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**UNIVERSITÉ DE ELOUED**
**FACULTÉ DE SCIENCES EXACTES**

**Thèse**

# Doctorat LMD

Domaine: Mathématiques et Informatique
Filière: Mathématiques
Spécialité: Mathématiques appliquées

**Thème**

# Etude asymptotique de quelques problèmes en thermoèlasticité poreuse avec simple ou double porosité

Présenté par

**Mme Aicha Nemsi**

Soutenu publiquement devant le jury composé de

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FACULTY OF EXACTES SCIENCES

Thesis

# Doctorat LMD

**Theme**

# A symptotic study of some problems in porous thermoelasticity with simple or double porosity

PhD. Thesis presented by: **Aicha Nemsi**

**Publicly defended before the jury composed**

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16 February 2022

## مـلـخص

في هذه الرسالة قمنا بدراسة وجود ووحدانية الحل والسلوك التقاربي لعدة جمل معادلات تعبر عن مسائل تنمذج لمواد مرنة مزدوجة المسامية في الفضاء المكاني ذو البعد الواحد.

الجملة الأولى فيها ثلاث حدود اضمحلال أحدها متعلق بالاستطالة واثنان متعلقين بالمسامية. ولقد أثبتنا وجود ووحدانية الحل وكذا الاستقرار الأسي للحل بغض النظر عن معاملات الجملة.

في الجملة الثانية حذفنا الاضمحلال الخاص بالاستطالة وأثبتنا استقرار أسي مشروط بعلاقة بين المعاملات، وفي حالة عدم توفرها يفقد الحل استقراره الأسي.

الجملة الثالثة تحتوي على عاملي اضمحلال لهما علاقة بالماضي (لهما ذاكرة) وأثبتنا الاستقرار الأسي مشروطا بعلاقة بين المعاملات والدوال الخاصة بحدود الذاكرة. وفي حالة عدم توفر هذا الشرط يفقد الحل استقراره الأسي كذلك.

الطرق التي استخدمناها هي النظرية الطيفية القائمة على نظرية أنصاف الزمر ونظرية Gerhart Pruss وطريقة الجداءات وعلى أساس تقدير الطاقة وطريقة Lyapunov المباشرة.

**الكلمات المفتاحية:** مزدوج المسامية, مسألة قابلة للحل, اضمحلال أسي, حل ضعيف, حد ذو ذاكرة, اضمحلال عام, غير مستقرة اسّيا.



## abstract

In this thesis, we studied the existence, uniqueness and the asymptotic behavior of some systems with two porous structures in one dimensional spatial space.

The first system is an elastic system with double porosity structure, we have shown the existence, uniqueness and the exponential stability of the solution in two cases of the viscoelastic dissipation.

The second system is an elastic system with two porous structures and memory effects in both porous equations. We prove that the weak dissipation generated by the memory terms produces a general rate of decay depending on the kernels of the memory terms and the coefficients of the system.

The methods we have used are the spectral method based on the semigroup theory and Gerhart Pruss theorem and the multiplier method based on the energy estimate and Lyapunov direct method.

**Key Words**: Double porosity, well-posedness, exponential decay, weak solution, memory term, general decay, lack of exponential decay.



## Résumé

Dans cette thèse, nous avons étudié l'existence, l'unicité et le comportement asymptotique de certains systèmes à deux structures poreuses dans un espace à une dimension.

Le premier système est un système élastique à structure en double porosité, nous avons montré l'existence, l'unicité et la stabilité exponentielle de la solution dans deux cas de dissipation viscoélastique.

Le deuxième système est un système élastique avec deux structures poreuses et des effets mémoire dans les deux équations poreuses. Nous montrons que la faible dissipation générée par les termes mémoire produit un taux de décroissance général dépend des noyaux des termes mémoire et des coefficients du système.

Les méthodes que nous avons utilisées sont la méthode spectrale basée sur la théorie du semi-groupe et le théorème de Gerhart Pruss et la méthode du multiplicateur basée sur l'estimation de l'énergie et la méthode directe de Lyapunov.

**Mots clés :** Double porosité, problème bien posé, décroissance exponentielle, solution de semi-groupes, terme mémoire, décroissance générale, absence de décroissance exponentielle.

# Dedication

With love and satisfaction, I dedicate this work to:

My mother: Henni Farida For her Love, for her Prayers, her encouragements and being the source of my motivation. Thank you for everything you have done and you still doing for my happiness and well being.

My father: Abdelaziz who taught me the way to become a strong woman. Thank you for everything you have done for me.

A special thanks to my husband; Youcef Tina, for all of the sacrifices that he has made on my behalf. Thank you for your motivation and encouragement.

To my lovely kids : Ouais, Jamel Eldine, I love you so much.

My dear brothers and sisters: Abderahman, Omar, Asma, Nour El-houda, Hana and Fatma. Thank you for supporting me for everything.

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A special thanks to my family. Words can not express how grateful I am to my mother and father for all of the sacrifices that they have made on my behalf. Their prayer for me were sustained me thus far. I also want to thank my dear husband T.Y, my dear kids, my dear brothers and sisters, my dear friends Thank you for supporting me for everything, and especially I can not thank you enough for encouraging me throughout this experience.

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# Notation

| | |
|---|---|
| $H^1, H_0^1, H^2, H^4, W^P, W_0^P$ | The sobolev spaces, |
| $\langle \cdot, \cdot \rangle$ | The inner product, |
| $\partial$ | The operator of partial differentiation, |
| $D(\mathcal{A})$ | The domain of the operator $\mathcal{A}$, |
| $\rho(\mathcal{A})$ | The resolvent set of the operator $\mathcal{A}$, |
| $\sigma(\mathcal{A})$ | The spectrum of operator $\mathcal{A}$, |
| $\lvert \cdot \rvert$ | The euclidean norm on $\mathbb{R}^d$, |
| $\lVert \cdot \rVert_X$ | The norm on a normed space $X$, |
| $\mathscr{L}(X), \mathcal{B}(X)$ | The space of all bounded linear operator over $X$, |
| $Re\langle ., . \rangle$ | The real part of the inner product, |
| $Im\langle ., . \rangle$ | The imaginary part of the inner product, |
| $C_0^\infty(\Omega)$ | The test functions space, |
| $(T(t))_{t \geqslant 0}$ | A semigroup, |
| $C(X, Y)$ | The espace of all continuous functions from $X$ into $Y$. |

# Published papers

- Exponential decay of the solution of a double porous elastic system, Nemsi Aicha, Fareh abdelfeteh, U.P.B. Sci. Bull., Series A, Vol. 83, Iss. 1, 2021, ISSN 1223-7027
- General decay of the solution of a one dimensional double porous elasticsystem with memory, Aicha Nemsi, Abdelfeteh Fareh,Bull. Math. Soc. Sci. Math. Roumanie Tome 64 (112), No. 4, 2021, 367-388

## Submitted paper

- On the stability of a double porous elastic system with visco-porous dampings

# Contents

# General introduction

TH he study of the asymptotic behavior of evolution equations and systems arising from mechanics and physics is an important issue, as it is essential, for practical applications, to be able to understand and predict the long time behavior of solutions of such equations or systems. The origin of the study of the asymptotic behavior of differential equations goes back to the works of Fubini [21], Sansone [37] and Bellman [10, 11]. Bellman [11] gave the definition of the asymptotic behavior and its relation with stability and boundlessness. We recall that the asymptotic behavior of a solution of a differential equation means the comparison of this solution or its norm with elementary functions as $t$ tends to infinity. The concept of asymptotic behavior is closely related to the stability concept developed by A. Lyapunov in 1899 which we will give the definition later.

The theory of porous materials which we are concerned here, was introduced first by Goodman and Cowin [23]. They extended the concept of mass distribution to admit granular materials. The basic premise underlying this theory is the concept that the mass at each point of the material is obtained as the product of the mass density of the material matrix by the volume fraction. This representation introduces an additional independent kenimatical variable: the volume fraction. Nunziato and Cowin [32, 16] employed this theory to study the behavior of materials which have interstitial voids or vacuous pores. This theory admits both finite deformations and nonlinear constitutive relations and the intended applications of this theory are to geological materials like rock and soils and to manufactured porous materials like ceramics and pressed powders. This theory differs significantly from the classical linear elasticity in that the volume fraction corresponding to the void volume is taken as an independent kinematical variable.

To be more precise, let us give a short description on the subject.

Let $\mathcal{B}$ be a material body that occupies the bounded domain $\Omega\,(\subset \mathbb{R}^3)$ in its configuration at time $t$ and let $x$ be the spatial position of a material point. The concept of mass distributed introduced by Goodman and Cowin asserts that for every $(x,t) \in \Omega \times [0,+\infty[$ the mass density $\rho$ has de decomposition $\rho = \gamma\nu$ where $\gamma$ is the density of the matrix material and $0 < \nu \leq 1$ is the volume fraction of the material.

The independent kinematic variables in the linear theory are the displacement field $u\,(x,t) = x\,(t) - x\,(t_0)$ from the reference configuration and the change in volume fraction $\phi\,(x,t) = \nu\,(x,t) - \nu\,(x\,(t_0)\,, t_0)\,.$ The infinitesimal strain tensor $e_{ij}\,(x,t)$ is determined from the displacement field, $u$ according to

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Assuming the region occupied by the body is regular, we have that the equations of motion governing a linear elastic continuum with voids are the balance of linear momentum

$$\rho\ddot{u}_i = T_{ji,j} + b_i, \tag{1}$$

and the balance of equilibrated force

$$\rho\kappa\ddot{\phi} = h_{i,i} + g + \rho\ell \tag{2}$$

where $T_{ji}$ is the symmetric stress tensor, $b_i$ is the body force vector, $h_i$ is the equilibrated stress vector, $\kappa$ is the equilibrated inertia, $g$ is the intrinsic equilibrated body force and $\ell$ is the extrinsic equilibrated body force. Here we have used the physicists convention where a superposed dot denotes the partial differentiation with respect to time, a subscript comma denotes the partial differentiation with respect to the spatial compenant of the followed index and the summation for the repeated subscript.

We require that $T_{ji}, h_i$ and $g$ satisfys the conditions for mechanical equilibrium in the absence of body forces in the reference state, the constitutive equations for $T_{ji}, h_i$ and $g$ are given by

$$T_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} + \beta\phi\delta_{ij},$$

$$h_i = \alpha\phi_{,i},$$

$$g=-\tau\dot{\phi}-\xi\phi-\beta e_{kk}$$

where $\lambda,\mu$ are Lamé modulis, $\alpha,\xi,\beta$ and $\tau$ are constitutive constants that satisfy the requirements

$$\mu\geq 0,\ \alpha\geq 0,\ \xi\geq 0,\beta\geq 0$$

and since the stored energy density associated with strain and void volume distortion from the reference configuration is a positive definite quadratic form then the coefficients must satisfy

$$3\lambda+2\mu\geq 0,\ (3\lambda+2\mu)>12\beta^2.$$

The coefficient $\tau$ and the equilibrated inertia $\kappa$ must be nonnegative in order to satisfy a dissipation inequality. By substituting the constitutive equations into (1)-(2) these later equation take the form

$$\begin{cases} \rho\ddot{u}_i=\lambda\delta_{ji}e_{kk,j}+2\mu e_{ji,j}+\beta\phi_{,j}\delta_{ij}+b_i, \\ \rho\kappa\ddot{\phi}=\alpha\phi_{,ii}-\tau\dot{\phi}-\xi\phi-\beta e_{kk}+\rho\ell. \end{cases} \tag{3}$$

As we are concerned by one-dimensional theory, then in the absence of body forces and the extrinsic equilibrated body force the system (3) becomes

$$\begin{cases} \rho u_{tt}=\mu u_{xx}+\beta\phi_x, \\ \rho\kappa\phi_{tt}=\alpha\phi_{xx}-\beta u_x-\xi\phi-\tau\phi_t, \end{cases} \tag{4}$$

where we write $\mu$ instead of $\lambda+2\mu$. System (4) was studied first by Quintanilla [35], he investigated the boundary conditions

$$u(0,t)=u(\pi,t)=\phi_x(0,t)=\phi_x(\pi,t)=0 \tag{5}$$

and proved that the solution can not be exponentially stable, only slow decay has been obtained. Various dissipative mechanisms have been added to stabilize system (4) exponentially. Magaña and Quintanilla [28] added the viscoelastic dissipation $\gamma u_{txx}$ to the first equation of (4) and established an exponential rate of decay. The same result was obtained by Casas and Quintanilla [15] when they added thermal dissipation to the system (4).

Apalara [2] replaced the porous damping in (4) by the viscoelastic damping of memory type $\int_0^t g(t-s)\,\phi_{xx}(x,s)\,ds$ where $g$ is a relaxation function that satisfies the property

$$g'(t)\leq -\xi(t)\,g(t)\,,$$

for a non increasing function $\xi$. He established a general decay result for which the exponential and the polynomial decay rates are only special cases.

The origin of the theory of double porosity goes back to the works of Barenblatt *et al.* [3, 4]. The authors distinguished the liquid pressure in the pores from the liquid pressure in the fissures and introduced a double porosity structure. This theory is an important generalization of Biot's theory [12] for porous materials with single porosity. Wilson and Aifentis [44] presented a theory of consolidation for elastic materials with double porosity which unifies the earlier models of Barenblatt and Biot. However, the theory proposed by Wilson and Aifentis ignored the cross-coupling effects between the volume change of the pores and fissures in the system. Khalili and Valliappan [26] modified Aifiens' theory and proposed a cross-coupling terms included in the equations of conservation of mass for the pores and fissures fluid. Barryman and coauthors [5, 6] included a cross-coupling in Darcy's law for solids with double porosity.

In [25] Ieşan and Quintanilla derived a double porosity model based on the Nunziato-Cowin theory for materials with voids [16, 32]. According to this theory the porosity structure in the equilibrium case is influenced by the displacement field, which is different from the theory based on Darcy's law.

In the case of isotropic solids the constitutive equations have the form

$$T_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\delta_{ij}\phi + d\delta_{ij}\psi,$$
$$\sigma_i = \alpha\phi_{,i} + b_1\psi_{,i}, \quad \chi_i = b_1\phi_{,i} + \gamma\psi_{,i},$$
$$\xi = -be_{jj} - \alpha_1\phi - \alpha_3\psi, \quad \zeta = -de_{jj} - \alpha_3\phi - \alpha_2\psi,$$

where $\delta_{ij}$ is Kronecker's delta, and $\lambda, \mu, b, d, b_1, \alpha, \beta, \gamma, \alpha_j$ are constitutive coefficients. The equations of motion are given by

$$\rho\ddot{u}_i = T_{ji,j},$$
$$\kappa_1\ddot{\phi} = \sigma_{j,j} + \xi + \rho g,$$
$$\kappa_2\ddot{\psi} = \chi_{j,j} + \zeta + \rho h.$$

In the context of double porous thermoelasticity, Bazarra *et al.* [8] considered the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x - \beta\theta_x, \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta - \varepsilon_1\varphi_t - \varepsilon_2\psi_t, \\ \kappa_2\psi_{tt} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta - \varepsilon_3\varphi_t - \varepsilon_4\psi_t, \\ c\theta_t = \kappa\theta_{xx} - \beta u_{tx} - \gamma_1\varphi_t - \gamma_2\psi_t, \end{cases}$$

with the following boundary conditions

$$u(x,t) = \varphi_x(x,t) = \psi_x(x,t) = \theta_x(x,t) = 0, \ x = 0, \ x = \pi \text{ and } t \geq 0.$$

They proved that the solution decays exponentially when porous dissipation is assumed for each porous equations. If the dissipation is considered only on one porous structures, the solution cannot be asymptotically stable in general. However, they give a sufficient conditions for which the solutions decay exponentially. See also [9].

The rest of this thesis is organized as follows: In Chapter 1, we give some preliminaries and fondamental tools that we used in the rest of the thesis. Chapter 2 we be devoted to the study of a double porous elastic system with dissipations in the elastic and porous equations. In Chapter 3 we study a double porous elastic system with weak dissipations of memory type. Finally, we summarize our results in a conclusion paragraph.

Chapter 1

# General preliminaries

In this chapter we will present some definitions, and results on $C_0$-semigroups, including some theorems on exponential stability. We will also collect some results on Sobolev spaces. Some results are given with proof and others are stated without proofs.

