

8. International conference on scientific research held on november 30 - december 1-2, 2021, krsehir ahi evran university.

9. Yarmouk mathematics conference on differential equations: analysis, modeling and numerical computations deamn, 18-20 september, 2021.

10. ALOP workshop on nonlocal models 12 july 2021 - 14 july 2021, Trier university, German.

11. 1^{st} International conference on pure and applied mathematics IC-PAM-21, may 26-27, 2021, Ouargla, Algeria.

12. The first national conference on pure and applied mathematics NC-PAM-2021, university of Laghouat, Algeria.

13. Conference of the southern africa mathematical sciences association (SAMSA), Auburn university, from 22^{nd} november to 24^{th} november 2021.

Notations

$ADRC$	:	Active disturbance rejection control.
$a.e$	:	Almost everywhere.
$\dfrac{\partial \varphi}{\partial t}, \varphi_t$	:	The partial derivative of φ.
$\mathrm{D}(\Omega)$	:	space of differentiable functions with compact support in Ω.
$\mathbb{R}^n$	:	Space of n-dimensional real vectors.
$\mathbb{R}_+$	:	The set of positive real numbers.
$C^k(\Omega)$	:	Space of functions k-times continuously differentiable in Ω.
$\mathbb{C}$	:	The set of complex numbers.
$\mathcal{D}(A)$	:	The domaine of A.
$\partial\Omega$	:	Boundary of Ω.
$+q'$	:	Conjugate of q.
$D'(\Omega)$	:	Distribution space.
$C_0(\Omega)$	:	Space of continuous functions null board in Ω.
$(Z, \\| \cdot \\|_Z)$	:	Banach space.
$L^p(\Omega)$	:	Space of functions p-th power integrated on Ω with measure of dx.
$C^k([0,T];Z)$	:	Space of functions k-times continuously differentiable .
$B_X = \{x \in X; \\|x\\| \leq 1\}$	:	Unit ball.
$\mathrm{Ran}(B)$	:	The range of B.
$\mathrm{Ker}(B)$	:	The kernel of B.
$\rho(B)$	:	The resolvent set of B.
$\mathrm{R}(\lambda, B)$	:	The resolvent set of B at point λ.
$\mathrm{Im}(B)$	:	Image of B.
$(M(t))_{t>0}$	:	Semigroup of operators.
B	:	Non Linear operator.
$\sigma(B)$	:	The spectrum of B.

Introduction

Axially moving continuous materials can be found in various engineering areas such as continuous material manufacturing lines and transport processes. Especially, the dynamics analysis and control for axially moving continuous materials which have received a growing attention due to the entrance of new applications in exible robotic manipulators and exible space structures. In all these applications, maximum conveying speed or conveying speed is desired in order to increase efficiency, optimize investment costs, the performance of the machines and its components. Installations are sometimes costly and complex. However, the dynamic behavior of these systems often prevent these objectives from being achieved. For more than 60 years, researchers have examined the dynamics of systems in axial movement. The early studies on the investigation of the transverse vibrations of a string in axial movement were published in 1950. Sack [39] and Archibald and Emslie [1] are pioneers in this field. Many scientists are interested in the case of a rope and a beam in axial movement. This interest promotes the emergence of new applications in a variety of disciplines.

One of the best studies devoted to the analysis of the vibrations of a axial moving string is that of Archibald and Emslie [1]. It deals with the case of transverse vibrations of a string moving with a constant speed C along the longitudinal direction, denoted by X.

Let $\varphi(x;t)$ be the transverse displacement of the string at position x, time t and l the length of the rope between the two ends, whereas T and ρ represent the tension and the mass per unit length of the string, respectively. Starting from Hamilton's principle (see [31]), we have

$$\int_{t_0}^{t_1} (\delta E_c - \delta E_p)\, dt = 0, \quad t_0 < t_1, \tag{1}$$

or

$$E_c := \frac{1}{2}\int_0^l \rho\left[c^2 + (\varphi_t + c\varphi_x)^2\right] dx, \quad E_p := \frac{1}{2}\int_0^l T\varphi_x^2 dx,$$

here E_c and E_p designate the kinetic energy and the potential energy, respectively between $x = 0$ and $x = l$ at time t. Integrate equation (1) by parts taking into account that $\delta\varphi\,(x, t_0) = \delta\varphi\,(x, t_1) = 0$ and $\delta\varphi(0, t) = \delta\varphi(l, t) = 0$ where the symbol δ indicates that it is in fact a tiny variation of the path taken to go from 0 to l. The transverse vibrations or transverse displacement of the string is given by

$$\rho\varphi_{tt} + 2\rho c\varphi_{xt} - \left(T - \rho c^2\right)\varphi_{xx} = 0,$$

where φ_{tt} is the local acceleration in the transverse direction of the chord, φ_{xt} is the Coriolis acceleration and $c^2\varphi_{xx}$ is the centripetal acceleration.

Different types of control, including domain control [8], parametric control [6] and variable structure control [22], have been studied. Boundary control provides an effective way to suppress transverse vibrations because it is easy to implement. It is also important to cite the work of [27] where the authors studied the following system

$$\begin{cases} \rho\varphi_{tt} + 2\rho c\varphi_{xt} - \left(T - \rho c^2\right)\varphi_{xx} = y(x,t) \\ \varphi_x(l,t) = y_c(t) \text{ ou } -\varphi_x(0,t) = y_c(t), \end{cases}$$

where $y(x,t)$ is a distributed force. Here the rope is controlled at the left and/or right end. The Lyapunov method was used to find control to stabilize vibration energy. In the case where the control acts on the right end $x = l$ with the left end fixed, they proposed the following control

$$y_c(t) = \varphi_x(l,t) = -k_1\varphi_t(l,t), k_1 > 0$$

In the case where the control is placed at the point $x = 0$, and the right endpoint $(x = l)$ is fixed, the authors proposed the following control

$$y_c(t) = -\varphi_x(0,t) = -k_2\varphi_t(0,t), k_2 > 0$$

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$$y_c(t) = -\varphi_x(0,t) = -k_3\left\{\varphi_t(l,t) + c\varphi_x(l,t)\right\}, k_3 > 0.$$

Note that in both cases, stabilization was obtained without energy decay rate.

Another type of control: Mass-Damper-Spring (MDS), was proposed in [22]. The MDS control mechanism includes a mass m, a viscous damper with coefficient d_m and a stiffness spring k_m. In [24], considered another model of a string in axial motion, known as "Kirchhoff", described by

$$\left\{ l\rho\varphi_{tt} + 2\rho c\varphi_{xt} - \left(T_0 - \rho c^2 + b\int_0^1 \varphi_x^2 dx\right)\varphi_{xx} = 0\varphi(0,t) = 0u_c(t) = \left(T_0 - \rho c^2 + b\int_0^1 \varphi_x^2 dx\right)\varphi_x(l,t).\varphi \right.$$

This model is characterized by the fact that the voltage depends only on the temporal variable

$$T(t) = T_0 + b\int_0^1 \varphi_x^2 dx.$$

The author proved that this nonlinear equation can be stabilized by the following control

$$u_c(t) = -k\varphi_t(l,t), k > 0$$

This work consists of five chapters:

First chapter, we introduced some notation and prepared some material needed for our work. The main results of this chapter such as: the L^p space, the Sobolev spaces , differential integral inequalities, nonlinear semigroups and their generators and nonlinear evolution with dissipative operator and other theorems of functional analysis.

Second chapter, is concerned with the following problem

$$\begin{cases} \varphi_{tt} + 2v\varphi_{xt} - \left(1 - v^2\right)\varphi_{xx} = 0, \;\; x \in (0,1)\,, \;\; t > 0, \\ \varphi(0,t) = 0, \quad t \geq 0, \\ \varphi_x(1,t) = u(t) + d(t), \;\; t \geq 0, \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \;\; x \in (0,1)\,, \\ \varphi_{out}(t) = (\varphi_t + v\varphi_x)(1,t), \;\; t > 0, \end{cases} \tag{2}$$

we proved the existence and uniqueness of solution of the closed loop system by using the semigroup theory and we proved the exponential stability by making use of the energy method.

Third chapter, we study the stabilization of an axially moving viscoelastic beam, the system is given by:

$$\begin{cases} \varrho\left(\varphi_{tt} + 2\mathcal{V}\varphi_{xt} + \mathcal{V}^2\varphi_{xx}\right) + EI\varphi_{xxxx} - EI\displaystyle\int_0^t \mathcal{Z}(t-s)\varphi_{xxxx}(s)ds = 0, \;\; x \in (0,l), \;\; t \geq 0, \\ \varphi(0,t) = \varphi_x(0,t) = \varphi_{xx}(l,t) = 0, \quad t \geq 0, \\ \varrho\mathcal{V}^2\varphi_x(l,t) + EI\varphi_{xxx}(l,t) - EI\displaystyle\int_0^t \mathcal{Z}(t-s)\varphi_{xxx}(l,s)ds = y(\varphi(l,t)), \quad t \geq 0, \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \;\; x \;\in (0,l) \end{cases} \tag{3}$$

where $\varphi = \varphi(x,t)$ is the beam transversal displacement, $\mathcal{V}$ is the axial speed (assumed constant here), EI is the beam flexural rigidity, ϱ is the beam mass per unit length, $\mathcal{Z}$ represents the kernel of the memory term or the relaxation function (nonnegative functions) see [20], $\varphi_0(x)$, $\varphi_1(x)$ are the initial data, and y represents the nonlinear term.

Fourth chapter, we studied the stabilization of an axially moving viscoelastic Kirchhoff string, the system is given by:

$$\begin{cases} \varphi_{tt} + 2\mathcal{V}\varphi_{xt} - \left(1 - \mathcal{V}^2 + q(t)\left\|\varphi_x\right\|^2\right)\varphi_{xx} + \left(1 - \mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}(s)\mathrm{d}s = 0 \\ x \in (0,1), \quad t > 0 \\ \varphi(0,t) = 0, \quad t \geq 0, \\ f_c(t) = m\varphi_{tt}(1,t) + (\eta_m - \mathcal{V})\varphi_t(1,t) + \left(1 - \mathcal{V}^2 + q(t)\left\|\varphi_x\right\|^2\right)\varphi_x(1,t) \\ -\left(1 - \mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\varphi_x(1,s)\mathrm{d}s, \quad t \geq 0, \end{cases} \tag{4}$$

Fifth chapter, we studied the stabilization of an axially moving viscoelastic beam with logarithmic source terms, the system is given by:

$$\begin{cases} \rho\left(\varphi_{tt}+2\mathcal{V}\varphi_{xt}+\mathcal{V}^2\varphi_{xx}\right)+EI\varphi_{xxxx}-EI\displaystyle\int_0^t\mathcal{Z}(t-s)\varphi_{xxxx}(s)ds=\kappa\varphi\ln|\varphi|, \quad x\in(0,l),\ t\geq 0,\\ \varphi(0,t)=\varphi_x(0,t)=\varphi_{xx}(l,t)=0, \quad t\geq 0,\\ \rho\mathcal{V}^2\varphi_x(l,t)+EI\varphi_{xxx}(l,t)-EI\displaystyle\int_0^t\mathcal{Z}(t-s)\varphi_{xxx}(l,s)ds=y(\varphi(l,t)),\ t\geq 0,\\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ x\ \in(0,l). \end{cases} \tag{5}$$

In this chapter, we proved the exponential stability by using the multiplier technique.

Chapter 1

Preliminary

In this chapter, we will introduce the necessary concepts for the good understanding of this thesis and shortly the basic results which concerning the Banach spaces, the L^p space, Sobolev spaces, semigroups of contractions and other theorems. The knowledge of all this notations and results are important for our study (See [5], [36], [33], [2], [29], [43]).

1.1 Banach Spaces

Definition 1. *A Banach space is a complete normed linear space Z. Its dual space Z' is the linear space of all continuous linear functional $g : Z \to \mathbb{R}$.*

Proposition 1. *([5] Z' equipped with the norm $\|\cdot\|_{Z'}$ defined by*

$$\|g\|_{Z'} = \sup\{|g(v)| : \|v\| \leq 1\},$$

is also a Banach space.

We shall denote the value of $g \in Z'$ at $v \in Z$ by either $g(v)$ or $\langle g, v\rangle_{Z',Z}$.

Remark 1. *([36]) From Z' we construct the bidual or second dual $Z'' = (Z')'$. Furthermore, with each $v \in Z$ we can define $\varphi(v) \in Z''$ by $\varphi(v)(g) = g(v), g \in Z'$, this satisfies clearly $\|\varphi(z)\| \leq \|v\|$. Moreover, for each $v \in Z$ there is an $g \in Z'$ with $g(v) = \|v\|$ and $\|g\| = 1$, so it follows that $\|\varphi(Z)\| = \|v\|$.*

Definition 2. *Since φ is linear we see that*

$$\varphi : Z \to Z''$$

is a linear isometry of Z onto a closed subspace of Z'', we denote this by

$$Z \hookrightarrow Z''.$$

Definition 3. *If φ (in the above definition) is onto Z'' we say Z is reflexive, $Z \approx Z''$.*

Definition 4. *Let Z be a Banach space, and let $(v_n)_{n\in\mathbb{N}}$ be a sequence in Z. Then v_n converges strongly to v in Z if and only if*

$$\lim_{n\to\infty} \|v_n - v\|_Z = 0,$$

and this is denoted by $v_n \to v$, or $\lim_{n\to\infty} v_n = v$.

1.1.1 The weak and weak star topologies

Let Z be a Banach space and $g \in Z'$. Denote by

$$\begin{aligned} \varphi_g : & Z \to \mathbb{R} \\ & z \mapsto \varphi_g(z), \end{aligned}$$

when g cover Z', we obtain a family $(\varphi_g)_{g\in Z'}$ of applications to Z in $\mathbb{R}$.

Definition 5. *The weak topology on Z, denoted by $\sigma(Z, Z')$, is the weakest topology on Z for which every $(\varphi_g)_{g\in Z'}$ is continuous.*

We will define the third topology on Z', the weak star topology, denoted by $\sigma(Z', Z)$. For all $z \in Z$. Denote by

$$\begin{aligned} \varphi_z : & Z' \to \mathbb{R} \\ & g \mapsto \varphi_z(g) = \langle g, z\rangle_{Z',Z}, \end{aligned}$$

when z cover Z, we obtain a family $(\varphi_z)_{z\in Z'}$ of applications to Z' in $\mathbb{R}$.

Definition 6. *The weak star topology on Z' is the weakest topology on Z' for which every $(\varphi_z)_{z\in Z'}$ is continuous.*

Definition 7. *A sequence (v_n) in Z is weakly convergent to Z if and only if*

$$\lim_{n\to\infty} g(v_n) = g(v)$$

for every $g \in Z'$ and this is denoted by $v_n \rightharpoonup v$.

Remark 2. *[38]*

1. *If the weak limit exist, it is unique.*
2. *If $v_n \to v \in Z$ (strongly), then $v_n \rightharpoonup v$ (weakly).*
3. *If $\dim Z < +\infty$, then the weak convergent implies the strong convergent.*

Proposition 2. *([5]) Let (g_n) be a sequence in Z'. We have:*

1. $\left[g_n \overset{*}{\rightharpoonup} g \text{ in } \sigma(Z', Z)\right] \Leftrightarrow [g_n(z) \to g(z), \forall z \in Z]$.

2. *If $g_n \to g$ (strongly), then $g_n \rightharpoonup g$, in $\sigma(Z', Z'')$.*

3. *If $g_n \rightharpoonup g$ in $\sigma(Z', Z'')$, then $g_n \overset{*}{\rightharpoonup} g$, in $\sigma(Z', Z)$.*

4. *If $g_n \overset{*}{\rightharpoonup} g$, in $\sigma(Z', Z)$, then $\|g_n\|$ is bounded and $\|g\| \leq \liminf \|g_n\|$.*

5. *If $g_n \overset{*}{\rightharpoonup} g$, in $\sigma(Z', Z)$ and $z_n \rightharpoonup z$ (strongly) in Z, then $g_n(z_n) \to g(z)$.*

1.1.2 Hilbert spaces

The most significant function space in contemporary physics and modern analysis, known as Hilbert spaces, turns out to be the appropriate framework for the formal theory of partial differential equation. Then, we must present some significant findings regarding these regions.

Definition 8. *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1. *([38]) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence which converges to u, in the weak topology and $(v_n)_{n\in\mathbb{N}}$ is an other sequence which converge weakly to v, then*

$$\lim_{n\to\infty} \langle v_n, u_n\rangle = \langle v, u\rangle.$$

Theorem 2. *([38]) Let Z be a normed space, then the unit ball*

$$B' = \left\{ l \in Z' : \|l\| \leq 1 \right\}$$

of Z' is compact in $\sigma(Z', Z)$.

1.2 Some Algebraic inequalities

1.2.1 Hölder's inequality

Lemma 1. *([2])Assume that $\varphi \in L^p$ and $g \in L^q$ with $1 \leqslant p \leqslant \infty$. Then $(\varphi g) \in L^1$ and*

$$\|\varphi g\| \leqslant \|\varphi\|_p \|g\|_q.$$

Lemma 2. *([2]) (Cauchy-Schwarz inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle Z_1, Z_2\rangle \leqslant \|Z_1\| \|Z_2\|.$$

The equality sign holds if and only if Z_1 and Z_2 are dependent.

1.2.2 Young's inequality

Lemma 3. *([2]) For all $a, b \geqslant 0$, the following inequality holds*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

whereas, $\frac{1}{p} + \frac{1}{q} = 1$.

1.2.3 Poincaré's inequality

Lemma 4. *([2]) Let $\Omega \subset \mathbb{R}^n$ is a bounded open subset. Then there exists a constant c, depending on Ω such that:*

$$\|\varphi\|_{L^2(\Omega)} \leqslant c\|\nabla\varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega).$$

Lemma 5. *([34, 41]) Let $w(z,t)$ a function in $\mathbb{R}$ defined for $z \in [0,l]$, $t \in \mathbb{R}_+$ that satisfies the boundary condition*

$$w(0,t) = 0, \ t \geq 0, \tag{1.1}$$

then the following inequalities hold

$$w^2(z,t) \leq l \left\|w_z\right\|^2, \ z \in [0,l], \ t \geq 0$$

and

$$\left\|w\right\|^2 \leq l^2 \left\|w_z\right\|^2, \ t \geq 0.$$

If in addition to (1.1) the function satisfies the boundary condition

$$w_z(0,t) = 0, \ t \geq 0,$$

then the following inequalities also hold

$$w_z^2(z,t) \leq l \left\|w_{zz}\right\|^2, \ z \in [0,l], \ t \geq 0$$

and

$$\left\|w_z\right\|^2 \leq l^2 \left\|w_{zz}\right\|^2, \ t \geq 0.$$

1.3 Functional Spaces

1.3.1 The $L^p(\Omega)$ spaces

Definition 9. *Let $1 \leq p < \infty$, and let Ω be an open domain in $\mathbb{R}^n, n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by $L^p(\Omega) = \left\{\varphi : \Omega \to \mathbb{R} : \varphi \text{ is measurable and } \int_\Omega |\varphi(z)|^p dz < \infty\right\}$.*

For $p \in \mathbb{R}$ *and* $1 \leq p < \infty$*, denote by*

$$\|\varphi\|_p = \left(\int_\Omega |\varphi(z)|^p dz\right)^{\frac{1}{p}}.$$

If $p = \infty$*, we have* $L^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi$ *is measurable and there exists a constant* C *such that,* $|\varphi(z)| \leq C$ *a.e in* $\Omega\}$.

Also, we denote by

$$\|\varphi\|_\infty = \text{Inf}\{C, |\varphi(z)| \leq C \text{ a.e in } \Omega\}.$$

1.3.2 The Sobolev space $\mathbf{W}^{m,p}(\boldsymbol{\Omega})$

Proposition 3. *([30]) Let* Ω *be an open domain in* $\mathbb{R}^N$*, Then the distribution* $T \in \mathcal{D}'(\Omega)$ *is in* $L^p(\Omega)$ *if there exists a function* $g \in L^p(\Omega)$ *such that*

$$\langle T, \varphi\rangle = \int_\Omega g(z)\varphi(z)dz, \text{ for all } \varphi \in \mathcal{D}(\Omega),$$

where $1 \leq p \leq \infty$*, and it's well-known that* φ *is unique.*

Definition 10. *Let* $m \in \mathbb{N}$ *and* $p \in [0, \infty]$*. The* $W^{m,p}(\Omega)$ *is the space of all* $\varphi \in L^p(\Omega)$*, defined as*

$$W^{m,p}(\Omega) = \left\{\varphi \in L^p(\Omega), \text{ such that } \partial^\gamma\varphi \in L^p(\Omega) \text{ for all } \gamma \in \mathbb{N}^m \text{ such that } |\gamma| = \sum_{j=1}^n \gamma_j \leq m, \text{ where } , \partial^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}..\partial_n^{\gamma_n}\right\}.$$

Theorem 3. *([38])* $W^{m,p}(\Omega)$ *is a Banach space with their usual norm*

$$\|\varphi\|_{W^{m,p}(\Omega)} = \sum_{|\gamma|\leq m} \|\partial^\gamma\varphi\|_{L^p}, 1 \leq p < \infty, \text{ for all } \varphi \in W^{m,p}(\Omega).$$

Definition 11. *Denote by* $W_0^{m,p}(\Omega)$ *the closure of* $D(\Omega)$ *in* $W^{m,p}(\Omega)$.

Definition 12. *When* $p = 2$*, we prefer to denote by* $W^{m,2}(\Omega) = H^m(\Omega)$ *and* $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ *supplied with the norm*

$$\|\varphi\|_{H^m(\Omega)} = \left(\sum_{|\gamma|\leq m} (\|\partial^\gamma\varphi\|_{L^2})^2\right)^{\frac{1}{2}}$$

which do at $H^m(\Omega)$ *a real Hilbert space with their usual scalar product*

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\gamma|\leq m} \int_\Omega \partial^\gamma u \partial^\gamma v dz.$$

1.3.3 The $L^p(0,T,Z)$ spaces

Definition 13. *Let Z be a Banach space, denote by $L^p(0,T,Z)$ the space of measurable functions*

$$\begin{aligned} \varphi: \quad &]0,T[\to Z, \\ & t \longmapsto \varphi(t), \end{aligned}$$

such that

$$\left(\int_0^T \|\varphi(t)\|_Z^p dt\right)^{\frac{1}{p}} = \|\varphi\|_{L^p(0,T,Z)} < \infty, \text{ for } 1 \leq p < \infty.$$

If $p = \infty$,

$$\|\varphi\|_{L^\infty(0,T,Z)} = \sup_{t\in]0,T[} \text{ess} \|\varphi(t)\|_Z.$$

Lemma 6. *([2]) Let Z_0, Z and Z_1 be three Banach spaces with $Z_0 \subseteq Z \subseteq Z_1$. Assume that Z_0 is compactly embedded in Z and that Z is continuously embedded in Z_1; assume also that Z_0 and Z_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; Z_0) / \quad \dot{u} \in L^q([0,T]; Z_1)\}.$$

Then the embedding of W into $L^p([0,T]; Z)$ is also compact.

1.4 Bounded and unbounded linear operators in Banach spaces

Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be two Banach spaces.

Definition 14. *B linear operator $B : X \to Z$ is a transformation which maps linearly X in Z, that is*

$$B(\alpha x + \beta z) = \alpha B(x) + \beta B(z), \forall x, z \in X \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 15. *B linear operator $B : X \to Z$ is bounded if there exists $C \geq 0$, such that*

$$\|Bv\|_Z \leq C\|v\|_X, \forall v \in X.$$

That is, we say that B is bounded if $\|Bv\|_Z$ remains bounded when $v \in \{x \in X, \|x\|_X \leq 1\}$. Otherwise, B is said to be unbounded.

Definition 16. *Let E and H be Banach spaces.*

1. *An operator $B : D(B) \subset X \to R(B) \subset H$ such that*

$$\|Bx - Bz\| \leq \|x - z\|, \quad x, z \in D(B),$$

is called a contraction operator.

Where $D(B), R(B)$ we denote domain and range of B, respectively.

2. *An operator $B : D(B) \subset E \to R(B) \subset H$, is called a strict contraction if there exists B constant α with $0 \leq \alpha < 1$ such that*

$$\|Bx - Bz\| \leq \alpha \|x - z\|, \quad x, z \in D(B).$$

We have the following fixed point theorem.

Theorem 4. *([38]) Let X be a closed subset of E. If $B : X \to X$ is B strict contraction operator, then B has a unique fixed point.*

1.5 Strongly continuous semigroups

Definition 17. *Let $(M(t))_{t>0}$ be a family of bounded linear operators on a Banach space X is called a strongly continuous (one-parameter) semigroup (or C_0-semigroup) if it satisfies the functional equation*

$$\begin{cases} M(t+s) = M(t) \cdot M(s), \text{ for all } t, s > 0 \\ M(0) = I, \end{cases}$$

and is strongly continuous in the following sense. For every $x \in X$ the orbit maps

$$\xi_x : t \to \xi_x(t) := M(t)x,$$

are continuous from $\mathbb{R}_+$ into X for every $x \in X$.

Definition 18. *(Infinitesimal generator of the semigroup) The linear operator B defined by*

$$\mathcal{D}(B) = \left\{ x \in X : \lim_{t \to 0} \frac{M(t)x - x}{t} \text{ exists} \right\},$$

and

$$Bx = \lim_{t \to 0} \frac{M(t)x - x}{t} = \left. \frac{d^+ M(t)x}{t} \right|_{t=0} \text{ for } x \in \mathcal{D}(B),$$

is the infinitesimal generator of the semigroup $M(t)$, $\mathcal{D}(B)$ is the domain of B.

Definition 19. *An unbounded linear operator $(B, \mathcal{D}(B))$ on X, is said to be dissipative if*

$$Re\langle Bv, v\rangle_X \leq 0, \forall v \in \mathcal{D}(B).$$

Definition 20. *An unbounded linear operator $(B, \mathcal{D}(B))$ on X, is said to be maximal dissipative (m-dissipative) if*

- *B is a dissipative operator.*
- $\exists \lambda_0 > 0$ *such that* $\operatorname{Im}(\lambda_0 I - B) = X$.

1.6 Nonlinear semigroups

Definition 21. *Let X be a real Banach space with norm $\|.\|$. Let X_0 be a closed subset of X. Let $M(t) : X_0 \to X_0$ be B nonlinear operator for every $t \geq 0$.*

The family $\{M(t), t \geq 0\}$ is called B nonlinear semigroup on X_0 of type α if:

1. $M(0)x = x$ *for every* $x \in X_0$,
2. $M(t+s)x = M(t)M(s)x$ *for every* $x \in X_0$ *and* $t, s \geq 0$,
3. $\lim_{t \to 0^+} M(t)x = x$ *for every* $x \in X_0$,
4. *For every* $x, z \in X_0, t \geq 0$ *and for some* $\alpha \in \mathbb{R}$

$$\|M(t)x - M(t)z\| \leq e^{\alpha t}\|x - z\|.$$

When $\alpha = 0$, $\{S(t), t \geq 0\}$ be a nonlinear semigroup of contraction on X_0, or simply a nonlinear semigroup on X_0.

Chapter 2

Stability of an axially moving string

2.1 Introduction

In this chapter, we study the stabilization of solutions of the closed-loop system of (2) given by

$$\begin{cases} \varphi_{tt} + 2v\varphi_{xt} - \left(1 - v^2\right)\varphi_{xx} = 0, \;\; x \in (0,1)\,, \;\; t > 0, \\ \varphi(0,t) = 0, \quad t \geq 0, \\ \varphi_x(1,t) = -r(t)\left(\varphi_t + v\varphi_x\right)(1,t) + \vartheta\,(t) + d(t), \;\; t \geq 0, \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \;\; x \in (0,1)\,, \\ r'(t) = p\left(\varphi_t + v\varphi_x\right)^2(1,t), \;\; p > 0, \;\; r(0) = r_0 > 0, \\ \varphi_{out}(t) = \left(\varphi_t + v\varphi_x\right)(1,t), \;\; t > 0, \end{cases} \tag{2.1}$$

whereas $\varphi = \varphi(x,t)$ is the transverse displacement of the string which is axially moving, v represents speed, $\varphi_{out}(t)$ stands for the measured signal of the system at that free end (output) and $d(t)$ represents the unknown external disturbance satisfy

$$d \in C\,(0,\infty) \;\; \text{and} \;\; |d\,(t)|\,, \left|d'\,(t)\right| \leq C_d e^{a_0 t}. \tag{2.2}$$

The total derivative operator with respect to time is given by

$$\frac{d}{dt}(.) = \dot{(.)} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}. \tag{2.3}$$

2.2 Existence Result

We introduce the following subspaces

$$V = \left\{\varphi \in H^1(0,1), \varphi(0) = 0\right\},$$

$$H = L^2(0,1),$$

equipped with the norm $\|\varphi\|_V = \|\varphi_x\|$. *Clearly* $V \subset H \subset V'$ *where* V' *denotes the dual of* V.