## 1.1 Functional spaces

### 1.1.1 Hilbert spaces

**Definition 1.1.1.** *[13] A Hilbert space is a vector space $\mathcal{H}$ equipped with an inner product $\langle .,. \rangle$ such that $\mathcal{H}$ is complete with respect to the norm associated to the inner product.*

*For example $L^2, \ell^2$ and $H^m$ are Hilbert spaces.*

### 1.1.2 Lax Milgram theorem

**Definition 1.1.2.** *Let $\mathcal{H}$ be a Hilbert space and $a(.,.) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form. $a(.,.)$ is said to be:*

- *continuous, if there exists a constant $C$ such that*

$$|a(u,v)| \leq C \|u\| \|v\| \quad \forall u, v \in \mathcal{H},$$

- *coercive, if there exists a constant $\alpha > 0$ such that*

$$|a(v,v)| \geq \alpha \|v\|^2 \quad \forall v \in \mathcal{H}.$$

**Theorem 1.1.1.** ***(Lax Milgram)***
*Assume that $a(.,.) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a continuous and coercive bilinear form, then for every linear continuous form $L \in \mathcal{H}'$, there exists a unique element $u \in \mathcal{H}$ such that*

$$a(u,v) = L(v) \quad \forall v \in \mathcal{H}.$$

### 1.1.3 $L^p$ spaces

**Definition 1.1.3.** *[13] Let $p \in \mathbb{R}$ with $1 < p < +\infty$; we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \quad \textit{is measurable and} \quad \int_\Omega |f|^p < +\infty \right\}$$

*with*

$$\|f\|_{L^p} = \|f\|_p = \left[ \int_\Omega |f(x)|^p \, d\mu \right]^{\frac{1}{p}}.$$

*we set*

$$L^\infty(\Omega) = \left\{ \begin{array}{c} f : \Omega \to \mathbb{R}; f \ \textit{is measurable and there exists a constant } C \\ \textit{suth that } |f(x)| \leq C \quad a.e. on \ \Omega \end{array} \right\}$$

*with*

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{C; \ |f(x)| \leq C \quad a.e. on \ \Omega\}.$$

**Notation 1.** *Let $1 \leq p \leq \infty$; we denote by $q$ the conjugate exponent of $p$*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

### 1.1.4 Sobolev spaces $W^{1,p}(\Omega)$

Let $\Omega$ be a bounded domain in $\mathbb{R}$; and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

**Definition 1.1.4.** *The Sobolev space $W^{1,p}(\Omega)$ is defined to be*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega); \exists g \in L^p(\Omega); \int_\Omega u\varphi' dx = -\int_\Omega g\varphi dx, \quad \forall \varphi \in C_c^1(\Omega) \right\}$$

*we set*

$$H^1(\Omega) = W^{1,2}(\Omega).$$

*For $u \in W^{1,p}(\Omega)$ we denote $u' = g$ and we say that $u'$ is the weak derivative of $u$.*

*The space $W^{1,p}$ is equipped with the norm*

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p}$$

or sometimes, if $1 < p < \infty$, with the equivalent norm $(\|u\|_{L^p}^p + \|u\|_{L^p}^p)^{1/p}$.
$H^1$ is equipped with the inner product

$$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle u', v'\rangle_{L^2} = \int_\Omega (uv + u'v')dx$$

and with the associated norm

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2\right)^{\frac{1}{2}}.$$

**Definition 1.1.5.** *Given $1 \leq p < \infty$, we denote by $W_0^{1,p}(\Omega)$ the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$. For $p = 2$, we note*

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

**Notation 2.** *The dual space of $W_0^{1,p}(\Omega)(1 \leq p < \infty)$ is denoted by $W^{-1,q}(\Omega)$ and the dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$.*

### 1.1.5 Some useful inequalities

The following inequalities are of great importance, we will use them frequency.

**Theorem 1.1.2.** ***(Young inequality)*** *Let $p$ and $q$ be two real conjugate, so*

$$\forall (a, b) \in \mathbb{R}_+^2 \ \ |ab| \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

*In particular if $u$, $v \in L^2(\Omega)$ we have*

$$\int_\Omega |uv| \leq \varepsilon \int_\Omega |u|^2 + \frac{1}{4\varepsilon} \int_\Omega |v|^2, \quad \forall \varepsilon > 0.$$

**Theorem 1.1.3.** *[13] **(Holder inequality).** Assume that $f \in L^p$ and $g \in L^q$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_p \|g\|_q .$$

**Theorem 1.1.4.** ***(Cauchy-Schwarz inequality)** Let $\mathcal{H}$ be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, we have,*

$$|\langle u, v\rangle| \leq \langle u, u\rangle^{\frac{1}{2}} \langle v, v\rangle^{\frac{1}{2}} \ \forall u, v \in \mathcal{H}.$$

**Theorem 1.1.5.** ***(Poincaré's inequality)** Let $\Omega$ be a bounded domain in $\mathbb{R}$ and $u \in H_0^1(\Omega)$, then there exists a positive constante $C$, depending only on mes$(\Omega)$ and $p$ such that*

$$\|u\|_{L^p(\Omega)} \leq C\| \bigtriangledown u\|_{L^p(\Omega)}, \qquad \forall u \in H_0^1(\Omega).$$

## 1.2 Some semigroup tools

**Definition 1.2.1.** *Let $\mathcal{A} \in \mathscr{L}(E)$.The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ is the set of all $\lambda$ in $\mathbb{C}$ such that, $(\mathcal{A} - \lambda I)$ is invertible*

$$\rho(\mathcal{A}) = \left\{\lambda \in \mathbb{C}; \quad (\mathcal{A} - \lambda I)^{-1} \in \mathscr{L}(E)\right\}.$$

*The spectrum of $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is the complement of the resolvent set, i.e.,*

$$\sigma(\mathcal{A}) = \mathbb{C} \backslash \rho(\mathcal{A}).$$

*A complex number $\lambda$ is said to be an eigenvalue of $\mathcal{A}$ if*

$$N(\mathcal{A} - \lambda I) \neq \{0\}.$$

**Theorem 1.2.1.** *Let $X$ be a Banach space. If $\mathcal{A} \in \mathscr{L}(E)$ is an operator with $\|\mathcal{A}\| < 1$ then $I - \mathcal{A}$ is invertible and the inverse is given by*

$$(I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n.$$

**Definition 1.2.2.** *[33] Let $\mathcal{H}$ be a Banach space. A one parameter family of bounded linear operators $T(t)$, $0 \le t < \infty$, from $\mathcal{H}$ into $\mathcal{H}$ is a semigroup of bounded linear operators on $\mathcal{H}$ if*

*$T(0) = I$, ($I$ is the identity operator on $\mathcal{H}$).*

*$T(t+s) = T(t)T(s)$ for every $t, s \ge 0$ (the semigroup property).*

*A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if*

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

*The infinitesimal generator of a semigroup $T(t)$, is the operator $\mathcal{A}$ defined on*

$$D(\mathcal{A}) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \quad \text{exists,} \right\}$$

*by*

$$\mathcal{A}x = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+ + T(t)x}{dt}\Big|_{t=1} \quad \text{for} \quad x \in D(\mathcal{A}).$$

**Definition 1.2.3.** *[13](**Maximal Monotone Operators**): An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is said to be monotone (accretive) if it satisfies*

$$Re \langle \mathcal{A}v, v \rangle \ge 0 \quad \forall v \in D(\mathcal{A}).$$

*It is called maximal monotone if in addition, $R(I + \mathcal{A}) = \mathcal{H}$, i.e., $\forall f \in \mathcal{H} \quad \exists u \in D(\mathcal{A})$ such that $u + \mathcal{A}u = f$.*

**Remark 1.2.1.** *If $-\mathcal{A}$ is monotone, we say that $\mathcal{A}$ is dissipative.*

**Proposition 1.2.1.** *[13] Let $\mathcal{A}$ be a maximal monotone operator. Then*

*(a) $D(\mathcal{A})$ is dense in $\mathcal{H}$,*

*(b) $\mathcal{A}$ is a closed operator,*

*(c) For every $\lambda > 0, (I + \lambda\mathcal{A})$ is bijective from $D(\mathcal{A})$ onto $\mathcal{H}$, $(I + \lambda\mathcal{A})^{-1}$ is a bounded operator, and $\|(I + \lambda\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le 1$.*

**Theorem 1.2.2.** ***(Hille-Yosida)*** *[13] Let $\mathcal{A}$ be a maximal monotone operator. Then, given any $u_0 \in D(\mathcal{A})$ there exists a unique function*

$$u \in C^1([0, +\infty); \mathcal{H}) \cap C([0, +\infty); D(\mathcal{A}))$$

*satisfying*

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}u = 0 \quad on[0, +\infty), \\ u(0) = u_0. \end{cases} \tag{1.1}$$

**Theorem 1.2.3.** ***(Lumer-Phillips)*** *[33, 43] Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be a densely defined operator. Then $\mathcal{A}$ generates a $C_0-$semigroup of contractions on $\mathcal{H}$ if and only if*

**i)** *$\mathcal{A}$ is dissipative,($-\mathcal{A}$ monotone).*

**ii)** *there exists $\lambda_0 > 0$ such that $\lambda_0 I - \mathcal{A}$ is surjective.*

**Remark 1.2.2.** *Suppose that assumptions (i) and (ii) of Theorem 1.2.3 are satisfied, then $D(\mathcal{A})$ is dense in $\mathcal{H}$.*

*Proof.* Indeed, let $f \in D(\mathcal{A})^{\perp}$ then $\langle f, v \rangle = 0$ for all $v \in D(\mathcal{A})$. Since $\lambda_0 I - \mathcal{A}I$ is surjective, there exists $v_0 \in D(\mathcal{A})$ such that $\lambda_0 v_0 - \mathcal{A}v_0 = f$. We have

$$0 = \langle f, v_0 \rangle = \langle \lambda_0 v_0 - \mathcal{A}v_0, v_0 \rangle = \lambda_0 \|v_0\|^2 - \langle \mathcal{A}v_0, v_0 \rangle \geq \lambda_0 \|v_0\|^2 .$$

Therefore, $v_0 = 0$ and consequently $f = 0$ and $D(\mathcal{A})$ is dense in $\mathcal{H}$. □

**Theorem 1.2.4.** *[43] Let $\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$ be the infintesimal generator of a $C_0$-semigroup $\{S(t); t \geq 0\}$. Then, for each $\xi \in D(\mathcal{A})$ and each $t \geq 0$, we have $S(t)\xi \in D(\mathcal{A})$, and the mapping*

$$t \longrightarrow S(t)\xi$$

*is of class $C^1$ on $[0, +\infty)$ and satisfies*

$$\frac{d}{dt}(S(t)\xi) = \mathcal{A}S(t)\xi = S(t)\mathcal{A}\xi.$$

**Theorem 1.2.5.** *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, , then $\mathcal{A}$ is the infinitesimal generator of a $C_0-$semigroup of contractions on $\mathcal{H}$.*

*Proof.* By the assumption $0 \in p(\mathcal{A})$, $\mathcal{A}$ is invertible and $\mathcal{A}^{-1}$ is a bounded linear operator. By the theorem 1.2.1 , it is easy to see that the operator $\lambda I - \mathcal{A} = \mathcal{A}(\lambda\mathcal{A}^{-1} - I)$ is invertible for $0 < \lambda < \|\mathcal{A}^{-1}\|$ . Therefore, it follows from Lumer-Phillips Theorem that $\mathcal{A}$ is the infinitesimal generator of a $C_0-$semigroup of contractions on $\mathcal{H}$. Thus the proof is complete. □

## 1.3 Cauchy abstract problem

Let $\mathcal{A}$ be a bounded operator, we consider the problem:

$$\begin{cases} U_t = \mathcal{A}U, \quad t \geq 0, \\ U(0) = U_0. \end{cases} \tag{*}$$

If $\mathcal{A}$ generated a $C_0$-semigroup. Then, the equation has a unique solution

$$U(t) = T(t)U_0 = e^{\mathcal{A}}U_0.$$

## 1.4 Stability notions

**Theorem 1.4.1.** *Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following condition: for all $T > 0$, there exist two constants $k, C$ such that*

$$|f(t,x) - f(t,y)| \leq k\,|x-y|\,, \quad \forall x, y \in \mathbb{R}^n, \quad \forall t \in [0,T],$$

$$|f(t,x_0)| \leq C, \quad \forall t \in [0,T],$$

*then the equation*

$$\begin{cases} x_t(t) = f(t,x), \forall t > 0, \\ x(t_0) = x_0 \end{cases} \tag{1.2}$$

*has a unique solution which we note $x(t,t_0,x_0)$. This solution satisfies*

$$x(t,t_1,x_1) = x(t,x_0), \forall t \geq t_1.$$

*where, $x_1 = x(t_1)$.*

**Definition 1.4.1.** *An equilibrium point of (1.2) is a point $x_0 \in \mathbb{R}^n$ which satisfies $f(t,x_0) =$ 0, for all $t \geq 0$.*

Clearly, the unique solution of (1.2) is $x(t,t_0,x_0) = x_0$ for all $t \geq t_0$. This fact means that if a solution starts at an equilibrium point it stays there. For the convenience we assume that the equilibrium point is $x_0 = 0$. that is

$$f(t,0) = 0, \forall t \geq 0,$$

or equivalently

$$x\left(t,t_0,0\right)=0,\forall t\geq 0.$$

## 1.5 Lyapunov stability notions

Lyapunov theory is concerned by the behavior of a solution which starts at $x_0 \neq 0$ but is close to it; $\|x_0\| < \delta$.

**Definition 1.5.1.** *[39] The equilibrium point* $0$ *is said to be stable if, for each* $\varepsilon > 0$ *and each* $t_0 \geq 0$*, there exists* $\delta(\varepsilon, t_0) > 0$*. such that*

$$\|x_0\| < \delta\left(\varepsilon, t_0\right) \Rightarrow \|x\left(t, t_0, x_0\right)\| < \varepsilon.$$

According to this definition, the equilibrium point is stable if the solution begining at $x_0$ is close at 0 as needed for all $x_0$ close at 0. In addition if depends only on $\varepsilon$, the equilibrium point is said to be uniformly stable.

**Definition 1.5.2.** *[39] The equilibrium point* $0$ *is asymptotically stable if it is stable and for each* $t_0 \geq 0$*, there exists* $\eta(t_0) > 0$ *such that*

$$\|x_0\| < \eta\left(t_0\right) \Rightarrow \lim_{t\to\infty} x\left(t, t_0, x_0\right) = 0.$$

In addition if $\eta$ doesn't depend on $t_0$ the equilibrium point is said to be asymptotically uniformly stable.

**Definition 1.5.3.** *[39] The equilibrium point is said to be exponentially stable if there exist positive constants* $C, \omega, r$ *such that*

$$\|x\left(t, t_0, x_0\right)\| < C\left\|x_0\right\| e^{-\omega t}, \forall t \geq t_0, \forall x_0, \|x_0\| < r.$$

*Moreover if*

$$\|x\left(t, t_0, x_0\right)\| < C\left\|x_0\right\| e^{-\omega t}, \forall t \geq t_0, \forall x_0 \in \mathbb{R},$$

*the equilibrium is globally exponentially stable.*

**Definition 1.5.4.** *The solution* $U(t) = e^{\mathcal{A}}U_0$ *is said to be exponentially stable if there exist two positive constants* $\alpha$ *and* $M > 1$ *such that*

$$\|U(t)\| \leq Me^{-\alpha t}, \quad \forall t \geq 0.$$

**Definition 1.5.5.** *[38] A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing; and $\phi(0) = 0$.*

**Definition 1.5.6.** *[38] A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is said to be a positive definite function if:*

i) $V$ is continuous,

ii) $V(t, 0) = 0, \forall t \geq 0$;

iii) there exists a function $\phi$ of class $\mathcal{K}$ such that

$$\phi(\|x\|) \leq V(t, x), \forall t \geq 0, \forall x \in \mathbb{R}^n.$$

**Definition 1.5.7.** *[38] A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is called decrescent function if there exists a function $\beta$ of class $\mathcal{K}$ such that*

$$V(t, x) \leq \beta(\|x\|), \forall t \geq 0, \forall x \in \mathbb{R}^n.$$

**Lemme 1.5.1.** *A function $W : \mathbb{R}^n \to \mathbb{R}$ is a positive definite function if and only if*

i) $W(0) = 0$,

ii) $W(x) > 0, \forall x \in \mathbb{R}^n - \{0\}$,

iii) there exists $r > 0$, such that $\underset{\|x\| \geq r}{inf}\, W(x) > 0$.