2.2.1 Abstract setting

In order to study the existence of solutions, we need to write our main problem as an abstract problem, for this we introduce:

$$A : D(A) \to H, A\varphi = -\varphi_{xx},$$

$$D(A) = \left\{\varphi \in H^1,\ \varphi_{xx} \in H,\ \varphi(0) = \varphi_x(1) = 0\right\}.$$

The operator A *is a positive operator with a compact inverse in* H*, we have*

$$A^{\frac{1}{2}}\varphi : D(A) \to H,\ A^{\frac{1}{2}}\varphi = \varphi_x,$$

with

$$\begin{aligned} d(A^{\frac{1}{2}}) &= \left\{\varphi \in H^1,\ \varphi(0) = 0\right\} = V, \\ (Af, g)_{V'\times V} &= \left(A^{\frac{1}{2}}f, A^{\frac{1}{2}}g\right), \\ (A\varphi, \varphi) &= (\varphi_x, \varphi_x) = \left\|A^{\frac{1}{2}}\varphi\right\|. \end{aligned} \tag{2.4}$$

By Poincaré inequality

$$\|\varphi\| \le \left\|A^{\frac{1}{2}}\varphi\right\|. \tag{2.5}$$

Then, (2.1) can be written as

$$\begin{cases} \dfrac{d^2\varphi}{dt^2} + r(t)KK^*\dfrac{d}{dt}\varphi + K(\vartheta + d) + A\varphi = 0,\ t > 0, \\ \varphi(0) = \varphi_0,\ \varphi_t(0) = \varphi_1, \\ \dot r(t) = pw^2(t),\ p > 0,\ r(0) = r_0 > 0,\ t > 0, \\ \varphi_{out}(t) = K^*\dot\varphi(t),\ t > 0. \end{cases} \tag{2.6}$$

$K : \mathbb{R} \to V'$, $K = \delta(x-1)$ *with* $K^* = (\delta(x-1), .)$ *where* δ *is the Dirac distribution (see [43]). We denote by* $\varphi(t) := \left(\varphi(t), \frac{d}{dt}\varphi(t), r(t)\right)$, *then* $\frac{d}{dt}\varphi := \left(\frac{d}{dt}\varphi(t), \frac{d^2\varphi}{dt^2}(t), r'(t)\right)$ *and* φ *satisfies*

$$\begin{cases} \frac{d}{dt}\varphi = \mathcal{A}\varphi + \mathcal{K}(d + \vartheta), \\ \varphi(0) = \varphi_0 = (\varphi_0, \varphi_1, r_0)^T, \end{cases}$$

where

$$\mathcal{A}\begin{pmatrix} \varphi \\ z \\ r \end{pmatrix} = \begin{pmatrix} z \\ -rkk^*z - A\varphi \\ p\left[K^*z\right]^2 \end{pmatrix}, \tag{2.7}$$

with domain

$$D(\mathcal{A}) = \left\{ (\varphi, z, r) \in D(A) \times V \times \mathbb{R}^+, rKK^*z + A\varphi \in H \right\},$$

and $\mathcal{K} = (0, -K)^T$. *Denote by* $\mathcal{H}$ *the Hilbert space*

$$\mathcal{H} = V \times H \times \mathbb{R}^+,$$

$$\left\langle (\varphi_1, z_1, r_1)^T, (\varphi_2, z_2, r_2)^T \right\rangle = \left(A^{\frac{1}{2}}\varphi_1, A^{\frac{1}{2}}\varphi_2 \right) + (z_1, z_2) + \frac{1}{2p} r_1 r_2.$$

In order to show the existence and uniqueness of solution of (2.6), see [29, 43].

Theorem 5. *For* $\varphi_0 \in D(A)$, *the system (2.1) admits a unique solution satisfying*

$$\varphi \in C([0,T), D(A)) \cap C^1([0,T), \mathcal{H}).$$

Proof. Let $(\varphi, z, r)^T \in D(\mathcal{A})$, then

$$\begin{aligned}
& \left\langle \mathcal{A}(\varphi_1, z_1, r_1) - \mathcal{A}(\varphi_2, z_2, r_2), (\varphi_1 - \varphi_2, z_1 - z_2, r_1 - r_2) \right\rangle \\
= & \left(A^{\frac{1}{2}}(z_1 - z_2), A^{\frac{1}{2}}(\varphi_1 - \varphi_2) \right) + \left((-r_1 KK^* z_1 - A\varphi_1) - (-r_2 KK^* z_2 - A\varphi_2), z_1 - z_2 \right) \\
& + \frac{1}{2}(r_1 - r_2)\left([K^* z_1]^2 - [K^* z_2]^2 \right) \\
= & \left((-r_1 KK^* z_1) - (-r_1 KK^* z_2), z_1 - z_2 \right) + \frac{1}{2}(r_1 - r_2)\left([K^* z_1]^2 - [K^* z_2]^2 \right) \\
= & \left[\frac{1}{2}(r_1 - r_2) - r_1 \right][K^* z_1]^2 + (r_1 + r_2) K^* z_1 K^* z_2 + \left[\frac{1}{2}(r_2 - r_1) - r_2 \right][K^* z_2]^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\langle \mathcal{A}(\varphi_1, z_1, r_1) - \mathcal{A}(\varphi_2, z_2, r_2), (\varphi_1 - \varphi_2, z_1 - z_2, r_1 - r_2) \right\rangle \\
= & -\frac{1}{2}(r_1 + r_2)\left([K^* z_1] - [K^* z_2] \right)^2.
\end{aligned}$$

We conclude that $\mathcal{A}$ is dissipative in $\mathcal{H}$.
Second, we show that $\lambda I - \mathcal{A}$ is surjective. Let $(\phi, \psi, \chi)^T \in \mathcal{H}$, $\lambda > 0$ and $(\varphi, z, r)^T \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A}) \begin{pmatrix} \varphi \\ z \\ r \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \\ \chi \end{pmatrix}.$$

This is equivalent to

$$\lambda\varphi - z = \phi, \tag{2.8}$$

$$\lambda z + rKK^* z + A\varphi = \psi, \tag{2.9}$$

$$\lambda r - p[K^* z]^2 = \chi. \tag{2.10}$$

Then, by (2.8)

$$z = \lambda\varphi - \phi \in V. \tag{2.11}$$

Moreover, from (2.8) and (2.11), r is given by

$$r = \frac{1}{\lambda}\left[\chi + p\left[K^* z\right]^2\right] = \frac{1}{\lambda}\left\{\chi + p\left[K^*\left(\lambda\varphi - \phi\right)\right]^2\right\}.$$

From (2.9) and (2.11), φ satisfies

$$\lambda^2\varphi + \lambda r K K^*\varphi + A\varphi = \psi + \lambda\phi + \lambda r K K^*\phi. \tag{2.12}$$

We set

$$A_\lambda = \lambda^2 + \lambda r K K^* + A$$

and

$$y = \psi + \lambda\phi + \lambda r K K^*\phi \in H \subset V'.$$

Let us introduce the operator

$$\Phi_\lambda : V \to V : \varphi \to \varphi - \frac{1}{\lambda^2}\left(A_\lambda\varphi - y\right).$$

The existence of a fixed point for Φ_λ is clearly equivalent to the existence of a solution to equation (2.12).

Using (2.4), we get

$$(\lambda r K K^*\varphi + A\varphi, \varphi)_{V',V} = \int_0^1 \varphi_x^2 dx + \frac{\lambda}{r}\varphi^2(1,t) \leq (1+\lambda r)\int_0^1 \varphi_x^2 dx = (1+\lambda r)\left\|\varphi\right\|_V^2$$

and

$$(\lambda r K K^*\varphi + A\varphi, \varphi)_{V',V} = \int_0^1 \varphi_x^2 dx + \lambda r\varphi^2(1,t) \geq \left\|\varphi\right\|_V^2.$$

we see that

$$\left|(\lambda r K K^*\varphi + A\varphi, u)_{V',V}\right| = \int_0^1 \varphi_x u_x dx + \lambda r\varphi(1,t)u(1,t) \leq (1+\lambda r)\left\|\varphi\right\|_V \left\|u\right\|_V,$$

which implies that

$$\left\|\lambda r K K^*\varphi + A\varphi\right\|_{V'} \leq (1+\lambda r)\left\|\varphi\right\|_V. \tag{2.13}$$

We calculate

$$\left\|\Phi_\lambda(\varphi_1) - \Phi_\lambda(\varphi_2)\right\|_V^2 = \frac{1}{\lambda^4}\left(\lambda r K K^*(\varphi_1-\varphi_2) + A(\varphi_1-\varphi_2), \lambda r K K^*(\varphi_1-\varphi_2) + A(\varphi_1-\varphi_2)\right).$$

From the estimate (2.13), we get

$$\|\Phi_\lambda(\varphi_1) - \Phi_\lambda(\varphi_2)\|_V^2 \le \frac{1}{\lambda^4}(1+\lambda r)^2 \|\varphi_1 - \varphi_2\|_V^2 .$$

For sufficiently large λ, Φ_λ is a contraction mapping from V into itself and has a unique fixed point φ in V which is a solution of (2.12). □

2.3 Disturbance estimate

This section is reserved to estimate the disturbance $d(t)$ in (2.1). We employ the active disturbance rejection control (ADRC) approach to investigate this problem (see [24]). Multiplying the first equation in (2.1) by $g(x) = x$ and integrating over $(0,1)$, we get

$$\int_0^1 x\frac{d}{dt}(\varphi_t + v\varphi_x)\,dx = \varphi_x(1,t) - \int_0^1 \varphi_x dx = -\varphi(1,t) + u(t) + d(t),$$

or

$$\frac{d}{dt}\int_0^1 x(\varphi_t + v\varphi_x)\,dx = \int_0^1 (\varphi_t + v\varphi_x)\,dx - \varphi(1,t) + u(t) + \vartheta(t) + d(t). \tag{2.14}$$

Set

$$y(t) = \int_0^1 x(\varphi_t + v\varphi_x)\,dx,\ \ y_0(t) = \int_0^1 (\varphi_t + v\varphi_x)\,dx - \varphi(1,t).$$

Then, (2.14) becomes

$$\frac{d}{dt}y(t) = y_0(t) + u(t) + d(t). \tag{2.15}$$

This is the first step to estimate the disturbance using the technique introduced in [24] for lumped parameter systems.

$$\begin{cases} \dot{\hat{y}}(t) = y_0(t) + u(t) + \hat{d}(t) - q(t)(\hat{y}(t) - y(t)), \\ \dot{\hat{d}}(t) = -q^2(t)(\hat{y}(t) - y(t)), \end{cases} \tag{2.16}$$

where $q \in C^1(\mathbb{R}_+)$ fulfills the conditions
(H1) $q(t), \dot{q}(t) > 0$ *and* $\sup_{t\ge 0}\frac{\dot{q}(t)}{q(t)} = N < \infty$,
(H2) $\lim_{t\to\infty}\frac{|d(t)|}{q(t)} = 0$.
Next, we introduce the errors which are formulated by

$$e_y = -q(t)(y(t) - \hat{y}(t)),\ \ e_d = d(t) - \hat{d}(t),\ \ t \ge 0.$$

Lemma 7. *Assume that* ***(H1)*** *and* ***(H2)*** *hold and $d(t)$ satisfy (2.2), then the solution of (2.16) satisfies*

$$\lim_{t\to\infty} e_y = \lim_{t\to\infty} e_d = 0.$$

Furthermore, if $q(t) = Re^{q_0 t}$ with $q > C_d$ and $q_0 > a_0$, then there exists a positive constant K such *that*

$$|e_y|\,,|e_d| \leq K e^{-(q_0-a_0)t},\ t\geq 0.$$

Proof. We have

$$\dot{e}_y(t) = -\dot{q}(t)\left(y(t)-\hat{y}(t)\right) - q(t)\left(\dot{y}(t)-\dot{\hat{y}}(t)\right),\ t\geq 0. \tag{2.17}$$

Then, we obtain

$$\begin{aligned}
\dot{e}_y(t) &= -\dot{q}(t)\left(y(t)-\hat{y}(t)\right) - q(t)\left[y_0(t)+u(t)+d(t)-y_0(t)-u(t)-\hat{d}(t)+q(t)\left(\hat{y}(t)-y(t)\right)\right]\\
&= -\dot{q}(t)\left(y(t)-\hat{y}(t)\right) - q(t)\left(d(t)-\hat{d}(t)\right) + q^2(t)\left(y(t)-\hat{y}(t)\right)\\
&= \frac{\dot{q}(t)}{q(t)}e_y(t) - q(t)\left(e_y(t)+e_d(t)\right),\ t\geq 0.
\end{aligned} \tag{2.18}$$

On the other hand

$$\dot{e}_d = \dot{d}(t) - \dot{\hat{d}}(t),\ t\geq 0. \tag{2.19}$$

Then, we get

$$\dot{e}_d = \dot{d}(t) - q^2(t)(y(t)-\hat{y}(t)) = q(t)e_y(t) + \dot{d}(t),\ t\geq 0. \tag{2.20}$$

Taking into account the previous computations, the following system is obtained

$$\begin{cases}
\dot{e}_y(t) = -q(t)\left(e_y(t)+e_d(t)\right) + \frac{\dot{q}(t)}{q(t)}e_y(t),\ t\geq 0,\\
\dot{e}_d(t) = q(t)e_y(t) + \dot{d}(t),\ t\geq 0,\\
e_y(0) = e_{y,0},\ e_d(0) = e_{d,0}.
\end{cases} \tag{2.21}$$

The system (2.21) can be written as

$$\begin{cases}
\dot{e}(t) = A\left(t\right)e(t) + f_d\left(t\right),\ t\geq 0,\\
e(0) = e_0 = \left(e_{\eta,0}, e_{d,0}\right),
\end{cases} \tag{2.22}$$

with

$$A\left(t\right) = \begin{pmatrix} -q(t)+\frac{\dot{q}(t)}{q(t)} & -q(t)\\ q(t) & 0 \end{pmatrix},\ f_d\left(t\right) = \begin{pmatrix} 0\\ \dot{d}(t)\end{pmatrix}.$$

We define the Lyapunov function for (2.21) by

$$\mathcal{L}(t) = e_y^2(t) + \frac{3}{2}e_d^2(t) + e_y(t)e_d(t),\ t\geq 0.$$

We see that

$$\frac{1}{2}\left(e_y^2(t)+e_d^2(t)\right) \leq \mathcal{L}(t) \leq 2\left(e_y^2(t)+e_d^2(t)\right),\ t\geq 0. \tag{2.23}$$

Differentiating the Lyapunov function $\dot{\mathcal{L}}(t)$, we get

$$\begin{aligned}\dot{\mathcal{L}}(t) &= 2\dot{e}_y(t)e_y(t) + 3\dot{e}_d(t)e_d(t) + \dot{e}_y(t)e_d(t) + e_y(t)\dot{e}_d(t)\\ &\leq -q(t)\left(e_y^2(t)+e_d^2(t)\right) + 2\frac{\dot{q}(t)}{q(t)}e_y^2(t) + \frac{\dot{q}(t)}{q(t)}e_y(t)e_d(t)\\ &\quad + \left|\dot{d}(t)\right|\left(3\left|e_d(t)\right| + \left|e_y(t)\right|\right),\ t \geq 0.\end{aligned}$$

System (2.21) and assumptions (2.2) imply that

$$\begin{aligned}\dot{\mathcal{L}}(t) &\leq \left(-q(t) + \frac{1}{2}\frac{\dot{q}(t)}{q(t)}\right)\left(e_y^2(t)+e_d^2(t)\right) + 2\frac{\dot{q}(t)}{q(t)}e_y^2(t) + 3C_d\left|d(t)\right|\left(\left|e_d(t)\right| + \left|e_y(t)\right|\right)\\ &\leq \left(-q(t) + \frac{5}{2}\frac{\dot{q}(t)}{q(t)}\right)\left(e_y^2(t)+e_d^2(t)\right) + 3\sqrt{2}C_d e^{a_0 t}\sqrt{e_y^2(t)+e_d^2(t)},\ t \geq 0.\end{aligned}$$

We have

$$-q(t) + \frac{5}{2}\frac{\dot{q}(t)}{q(t)} < -q(t) + \frac{5}{2}N < 0.$$

This is owing to

$$\dot{\mathcal{L}}(t) \leq \left(-q(t) + \frac{5}{2}N\right)\mathcal{L}(t) + 3\sqrt{2}C_d e^{a_0 t}\sqrt{\mathcal{L}(t)},\ t \geq 0,$$

which gives

$$\frac{d}{dt}\sqrt{\mathcal{L}(t)} \leq \frac{1}{2}\left(-q(t) + \frac{5}{2}N\right)\sqrt{\mathcal{L}(t)} + \frac{3}{\sqrt{2}}C_d e^{a_0 t},\ t \geq 0.$$

Integrating over (t_0, t), we obtain

$$\sqrt{\mathcal{L}(t)} \leq \sqrt{\mathcal{L}(t_0)}e^{\frac{1}{2}\int_{t_0}^t\left(-q(s)+\frac{5}{2}N\right)ds} + \frac{3C_d}{\sqrt{2}}\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_\tau^t\left(-q(s)+\frac{5}{2}N\right)ds}d\tau,\ t > t_0.$$

Using L'Hospital rule and (**H2**), we get

$$\begin{aligned}\lim_{t\to\infty}\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_\tau^t\left(-q(s)+\frac{5}{2}N\right)ds}d\tau &= \lim_{t\to\infty}\frac{\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_{t_0}^\tau\left(q(s)-\frac{5}{2}N\right)ds}d\tau}{e^{\frac{1}{2}\int_{t_0}^t\left(q(s)-\frac{5}{2}N\right)ds}}\\ &= 2\lim_{t\to\infty}\frac{e^{a_0 t}}{2r(t)-5N} = 0,\end{aligned}$$

then $\lim_{t\to\infty}\sqrt{\mathcal{L}(t)} = 0$. This leads by the equivalence result (2.23) to

$$\lim_{t\to\infty}\left(e_\eta^2(t) + e_d^2(t)\right) = 0.$$

This completes the proof of the first assertion of Lemma 7. Furthermore, since

$$\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_\tau^t\left(-q(s)+\frac{5}{2}N\right)ds}d\tau = \frac{\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_{t_0}^\tau\left(q(s)-\frac{5}{2}N\right)ds}d\tau}{e^{\frac{1}{2}\int_{t_0}^t\left(q(s)-\frac{5}{2}N\right)ds}},\ t > t_0.$$

We compute

$$\lim_{t\to\infty}\frac{\int_{t_0}^{t}e^{a_0\tau}e^{\frac{1}{2}\int_{\tau}^{t}\left(-q(s)+\frac{5}{2}N\right)ds}d\tau}{e^{-(q_0-a_0)t}}=\lim_{t\to\infty}\frac{\int_{t_0}^{t}e^{a_0\tau}e^{\frac{1}{2}\int_{t_0}^{\tau}\left(q(s)-\frac{5}{2}N\right)ds}d\tau}{e^{-(q_0-a_0)t+\frac{1}{2}\int_{t_0}^{t}\left(q(s)-\frac{5}{2}N\right)ds}}.$$

Again L'Hospital rule

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_{t_0}^{t}e^{a_0\tau}e^{\frac{1}{2}\int_{\tau}^{t}\left(-q(s)+\frac{5}{2}N\right)ds}d\tau}{e^{-(q_0-a_0)t}} &= 2\lim_{t\to\infty}\frac{e^{a_0t}}{\left(Re^{q_0t}-\frac{5}{2}N-2\left(q_0-a_0\right)\right)e^{-(q_0-a_0)t}}\\ &= 2\lim_{t\to\infty}\frac{1}{e^{-q_0t}\left(Re^{q_0t}-\frac{5}{2}N-2b\right)}=\frac{2}{q}.\end{aligned}$$

This shows that there exists a positive constant A_1 such that

$$\frac{3C_d}{\sqrt{2}}\int_{t_0}^{t}e^{a_0\tau}e^{\frac{1}{2}\int_{\tau}^{t}\left(-\frac{q(s)}{\beta}+\frac{5}{2}N\right)ds}d\tau\leq A_1e^{-(q_0-a_0)t},\ t>0. \tag{2.24}$$

Assumption **(H1)** implies for t sufficiently large that

$$e^{\frac{1}{2}\int_{t_0}^{t}\left(-\frac{q(s)}{\beta}+\frac{5}{2}N\right)ds}\leq e^{-(q_0-a_0)t},\ t>0. \tag{2.25}$$

Now, the identities (2.24) and (2.25) together lead to

$$\sqrt{\mathcal{L}(t)}\leq Ae^{-(q_0-a_0)t},\ t>0.$$

So,

$$\sqrt{e_y^2(t)+e_d^2(t)}\leq A\sqrt{2}e^{-(q_0-a_0)t},\ t>0.$$

$K=A\sqrt{2}$. □

We take the feedback control law $\vartheta(t)=-\hat{d}(t)$. Then, the new closed-loop system is written as

$$\begin{cases}\varphi_{tt}+2v\varphi_{xt}-\left(1-v^2\right)\varphi_{xx}=0,\ \ x\in(0,1)\,,\ t>0,\\ \varphi(0,t)=0,\quad t\geq 0,\\ \varphi_x(1,t)=-r(t)\left(\varphi_t+v\varphi_x\right)(1,t)+e_d,\ t\geq 0,\\ \dot{e}_y(t)=-q(t)\left(e_y(t)+e_d(t)\right)+\frac{\dot{q}(t)}{q(t)}e_y(t),\\ \dot{e}_d(t)=q(t)e_y(t)+\dot{d}(t),\ t\geq 0,\\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \ x\in(0,1)\,,\\ e_y(0)=e_{y,0},\ e_d(0)=e_{d,0}.\\ r'(t)=p\left(\varphi_t+v\varphi_x\right)^2(1,t),\ p>0,\ r(0)=r_0>0.\end{cases} \tag{2.26}$$

2.4 Exponential Stability

We define the classical energy of problem by

$$E(t) = E_{\varphi}(t) + E_{y,d}(t), \ t \geq 0,$$

where $E_{\varphi}(t)$ *and* $E_{y,d}(t)$ *are given by*

$$E_{\varphi}(t) = \frac{1}{2}\left\|\varphi_t + v\varphi_x\right\|^2 + \frac{1}{2}\left\|\varphi_x\right\|^2, \ t \geq 0$$

and

$$E_{y,d}(t) = e_y^2(t) + e_d^2(t), \ t \geq 0.$$

Theorem 6. *Under assumptions **(H1)** and **(H2)**, the solution of (2.26) satisfies* $(\varphi_t + v\varphi_x)(1) \in L^2(0,\infty)$ *such that*

$$E_{\varphi}(t) \leq Ne^{-\delta t}, \ t \geq 0,$$

where N *and* δ*, are positive constants*

Proof. The differentiation of E gives

$$\frac{d}{dt}E(t) = \int_0^1 (\varphi_t + v\varphi_x)\left(\varphi_{tt} + 2v\varphi_{xt} + v^2\varphi_{xx}\right)dx + \int_0^1 \varphi_x\left(\varphi_{xt} + v\varphi_{xx}\right)dx, \ t \geq 0.$$

That is

$$\frac{d}{dt}E(t) = \left[(\varphi_t + v\varphi_x)\varphi_x\right]_0^1, \ t \geq 0. \tag{2.27}$$

Taking into account the boundary conditions, the following formula is obtained

$$\frac{d}{dt}E(t) = -r(t)\left(\varphi_t + v\varphi_x\right)^2(1) - v\varphi_x^2(0) + \left(\varphi_t + v\varphi_x\right)(1)\, e_d(t), \ t \geq 0.$$

If $q(t) = Re^{q_0 t}$ with $q > C_d$ and $q_0 > a_0$, then there exist a positive constant K such that

$$\frac{d}{dt}E(t) \leq -r(t)\left(\varphi_t + v\varphi_x\right)^2(1) - v\varphi_x^2(0) + K\left|(\varphi_t + v\varphi_x)(1)\right| e^{-(q_0 - a_0)t}, \ t \geq 0. \tag{2.28}$$

$$\frac{d}{dt}E(t) \leq \frac{1}{2}\left(r_0 - 2k(t)\right)\left(\varphi_t + v\varphi_x\right)^2(1) - v\varphi_x^2(0,t) + \frac{2B^2}{r_0}e^{-2(q_0 - a_0)t}, \ t \geq 0.$$

In order to deal with the stability of the main problem , we introduce this new functional

$$L(t) = E(t) + \frac{1}{2p}r^2(t), \ t \geq 0,$$

which gives after differentiation

$$\begin{aligned}\frac{d}{dt}L(t) &= \frac{d}{dt}E(t)+\frac{1}{2}r(t)\left(\varphi_t+v\varphi_x\right)^2(1)\\ &\le \frac{1}{2}\left(r_0-r(t)\right)\left(\varphi_t+v\varphi_x\right)^2(1)-v\varphi_x^2(0)+\frac{2B^2}{r_0}e^{-2(q_0-a_0)t},\ t\ge 0.\end{aligned}$$

Since r is increasing, we obtain

$$\frac{d}{dt}L(t)\le \frac{2B^2}{r_0}e^{-2(q_0-a_0)t},\ t\ge 0.$$

By integrating, we arrive at

$$L(t)\le E(0)+\frac{K^2}{(q_0-a_0)\,r_0}=L_1,\ t\ge 0.$$

This implies that

$$\sup_{t\ge 0}\left[E(t)+\frac{1}{2p}r^2(t)\right]\le L_1,\ t\ge 0. \tag{2.29}$$

It results that

$$r(t)<\sqrt{2pL_1},\ t\ge 0. \tag{2.30}$$

Since $r'(t)=pw^2(t)$, $q>0$, we define the functional

$$\mathcal{V}(t)=E(t)+\epsilon\Phi(t),\ t\ge 0$$

where

$$\Phi(t)=\int_0^1 x\varphi_x\left(\varphi_t+v\varphi_x\right)dx,\ t\ge 0.$$

This functional satisfies

$$\begin{aligned}|\Phi(t)| &\le \int_0^1\left|\varphi_x\left(\varphi_t+v\varphi_x\right)\right|dx\\ &\le \frac{1}{2}\left(\|\varphi_t+v\varphi_x\|^2+\|\varphi_x\|^2\right)\le E(t),\ t\ge 0.\end{aligned}$$

Then, the following relation holds

$$\beta_1E(t)\le\mathcal{V}(t)\le\beta_2E(t),\ t\ge 0 \tag{2.31}$$

where $\beta_1=1-\epsilon$ and $\beta_2=1+\epsilon$ with $\epsilon<1$. We have

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= \int_0^1 x\left(\varphi_{xt}+v\varphi_{xx}\right)\left(\varphi_t+v\varphi_x\right)dx+v\int_0^1\varphi_x\left(\varphi_t+v\varphi_x\right)dx\\ &\quad+\int_0^1 x\varphi_x\left(\varphi_{tt}+2v\varphi_{xt}+v^2\varphi_{xx}\right)dx,\ t\ge 0,\end{aligned} \tag{2.32}$$

$$\frac{d}{dt}\Phi(t) = \int_0^1 x\left(\varphi_{xt}+v\varphi_{xx}\right)\left(\varphi_t+v\varphi_x\right)dx + v\int_0^1 \varphi_x\left(\varphi_t+v\varphi_x\right)dx + \int_0^1 x\varphi_x\varphi_{xx}dx,\ t\geq 0, \tag{2.33}$$

$$\int_0^1 x\left(\varphi_{xt}+v\varphi_{xx}\right)\left(\varphi_t+v\varphi_x\right)dx = \frac{1}{2}\left(\varphi_t+v\varphi_x\right)^2(1) - \frac{1}{2}\left\|\varphi_t+v\varphi_x\right\|^2,\ t\geq 0, \tag{2.34}$$

$$\int_0^1 \varphi_x\left(\varphi_t+v\varphi_x\right)dx \leq \frac{1}{2}\left\|\varphi_t+v\varphi_x\right\|^2 + \frac{1}{2}\left\|\varphi_x\right\|^2,\ t\geq 0, \tag{2.35}$$

$$\int_0^1 x\varphi_x\varphi_{xx}dx = \frac{1}{2}\varphi_x^2(1,t) - \frac{1}{2}\left\|\varphi_x\right\|^2,\ t\geq 0. \tag{2.36}$$

Gathering the estimates (2.34)-(2.36) in (2.33), we obtain

$$\frac{d}{dt}\Phi(t) \leq -\frac{1}{2}(1-v)\left\|\varphi_t+v\varphi_x\right\|^2 - \frac{1}{2}(1-v)\left\|\varphi_x\right\|^2 + \frac{1}{2}\left(\varphi_t+v\varphi_x\right)^2(1) + \frac{1}{2}\varphi_x^2(1),\ t\geq 0.$$

We have

$$\varphi_x^2(1,t) = 2k^2(t)\left(\varphi_t+v\varphi_x\right)^2(1) + 2e_d^2 \leq 2k^2(t)\left(\varphi_t+v\varphi_x\right)^2(1) + 2B^2e^{-2(q_0-a_0)t},\ t\geq 0.$$

Then, we obtain

$$\frac{d}{dt}\Phi(t) \leq -\frac{1}{2}(1-v)\left\|\varphi_t+v\varphi_x\right\|^2 - \frac{1}{2}(1-v)\left\|\varphi_x\right\|^2 + \frac{1}{2}\left(1+2k^2(t)\right)\left(\varphi_t+v\varphi_x\right)^2(1) + K^2e^{-2(q_0-a_0)t},\ t\geq 0. \tag{2.37}$$

(2.28) and (2.37), it holds

$$\frac{d}{dt}\mathcal{V}(t) \leq -\frac{\epsilon}{2}(1-v)\left\|\varphi_t+v\varphi_x\right\|^2 - \frac{\epsilon}{2}(1-v)\left\|\varphi_x\right\|^2 - \left[r(t) - \frac{\epsilon}{2}\left(1+2k^2(t)\right)\right]\left(\varphi_t+v\varphi_x\right)^2(1) + \epsilon K^2e^{-2(q_0-a_0)t} + K\left|\left(\varphi_t+v\varphi_x\right)(1)\right|e^{-(q_0-a_0)t},\ t\geq 0. \tag{2.38}$$

Applying the Young inequality to (2.38), we obtain

$$\frac{d}{dt}\mathcal{V}(t) \leq -\epsilon(1-v)E(t) - \left[r(t) - \frac{\epsilon}{2}\left(1+2k^2(t)\right) - \eta\right]\left(\varphi_t+v\varphi_x\right)^2(1) + \left(\epsilon+\frac{1}{4\eta}\right)K^2e^{-2(q_0-a_0)t},\ t\geq 0, \tag{2.39}$$

for some $\eta > 0$. Since $r(t) < \sqrt{2pL_1}$ (see (2.30)) and $r(t) \geq r_0$, it follows that

$$r(t) - \frac{\epsilon}{2}\left(1+2k^2(t)\right) \geq r_0 - \frac{\epsilon}{2}\left(1+4pL_1\right),\ t\geq 0. \tag{2.40}$$

By virtue of (2.31), it results from (2.39) and (2.40) that

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) &\leq -\frac{\epsilon}{\beta_2}\mathcal{V}(t) - \left[r_0 - \frac{\epsilon}{2}\left(1+4pL_1\right) - \eta\right]\left(\varphi_t + v\varphi_x\right)^2(1) \\ &\quad + \left(\epsilon + \frac{1}{4\eta}\right)K^2 e^{-2(q_0-a_0)t}, \ t \geq 0. \end{aligned} \tag{2.41}$$

Choosing ϵ and η small enough, we obtain

$$\frac{d}{dt}\mathcal{V}(t) \leq -\frac{\epsilon}{\beta_2}\mathcal{V}(t) + \left(\epsilon + \frac{1}{4\eta}\right)K^2 e^{-2(q_0-a_0)t}, \ t \geq 0. \tag{2.42}$$

Integrating (2.42) over $(0,t), \ t \geq 0$, we entail that

$$\begin{aligned}\mathcal{V}(t) &\leq \mathcal{V}(0)e^{-\frac{\epsilon}{\beta_2}t} + \left(\epsilon + \frac{1}{4\eta}\right)K^2 e^{-\frac{\epsilon}{\beta_2}t}\int_0^t e^{\left[-2(q_0-a_0)+\frac{\epsilon}{\beta_2}\right]s}ds \\ &\leq \mathcal{V}(0)e^{-\frac{\epsilon}{\beta_2}t} + \left(\epsilon + \frac{1}{4\eta}\right)\frac{K^2}{\left[2\left(q_0-a_0\right)-\frac{\epsilon}{\beta_2}\right]}\left(e^{-\frac{\epsilon}{\beta_2}t} - e^{-2(q_0-a_0)t}\right). \end{aligned} \tag{2.43}$$

Choosing again ϵ sufficiently small so that $2\left(q_0-a_0\right) - \frac{\epsilon}{\beta_2} > 0$ and exploiting (2.31), we obtain

$$E_\varphi(t) \leq Ne^{-\delta t}, \ t \geq 0,$$

where $N = \frac{\mathcal{V}(0)}{\beta_1} + \left(\epsilon + \frac{1}{4\eta}\right)\frac{K^2}{\beta_1\left[2(q_0-a_0)-\frac{\epsilon}{\beta_2}\right]}$ and $\delta = \frac{\epsilon}{\beta_2}$. Then, the assertion of the theorem is established. □

Chapter 3

General decay for an axially moving viscoelastic beam

3.1 Introduction

We consider the following initial boundary value problem in this chapter, which is

$$\begin{cases} \varrho\left(\varphi_{tt} + 2\mathcal{V}\varphi_{xt} + \mathcal{V}^2\varphi_{xx}\right) + EI\varphi_{xxxx} - EI\displaystyle\int_0^t \mathcal{Z}(t-s)\varphi_{xxxx}(s)ds = 0, \\ \quad x \in (0,l),\ t \geq 0, \\ \varphi(0,t) = \varphi_x(0,t) = \varphi_{xx}(l,t) = 0, \quad t \geq 0, \\ \varrho\mathcal{V}^2\varphi_x(l,t) + EI\varphi_{xxx}(l,t) - EI\displaystyle\int_0^t \mathcal{Z}(t-s)\varphi_{xxx}(l,s)ds = y(\varphi(l,t)), \\ \ t \geq 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ x\ \in (0,l) \end{cases} \tag{3.1}$$

whereas $\varphi = \varphi(x,t)$ *is the beam transversal displacement.*
$\mathcal{V}$ *is the axial speed .*
EI *is the beam flexural rigidity.*
ϱ *is the beam mass per unit length.*
$\mathcal{Z}$ *(nonnegative functions) see [20].*
$\varphi_0(x)$, $\varphi_1(x)$ *are the initial data.*
y *represents the nonlinear term.*

3.2 Preliminaries

In this section, we give some material needed in the proof of our results.
For every measurable set $\mathcal{A} \subset \mathbb{R}_+$, *we define for all* $t \geq 0$

$$\hat{\mathcal{Z}}(\mathcal{A}) = \frac{1}{\mathcal{K}} \int_{\mathcal{A}} \mathcal{Z}(s)ds \tag{3.2}$$

and

$$\mathcal{A}_t = \mathcal{A} \cap [0, t].$$

The set of flatness and the flatness rate of $\mathcal{Z}$ *are defined by*

$$\mathcal{Y}_{\mathcal{Z}} = \{s \in \mathbb{R}_+ : \mathcal{Z}(s) > 0 \text{ and } \mathcal{Z}'(s) = 0\} \tag{3.3}$$

and

$$\mathcal{R}_{\mathcal{Z}} = \hat{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}}),$$

$$\tilde{y}_{\mathcal{Z}} = \{s \in \mathbb{R}_+ : \ \mathcal{Z}(t-s) > 0 \text{ and } \mathcal{Z}'(t-s) = 0\}.$$

The following hypotheses are taken into consideration:

(H1) $\mathcal{Z}(t) \geq 0$, $t \geq 0$, $0 < \mathcal{K} = \displaystyle\int_0^\infty \mathcal{Z}(s)ds < 1.$

(H2) $\mathcal{Z}'(t) \leq 0$ *for almost all* $t > 0$.