**Lemme 1.5.2.** *A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is a positive definite function if and only if : i)$V(t, 0) = 0, \forall t \geq 0$, ii) there exists a positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that*
*$V(x, t) \geq W(x), \forall t \geq 0, \forall x \in \mathbb{R}^n$.*

### 1.5.1 Lyapunov direct method

**Theorem 1.5.1.** *The equilibrium point $0$ of the equation*

$$x_t = f(t, x), t > 0,$$

*is stable if there exists $r > 0$ and a locally positive definite function $V$ of class $C^1$ such that*

$$V_t(t, x) \leq 0, \forall t \geq t_0, \forall x, \|x\| \leq r.$$

**Theorem 1.5.2.** *The equilibrium point* $0$ *of the equation*

$$x_t = f(t, x), t > 0,$$

*is uniformly stable if there exist* $r > 0$ *and a locally positive definite decrescent function* $V$ *of class* $C^1$ *such that*

$$V_t(t, x) \leq 0, \forall t \geq t_0, \forall x, \|x\| \leq r.$$

**Theorem 1.5.3.** *The equilibrium point* $0$ *of the equation*

$$x_t = f(t, x), t > 0,$$

*is uniformly asymptotically stable, if there exist a* $C^1$ *decreasing and locally positive definite function* $V$ *such that* $-V_t$ *is locally positive definite.*

**Theorem 1.5.4.** *Suppose that there exist constants,* $a, b, c, r > 0, p \geq 1$ *and a* $C^1$ *function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *such that:*

i) $a \|x\|^p \leq V(t; x) \leq b \|x\|^p, \forall t \geq 0, \forall x, \|x\| \leq r,$

ii) $V_t(t, x) \leq -c \|x\|^p, \forall t \geq 0, \forall x, \|x\| \leq r.$

Then, the equilibrium point of the equation $x_t = f(t, x)$, is exponentially stable.

**Theorem 1.5.5.** *Suppose that there exist constants,* $a, b, c, r > 0, p \geq 1$ *and a* $C^1$ *function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *such that:*

i) $a \|x\|^p \leq V(t; x) \leq b \|x\|^p, \forall t \geq 0, \forall x \in \mathbb{R}^n,$

ii) $V_t(t, x) \leq -c \|x\|^p, \forall t \geq 0, \forall x \in \mathbb{R}^n.$

Then, the equilibrium point of the equation $x_t = f(t, x)$, is globally exponentially stable.

### 1.5.2 Stability of linear systems

In this subsection we are concerned with the stability of the solution of the linear system

$$\begin{cases} U_t(t) = \mathcal{A}U(t), t \geq 0, \\ U(0) = U_0. \end{cases} \tag{1.3}$$

Clearly, $U(t) = 0$ is an equilibrium point.

From the semigroup theory, we infer that if $\mathcal{A}$ is a generator of a stongly continuous semigroup $(T(t))_{t\geq 0}$, then for every $U_0 \in D(\mathcal{A}), U(t) = T(t)U_0$ is the unique classical solution of (1.3).

**Theorem 1.5.6.** *[27] A $C_0-$semigroup of contractions $S(t) = e^{-\mathcal{A}t}$, generated by an operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, is exponentially stable if and only if*

$$i) i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}),$$

*and*

$$ii) \overline{\lim_{|\lambda|\to\infty}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\| < \infty.$$

*hold.*

**Theorem 1.5.7.** *[27] A $C_0-$semigroup of contractions $S(t) = e^{-\mathcal{A}t}$, generated by an operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, is exponentially stable if and only if*

$$i) \sup\{Re\lambda, \lambda \in \sigma(\mathcal{A})\} < 0,$$

*and*

$$ii) \overline{\lim_{Re\lambda\geq 0}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\| < \infty.$$

*hold.*

**Remark 1.5.1.** *According to Gerhart-Prüss-Huang theorem, to prove that a solution $U$ of a linear system*

$$\begin{cases} U_t(t) = \mathcal{A}U(t), t \geq 0, \\ U(0) = U_0. \end{cases}$$

*is not exponentialy stable it suffices to prove that there exists a sequence $(F_n) \subset \mathcal{H}$ with bounded norm $\|F_n\| < 1$ such that*

$$\overline{\lim_{|\lambda|\to\infty}} \left\|(i\lambda I - \mathcal{A})^{-1}F_n\right\|_{\mathcal{H}} = \overline{\lim_{|\lambda|\to\infty}} \|U_n\|_{\mathcal{H}} < \infty.$$

Chapter 2

# Exponential decay of a double porous elastic system

In this chapter, we consider a one-dimensional double porous elastic system with two dissipative mechanisms : a viscoelastic dissipation in the displacement field and visco-porous dissipations. we study the system in the presence of the viscoelastic dissipation given by the damping $\lambda u_{txx}$.

We prove the existence and uniqueness of a solution, by means of semigroup theory. Exponential decay of the solutions is obtained when porous dissipation is assumed for each porous structure. The proof will be given by the use of Gearhart Prüss theorem and the mothed developed by Zheng and Liu [27].

## 2.1 Setting of the problem

We consider a double porous elastic solid in the framework of Ieşan and Quintanilla theory [25]. In the one-dimensional case the evolution equations are

$$\begin{aligned}
\rho u_{tt} &= \mathbb{T}_x, \\
\kappa_1 \varphi_{tt} &= \sigma_x + \xi, \\
\kappa_2 \psi_{tt} &= \chi_x + \zeta,
\end{aligned}$$

where $u$ is the displacement, $\varphi$ and $\psi$ are the porous variables, $\rho, \kappa_1$ and $\kappa_2$ are positive constants. $\mathbb{T}$ is the first Piola-Kirchhoff stress tensor, $\sigma$, $\chi$ are equilibrated stress vectors, $\xi$ and $\zeta$ are the intrinsic equilibrated body forces that they must be given by constitutives assumptions. Following Ieşan and Quintanilla [25], we assume that

$$\begin{gathered}\mathbb{T} = \mu u_x + b\varphi + d\psi + \lambda u_{tx},\\ \sigma = \alpha\varphi_x + b_1\psi_x,\ \chi = b_1\varphi_x + \gamma\psi_x,\\ \xi = -bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t,\\ \zeta = -du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t.\end{gathered}$$

Here $\mu, b, d, \lambda, \alpha, \alpha_1, \alpha_2, \alpha_3, b_1, b_2, \gamma, \tau_1$ and $\tau_2$ are constants.

If we introduce the constitutive equations into the evolution equations we obtain the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x + \lambda u_{txx} & \text{in } (0,\infty)\times(0,L),\\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t & \text{in } (0,\infty)\times(0,L),\\ \kappa_2\psi_{tt} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t & \text{in } (0,\infty)\times(0,L).\end{cases} \tag{2.1}$$

In this chapter, we study the system (2.1) in the presence of the viscoelastic dissipation given by the damping $\lambda u_{txx}$. We consider the following boundary and initial conditions

$$u(t,0) = u(t,L) = \varphi_x(t,0) = \varphi_x(t,L) = \psi_x(t,0) = \psi_x(t,L) = 0 \quad \text{in } (0,\infty), \tag{2.2}$$

$$\begin{gathered} u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \varphi(0,x) = \varphi_0(x),\ \varphi_t(0,x) = \varphi_1(x),\\ \psi(0,x) = \psi_0(x),\ \psi_t(0,x) = \psi_1(x),\ x \in [0,L]. \end{gathered} \tag{2.3}$$

There are solutions (uniform in the variable x) that do not decay. To avoid this case, we also assume that

$$\int_0^L \varphi_0(x)\,dx = \int_0^L \varphi_1(x)\,dx = \int_0^L \psi_0(x)\,dx = \int_0^L \psi_1(x)\,dx = 0.$$

We introduce the energy associated with the system (2.1)-(2.3) as

$$\begin{aligned} E(t) :=& \frac{1}{2}\int_0^L \left[\rho|u_t|^2 + \kappa_1|\varphi_t|^2 + \kappa_2|\psi_t|^2 + \mu|u_x|^2 + \alpha|\varphi_x|^2 + \gamma|\psi_x|^2 + \alpha_1|\varphi|^2 + \alpha_2|\psi|^2\right]dx\\ &+ b\int_0^L Re\left(u_x\overline{\varphi}\right)dx + d\int_0^L Re\left(u_x\overline{\psi}\right)dx + \alpha_3\int_0^L Re\left(\varphi\overline{\psi}\right)dx + b_1\int_0^L Re\left(\varphi_x\overline{\psi_x}\right)dx, \end{aligned}$$

which can be written as

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^L \Big[&(u_t,\varphi_t,\psi_t)\,A\left(\overline{u_t},\overline{\varphi_t},\overline{\psi_t}\right)^T + (u_x,\varphi,\psi)\,B\left(\overline{u_x},\overline{\varphi},\overline{\psi}\right)^T\\ &+ (\varphi_x,\psi_x)\,C\left(\overline{\varphi_x},\overline{\psi_x}\right)^T\Big], \end{aligned}$$

where,

$$A=\begin{pmatrix}\rho & 0 & 0\\ 0 & \kappa_1 & 0\\ 0 & 0 & \kappa_2\end{pmatrix}, B=\begin{pmatrix}\mu & b & d\\ b & \alpha_1 & \alpha_3\\ d & \alpha_3 & \alpha_2\end{pmatrix}, C=\begin{pmatrix}\alpha & b_1\\ b_1 & \gamma\end{pmatrix}.$$

It is assumed that the internal mechanical energy density is a positive definite form. Thus the matrix $A, B$ and $C$ must be positive definite.

We have the following result:

**Lemme 2.1.1.** *If $(u,\varphi,\psi)$ is the solution of (2.1)-(2.3), then the energy $E(t)$ satisfies the estimate*

$$E'(t)\leq-\tau_1\int_0^L|\varphi_t|^2\,dx-\tau_2\int_0^L|\psi_t|^2\,dx-\lambda\int_0^L|u_{xt}|^2\,dx.$$

*Proof.* Taking the $L^2-$product of $(2.1)_1$ by $u_t$, $(2.1)_2$ by $\varphi_t$ and $(2.1)_3$ by $\psi_t$ and summing up we obtain

$$\rho\int_0^L u_{tt}\overline{u}_t dx+\kappa_1\int_0^L\varphi_{tt}\overline{\varphi}_t dx+\kappa_2\int_0^L\psi_{tt}\overline{\psi_t}dx$$

$$=\mu\int_0^L u_{xx}\overline{u}_t dx+b\int_0^L\varphi_x\overline{u}_t dx+d\int_0^L\psi_x\overline{u}_t dx+\lambda\int_0^L u_{txx}\overline{u}_t dx+\alpha\int_0^L\varphi_{xx}\overline{\varphi}_t dx$$

$$+b_1\int_0^L\psi_{xx}\overline{\varphi}_t dx-b\int_0^L u_x\overline{\varphi}_t dx-\alpha_1\int_0^L\varphi\overline{\varphi}_t dx-\alpha_3\int_0^L\psi\overline{\varphi}_t dx-\tau_1\int_0^L\varphi_t\overline{\varphi}_t dx+b_1\int_0^L\varphi_{xx}\overline{\psi_t}dx$$

$$+\gamma\int_0^L\psi_{xx}\overline{\psi_t}dx-d\int_0^L u_x\overline{\psi_t}dx-\alpha_3\int_0^L\varphi\overline{\psi_t}dx-\alpha_2\int_0^L\psi\overline{\psi_t}dx-\tau_2\int_0^L\psi_t\overline{\psi_t}dx,$$

then, we get

$$\frac{d}{dt}\left[\frac{\rho}{2}\int_0^L|u_t|^2\,dx+\frac{\kappa_1}{2}\int_0^L|\varphi_t|^2\,dx+\frac{\kappa_2}{2}\int_0^L|\psi_t|^2\,dx+\frac{\gamma}{2}\int_0^L|\psi_x|^2\,dx\right.$$

$$\left.+\frac{\alpha_2}{2}\int_0^L|\psi|^2\,dx+\frac{\mu}{2}\int_0^L|u_x|^2\,dx+\alpha\int_0^L|\varphi_x|^2\,dx+\alpha_1\int_0^L|\varphi|^2\,dx\right]$$

$$+\alpha_3\int_0^L\left(\psi\overline{\varphi}_t+\varphi\overline{\psi_t}\right)dx+b_1\int_0^L\left(\psi_x\overline{\varphi}_{xt}+\varphi_x\overline{\psi}_{xt}\right)dx+d\int_0^L\left(\psi\overline{u}_{xt}+u_x\overline{\psi}_t\right)dx$$

$$+b\int_0^L\left(u_x\overline{\varphi}_t+\varphi\overline{u}_{xt}\right)dx=-\lambda\int_0^L|u_{tx}|\,dx-\tau_1\int_0^L|\varphi_t|^2\,dx-\tau_2\int_0^L|\psi_t|^2\,dx,$$

$$E(t)\;=\;\frac{1}{2}\left[\rho\int_0^L|u_t|^2\,dx+\kappa_1\int_0^L|\varphi_t|^2\,dx+\kappa_2\int_0^L|\psi_t|^2\,dx+\gamma\int_0^L|\psi_x|^2\,dx\right.$$

$$+\alpha_2 \int_0^L |\psi|^2\, dx + \mu \int_0^L |u_x|^2\, dx + \alpha \int_0^L |\varphi_x|^2\, dx + \alpha_1 \int_0^L |\varphi|^2\, dx\Big]$$
$$+2Re\left[\alpha_3 \int_0^L \psi\overline{\varphi}dx + b_1 \int_0^L \psi_x\overline{\varphi}_x dx + d\int_0^L \psi\overline{u}_x dx + b\int_0^L u_x\overline{\varphi}dx\right].$$

□

The aim of this chapter is to prove that the problem determined by (2.1)-(2.3) has a unique solution that decays exponentially in time. For the well-posedness we use the Lumer-Phillips theorem and for the exponential stability we use the method developed by Liu and Zheng [27].

To the best of our knowledge the problem is novel and no study has been done to determine the rate of decay of the solution of problems in double porous elasticity.

## 2.2 Well posedness

The aim of this section is to prove that the problem (2.1)-(2.3) has a unique solution. Our main tools are the two following theorems from the theory of semigroups of operators in Hilbert spaces.

In order to rewrite the problem (2.1)-(2.3) in the semigroup setting we introduce the Hilbert space

$$\mathcal{H} := H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L^2(0,L),$$

where $H^1(0,L)$, $H^2(0,L)$ are the usual Sobolev spaces, $H_0^1(0,L)$ is the closure of $C_0^\infty(0,L)$ in $H^1(0,L)$ [13, 27] and

$$H_*^1(0,L) := \left\{\varphi \in H^1(0,L);\ \int_0^L \varphi(t,x)dx = 0\right\}.$$

The space $\mathcal{H}$ is endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined for $U = (u, v, \varphi, \phi, \psi, \omega)^T$ and $U^* = (u^*, v^*, \varphi^*, \phi^*, \psi^*, \omega^*)^T$ by

$$\begin{aligned}\langle U, U^*\rangle_{\mathcal{H}} = & \int_0^L [\rho v\overline{v^*} + \mu u_x\overline{u^*}_x + \kappa_1\phi\overline{\phi^*} + \alpha\varphi_x\overline{\varphi^*}_x + \alpha_1\varphi\overline{\varphi^*} + \kappa_2\omega\overline{\omega^*} + \gamma\psi_x\overline{\psi^*}_x \\ & +\alpha_2\psi\overline{\psi^*} + b(u_x\overline{\varphi^*} + \varphi\overline{u^*}_x) + d(u_x\overline{\psi^*} + \psi\overline{u^*}_x) + b_1(\psi_x\overline{\varphi^*}_x + \varphi_x\overline{\psi^*}_x) \\ & +\alpha_3(\psi\overline{\varphi^*} + \varphi\overline{\psi^*})]dx.\end{aligned}$$

By introducing the new variables $v=u_t, \phi=\varphi_t$ and $\omega=\psi_t$, system (2.1) becomes

$$\begin{cases} u_t = v, \\ v_t = \frac{1}{\rho}\left(\mu u_{xx} + b\varphi_x + d\psi_x + \lambda u_{txx}\right), \\ \varphi_t = \phi, \\ \phi_t = \frac{1}{\kappa_1}\left(\alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t\right), \\ \psi_t = \omega, \\ \omega_t = \frac{1}{\kappa_2}\left(b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t\right), \end{cases}$$

then (2.1) can be written under form

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = \left(u_0, u_1, \varphi_0, \varphi_1, \psi_0, \psi_1\right)^T, \end{cases} \tag{2.4}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the operator defined by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}\partial_{xx} & \frac{\lambda}{\rho}\partial_{xx} & \frac{b}{\rho}\partial_x & 0 & \frac{d}{\rho}\partial_x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{b}{\kappa_1}\partial_x & 0 & \frac{\alpha}{\kappa_1}\partial_{xx} - \frac{\alpha_1}{\kappa_1} & -\frac{\tau_1}{\kappa_1} & \frac{b_1}{\kappa_1}\partial_{xx} - \frac{\alpha_3}{\kappa_\&} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{d}{\kappa_2}\partial_x & 0 & \frac{b_1}{\kappa_2}\partial_{xx} - \frac{\alpha_3}{\kappa_2} & 0 & \frac{\gamma}{\kappa_2}\partial_{xx} - \frac{\alpha_2}{\kappa_2} & -\frac{\tau_2}{\kappa_2} \end{pmatrix} \tag{2.5}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} (u, v, \varphi, \phi, \psi, w) \in \mathcal{H} \; |v \in H_0^1(0,L), \phi, \omega \in H_*^1(0,L), \\ \mu u + \lambda v \in H^2 \cap H_0^1, \varphi, \psi \in H^2, \; \varphi_x(t,x) = \; \psi_x(t,x) = 0, \; x = 0, L \end{array} \right\}.$$

Our existence and uniqueness result reads as follows.