(H3) y *needs to satisfy the following conditions:*

$$y(0) = 0, \quad |y(x) - y(s)| \leq m(1 + |x|^\alpha + |s|^\alpha)\,|x - s|\,, \quad \forall x, s \in \mathbb{R},\ \alpha \in \mathbb{R}_+,$$

$$0 \leq Y(x) \leq xy(x), \quad \forall x \in \mathbb{R},$$

and $Y(z) = \displaystyle\int_0^z y(s)ds.$

(H4) $\int_t^\infty \mathcal{Z}(s)ds \leq \sigma(t)$, $t \geq 0$ *where*

$\sigma : [0, \infty) \to \mathbb{R}^+$ *is absolutely continuous and satisfies the condition*

$$\sigma'(t) + \zeta(t)\,\chi(\sigma(t)) \leq 0, \ a, e, t > 0, \tag{3.4}$$

where $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ *in the neighborhood of* ∞ *is decreasing, continuous and non-summable, while* $\chi : [0, \infty) \to [0, \infty)$ *is a continuous function and for an arbitrary, fixed* $r > 0$, *let*

$$B(x) = \int_x^r \frac{ds}{\chi(s)}.$$

Lemma 8. *Let $\bar{t} > 0$ be given and $\varphi : [\bar{t}, \infty) \to \mathbb{R}^{+}$ be an absolutely continuous function satisfying*

$$\varphi'(t) + a\zeta(t)\chi(\varphi(t)) \leq 0, \ a.e.t > \bar{t}, a > 0.$$

Then, there exists $\hat{t} \geq \bar{t}$, depending on $\varphi(\bar{t})$, such that

$$\varphi(t) \leq B^{-1}\left(a\int_{\hat{t}}^{t}\zeta(s)\,ds\right), \ \forall t \geq \hat{t}.$$

Lemma 9. *(see [32])*
i) Let $y : (0,t) \to [a,b]$ be an integrable function on $(0,t)$ and $\mathcal{Z}(s) \geq 0$ with $\int_0^t \mathcal{Z}(s)ds = 1$, then Jensen's inequality states that

$$\chi\left(\int_0^t y(s)\mathcal{Z}(s)ds\right) \leq \int_0^t \chi(y(s))\,\mathcal{Z}(s)ds.$$

ii) For real numbers $x_1, x_2, ..., x_n$ in the domain of χ, and positive real numbers $a_1, a_2, ..., a_n$, Jensen's inequality is written as

$$\frac{\sum_{i=1}^{n} a_i\chi(x_i)}{\sum_{i=1}^{n} a_i} \geq \chi\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right).$$

We define the modified energy of problem (3.1) by

$$\begin{aligned} E(t) &= \frac{\varrho}{2}\|\varphi_t\|^2 - \frac{\varrho\mathcal{V}^2}{2}\|\varphi_x\|^2 + \frac{EI}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\|\varphi_{xx}\|^2 \\ &\quad + \frac{EI}{2}\int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx + Y(\varphi(l)), \ t \geq 0 \end{aligned} \tag{3.5}$$

where $\|.\|$ is the L^2-norm and

$$(\mathcal{Z}\circ\varphi)(t) = \int_0^t \mathcal{Z}(t-s)\,|\varphi(t) - \varphi(s)|^2\,ds, \ t \geq 0.$$

Proposition 4. *Under the hypotheses (H1)-(H3) and $\mathcal{V}^2 < EI(1-\mathcal{K})/\varrho l^2$, we have*

$$E(t) \geq 0, \ t \geq 0.$$

Proof. Using Poincaré's inequality, we have

$$\begin{aligned} E(t) &\geq \frac{\varrho}{2}\left\|\varphi_t\right\|^2+\frac{1}{2}\left[EI-\varrho\mathcal{V}^2 l^2-EI\int_0^t \mathcal{Z}(s)ds\right]\left\|\varphi_{xx}\right\|^2 \\ &+\frac{EI}{2}\int_0^l\left(\mathcal{Z}\circ\varphi_{xx}\right)(t)dx+Y(\varphi(l)),\ t\geq 0. \end{aligned} \tag{3.6}$$

Since $\mathcal{V}^2 < EI(1-\mathcal{K})/\varrho l^2$, we end the proof of the Proposition 4. □

Lemma 10. *(See [42]) We have for $\mathcal{Z}\in\mathcal{C}(0,\infty)$ and $\varphi\in\mathcal{C}\left((0,\infty);L^2(0,l)\right)$*

$$\begin{aligned} &\int_0^l \varphi\int_0^t \mathcal{Z}(t-s)\varphi(s)dsdx \\ =\ &\frac{1}{2}\left(\int_0^t\mathcal{Z}(s)ds\right)\|\varphi\|^2+\frac{1}{2}\int_0^t\mathcal{Z}(t-s)\int_0^l\varphi^2(s)dxds-\frac{1}{2}\int_0^l\left(\mathcal{Z}\circ\varphi\right)dx,\ t\geq 0. \end{aligned}$$

3.3 Well posedness

We define the following spaces

$$V=\left\{u\in H^2(0,1),u(0)=u_x(0)=0\right\}$$

and

$$W=\left\{u\in V\cap H^4(0,1),u_{xx}(1)=0\right\}.$$

Theorem 7. *Let $(\varphi_0,\varphi_1)\in V\times L^2(0,1)$. Under the hypotheses (H1)-(H4) and $\mathcal{V}^2<lEI/\varrho$, the problem (3.1) has a unique (weak) solution such that*

$$\varphi\in L^\infty(0,\infty;V),\varphi_t\in L^\infty\left(0,\infty;L^2(0,1)\right),\varphi_{tt}\in L^2\left(0,\infty;V^*\right)$$

where V^* is the dual of V.

Proof. Let us solve the variational problem associated with equation (3.1)

$$\begin{aligned} &\varrho\left(\varphi_{tt},w\right)+2\varrho\mathcal{V}\left(\varphi_{xt},w\right)-\varrho\mathcal{V}^2\left(\varphi_x,w_x\right)+EI\left(\varphi_{xx},w_{xx}\right) \\ &-EI\int_0^t\mathcal{Z}(t-s)\left(\varphi_{xx}(s),w_{xx}\right)ds+y(\varphi(1,t))w(1)=0,\ \text{for all } w\in\mathcal{V} \end{aligned}$$

Let $\left\{w^i\right\}_{i=1}^\infty$, we denote $W_m=\operatorname{span}\left\{w^1,w^2,\ldots,w^m\right\}$.

$$\varphi^m(x,t)=\sum_{i=1}^k c_m^i(t)w^i(x),x\in(0,1),t\geq 0$$

for any $w \in W_m$, we get

$$\begin{cases} \varrho\left(\varphi_{tt}^m, w\right) + 2\varrho\mathcal{V}\left(\varphi_{xt}^m, w\right) - \varrho\mathcal{V}^2\left(\varphi_x^m, w_x\right) + EI\left(\varphi_{xx}^m, w_{xx}\right) \\ -EI\int_0^t \mathcal{Z}(t-s)\left(\varphi_{xx}^m(s), w_{xx}\right)ds + y\left(\varphi^m(1,t)\right)w(1) = 0, \\ \varphi^m(x,0) = \varphi_{0,m}(x) \to \varphi_0 \text{ in } \mathcal{V}, \varphi_t^m(x,0) = \varphi_{1,m}(x) \to \varphi_1 \text{ in } L^2(0,1). \end{cases} \tag{3.7}$$

We recall

$$E^m(t) = \frac{\varrho}{2}\left\|\varphi_t^m\right\|^2 - \frac{\varrho\mathcal{V}^2}{2}\left\|\varphi_x^m\right\|^2 + \frac{EI}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\left\|\varphi_{xx}^m\right\|^2$$
$$+ \frac{EI}{2}\int_0^1\left(\mathcal{Z} \circ \varphi_{xx}^m\right)dx + y\left(\varphi^m(1,t)\right), t \in [0, t_m)$$

and define

$$X^m(t) = \frac{\varrho}{2}\left\|\varphi_t^m\right\|^2 + \frac{1}{2}\left[EIl - \varrho\mathcal{V}^2\right]\left\|\varphi_{xx}^m\right\|^2 + \frac{EI}{2}\int_0^1\left(\mathcal{Z} \circ \varphi_{xx}^m\right)dx$$
$$+ y\left(\varphi^m(1,t)\right), t \in [0, t_m).$$

As $\varphi^m \in \mathcal{V}$, we get $\left\|\varphi_x^m\right\|^2 \le \left\|\varphi_{xx}^m\right\|^2, t \in [0, t_m)$ we have $\mathcal{V}^2 < lEI/\varrho$, we have

$$0 < X^m(t) < E^m(t), t \in [0, t_m)$$

where φ^m is a solution of equation (3.7) we hve (see [21])

$$\frac{d}{dt}E^m(t) = \frac{d}{dt}\int_0^1 \widetilde{E^m}(x,t)dx + \frac{d}{dt}y\left(\varphi^m(1,t)\right)$$

$$= \int_0^1\left[\frac{\partial}{\partial t}\widetilde{E^m}(x,t) + \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\widetilde{E^m}(x,t)\right]dx + \frac{d}{dt}y\left(\varphi^m(1,t)\right)$$

$$= \int_0^1 \frac{\partial}{\partial t}\widetilde{E^m}(x,t)dx + \mathcal{V}\widetilde{E^m}(x,t)\Big|_0^1 + \frac{d}{dt}y\left(\varphi^m(1,t)\right), t \in [0, t_m),$$

where

$$\widetilde{E^m}(x,t) = \frac{\varrho}{2}\left[\varphi_t^m(x,t)\right]^2 - \frac{\varrho\mathcal{V}^2}{2}\left[\varphi_x^m(x,t)\right]^2 + \frac{EI}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\left[\varphi_{xx}^m(x,t)\right]^2$$
$$+ \frac{EI}{2}\left(\mathcal{Z} \circ \varphi_{xx}^m\right)(x,t), t \in [0, t_m),$$

we have

$$\widetilde{E^m}(x,t)\Big|_0^1 = \frac{\varrho}{2}\left[\varphi_t^m(1,t)\right]^2 - \frac{\varrho\mathcal{V}^2}{2}\left[\varphi_x^m(1,t)\right]^2 - \frac{EI}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\left[\varphi_{xx}^m(0,t)\right]^2$$
$$- \frac{EI}{2}\left(\mathcal{Z} \circ \varphi_{xx}^m\right)(0,t), t \in [0, t_m). \tag{3.8}$$

We have

$$\frac{\partial}{\partial t}\left(\mathcal{Z} \circ \varphi_{xx}^m\right)(t) = \left(\mathcal{Z}' \circ \varphi_{xx}^m\right)(t) - 2\int_0^t \mathcal{Z}(t-s)\left(\varphi_{xx}^m(s), \varphi_{xxt}^m\right)ds$$
$$+ 2\left(\int_0^t \mathcal{Z}(s)ds\right)\varphi_{xxt}^m(t)\varphi_{xx}^m(t), t \in [0, t_m). \tag{3.9}$$

We obtain

$$\begin{aligned}&\varrho\left(\varphi_{tt}^{m},\varphi_{t}^{m}\right)-\varrho\mathcal{V}^{2}\left(\varphi_{x}^{m},\varphi_{xt}^{m}\right)+EI\left(\varphi_{xx}^{m},\varphi_{xxt}^{m}\right)-EI\int_{0}^{t}\mathcal{Z}(t-s)\left(\varphi_{xx}^{m}(s),\varphi_{xxt}^{m}\right)ds\\&+y\left(\varphi^{m}(1,t)\right)\varphi_{t}^{m}(1,t)=-\varrho\mathcal{V}\left[\varphi_{t}^{m}(1,t)\right]^{2},t\in[0,t_{m})\end{aligned}\tag{3.10}$$

We deduce that

$$\begin{aligned}\int_{0}^{1}\frac{\partial}{\partial t}\widetilde{E}^{m}(x,t)dx+\frac{d}{dt}y\left(\varphi^{m}(1,t)\right)=-\,\varrho\mathcal{V}\left[\varphi_{t}^{m}(1,t)\right]^{2}-\frac{EI}{2}\mathcal{Z}(t)\left\|\varphi_{xx}^{m}\right\|^{2}\\+\frac{EI}{2}\left(\mathcal{Z}'\circ\varphi_{xx}^{m}\right)(t),t\in[0,t_{m})\end{aligned}$$

Therefore, we get

$$\begin{aligned}\frac{d}{dt}E^{m}(t)=-\frac{EI}{2}\mathcal{Z}(t)\left\|\varphi_{xx}^{m}\right\|^{2}+\frac{EI}{2}\int_{0}^{1}\left(\mathcal{Z}'\circ\varphi_{xx}^{m}\right)dx-\frac{\varrho\mathcal{V}}{2}\left|\varphi_{t}^{m}(1,t)\right|^{2}\\-\frac{\varrho\mathcal{V}^{3}}{2}\left|\varphi_{x}^{m}(1,t)\right|^{2}-\frac{EI\mathcal{V}}{2}\left(1-\int_{0}^{t}\mathcal{Z}(s)ds\right)\left|\varphi_{xx}^{m}(0,t)\right|^{2}-\frac{EI\mathcal{V}}{2}\left(\mathcal{Z}\circ\varphi_{xx}^{m}\right)(0,t),\end{aligned}$$

for all $t\in[0,t_m)$.

$$X^{m}(t)<E^{m}(t)\leq E^{m}(0)\leq K_{1},\tag{3.11}$$

where $K_1=(\|\varphi_1\|,\|\varphi_{0xx}\|,\|\varphi_{0x}\|)$. The estimate equation (3.11) implies

$$\begin{cases}\varphi^{m}\text{ is uniformly bounded in }L^{\infty}(0,T;V),\\\varphi_{t}^{m}\text{ is uniformly bounded in }L^{\infty}\left(0,T;L^{2}(0,1)\right).\end{cases}\tag{3.12}$$

□

3.4 Passage to the limit

Consequently

$$\begin{cases}\varphi^{\mu}\rightharpoonup\varphi\ \text{ in } L^{\infty}(0,T;V)\ \text{ weakly star},\\\varphi_{t}^{\mu}\rightharpoonup\varphi_{t}\ \text{ in } L^{\infty}\left(0,T;L^{2}(0,1)\right)\ \text{ weakly star}.\end{cases}$$

We easily deduce from equation (3.12) that φ^m and φ_t^m are bounded in $L^2(0,T;V)$ and $L^2\left(0,T;L^2(0,1)\right)$, respectively.

Since the embedding $\mathcal{Z}^1(0,T;V)\hookrightarrow L^2\left(0,T;L^2(0,1)\right)$ is compact, we deduce that $\varphi^\mu\to\varphi$ strongly in $L^2\left(0,T;L^2(0,1)\right)$. Therefore, $\varphi^\mu\to\varphi$ strongly and a.e. on $(0,T)\times\,(0,1)$. The assumption (**H**1) *and Lemma 5 lead to*

$$\begin{aligned}|y(\varphi^m(1,t))| &\leq k\left(|\varphi^m(1,t)| + |\varphi^m(1,t)|^{(\alpha+1)}\right)\\ &\leq k\left(\|\varphi^m_{xx}\| + \|\varphi^m_{xx}\|^{(\alpha+1)}\right), \quad t\in[0,T).\end{aligned}$$

The previous relation implies (see [21])

$$y(\varphi^m(1,t)) \rightharpoonup y(\varphi(1,t)) \text{ in } L^\infty(0,T) \text{ weakly star.}$$

3.5 Uniqueness

Let u and z be two solutions of problem (3.1) and set $\varphi = u - z$. Then, φ satisfies

$$\begin{cases} \varrho(\varphi_{tt}, w) + 2\varrho\mathcal{V}(\varphi_{xt}, w) - \varrho\mathcal{V}^2(\varphi_x, w_x) + EI(\varphi_{xx}, w_{xx}) \\ -EI\int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(s), w_{xx})\,ds + [y(u(1,t)) - y(z(1,t))]w(1) = 0, \\ \varphi(x,0) = 0, \quad \varphi_t(x,0) = 0. \end{cases}$$

We define

$$\varphi(t) = \frac{\varrho}{2}\|\varphi_t\|^2 - \frac{\varrho\mathcal{V}^2}{2}\|\varphi_x\|^2 + \frac{EI}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\|\varphi_{xx}\|^2 + \frac{EI}{2}\int_0^1 (\mathcal{Z}\circ\varphi_{xx})\,dx,\ t\in[0,T),$$

we have

$$\begin{aligned}\frac{d}{dt}\varphi(t) = &-\frac{EI}{2}\mathcal{Z}(t)\|\varphi_{xx}\|^2 + \frac{EI}{2}\int_0^1 (\mathcal{Z}'\circ\varphi_{xx})\,dx - \frac{\varrho\mathcal{V}}{2}\varphi_t^2(1,t) - \frac{\varrho\mathcal{V}^3}{2}\varphi_x^2(1,t)\\ &-\frac{EI\mathcal{V}}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\varphi_{xx}^2(0,t) - \frac{EI\mathcal{V}}{2}(\mathcal{Z}\circ\varphi_{xx})(0,t)\\ &+ [y(\varphi_2(1,t)) - y(\varphi_1(1,t))]\varphi_t(1,t), t\in[0,T).\end{aligned} \tag{3.13}$$

Next, we have

$$\begin{aligned}[y(z(1,t)) - y(u(1,t))]\varphi_t(1) &\leq k(1 + |z(1,t)|^\alpha + |u(1,t)|^\alpha)|\varphi(1,t)||\varphi_t(1,t)|\\ &\leq k(1 + \|z_{xx}\|^\alpha + \|u_{xx}\|^\alpha)\|\varphi_{xx}\||\varphi_t(1,t)|\\ &\leq K_3\|\varphi_{xx}\||\varphi_t(1,t)| \leq \mu K_3\varphi_t^2(1,t) + \frac{K_3}{4\mu}\|\varphi_{xx}\|^2.\end{aligned} \tag{3.14}$$

We arrive at

$$\varphi(t) \leq \left(\mu K_3 - \frac{\varrho\mathcal{V}}{2}\right)\int_0^t \varphi_t^2(1,s)ds + \frac{K_3}{4\mu}\int_0^t \|\varphi_{xx}(s)\|^2 ds, t\in[0,T),$$

Now, for sufficiently small μ, it is clear that

$$\varphi(t) \leq K_4 \int_0^t |\varphi(s)| ds, t \in [0, T).$$

for some positive constant $K_4 > 0$. Using Gronwall's Lemma, we deduce that

$$\varphi(t) = 0, t \in [0, T),$$

which means that $u = z$.

3.6 Analysis of stability

In this section, we shall state and prove our main result.

Lemma 11. *(see [40]) If φ is a solution of equation (3.1), then the energy $E(t)$ satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) \leq \frac{EI}{2} \int_0^l \left(\mathcal{Z}' \circ \varphi_{xx}\right)(t) dx - \frac{\varrho \mathcal{V}}{2} \varphi_t^2(l) - \frac{\varrho \mathcal{V}^3}{2} \varphi_x^2(l) \\ -\frac{EI\mathcal{V}}{2} \left(1 - \int_0^t \mathcal{Z}(s) ds\right) \varphi_{xx}^2(0) - \frac{EI\mathcal{V}}{2} \left(\mathcal{Z} \circ \varphi_{xx}\right)(0), \ t \geq 0. \end{aligned} \quad (3.15)$$

Proof. We have

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{d}{dt} \int_0^l \widetilde{E}(x,t) dx + \frac{d}{dt} Y(\varphi(l)) = \int_0^l \frac{d}{dt} \widetilde{E}(x,t) dx + \frac{d}{dt} Y(\varphi(l)) \\ &= \int_0^l \left[\frac{\partial}{\partial t} \widetilde{E}(x,t) + \frac{\partial x}{\partial t} \frac{\partial}{\partial x} \widetilde{E}(x,t)\right] dx + \frac{d}{dt} Y(\varphi(l)) \\ &= \int_0^l \frac{\partial}{\partial t} \widetilde{E}(x,t) dx + \mathcal{V} \left. \widetilde{E}(x,t) \right|_0^l + \frac{d}{dt} Y(\varphi(l)), \end{aligned} \quad (3.16)$$

where

$$\begin{aligned} \widetilde{E}(x,t) &= \frac{\varrho}{2} \varphi_t^2(x,t) - \frac{\varrho \mathcal{V}^2}{2} \varphi_x^2(x,t) + \frac{EI}{2} \left(1 - \int_0^t \mathcal{Z}(s) ds\right) \varphi_{xx}^2(x,t) \\ &\quad + \frac{EI}{2} \left(\mathcal{Z} \circ \varphi_{xx}\right)(x,t), \ x \in [0,l], \ t \geq 0. \end{aligned}$$

Using (3.1) and (3.16) together gives

$$\frac{d}{dt}E(t) = -\int_0^l \varphi_t \left[2\rho \mathcal{V} \varphi_{xt} + \varrho \mathcal{V}^2 \varphi_{xx} + EI \left(\varphi_{xxxx} - \int_0^t \mathcal{Z}(t-s) \varphi_{xxxx}(s) ds\right)\right] dx$$

$$-\varrho\mathcal{V}^2\int_0^l \varphi_x\varphi_{xt}dx + EI\left(1-\int_0^t \mathcal{Z}(s)ds\right)\int_0^l \varphi_{xxt}\varphi_{xx}dx - \frac{EI}{2}\mathcal{Z}(t)\int_0^l \varphi_{xx}^2dx$$
$$+\frac{EI}{2}\int_0^l \frac{\partial}{\partial t}\left(\mathcal{Z}\circ\varphi_{xx}\right)(t)dx + \varphi_t(l)y(\varphi(l)) + \mathcal{V}\,\widetilde{E}(x,t)\Big|_0^l,\ t\geq 0.$$

Integrating by parts from 0 to l and boundary conditions in (3.1), we obtain

$$\begin{aligned}
&\frac{d}{dt}E(t) = -\varrho\mathcal{V}\varphi_t^2(l) - \varphi_t(l)\left[\varrho\mathcal{V}^2\varphi_x(l) + EI\varphi_{xxx}(l)\right.\\
&\left.-EI\int_0^t \mathcal{Z}(t-s)\varphi_{xxx}(l,s)ds\right] + EI\int_0^l \varphi_{xxt}\int_0^t \mathcal{Z}(t-s)\varphi_{xx}(s)dsdx\\
&-EI\left(\int_0^t \mathcal{Z}(s)ds\right)\int_0^l \varphi_{xxt}\varphi_{xx}dx - \frac{EI}{2}\mathcal{Z}(t)\int_0^1 \varphi_{xx}^2dx\\
&+\frac{EI}{2}\int_0^l \frac{\partial}{\partial t}\left(\mathcal{Z}\circ\varphi_{xx}\right)(t)dx + \varphi_t(l)y(\varphi(l)) + \mathcal{V}\,\widetilde{E}(x,t)\Big|_0^l,\ t\geq 0.
\end{aligned}$$

Using the definition of $\widetilde{E}(x,t)$ and the boundary conditions in (3.1) to find

$$\begin{aligned}
\widetilde{E}(x,t)\Big|_0^l &= \frac{\varrho}{2}\varphi_t^2(l) - \frac{\varrho\mathcal{V}^2}{2}\varphi_x^2(l) - \frac{EI}{2}\left(1-\int_0^t \mathcal{Z}(s)ds\right)\varphi_{xx}^2(0)\\
&\quad -\frac{EI}{2}\left(\mathcal{Z}\circ\varphi_{xx}\right)(0),\ t\geq 0.
\end{aligned} \tag{3.17}$$

Clearly

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\mathcal{Z}\circ\varphi_{xx}\right)(x,t) &= \left(\mathcal{Z}'\circ\varphi_{xx}\right)(x,t) - 2\varphi(x,t)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}(x,s)ds\\
&\quad +2\left(\int_0^t \mathcal{Z}(s)ds\right)\varphi_{xxt}(x,t)\varphi_{xx}(x,t),\ t\geq 0.
\end{aligned} \tag{3.18}$$

Consequently, by using (3.17), (3.18) and the boundary conditions in (3.1). The proof is completed.

We define the functionals in order to prepare the next section

$$\Theta_1(t) = \varrho\int_0^l \varphi_t dx + \frac{\varrho\mathcal{V}}{2}\varphi^2(l), t\geq 0,$$
$$\Theta_2(t) = -\varrho\int_0^l \varphi_t\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))dsdx, t\geq 0,$$
$$\Theta_3(t) = \int_0^t\left(\int_t^t \mathcal{Z}(\tau-s)d\tau\right)\left\|\varphi_{xx}(s)\right\|^2 ds, t\geq 0,$$
$$\Theta_4(t) = \int_0^t\left(\int_t^\infty \mathcal{Z}(\tau-s)d\tau\right)\varphi_{xx}^2(l,s)ds, t\geq 0,$$

and

$$L(t) = \mathcal{M}E(t) + \eta\Theta_1(t) + \Theta_2(t) + \mu\Theta_3(t) + \varsigma\Theta_4(t),\ t \geq 0$$

where η, μ, $\mathcal{M}$ and ς are positive constants. □

Proposition 5. *Let the hypotheses (H1)- (H4) be satisfied and $\mathcal{V}^2 < EI(1-\mathcal{K})/l^2\varrho$, then we have*

$$L(t) \leq E(t) \leq \mathcal{C}(L(t) + \Theta_3(t) + \Theta_4(t)),\ t \geq 0, \tag{3.19}$$

where $\mathcal{M}$, η and μ and ς are positive constants to be specified later and

$$L'(t) + \alpha_1 E(t) \leq 0,\ t \geq 0 \tag{3.20}$$

for some $\alpha_1 > 0$.

Proof. Using Young and Poincaré inequalities, we obtain

$$\begin{aligned}\Theta_1(t) &\leq \frac{\varrho}{2}\|\varphi_t\|^2 + \frac{\varrho}{2}\|\varphi\|^2 + \frac{\varrho\mathcal{V}}{2}\varphi^2(l)\\ &\leq \frac{\varrho}{2}\|\varphi_t\|^2 + \frac{l\rho}{2}(l+\mathcal{V})\|\varphi_x\|^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Theta_2(t) &\leq \frac{\varrho}{2}\|\varphi_t\|^2 + \frac{\varrho}{2}\int_0^l\left(\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))ds\right)^2 dx\\ &\leq \frac{\varrho}{2}\|\varphi_t\|^2 + \frac{\varrho k l^4}{2}\int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx,\end{aligned}$$

we find

$$\begin{aligned}L(t) \leq\ & \mathcal{M}E(t) + \frac{\varrho}{2}(\eta+1)\|\varphi_t\|^2 + \frac{\eta l}{2}\varrho(l+\mathcal{V})\|\varphi_x\|^2\\ &+\frac{\varrho k l^4}{2}\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx + \mu\Theta_3(t) + \eta_1\Theta_4(t),\ t \geq 0.\end{aligned}$$

This implies that $E(t) \leq \mathcal{C}\left(L(t) + \Theta_3(t) + \Theta_4(t)\right),\ t \geq 0$. For some $\mathcal{C} \geq 0$.

On the other side

$$\begin{aligned} L(t)-E(t) &= (\mathcal{M}-1)E(t)+\eta\Theta_1(t)+\Theta_2(t)+\mu\Theta_3(t)+\varsigma\Theta_4(t)\\ &\geq \frac{\varrho}{2}(\mathcal{M}-\eta-1)\|\varphi_t\|^2+(\mathcal{M}-1)Y(\varphi(l)\\ &+\frac{1}{2}\left\{\left[((\mathcal{M}-1)\left(\frac{EI}{l^2}(1-\mathcal{K})\right)-\varrho\mathcal{V}^2)\right]-\eta l\rho(l+\mathcal{V})\right\}\|\varphi_x\|^2\\ &+\left[(\mathcal{M}-1)\frac{EI}{2}-\frac{\varrho k l^4}{2}\right]\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx+\mu\Theta_3(t)+\varsigma\Theta_4(t),\ t\geq 0. \end{aligned}$$

Provided that $\mathcal{V}^2<EI(1-\mathcal{K})/\varrho l^2$ and choosing η small such that $\eta\leq\min\{\mathcal{M}-1, \left[(\mathcal{M}-1)EI(1-\mathcal{K})-l^2\varrho\mathcal{V}^2\right]/l^3\varrho(1+\mathcal{V})\}$ and $\mathcal{M}\geq\max\{1, \frac{\eta l^3 p(l+\mathcal{V})}{EI(1-\mathcal{K})-l^2\varrho^2}+1, \frac{\varrho k l^4}{EI}+1, \eta+1\}$.

We will now show the second assertion.

By taking the total derivative of $\Theta_1(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}\Theta_1(t) &= \int_0^l\frac{d}{dt}\widetilde{\Theta_1}(x,t)dx=\int_0^l\left(\frac{\partial}{\partial t}\widetilde{\Theta_1}(x,t)\right)dx+\mathcal{V}\widetilde{\Theta_1}(x,t)\Big|_0^l\\ &= \varrho\|\varphi_t\|^2+\varrho\int_0^l\varphi\varphi_{tt}dx-\varrho\mathcal{V}\varphi_t(l)\varphi(l)+\mathcal{V}\widetilde{\Theta_1}(x,t)\Big|_0^l,\ t\geq 0 \end{aligned} \tag{3.21}$$

where

$$\widetilde{\Theta_1}(x,t)=\varrho\varphi(x,t)\varphi_t(x,t),\ t\geq 0.$$

From (3.1), we see that

$$\widetilde{\Theta_1}(x,t)\Big|_0^l=\varrho\varphi_t(l)\varphi(l),\ t\geq 0. \tag{3.22}$$

When we integrate by parts and taking into account the boundary conditions in (3.1), we get

$$\begin{aligned} &\frac{d}{dt}\Theta_1(t)\leq\varrho\|\varphi_t\|^2+2\rho\mathcal{V}\int_0^l\varphi_x\varphi_t dx+\varrho\mathcal{V}^2\|\varphi_x\|^2-EI\|\varphi_{xx}\|^2\\ &+EI\int_0^l\varphi_{xx}\int_0^t\mathcal{Z}(t-s)\varphi_{xx}(s)dsdx-\varphi(l)y(\varphi(l)),\ t\geq 0. \end{aligned} \tag{3.23}$$

The second term in the right hand side of (3.23) is now estimated as follows

$$2\rho\mathcal{V}\int_0^l\varphi_x\varphi_t dx\leq\varrho\|\varphi_t\|^2+\varrho\mathcal{V}^2\|\varphi_x\|^2,\ t\geq 0. \tag{3.24}$$

Collecting the estimates (3.24) into (3.23) and the use of Lemma 10, we find

$$\frac{d}{dt}\Theta_1(t) \leq 2\rho \|\varphi_t\|^2 + 2\rho\mathcal{V}^2 \|\varphi_x\|^2 - EI\left(1-\frac{\mathcal{K}}{2}\right)\|\varphi_{xx}\|^2$$
$$+\frac{EI}{2}\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds - \frac{EI}{2}\int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx - \varphi(l)y(\varphi(l)). \tag{3.25}$$

For $\Theta_2(t)$, we have

$$\frac{d}{dt}\Theta_2(t) = \int_0^l \frac{d}{dt}\widetilde{\Theta_2}(x,t)dx = \int_0^l \left(\frac{\partial}{\partial t}\widetilde{\Theta_2}(x,t)\right)dx + \mathcal{V}\widetilde{\Theta_2}(x,t)\Big|_0^l$$
$$= -\varrho\int_0^l \varphi_{tt}\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))dsdx - \varrho\left(\int_0^t \mathcal{Z}(s)ds\right)\|\varphi_t\|^2$$
$$-\varrho\int_0^l \varphi_t\int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))dsdx + \mathcal{V}\widetilde{\Theta_2}(x,t)\Big|_0^l, \ t\geq 0, \tag{3.26}$$

where

$$\widetilde{\Theta_2}(x,t) = -\varrho\varphi_t(x,t)\int_0^t \mathcal{Z}(t-s)(\varphi(x,t)-\varphi(x,s))ds, \ t\geq 0. \tag{3.27}$$

Using our boundary conditions in (3.1), yields

$$\widetilde{\Theta_2}(x,t)\Big|_0^l = -\varrho\varphi_t(l)\int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds, \ t\geq 0. \tag{3.28}$$

By taking the total derivative of $\Theta_2(t)$, using integration by parts and (3.28), we obtain

$$\begin{aligned}
\frac{d}{dt}\Theta_2(t) &= EI\left(1-\int_0^t \mathcal{Z}(s)ds\right)\int_0^l \varphi_{xx}\int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))ds \\
&+EI\int_0^l \left|\int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))ds\right|^2 dx \\
&-\varrho\mathcal{V}^2\int_0^l \varphi_x\int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
&-2\rho\mathcal{V}\int_0^l \varphi_t\int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
&-\varrho\int_0^l \varphi_t\int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))dsdx - \varrho\left(\int_0^t \mathcal{Z}(s)ds\right)\|\varphi_t\|^2 \\
&+y(\varphi(l))\int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds \\
&+\varrho\mathcal{V}\varphi_t(l)\int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds, \ t\geq 0.
\end{aligned} \tag{3.29}$$

We estimate the terms on the right-hand side of expression (3.29), we start by the first term. For all measurable sets $\mathcal{A}$ and $\mathcal{Y}$ such that $\mathcal{A} = \mathrm{R}^+ \backslash \mathcal{Y}$, we obtain

$$\begin{aligned}
& \int_0^l \varphi_{xx} \int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t) - \varphi_{xx}(s))dsdx \\
= & \int_0^l \varphi_{xx} \left(\int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t) - \varphi_{xx}(s))ds + \int_{\mathcal{Y}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t) - \varphi_{xx}(s))ds \right) dx \\
\leq & \int_0^l \varphi_{xx} \int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t) - \varphi_{xx}(s))dsdx + \left(\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)ds \right) \|\varphi_{xx}\|^2 \\
& - \int_0^l \varphi_{xx} \int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\varphi_{xx}(s)dsdx, \ t \geq 0.
\end{aligned} \tag{3.30}$$

We easily see that

$$\begin{aligned}
& \int_0^l \varphi_{xx} \int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t) - \varphi_{xx}(s))dsdx \\
\leq & \ \xi_1 \|\varphi_{xx}\|^2 + \frac{\mathcal{K}}{4\xi_1} \int_0^l \int_{\mathcal{A}_t} \mathcal{Z}(t-s) |\varphi_{xx}(t) - \varphi_{xx}(s)|^2 dsdx, \ \xi_1 > 0,
\end{aligned}$$

and

$$\begin{aligned}
& - \int_0^l \varphi_{xx} \int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\varphi_{xx}(s)dsdx \\
\leq & \ \frac{1}{2} \|\varphi_{xx}\|^2 + \frac{\mathcal{K}}{2} \int_0^t \mathcal{Z}(t-s) \|\varphi_{xx}(s)\|^2 ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_0^l \varphi_{xx} \int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t) - \varphi_{xx}(s))dsdx \\
\leq & \left((\frac{1}{2} + \xi_1) + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) \right) \|\varphi_{xx}\|^2 + \frac{\mathcal{K}}{4\xi_1} \int_0^l \int_{\mathcal{A}_t} \mathcal{Z}(t-s) |\varphi_{xx}(t) - \varphi_{xx}(s)|^2 dsdx \\
& + \frac{\mathcal{K}}{2} \int_0^t \mathcal{Z}(t-s) \|\varphi_{xx}(s)\|^2 ds, \ t \geq 0.
\end{aligned} \tag{3.31}$$

It is obvious that by the definition of $\hat{\mathcal{Z}}(\mathcal{Y})$ in (3.2), we have

$$\int_0^l \left| \int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t) - \varphi_{xx}(s))ds \right|^2 dx$$

$$\begin{aligned}
&\leq \left(1+\frac{1}{\xi_2}\right)\mathcal{K}\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx \\
&\quad+(1+\xi_2)\,\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\int_0^l\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx,\ \xi_2>0. \qquad (3.32)
\end{aligned}$$

For the 3^{rd} term, can be estimated as follows

$$\begin{aligned}
&-\int_0^l\varphi_x\int_0^t\mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
&= -\left(\int_0^t\mathcal{Z}(s)ds\right)\|\varphi_x\|^2+\int_0^l\varphi_x\int_0^t\mathcal{Z}(t-s)\varphi_x(s)dsdx \\
&\leq \left(\frac{1}{2}-\mathcal{Z}_*\right)\|\varphi_x\|^2+\frac{l^2\mathcal{K}}{2}\int_0^t\mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds,\ t\geq t_*,\ \xi_3>0. \qquad (3.33)
\end{aligned}$$

The fourth term is estimated as follows

$$\begin{aligned}
&-2\rho\mathcal{V}\int_0^l\varphi_t\int_0^t\mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
&= -2\rho\mathcal{V}\int_0^l\varphi_t\left(\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))ds+\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))ds\right)dx \\
&= -2\rho\mathcal{V}\int_0^l\varphi_t\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx-2\rho\mathcal{V}\left(\int_{\dagger_t}\mathcal{Z}(t-s)ds\right)\int_0^l\varphi_t\varphi_x dx \\
&\quad+2\rho\mathcal{V}\int_0^l\varphi_t\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)\varphi_x(s)dsdx, \qquad (3.34)
\end{aligned}$$

or

$$-2\rho\mathcal{V}\int_0^l\varphi_t\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx$$

$$\leq \xi_3\varrho\|\varphi_t\|^2+\frac{\varrho\mathcal{V}^2l^2\mathcal{K}}{\xi_3}\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx,\ t\geq 0 \qquad (3.35)$$

and

$$-2\rho\mathcal{V}\left(\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)ds\right)\int_0^l\varphi_t\varphi_x dx\leq\varrho\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\|\varphi_t\|^2+\varrho\mathcal{V}^2\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\|\varphi_x\|^2, \qquad (3.36)$$

$$2\rho\mathcal{V}\int_0^l\varphi_t\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)\varphi_x(s)dsdx\leq\varrho\|\varphi_t\|^2+\varrho\mathcal{V}^2l^2\mathcal{K}\int_0^t\mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds, \qquad (3.37)$$

hence, we have

$$\begin{aligned}
&-2\rho\mathcal{V}\int_0^l \varphi_t \int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
\leq\ & \varrho\left(1+\xi_3+\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right)\|\varphi_t\|^2+\varrho\mathcal{V}^2\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\|\varphi_x\|^2+\varrho\mathcal{V}^2l^2\mathcal{K}\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds \\
&+\frac{\varrho\mathcal{V}^2l^2\mathcal{K}}{\xi_3}\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx. \qquad (3.38)
\end{aligned}$$

For the fifth term, we have

$$\begin{aligned}
&-\varrho\int_0^l \varphi_t\int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))dsdx \\
\leq\ & \xi_4\varrho\|\varphi_t\|^2-\frac{\varrho\mathcal{Z}(0)l^4}{4\xi_4}\int_0^l\left(\mathcal{Z}'\circ\varphi_{xx}\right)(t)dx,\ \xi_4>0,\ t\geq 0. \qquad (3.39)
\end{aligned}$$

For the 6^{th} term, we find

$$\begin{aligned}
&y(\varphi(l))\int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds \\
&=y(\varphi(l))\left(\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds+\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds\right) \\
&=y(\varphi(l))\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds-y(\varphi(l))\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)\varphi(l,s)ds \\
&+\left(\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)ds\right)\varphi(l)y(\varphi(l)), \qquad (3.40)
\end{aligned}$$

or

$$y(\varphi(l))\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds$$

$$\leq\ \xi_5\left|y(\varphi(l))\right|^2+\frac{l^3\mathcal{K}}{4\xi_5}\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx,\ t\geq 0$$

and

$$\begin{aligned}
&-y(\varphi(l))\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)\varphi(l,s)ds \\
\leq\ & \frac{\left|y(\varphi(l))\right|^2}{2}+\frac{l^3\mathcal{K}}{2}\int_0^t\mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds,\ \xi_4>0,\ t\geq 0.
\end{aligned}$$

By using assumption **(H3)** and Lemma 5 with the help of (3.6), we get

$$
\begin{aligned}
&|y(\varphi(l))|^2 \leq 2m^2(|\varphi(l)|^2 + |\varphi(l)|^{2(\alpha+1)}) \\
\leq\ & 2m^2 l^3 \left[\|\varphi_{xx}\|^2 + \left(\frac{2}{EIk - \varrho\mathcal{V}^2 l^2} E(0) \right)^{\alpha} \|\varphi_{xx}\|^2 \right] = \beta \|\varphi_{xx}\|^2 .
\end{aligned}
$$

Hence, (3.40) becomes

$$
\begin{aligned}
& y(\varphi(l)) \int_0^t \mathcal{Z}(t-s)(\varphi(l,t) - \varphi(l,s))ds \\
\leq\ & (\xi_5 + \frac{\beta}{2}) \|\varphi_{xx}\|^2 + \frac{l^3\mathcal{K}}{2} \int_0^t \mathcal{Z}(t-s) \|\varphi_{xx}(s)\|^2 ds \\
& + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\varphi(l)y(\varphi(l)) + \frac{l^3\mathcal{K}}{4\xi_5} \int_0^l \int_{\mathcal{A}_t} \mathcal{Z}(t-s) |\varphi_{xx}(t) - \varphi_{xx}(s)|^2 dsdx. \qquad (3.41)
\end{aligned}
$$

For the seventh term, it is easy to see that

$$
\begin{aligned}
& \varrho\mathcal{V}\varphi_t(l) \int_0^t \mathcal{Z}(t-s)(\varphi(l,t) - \varphi(l,s))ds \\
\leq\ & \frac{\varrho\mathcal{V}}{4\xi_7}\varphi_t^2(l) + \xi_7 l^3\mathcal{K} \int_0^l (\mathcal{Z} \circ \varphi_{xx})\, dx,\ \ \xi_7 > 0,\ \ t \geq 0. \qquad (3.42)
\end{aligned}
$$

Taking into account the estimates (3.31)-(3.33), (3.38), (3.39), (3.41) and (3.42), we can write

$$
\begin{aligned}
& \frac{d}{dt}\Theta_2(t) \leq -\frac{\mathcal{Z}(0)l^4}{4\xi_4} \int_0^l (\mathcal{Z}' \circ \varphi_{xx})(t)dx + \varrho \left(-\mathcal{Z}_* + \xi_4 + \frac{1}{2} + \xi_3 + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) \right) \|\varphi_t\|^2 \\
& + \varrho\mathcal{V}^2 \left(-\mathcal{Z}_* + \frac{1}{2} + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) \right) \|\varphi_x\|^2 + \{ (\frac{1}{2} + \xi_1)(1 - \mathcal{Z}_*) EI + (\frac{\beta}{2} + \xi_5 \frac{\beta}{2}) \\
& + EI(1 - \mathcal{Z}_*)\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) \} \|\varphi_{xx}\|^2 + \xi_7 l^3 \mathcal{K} \int_0^l (\mathcal{Z} \circ \varphi_{xx})(t)dx \\
& + \frac{1}{2}\left((1 - \mathcal{Z}_*) kEI + 5\rho\mathcal{V}^2 l^2 \mathcal{K} + l^3\mathcal{K} \right) \int_0^t \mathcal{Z}(t-s) \|\varphi_{xx}(s)\|^2 ds \\
& + (1 + \xi_2) EIk\hat{\mathcal{Z}}(\mathcal{Y}) \int_0^l \int_{\mathcal{Y}_t} \mathcal{Z}(t-s) |\varphi_{xx}(t) - \varphi_{xx}(s)|^2 dsdx + \mathcal{K} \Big\{ (1 - \mathcal{Z}_*) \frac{EI}{4\xi_1} \\
& + \left(1 + \frac{1}{\xi_2} \right) EI + \frac{\varrho\mathcal{V}^2 l^2}{\xi_3} + \frac{l^3}{4\xi_5} \Big\} \int_0^l \int_{\mathcal{A}_t} \mathcal{Z}(t-s) |\varphi_{xx}(t) - \varphi_{xx}(s)|^2 dsdx \\
& + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\varphi(l)y(\varphi(l)) + \frac{\varrho\mathcal{V}}{4\xi_7}\varphi_t^2(l),\ \ t \geq t_* > 0, \qquad (3.43)
\end{aligned}
$$

where $\beta = 2m^2 l^3 \left[1 + \left(\frac{2E(0)}{kEI - l^2\varrho\mathcal{V}^2} \right)^{\alpha} \right]$ for all $\xi_i > 0$, $i = 1, ..., 7$.

A differentiation of $\Theta_3(t)$ with respect to t along the solution of (3.1) yields

$$
\frac{d}{dt}\Theta_3(t) = \mathcal{K} \|\varphi_{xx}\|^2 - \int_0^t \mathcal{Z}(t-s) \|\varphi_{xx}(s)\|^2 ds,\ t \geq 0. \qquad (3.44)
$$

By taking the total derivative of $\Theta_3(t)$, we obtain

$$\frac{d}{dt}\Theta_3(t) = \int_0^l \left(\frac{\partial}{\partial t}\widetilde{\Theta_3}(x,t)\right) dx + \mathcal{V}\widetilde{\Theta_3}(x,t)\Big|_0^l, \; t \geq 0, \tag{3.45}$$

where

$$\tilde{\Theta}_3(x,t) = \int_0^t \left(\int_t^\infty \mathcal{Z}(\tau - s)d\tau\right) \varphi_{xx}^2(x,s)ds, \; t \geq 0.$$

Clearly

$$\frac{d}{dt}\Theta_3(t) \leq \mathcal{K} \left\|\varphi_{xx}\right\|^2 - \int_0^t \mathcal{Z}(t-s) \int_0^l \varphi_{xx}^2(s)dxds + \mathcal{V}\int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds, \; t \geq 0.$$

A differentiation of $\Theta_4(t)$ gives

$$\frac{d}{dt}\Theta_4(t) = \mathcal{K}\varphi^2(l,s) - \int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds, \; t \geq 0. \tag{3.46}$$

Taking into account the estimates (3.15), (3.25), (3.43) and (3.44), we can write, for all $t \geq t_*$

$$\begin{aligned}
&\frac{d}{dt}L(t) \leq \left[\frac{\mathcal{M}EI}{2} - \frac{\mathcal{Z}(0)l^4}{4\xi_4}\right] \int_0^l \left(\mathcal{Z}' \circ \varphi_{xx}\right)(t)dx \\
&+\varrho\left[2\eta - \mathcal{Z}_* + \xi_4 + (1+\xi_3) + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right] \|\varphi_t\|^2 \\
&+\varrho\mathcal{V}^2\left[2\eta - \mathcal{Z}_* + \frac{1}{2} + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right] \|\varphi_x\|^2 \\
&+\left\{[(\frac{1}{2}+\xi_1)(1-\mathcal{Z}_*)EI + (\xi_5\frac{\beta}{2}+\frac{\beta}{2}) + EI(1-\mathcal{Z}_*)\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})] - \eta EI\left(1 - \frac{\mathcal{K}}{2}\right) + (\mu+\varsigma)\mathcal{K}\right\} \\
&\times \|\varphi_{xx}\|^2 + \left(-\eta\frac{EI}{2} + \xi_7 l^3\mathcal{K}\right) \times \int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx + (\mu\mathcal{V} - \varsigma)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds \\
&+\left\{\eta\frac{EI}{2} + \frac{\mathcal{K}}{2}\left[(1-\mathcal{Z}_*)EI + 5\rho\mathcal{V}^2l^2 + l^3\right] - \mu\right\}\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds + \\
&(1+\xi_2)EIk\hat{\mathcal{Z}}(\mathcal{Y}) \times \int_0^l\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2 dsdx + \mathcal{K}\left[(1-\mathcal{Z}_*)\frac{EI}{4\xi_1} + \left(1+\frac{1}{\xi_2}\right)EI\right. \\
&\left.+\left(1+\frac{1}{\xi_2}\right)EI + \frac{\varrho\mathcal{V}^2l^2}{\xi_3} + \frac{l^3}{4\xi_5}\right]\int_0^l\int_{\mathcal{A}_t} \mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2 dsdx
\end{aligned}$$

$$+\left[-\eta + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right]\varphi(l)y(\varphi(l)) + \frac{\varrho\mathcal{V}}{2}\left(\frac{1}{2\xi_7} - 1\right)\varphi_t^2(l). \tag{3.47}$$

We choose $\mathcal{M}$ large enough so that

$$\frac{\mathcal{M}EI}{2} - \frac{\mathcal{Z}(0)l^4}{4\xi_4} \geq \frac{\mathcal{M}EI}{4}. \tag{3.48}$$

As in [35], we introduce the sets

$$\mathcal{A}_n = \{s \in \mathbb{R}_+ : n\mathcal{Z}'(s) + \mathcal{Z}(s) \leq 0\}, \, \mathcal{Y}_n = \mathbb{R}_+ \backslash \mathcal{A}_n,$$

and

$$\tilde{A}_{nt} = \{s \in \mathbb{R}_+ : 0 \leq s \leq t, \; n\mathcal{Z}'(t-s) + \mathcal{Z}(t-s) \leq 0\}, \; n \in \mathbb{N}.$$

Observe that

$$\bigcup_n \mathcal{A}_n = \mathbb{R}_+ \backslash \{\mathcal{Y}_{\mathcal{Z}} \cup \mathcal{N}_{\mathcal{Z}}\},$$

where $\mathcal{N}_{\mathcal{Z}}$ is the set where $\mathcal{Z}'$ is not defined and $\mathcal{Y}_{\mathcal{Z}}$ is defined in (3.3). Since $\mathcal{Y}_{n+1} \subset \mathcal{Y}_n$ for all n and $\bigcap_n \mathcal{Y}_n = \mathcal{Y}_{\mathcal{Z}} \cup \mathcal{N}_{\mathcal{Z}}$, then

$\lim_{n\to\infty} \hat{\mathcal{Z}}(\mathcal{Y}_n) = \hat{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}})$. Taking $\mathcal{A}_t := \tilde{A}_{nt}$, $\mathcal{Y}_t := \tilde{y}_{nt}$, it follows from (3.49) that

$$\begin{aligned}
\frac{d}{dt}L(t) \leq& \varrho \left[2\eta - \mathcal{Z}_* + \xi_4 + (1+\xi_3) + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right] \|\varphi_t\|^2 \\
&+ \varrho\mathcal{V}^2 \left[2\eta - \mathcal{Z}_* + \frac{1}{2} + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right] \|\varphi_x\|^2 \\
&- \{\left[\eta EI \left(1 - \frac{\mathcal{K}}{2}\right) - (\frac{1}{2} + \xi_1)(1-\mathcal{Z}_*) EI - (\frac{\beta}{2}\xi_5 + \frac{\beta}{2}) - EI(1-\mathcal{Z}_*)\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right] \\
&- (\mu + \varsigma)\mathcal{K}\} \times \|\varphi_{xx}\|^2 \\
&+ \left(-\eta\frac{EI}{2} + \xi_7 l^3 \mathcal{K}\right) \times \int_0^l (\mathcal{Z} \circ \varphi_{xx})(t)dx + (\mu\mathcal{V} - \varsigma) \int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds \\
&+ \left\{\eta\frac{EI}{2} + \frac{\mathcal{K}}{2}\left[(1-\mathcal{Z}_*)EI + 5\rho\mathcal{V}^2 l^2 + l^3\right] - \mu\right\} \int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds + \\
&(1+\xi_2) EIk\hat{\mathcal{Z}}(\mathcal{Y}) \times \int_0^l \int_{\mathcal{Y}_t} \mathcal{Z}(t-s)|\varphi_{xx}(t) - \varphi_{xx}(s)|^2 dsdx \\
&+ \mathcal{K}\left[(1-\mathcal{Z}_*)\frac{EI}{4\xi_1} + \left(1 + \frac{1}{\xi_2}\right)EI + \left(1 + \frac{1}{\xi_2}\right)EI + \frac{\varrho\mathcal{V}^2 l^2}{\xi_3} + \frac{l^3}{4\xi_5} - \frac{\mathcal{M}EI}{4n}\right] \\
&\times \int_0^l \int_{\mathcal{A}_t} \mathcal{Z}(t-s)|\varphi_{xx}(t) - \varphi_{xx}(s)|^2 dsdx + \left[-\eta + \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y})\right]\varphi(l)y(\varphi(l)) \\
&+ \frac{\varrho\mathcal{V}}{2}\left(\frac{1}{2\xi_7} - 1\right)\varphi_t^2(l)
\end{aligned} \tag{3.49}$$

In (3.49), for $\mathcal{Z}_* > \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) + \frac{1}{2}$ sufficiently large we take $\eta = \frac{1}{2}\left(\mathcal{Z}_* - \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) - \frac{1}{2} - \upsilon\right)$. We infer that

$$-\frac{\eta EI}{2} + (1+\xi_2) EIk\hat{\mathcal{Z}}(\mathcal{Y}_n) + \xi_7 l^3 \mathcal{K} \leq 0$$

and

$$\frac{1}{2\xi_7} - 1 \leq 0.$$

At this point, we will need (we neglect $\xi_5$and ξ_1 as will be chosen small enough)

$$\frac{1}{2}\left(1-\mathcal{Z}_*\right)EI + \frac{\beta}{2} + EI\left(1-\mathcal{Z}_*\right)\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) - \eta EI\left(1-\frac{\mathcal{K}}{2}\right) + (\mu+\varsigma)\mathcal{K} < 0.$$

Adding and subtracting $\frac{\delta}{2}\left(\mathcal{Z}_* - \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) - \frac{1}{2} - \upsilon\right)\left(1-\frac{\mathcal{K}}{2}\right)EI$, we obtain

$$\left\{\left(1-\frac{\delta}{2}\right)\left[\mathcal{Z}_* - \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) - \frac{1}{2} - \upsilon\right]EI\left(1-\frac{\mathcal{K}}{2}\right) - \mu\mathcal{K}\right\} +$$

$$[\frac{\delta}{2}\left(\mathcal{Z}_* - \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) - \frac{1}{2} - \upsilon\right)(1-\frac{\mathcal{K}}{2})EI - \frac{1}{2}\left(1-\mathcal{Z}_*\right)EI - \frac{\beta}{2} - \left(1-\mathcal{Z}_*\right)\mathcal{K}\hat{\mathcal{Z}}\left(\mathcal{Y}_n\right) - \varsigma\mathcal{K}] \quad > \quad 0$$

This term is devided into parts.
For the second part, we need

$$\begin{aligned} & \frac{1}{2}\left(1-\mathcal{Z}_*\right)EI + \frac{\beta}{2} + EI\left(1-\mathcal{Z}_*\right)\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}_n) + \varsigma\mathcal{K} \\ < \ & \frac{\delta}{2}\left[\mathcal{Z}_* - \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}) - \frac{1}{2} - \upsilon\right]\left(1-\frac{\mathcal{K}}{2}\right)EI, \end{aligned}$$

where

$$\delta = \frac{2\left(1-\mathcal{Z}_*\right)EI(\frac{2k}{3}+1) + 2\beta + 4\varsigma\mathcal{K}}{(\mathcal{Z}_* - \frac{\mathcal{K}}{3} - \frac{1}{2})\left(2-\mathcal{K}\right)}EI.$$

and $\hat{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}}) < 1/3$.