**Theorem 2.2.1.** *For any* $(u_0, u_1, \varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H}$ *the problem (2.1)-(2.3) has a unique weak solution* $(u, \varphi, \psi)$ *that satisfies*

$$u \in C\left(0,+\infty; H_0^1(0,L)\right), \; \varphi \in C\left(0,+\infty; H_*^1(0,L)\right), \; \psi \in C\left(0,+\infty; H_*^1(0,L)\right).$$

*Moreover, if* $(u_0, u_1, \varphi_0, \varphi_1, \psi_0, \psi_1) \in D(\mathcal{A})$ *then the solution* $(u, \varphi, \psi)$ *satisfies*

$$u \in C\left(0,+\infty; H^2 \cap H_0^1(0,L)\right) \cap C^1\left(0,+\infty; H_0^1(0,L)\right),$$

$$\varphi, \psi \in C\left(0,+\infty; H^2 \cap H_*^1(0,L)\right) \cap C^1\left(0,+\infty; H_*^1(0,L)\right).$$

*Proof.* First, for any $(u,v,\varphi,\phi,\psi,\omega)\in D(\mathcal{A})$ we have

$$Re\,\langle \mathcal{A}U,U\rangle_{\mathcal{H}}=-\int_0^L|v_x|^2dx-\tau_1\int_0^L|\phi|^2dx-\tau_2\int_0^L|w|^2dx\le 0.$$

Thus, $\mathcal{A}$ is dissipative.

Secondly, we prove that $0\in\rho(\mathcal{A})$. Indeed, let $F=(f,g,h,l,q,k)\in\mathcal{H}$ and find $U=(u,v,\varphi,\phi,\psi,\omega)\in D(\mathcal{A})$ such that $\mathcal{A}U=F$, that is

$$\begin{cases} v=f,\\ \mu u_{xx}+b\varphi_x+d\psi_x+\lambda v_{xx}=\rho g,\\ \phi=h,\\ \alpha\varphi_{xx}+b_1\psi_{xx}-bu_x-\alpha_1\varphi-\alpha_3\psi-\tau_1\phi=\kappa_1 l,\\ \omega=q,\\ b_1\varphi_{xx}+\gamma\psi_{xx}-du_x-\alpha_3\varphi-\alpha_2\psi-\tau_2\omega=\kappa_2 k. \end{cases} \tag{2.6}$$

From $(2.6)_1$, $(2.6)_3$ and $(2.6)_5$ we have $v\in H_0^1(0,L)$, $\phi,\omega\in H_*^1(0,L)$.

Replacing $v,\phi$ and $\omega$ by $f,h$ and $l$ respectively we obtain

$$\begin{cases} \mu u_{xx}+b\varphi_x+d\psi_x=\rho g-\lambda f_{xx}=g_1\in H^{-1}(0,L),\\ \alpha\varphi_{xx}+b_1\psi_{xx}-bu_x-\alpha_1\varphi-\alpha_3\psi=\kappa_1 l+\tau_1 h=g_2\in L^2(0,L),\\ b_1\varphi_{xx}+\gamma\psi_{xx}-du_x-\alpha_3\varphi-\alpha_2\psi=\kappa_2 k+\tau_2 l=g_3\in L^2(0,L). \end{cases} \tag{2.7}$$

Taking the duality product of $(2.7)_1$ by $u^*$ and the $L^2-$product of $(2.7)_2$ and $(2.7)_3$ by $\varphi^*$ and $\psi^*$, respectively,we get

$$\begin{cases} \mu\int_0^L u_{xx}\overline{u^*}dx+b\int_0^L\varphi_x\overline{u^*}dx+d\int_0^L\psi_x\overline{u^*}dx=\langle g_1,u^*\rangle_{H^{-1}\times H_0^1},\\ \alpha\int_0^L\varphi_{xx}\overline{\varphi^*}dx+b_1\int_0^L\psi_{xx}\overline{\varphi^*}dx-b\int_0^L u_x\overline{\varphi^*}dx-\alpha_1\int_0^L\varphi\overline{\varphi^*}dx\quad-\alpha_3\int_0^L\psi\overline{\varphi^*}dx\\ \qquad=\int_0^L g_2\overline{\varphi^*}dx,\\ b_1\int_0^L\varphi_{xx}\overline{\psi^*}dx+\gamma\int_0^L\psi_{xx}\overline{\psi^*}dx-d\int_0^L u_x\overline{\psi^*}dx-\alpha_3\int_0^L\varphi\overline{\psi^*}dx\quad-\alpha_2\int_0^L\psi\overline{\psi^*}dx\\ \qquad=\int_0^L g_3\overline{\psi^*}dx, \end{cases}$$

then summing up we obtain

$$a(U,U^*)=L(U^*), \tag{2.8}$$

where $U=(u,\varphi,\psi)^T$, $U^*=(u^*,\varphi^*,\psi^*)^T$ and

$$a(U,U^*)=\mu\int_0^L u_x\overline{u_x^*}dx+b\int_0^L\varphi\overline{u_x^*}dx+d\int_0^L\psi\overline{u_x^*}dx$$

$$+\alpha\int_0^L \varphi_x\overline{\varphi_x^*}dx+b_1\int_0^L \psi_x\overline{\varphi_x^*}dx+b\int_0^L u_x\overline{\varphi^*}dx+\alpha_1\int_0^L \varphi\overline{\varphi^*}dx+\alpha_3\int_0^L \psi\overline{\varphi^*}dx$$

$$+b_1\int_0^L \varphi_x\overline{\psi_x^*}dx+\gamma\int_0^L \psi_x\overline{\psi_x^*}dx+d\int_0^L u_x\overline{\psi^*}dx+\alpha_3\int_0^L \varphi\overline{\psi^*}dx+\alpha_2\int_0^L \psi\overline{\psi^*}dx,$$

$$L\left(U^*\right)=\left\langle -g_1,u^*\right\rangle_{H^{-1}\times H_0^1}-\int_0^L g_2\overline{\varphi^*}dx-\int_0^L g_3\overline{\psi^*}dx,$$

are a bilinear and linear forms over the Hilbert space $\mathcal{W}=H_0^1\left(0,L\right)\times H_*^1\left(0,L\right)\times H_*^1\left(0,L\right)$ respectively.

New we prouve that $a\left(\cdot,\cdot\right)$ and $L$ are continuous.

$$\begin{aligned}\left|a\left(U,U^*\right)\right| &\leq \left|\mu\right|\int_0^L \left|u_x\overline{u_x^*}\right|dx+\left|b\right|\int_0^L \left|\varphi\overline{u_x^*}\right|dx+\left|d\right|\int_0^L \left|\psi\overline{u_x^*}\right|dx\\ &+\left|\alpha\right|\int_0^L \left|\varphi_x\overline{\varphi_x^*}\right|dx+\left|b_1\right|\int_0^L \left|\psi_x\overline{\varphi_x^*}\right|dx+\left|b\right|\int_0^L \left|u_x\overline{\varphi^*}\right|dx+\left|\alpha_1\right|\int_0^L \left|\varphi\overline{\varphi^*}\right|dx\\ &+\left|\alpha_3\right|\int_0^L \left|\psi\overline{\varphi^*}\right|dx+\left|b_1\right|\int_0^L \left|\varphi_x\overline{\psi_x^*}\right|dx+\left|\gamma\right|\int_0^L \left|\psi_x\overline{\psi_x^*}\right|dx\\ &+\left|d\right|\int_0^L \left|u_x\overline{\psi^*}\right|dx+\left|\alpha_3\right|\int_0^L \left|\varphi\overline{\psi^*}\right|dx+\left|\alpha_2\right|\int_0^L \left|\psi\overline{\psi^*}\right|dx,\end{aligned}$$

then, we use Cauchy-Schwarz inequality

$$\left|a\left(U,U^*\right)\right|\leq\left|\mu\right|\left\|u_x\right\|_{L^2}\left\|\overline{u_x^*}\right\|_{L^2}+\left|b\right|\left\|\varphi\right\|_{L^2}\left\|\overline{u_x^*}\right\|_{L^2}+\left|d\right|\left\|\psi\right\|_{L^2}\left\|\overline{u_x^*}\right\|_{L^2}$$

$$+\left|d\right|\left\|\psi\right\|_{L^2}\left\|\overline{u_x^*}\right\|_{L^2}+\left|\alpha\right|\left\|\varphi_x\right\|_{L^2}\left\|\overline{\varphi_x^*}\right\|_{L^2}+\left|b_1\right|\left\|\psi_x\right\|_{L^2}\left\|\overline{\varphi_x^*}\right\|_{L^2}+\left|b\right|\left\|u_x\right\|_{L^2}\left\|\overline{\varphi^*}\right\|_{L^2}$$

$$+\left|\alpha_1\right|\left\|\varphi\right\|_{L^2}\left\|\overline{\varphi^*}\right\|_{L^2}+\left|\alpha_3\right|\left\|\psi\right\|_{L^2}\left\|\overline{\varphi^*}\right\|_{L^2}+\left|b_1\right|\left\|\varphi_x\right\|_{L^2}\left\|\overline{\psi_x^*}\right\|_{L^2}+\left|\gamma\right|\left\|\psi_x\right\|_{L^2}\left\|\overline{\psi_x^*}\right\|_{L^2}$$

$$+\left|d\right|\left\|u_x\right\|_{L^2}\left\|\overline{\psi^*}\right\|_{L^2}+\left|\alpha_3\right|\left\|\varphi\right\|_{L^2}\left\|\overline{\psi^*}\right\|_{L^2}+\left|\alpha_2\right|\left\|\psi\right\|_{L^2}\left\|\overline{\psi^*}\right\|_{L^2},$$

by estimete we get

$$\left|a\left(U,U^*\right)\right|\leq C\left\|U\right\|_{\mathcal{W}}\left\|U^*\right\|_{\mathcal{W}}$$

and

$$\begin{aligned}\left|L\left(U^*\right)\right| &\leq \left|\left\langle -g_1,u^*\right\rangle\right|_{H^{-1}\times H_0^1}+\int_0^L\left|g_2\overline{\varphi^*}\right|dx+\int_0^L\left|g_3\overline{\psi^*}\right|dx\\ &\leq \left\|g_1\right\|_{H^1}\left\|u^*\right\|_{H_0^1}+\left\|g_2\right\|_{L^2}\left\|\varphi^*\right\|_{L^2}+\left\|g_3\right\|_{L^2}\left\|\psi^*\right\|_{L^2},\end{aligned}$$

$$\left|L\left(U^*\right)\right|\leq C\left\|U^*\right\|_{\mathcal{W}},$$

Thus $a(\cdot,\cdot)$ and $L$ are continuous. Moreover, straightforward calculations show that

$$a(U,U)=\mu\int_0^L |u_x|^2\,dx+b\int_0^L \varphi\overline{u}_x dx+d\int_0^L \psi\overline{u}_x dx+d\int_0^L u_x\overline{\psi}dx$$
$$+b\int_0^L u_x\overline{\varphi}dx+\alpha_1\int_0^L |\varphi|^2\,dx+\alpha_3\int_0^L \psi\overline{\varphi}dx+\alpha_3\int_0^L \varphi\overline{\psi}dx+\alpha_2\int_0^L |\psi|^2 dx$$
$$+\frac{1}{2}\int_0^L\left[\alpha\left|\varphi_x+\frac{b_1}{\alpha}\psi_x\right|^2+\gamma\left|\psi_x+\frac{b_1}{\gamma}\varphi_x\right|^2\right]$$
$$+\frac{1}{2}\left(\alpha-\frac{b_1^2}{\gamma}\right)\int_0^L |\varphi_x|^2\,dx+\frac{1}{2}\left(\gamma-\frac{b_1^2}{\alpha}\right)\int_0^L |\psi_x|^2\,dx.$$

On the other hand, there exists $\eta>0$ such that the matrix

$$B'=\begin{pmatrix}\mu-\eta & b & d\\ b & \alpha_1-\eta & \alpha_3\\ d & \alpha_3 & \alpha_2-\eta\end{pmatrix}$$

still positive definite. Therefore,

$$(u_x,\varphi,\psi)\,B'\left(\overline{u}_x,\overline{\varphi},\overline{\psi}\right)^T=\mu\int_0^L |u_x|^2\,dx+b\int_0^L \varphi\overline{u}_x dx+d\int_0^L \psi\overline{u}_x dx+d\int_0^L u_x\overline{\psi}dx$$
$$+b\int_0^L u_x\overline{\varphi}dx+\alpha_1\int_0^L |\varphi|^2\,dx+\alpha_3\int_0^L \psi\overline{\varphi}dx+\alpha_3\int_0^L \varphi\overline{\psi}dx+\alpha_2\int_0^L |\psi|^2 dx>0$$

then we can wrriten $a(U,U)$ as

$$a(U,U)=(u_x,\varphi,\psi)\,B'\left(\overline{u}_x,\overline{\varphi},\overline{\psi}\right)^T+\eta\left(\|u_x\|^2+\|\varphi\|^2+\|\psi\|^2\right)$$
$$+\frac{1}{2}\int_0^L\left[\alpha\left|\varphi_x+\frac{b_1}{\alpha}\psi_x\right|^2+\gamma\left|\psi_x+\frac{b_1}{\gamma}\varphi_x\right|^2\right]+\frac{1}{2}\left(\alpha-\frac{b_1^2}{\gamma}\right)\int_0^L |\varphi_x|^2\,dx$$
$$+\frac{1}{2}\left(\gamma-\frac{b_1^2}{\alpha}\right)\int_0^L |\psi_x|^2\,dx\geq+\eta\left(\|u_x\|^2+\|\varphi\|^2+\|\psi\|^2\right).$$

Thus

$$a(U,U)\geq\widetilde{\eta}\,\|U\|_{\mathcal{W}}^2,$$

for a positive constant $\widetilde{\eta}>0$, which shows the coercivity of $a(\cdot,\cdot)$.

Thus, Lax-Milgram theorem ensures the existence of unique $(u,\varphi,\psi)\in\mathcal{W}$ satisfying

$$a(U,U^*)=L(U^*),\ \forall U^*\in\mathcal{W}.$$

Now, taking $\varphi^* = \psi^* = 0$ in (2.8) and replacing $f$ by $v$ we get

$$\mu \int_0^L u_x \overline{u^*}_x dx + b \int_0^L \varphi \overline{u^*}_x dx + d \int_0^L \psi \overline{u^*}_x dx = \int_0^L (\lambda v_{xx} - \rho g) \overline{u^*} dx,$$

and integration by parts gives

$$\int_0^L (\mu u_x + \lambda v_x) \overline{u^*}_x dx = \int_0^L (b\varphi_x + d\psi_x - \rho g) \overline{u^*} dx, \ \ \forall u^* \in H_0^1(0, L),$$

therefore,

$$\mu u + \lambda v \in H^2(0, L).$$

Next, let $\varphi^* \in H_0^1(0, L)$ and define

$$\varphi_1^*(x) = \varphi^*(x) - \int_0^L \varphi^*(x) dx.$$

Observing that $\varphi_1^* \in H_*^1(0, L)$ and taking $u^* = \psi^* = 0$ in (2.8) we obtain

$$\int_0^L (\alpha\varphi_x + b_1\psi_x) \overline{\varphi_{1x}^*} dx = -\int_0^L (bu_x + \alpha_1\varphi + \alpha_3\psi + g_2) \overline{\varphi_1^*} dx, \ \forall \varphi^* \in H_0^1(0, L),$$

therefore,

$$\alpha\varphi + b_1\psi \in H^2(0, L). \tag{2.9}$$

Moreover, integration by parts gives

$$(\alpha\varphi_x(L) + b_1\psi_x(L)) \overline{\varphi_1^*}(L) - (\alpha\varphi_x(0) + b_1\psi_x(0)) \overline{\varphi_1^*}(0) = 0, \ \forall \varphi^* \in H_0^1(0, L).$$

Since $\varphi^*$ is arbitrary, taking $\varphi^*(0) = 0$ and $\varphi^*(L) \neq 0$ we obtain

$$\alpha\varphi_x(L) + b_1\psi_x(L) = 0,$$

then taking $\varphi^*(0) \neq 0$ and $\varphi^*(L) = 0$ we get

$$\alpha\varphi_x(0) + b_1\psi_x(0) = 0.$$

Similarly,

$$b_1\varphi + \gamma\psi \in H^2(0, L) \tag{2.10}$$

and

$$b_1\varphi_x(0) + \gamma\psi_x(0) = b_1\varphi_x(L) + \gamma\psi_x(L) = 0.$$

Thus,

$$\varphi, \psi \in H^2(0,L) \quad \text{and} \quad \varphi_x = \psi_x = 0, \quad \text{for} \quad x = 0, L.$$

Therefore, $U \in D(\mathcal{A})$ and $0 \in \rho(\mathcal{A})$. Moreover, using a geometric series argument we prove that $\lambda I - \mathcal{A} = \mathcal{A}(\lambda\mathcal{A}^{-1} - I)$ is invertible for $|\lambda| < \|\mathcal{A}^{-1}\|$, then $\lambda \in \rho(\mathcal{A})$, which completes the proof that $\mathcal{A}$ is the infinitesimal generator of a $C_0-$semigroup, then the Lumer-Phillips theorem ensures the existence of unique solution to the problem (2.1)-(2.3) satisfying the statements of Theorem 1. □

**Remark 2.1.** We note that if $U_0 \in D(\mathcal{A})$ then the solution

$$U(t) = e^{t\mathcal{A}}U_0 \in C((0,\infty); D(\mathcal{A})) \cap C^1((0,\infty); \mathcal{H})$$

and (2.4) is satisfied in $\mathcal{H}$ for every $t > 0$. It turns out that $u, \varphi, \psi$ satisfy (2.1) in the strong sense.

If $U_0 \in \mathcal{H}$ there exists a sequence $U_{0n} \in D(\mathcal{A})$ converging to $U_0$ in $\mathcal{H}$. Accordingly, there exists a sequence of solutions $U_n(t) = e^{t\mathcal{A}}U_{0n}$ such that $u_n, \varphi_n, \psi_n$ satisfy (2.1) in $L^2$ for every $t > 0$, and for any $T > 0$, $u_n \to u$ in $C((0,T), H_0^1) \cap C^1((0,T); L^2)$, $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $C((0,T), H_*^1) \cap C^1((0,T); L^2)$. Therefore, if we multiply the equations of (2.1) for $u_n, \varphi_n, \psi_n$ by $u^* \in H_0^1$ and $\varphi^*, \psi^* \in H_*^1$, respectively, then integrate by parts with respect to $x$ and integrate with respect to $t$, finally passing to the limit, we find that $u, \varphi$ and $\psi$ are weak solutions to the variational form of system (2.1).

## 2.3 Exponential stability

In this section we establish an exponential decay of the solution of the system (2.1)-(2.3). The theorem, due to Gearhart and Prüss [27], gives the necessary and sufficient conditions of exponential stability of a $C_0$-semigroup generated by an operator $\mathcal{A}$.

The main result of this section is given by the following theorem:

**Theorem 2.3.1.** (Gearhart Prüss) *For any* $(u_0, u_1, \varphi_0, \varphi_1, \psi_0, \psi_1) \in D(\mathcal{A})$, *the energy associated with the solution of the problem (2.4) satisfies the estimate*

$$E(t) \leq \beta e^{-\omega t}, \quad \forall t \geq 0,$$

*where $\beta, \omega$ are two positive constants.*

*Proof.* The proof of this theorem will be established through the two following lemmas. $\square$

**Lemme 2.3.1.** *Let $\mathcal{A}$ be the operator defined by (2.5). Then,*

$$i\mathbb{R} = \{i\lambda; \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}),$$

*where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.*

*Proof.* The proof of the lemma will be established in 3 steps:
(i) Let $\lambda \in \mathbb{R}$

$$i\lambda I - \mathcal{A} = \mathcal{A}\left(i\lambda\mathcal{A}^{-1} - I\right),$$

$i\lambda I - \mathcal{A}$ is invertible if $\mathcal{A}$ is invertible and $(i\lambda\mathcal{A}^{-1} - I)$ is invertible.

Using the fact that $0 \in \rho(\mathcal{A})$ we have $\mathcal{A}$ is invertible.

Using a geometric series convergence argument it follows that $(i\lambda\mathcal{A}^{-1} - I)$ is invertible if $\|i\lambda\mathcal{A}^{-1}\| < 1$, then $|\lambda| < \|\mathcal{A}^{-1}\|^{-1}$.

the operator $i\lambda I - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - I)$ is invertible for $\lambda \in \left]-\|\mathcal{A}^{-1}\|^{-1}, \|\mathcal{A}^{-1}\|^{-1}\right[$.

Moreover $\|(i\lambda I - \mathcal{A})^{-1}\|$ is a continuous function of $\lambda$ in the interval $(-\|\mathcal{A}^{-1}\|^{-1}, \|\mathcal{A}^{-1}\|^{-1})$.

(ii) If there exists a constant $M > 0$, such that

$$\sup\left\{\left\|(i\lambda I - \mathcal{A})^{-1}\right\|, |\lambda| < \left\|\mathcal{A}^{-1}\right\|^{-1}\right\} = M < \infty, \tag{2.11}$$

then, for $|\lambda_0| < \|\mathcal{A}^{-1}\|^{-1}$,

$$\begin{aligned} i\lambda I - \mathcal{A} &= i\lambda I - \mathcal{A} + i\lambda_0 I - i\lambda_0 I \\ &= (i\lambda_0 I - \mathcal{A}) + (i\lambda - i\lambda_0) \\ &= (i\lambda_0 I - \mathcal{A})(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1}) \end{aligned}$$

$i\lambda I - \mathcal{A}$ is invertible if $(i\lambda_0 I - \mathcal{A})$ is invertible and $(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1})$ is invertible.

$(i\lambda_0 I - \mathcal{A})$ is invertible because $|\lambda_0| < \|\mathcal{A}^{-1}\|^{-1}$,

again the geometric series argument ensures that the operator $(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1})$ is invertible if $\|i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1}\| < 1$,

$$\begin{aligned} |\lambda - \lambda_0| \left\|(i\lambda_0 I - \mathcal{A})^{-1}\right\| &< 1, \\ \frac{1}{M} &\leq \frac{1}{\|(i\lambda_0 I - \mathcal{A})^{-1}\|}. \end{aligned}$$

$$|\lambda - \lambda_0| \sup \left\|(i\lambda_0 I - \mathcal{A})^{-1}\right\| < 1,$$

$$|\lambda - \lambda_0| M < 1,$$

$$|\lambda| - \left\|\mathcal{A}^{-1}\right\|^{-1} \leq |\lambda| - |\lambda_0| \leq |\lambda - \lambda_0| < \frac{1}{M},$$

$$|\lambda| \leq \frac{1}{M} + \left\|\mathcal{A}^{-1}\right\|^{-1}.$$

It turns out that if we choose $|\lambda_0|$ as close as possible to $\|\mathcal{A}^{-1}\|^{-1}$, we have that $i\lambda I - \mathcal{A}$ is invertible for $|\lambda| < \|\mathcal{A}^{-1}\|^{-1} + \frac{1}{M}$. Therefore,

$$\left\{i\lambda; |\lambda| < \left\|\mathcal{A}^{-1}\right\|^{-1} + \frac{1}{M}\right\} \subset \rho(\mathcal{A})$$

and $\|(i\lambda I - \mathcal{A})^{-1}\|$ is a continuous function of $\lambda$ in the interval
$\left(-\|\mathcal{A}^{-1}\|^{-1} - \frac{1}{M}, \|\mathcal{A}^{-1}\|^{-1} + \frac{1}{M}\right)$. So on

$$\left\{i\lambda; |\lambda| < \left\|\mathcal{A}^{-1}\right\|^{-1} + \frac{1}{M} + \frac{1}{M'}\right\} \subset \rho(\mathcal{A})$$

provided that

$$\sup\left\{\left\|(i\lambda I - \mathcal{A})^{-1}\right\|, |\lambda| < \left\|\mathcal{A}^{-1}\right\|^{-1} + M^{-1}\right\} = M' < \infty.$$

The interval of imaginary axis included in the resolvent set can be extended indefinitely until it coincides with $i\mathbb{R}$.
(iii) If $i\mathbb{R} \not\subset \rho(\mathcal{A})$ then from the argument (ii) above, there exists $\sigma \in \mathbb{R}$ with $\|\mathcal{A}^{-1}\|^{-1} \leq |\sigma| < \infty$ such that

$$\{i\lambda; |\lambda| < |\sigma|\} \subset \rho(\mathcal{A})$$

and

$$\sup\left\{\left\|(i\lambda I - \mathcal{A})^{-1}\right\|, |\lambda| < |\sigma|\right\} = \infty.$$

we have

$$(i\lambda_n I - \mathcal{A}) U_n = F_n, \tag{2.12}$$

multiplying (2.12) by $(i\lambda_n I - \mathcal{A})^{-1}$ we obtain

$$U_n = (i\lambda_n I - \mathcal{A})^{-1}(i\lambda_n I - \mathcal{A}) U_n = (i\lambda_n I - \mathcal{A})^{-1} F_n$$

then

$$\begin{aligned}\left\|(i\lambda I-\mathcal{A})^{-1}\right\| &= \sup\frac{\|(i\lambda_n I-\mathcal{A})^{-1}F_n\|}{\|F_n\|},\\ &= \sup\frac{\|U_n\|}{\|(i\lambda_n I-\mathcal{A})\,U_n\|},\\ &= \sup\frac{1}{\|(i\lambda_n I-\mathcal{A})\,U_n\|}.\end{aligned}$$

Thus, there exist a sequence $(\lambda_n)\subset\mathbb{R}$ such that $|\lambda_n|<|\sigma|\,,\lambda_n\longrightarrow\sigma$ and a sequence of unit vectors $(U_n)\,,U_n=(u_n,v_n,\phi_n,\varphi_n,\theta_n)\in D\,(\mathcal{A})\,,$ such that

$$\lim_{n\longrightarrow\infty}\|(i\lambda_n I-\mathcal{A})\,U_n\|=0,$$

that is

$$i\lambda_n u_n-v_n\to 0,\ \text{in}\ H_0^1\ , \tag{2.13}$$

$$i\lambda_n\rho v_n-\mu D^2u_n-bD\varphi_n-dD\psi_n-\lambda D^2v_n\longrightarrow 0,\ \text{in}\ L^2, \tag{2.14}$$

$$i\lambda_n\varphi_n-\phi_n\longrightarrow 0,\ \text{in}\ H_*^1, \tag{2.15}$$

$$i\lambda_n\kappa_1\phi_n-\alpha D^2\varphi_n-b_1D^2\psi_n+bDu_n+\alpha_1\varphi_n+\alpha_3\psi_n+\tau_1\phi_n\longrightarrow 0,\ \text{in}\ L^2, \tag{2.16}$$

$$i\lambda_n\psi_n-\omega_n\longrightarrow 0,\ \text{in}\ H_*^1, \tag{2.17}$$

$$i\lambda_n\kappa_2\omega_n-b_1D^2\varphi_n-\gamma D^2\psi_n+dDu_n+\alpha_3\varphi_n+\alpha_2\psi_n+\tau_2\omega_n\longrightarrow 0,\ \text{in}\ L^2. \tag{2.18}$$

First we have

$$\langle(i\lambda_n I-\mathcal{A})U_n,U_n\rangle_{\mathcal{H}}=\langle i\lambda_n U_n,U_n\rangle_{\mathcal{H}}-\langle\mathcal{A}U_n,U_n\rangle_{\mathcal{H}}\longrightarrow 0.$$

Take the reel part,Thus

$$\begin{aligned}Re\langle(i\lambda_n I-\mathcal{A})U_n,U_n\rangle_{\mathcal{H}} &= -Re\langle\mathcal{A}U_n,U_n\rangle_{\mathcal{H}},\\ &= \tau_2\int_0^L|\omega_n|^2dx+\tau_1\int_0^L|\phi_n|^2dx+\gamma\int_0^L|Dv_n|^2dx\longrightarrow 0.\end{aligned}$$

Therefore

$$\|\phi_n\|_{L^2}\quad\|\omega_n\|_{L^2}\longrightarrow 0, \tag{2.19}$$

and

$$\|Dv_n\|_{L^2}\longrightarrow 0. \tag{2.20}$$

Moreover, Poincaré's inequality leads to

$$\|v_n\|_{L^2} \to 0. \tag{2.21}$$

Since $\lambda_n$ is bounded, taking the $L^2-$product of (2.13), (2.15) and (2.17) by $u, \varphi$ and $\psi$, respectively, we get

$$i\lambda_n u_n - v_n \to 0, \text{ in } H_0^1,$$

$$i\lambda_n \varphi_n - \phi_n \longrightarrow 0, \text{ in } H_*^1,$$

$$i\lambda_n \psi_n - \omega_n \longrightarrow 0, \text{ in } H_*^1,$$

then

$$i\lambda_n \|u_n\|_{L^2} - \langle v_n, u_n\rangle \to 0, \text{ in } H_0^1,$$

$$i\lambda_n \|\varphi_n\|_{L^2} - \langle \phi_n, \varphi_n\rangle \longrightarrow 0, \text{ in } H_*^1,$$

$$i\lambda_n \|\psi_n\|_{L^2} - \langle \omega_n, \psi_n\rangle \longrightarrow 0, \text{ in } H_*^1,$$

using Cauchy-Schwarz inequality then the arguments (2.21), (2.20) and (2.19) and the fact that $U_n$ unit vectors give

$$|\langle v_n, u_n\rangle| \leq \|v_n\| \, \|u_n\| \to 0, \text{ in } H_0^1,$$

$$|\langle \phi_n, \varphi_n\rangle| \leq \|\phi_n\| \, \|\varphi_n\| \to 0, \text{in } H_*^1,$$

$$|\langle \omega_n, \psi_n\rangle| \leq \|\omega_n\| \, \|\psi_n\| \to 0, \text{in } H_*^1,$$

then we obtain

$$\|u_n\|_{L^2} \to 0, \ \|Du_n\|_{L^2} \to 0. \tag{2.22}$$

$$\|\varphi_n\|_{L^2} \to 0, \ \|\psi_n\|_{L^2} \to 0. \tag{2.23}$$

Removing the terms that tend to 0 from (2.16) and (2.18), it remains

$$\begin{cases} \alpha D^2\varphi_n + b_1 D^2\psi_n \longrightarrow 0, \text{ in } L^2, \\ b_1 D^2\varphi_n + \gamma D^2\psi_n \longrightarrow 0, \text{ in } L^2. \end{cases} \tag{2.24}$$

Multiplying $(2.24)_1$ by $\gamma\varphi_n$, $(2.24)_2$ by $b_1\varphi_n$ and subtracting we obtain

$$\|D\varphi_n\|_{L^2} \to 0. \tag{2.25}$$

Similarly,

$$\|D\psi_n\|_{L^2} \to 0. \tag{2.26}$$

By combining (3.19),(4.14),(2.22),(2.23),(2.25) and (2.26) we obtain that $\|U_n\|_{\mathcal{H}} \to 0$, which contradicts the fact that $\|U_n\|_{\mathcal{H}} = 1$. Thus, the proof of Lemma 2.3.1 is completed. □

**Lemme 2.3.2.** *The operator $\mathcal{A}$ defined by (2.5) satisfies*

$$\lim_{|\lambda|\to\infty} \sup \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.27}$$

*Proof.* To prove the lemma statement we use a contradiction argument. Suppose that (2.27) does not hold, that is

$$\lim_{|\lambda|\to\infty} \sup \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

Then, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and a sequence of unit vectors $U_n = (u_n, v_n, \varphi_n, \phi_n, \psi_n, \omega_n) \in D(\mathcal{A})$ such that

$$\lim_{|\lambda_n|\longrightarrow+\infty} \|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} \longrightarrow 0.$$

As in the proof of the previous lemma, (2.13)-(2.18) hold. Consequently,

$$\|\omega_n\|_{L^2} \longrightarrow 0, \|\phi_n\|_{L^2} \longrightarrow 0, \|Dv_n\|_{L^2} \longrightarrow 0, \ \|v_n\|_{L^2} \longrightarrow 0. \tag{2.28}$$

By dividing (2.13),(2.15) and (2.17) by $i\lambda_n$, and taking $L^2-$product by $u_n$, $\varphi_n$ and $\psi_n$, respectively, we obtain

$$\left\langle u_n + i\frac{v_n}{\lambda_n}, u_n \right\rangle \to 0, \text{ in } H_0^1,$$

$$\left\langle \varphi_n + i\frac{\phi_n}{\lambda_n}, \varphi_n \right\rangle \longrightarrow 0, \text{ in } H_*^1,$$

$$\left\langle \psi_n + i\frac{\omega_n}{\lambda_n}, \psi_n \right\rangle \longrightarrow 0, \text{ in } H_*^1,$$

then

$$\|u_n\|_{L^2} - \frac{1}{\lambda_n}\langle v_n, iu_n\rangle \to 0, \text{ in } H_0^1,$$

$$\|\varphi_n\|_{L^2} - \frac{1}{\lambda_n}\langle \phi_n, i\varphi_n\rangle \longrightarrow 0, \text{ in } H_*^1,$$

$$\|\psi_n\|_{L^2} - \frac{1}{\lambda_n}\langle \omega_n, i\psi_n\rangle \longrightarrow 0, \text{ in } H_*^1,$$