For the first part, we need

$$\mu < \left(1-\frac{\delta}{2}\right)\frac{\left[\mathcal{Z}_* - \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}_n) - \frac{1}{2} - \upsilon\right]}{\mathcal{K}}\left(1-\frac{\mathcal{K}}{2}\right)EI.$$

We choose μ such that

$$\frac{\mathcal{Z}_* - \upsilon}{2} + \frac{2-\mathcal{Z}_*}{2} < \mu < \left(1-\frac{\delta}{2}\right)\frac{\left[\mathcal{Z}_* - \mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}_n) - \frac{1}{2} - \upsilon\right]}{\mathcal{K}}\left(1-\frac{\mathcal{K}}{2}\right)EI. \tag{3.50}$$

Note that (υ is ignored)

$$\left(1-\frac{\delta}{2}\right)\frac{\left[\mathcal{Z}_*\mathcal{K}\hat{\mathcal{Z}}(\mathcal{Y}_n)-\frac{1}{2}-\upsilon\right]}{\mathcal{K}}\left(1-\frac{\mathcal{K}}{2}\right)EI$$
$$= \frac{(\mathcal{Z}_*-\frac{\mathcal{K}}{3}-\frac{1}{2})\,(2-\mathcal{K})-(1-\mathcal{Z}_*)EI(\frac{-2k}{3}+1)-2\varsigma\mathcal{K}-\beta)}{2k}.$$

By replacing δ and neglecting υ,the choice of μ in the relation (3.50) is possible if

$$1<\frac{(\mathcal{Z}_*-\frac{\mathcal{K}}{3}-\frac{1}{2})\,(2-\mathcal{K})\;-(1-\mathcal{Z}_*)EI(\frac{-2k}{3}+1)-2\varsigma\mathcal{K}-\beta)}{2k}, \tag{3.51}$$

that is when

$$\mathcal{Z}_*>\frac{2k\varsigma+2\beta+(\frac{\mathcal{K}}{3}+\frac{3}{2})(2-\mathcal{K})+EI(2\frac{\mathcal{K}}{3}+1)}{(2-\mathcal{K})+(2\frac{\mathcal{K}}{3}+1)EI}. \tag{3.52}$$

We know that $\mathcal{Z}_*<\mathcal{K}$ and to make (3.52) possible, we must have

$$\frac{2k\varsigma+2\beta+(\frac{\mathcal{K}}{3}+\frac{3}{2})(2-\mathcal{K})+EI(2\frac{\mathcal{K}}{3}+1)}{(2-\mathcal{K})+(2\frac{\mathcal{K}}{3}+1)EI}<\mathcal{Z}_*<\mathcal{K}. \tag{3.53}$$

Provided that $E\,(0)$ is so small that

$$\beta<\frac{(2-\mathcal{K})(\frac{1}{2}-\frac{\mathcal{K}}{3})-\varsigma\mathcal{K}}{2}.$$

Hence, $L'(t)+\alpha_1E(t)\le 0,\ t\ge t_*$. for some constant α_1, provided that $\varsigma>\mu\mathcal{V}$. □

We introduce the functional

$$\Theta_5(t)=\int_0^t\varsigma\,(t-s)\,\|\varphi_{xx}\|^2\,ds,\ t\ge 0.$$

Lemma 12. *we have*

$$L(t)\le\mathcal{C}_1\left(E(t)+\Theta_4+\Theta_5(t)\right),\ t\ge 0 \tag{3.54}$$

for some $\mathcal{C}_1>0$. Moroever, there exists $t_0\ge t_*$ such that

$$E(t)\le 1,\ t\ge t_0. \tag{3.55}$$

Proof. Integrating (3.20) on $[t_*,t]$, we get

$$\int_{t_*}^t E(s)ds\le\frac{L(t_*)}{\alpha_1}=\mathcal{C}_2,\quad t\ge t_*.$$

This yields the integral control

$$\omega=\int_{t_*}^{\infty}\|\varphi_{xx}(s)\|^2\,ds<\infty.$$

We have

$$(t-t_*)\,E(t)\leq\mathcal{C}_2,\ t\geq t_*$$

Hence

$$E(t)\leq\frac{\mathcal{C}_2}{1+t-t_*},\ t\geq t_*.$$

Thus, there exists $t_0\geq t_*$ such that

$$E(t)\leq 1,\ t\geq t_0.$$

□

Lemma 13. *Along solutions of (3.1), we have*

$$\Theta_5'(t)+\omega\zeta(t)\,\chi\left(\frac{\Theta_4(t)}{\omega}\right)\leq\frac{2\varsigma(0)}{1-\mathcal{K}}\mathcal{E}(t),\ t\geq 0. \tag{3.56}$$

Proof. Since ζ is decreasing , and σ fulfills the differential inequality (3.4), we get

$$\begin{aligned}\Theta_5'(t) &= \int_0^t\sigma'(t-s)\,\|\varphi_{xx}\|^2\,ds+\sigma(0)\,\|\varphi_{xx}\|^2\\ &\leq -\int_0^t\zeta(t-s)\,\chi(\sigma(t-s))\,\|\varphi_{xx}\|^2\,ds+\frac{2\varsigma(0)}{1-\mathcal{K}}E(t)\\ &\leq -\zeta(t)\int_0^t\chi(\sigma(t-s))\,\|\varphi_{xx}\|^2\,ds+\frac{2\varsigma(0)}{1-\mathcal{K}}E(t),\ t\geq 0.\end{aligned}$$

Next, we define

$$I(t)=\int_0^t\|\varphi_{xx}(s)\|^2\,ds,\ t\geq 0.$$

Applying now the Jensen inequality with $y(s)=\sigma(t-s)$ and $\mathcal{Z}(s)=\|\varphi_{xx}(s)\|^2/I(t)$, we get

$$\begin{aligned}\int_0^t\chi(\sigma(t-s))\,\|\varphi_{xx}(s)\|^2\,ds &= I(t)\int_0^t\chi(\sigma(t-s))\,\frac{\|\varphi_{xx}(s)\|^2}{I(t)}ds,\\ &\geq I(t)\,\chi\left(\int_0^t\sigma(t-s)\,\frac{\|\varphi_{xx}(s)\|^2}{I(t)}ds\right),\ t\geq 0.\end{aligned}$$

Since χ is convex with $\chi(0) = 0$, then $\theta\chi(x) \geq \chi(\theta x)$ for $\theta \in [0,1]$. This implies by taking $\theta = I(t)/\omega$ that

$$\int_0^t \chi(\sigma(t-s)) \|\varphi_{xx}(s)\|^2 ds \geq \omega\chi\left(\frac{I(t)}{\omega}\int_0^t \sigma(t-s)\frac{\|\varphi_{xx}(s)\|^2}{I(t)}ds\right)$$

$$= \omega\chi\left(\frac{\Theta_5(t)}{\omega}\right), \ t \geq 0.$$

The proof of Lemma 13 is now complete. □

Theorem 8. *Assume that **(H1)**-**(H4)** hold, $\varphi_0, \varphi_1 \in H_0^1(\Omega)$ and that $\mathcal{R}_g < 1/3$. Then there exist positive constants $a < 1$ and K such that*

$$E(t) \leq K\mathcal{B}^{-1}\left(a\int_{\hat{t}}^t \zeta(s)\,ds\right), \ \forall t \geq \hat{t}$$

where K and $\hat{t}$ depend on $E(0)$.

Proof. Choosing $\alpha_2 > 0$, we introduce the functional

$$y(t) = L(t) + \Theta_4(t) + \alpha_2\Theta_5(t), \ t \geq 0$$

which, in light of (3.54), satisfies

$$E(t) \leq y(t) \leq (\mathcal{C}_1 + \alpha_2)(E(t) + \Theta_4 + \Theta_5(t)), \ t \geq 0. \tag{3.57}$$

$$y'(t) + \left(\alpha_1 - \frac{2\varsigma(0)}{1-\mathcal{K}}\alpha_2\right)E(t) + \alpha_2\omega\zeta(t)\chi\left(\frac{\Theta_5(t)}{\omega}\right) \leq 0, \ t \geq t_*.$$

$\alpha_1 \geq \left(\frac{2\varsigma(0)}{1-\mathcal{K}} + 1\right)\alpha_2$ to obtain

$$y'(t) + \alpha_2 E(t) + \alpha_2\omega\zeta(t)\chi\left(\frac{\Theta_5(t)}{\omega}\right) \leq 0.$$

Let

$$t \geq t_0 \geq t_* \geq 1.$$

$$\zeta(t)\chi(E(t)) \leq \zeta(1)\chi(E(t)) \leq \zeta(1)\chi(1)E(t) \leq \varrho_1 E(t), \ t \geq t_0 \geq t_*,$$

where $\varrho_1 > 1$. Accordingly,

$$\alpha_2 E(t) + \alpha_2 \omega \zeta(t) \chi\left(\frac{\Theta_5(t)}{\omega}\right) \geq \alpha_2 \zeta(t)\left[\frac{1}{\varrho}\chi(E(t)) + \omega\chi\left(\frac{\Theta_5(t)}{\omega}\right)\right].$$

Exploiting the Jensen inequality, we obtain

$$\alpha_2 E(t) + \alpha_2 \omega \zeta(t) \chi\left(\frac{\Theta_5(t)}{\omega}\right) \geq \alpha_2 \mathcal{C}_3 \zeta(t) \chi\left(\frac{\frac{1}{\varrho_1}E(t) + \Theta_4 + \Theta_5(t)}{\mathcal{C}_3}\right),$$

where $\mathcal{C}_3 = \left(\frac{1}{\varrho_1} + \omega\right)$.
Using (3.57) and χ is increasing, to get

$$\chi\left(\frac{\frac{1}{\varrho_1}E(t) + \Theta_4 + \Theta_5(t)}{\mathcal{C}_3}\right) \geq \chi\left(\frac{y(t)}{\varrho_1(\mathcal{C}_1 + \alpha_2)\mathcal{C}_3}\right).$$

In conclusion

$$y'(t) + \alpha_2 \mathcal{C}_3 \zeta(t) \chi\left(\frac{y(t)}{\varrho_1(\mathcal{C}_1 + \alpha_2)\mathcal{C}_3}\right) \leq 0.$$

Next, we divide by $\varrho_1(\mathcal{C}_1 + \alpha_2)\mathcal{C}_3$ and we write

$$G'(t) + a\zeta(t)\chi(G(t)) \leq 0, \tag{3.58}$$

where

$$G(t) = \frac{y(t)}{\varrho_1(\mathcal{C}_1 + \alpha_2)\mathcal{C}_3} \text{ and } a = \frac{\alpha_2 \mathcal{C}_3}{\varrho_1(\mathcal{C}_1 + \alpha_2)\mathcal{C}_3} = \frac{\alpha_2}{\varrho_1(\mathcal{C}_1 + \alpha_2)} < 1.$$

There exists $\hat{t} \geq t_0$, depending on $G(t_0)$, such that

$$G(t) \leq B^{-1}\left(a\int_{\hat{t}}^{t} \zeta(s)\, ds\right), \quad \forall t \geq t_0$$

and since $E(t) \leq \mathcal{C}_4 G(t)$ for some $\mathcal{C}_4 > 0$, it holds that

$$E(t) \leq K\mathcal{B}^{-1}\left(a\int_{\hat{t}}^{t} \zeta(s)\, ds\right), \quad \forall t \geq \hat{t}.$$

Chapter 4

General decay for an axially moving viscoelastic Kirchhoff string

4.1 Introduction

In this chapter, we study the stability of the equation of an axially moving viscoelastic Kirchhoff string described by the following nonlinear partial differential equation (PDE):

$$\begin{cases} \varphi_{tt} + 2\mathcal{V}\varphi_{xt} - \left(1 - \mathcal{V}^2 + q(t)\left\|\varphi_x\right\|^2\right)\varphi_{xx} + \left(1 - \mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}(s)\mathrm{d}s = 0, \\ x \in (0,1), \quad t > 0, \end{cases} \tag{4.1}$$

$$\begin{cases} \varphi(0,t) = 0, \quad t \geq 0, \\ f_c(t) = m\varphi_{tt}(1,t) + (\eta_m - \mathcal{V})\varphi_t(1,t) + \left(1 - \mathcal{V}^2 + q(t)\left\|\varphi_x\right\|^2\right)\varphi_x(1,t) \\ -\left(1 - \mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\varphi_x(1,s)\mathrm{d}s, \quad t \geq 0, \end{cases} \tag{4.2}$$

where $\varphi = \varphi(x,t)$ the transversal displacement of the string,
$\mathcal{V}$ is the axial speed (assumed constant here),
m and η_m are the mass and the damping coefficient of the dynamic actuator, respectively.
$\mathcal{Z}$ represents the kernel of the memory term or the relaxation function (nonnegative functions) see [20], $f_c(t)$ represents the control force being applied by the hydraulic actuator. This essay's tension is described by

$$T(t) = 1 + q(t)\left\|\varphi_x\right\|^2, \quad t \geq 0.$$

The well-posedeness of the system can be derived, for instance, from [7, 6] combined with arguments from [28].

4.2 Fundamental knowledge

The hypotheses that we make for $\mathcal{Z}, \zeta, \chi, \sigma$ and B are the same as those which we made in the previous chapter.
and

(H5) The function $q(t)$ is continuously differentiable and $q(t) \geq 0$.

We modify the classical energy of the system (4.1) as follows

$$\begin{aligned} E(t) =& \frac{1}{2} \|\varphi_t\|^2 + \frac{1}{2}\left(1-\mathcal{V}^2\right)\left(1-\int_0^t \mathcal{Z}(s)\mathrm{d}s\right)\|\varphi_x\|^2 + \frac{q(t)}{4}\|\varphi_x\|^4 \\ &+ \frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^1 \left(\mathcal{Z}\circ\varphi_x\right)\mathrm{d}x + \frac{m}{2}\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right)^2, \quad t\geq 0. \end{aligned}$$

Where

$$(\mathcal{Z}\circ w)(t) = \int_0^t \mathcal{Z}(t-s)|w(t)-w(s)|^2 \,\mathrm{d}s, \quad t\geq 0.$$

We assume that the kernel is such that

$$\int_0^\infty \mathcal{Z}(s)\mathrm{d}s =: k < 1.$$

Since the string moves at a constant velocity $\mathcal{V}$, the overall time derivative operator is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathcal{V}\frac{\partial}{\partial x}.$$

Therefore, the total derivative of $E(t)$ is equal to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}(x,t)\mathrm{d}x + \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right)^2\right] \\ &= \int_0^1 \left(\frac{\partial}{\partial t}\widetilde{E}(x,t)\right)\mathrm{d}x + \mathcal{V}\widetilde{E}(x,t)\Big|_0^1 + \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right)^2\right], \quad t\geq 0 \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \widetilde{E}(x,t) =& \frac{1}{2}\varphi_t^2(x,t) + \frac{1}{2}\left(1-\mathcal{V}^2\right)\left[\left(1-\int_0^t \mathcal{Z}(s)\mathrm{d}s\right)\varphi_x^2(x,t) + \left(\mathcal{Z}\circ\varphi_x\right)(x,t)\right] \\ &+ \frac{1}{4}q(t)\varphi_x^2(x,t)\|\varphi_x\|^2, \quad t\geq 0. \end{aligned}$$

The explanation is as follows: Since the system is mass flow, flowing in and out at the boundaries, the net change in total energy is the sum of the changes in the control volumes.

According to (4.3), the total derivatives of $E(t)$ are equal

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \int_0^1 \varphi_t\varphi_{tt}\ \mathrm{d}x + \left\{\left(1-\mathcal{V}^2\right)\left(1-\int_0^t \mathcal{Z}(s)\mathrm{d}s\right) + q(t)\left\|\varphi_x\right\|^2\right\}_0^1 \int_x \varphi_{xt}\ \mathrm{d}x \\ &- \frac{1}{2}\left[\left(1-\mathcal{V}^2\right)\mathcal{Z}(t) - \frac{q'(t)}{2}\left\|\varphi_x\right\|^2\right]\left\|\varphi_x\right\|^2 + \frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^1\left(\mathcal{Z}'\circ\varphi_x\right)\mathrm{d}x \\ &+ \left(1-\mathcal{V}^2\right)\int_0^1 \varphi_{xt}\int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s\ \mathrm{d}x \\ &+ m\left(\varphi_t+\mathcal{V}\varphi_x\right)\left(\varphi_{tt}+\mathcal{V}\varphi_{xt}\right)(1,t) + \mathcal{V}\tilde{E}(x,t)\Big|_0^1 \quad t\geq 0.\end{aligned} \tag{4.4}$$

Substituting φ_{tt} and $\varphi_{tt}(1,t)$ from equation (4.1) into equation (4.4), we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \int_0^1 \varphi_t\left\{-2\mathcal{V}\varphi_{xt} - \left(1-\mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}(s)\mathrm{d}s\right. \\ &\left. + \left(1-\mathcal{V}^2+q(t)\left\|\varphi_x\right\|^2\right)\varphi_{xx}\right\}\mathrm{d}x - \frac{1}{2}\left(1-\mathcal{V}^2\right)\mathcal{Z}(t)\left\|\varphi_x\right\|^2 + \frac{q'(t)}{4}\left\|\varphi_x\right\|^4 \\ &+ \left\{\left(1-\mathcal{V}^2\right)\left(1-\int_0^t \mathcal{Z}(s)\mathrm{d}s\right) + q(t)\left\|\varphi_x\right\|^2\right\}\int_0^1 \varphi_x\varphi_{xt}\ \mathrm{d}x \\ &+ \frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^1\left(\mathcal{Z}'\circ\varphi_x\right)\mathrm{d}x + \left(1-\mathcal{V}^2\right)\int_0^1\varphi_{xt}\int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s\ \mathrm{d}x \\ &+ \left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\left[f_c(t) - \left(\eta_m-\mathcal{V}\right)\varphi_t(1,t) + m\mathcal{V}\varphi_{xt}(1,t)\right] \\ &+ \left(1-\mathcal{V}^2\right)\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\left(-\varphi_x(1,t) + \int_0^t \mathcal{Z}(t-s)\varphi_x(1,s)\mathrm{d}s\right) \\ &- q(t)\varphi_x(1,t)\left(\varphi_t+\mathcal{V}\varphi_x\right)\left\|\varphi_x\right\|^2 + \mathcal{V}\tilde{E}(x,t)\Big|_0^1 t\geq 0.\end{aligned}$$

Next, given the definition of $\tilde{E}(x,t)$ and extending the convolution term of $t\geq 0$, we see

$$\begin{aligned}\tilde{E}(x,t)\Big|_0^1 \leq &\frac{1}{2}\varphi_t^2(1,t) + \frac{1}{2}\left(1-\mathcal{V}^2\right)\left(\varphi_x^2(1,t) - 2\varphi_x(1,t)\int_0^t \mathcal{Z}(t-s)\varphi_x(1,s)\mathrm{d}s\right) \\ &+ \frac{1}{4}q(t)\varphi_x^2(1,t)\left\|\varphi_x\right\|^2 + \frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\varphi_x^2(1,s)\mathrm{d}s.\end{aligned} \tag{4.5}$$

If we apply integration by parts, and consider the boundary conditions in (4.1) and the relation (4.5), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\frac{\mathcal{V}}{2}\varphi_t^2(1,t) + \frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^1\left(\mathcal{Z}'\circ\varphi_x\right)\mathrm{d}x + \frac{q'(t)}{4}\left\|\varphi_x\right\|^4$$

$$
\begin{aligned}
&-\frac{1}{2}\left(1-\mathcal{V}^2\right)\mathcal{Z}(t)\left\|\varphi_x\right\|^2-\frac{\mathcal{V}}{2}\left(1-\mathcal{V}^2\right)\varphi_x^2(1,t)\\
&-\frac{3}{4}\mathcal{V}q(t)\varphi_x^2(1,t)\left\|\varphi_x\right\|^2+\frac{\mathcal{V}}{2}\left(1-\mathcal{V}^2\right)\int_0^t\mathcal{Z}(t-s)\varphi_x^2(1,s)\mathrm{d}s\\
&+\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\left[f_c(t)-\left(\eta_m-\mathcal{V}\right)\varphi_t(1,t)+m\mathcal{V}\varphi_{xt}(1,t)\right],\quad t\geq 0
\end{aligned}
\tag{4.6}
$$

a boundary control law is then proposed as follows

$$
f_c(t)=\left(\eta_m-\mathcal{V}\right)\varphi_t(1,t)-m\mathcal{V}\varphi_{xt}(1,t),\quad t\geq 0. \tag{4.7}
$$

Then, (4.6) becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t)\leq&-\frac{\mathcal{V}}{2}\varphi_t^2(1,t)+\frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^1\left(\mathcal{Z}'\circ\varphi_x\right)\mathrm{d}x+\frac{q'(t)}{4}\left\|\varphi_x\right\|^4\\
&-\frac{1}{2}\left(1-\mathcal{V}^2\right)\mathcal{Z}(t)\left\|\varphi_x\right\|^2-\frac{\mathcal{V}}{2}\left(1-\mathcal{V}^2\right)\varphi_x^2(1,t)\\
&-\frac{3\mathcal{V}}{4}q(t)\varphi_x^2(1,t)\left\|\varphi_x\right\|^2+\frac{\mathcal{V}}{2}\left(1-\mathcal{V}^2\right)\int_0^t\mathcal{Z}(t-s)\varphi_x^2(1,s)\mathrm{d}s,\quad t\geq 0
\end{aligned}
\tag{4.8}
$$

4.3 Analysis of stability

In this section we state and prove our decay result.

Firstly, we define for all $t\geq 0$ the functionals

$$
\begin{aligned}
\Theta_1(t)&=\int_0^1\varphi\varphi_t\ \mathrm{d}x+m\varphi(1,t)\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t),\\
\Theta_2(t)&=-\int_0^1\varphi_t\int_0^t\mathcal{Z}(t-s)(\varphi(t)-\varphi(s))\mathrm{d}s\ \mathrm{d}x\\
&\quad-m\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\int_0^t\mathcal{Z}(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s,\\
\Theta_3(t)&=\int_0^t\left(\int_t^\infty\mathcal{Z}(\tau-s)d\tau\right)\varphi_x^2(l,s)ds,t\geq 0,\\
\Theta_4(t)&=\int_0^t\left(\int_t^\infty\mathcal{Z}(\tau-s)d\tau\right)\|\varphi_x(s)\|^2ds,t\geq 0,
\end{aligned}
$$

and

$$
L(t)=NE(t)+\lambda\Theta_1(t)+\Theta_2(t)+M\Theta_3(t)+\varsigma\Theta_4(t),\quad t\geq 0,
$$

where N, λ, M and ς are positive constants.

Proposition 6. *Let the hypotheses (H1)-(H5) be satisfied. If $\mathcal{R}_h$ is small enough and for some*

$\mathcal{C} > 0$, and $4q(t) + q'(t) \leq 0$ then we have

$$E(t) \leq L(t) \leq \mathcal{C}(E(t) + \Theta_3(t)),\ t \geq 0, \tag{4.9}$$

$$L'(t) + \alpha E(t) \leq 0,\ t \geq 0 \tag{4.10}$$

for some $\alpha > 0$.

Proof. By using Lemmas 3 and 5, we obtain

$$\begin{aligned}\Theta_1(t) &\leq \frac{1}{2}\left(\|\varphi_t\|^2 + \|\varphi\|^2\right) + \frac{m}{2}\left[\varphi^2(1,t) + (\varphi_t(1,t) + \mathcal{V}\varphi_x(1,t))^2\right]\\ &\leq \frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}(1+m)\|\varphi_x\|^2 + \frac{m}{2}(\varphi_t(1,t) + \mathcal{V}\varphi_x(1,t))^2, \quad t \geq 0.\end{aligned}$$

Similarly, we get

$$\Theta_2(t) \leq \frac{1}{2}\|\varphi_t\|^2 + \frac{k}{2}(1+m)\int_0^1 (\mathcal{Z} \circ \varphi_x)\,\mathrm{d}x + \frac{m}{2}(\varphi_t(1,t) + \mathcal{V}\varphi_x(1,t))^2, \quad t \geq 0.$$

Then

$$\begin{aligned}L(t) \leq& \frac{1}{2}(1+\lambda+N)\|\varphi_t\|^2 + \frac{Nq(t)}{4}\|\varphi_x\|^4 + \frac{1}{2}\left\{N\left(1-\mathcal{V}^2\right)\left(1 - \int_0^t \mathcal{Z}(s)\mathrm{d}s\right)\right.\\ &\left. + \lambda(1+m)\right\}\|\varphi_x\|^2 + \frac{1}{2}\left[N\left(1-\mathcal{V}^2\right) + k(1+m)\right]\int_0^1 (\mathcal{Z} \circ \varphi_x)\,\mathrm{d}x\\ &+ \frac{m}{2}(1+\lambda+N)(\varphi_t(1,t) + \mathcal{V}\varphi_x(1,t))^2 + M\Theta_3(t) + \varsigma\Theta_4(t), t \geq 0.\end{aligned}$$

This implies that $L(t) \leq \mathcal{C}\left(E(t) + \Theta_3(t)\right), t \geq 0.$ for some $\mathcal{C} \geq 0$.
On the other side

$$\begin{aligned}L(t) - E(t) &= (N-1)E(t) + \lambda\Theta_1(t) + \Theta_2(t) + M\Theta_3(t)\\ &\geq \frac{1}{2}(N-2-\lambda)\|\varphi_t\|^2 + \frac{1}{2}\left[(N-1)\left(1-\mathcal{V}^2\right)(1-k) - \lambda(1+m)\right]\|\varphi_x\|^2\\ &\quad + \frac{(N-1)q(t)}{4}\|\varphi_x\|^4 + \frac{1}{2}\left[(N-1)\left(1-\mathcal{V}^2\right) - k(1+m)\right]\int_0^1 (h \circ \varphi_x)\,\mathrm{d}x\\ &\quad + \frac{m}{2}(N-2-\lambda)(\varphi_t(1,t) + \mathcal{V}\varphi_x(1,t))^2 + M\Theta_3(t) + \varsigma\Theta_4(t), \quad t \geq 0.\end{aligned}$$

Choosing small N, λ, M such that $\lambda < \min\left\{2, (N-1)\left(1-\mathcal{V}^2\right)(1-k)/(1+m)\right\}$ and $N < \min\left\{\left(1-\mathcal{V}^2\right)/k(1+m) + 1, 2-\lambda\right\}$., we find that $L(t) - \mathcal{E}(t) \geq 0$. Thus, the first assertion is proved. We will now demonstrate the second assertion.

The total derivative of $\Theta_1(t)$ is equal to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1(t) &= \int_0^1 \left(\frac{\partial}{\partial t}\widetilde{\Theta_1}(x,t)\right)\mathrm{d}x + \mathcal{V}\widetilde{\Theta}_1(x,t)\Big|_0^1 + \frac{\mathrm{d}}{\mathrm{d}t}\left[m\varphi(1,t)\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\right] \\ &= \|\varphi_t\|^2 + \int_0^1 \varphi\varphi_{tt}\ \mathrm{d}x + \mathcal{V}\widetilde{\Theta}_1(x,t)\Big|_0^1 + \left[m\varphi_t(1,t)\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\right] \\ &\quad + \left[m\varphi(1,t)\left(\varphi_{tt}+\mathcal{V}\varphi_{xt}\right)(1,t)\right], \quad t\geq 0.\end{aligned} \tag{4.11}$$

Where

$$\widetilde{\Theta_1}(x,t) = \varphi(x,t)\varphi_t(x,t), \quad t\geq 0.$$

Clearly,

$$\widetilde{\Theta}_1(x,t)\Big|_0^1 = \varphi(1,t)\varphi_t(1,t), t\geq 0. \tag{4.12}$$

Substituting φ_{tt} and $\varphi_{tt}(1,t)$ from equation (4.1) into equation (4.11) and taking into account the relation (4.12), we entail

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1(t) = \|\varphi_t\|^2 &+ \int_0^1 \varphi\Big\{-2\mathcal{V}\varphi_{xt} + \left(1-\mathcal{V}^2\right)\left(\varphi_{xx} - \int_0^t \mathcal{Z}(t-s)\varphi_{xx}(s)\mathrm{d}s\right) \\ &+q(t)\|\varphi_x\|^2\varphi_{xx}\Big\}\mathrm{d}x + \mathcal{V}\varphi(1,t)\varphi_t(1,t) + m\varphi_t(1,t)\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right) \\ &+\varphi(1,t)\Big\{-\left(1-\mathcal{V}^2\right)\varphi_x(1,t) + \left(1-\mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\varphi_x(1,s)\mathrm{d}s \\ &-q(t)\|\varphi_x\|^2\varphi_x(1,t) + f_c(t) - \left(\eta_m-\mathcal{V}\right)\varphi_t(1,t) + m\mathcal{V}\varphi_{xt}(1,t)\Big\}, \quad t\geq 0.\end{aligned}$$

Integrating by parts, taking into account the boundary conditions in (4.1) and the expression (4.7), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1(t) =& 2\mathcal{V}\int_0^1 \varphi_t\varphi_x\ \mathrm{d}x + \|\varphi_t\|^2 - \left(1-\mathcal{V}^2\right)\|\varphi_x\|^2 - q(t)\|\varphi_x\|^4 \\ &+ \left(1-\mathcal{V}^2\right)\int_0^1 \varphi_x \int_0^t \mathcal{Z}(t-s)\varphi_x(s)\mathrm{d}s\ \mathrm{d}x - \mathcal{V}\varphi(1,t)\varphi_t(1,t) \\ &+ m\varphi_t(1,t)\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right), t\geq 0.\end{aligned} \tag{4.13}$$

The first term in the right-hand side of(4.13) can be rewritten as

$$2\mathcal{V}\int_0^1 \varphi_t\varphi_x\ \mathrm{d}x \leq \frac{\mathcal{V}^2}{\beta}\|\varphi_t\|^2 + \beta\|\varphi_x\|^2, \quad t\geq 0 \tag{4.14}$$

The last two terms are estimated with the help of Lemmas 5 and 3 as follows

$$\mathcal{V}\varphi(1,t)\varphi_t(1,t) \leq \mathcal{V}\delta_1\|\varphi_x\|^2 + \frac{\mathcal{V}}{4\delta_1}\varphi_t^2(1,t), \quad \delta_1>0, \quad t\geq 0, \tag{4.15}$$

$$m\varphi_t(1,t)\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right) \leq m(1+\mathcal{V})\varphi_t^2(1,t) + \frac{m\mathcal{V}}{4}\varphi_x^2(1,t), \quad t\geq 0. \tag{4.16}$$

Inserting the estimates (4.14)-(4.16) in (4.13) and applying Lemma 10 we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1(t) \leq & \left(1+\frac{\mathcal{V}^2}{\beta}\right)\|\varphi_t\|^2 - \left[\left(1-\mathcal{V}^2\right)\left(1-\frac{k}{2}\right)-\delta_1\mathcal{V}-\beta\right]\|\varphi_x\|^2 \\ & - q(t)\|\varphi_x\|^4 + \frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^t \mathcal{Z}(t-s)\|\varphi_x(s)\|^2 \,\mathrm{d}s - \frac{1}{2}\left(1-\mathcal{V}^2\right)\int_0^1 (\mathcal{Z}\circ\varphi_x)\,\mathrm{d}x \\ & + \left(m(1+\mathcal{V})+\frac{\mathcal{V}}{4\delta_1}\right)\varphi_t^2(1,t) + \frac{m\mathcal{V}}{4}\varphi_x^2(1,t), t\geq 0.\end{aligned} \tag{4.17}$$