Sience $\frac{1}{\lambda_n} \to 0$ and the arguments (2.28) gives

$$\|u_n\|_{L^2}, \|Du_n\|_{L^2} \longrightarrow 0, \tag{2.29}$$

$$\|\varphi_n\|_{L^2}, \|\psi_n\|_{L^2} \longrightarrow 0. \tag{2.30}$$

The $L^2$ product of (2.15) by $\phi_n$ gives

$$\langle i\lambda_n \varphi_n - \phi_n, \phi_n \rangle \longrightarrow 0, \text{ in } H^1_*, \tag{2.31}$$

$$i\lambda_n \langle \varphi_n, \phi_n \rangle - \|\phi_n\|^2 \longrightarrow 0.$$

Therefore,

$$i\lambda_n \langle \varphi_n, \phi_n \rangle \longrightarrow 0. \tag{2.32}$$

Taking the inner product of (2.17) by $\varphi_n$ we get

$$i\lambda_n \langle \psi_n, \varphi_n \rangle - \langle \omega_n, \varphi_n \rangle \longrightarrow 0.$$

The fact that $|\langle \varphi_n, \omega_n \rangle| \leq \|\varphi_n\| \|\omega_n\| \longrightarrow 0$ yields

$$i\lambda_n \langle \varphi_n, \psi_n \rangle \longrightarrow 0. \tag{2.33}$$

Removing the terms that tend to 0 from (2.16), (2.18) it remains

$$i\lambda_n \kappa_1 \phi_n - \alpha D^2 \varphi_n - b_1 D^2 \psi_n \longrightarrow 0, \text{ in } L^2. \tag{2.34}$$

and

$$i\lambda_n \kappa_2 \omega_n - b_1 D^2 \varphi_n - \gamma D^2 \psi_n \longrightarrow 0, \text{ in } L^2. \tag{2.35}$$

Multiplying (2.34) by $\gamma\varphi_n$, (2.35) by $-b_1\varphi_n$, we get

$$\kappa_1 \gamma i\lambda_n \langle \phi_n, \varphi_n \rangle + \alpha\gamma \|\varphi_n\|^2 + b_1\gamma \langle D\psi_n, D\varphi_n \rangle \longrightarrow 0, \text{ in } L^2.$$

$$-b_1 \kappa_2 i\lambda_n \langle \omega_n, \varphi_n \rangle - b_1^2 \|\varphi_n\|^2 - \gamma b_1 \langle D\psi_n, D\varphi_n \rangle \longrightarrow 0, \text{ in } L^2.$$

summing up and taking into account (2.32), (2.33), we obtain

$$\left(\alpha\gamma - b_1^2\right) \|D\varphi_n\|^2 \longrightarrow 0.$$

Therefore,

$$\|D\varphi_n\| \longrightarrow 0. \tag{2.36}$$

The $L^2-$inner product of (2.17) by $\phi_n$ gives

$$i\lambda_n \langle \phi_n, \psi_n \rangle - \langle \phi_n, \omega_n \rangle \longrightarrow 0.$$

Recalling that $\|\phi_n\|, \|\omega_n\| \longrightarrow 0$ we arrive at

$$i\lambda_n \langle \phi_n, \psi_n \rangle \longrightarrow 0.$$

Similarly, multiplying (2.17) by $\omega_n$ we get

$$i\lambda_n \langle \omega_n, \psi_n \rangle \longrightarrow 0.$$

At this point we multiply (2.34) by $b_1\psi_n$, (2.35) by $-\alpha\psi_n$

$$\kappa_1 b_1 i\lambda_n \langle \phi_n, \psi_n \rangle + b_1\alpha \langle D\varphi_n, D\psi_n \rangle + b_1^2 \|D\psi_n\|^2 \longrightarrow 0, \text{ in } L^2.$$

and

$$-\alpha\kappa_2 i\lambda_n \langle \omega_n, \psi_n \rangle - b_1\alpha \langle D\varphi_n, D\psi_n \rangle - \gamma\alpha \|D\psi_n\|^2 \longrightarrow 0, \text{ in } L^2.$$

summing up and recalling that $\alpha\gamma - b_1^2 > 0$ we obtain

$$\|D\psi_n\| \longrightarrow 0. \tag{2.37}$$

From (2.28), (2.29), (2.30), (2.36) and (2.37) we have $\|U\|_{\mathcal{H}} \longrightarrow 0$ which contradicts the fact that $\|U\|_{\mathcal{H}} = 1$. Therefore, (2.27) holds and the proof of Lemma 4.2.4 is completed. $\square$

Combining the results of Lemmas 2.3.1, 4.2.4 and Theorem **??** the proof of Theorem 2.3.1 is completed.

Chapter 3

# Double porous elastic problem without visco-elastic dissipation

## 3.1 Introduction

In this chapter, we consider a one-dimensional double porous elastic system with one dissipative mechanisms: visco-porous dissipations in the porous equations. we study the system in the last chapter without viscoelastic dissipation given by the damping $\lambda u_{txx}$.

We prove the existence and uniqueness of a solution, by means of semigroup theory, and extend the exponential stability result to this problem provided that the coefficients satisfy some assumptions by use multiplier methode and Lyapunov functional, otherwise we prove the lack of the exponential decay.

In this chapter, we study the system (1.1) in the elimination of the viscoelastic dissipation, namely;

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x, & \text{in } (0,\infty)\times(0,\pi), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t - \tau_2\psi_t, & \text{in } (0,\infty)\times(0,\pi), \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_3\varphi_t - \tau_4\psi_t, & \text{in } (0,\infty)\times(0,\pi), \end{cases} \tag{3.1}$$

where $u$ is the transversal displacement of a one-dimensional porous elastic solid material of length $\pi$, $\varphi$ and $\psi$ are the porous unknown functions one associated to the pores in the skeleton and the other associated with the fissures in the material body, $\rho$ is the mass density, $\kappa_1, \kappa_2$ are the products of the mass density by the equilibrated inertias and the coefficients

$\mu, \alpha, \beta, \gamma, \alpha_1, \alpha_2, \alpha_3, b, d, \tau_1, \tau_2, \tau_3, \tau_4$ are parameters that related on the properties of the material and have to satisfy some restrictions that will be specifed later. The system (3.1) is endowed with the initial conditions

$$
\begin{aligned}
&u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\
&\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \\
&\psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x),
\end{aligned}
\tag{3.2}
$$

for all $x \in (0, \pi)$ and the following boundary conditions

$$
u_x(0,t) = u_x(\pi,t) = \varphi(0,t) = \varphi(\pi,t) = \psi(0,t) = \psi(\pi,t) = 0, \ t \geq 0, \tag{3.3}
$$

or

$$
u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \psi_x(0,t) = \psi_x(\pi,t) = 0, \ t \geq 0. \tag{3.4}
$$

New, we introduce some preliminaries and prove some technical lemmas which will be needed later. First, we note that the mass density $\rho$ and the inertias coefficients $\kappa_1$ and $\kappa_2$ are always positive. As coupling is considered the coefficients $b, d$ must not be zero simultaneously. We assume further that the constitutive cofficients $\mu, \alpha, \beta, \gamma, \alpha_1, \alpha_2$ are positive. Next we define the internal energy associated to the solution of (3.1) by:

$$
\begin{aligned}
E(t) := \frac{1}{2} \int_0^{\pi} & \left[ \rho u_t^2 + \kappa_1 \varphi_t^2 + \kappa_2 \psi_t^2 + \mu u_x^2 + \alpha \varphi_x^2 + \gamma \psi_x^2 + \alpha_1 \varphi^2 + \alpha_2 \psi^2 \right. \\
& \left. + 2\beta \varphi_x \psi_x + 2b u_x \varphi + 2d u_x \psi + 2\alpha_3 \varphi \psi \right] dx.
\end{aligned}
\tag{3.5}
$$

**Remark 3.1.1.** *To guarantee that the energy $E(t)$ is a positive definite form, we essume that that the matrix*

$$
A = \begin{pmatrix} \mu & b & d & 0 & 0 \\ b & \alpha_1 & \alpha_3 & 0 & 0 \\ d & \alpha_3 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \beta & \gamma \end{pmatrix}
$$

*is positive definite.*

**Remark 3.1.2.** *It is well known that any principal submatrix of a positive definite matrix is also positive definite, then we have*

$$\left(\alpha_1-\frac{b^2}{\mu}\right)\left(\alpha_2-\frac{d^2}{\mu}\right)-\left(\alpha_3-\frac{bd}{\mu}\right)^2>0, \tag{3.6}$$

$$\alpha_1\mu-b^2>0,\ \ \alpha_2\mu-d^2>0,\ \alpha_1\alpha_2-\alpha_3^2 \text{ and } \alpha\gamma-\beta^2>0. \tag{3.7}$$

*Therefore,*

$$\alpha\varphi_x^2+\gamma\psi_x^2+2\beta\varphi_x\psi_x=\frac{1}{2}\left(\alpha-\frac{\beta^2}{\gamma}\right)\varphi_x^2+\frac{1}{2}\left(\gamma-\frac{\beta^2}{\alpha}\right)\psi_x^2$$
$$+\frac{\alpha}{2}\left(\varphi_x+\frac{\beta}{\alpha}\psi_x\right)^2+\frac{\gamma}{2}\left(\psi_x+\frac{\beta}{\gamma}\varphi_x\right)^2\geq 0,$$

*and there exists a $\varepsilon>0$ such that the matrix*

$$B=\begin{pmatrix}\mu-\varepsilon & b & d\\ b & \alpha_1-\varepsilon & \alpha_3\\ d & \alpha_3 & \alpha_2-\varepsilon\end{pmatrix}$$

*still positive definite. Thus,*

$$E(t)=\frac{1}{2}\int_0^\pi\left(\rho u_t^2+\kappa_1\varphi_t^2+\kappa_2\psi_t^2+\varepsilon u_x^2+\varepsilon\varphi^2+\varepsilon\psi^2\right)dx$$
$$+\frac{1}{2}\int_0^\pi\left[(\mu-\varepsilon)u_x^2+2bu_x\varphi+2du_x\psi+(\alpha_1-\varepsilon)\varphi^2+2\alpha_3\varphi\psi+(\alpha_2-\varepsilon)\psi^2\right]dx$$
$$+\frac{1}{2}\int_0^L\left[\alpha\left(\varphi_x+\frac{b_1}{\alpha}\psi_x\right)^2+\gamma\left(\psi_x+\frac{b_1}{\gamma}\varphi_x\right)^2\right]dx$$
$$+\frac{1}{2}\left(\alpha-\frac{b_1^2}{\gamma}\right)\int_0^L|\varphi_x|^2\,dx+\frac{1}{2}\left(\gamma-\frac{b_1^2}{\alpha}\right)\int_0^L|\psi_x|^2\,dx\geq 0.$$

## 3.2 Existence and uniqueness

*In this section we present an existence and uniqueness result for the solutions of the problem determined by system and conditions (3.1),(3.2) and (3.4), the case of boundary conditions (3.3) is similar.*

*As Neumann boundary condutions are considered for $\varphi$ and $\psi$, Poincaré's inequality cannot be applied. From the second and the third equations of (3.1) and boundary conditions (3.4), we have*

$$\begin{aligned} \frac{d^2}{dt^2}\int_0^\pi \varphi dx &= -\frac{\alpha_1}{\kappa_1}\int_0^\pi \varphi dx - \frac{\alpha_3}{\kappa_1}\int_0^\pi \psi dx - \frac{\tau_1}{\kappa_1}\frac{d}{dt}\int_0^\pi \varphi dx - \frac{\tau_2}{\kappa_1}\frac{d}{dt}\int_0^\pi \psi dx, \\ \frac{d^2}{dt^2}\int_0^\pi \psi dx &= -\frac{\alpha_3}{\kappa_2}\int_0^\pi \varphi dx - \frac{\alpha_2}{\kappa_2}\int_0^\pi \psi dx - \frac{\tau_3}{\kappa_2}\frac{d}{dt}\int_0^\pi \varphi_t dx - \frac{\tau_4}{\kappa_2}\frac{d}{dt}\int_0^\pi \psi_t dx. \end{aligned} \tag{3.8}$$

*So if we set $X = \left(\int_0^\pi \varphi dx, \int_0^\pi \varphi_t dx, \int_0^\pi \psi dx, \int_0^\pi \psi_t dx\right)^T$ then (3.8) can be written*

$$X_t(t) = MX(t), \; X(0) = X_0, \tag{3.9}$$

*where*

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\alpha_1}{\kappa_1} & -\frac{\tau_1}{\kappa_1} & -\frac{\alpha_3}{\kappa_1} & -\frac{\tau_2}{\kappa_1} \\ 0 & 0 & 0 & 1 \\ -\frac{\alpha_3}{\kappa_2} & -\frac{\tau_3}{\kappa_2} & -\frac{\alpha_2}{\kappa_2} & -\frac{\tau_4}{\kappa_2} \end{pmatrix}$$

*and*

$$X_0 = \left(\int_0^\pi \varphi_0 dx, \int_0^\pi \varphi_1 dx, \int_0^\pi \psi_0 dx, \int_0^\pi \psi_1 dx\right)^T.$$

*Solving (3.9) we get*

$$X(t) = \exp(tM) X_0,$$

*in particular,*

$$\int_0^\pi \varphi dx = \sum_{k=1}^4 (\exp(tM))_{1k} X_{0k}, \quad \int_0^\pi \psi dx = \sum_{j=1}^4 (\exp(tM))_{3k} X_{0k}.$$

*Therefore, if we set*

$$\bar{\varphi} = \varphi - \sum_{k=1}^4 (\exp(tM))_{1k} X_{0k}, \quad \bar{\psi} = \psi - \sum_{j=1}^4 (\exp(tM))_{3k} X_{0k},$$

*then $\left(u, \overline{\varphi}, \overline{\psi}\right)$ solves (3.1) with boundary conditions (3.4), and we have*

$$\int_0^\pi \overline{\varphi} dx = \int_0^\pi \overline{\psi} dx = 0,$$

*which allows to apply Poincaré's inequality. In the sequel we will work with $\bar{\varphi}$ and $\bar{\psi}$ but for convenience, we write $\varphi, \psi$ instead of $\bar{\varphi}, \bar{\psi}$ respectively.*

*Furthermore, as porous dissipations are considered, the weights of porous dampings $\tau_1, \tau_2, \tau_3, \tau_4$ are supposed to satisfy*

$$4\tau_1\tau_4 > (\tau_2 + \tau_3)^2. \tag{3.10}$$

**Lemme 3.2.1.** *The energy $E(t)$satisfies along the solution $(u, \varphi, \psi)$ of (3.1)-(3.3) the estimate*

$$E'(t) = -\tau_1 \int_0^\pi \varphi_t^2 dx - \tau_4 \int_0^\pi \psi_t^2 dx - (\tau_2 + \tau_3) \int_0^\pi \varphi_t \psi_t dx \tag{3.11}$$

*and we have*

$$E'(t) = -\frac{1}{2}\left(\tau_1 - \frac{(\tau_2 + \tau_3)^2}{4\tau_2}\right) \int_0^\pi \varphi_t^2 dx - \frac{1}{2}\left(\tau_2 - \frac{(\tau_2 + \tau_3)^2}{4\tau_1}\right) \int_0^\pi \psi_t^2 dx$$

$$-\frac{\tau_1}{2} \int_0^\pi \left(\varphi_t + \frac{(\tau_2 + \tau_3)}{2\tau_1}\psi_t\right)^2 dx - \frac{\tau_2}{2} \int_0^\pi \left(\psi_t + \frac{(\tau_2 + \tau_3)}{2\tau_2}\varphi_t\right)^2 dx \leq 0. \tag{3.12}$$

*Proof.* Multiplying the equations of (3.1) by $u_t, \varphi_t$ and $\psi_t$ respectively, then integrating with respect to $x$ over $(0, \pi)$ and using integration by parts and boundary conditions (3.4), the estimate (3.11) follows immediately. □

*To prove the well-posedness, we use a semigroup approach. First, we introduce the Hilbert space*

$$\mathcal{H} = H_0^1(0, \pi) \times L^2(0, \pi) \times H_*^1(0, \pi) \times L_*^2(0, \pi) \times H_*^1(0, \pi) \times L_*^2(0, \pi),$$

*where,*

$$H_*^1(0, \pi) = \left\{\phi \in H^1(0, \pi) : \int_0^\pi \phi(x)\, dx = 0\right\},$$
$$L_*^2(0, \pi) = \left\{\phi \in L^2(0, \pi) : \int_0^\pi \phi(x)\, dx = 0\right\}.$$