For $\Theta_2(t)$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta_2(t) = & \int_0^1 \left(\frac{\partial}{\partial t}\widetilde{\Theta_2}(x,t)\right)\mathrm{d}x + \mathcal{V}\widetilde{\Theta}_2(x,t)\Big|_0^1 \\ & - \frac{\mathrm{d}}{\mathrm{d}t}\left[m\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\int_0^t \mathcal{Z}(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s\,\mathrm{d}x\right], t\geq 0\end{aligned} \tag{4.18}$$

Where

$$\tilde{\Theta}_2(x,t) = -\varphi_t(x,t)\int_0^t \mathcal{Z}(t-s)(\varphi(x,t)-\varphi(x,s))\mathrm{d}s, t\geq 0.$$

Together, this concept and the border condition in (4.1) result in

$$\widetilde{\Theta_2}(x,t)\Big|_0^1 = -\varphi_t(1,t)\int_0^t \mathcal{Z}(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s, t\geq 0.$$

In view of the expression 4.18, the total derivative of $\Theta_2(t)$ is then equal to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(t) = & -\int_0^1 \varphi_{tt}\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))\mathrm{d}s\,\mathrm{d}x \\ & - \int_0^1 \varphi_t\left[\int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))\mathrm{d}s\,\mathrm{d}x + \varphi_t\left(\int_0^t \mathcal{Z}(s)\mathrm{d}s\right)\right] \\ & - m\left(\varphi_{tt}(1,t)+v\varphi_{xt}(1,t)\right)\int_0^t \mathcal{Z}(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s. \\ & - m\left(\varphi_t(1,t)+v\varphi_x(1,t)\right)\left[\int_0^t \mathcal{Z}'(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s\right] \\ & - m\left(\varphi_t(1,t)+v\varphi_x(1,t)\right)\varphi_t(1,t)\left(\int_0^t \mathcal{Z}(s)\mathrm{d}s\right) \\ & - v\varphi_t(1,t)\int_0^t \mathcal{Z}(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s.\end{aligned}$$

Substituting φ_{tt} and $\varphi_{tt}(1,t)$ from equation (4.1) into equation (4.2), Using integrating by parts and taking into account the boundary conditions in (4.1) and (4.7), the result is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Theta_2(t) = & \left(1-\mathcal{V}^2\right)\left[\left(1-\int_0^t \mathcal{Z}(s)\mathrm{d}s\right)\int_0^1 \varphi_x \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s\right] \\
& + \left(1-\mathcal{V}^2\right)\int_0^1 \left|\int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s\right|^2 \,\mathrm{d}x \\
& + q(t)\left\|\varphi_x\right\|^2 \int_0^1 \varphi_x \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s \\
& - 2\mathcal{V}\int_0^1 \varphi_t \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s\ \mathrm{d}x \\
& - \int_0^1 \varphi_t \int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))\mathrm{d}s\ \mathrm{d}x - \left(\int_0^t \mathcal{Z}(s)\mathrm{d}s\right)\left\|\varphi_t\right\|^2 \\
& - m\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\int_0^t \mathcal{Z}'(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s\ \mathrm{d}x \\
& - m\varphi_t(1,t)\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\left(\int_0^t \mathcal{Z}(s)\mathrm{d}s\right) \\
& + \mathcal{V}\varphi_t(1,t)\int_0^t \mathcal{Z}(t-s)(\varphi(1,t)-\varphi(1,s))\mathrm{d}s, \quad t \geq 0.
\end{aligned} \tag{4.19}$$

We have

$$\begin{aligned}
& \int_0^1 \varphi_x \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s \\
& \quad = \int_0^1 \varphi_x \left(\int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s + \int_{\mathcal{F}_t} \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s\right) \\
& \quad \leq \int_0^1 \varphi_x \int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)\mathrm{d}s\ \mathrm{d}x + \left(\int_{\mathcal{F}_t} \mathcal{Z}(t-s)\mathrm{d}s\right)\left\|\varphi_x\right\|^2 .
\end{aligned}$$

$$-\int_0^1 \varphi_x \int_{\mathcal{F}_t} \mathcal{Z}(t-s)\varphi_x(s)\mathrm{d}s\ \mathrm{d}x \tag{4.20}$$

For $\delta_2 > 0$, we see that

$$\begin{aligned}
& \int_0^1 \varphi_x \int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left(\varphi_x(s)-\varphi_x(t)\right)\mathrm{d}s\ \mathrm{d}x \\
& \quad \leq \delta_2\left\|\varphi_x\right\|^2 + \frac{k}{4\delta_2}\int_{\mathcal{A}_t} \mathcal{Z}(t-s)\int_0^1 \left(\varphi_x(t)-\varphi_x(s)\right)^2 \,\mathrm{d}x\ \mathrm{d}s, t \geq 0
\end{aligned} \tag{4.21}$$

and for $\eta > 0$

$$\begin{aligned}
& \int_0^1 \varphi_x \int_{\mathcal{F}_t} \mathcal{Z}(t-s)\varphi_x(s)\mathrm{d}s\ \mathrm{d}x \\
& \quad \leq \frac{\eta}{2}\left(\int_{\mathcal{F}_t} \mathcal{Z}(t-s)\mathrm{d}s\right)\left\|\varphi_x\right\|^2 + \frac{1}{2\eta}\int_{\mathcal{F}_t} \mathcal{Z}(t-s)\left\|\varphi_x(s)\right\|^2 \,\mathrm{d}s, t \geq 0
\end{aligned} \tag{4.22}$$

Therefore, (4.20) becomes

$$\begin{aligned}
&\int_0^1 \varphi_x \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(s)-\varphi_x(t)\right) \mathrm{d}s\ \mathrm{d}x \\
&\leq \delta_2 \left\|\varphi_x\right\|^2 + \frac{k}{4\delta_2} \int_{\mathcal{A}_t} \mathcal{Z}(t-s) \int_0^1 \left(\varphi_x(t)-\varphi_x(s)\right)^2\ \mathrm{d}x\ \mathrm{d}s \\
&+ \left(1+\frac{\eta}{2}\right)\left(\int_{\mathcal{F}_t} \mathcal{Z}(t-s)\mathrm{d}s\right)\left\|\varphi_x\right\|^2 + \frac{1}{2\eta}\int_{\mathcal{F}_t} \mathcal{Z}(t-s)\left\|\varphi_x(s)\right\|^2\ \mathrm{d}s, t \geq 0.
\end{aligned} \tag{4.23}$$

The second term in the right hand side of (4.19) can be estimated for $\delta_3 > 0$ as follows

$$\begin{aligned}
&\int_0^1 \left|\int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right) \mathrm{d}s\right|^2\ \mathrm{d}x \\
&\leq \left(1+\frac{1}{\delta_3}\right) k \int_0^1 \int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\ \mathrm{d}s\ \mathrm{d}x \\
&+ (1+\delta_3)\left(\int_{\mathcal{F}_t} \mathcal{Z}(t-s)\mathrm{d}s\right)\int_0^1 \int_{\mathcal{F}_t} \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\ \mathrm{d}s\ \mathrm{d}x, t \geq 0
\end{aligned} \tag{4.24}$$

For the third term, we can write

$$\begin{aligned}
q(t)\left\|\varphi_x\right\|^2 &\int_0^1 \varphi_x \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right) \mathrm{d}s \\
&\leq \frac{q(t)}{2}\left\|\varphi_x\right\|^2\left[\left\|\varphi_x\right\|^2 + \left(\int_0^t \mathcal{Z}(t-s)\mathrm{d}s\right)\int_0^1 \left(\mathcal{Z}\circ\varphi_x\right)\mathrm{d}x\right] \\
&\leq \frac{q(t)}{2}\left\|\varphi_x\right\|^4 + \frac{q(t)}{2}\left(\int_0^t \mathcal{Z}(s)\mathrm{d}s\right)\left\|\varphi_x\right\|^2 \int_0^1 \left(\mathcal{Z}\circ\varphi_x\right)\mathrm{d}x \\
&\leq \frac{2q(t)}{(1-k)^2\left(1-\mathcal{V}^2\right)^2}E^2(t) + \frac{2q(t)}{(1-k)^2\left(1-\mathcal{V}^2\right)^2}E^2(t) \\
&= \frac{4q(t)}{(1-k)^2\left(1-\mathcal{V}^2\right)^2}E^2(t), t \geq 0.
\end{aligned} \tag{4.25}$$

The fourth term can be written

$$\begin{aligned}
&-2\mathcal{V}\int_0^1 \varphi_t \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right) \mathrm{d}s\ \mathrm{d}x \\
&\leq \delta_4 \mathcal{V}\left\|\varphi_t\right\|^2 + \frac{\mathcal{V}}{\delta_4}\int_0^1 \left|\int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right) \mathrm{d}s\right|^2\ \mathrm{d}x
\end{aligned}$$

or

$$\begin{aligned}
&-2\mathcal{V}\int_0^1 \varphi_t \int_0^t \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right) \mathrm{d}s\ \mathrm{d}x \\
&\le \delta_4 \mathcal{V}\left\|\varphi_t\right\|^2+\frac{2\mathcal{V}}{\delta_4} k \int_0^1 \int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\ \mathrm{d}s\ \mathrm{d}x \\
&+\frac{2\mathcal{V}}{\delta_4}\left(\int_{\mathcal{F}_t} \mathcal{Z}(t-s) \mathrm{d}s\right) \int_0^1 \int_{\mathcal{F}_t} \mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\ \mathrm{d}s\ \mathrm{d}x, \quad \delta_4>0
\end{aligned} \tag{4.26}$$

Similarly, we find

$$\begin{aligned}
&-\int_0^1 \varphi_t \int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s)) \mathrm{d}s\ \mathrm{d}x \\
&\le \delta_5\left\|\varphi_t\right\|^2+\frac{1}{4\delta_5} \int_0^t \left|\mathcal{Z}'(s)\right| \mathrm{d}s \int_0^1 \left(\left|\mathcal{Z}'\right| \circ \varphi_x\right) \mathrm{d}x \\
&\le \delta_5\left\|\varphi_t\right\|^2-\frac{\mathcal{Z}(0)}{4\delta_5} \int_0^1\left(\mathcal{Z}' \circ \varphi_x\right) \mathrm{d}x, \quad \delta_5>0, \quad t \ge 0 .
\end{aligned} \tag{4.27}$$

For the remaining terms we have the evaluations

$$\begin{aligned}
&-m\left(\varphi_t+\mathcal{V}\varphi_x\right)(1,t)\left(\int_0^t \mathcal{Z}'(t-s)(\varphi(1,t)-\varphi(1,s)) \mathrm{d}s\ \mathrm{d}x\right) \\
&\le \frac{m}{2\delta_6}\left(\varphi_t^2+\mathcal{V}^2\varphi_x^2\right)(1,t)+m\delta_6 \int_0^t\left|\mathcal{Z}'(s)\right| \mathrm{d}s\left(\left|\mathcal{Z}'\right| \circ \varphi\right)(1,t) \\
&\le \frac{m}{2\delta_6}\left(\varphi_t^2(1,t)+\mathcal{V}^2\varphi_x^2(1,t)\right)-\delta_6 m \mathcal{Z}(0) \int_0^1\left(\mathcal{Z}' \circ \varphi_x\right) \mathrm{d}x, \quad \delta_6>0, \quad t \ge 0 \\
&-m\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right) \varphi_t(1,t)\left(\int_0^t \mathcal{Z}(s) \mathrm{d}s\right) \\
&\le -m\left(\int_0^t \mathcal{Z}(s) \mathrm{d}s\right) \varphi_t^2(1,t)+\frac{m\mathcal{V}}{2}\left(\int_0^t \mathcal{Z}(s) \mathrm{d}s\right)\left(\varphi_t^2+\varphi_x^2\right)(1,t) \\
&\le -m\mathcal{Z}_*\left(1-\frac{\mathcal{V}}{2}\right) \varphi_t^2(1,t)+\frac{m\mathcal{V}}{2} k \varphi_x^2(1,t), \quad t \ge t_*
\end{aligned}$$

and

$$\mathcal{V}\varphi_t(1,t) \int_0^t \mathcal{Z}(t-s)(\varphi(1,t)-\varphi(1,s)) \mathrm{d}s \le \frac{\mathcal{V}}{4\delta_7} \varphi_t^2(1,t)+\delta_7 \mathcal{V} k \int_0^1\left(\mathcal{Z} \circ \varphi_x\right) \mathrm{d}x \tag{4.28}$$

Collecting the previous estimates (4.20)-(4.28) and inserting them in (4.19), we obtain for $t \geq t_*$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Theta_2(t) \leq (\delta_5 + \delta_4\mathcal{V} - \mathcal{Z}_*)\|\varphi_t\|^2 \\
&+ \left(1-\mathcal{V}^2\right)(1-\mathcal{Z}_*)\left[\delta_2 + \left(1+\frac{\eta}{2}\right)\left[\int_{\mathcal{F}_t}\mathcal{Z}(t-s)\mathrm{d}s\right)\right]\|\varphi_x\|^2 \\
&+ \frac{4q(t)}{(1-k)^2\left(1-\mathcal{V}^2\right)^2}E^2(t) + \delta_7\mathcal{V}k\int_0^1(\mathcal{Z}\circ\varphi_x)\,\mathrm{d}x \\
&+ k\left[\left(1-\mathcal{V}^2\right)\left(1+\frac{1}{\delta_3}+\frac{1-\mathcal{Z}_*}{4\delta_2}\right)+\frac{2\mathcal{V}}{\delta_4}\right]\int_0^1\int_{\mathcal{A}_t}\mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\,\mathrm{d}s\,\mathrm{d}x \\
&+ \frac{1}{2\eta}(1-\mathcal{Z}_*)\int_{\mathcal{F}_t}\mathcal{Z}(t-s)\|\varphi_x(s)\|^2\,\mathrm{d}s - \frac{\mathcal{Z}(0)}{4}\left(\frac{1}{\delta_5}+\frac{m}{\delta_6}\right)\int_0^1\left(\mathcal{Z}'\circ\varphi_x\right)\mathrm{d}x \\
&+ \left[\frac{2\mathcal{V}}{\delta_4}+\left(1-\mathcal{V}^2\right)(1+\delta_3)\right]\left(\int_{\mathcal{F}_t}\mathcal{Z}(t-s)\mathrm{d}s\right)\int_0^1\int_{\mathcal{F}_t}\mathcal{Z}(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\,\mathrm{d}s\,\mathrm{d}x \\
&+ m\left[-\mathcal{Z}_*\left(1-\frac{\mathcal{V}}{2}\right)+\frac{1}{2\delta_6}+\frac{\mathcal{V}}{4\delta_7 m}\right]\varphi_t^2(1,t) + \frac{m}{2}\left(k\mathcal{V}+\frac{\mathcal{V}^2}{\delta_6}\right)\varphi_x^2(1,t)
\end{aligned} \tag{4.29}$$

a differentiation of $\Theta_3(t)$ gives

$$\frac{d}{dt}\Theta_3(t) = \mathcal{K}\varphi_x^2(l,s) - \int_0^t \mathcal{Z}(t-s)\varphi_x^2(l,s)ds,\ t \geq 0. \tag{4.30}$$

a differentiation of $\Theta_4(t)$ gives

$$\frac{d}{dt}\Theta_4(t) = \mathcal{K}\|\varphi_x(s)\|^2 - \int_0^t \mathcal{Z}(t-s)\|\varphi_x(s)\|^2 ds,\ t \geq 0. \tag{4.31}$$

Taking into account the estimates (4.8), (4.17), (4.29), (4.30) and (4.31), we infer that for $t \geq t_*$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}L(t) \leq\ & \left\{\tfrac{N}{2}\left(1-\mathcal{V}^2\right) - \mathcal{Z}(0)\left(\tfrac{1}{4\delta_5}+m\delta_6\right)\right\}\textstyle\int_0^1\left(\mathcal{Z}'\circ\varphi_x\right)\mathrm{d}x \\
&+\Big\{(1-\mathcal{Z}_*)\left(1-\mathcal{V}^2\right)\left(\delta_2+\tfrac{1}{2}(2+\eta)\textstyle\int_{\mathcal{F}_t}\mathcal{Z}(t-s)\mathrm{d}s\right) \\
&-\lambda\left[\left(1-\mathcal{V}^2\right)\left(1-\tfrac{k}{2}\right)-\beta-\delta_1\mathcal{V}\right]-k\varsigma\Big\}\|\varphi_x\|^2 \\
&+\left\{\tfrac{\beta+\mathcal{V}^2}{\beta}\lambda+(\delta_5+\delta_4\mathcal{V}-\mathcal{Z}_*)\right\}\|\varphi_t\|^2 \\
&+\tfrac{4q(t)}{(1-k)^2(1-\mathcal{V}^2)^2}E^2(t)+\left(\tfrac{Nq'(t)}{4}-\lambda q(t)\right)\|\varphi_x\|^4 \\
+\ & \left\{\tfrac{\lambda}{2}\left(1-\mathcal{V}^2\right)+\tfrac{}{2\eta}\left(1-\mathcal{V}^2\right)(1-\mathcal{Z}_*)-\varsigma\right\}\textstyle\int_{\mathcal{F}_t}\mathcal{Z}(t-s)\|\varphi_x(s)\|^2\,\mathrm{d}s
\end{aligned}$$

$$+\left\{\left[\frac{2\mathcal{V}}{\delta_4}+\left(1-\mathcal{V}^2\right)(1+\delta_3)\right]\left(\int_{\mathcal{F}_t} h(t-s)\mathrm{d}s\right)+\delta_7\mathcal{V}k\right.$$
$$\left.-\frac{\lambda}{2}\left(1-\mathcal{V}^2\right)\right\}\int_0^1\left(h\circ\varphi_x\right)\mathrm{d}x+k\left\{\left(1-\mathcal{V}^2\right)\left(1+\frac{1}{\delta_3}+\frac{1-h_*}{4\delta_2}\right)+\frac{2\mathcal{V}}{\delta_4}\right\}$$
$$\times\int_0^1\int_{\mathcal{A}_t} h(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\ \mathrm{d}s\ \mathrm{d}x$$
$$+\left\{\frac{m}{2}\left(k\mathcal{V}+\frac{\mathcal{V}^2}{\delta_6}\right)+\lambda\frac{m\mathcal{V}}{4}+Mk-\frac{N\mathcal{V}}{2}\left(1-\mathcal{V}^2\right)\right\}\varphi_x^2(1,t)$$
$$+\left\{m\left[-h_*\left(1-\frac{\mathcal{V}}{2}\right)+\frac{1}{2\delta_6}+\frac{\mathcal{V}}{4\delta_7 m}\right]+\lambda\left[m(1+\mathcal{V})+\frac{\mathcal{V}}{4\delta_1}\right]-\frac{Nv}{2}\right\}\varphi_t^2(1,t)$$
$$-\frac{N3}{4}\mathcal{V}q(t)\left\|\varphi_x\right\|^2\varphi_x^2(1,t)+\left\{\frac{\mathcal{V}}{2}\left(1-\mathcal{V}^2\right)-M\right\}\int_0^t h(t-s)\varphi_x^2(1,s)\mathrm{d}s.$$

Notice that

$$\frac{q'(t)}{4}\left\|\varphi_x\right\|^4\leq\frac{q_{\prime(t)}}{(1-k)^2\left(1-\mathcal{V}^2\right)^2}E^2(t).$$

As in [45], we introduce

$$\mathcal{A}_n=\left\{s\in\mathbb{R}_+: n\mathcal{Z}'(s)+\mathcal{Z}(s)\leq 0\right\},\quad n\in\mathbb{N}$$

and also

$$\tilde{A}_{nt}=\left\{s\in\mathbb{R}_+: 0\leq s\leq t, n\mathcal{Z}'(t-s)+\mathcal{Z}(t-s)\leq 0\right\},\quad n\in\mathbb{N}.$$

Noting that

$$\bigcup_n\mathcal{A}_n=\mathbb{R}_+\backslash\left\{\mathcal{F}_{\mathcal{Z}}\cup\mathcal{N}_{\mathcal{Z}}\right\}$$

where $\mathcal{N}_{\mathcal{Z}}$ is the set where $\mathcal{Z}'$ is not defined and $\mathcal{F}_{\mathcal{Z}}$ is defined in (15), if we denote $\mathcal{F}_n=\mathbb{R}_+\backslash\mathcal{A}_n$, then $\lim_{n\to\infty}\hat{\mathcal{Z}}\left(\mathcal{F}_n\right)=\hat{\mathcal{Z}}\left(\mathcal{F}_{\mathcal{Z}}\right)$ because $\mathcal{F}_{n+1}\subset\mathcal{F}_n$ for all n and $\bigcap_n\mathcal{F}_n=\mathcal{F}_{\mathcal{Z}}\cup\mathcal{N}_{\mathcal{Z}}$. In (39), we take $\mathcal{A}_t:=\tilde{A}_{nt},\mathcal{F}_t:=\tilde{F}_{nt}$. We choose

$$\lambda=\frac{\beta}{\beta+\mathcal{V}^2}\left(\mathcal{Z}_*-\varepsilon\right),\quad\eta=\frac{\beta+\mathcal{V}^2}{\beta},\quad M=\frac{Nv}{2}\left(1-\mathcal{V}^2\right),N=\frac{2\mathcal{G}(0)\left(\frac{1}{4\delta_5}+m\delta_6\right)}{\left(1-\mathcal{V}^2\right)},$$

$$\varsigma=\frac{\lambda}{2}\left(1-\mathcal{V}^2\right)+\frac{}{2\eta}\left(1-\mathcal{V}^2\right)\left(1-\mathcal{Z}_*\right).$$

to get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} L(t) \leq & \left\{ \left(1-\mathcal{Z}_*\right)\left(1-\mathcal{V}^2\right)\left(\delta_2+\frac{1}{2}(2+\eta)\int_{\mathcal{F}_t}\mathcal{Z}(t-s)\mathrm{d}s\right)\right. \\
& \left. -\lambda\left[\left(1-\mathcal{V}^2\right)\left(1-\frac{k}{2}\right)-\beta-\delta_1\mathcal{V}\right]-k\varsigma\right\}\left\|\varphi_x\right\|^2 \\
& +\left\{\frac{\beta+\mathcal{V}^2}{\beta}\lambda+\left(\delta_5+\delta_4\mathcal{V}-\mathcal{Z}_*\right)\right\}\left\|\varphi_t\right\|^2 \\
& +\frac{4q(t)}{(1-k)^2\left(1-\mathcal{V}^2\right)^2}E^2(t)+\left(\frac{Nq'(t)}{4}-\lambda q(t)\right)\left\|\varphi_x\right\|^4 \\
& +\left\{\left[\frac{2\mathcal{V}}{\delta_4}+\left(1-\mathcal{V}^2\right)\left(1+\delta_3\right)\right]\left(\int_{\mathcal{F}_t}h(t-s)\mathrm{d}s\right)+\delta_7\mathcal{V}k\right. \\
& \left.-\frac{\lambda}{2}\left(1-\mathcal{V}^2\right)\right\}\int_0^1\left(h\circ\varphi_x\right)\mathrm{d}x+k\left\{\left(1-\mathcal{V}^2\right)\left(1+\frac{1}{\delta_3}+\frac{1-h_*}{4\delta_2}\right)+\frac{2\mathcal{V}}{\delta_4}\right\} \\
& \times\int_0^1\int_{\mathcal{A}_t}h(t-s)\left(\varphi_x(t)-\varphi_x(s)\right)^2\ \mathrm{d}s\ \mathrm{d}x \\
& +\left\{\frac{m}{2}\left(k\mathcal{V}+\frac{\mathcal{V}^2}{\delta_6}\right)+\lambda\frac{m\mathcal{V}}{4}+Mk-\frac{N\mathcal{V}}{2}\left(1-\mathcal{V}^2\right)\right\}\varphi_x^2(1,t) \\
& +\left\{m\left[-h_*\left(1-\frac{\mathcal{V}}{2}\right)+\frac{1}{2\delta_6}+\frac{\mathcal{V}}{4\delta_7 m}\right]+\lambda\left[m(1+\mathcal{V})+\frac{\mathcal{V}}{4\delta_1}\right]-\frac{Nv}{2}\right\}\varphi_t^2(1,t).
\end{aligned}
$$

Now for $\beta<\frac{1}{2}\left(1-\mathcal{V}^2\right)(2-k)$, we choose $\hat{\mathcal{Z}}\left(\mathcal{F}_n\right)$ small enough such that

$$
\frac{1}{2}(2+\eta)\left(1-\mathcal{Z}_*\right)\left(1-\mathcal{V}^2\right)k\hat{\mathcal{Z}}\left(\mathcal{F}_n\right)<\delta\frac{\beta\left(1-\mathcal{V}^2\right)}{\left(\beta+\mathcal{V}^2\right)}\mathcal{Z}_*\left[\left(1-\frac{k}{2}\right)-\frac{\beta}{1-\mathcal{V}^2}\right]
$$

Where

$$
\delta=\frac{\left(1-\mathcal{Z}_*\right)k}{4\left[(2-k)-2\beta/\left(1-\mathcal{V}^2\right)\right]\mathcal{Z}_*}
$$

and

$$
\left[\frac{2\mathcal{V}}{\delta_4}+\left(1-\mathcal{V}^2\right)\right]k\hat{\mathcal{Z}}\left(\mathcal{F}_n\right)-\frac{\mathcal{Z}_*}{2}\frac{\beta\left(1-\mathcal{V}^2\right)}{\left(\beta+\mathcal{V}^2\right)}\leq 0
$$

Note that $\delta<1$. For the remaining $1-\delta$ we require that

$$
\frac{(1-\varepsilon)}{2}<(1-\delta)\left(\mathcal{Z}_*-\varepsilon\right)\left[\left(1-\frac{k}{2}\right)-\frac{\beta}{1-\mathcal{V}^2}\right].
$$

To achieve this, we take

$$
\mathcal{Z}_*>\frac{5k}{8\left[1-\beta/\left(1-\mathcal{V}^2\right)\right]-3k}.
$$

We state that β is selected so that $\mathcal{Z}_*<k<1$ and this is true even if we take

$$
\beta<\frac{3}{8}\left(1-\mathcal{V}^2\right)(1-k)
$$

We also need

$$\begin{cases} k\left\{\left(1-\mathcal{V}^2\right)\left(1+\frac{1}{\delta_3}+\frac{1-\mathcal{Z}_*}{4\delta_2}\right)+\frac{2\mathcal{V}}{\delta_4}\right\}<\frac{1}{4n}\left(1-\mathcal{V}^2\right), \\ \frac{m}{2}\left(k\mathcal{V}+\frac{\mathcal{V}^2}{\delta_6}\right)+\frac{\beta}{(\beta+\mathcal{V}^2)}\mathcal{Z}_*\frac{m\mathcal{V}}{4}+Mk-\frac{Nv}{2}\left(1-\mathcal{V}^2\right)<0, \\ m\left[-\mathcal{Z}_*\left(1-\frac{\mathcal{V}}{2}\right)+\frac{1}{2\delta_6}+\frac{\mathcal{V}}{4\delta_7\bar{m}}\right]+\frac{\beta}{(\beta+\mathcal{V}^2)}\mathcal{Z}_*\left(m(1+\mathcal{V})+\frac{\mathcal{V}}{4\delta_1}\right)-\frac{\mathcal{V}}{2}<0. \end{cases}$$

We obtain for $t \geq t_*$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}L(t) \leq & -\alpha_1\left\|\varphi_t\right\|^2-\alpha_2\left\|\varphi_x\right\|^2-\alpha_3\int_0^1\left(\mathcal{Z}\circ\varphi_x\right)\mathrm{d}x-\lambda q(t)\left\|\varphi_x\right\|^4 \\ & +\frac{4q(t)+q_{\prime(t)}}{(1-k)^2\left(1-\mathcal{V}^2\right)^2}E^2(t)-\alpha_4\left(\varphi_t(1,t)+\mathcal{V}\varphi_x(1,t)\right)^2 \end{aligned}$$

for some positive constants $\alpha_1, \alpha_2, \alpha_3$ and α_4.