*We note that $L_*^2(0, \pi)$ and $H_*^1(0, \pi)$ are closed subspaces of $L^2(0, \pi)$ and $H^1(0, \pi)$ respectively. Thus, they are Hilbert spaces and so $\mathcal{H}$ is.*

*Next, we rewrite the system (3.1) in the setting of Lumer-Phillips theorem, to do so we* *introduce the new variables $v = u_t, \phi = \varphi_t$ and $w = \psi_t$ the system (3.1) becomes*

$$\begin{cases} u_t = v \\ v_t = \dfrac{1}{\rho}\left(\mu u_{xx} + b\varphi_x + d\psi_x\right), \\ \varphi_t = \phi \\ \phi_t = \dfrac{1}{\kappa_1}\left(\alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\phi - \tau_2 w\right), \\ \psi_t = w \\ w_t = \dfrac{1}{\kappa_2}\left(\beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_3\phi - \tau_4 w\right), \end{cases}$$

*which can be written*

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{3.13}$$

*where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the operator defined by*

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}\partial_{xx} & 0 & \frac{b}{\rho}\partial_x & 0 & \frac{d}{\rho}\partial_x & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{b}{\kappa_1}\partial_x & 0 & \frac{\alpha}{\kappa_1}\partial_{xx} - \frac{\alpha}{\kappa_1} & -\frac{\tau_1}{\kappa_1} & \frac{\beta}{\kappa_1}\partial_{xx} - \frac{\alpha_3}{\kappa_1} & -\frac{\tau_2}{\kappa_1} \\ 0 & 0 & 0 & 0 & 0 & I \\ -\frac{d}{\kappa_2}\partial_x & 0 & \frac{\beta}{\kappa_2}\partial_{xx} - \alpha_3 & -\frac{\tau_3}{\kappa_2} & \frac{\gamma}{\kappa_2}\partial_{xx} - \frac{\alpha_2}{\kappa_2} & -\frac{\tau_4}{\kappa_2} \end{pmatrix}$$

*with domain*

$$D(\mathcal{A}) = \left(H^2(0,\pi) \cap H_0^1(0,\pi)\right) \times H_0^1(0,\pi) \times H_*^2(0,\pi) \times H_*^1(0,\pi) \times H_*^2(0,\pi) \times H_*^1(0,\pi).$$

*Here $I$ is the identity operator, $\partial$ denotes the derivative with respect to $x$ and*

$$H_*^2(0,\pi) = \left\{\phi \in H^2(0,\pi) : \phi(0) = \phi(\pi) = 0\right\}.$$

*Now, we state and prove the well-posedness theorem of the problem (3.1), (3.2) and (3.4).*

**Theorem 3.2.1.** *For any $U_0 = (u_0, u_1, \varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H}$, the problem (3.1),(3.2) and (3.4) has a unique weak solution $(u, \varphi, \psi)$ satisfies the property:*

$$u \in C\left([0,+\infty[\,; H_0^1(0,\pi)\right) \cap C^1\left([0,+\infty[\,; L^2(0,\pi)\right),$$
$$\varphi, \psi \in C\left([0,+\infty[\,; H_*^1(0,\pi)\right) \cap C^1\left([0,+\infty[\,; L_*^2(0,\pi)\right).$$

*Moreover, if* $U_0 \in D(\mathcal{A})$, *the solution* $(u, \varphi, \psi)$ *satisfies*

$$u \in C\left([0,+\infty[\,; H^2(0,\pi) \cap H_0^1(0,\pi)\right) \cap C^1\left([0,+\infty[\,; H_0^1(0,\pi)\right) \cap C^2\left([0,+\infty[\,; L^2(0,\pi)\right),$$
$$\varphi, \psi \in C\left([0,+\infty[\,; H_*^2(0,\pi)\right) \cap C^1\left([0,+\infty[\,; H_*^1(0,\pi)\right) \cap C^2\left([0,+\infty[\,; L_*^2(0,\pi)\right).$$

*Proof.* According to the Lumer-Phillips theorem, it suffices to prove that the operator $\mathcal{A}$ is dissipative and maximal.

First, we have for any $U \in D(\mathcal{A})$,

$$Re\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\tau_1 \int_0^\pi \varphi_t^2 dx - (\tau_2 + \tau_3) \int_0^\pi \varphi_t \psi_t dx - \tau_4 \int_0^\pi \psi_t^2 dx \leq 0.$$

Therefore, $\mathcal{A}$ is dissipative.

Secondly, let $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$, and find $U \in D(\mathcal{A})$ such that $\mathcal{A}U = \mathcal{F}$,

that is,

$$\begin{cases} v = f_1 \in H_0^1, \\ \mu u_{xx} + b\varphi_x + d\psi_x = \rho f_2 \in L^2, \\ \phi = f_3 \in H_*^1, \\ \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\phi - \tau_2 w = \kappa_1 f_4 \in L_*^2, \\ w = f_5 \in H_*^1, \\ \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_3\phi - \tau_4 w = \kappa_2 f_6 \in L_*^2. \end{cases}$$

From the first, the third and the fifth equations we have $v \in H_0^1(0,\pi)$ and $\phi, w \in H_*^1(0,\pi)$.

Substituting $v, \phi$ and $w$ by $f_1, f_3$ and $f_5$ we get

$$\begin{cases} \mu u_{xx} + b\varphi_x + d\psi_x = \rho f_2 \in L^2, \\ \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi = \kappa_1 f_4 + \tau_1 f_3 + \tau_2 f_5 = g_1 \in L_*^2, \\ \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi = \kappa_2 f_6 + \tau_3 f_3 + \tau_4 f_5 = g_2 \in L_*^2, \end{cases} \tag{3.14}$$

Taking the $L^2$-product of $(3.14)_1$ ,$(3.14)_2$ and $(3.14)_3$ by $u^*, \varphi^*$ and $\psi^*$ respectively, using integration by parts and adding the obtained equations, we arrive at

$$a(V, V^*) = L(V^*), \tag{3.15}$$

where, $a$ is the bilinear form defined over $\mathcal{W} = (H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi))$ by

$$a(V, V^*) = \mu \int_0^\pi u_x u_x^* dx + b \int_0^\pi \varphi u_x^* dx + d \int_0^\pi \psi u_x^* dx + \alpha \int_0^\pi \varphi_x \varphi_x^* dx + \beta \int_0^\pi \psi_x \varphi_x^* dx$$

$$+b\int_0^\pi u_x\varphi^* dx+\alpha_1\int_0^\pi \varphi\varphi^* dx+\alpha_3\int_0^\pi \psi\varphi^* dx+\beta\int_0^\pi \varphi_x\psi_x^* dx$$
$$+\gamma\int_0^\pi \psi_x\psi_x^* dx+d\int_0^\pi u_x\psi^* dx+\alpha_3\int_0^\pi \varphi\psi^* dx+\alpha_2\int_0^\pi \psi\psi^* dx$$

and $L$ is the linear form defined by

$$L(V^*)=-\rho\int_0^\pi f_2u^* dx-\int_0^\pi g_1\varphi^* dx-\int_0^\pi g_2\psi^* dx.$$

Clearly, $a$ and $L$ are continuous. Furthermore, from Remark 3.1.2, there exists $\varepsilon>0$, such that

$$a(V,V)=\mu\int_0^\pi u_x^2 dx+\alpha\int_0^\pi \varphi_x^2 dx+\gamma\int_0^\pi \psi_x^2 dx+\alpha_1\int_0^\pi \varphi^2 dx+\alpha_2\int_0^\pi \psi^2 dx$$
$$+2b\int_0^\pi u_x\varphi dx+2d\int_0^\pi \psi u_x dx+2\beta\int_0^\pi \varphi_x\psi_x dx+2\alpha_3\int_0^\pi \psi\varphi dx$$
$$\geq \frac{1}{2}\left(\alpha-\frac{\beta^2}{\gamma}\right)\varphi_x^2+\frac{1}{2}\left(\gamma-\frac{\beta^2}{\alpha}\right)\psi_x^2+\varepsilon\left(u_x^2+\varphi^2+\psi^2\right).$$

Thus,

$$a(V,V)\geq c\|V\|_{\mathcal{W}}^2$$

for $c=\frac{1}{2}\min\{\alpha-\frac{\beta^2}{\gamma},\gamma-\frac{\beta^2}{\alpha},2\varepsilon\}$, which infer that $a$ is coercive. Therefore, Lax-milgram theorem ensures the existence of a unique $V=(u,\varphi,\psi)\in\mathcal{W}$ satifying

$$a(V,V^*)=L(V^*),\quad \forall V^*\in\mathcal{W}.$$

Now, taking $\varphi^*=\psi^*=0$ in (3.15) we get

$$\mu\int_0^\pi u_xu_x^* dx=-\int_0^\pi(\rho f_2-b\varphi_x-d\psi_x)u^* dx,\ \forall u^*\in H_0^1, \tag{3.16}$$

therefore,

$$u\in H^2(0,\pi).$$

Next, let $\varphi^*\in H_0^1(0,\pi)$ and define

$$\widetilde{\varphi}(x)=\varphi^*(x)-\int_0^\pi \varphi^*(x)\,dx,$$

clearly, $\widetilde{\varphi}\in H_*^1(0,\pi)$. Taking $(u^*,\varphi^*,\psi^*)=(0,\widetilde{\varphi},0)$ in (3.15) we get

$$\int_0^\pi(\alpha\varphi_x+\beta\psi_x)\widetilde{\varphi}_x dx=-\int_0^\pi(g_1+bu_x+\alpha_1\varphi+\alpha_3\psi)\widetilde{\varphi}dx,\ \forall\widetilde{\varphi}\in H_*^1, \tag{3.17}$$

which means that

$$\alpha\varphi+\beta\psi\in H^{2}\left(0,\pi\right), \tag{3.18}$$

then, $\widetilde{\varphi}\in H_{*}^{1}\left(0,\pi\right).$ Taking $\left(u^{*},\varphi^{*},\psi^{*}\right)=\left(0,0,\widetilde{\psi}\right)$ in (3.15) we get

$$\int_{0}^{\pi}\left(\beta\varphi_{x}+\gamma\psi_{x}\right)\psi_{x}^{*}dx=-\int_{0}^{\pi}\left(g_{2}+ud_{x}+\alpha_{3}\varphi+\alpha_{2}\psi\right)\psi^{*}dx,$$

which means that

$$\beta\varphi+\gamma\psi\in H^{2}\left(0,\pi\right). \tag{3.19}$$

From (3.18) and (3.19) we get

$$\varphi,\psi\in H^{2}\left(0,\pi\right).$$

To show that $\varphi$ belongs to $H_{*}^{2}\left(0,\pi\right)$ we take $\varphi^{*}\in C^{1}\left(0,\pi\right)$ in (3.17) and define $\widetilde{\varphi}$ as above, then using integration by parts, we obtain,

$$\left[\left(\alpha\varphi_{x}+\beta\psi_{x}\right)\widetilde{\varphi}\right]_{0}^{\pi}-\int_{0}^{\pi}\left(\alpha\varphi_{xx}+\beta\psi_{xx}-g_{1}-bu_{x}-\alpha_{1}\varphi-\alpha_{3}\psi\right)\widetilde{\varphi}dx=0,\ \forall\widetilde{\varphi}\in H_{*}^{1}. \tag{3.20}$$

First, we take $\widetilde{\varphi}\in C_{0}^{1}\left(0,\pi\right),$ we get

$$\alpha\varphi_{xx}+\beta\psi_{xx}=g_{1}+bu_{x}+\alpha_{1}\varphi+\alpha_{3}\psi,\quad a.e.\text{ in }\left(0,\pi\right).$$

Back to (3.20), we get

$$\left(\alpha\varphi_{x}\left(\pi\right)+\beta\psi_{x}\left(\pi\right)\right)\widetilde{\varphi}\left(\pi\right)-\left(\alpha\varphi_{x}\left(0\right)+\beta\psi_{x}\left(0\right)\right)\widetilde{\varphi}\left(0\right)=0,\ \forall\widetilde{\varphi}\in H_{*}^{1}.$$

As $\widetilde{\varphi}$ is arbitrary in $H_{*}^{1}\left(0,\pi\right)$, we obtain

$$\alpha\varphi_{x}\left(\pi\right)+\beta\psi_{x}\left(\pi\right)=0\text{ in }\alpha\varphi_{x}\left(0\right)+\beta\psi_{x}\left(0\right)=0.$$

Similarly, we obtain

$$\beta\varphi_{x}\left(\pi\right)+\gamma\psi_{x}\left(\pi\right)=0\text{ in }\beta\varphi_{x}\left(0\right)+\gamma\psi_{x}\left(0\right)=0.$$

Therefore, $\varphi,\psi\in H_{*}^{2}\left(0,\pi\right),$ consequently $U\in D\left(\mathcal{A}\right),$ and $0\in\rho\left(\mathcal{A}\right).$
Moreover, using a geometric series argument we prove that $\lambda I-\mathcal{A}=\mathcal{A}(\lambda\mathcal{A}^{-1}-I)$ is invertible for $|\lambda|<\|\mathcal{A}^{-1}\|$,
then $\lambda\in\rho(\mathcal{A})$, which completes the proof that $\mathcal{A}$ is the infinitesimal generator of a $C_0-$semigroup, then the Lumer-Phillips theorem ensures the existence of unique solution to the problem (3.1),(3.2) and (3.4) satisfying the statements of Theorem 3.2.1. □

**Remark 3.2.1.** *We note that if $U_0 \in D(\mathcal{A})$ then the solution*

$$U(t) = e^{t\mathcal{A}}U_0 \in C((0,\infty); D(\mathcal{A})) \cap C^1((0,\infty); \mathcal{H})$$

*and (3.13) is satisfied in $\mathcal{H}$ for every $t > 0$. It turns out that $u, \varphi, \psi$ satisfy (3.1) in the strong sense.*

If $U_0 \in \mathcal{H}$ there exists a sequence $U_{0n} \in D(\mathcal{A})$ converging to $U_0$ in $\mathcal{H}$. Accordingly, there exists a sequence of solutions $U_n(t) = e^{t\mathcal{A}}U_{0n}$ such that $u_n, \varphi_n, \psi_n$ satisfy (3.1) in $L^2$ for every $t > 0$, and for any $T > 0$, $u_n \to u$ in $C((0,T), H_0^1) \cap C^1((0,T); L^2)$, $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $C((0,T), H_*^1) \cap C^1((0,T); L^2)$. Therefore, if we multiply the equations of (3.1) for $u_n, \varphi_n, \psi_n$ by $u^* \in H_0^1$ and $\varphi^*, \psi^* \in H_*^1$, respectively, then integrate by parts with respect to $x$ and integrate with respect to $t$, finally passing to the limit, we find that $u, \varphi$ and $\psi$ are weak solutions to the variational form of system (3.1).

## 3.3 Exponential stability

In the present section we tackle the main objective of the paper, that is the prove of the exponential decay of the solution of (3.1). Fisrt we introduce the two following constants

$$\chi_0 = \left(\frac{\mu\kappa_1}{\rho} - \alpha\right)\left(\frac{\mu\kappa_2}{\rho} - \gamma\right) - \beta^2,$$

and

$$\chi_1 = d^2\left(\frac{\mu\kappa_1}{\rho} - \alpha\right) + b^2\left(\frac{\mu\kappa_2}{\rho} - \gamma\right) + 2bd\beta.$$

Our stability result reads as follow:

**Theorem 3.3.1.** *Let $(u, \varphi, \psi)$ be a solution of problem (3.1) with boundary condutions (3.4). Assume that*

$$\chi_0 = 0 \quad \chi_1 \neq 0. \tag{3.21}$$

*Then the energy functional $E(t)$ defined by (3.5) satisfies*

$$E(t) \le \lambda e^{-\xi t}, \ \forall t \ge 0, \tag{3.22}$$

*where $\lambda$ and $\xi$ are two positive constants.*

**Remark 3.3.1.** *The hypothesis (3.21) is equivalent to the following:*

*There exist two constants $\sigma, \omega \in \mathbb{R}^*$, such that*

$$\begin{aligned} \frac{\mu}{\rho} = \frac{\sigma\alpha + \omega\beta}{\sigma\kappa_1} = \frac{\sigma\beta + \omega\gamma}{\omega\kappa_2}, \qquad & \text{if } \beta \neq 0, \\ \left(\frac{\mu}{\rho} = \frac{\alpha}{\kappa_1} \text{ and } b \neq 0\right) \text{ or } \left(\frac{\mu}{\rho} = \frac{\gamma}{\kappa_2} \text{ and } d \neq 0\right), \quad & \text{if } \beta = 0. \end{aligned} \tag{3.23}$$

*It is clear that in the case where $\beta \neq 0$, if $\sigma = b, \omega = d$ solve (3.23), then $\alpha, \beta, \gamma, \mu, \rho, \kappa_1$ and $\kappa_2$ solve (3.21).*

The proof of Theorem 3.3.1, will be established through several lemmas.