This implies that

$$L'(t)+\alpha E(t) \leq 0,\ t \geq t_*.$$

for some positive constant α. □

we introduce the functional

$$\Theta_5(t)=\int_0^t \varsigma(t-s)\left\|\varphi_{xx}\right\|^2 ds,\ t \geq 0.$$

Lemma 14. *We have*

$$L(t) \leq \mathcal{C}_1\left(E(t)+\Theta_4+\Theta_5(t)\right),\ t \geq 0 \tag{4.32}$$

for some $\mathcal{C}_1>0$. Moroever, there exists $t_0 \geq t_*$ such that

$$E(t) \leq 1,\ t \geq t_0. \tag{4.33}$$

Proof. Integrating (4.10) on $[t_*, t]$, we get

$$\int_{t_*}^t E(s)ds \leq \frac{L(t_*)}{\alpha_1}=\mathcal{C}_2,\quad t \geq t_*.$$

This yields the integral control

$$\omega=\int_{t_*}^{\infty}\left\|\varphi_{xx}(s)\right\|^2 ds<\infty.$$

We have

$$(t - t_*)\, E(t) \leq \mathcal{C}_2, \;\; t \geq t_*$$

Hence

$$E(t) \leq \frac{\mathcal{C}_2}{1 + t - t_*}, \;\; t \geq t_*.$$

Therefore, there is a condition where $t_0 \geq t_*$

$$E(t) \leq 1, \;\; t \geq t_0.$$

□

Lemma 15. *Along solutions of (4.1), we have*

$$\Theta_5'(t) + \omega \zeta\,(t)\, \chi \left(\frac{\Theta_4(t)}{\omega} \right) \leq \frac{2\varsigma\,(0)}{1 - \mathcal{K}} \mathcal{E}(t), \;\; t \geq 0. \tag{4.34}$$

Proof. Since ζ is decreasing , and σ fulfills the differential inequality (3.4) , we get

$$\begin{aligned}
\Theta_5'(t) &= \int_0^t \sigma'\,(t-s)\, \|\varphi_{xx}\|^2\, ds + \sigma\,(0)\, \|\varphi_{xx}\|^2 \\
&\leq -\int_0^t \zeta\,(t-s)\, \chi\,(\sigma\,(t-s))\, \|\varphi_{xx}\|^2\, ds + \frac{2\varsigma\,(0)}{1-\mathcal{K}} E(t) \\
&\leq -\zeta\,(t) \int_0^t \chi\,(\sigma\,(t-s))\, \|\varphi_{xx}\|^2\, ds + \frac{2\varsigma\,(0)}{1-\mathcal{K}} E(t), \;\; t \geq 0.
\end{aligned}$$

Next, we define

$$I\,(t) = \int_0^t \|\varphi_{xx}\,(s)\,\|^2\, ds, \;\; t \geq 0$$

Applying now the Jensen inequality with $y(s) = \sigma\,(t-s)$ and $\mathcal{Z}(s) = \|\varphi_{xx}\,(s)\|^2 / I\,(t)\,$, we get

$$\begin{aligned}
\int_0^t \chi\,(\sigma\,(t-s))\, \|\varphi_{xx}\,(s)\|^2\, ds &= I\,(t) \int_0^t \chi\,(\sigma\,(t-s))\, \frac{\|\varphi_{xx}\,(s)\|^2}{I\,(t)} ds \\
&\geq I\,(t)\, \chi \left(\int_0^t \sigma\,(t-s)\, \frac{\|\varphi_{xx}\,(s)\|^2}{I\,(t)} ds \right), \;\; t \geq 0.
\end{aligned}$$

Since χ is convex with $\chi\,(0) = 0$, then $\theta\chi(x) \geq \chi(\theta x)$ for $\theta \in [0, 1]$. This implies by taking $\theta = I\,(t)\,/\omega$ that

$$\begin{aligned}
\int_0^t \chi\,(\sigma\,(t-s))\, \|\varphi_{xx}\,(s)\|^2\, ds &\geq \omega\chi \left(\frac{I\,(t)}{\omega} \int_0^t \sigma\,(t-s)\, \frac{\|\varphi_{xx}\,(s)\|^2}{I\,(t)} ds \right) \\
&= \omega\chi \left(\frac{\Theta_5(t)}{\omega} \right), \;\; t \geq 0.
\end{aligned}$$

The proof of Lemma 15 is now complete. □

Theorem 9. *Assume that (**H1**)-(**H5**) hold, $\varphi_0, \varphi_1 \in H_0^1(\Omega)$ and that $\mathcal{R}_g < 1/3$. Then there exist positive constants $a < 1$ and K such that*

$$E(t) \le K\mathcal{B}^{-1}\left(a\int_{\hat{t}}^{t} \zeta\left(s\right) ds\right), \ \forall t \ge \hat{t}$$

where K and $\hat{t}$ depend on $E(0)$.

Proof. Choosing $\alpha_2 > 0$, we present the functional

$$y(t) = L\left(t\right) + \Theta_4(t) + \alpha_2\Theta_5(t), \ t \ge 0$$

which, in light of (4.32), satisfies

$$E(t) \le y(t) \le \left(\mathcal{C}_1 + \alpha_2\right)\left(E(t) + \Theta_4 + \Theta_5(t)\right), \ t \ge 0. \tag{4.35}$$

$$y'(t) + \left(\alpha_1 - \frac{2\varsigma\left(0\right)}{1-\mathcal{K}}\alpha_2\right)E(t) + \alpha_2\omega\zeta\left(t\right)\chi\left(\frac{\Theta_5(t)}{\omega}\right) \le 0, \ t \ge t_*.$$

$\alpha_1 \ge \left(\frac{2\varsigma(0)}{1-\mathcal{K}} + 1\right)\alpha_2$ to obtain

$$y'(t) + \alpha_2 E(t) + \alpha_2\omega\zeta\left(t\right)\chi\left(\frac{\Theta_5(t)}{\omega}\right) \le 0.$$

let

$$t \ge t_0 \ge t_* \ge 1.$$

$$\zeta\left(t\right)\chi\left(E(t)\right) \le \zeta\left(1\right)\chi\left(E(t)\right) \le \zeta\left(1\right)\chi(1)E(t) \le \varrho_1 E(t), \ t \ge t_0 \ge t_*$$

where $\varrho_1 > 1$. Accordingly,

$$\alpha_2 E(t) + \alpha_2\omega\zeta\left(t\right)\chi\left(\frac{\Theta_5(t)}{\omega}\right) \ge \alpha_2\zeta\left(t\right)\left[\frac{1}{\varrho}\chi\left(E(t)\right) + \omega\chi\left(\frac{\Theta_5(t)}{\omega}\right)\right].$$

Exploiting the Jensen inequality, we obtain

$$\alpha_2 E(t) + \alpha_2\omega\zeta\left(t\right)\chi\left(\frac{\Theta_5(t)}{\omega}\right) \ge \alpha_2\mathcal{C}_3\zeta\left(t\right)\chi\left(\frac{\frac{1}{\varrho_1}E(t) + \Theta_4 + \Theta_5(t)}{\mathcal{C}_3}\right)$$

where $\mathcal{C}_3 = \left(\frac{1}{\varrho_1} + \omega\right)$.
Using (4.35) and χ is increasing, to get

$$\chi\left(\frac{\frac{1}{\varrho_1}E(t) + \Theta_4 + \Theta_5(t)}{\mathcal{C}_3}\right) \geq \chi\left(\frac{y(t)}{\varrho_1\left(\mathcal{C}_1 + \alpha_2\right)\mathcal{C}_3}\right).$$

In conclusion

$$y'(t) + \alpha_2\mathcal{C}_3\zeta\left(t\right)\chi\left(\frac{y(t)}{\varrho_1\left(\mathcal{C}_1 + \alpha_2\right)\mathcal{C}_3}\right) \leq 0.$$

Next, we divide by $\varrho_1\left(\mathcal{C}_1 + \alpha_2\right)\mathcal{C}_3$, and we write

$$G'(t) + a\zeta\left(t\right)\chi\left(G(t)\right) \leq 0 \tag{4.36}$$

where

$$G(t) = \frac{y(t)}{\varrho_1\left(\mathcal{C}_1 + \alpha_2\right)\mathcal{C}_3} \text{ and } a = \frac{\alpha_2\mathcal{C}_3}{\varrho_1\left(\mathcal{C}_1 + \alpha_2\right)\mathcal{C}_3} = \frac{\alpha_2}{\varrho_1\left(\mathcal{C}_1 + \alpha_2\right)} < 1.$$

there exists $\hat{t} \geq t_0$, depending on $G(t_0)$, such that

$$G(t) \leq B^{-1}\left(a\int_{\hat{t}}^{t}\zeta\left(s\right)ds\right), \ \forall t \geq t_0$$

and since $E(t) \leq \mathcal{C}_4 G(t)$ for some $\mathcal{C}_4 > 0$, it holds that

$$E(t) \leq K\mathcal{B}^{-1}\left(a\int_{\hat{t}}^{t}\zeta\left(s\right)ds\right), \ \forall t \geq \hat{t}$$

□

Chapter 5

General decay for an axially moving with a logarithmic nonlinearity

5.1 Introduction

In this chapter, we take into account the initial boundary value problem that is

$$\begin{cases} \rho\left(\varphi_{tt} + 2\mathcal{V}\varphi_{xt} + \mathcal{V}^2\varphi_{xx}\right) + EI\varphi_{xxxx} - EI\displaystyle\int_0^t \mathcal{Z}(t-s)\varphi_{xxxx}(s)ds = \kappa\varphi\ln|\varphi|, \\ \quad x \in (0,l),\ t \geq 0, \\ \varphi(0,t) = \varphi_x(0,t) = \varphi_{xx}(l,t) = 0, \quad t \geq 0, \\ \rho\mathcal{V}^2\varphi_x(l,t) + EI\varphi_{xxx}(l,t) - EI\displaystyle\int_0^t \mathcal{Z}(t-s)\varphi_{xxx}(l,s)ds = y(\varphi(l,t)),\ t \geq 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ x\ \in (0,l) \end{cases} \tag{5.1}$$

whereas $\varphi = \varphi(x,t)$ is the beam transversal displacement,
$\mathcal{V}$ is the axial speed ,
EI is the beam flexural rigidity,
ϱ is the beam mass per unit length,
$\mathcal{Z}$ nonnegative functions see [20],
$\varphi_0(x)$, $\varphi_1(x)$are the initial data,
y represents the nonlinear term,
κ ia a small postive real number.

5.2 Preliminaries

The hypotheses that we make for $\mathcal{Z}$ and y are the same as those which we made in the previous chapter.

We present the modified energy linked to (5.1) by

$$\begin{aligned}\mathcal{E}(t) &= \frac{\rho}{2}\|\varphi_t\|^2 - \frac{\rho\mathcal{V}^2}{2}\|\varphi_x\|^2 + \frac{EI}{2}\left(1-\int_0^t \mathcal{Z}(s)ds\right)\|\varphi_{xx}\|^2 \\ &+\frac{EI}{2}\int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx + y(\varphi(l)) - \frac{\kappa}{2}\int_0^l |\varphi(s)|^2\ln|\varphi(s)|ds + \frac{\kappa}{4}\|\varphi\|^2, \end{aligned} \tag{5.2}$$

where $\|.\|$ is the L^2-norm and

$$(\mathcal{Z}\circ\varphi)(t) = \int_0^t \mathcal{Z}(t-s)|\varphi(t)-\varphi(s)|^2 ds,\ t\geq 0.$$

Lemma 16. *For $\epsilon_0 \in (0,1)$ then, there exists $d_{\epsilon_0} > 0$ such that*

$$s|\ln s| \leq s^2 + d_{\epsilon_0}s^{1-\epsilon_0}, \forall s > 0. \tag{5.3}$$

Lemma 17. *(See [23]) Let φ be any function in $H_0^1(0,l)$ and $a>0$. Then,*

$$\int_0^l \varphi^2\ln|\varphi(s)|dx \leq \frac{1}{2}\|\varphi\|^2 ln\|\varphi\|^2 + \frac{a^2}{2\pi}\|\varphi_t\|^2 - (1+lna)\|\varphi\|^2. \tag{5.4}$$

Proposition 7. *Under the hypotheses (H1)-(H3) and*

$$\mathcal{V}^2 < \frac{EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|\varphi\|^2}{2}+l^4(1+\ln a)\right]}{\rho l^2}$$

and then

$$EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|\varphi\|^2}{2}+l^4(1+\ln a)\right] > 0,$$

we have

$$\mathcal{E}(t)\geq 0,\ t\geq 0.$$

Proof. We apply lemma 17 and Poincaré inequality leads to

$$\begin{aligned}\mathcal{E}(t) &\geq \frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|\varphi\|^2}{2}+l^4(1+\ln a)+EI-\rho\mathcal{V}^2l^2-EI\int_0^t\mathcal{Z}(s)ds\right]\|\varphi_{xx}\|^2 \\ &+\frac{EI}{2}\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx+\frac{\rho}{2}\|\varphi_t\|^2+y(\varphi(l)),\ t\geq 0.\end{aligned} \tag{5.5}$$

Since $\mathcal{V}^2 < \frac{EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|\varphi\|^2}{2}+l^4(1+\ln a)\right]}{\rho l^2}$,
and $EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|\varphi\|^2}{2}+l^4(1+\ln a)\right] > 0$. We get

$$\mathcal{E}(t)\geq 0,\ t\geq 0.$$

□

Lemma 18. *Let $y \in L^p(\mathbf{R})$ and $\mathcal{Z} \in L^q(\mathbf{R})$ with $1 \leq p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then $(y * \mathcal{Z}) \in L^r(\mathbf{R})$ and*

$$\|y * \mathcal{Z}\|_{L^r} \leq \|y\|_{L^p} \|\mathcal{Z}\|_{L^q}.$$

Using the Faedo Galerkin method, the problem's well-posedness(5.1)can be demonstrated (see [34]).

5.3 Exponential Stability

In this section, we shall state and prove our main result.

Lemma 19. *The modified energy $\mathcal{E}(t)$ satisfies*

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &\leq \frac{EI}{2}\int_0^l \left(\mathcal{Z}' \circ \varphi_{xx}\right)(t)dx - \frac{\rho\mathcal{V}}{2}\varphi_t^2(l) - \frac{\rho\mathcal{V}^3}{2}\varphi_x^2(l) \\ &- \frac{EIv}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\varphi_{xx}^2(0) - \frac{EIv}{2}\left(\mathcal{Z} \circ \varphi_{xx}\right)(0), \ t \geq 0. \end{aligned} \tag{5.6}$$

Proof.

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\int_0^l \widetilde{\mathcal{E}}(x,t)dx + \frac{d}{dt}y(\varphi(l)) = \int_0^l \frac{d}{dt}\widetilde{\mathcal{E}}(x,t)dx + \frac{d}{dt}y(\varphi(l)) \\ &= \int_0^l \left[\frac{\partial}{\partial t}\widetilde{\mathcal{E}}(x,t) + \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\widetilde{\mathcal{E}}(x,t)\right]dx + \frac{d}{dt}y(\varphi(l)) \\ &= \int_0^l \frac{\partial}{\partial t}\widetilde{\mathcal{E}}(x,t)dx + \mathcal{V}\,\widetilde{\mathcal{E}}(x,t)\Big|_0^l + \frac{d}{dt}y(\varphi(l)) \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} \widetilde{\mathcal{E}}(x,t) &= \frac{\rho}{2}\varphi_t^2(x,t) - \frac{\rho\mathcal{V}^2}{2}\varphi_x^2(x,t) + \frac{EI}{2}\left(1 - \int_0^t \mathcal{Z}(s)ds\right)\varphi_{xx}^2(x,t) \\ &+ \frac{EI}{2}\left(\mathcal{Z} \circ \varphi_{xx}\right)(x,t) - \frac{\kappa}{2}\varphi^2 \ln|\varphi| + \frac{\kappa}{4}\varphi^2, \ x \in [0,l], \ t \geq 0. \end{aligned}$$

Using (5.1)and (5.7) together means

$$\frac{d}{dt}\mathcal{E}(t) = -\int_0^l \varphi_t \left[2\rho\mathcal{V}\varphi_{xt} + \rho\mathcal{V}^2\varphi_{xx} + EI\left(\varphi_{xxxx} - \int_0^t \mathcal{Z}(t-s)\varphi_{xxxx}(s)ds\right)\right]dx$$

$$-\rho\mathcal{V}^2\int_0^l \varphi_x\varphi_{xt}dx + EI\left(1-\int_0^t \mathcal{Z}(s)ds\right)\int_0^l \varphi_{xxt}\varphi_{xx}dx - \frac{EI}{2}\mathcal{Z}(t)\int_0^l \varphi_{xx}^2 dx$$
$$+\frac{EI}{2}\int_0^l \frac{\partial}{\partial t}\left(\mathcal{Z}\circ\varphi_{xx}\right)(t)dx - \frac{\partial}{\partial t}\left(\frac{\kappa}{2}\int_0^l |\varphi(s)|^2\ln|\varphi(s)|\right)ds$$
$$+\frac{\partial}{\partial t}\left(\frac{\kappa}{4}\|\varphi\|^2\right) + \varphi_t(l)y(\varphi(l)) + \mathcal{V}\,\widetilde{\mathcal{E}}(x,t)\Big|_0^l,\ t\geq 0.$$

Integrating by parts from 0 to l and boundary conditions in (5.1), we obtain

$$\frac{d}{dt}\mathcal{E}(t) = -\rho\mathcal{V}\varphi_t^2(l) - \varphi_t(l)\left[\rho\mathcal{V}^2\varphi_x(l) + EIw_{xxx}(l)\right.$$
$$\left.-EI\int_0^t \mathcal{Z}(t-s)\varphi_{xxx}(l,s)ds\right] + EI\int_0^l \varphi_{xxt}\int_0^t \mathcal{Z}(t-s)\varphi_{xx}(s)dsdx$$
$$-EI\left(\int_0^t \mathcal{Z}(s)ds\right)\int_0^l \varphi_{xxt}\varphi_{xx}dx - \frac{EI}{2}\mathcal{Z}(t)\int_0^1 \varphi_{xx}^2dx + \varphi_t(l)y(\varphi(l))$$
$$+\frac{EI}{2}\int_0^l \frac{\partial}{\partial t}\left(\mathcal{Z}\circ\varphi_{xx}\right)(t)dx - \frac{\partial}{\partial t}\left(\frac{\kappa}{2}\int_0^l |\varphi(s)|^2\ln|\varphi(s)|\right)ds + \frac{\partial}{\partial t}\left(\frac{\kappa}{4}\|\varphi\|^2\right) + \mathcal{V}\,\widetilde{\mathcal{E}}(x,t)\Big|_0^l,\ t\geq 0.$$

We have

$$\widetilde{\mathcal{E}}(x,t)\Big|_0^l = \frac{\rho}{2}\varphi_t^2(l) - \frac{\rho\mathcal{V}^2}{2}\varphi_x^2(l) - \frac{EI}{2}\left(1-\int_0^t \mathcal{Z}(s)ds\right)\varphi_{xx}^2(0)$$
$$-\frac{EI}{2}\left(\mathcal{Z}\circ\varphi_{xx}\right)(0),\ t\geq 0. \tag{5.8}$$

Clearly

$$\frac{\partial}{\partial t}\left(\mathcal{Z}\circ\varphi_{xx}\right)(x,t) = \left(\mathcal{Z}'\circ\varphi_{xx}\right)(x,t) - 2w_{xxt}(x,t)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}(x,s)ds$$
$$+2\left(\int_0^t \mathcal{Z}(s)ds\right)\varphi_{xxt}(x,t)\varphi_{xx}(x,t),\ t\geq 0. \tag{5.9}$$

Consequently, by using (5.8), (5.9) and the boundary conditions in (5.1). The proof is completed.
We define

$$\Psi_1(t) = \rho\int_0^l \varphi_t dx + \frac{\rho\mathcal{V}}{2}\varphi^2(l),\ t\geq 0,$$

$$\Psi_2(t) = -\rho\int_0^l \varphi_t\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))dsdx,\ t\geq 0,$$

$$\Psi_3(t) = \int_0^t\left(\int_t \mathcal{Z}(\tau-s)d\tau\right)\left\|\varphi_{xx}(s)\right\|^2 ds,\ t\geq 0.$$

$$\Psi_4(t) = \int_0^t\left(\int_t^\infty \mathcal{Z}(\tau-s)d\tau\right)\varphi_{xx}^2(l,s)ds,\ t\geq 0,$$

and

$$L(t) = \mathcal{M}\mathcal{E}(t) + \lambda\Psi_1(t) + \Psi_2(t) + \mu\Psi_3(t) + \gamma\Psi_4(t),\ t \geq 0$$

where λ, μ and γ are all positive constants.. □

Proposition 8. *If*

$$\mathcal{V}^2 < \frac{EI(1-k) + \frac{\kappa}{2}\left[\frac{l^4}{2} - \frac{a^2l^2}{2\pi} - \frac{l^4 \ln\|\varphi\|^2}{2} + l^4(1+\ln a)\right]}{\rho l^2},$$

then we have

$$\mathcal{E}(t) \leq L(t) \leq C(\mathcal{E}(t) + \Psi_3(t) + \gamma\Psi_4(t)),\ t \geq 0$$

for some $C > 0$ *and*

$$L'(t) + \alpha_1\mathcal{E}(t) \leq 0,\ t \geq 0 \tag{5.10}$$

for some $\alpha_1 > 0$.

Proof. First, Young inequality and Poincaré inequality leads to

$$\begin{aligned}\Psi_1(t) &\leq \frac{\rho}{2}\|\varphi_t\|^2 + \frac{\rho}{2}\|\varphi\|^2 + \frac{\rho\mathcal{V}}{2}\varphi^2(l)\\ &\leq \frac{\rho}{2}\|\varphi_t\|^2 + \frac{l\rho}{2}(l+\mathcal{V})\|\varphi_x\|^2.\end{aligned}$$

Similarly , we get

$$\begin{aligned}\Psi_2(t) &\leq \frac{\rho}{2}\|\varphi_t\|^2 + \frac{\rho}{2}\int_0^l\left(\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))ds\right)^2 dx\\ &\leq \frac{\rho}{2}\|\varphi_t\|^2 + \frac{\rho k l^4}{2}\int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx.\end{aligned}$$

Then, Lemma 3 and Lemma 5 allow us to write

$$\begin{aligned}L(t) \leq\ & \frac{\rho}{2}(\mathcal{M}+\lambda)\|\varphi_t\|^2 + \frac{\rho}{2}\left[\lambda\rho(l+\mathcal{V}) - \mathcal{M}\mathcal{V}^2 + \frac{\mathcal{M}\kappa l^2}{2}\right]\|\varphi_x\|^2 + \frac{MEI}{2}\int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx\\ &+ \left(\frac{\rho k l^4}{2} + MEI\right)\int_0^l (\mathcal{Z}\circ\varphi_{xx})(t)dx - \frac{\mathcal{M}\kappa}{2}\int_0^l |\varphi(s)|^2\ln|\varphi(s)|ds + \mu\Psi_3(t) + \gamma\Psi_4(t),\ t \geq 0.\end{aligned}$$

This implies that $\mathcal{E}(t) \leq C\left(L(t) + \Psi_3(t) + \Psi_4(t)\right),\ t \geq 0$,
and

$$L(t) - \mathcal{E}(t) = (\mathcal{M}-1)\mathcal{E}(t) + \lambda\Psi_1(t) + \Psi_2(t) + \mu\Psi_3(t) + \gamma\Psi_4(t)$$

$$\geq \frac{\rho}{2}(\mathcal{M}+\lambda)\|\varphi_t\|^2+(\mathcal{M}-1)y(\varphi(l))+\left[(\mathcal{M}-1)\frac{EI}{2}-\frac{\rho k l^4}{2}\right]\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx$$
$$+\frac{1}{2}\left\{\left[(\mathcal{M}-1)\left(\frac{EI}{l^2}(1-K)-\rho\mathcal{V}^2+\frac{\mathcal{M}\kappa l^2}{2}\right)\right]-\lambda l\rho(l+\mathcal{V})\right\}\|\varphi_x\|^2$$
$$-\frac{\kappa(\mathcal{M}-1)}{2}\int_0^l|\varphi(s)|^2\ln|\varphi(s)|ds+\mu\Psi_3(t)+\gamma\Psi_4(t),\ t\geq 0.$$

Provided that $\mathcal{V}^2 < EI(1-\mathcal{K})/\varrho l^2$ and choose the smaller λ , such that $\lambda \leq \min\{\mathcal{M}-1,$ $\left[(\mathcal{M}-1)EI(1-\mathcal{K})-l^2\varrho\mathcal{V}^2\right]/l^3\varrho(1+\mathcal{V})\}$ and $\mathcal{M}\geq\max\{1, \frac{\lambda l^3 p(l+\mathcal{V})}{EI(1-\mathcal{K})-l^2\varrho^2}+1, \frac{\varrho k l^4}{EI}+1, \lambda+1\}$, This implies that $L(t)-\mathcal{E}(t)\geq 0$.

We will now show the second assertion.

$$\begin{aligned}\frac{d}{dt}\Psi_1(t) &= \int_0^l\frac{d}{dt}\widetilde{\Psi_1}(x,t)dx=\int_0^l\left(\frac{\partial}{\partial t}\widetilde{\Psi_1}(x,t)\right)dx+\mathcal{V}\widetilde{\Psi_1}(x,t)\Big|_0^l\\ &= \rho\|\varphi_t\|^2+\rho\int_0^l ww_{tt}dx-\rho\mathcal{V}\varphi_t(l)\varphi(l)+\mathcal{V}\widetilde{\Psi_1}(x,t)\Big|_0^l,\ t\geq 0,\end{aligned}\tag{5.11}$$

where

$$\widetilde{\Psi_1}(x,t)=\rho\varphi(x,t)\varphi_t(x,t),\ t\geq 0.$$

The equation in (5.1) and an integration by parts we find

$$\frac{d}{dt}\Psi_1(t)\leq\rho\|\varphi_t\|^2+2\rho\mathcal{V}\int_0^l\varphi_x\varphi_t dx+\rho\mathcal{V}^2\|\varphi_x\|^2-EI\|\varphi_{xx}\|^2-\varphi(l)y(\varphi(l))$$
$$+EI\int_0^l\varphi_{xx}\int_0^t\mathcal{Z}(t-s)\varphi_{xx}(s)dsdx+\frac{\kappa}{2}\int_0^l\varphi^2(s)\ln|\varphi(s)|ds,\ t\geq 0.\tag{5.12}$$

$$2\rho\mathcal{V}\int_0^l\varphi_x\varphi_t dx\leq\rho\|\varphi_t\|^2+\rho\mathcal{V}^2\|\varphi_x\|^2,\ t\geq 0.\tag{5.13}$$

The insertion of (5.13) into (5.12) and Lemma 10, we find

$$\frac{d}{dt}\Psi_1(t)\leq 2\rho\|\varphi_t\|^2+2\rho\mathcal{V}^2\|\varphi_x\|^2-EI\left(1-\frac{k}{2}\right)\|\varphi_{xx}\|^2$$
$$+\frac{EI}{2}\int_0^t\mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2ds-\frac{EI}{2}\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx$$
$$-\varphi(l)y(\varphi(l))+\frac{\kappa}{2}\int_0^l\varphi^2(s)\ln|\varphi(s)|ds\tag{5.14}$$

For $\Psi_2(t)$, we have

$$\frac{d}{dt}\Psi_2(t)=\int_0^l\frac{d}{dt}\widetilde{\Psi_2}(x,t)dx=\int_0^l\left(\frac{\partial}{\partial t}\widetilde{\Psi_2}(x,t)\right)dx+\mathcal{V}\widetilde{\Psi_2}(x,t)\Big|_0^l$$

$$= -\rho \int_0^l \varphi_{tt} \int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))dsdx - \rho \left(\int_0^t \mathcal{Z}(s)ds \right) \|\varphi_t\|^2$$

$$-\rho \int_0^l \varphi_t \int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))dsdx + \mathcal{V}\widetilde{\Psi_2}(x,t)\Big|_0^l, \ t \geq 0 \tag{5.15}$$

where

$$\widetilde{\Psi_2}(x,t) = -\rho\varphi_t(x,t) \int_0^t \mathcal{Z}(t-s)(\varphi(x,t)-\varphi(x,s))ds, \ t \geq 0. \tag{5.16}$$

$$\widetilde{\Psi_2}(x,t)\Big|_0^l = -\rho\varphi_t(l) \int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds, \ t \geq 0. \tag{5.17}$$

By taking the total derivative of $\Theta_2(t)$, using integration by parts and (5.17), we obtain

$$\begin{aligned}
\frac{d}{dt}\Psi_2(t) &= EI\left(1 - \int_0^t \mathcal{Z}(s)ds\right) \int_0^l \varphi_{xx} \int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))ds \\
&\quad + EI \int_0^l \left| \int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))ds \right|^2 dx \\
&\quad - \rho\mathcal{V}^2 \int_0^l \varphi_x \int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
&\quad - 2\rho\mathcal{V} \int_0^l \varphi_t \int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
&\quad - \rho \int_0^l \varphi_t \int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))dsdx - \rho\left(\int_0^t \mathcal{Z}(s)ds\right)\|\varphi_t\|^2 \\
&\quad + y(\varphi(l)) \int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds \\
&\quad + \rho\mathcal{V}\varphi_t(l) \int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds \\
&\quad - \kappa \int_0^l \varphi \ln|\varphi| \int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))dsdx, \ t \geq 0,
\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}
&\int_0^l \varphi_{xx} \int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))dsdx \\
= &\int_0^l \varphi_{xx} \left(\int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))ds + \int_{\mathcal{Y}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))ds \right) dx \\
\leq &\int_0^l \varphi_{xx} \int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))dsdx
\end{aligned}$$

$$+\left(\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)ds\right)\|\varphi_{xx}\|^2 - \int_0^l \varphi_{xx}\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\varphi_{xx}(s)dsdx,\ t\geq 0. \tag{5.19}$$

We easily see that

$$\begin{aligned}
&\int_0^l \varphi_{xx}\int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))dsdx \\
\leq\ & \eta_1\|\varphi_{xx}\|^2 + \frac{k}{4\eta_1}\int_0^l\int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx,\ \eta_1>0.
\end{aligned}$$

and

$$\begin{aligned}
&-\int_0^l \varphi_{xx}\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\varphi_{xx}(s)dsdx \\
\leq\ & \frac{1}{2}\|\varphi_{xx}\|^2 + \frac{k}{2}\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^l \varphi_{xx}\int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))dsdx \\
\leq\ & \left((\frac{1}{2}+\eta_1)+k\hat{\mathcal{Z}}(\mathcal{Y})\right)\|\varphi_{xx}\|^2 + \frac{k}{4\eta_1}\int_0^l\int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx
\end{aligned}$$

$$+\frac{k}{2}\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds,\ t\geq 0 \tag{5.20}$$

It is obvious that by the definition of $\hat{\mathcal{Z}}(\mathcal{Y})$ in (3.2), we have

$$\begin{aligned}
&\int_0^l \left|\int_0^t \mathcal{Z}(t-s)(\varphi_{xx}(t)-\varphi_{xx}(s))ds\right|^2 dx \\
\leq\ & \left(1+\frac{1}{\eta_2}\right)k\int_0^l\int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx \\
&+(1+\eta_2)\,k\hat{\mathcal{Z}}(\mathcal{Y})\int_0^l\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2 dsdx,\ \eta_2>0.
\end{aligned} \tag{5.21}$$

For the 3^{rd} term

$$\begin{aligned}
&-\int_0^l \varphi_x\int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
=\ & -\left(\int_0^t \mathcal{Z}(s)ds\right)\|\varphi_x\|^2 + \int_0^l \varphi_x\int_0^t \mathcal{Z}(t-s)\varphi_x(s)dsdx
\end{aligned}$$

$$\leq \left(\frac{1}{2}-\mathcal{Z}_*\right)\|\varphi_x\|^2+\frac{l^2k}{2}\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds,\ t\geq t_*,\ \eta_3>0. \tag{5.22}$$

The fourth term is estimated as :

$$\begin{aligned}
& -2\rho\mathcal{V}\int_0^l \varphi_t \int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
= & -2\rho\mathcal{V}\int_0^l \varphi_t \left(\int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))ds+\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))ds\right)dx
\end{aligned}$$

$$\begin{aligned}
= & -2\rho\mathcal{V}\int_0^l \varphi_t \int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
& -2\rho\mathcal{V}\left(\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)ds\right)\int_0^l \varphi_t\varphi_x dx+2\rho\mathcal{V}\int_0^l \varphi_t\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\varphi_x(s)dsdx,
\end{aligned} \tag{5.23}$$

or

$$\begin{aligned}
& -2\rho\mathcal{V}\int_0^l \varphi_t \int_{\mathcal{A}_t} \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
\leq & \ \eta_3\rho\|\varphi_t\|^2+\frac{\rho\mathcal{V}^2l^2k}{\eta_3}\int_0^l\int_{\mathcal{A}_t} \mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2 dsdx,\ t\geq 0,
\end{aligned} \tag{5.24}$$

$$-2\rho\mathcal{V}\left(\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)ds\right)\int_0^l \varphi_t\varphi_x dx \leq \rho k\hat{\mathcal{Z}}(\mathcal{Y})\|\varphi_t\|^2+\rho\mathcal{V}^2k\hat{\mathcal{Z}}(\mathcal{Y})\|\varphi_x\|^2, \tag{5.25}$$

and

$$2\rho\mathcal{V}\int_0^l \varphi_t\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\varphi_x(s)dsdx \leq \rho\|\varphi_t\|^2+\rho\mathcal{V}^2l^2k\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds, \tag{5.26}$$

Therefore (5.23) becomes

$$\begin{aligned}
& -2\rho\mathcal{V}\int_0^l \varphi_t \int_0^t \mathcal{Z}(t-s)(\varphi_x(t)-\varphi_x(s))dsdx \\
\leq & \ \rho\left(1+\eta_3+k\hat{\mathcal{Z}}(\mathcal{Y})\right)\|\varphi_t\|^2+\rho\mathcal{V}^2k\hat{\mathcal{Z}}(\mathcal{Y})\|\varphi_x\|^2+\rho\mathcal{V}^2l^2k\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2 ds \\
& +\frac{\rho\mathcal{V}^2l^2k}{\eta_3}\int_0^l\int_{\mathcal{A}_t} \mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2 dsdx.
\end{aligned} \tag{5.27}$$

For the fifth term, we have

$$\begin{aligned}
& -\rho\int_0^l \varphi_t\int_0^t \mathcal{Z}'(t-s)(\varphi(t)-\varphi(s))dsdx \\
\leq & \ \eta_4\rho\|\varphi_t\|^2-\frac{\rho\mathcal{Z}(0)l^4}{4\eta_4}\int_0^l \left(\mathcal{Z}'\circ\varphi_{xx}\right)(t)dx,\ \eta_4>0,\ t\geq 0.
\end{aligned} \tag{5.28}$$

For the 6^{th} term, find

$$\begin{aligned}
&y(\varphi(l))\int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds\\
&= y(\varphi(l))\left(\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds+\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds\right)\\
&= y(\varphi(l))\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds-y(\varphi(l))\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)\varphi(l,s)ds\\
&+\left(\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)ds\right)\varphi(l)y(\varphi(l)) \qquad (5.29)
\end{aligned}$$

or

$$\begin{aligned}
&y(\varphi(l))\int_{\mathcal{A}_t}\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds\\
\leq\ &\eta_5\left|y(\varphi(l))\right|^2+\frac{l^3k}{4\eta_5}\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2dsdx,\ t\geq 0
\end{aligned}$$

and

$$\begin{aligned}
&-y(\varphi(l))\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)\varphi(l,s)ds\\
\leq\ &\frac{\left|y(\varphi(l))\right|^2}{2}+\frac{l^3k}{2}\int_0^t\mathcal{Z}(t-s)\left\|\varphi_{xx}(s)\right\|^2ds,\ \eta_4>0,\ t\geq 0.
\end{aligned}$$

and

$$\begin{aligned}
&\left|y(\varphi(l))\right|^2\leq 2m^2(\left|\varphi(l)\right|^2+\left|\varphi(l)\right|^{2(\alpha+1)})\\
\leq\ &2m^2l^3\left[\left\|\varphi_{xx}\right\|^2+\left(\frac{2}{EIk-\rho\mathcal{V}^2l^2}E(0)\right)^{\alpha}\left\|\varphi_{xx}\right\|^2\right]=\beta\left\|\varphi_{xx}\right\|^2.
\end{aligned}$$

Hence, (5.29) becomes

$$\begin{aligned}
&y(\varphi(l))\int_0^t\mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds\\
\leq\ &(\eta_5+\frac{\beta}{2})\left\|\varphi_{xx}\right\|^2+\frac{l^3k}{2}\int_0^t\mathcal{Z}(t-s)\left\|\varphi_{xx}(s)\right\|^2ds\\
&+k\hat{\mathcal{Z}}(\mathcal{Y})\varphi(l)y(\varphi(l))+\frac{l^3k}{4\eta_5}\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)\left|\varphi_{xx}(t)-\varphi_{xx}(s)\right|^2dsdx. \qquad (5.30)
\end{aligned}$$

The seventh term can be written as follows

$$\begin{aligned} &\rho\mathcal{V}\varphi_t(l)\int_0^t \mathcal{Z}(t-s)(\varphi(l,t)-\varphi(l,s))ds \\ \leq\ & \frac{\rho\mathcal{V}}{4\eta_7}\varphi_t^2(l)+\eta_7 l^3 k\int_0^l (\mathcal{Z}\circ\varphi_{xx})\,dx,\ \eta_7>0,\ t\geq 0, \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} &-\kappa\int_0^l \varphi\ln|\varphi|\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))dsdx \\ \leq\ & \kappa\int_0^l \left(\varphi^2+d_{\epsilon_0}\sqrt{|\varphi|}\right)\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))dsdx \\ \leq\ & \kappa\left(\eta_8\int_0^l(\varphi^2+d_{\epsilon_0}\sqrt{|\varphi|})^2dx+\frac{1}{4\eta_8}\int_0^l|\int_0^t \mathcal{Z}(t-s)(\varphi(t)-\varphi(s))ds|^2dx\right) \\ \leq\ & \kappa\eta_8(l^8+l^4)\|\varphi_{xx}\|^2+\frac{\kappa k l^4}{4\eta_8}\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx. \end{aligned} \tag{5.32}$$

The insertion of (5.20)-(5.22), (5.27), (5.28), (5.30), (5.32) and (5.31) into (5.18), we obtain

$$\begin{aligned} &\frac{d}{dt}\Psi_2(t)\leq-\frac{\mathcal{Z}(0)l^4}{4\eta_4}\int_0^l(\mathcal{Z}'\circ\varphi_{xx})(t)dx+\rho\left(-\mathcal{Z}_*+\eta_4+\frac{1}{2}+\eta_3+k\hat{\mathcal{Z}}(\mathcal{Y})\right)\|\varphi_t\|^2 \\ &+\rho\mathcal{V}^2\left(-\mathcal{Z}_*+\frac{1}{2}+k\hat{\mathcal{Z}}(\mathcal{Y})\right)\|\varphi_x\|^2+\{(\frac{1}{2}+\eta_1)(1-\mathcal{Z}_*)EI+(\frac{\beta}{2}+\eta_5\frac{\beta}{2}) \\ &+\kappa\eta_8(l^8+l^4)+EI(1-\mathcal{Z}_*)k\hat{\mathcal{Z}}(\mathcal{Y})\}\|\varphi_{xx}\|^2+(\frac{\kappa k l^4}{4\eta_8}+\eta_7 l^3 k)\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx \\ &+\frac{1}{2}\left((1-\mathcal{Z}_*)kEI+5\rho\mathcal{V}^2l^2k+l^3k\right)\int_0^t\mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2ds \\ &+(1+\eta_2)EIk\hat{\mathcal{Z}}(\mathcal{Y})\int_0^l\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2dsdx+k\left\{(1-\mathcal{Z}_*)\frac{EI}{4\eta_1}\right. \\ &\left.+\left(1+\frac{1}{\eta_2}\right)EI+\frac{\rho\mathcal{V}^2l^2}{\eta_3}+\frac{l^3}{4\eta_5}\right\}\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2dsdx \\ &+k\hat{\mathcal{Z}}(\mathcal{Y})\varphi(l)y(\varphi(l))+\frac{\rho\mathcal{V}}{4\eta_7}\varphi_t^2(l),\ t\geq t_*>0, \end{aligned} \tag{5.33}$$

where $\beta=2m^2l^3\left[1+\left(\frac{2E(0)}{kEI-l^2\rho\mathcal{V}^2}\right)^{\alpha}\right]$ for some positive constants η_i, $i=1,...7$.

A differentiation of $\Psi_3(t)$ gives

$$\frac{d}{dt}\Psi_3(t)=k\|\varphi_{xx}\|^2-\int_0^t\mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2ds,\ t\geq 0. \tag{5.34}$$

The total derivative of $\Psi_3(t)$ it is calculated as follows

$$\frac{d}{dt}\Psi_3(t) = \int_0^l \left(\frac{\partial}{\partial t}\widetilde{\Psi_3}(x,t)\right) dx + \mathcal{V}\widetilde{\Psi_3}(x,t)\Big|_0^l, \; t \geq 0 \tag{5.35}$$

where

$$\tilde{\Psi}_3(x,t) = \int_0^t \left(\int_t^\infty \mathcal{Z}(\tau - s)d\tau\right) \varphi_{xx}^2(x,s)ds, \; t \geq 0.$$

Clearly

$$\frac{d}{dt}\Psi_3(t) \leq k\left\|\varphi_{xx}\right\|^2 - \int_0^t \mathcal{Z}(t-s)\int_0^l \varphi_{xx}^2(s)dxds + \mathcal{V}\int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds, \; t \geq 0,$$

and

$$\frac{d}{dt}\Psi_4(t) = kw_{xx}^2(l,s) - \int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds, \; t \geq 0. \tag{5.36}$$

According to the relations (5.6), (5.14), (5.33) and (5.36) we have

$$\begin{aligned}
&\frac{d}{dt}L(t) \leq \left(\frac{MEI}{2} - \frac{\mathcal{Z}(0)l^4}{4\eta_4}\right)\int_0^l \left(\mathcal{Z}' \circ \varphi_{xx}\right)(t)dx \\
&+\rho\left[2\lambda + \left(-\mathcal{Z}_* + \eta_4 + (1+\eta_3) + k\hat{\mathcal{Z}}(\mathcal{Y})\right)\right]\left\|\varphi_t\right\|^2 \\
&+\rho\mathcal{V}^2\left[2\lambda + \left(-\mathcal{Z}_* + \frac{1}{2} + k\hat{\mathcal{Z}}(\mathcal{Y})\right)\right]\left\|\varphi_x\right\|^2 \\
&+\Big\{[(\frac{1}{2}+\eta_1)\left(1-\mathcal{Z}_*\right)EI + (\eta_5\frac{\beta}{2} + \frac{\beta}{2}) + \kappa\eta_8(l^8+l^4) + EI\left(1-\mathcal{Z}_*\right)k\hat{\mathcal{Z}}(\mathcal{Y})] - \lambda EI\left(1-\frac{k}{2}\right) \\
&+(\mu+\gamma)k\Big\}\left\|\varphi_{xx}\right\|^2 \\
&+\left(\frac{\kappa k l^4}{4\eta_8} - \lambda\frac{EI}{2} + \eta_7 l^3 k\right) \times \int_0^l \left(\mathcal{Z}\circ\varphi_{xx}\right)(t)dx + (\mu\mathcal{V} - \gamma)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds \\
&+\left\{\lambda\frac{EI}{2} + \frac{k}{2}\left[\left(1-\mathcal{Z}_*\right)EI + 5\rho\mathcal{V}^2l^2 + l^3\right] - \mu\right\}\int_0^t \mathcal{Z}(t-s)\left\|\varphi_{xx}(s)\right\|^2 ds + \\
&(1+\eta_2)EIk\hat{\mathcal{Z}}(\mathcal{Y}) \times \int_0^l\int_{\mathcal{Y}_t} \mathcal{Z}(t-s)\left|\varphi_{xx}(t) - \varphi_{xx}(s)\right|^2 dsdx + k\left[\left(1-\mathcal{Z}_*\right)\frac{EI}{4\eta_1} + \left(1+\frac{1}{\eta_2}\right)EI\right. \\
&\left.+\left(1+\frac{1}{\eta_2}\right)EI + \frac{\rho\mathcal{V}^2l^2}{\eta_3} + \frac{l^3}{4\eta_5}\right]\int_0^l\int_{\mathcal{A}_t} \mathcal{Z}(t-s)\left|\varphi_{xx}(t) - \varphi_{xx}(s)\right|^2 dsdx \\
&+\left\{-\lambda + k\hat{\mathcal{Z}}(\mathcal{Y})\right\}\varphi(l)y(\varphi(l)) + \frac{\rho\mathcal{V}}{2}\left(\frac{1}{2\eta_7} - 1\right)\varphi_t^2(l).
\end{aligned} \tag{5.37}$$

We choose $\mathcal{M}$ to be so large that

$$\frac{MEI}{2} - \frac{\mathcal{Z}(0)l^4}{4\eta_4} \geq \frac{MEI}{4}. \tag{5.38}$$

As in [35], we introduced the collection

$$\mathcal{A}_n = \{s \in \mathbb{R}_+ : n\mathcal{Z}'(s) + \mathcal{Z}(s) \le 0\},\ \mathcal{Y}_n = \mathbb{R}_+ \backslash \mathcal{A}_n$$

and

$$\tilde{A}_{nt} = \{s \in \mathbb{R}_+ : 0 \le s \le t,\ n\mathcal{Z}'(t-s) + \mathcal{Z}(t-s) \le 0\},\ n \in \mathbb{N}.$$

Observe that

$$\bigcup_n \mathcal{A}_n = \mathbb{R}_+ \backslash \{\mathcal{Y}_{\mathcal{Z}} \cup \mathcal{N}_{\mathcal{Z}}\}$$

where $\mathcal{N}_{\mathcal{Z}}$ is the set where $\mathcal{Z}'$ is not defined and $\mathcal{Y}_{\mathcal{Z}}$ is defined in (3.3). Since $\mathcal{Y}_{n+1} \subset \mathcal{Y}_n$ for all n and $\bigcap_n \mathcal{Y}_n = \mathcal{Y}_{\mathcal{Z}} \cup \mathcal{N}_{\mathcal{Z}}$, then $\lim_{n\to\infty} \hat{\mathcal{Z}}(\mathcal{Y}_n) = \hat{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}})$. Taking $\mathcal{A}_t := \tilde{A}_{nt}$, $\mathcal{Y}_t := \tilde{y}_{nt}$, it follows from (3.49) that

$$\begin{aligned}
&\frac{d}{dt}L(t) \le \rho\left[2\lambda + \left(-\mathcal{Z}_* + \eta_4 + (1+\eta_3) + k\hat{\mathcal{Z}}(\mathcal{Y})\right)\right]\|\varphi_t\|^2 \\
&+\rho\mathcal{V}^2\left[2\lambda + \left(-\mathcal{Z}_* + \frac{1}{2} + k\hat{\mathcal{Z}}(\mathcal{Y})\right)\right]\|\varphi_x\|^2 \\
&-\left\{[\lambda EI\left(1-\frac{k}{2}\right) - (\frac{1}{2}+\eta_1)(1-\mathcal{Z}_*)EI - (\frac{\beta}{2}\eta_5 + \frac{\beta}{2}) - \kappa\eta_8(l^8+l^4) - EI(1-\mathcal{Z}_*)k\hat{\mathcal{Z}}(\mathcal{Y})]\right. \\
&\left.-(\mu+\gamma)k\right\}\|\varphi_{xx}\|^2 \\
&+\left(\frac{\kappa k l^4}{4\eta_8} - \lambda\frac{EI}{2} + \eta_7 l^3 k\right)\times\int_0^l(\mathcal{Z}\circ\varphi_{xx})(t)dx + (\mu\mathcal{V}-\gamma)\int_0^t \mathcal{Z}(t-s)\varphi_{xx}^2(l,s)ds \\
&+\left\{\lambda\frac{EI}{2} + \frac{k}{2}\left[(1-\mathcal{Z}_*)EI + 5\rho\mathcal{V}^2l^2 + l^3\right] - \mu\right\}\int_0^t \mathcal{Z}(t-s)\|\varphi_{xx}(s)\|^2ds + \\
&(1+\eta_2)EIk\hat{\mathcal{Z}}(\mathcal{Y})\times\int_0^l\int_{\mathcal{Y}_t}\mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2dsdx + k\left[(1-\mathcal{Z}_*)\frac{EI}{4\eta_1} + \left(1+\frac{1}{\eta_2}\right)EI\right. \\
&\left.+\left(1+\frac{1}{\eta_2}\right)EI + \frac{\rho\mathcal{V}^2l^2}{\eta_3} + \frac{l^3}{4\eta_5} - \frac{MEI}{4n}\right]\int_0^l\int_{\mathcal{A}_t}\mathcal{Z}(t-s)|\varphi_{xx}(t)-\varphi_{xx}(s)|^2dsdx \\
&+\left\{-\lambda + k\hat{\mathcal{Z}}(\mathcal{Y})\right\}\varphi(l)y(\varphi(l)) + \frac{\rho\mathcal{V}}{2}\left(\frac{1}{2\eta_7}-1\right)\varphi_t^2(l).
\end{aligned} \tag{5.39}$$

In (5.39) ,for $\mathcal{Z}_* > k\hat{\mathcal{Z}}(\mathcal{Y}) + \frac{1}{2}$ sufficiently large we take $\lambda = \frac{1}{2}\left(\mathcal{Z}_* - k\hat{\mathcal{Z}}(\mathcal{Y}) - \frac{1}{2} - \varepsilon\right)$. We infer that

$$\frac{\kappa k l^4}{4\eta_8} - \frac{\lambda EI}{2} + (1+\eta_2)EIk\hat{\mathcal{Z}}(\mathcal{Y}_n) + \eta_7 l^3 k \le 0$$

and

$$\frac{1}{2\eta_7} - 1 \le 0.$$

We will need (we neglect η_5 and $\eta_1$as will be chosen small enough)

$$\frac{1}{2}\left(1-\mathcal{Z}_*\right)EI+\frac{\beta}{2}+\kappa\eta_8(l^8+l^4)+EI\left(1-\mathcal{Z}_*\right)k\hat{\mathcal{Z}}(\mathcal{Y})-\lambda EI\left(1-\frac{k}{2}\right)+(\mu+\gamma)k<0.$$

Adding and substraction the term $\frac{\delta}{2}\left(\mathcal{Z}_*-k\hat{\mathcal{Z}}(\mathcal{Y})-\frac{1}{2}-\varepsilon\right)\left(1-\frac{k}{2}\right)EI$, the previous term becomes

$$\left[\left(1-\frac{\delta}{2}\right)\left(\mathcal{Z}_*-k\hat{\mathcal{Z}}(\mathcal{Y})-\frac{1}{2}-\varepsilon\right)EI\left(1-\frac{k}{2}\right)-\mu k\right]$$
$$+\left[\frac{\delta}{2}\left(\mathcal{Z}_*-k\hat{\mathcal{Z}}(\mathcal{Y})-\frac{1}{2}-\varepsilon\right)\left(1-\frac{k}{2}\right)EI-\frac{1}{2}\left(1-\mathcal{Z}_*\right)EI-\frac{\beta}{2}-\left(1-\mathcal{Z}_*\right)k\hat{\mathcal{Z}}(\mathcal{Y}_n)-\gamma k\right]>0.$$

The term is divided into several parts.
The second part has what you need

$$\frac{1}{2}\left(1-\mathcal{Z}_*\right)EI+\frac{\beta}{2}+EI\left(1-\mathcal{Z}_*\right)k\hat{\mathcal{Z}}(\mathcal{Y}_n)+\gamma k<\frac{\delta}{2}\left(\mathcal{Z}_*-k\hat{\mathcal{Z}}(\mathcal{Y})-\frac{1}{2}-\varepsilon\right)\left(1-\frac{k}{2}\right)EI$$

where

$$\delta=\frac{2\left(1-\mathcal{Z}_*\right)EI(\frac{2k}{3}+1)+2\beta+4\gamma k}{\left(\mathcal{Z}_*-\frac{k}{3}-\frac{1}{2}\right)(2-k)}EI.$$

and

$$\mu<\left(1-\frac{\delta}{2}\right)\frac{\left(\mathcal{Z}_*-k\hat{\mathcal{Z}}(\mathcal{Y}_n)-\frac{1}{2}-\varepsilon\right)}{k}\left(1-\frac{k}{2}\right)EI$$

$$\mu<\left(1+\frac{\delta}{2}\right)\frac{\left(\mathcal{Z}_*-k\hat{\mathcal{Z}}(\mathcal{Y}_n)-\frac{1}{2}-\varepsilon\right)}{k}\left(1-\frac{k}{2}\right)EI. \tag{5.40}$$

Note that (ε is ignored)

$$\left(1-\frac{\delta}{2}\right)\frac{\left(\mathcal{Z}_*k\hat{\mathcal{Z}}(\mathcal{Y}_n)-\frac{1}{2}-\varepsilon\right)}{k}\left(1-\frac{k}{2}\right)EI=$$

$$\frac{\left(\mathcal{Z}_*-\frac{k}{3}-\frac{1}{2}\right)(2-k)-(1-\mathcal{Z}_*)EI(\frac{-2k}{3}+1)-2\gamma k-\beta)}{2k}$$

and

$$\frac{2k\gamma+2\beta+(\frac{k}{3}+\frac{3}{2})(2-k)+EI(2\frac{k}{3}+1)}{(2-k)+(2\frac{K}{3}+1)EI}<\mathcal{Z}_*<k. \tag{5.41}$$

$$\beta<\frac{(2-k)(\frac{1}{2}-\frac{k}{3})-\gamma k}{2}.$$

We select at the end $\gamma > \mu\mathcal{V}$. This suggests that

$$L'(t) + \alpha_1 \mathcal{E}(t) \leq 0,\ t \geq t_*.$$

□

Theorem 10. *Assume that* ***(A1)-(A3)*** *and if* $\mathcal{V}^2 < \frac{EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|\varphi\|^2}{2}+l^4(1+\ln a)\right]}{\rho l^2}$*,is sufficiently small,where* A *and* ϱ *are two positive constants that*

$$\mathcal{E}(t) \leq Ae^{-\varrho t},\ t \geq 0.$$

Proof. We can observe that based on the equivalence result (5.10)

$$\frac{d}{dt}L(t) \leq -\frac{\alpha_1}{C}L(t),\ t \geq \bar{t}. \tag{5.42}$$

Can get to by integrating (5.42) over $(\bar{t}, t)$ conducts

$$L(t) \leq L(\bar{t})e^{-\frac{\alpha_1}{C}(t-\bar{t})},\ t \geq \bar{t}.$$

The major outcome so follows once more as a result of (5.10). □

General conclusion

In this thesis, we considered Axially moving systems and analyzed the asymptotic behavior of the solutions of these systems.

In the second chapter, we have dealt with the stabilization of an axially moving string subject to a boundary disturbances. We employ the active disturbance rejection control (ADRC) approach to estimate the disturbance.

In the third chapter, we are proved a general decay of an axially moving viscoelastic beam with a boundary non linear term. Our purpose was to extend the class of the relaxation functions $\mathcal{Z}$ that guaranteeing a general deacy. This type of functions were developed by Conti and Pata in [14] and improved by Kelleche and Feng in [44]. The obtained result improved the previous ones [26]

In the fourth chapter, we studied the stabilization of an axially moving viscoelastic Kirchhoff string.

In the fifth chapter, we studied the stabilization of an axially moving viscoelastic beam with Logarithmic Source Terms. We obtained an asymptotic stability result of global solution, for certain class of relaxation functions.

In the future, it would be interesting to extend the results of the present thesis by the existence and the uniqueness of these problems as well as the we prove the exponential stability.

Bibliography

[1] F. R. Archibald and A. G. Emslie, The vibrations of a string having a uniform motion along its length, J. Appl. Mech. ASME. 25, 347348, (1958).

[2] J. Banasiak and A. Bobrowski, Semigroups of Operators:Theory and Applications, Springer Proceedingsin Mathematics and Statistics.Volume 113, Bedlewo,Poland, October (2013).

[3] J. Barrow and P. Parsons, Inflationary models with logarithmic potentials, Phys. Rev. D 52, 5576–5587 (1995).

[4] S. Berrimi and S. Messaoudi, Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping, Elect J. Diff. Eqns. Vol. 88, 1–10 (2004).

[5] H. Brézis, Analyse Fonctionnelle: Théorie et applications, Dunod, Paris (1999).

[6] M. M. Cavalcanti, V. N. Domingos Cavalcanti and P. Martinez, General decay rate estimates for viscoelastic dissipative systems. Nonlinear Anal TMA. 68(1):177–193 (2008).

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka and FAF Nascimento, Intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional damping. Disc Cont Dyn Syst. 19(7):1987–2012 (2014).

[8] C. H. Chung and C. A. Tan, Active vibration control of the axially moving string by wave cancellation, J. Vib. Acoust. 117, 49–55 (1995).

[9] R. F. Curtain and A. J. Pritchard, Infinite-dimensional linear systems theory. Lecture Notes in Control and Information Sciences, vol. 8. Springer: Berlin, New York, (1978).

[10] J. W. Choi, K. S. Hong and K. J. Yang, Exponential stabilization of an axially moving tensioned strip by passive damping and boundary control. J. Vib. Acoust. 10 (5), 661-682 (2004).

[11] B.D. Coleman and W. Noll, Foundations of linear viscoelasticity. Rev. Modern Phys. 33, 239-249 (1961).

[12] B.D. Coleman and V. J. Mizel, On the general theory of fading memory. Arch. Ration. Mech. Anal. 29, 18-31 (1968).

[13] J. M. Coron and Brigitte dAndrea-Novel, Stabilization of a rotating rody beam without damping. IEEE Trans. Auto. Control 43 (5), 608–619 (1998).

[14] M. Conti and V. Pata, General decay properties of abstract linear viscoelasticity, Z. Angew. Math. Phys. 71(6) (2020).

[15] N. Doudi, Existence and asymptotic behavior of solutions for some hyperbolic equations with time delay. thesis phd. Univ Biskra, (2021).

[16] K. Enqvist and J. McDonald, Q-balls and baryogenesis in the MSSM, Physics Letters B, vol. 425, no. 3-4, pp. 309–321, (1998).

[17] R. F. Fung and C. C. Tseng, Boundary control of an axially moving string via Lyapunov method, J. Dyn. Syst. Meas. Control, 121, 105–110 (1999).

[18] F .R. Fung, J. W. Wu and S. L. Wu, Stabilization of an Axially Moving String by Nonlinear Boundary Feedback, ASME J. Dyn. Syst. Meas. Control, 121, 117–121 (1999).

[19] F. R. Fung, J. W. Wu and S. L. Wu, Exponential stabilization of an axially moving string by linear boundary feedback, Automatica 35(1), 177–181.B.Z (1999).

[20] M. Fabrizio and A. Morro, Mathematical Problems in Linear Viscoelasticity. SIAM Stud. Appl. Math. Philadelphia (1992).

[21] R. F. Fung, J. S. Huang and Y. C. Chen, The transient amplitude of the viscoelastic traveling string: an integral constitutive law. J. Sound Vib. 201(2), 153–167 (1997).

[22] R. F. Fung and C. C. Liao, Application of variable structure control in the nonlinear system. I. J. Mech. Sci. 37, 985–993 (1995).

[23] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97(4), 1061-1083 (1975).

[24] B. Z. Guo and H. C. Zhao, Active disturbance rejection control for rejecting boundary disturbance from multidimensional Kirchhoff plate via boundary control, SIAM J. Control Optim. 52(5):2800–2830 (2014).

[25] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities. Cambridge, UK: Cambridge University Press, (1959).

[26] A. Kelleche, N. E. Tatar and A. Khemmoudj, Stability of an axially moving viscoelastic beam, J. Dyn. Control Syst. 23(2), 283–299 (2017).

[27] S. Y. Lee and C. D. Mote, Vibration control of an axially moving string by boundary control, J. Dyn. Sys. Meas. Control 118(1), 66–74 (1996).

[28] I. Lasiecka and N. Fourrier, Regularity and stability of a wave equation with strong damping and dynamic boundary conditions. Evol Equ Contr Theory. 2(4):631–667 (2013).

[29] Z. H. Luo, O. Guo and O. Morgul, Stability and stabilization of infinite dimensional systems with applications. London: Springer (1999).

[30] L. J. Lions, quelques méthodes de résolution des problemes aux limites non lineaires, Dunod, Gaulthier-Villars, Paris (1969).

[31] D. B. McIver, Hamilton's principle for systems of changing mass, J. Eng. Math. 7(3), 249–261 (1973).

[32] D. S. Mitrinovic, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis. Kluwer Academic, Dordrecht (1993).

[33] A. Pazy, Semigroups of linear operators and applications to partial differential equations. volume 44 of Applied Math.Sciences. Springer-Verlag, New York, (1983).

[34] J. W. Park and J. A. Kim, Existence and uniform decay for Euler–Bernoulli beam equation with memory term, Maths. Method in Appl. Sci. 27(14), 1629–1640 (2004).

[35] V. Pata, Exponential stability in linear viscoelasticity. Quart. Appl. Math. LXIV(3), 499–513 (2006).

[36] R. E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Diďerential Equation, By the American Mathematical Society, (1997).

[37] W. Walter, Ordinary Diďerential Equations, Springer-Verlage, New York, Inc, (1998).

[38] B. Said-Houari, Etude de l'interaction enter un terme dissipatif et un terme d'explosion pour un probleme hyperbolique, Memoire de magister, Université de Annaba (2002).

[39] R. A.Sack, Transverse oscillations in travelling strings, Br. J. Appl. Phys. 5(6), 224–226 (1954).

[40] B. Tabarrok, Leech C. M. and Kim W. I., On the dynamics of an axially moving beam. J. Franklin Inst 297 (3), 201–220 (1974).

[41] N. E. Tatar, Exponential decay for a viscoelastic problem with singular kernel. Zeit. Angew. Math. Phys. 60(4), 640–650 (2009).

[42] N. E. Tatar, A new class of kernels leading to an arbitrary decay in viscoelasticity. Meditter. J. Math. 10, 213–226 (2013).

[43] M.Tucsnak and G. Weiss, Observation and Control for Operator Semigroups. Basel, Switzerland: Birkhäuser, (2009).

[44] A. Kelleche and B. Feng, On general decay for a nonlinear viscoelastic equation. Applicable analysis. 102, (2023). Doi: 10.1080/00036811.2021.1992394