**Lemme 3.3.1.** *For $(u, \varphi, \psi)$ solution of (3.1), there exist positive constants $\hat{\alpha}, \hat{\gamma}, \hat{\alpha}_1$ and $\hat{\alpha}_2$, such that the functional*

$$\begin{aligned} F_1(t) = {} & \kappa_1 \int_0^1 \varphi_t \varphi dx + \kappa_2 \int_0^1 \psi_t \psi dx + \frac{\tau_1}{2} \int_0^1 \varphi^2 dx + \frac{\tau_4}{2} \int_0^1 \psi^2 dx \\ & - \frac{\rho}{\mu} \int_0^1 u_t \left( \int_0^x (b\varphi + d\psi)(y)\, dy \right) dx \end{aligned}$$

*satisfies for any $\delta > 0$, the estimate*

$$\begin{aligned} F_1'(t) \leq {} & -\hat{\alpha} \int_0^1 \varphi_x^2 dx - \hat{\gamma} \int_0^1 \psi_x^2 dx - \frac{\hat{\alpha}_1}{2} \int_0^1 \varphi^2 dx - \frac{\hat{\alpha}_2}{2} \int_0^1 \psi^2 dx \\ & + \delta \int_0^1 u_t^2 dx + m_\delta \int_0^1 \varphi_t^2 dx + m_\delta \int_0^1 \psi_t^2 dx. \end{aligned} \tag{3.24}$$

*Proof.* The differentiation of $F_1(t)$ gives

$$\begin{aligned} F_1'(t) = {} & \kappa_1 \int_0^1 \varphi_{tt} \varphi dx + \kappa_1 \int_0^1 \varphi_t^2 dx + \kappa_2 \int_0^1 \psi_{tt} \psi dx + \kappa_2 \int_0^1 \psi_t^2 dx \\ & + \tau_1 \int_0^1 \varphi \varphi_t dx + \tau_4 \int_0^1 \psi_t \psi dx - \frac{\rho}{\mu} \int_0^1 u_{tt} \left( \int_0^x (b\varphi + d\psi)(y)\, dy \right) dx \\ & - \frac{\rho}{\mu} \int_0^1 u_t \left( \int_0^x (b\varphi + d\psi)_t (y)\, dy \right) dx. \end{aligned}$$

By exploiting the equations of (3.1) and using integration by parts, we get

$$\begin{aligned} F_1'(t) = {} & -\alpha \int_0^1 \varphi_x^2 dx - 2\beta \int_0^1 \psi_x \varphi_x dx - \gamma \int_0^1 \psi_x^2 dx \\ & - \left(\alpha_1 - \frac{b^2}{\mu}\right) \int_0^1 \varphi^2 dx - 2\left(\alpha_3 - \frac{bd}{\mu}\right) \int_0^1 \psi \varphi dx - \left(\alpha_2 - \frac{d^2}{\mu}\right) \int_0^1 \psi^2 dx \\ & + \kappa_1 \int_0^1 \varphi_t^2 dx + \kappa_2 \int_0^1 \psi_t^2 dx - \tau_2 \int_0^1 \psi_t \varphi dx - \tau_3 \int_0^1 \varphi_t \psi dx \\ & - \frac{\rho}{\mu} \int_0^1 u_t \left( \int_0^x (b\varphi_t + d\psi_t)(y)\, dy \right) dx. \end{aligned} \tag{3.25}$$

Then, using Young's and Cauchy Schwarz inequalities, we obtain

$$-2\beta\int_0^1 \psi_x\varphi_x dx \le \varepsilon\beta\int_0^1 \varphi_x^2 dx + \frac{\beta}{\varepsilon}\int_0^1 \psi_x^2 dx, \tag{3.26}$$

$$-2\left(\alpha_3 - \frac{bd}{\mu}\right)\int_0^1 \psi\varphi dx \le \left(\alpha_3 - \frac{bd}{\mu}\right)\eta\int_0^1 \varphi^2 dx + \frac{1}{\eta}\left(\alpha_3 - \frac{bd}{\mu}\right)\int_0^1 \psi^2 dx, \tag{3.27}$$

$$-\tau_3\int_0^1 \varphi_t\psi dx \le \epsilon\int_0^1 \psi^2 dx + \frac{\tau_3^2}{4\epsilon}\int_0^1 \varphi_t^2 dx, \tag{3.28}$$

$$-\tau_2\int_0^1 \psi_t\varphi dx \le \epsilon\int_0^1 \varphi dx + \frac{\tau_2}{4\epsilon}\int_0^1 \psi_t^2 dx, \tag{3.29}$$

we have

$$\begin{aligned}\int_0^1\left(\int_0^x (b\varphi_t + d\psi_t)(y)\,dy\right)^2 dx &\le \int_0^1\int_0^x dy\int_0^x (b\varphi_t(y) + d\psi_t(y))^2\,dydx \\ &\le \int_0^1 x\int_0^x (b\varphi_t(y) + d\psi_t(y))^2\,dydx \\ &\le \int_0^1 (b\varphi_t(y) + d\psi_t(y))^2\,dydx \times \int_0^1 xdx \\ &\le \int_0^1 (b\varphi_t + d\psi_t)^2\,dx,\end{aligned} \tag{3.30}$$

and

$$2bd\int_0^1 \varphi_t\psi_t dx \le bd\int_0^1 \varphi_t^2 dx + bd\int_0^1 \psi_t^2 dx \tag{3.31}$$

then

$$\begin{aligned}&-\frac{\rho}{\mu}\int_0^1 u_t\left(\int_0^x (b\varphi_t + d\psi_t)(y)\,dy\right)dx \\ &\le \delta\int_0^1 u_t^2 dx + \left(\frac{\rho}{\mu}\right)^2\frac{1}{4\delta}\int_0^1\left(\int_0^x (b\varphi_t + d\psi_t)(y)\,dy\right)^2 dx \\ &\le \delta\int_0^1 u_t^2 dx + \left(\frac{\rho}{\mu}\right)^2\frac{1}{4\delta}\int_0^1 (b0\varphi_t + d\psi_t)^2\,dx, \\ &\le \delta\int_0^1 u_t^2 dx. + \left(\frac{\rho}{\mu}\right)^2\frac{b}{4\delta}(b+d)\int_0^1 \varphi_t^2 dx + \left(\frac{\rho}{\mu}\right)^2\frac{d}{4\delta}(d+b)\int_0^1 \psi_t^2 dx \\ &\le \delta\int_0^1 u_t^2 dx. + \frac{1}{\delta}\left[\left(\frac{\rho}{\mu}\right)^2\frac{b}{4}(b+d)\right]\int_0^1 \varphi_t^2 dx + \frac{1}{\delta}\left[\left(\frac{\rho}{\mu}\right)^2\frac{d}{4}(d+b)\right]\int_0^1 \psi_t^2 dx\end{aligned} \tag{3.32}$$

By substituting (3.26)-(3.32) in (3.25) we get

$$
\begin{aligned}
F_1'(t) \leq & -(\alpha-\varepsilon\beta)\int_0^1 \varphi_x^2 dx - \left(\gamma-\frac{\beta}{\varepsilon}\right)\int_0^1 \psi_x^2 dx + \delta\int_0^1 u_t^2 dx \\
& - \left[\left(\alpha_1-\frac{b^2}{\mu}\right) - \left(\alpha_3-\frac{bd}{\mu}\right)\eta - \epsilon\right]\int_0^1 \varphi^2 dx \\
& - \left[\left(\alpha_2-\frac{d^2}{\mu}\right) - \frac{1}{\eta}\left(\alpha_3-\frac{bd}{\mu}\right) - \epsilon\right]\int_0^1 \psi^2 dx \\
& + \left(\kappa_1+\frac{\tau_3^2}{4\epsilon}+\frac{1}{\delta}\left[\left(\frac{\rho}{\mu}\right)^2\frac{(b+d)}{4}\right]b\right)\int_0^1 \varphi_t^2 dx \\
& + \left(\kappa_2+\frac{\tau_2^2}{4\epsilon}+\frac{1}{\delta}\left[\left(\frac{\rho}{\mu}\right)^2\frac{(d+b)}{4}\right]d\right)\int_0^1 \psi_t^2 dx
\end{aligned}
$$

we choose $m=\max\left(\kappa_1,\kappa_2,\frac{\tau_3^2}{4\epsilon},\frac{\tau_2^2}{4\epsilon},\left[\left(\frac{\rho}{\mu}\right)^2\frac{(b+d)}{4}\right]b,\left[\left(\frac{\rho}{\mu}\right)^2\frac{(d+b)}{4}\right]d\right)$, we arrive at

$$
\begin{aligned}
F_1'(t) \leq & -(\alpha-\beta\varepsilon)\int_0^1 \varphi_x^2 dx - \left(\gamma-\frac{b}{\varepsilon}\right)\int_0^1 \psi_x^2 dx \\
& - \left[\left(\alpha_1-\frac{b^2}{\mu}\right) - \left(\alpha_3-\frac{bd}{\mu}\right)\eta - \epsilon\right]\int_0^1 \varphi^2 dx \\
& - \left[\left(\alpha_2-\frac{d^2}{\mu}\right) - \frac{1}{\eta}\left(\alpha_3-\frac{bd}{\mu}\right) - \epsilon\right]\int_0^1 \psi^2 dx \\
& + m\left(1+\frac{1}{\epsilon}+\frac{1}{\delta}\right)\int_0^1 \varphi_t^2 dx + m\left(1+\frac{1}{\epsilon}+\frac{1}{\delta}\right)\int_0^1 \psi_t^2 dx + \delta\int_0^1 u_t^2 dx,
\end{aligned}
$$

for any $\varepsilon, \eta, \epsilon, \delta > 0$.

First, by virtue of (3.7), we can choose $\varepsilon > 0$ such that

$$\hat{\alpha} = \alpha - \beta\varepsilon > 0, \text{and} \quad \hat{\gamma} = \gamma - \frac{b}{\varepsilon} > 0.$$

Similarly, (3.6) allows us to choose $\eta > 0$ such that

$$\hat{\alpha}_1 = \left(\alpha_1-\frac{b^2}{\mu}\right) - \left(\alpha_3-\frac{bd}{\mu}\right)\eta > 0,$$

and

$$\hat{\alpha}_2 = \left(\alpha_2-\frac{d^2}{\mu}\right) - \frac{1}{\eta}\left(\alpha_3-\frac{bd}{\mu}\right) > 0.$$

Finally, we choose $\epsilon > 0$ so that

$$\hat{\alpha}_1 - \epsilon \geq \frac{\hat{\alpha}_1}{2}, \text{ and } \quad \hat{\alpha}_2 - \epsilon \geq \frac{\hat{\alpha}_2}{2}.$$

we put

$$m\left(1+\frac{1}{\epsilon}+\frac{1}{\delta}\right)=m_{\delta}$$

Consequently, the estimate (**??**) follows.

$$F_1'(t)\leq -\hat{\alpha}\int_0^1\varphi_x^2dx-\hat{\gamma}\int_0^1\psi_x^2dx-\frac{\hat{\alpha}_1}{2}\int_0^1\varphi^2dx-\frac{\hat{\alpha}_2}{2}\int_0^1\psi^2dx$$
$$+m_{\delta}\int_0^1\varphi_t^2dx+m_{\delta}\int_0^1\psi_t^2dx+\delta\int_0^1u_t^2dx,$$

*Proof.* Differentiating $F_2(t)$, using integration by parts and boundary conditions (3.4) we get

$$F_2'(t):=\rho(\sigma\alpha+\omega\beta)\int_0^{\pi}\varphi_{xt}u_tdx+\rho(\sigma\alpha+\omega\beta)\int_0^{\pi}\varphi_xu_{tt}dx$$
$$+\rho(\sigma\beta+\omega\gamma)\int_0^{\pi}\psi_{xt}u_tdx+\rho(\sigma\beta+\omega\gamma)\int_0^{\pi}\psi_xu_{tt}dx+\mu\sigma\kappa_1\int_0^{\pi}\varphi_{tt}u_xdx$$
$$+\mu\sigma\kappa_1\int_0^{\pi}\varphi_tu_{xt}dx+\mu\omega\kappa_2\int_0^{\pi}\psi_{tt}u_xdx+\mu\omega\kappa_2\int_0^{\pi}\psi_tu_{xt}dx$$

By exploiting the equations of (3.1) and using integration by parts, we get

$$F_2'(t):=\rho\left[\frac{\mu\sigma\kappa_1}{\rho}-(\sigma\alpha+\omega\beta)\right]\int_0^{\pi}u_{xt}\varphi_tdx+\rho\left[\frac{\mu\omega\kappa_2}{\rho}-(\sigma\beta+\omega\gamma)\right]\int_0^{\pi}u_{xt}\psi_tdx$$
$$-\mu(b\sigma+d\omega)\int_0^{\pi}u_x^2dx+b(\sigma\alpha+\omega\beta)\int_0^{\pi}\varphi_x^2dx+d(\sigma\beta+\omega\gamma)\int_0^{\pi}\psi_x^2dx$$
$$+[\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma)]\int_0^{\pi}\varphi_x\psi_xdx-\mu(\sigma\alpha_1+\omega\alpha_3)\int_0^{\pi}\varphi u_xdx$$
$$-\mu(\sigma\alpha_3+\omega\alpha_2)\int_0^{\pi}\psi u_xdx-\mu(\sigma\tau_1+\omega\tau_3)\int_0^{\pi}\varphi_tu_xdx-\mu(\sigma\tau_2+\omega\tau_4)\int_0^{\pi}\psi_tu_xdx$$
$$-(\mu(\sigma\alpha+\omega\beta)-\mu\sigma\alpha-\mu\omega\beta)\int_0^{\pi}\varphi_{xx}u_xdx-(\mu(\sigma\beta+\omega\gamma)-\mu\sigma\beta-\mu\omega\gamma)\int_0^{\pi}\psi_{xx}u_xdx$$

then

$$F_2'(t)=\rho\left(\frac{\mu\sigma\kappa_1}{\rho}-(\sigma\alpha+\omega\beta)\right)\int_0^1u_{xt}\varphi_tdx+\rho\left(\frac{\mu\omega\kappa_2}{\rho}-(\sigma\beta+\omega\gamma)\right)\int_0^1u_{xt}\psi_tdx \quad (3.33)$$
$$-\mu(b\sigma+d\omega)\int_0^{\pi}u_x^2dx+b(\sigma\alpha+\omega\beta)\int_0^{\pi}\varphi_x^2dx+d(\sigma\beta+\omega\gamma)\int_0^{\pi}\psi_x^2dx$$
$$+(\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma))\int_0^{\pi}\varphi_x\psi_xdx-\mu(\sigma\alpha_1+\omega\alpha_3)\int_0^{\pi}\varphi u_xdx$$
$$-\mu(\sigma\alpha_3+\omega\alpha_2)\int_0^{\pi}\psi u_xdx-\mu(\sigma\tau_1+\omega\tau_3)\int_0^{\pi}\varphi_tu_xdx-\mu(\sigma\tau_2+\omega\tau_4)\int_0^{\pi}\psi_tu_xdx.$$

by taking into account the (3.23),we have

$$\rho\left(\tfrac{\mu\sigma\kappa_1}{\rho}-(\sigma\alpha+\omega\beta)\right)=0 \quad \text{and,} \quad \rho\left(\tfrac{\mu\omega\kappa_2}{\rho}-(\sigma\beta+\omega\gamma)\right)=0$$

using Young's we have

$$(\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma))\int_0^\pi \varphi_x\psi_x dx \le \frac{1}{2}(\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma))\int_0^\pi \varphi_x^2 dx$$

$$+\frac{1}{2}(\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma))\int_0^\pi \psi_x^2 dx,$$

$$-\mu(\sigma\tau_1+\omega\tau_3)\int_0^\pi \varphi_t u_x dx \le \frac{1}{2}\mu(\sigma\tau_1+\omega\tau_3)^2\int_0^\pi \varphi_t^2 dx+\frac{1}{5}\mu(b\sigma+d\omega)\int_0^\pi u_x^2 dx,$$

$$-\mu(\sigma\tau_2+\omega\tau_4)\int_0^\pi \psi_t u_x dx \le \frac{1}{2}\mu(\sigma\tau_2+\omega\tau_4)^2\int_0^\pi \psi_t^2 dx+\frac{1}{5}\mu(b\sigma+d\omega)\int_0^\pi u_x^2 dx,$$

using Young's and Poincaré's inequalities, we get

$$\begin{aligned}-\mu(\sigma\alpha_1+\omega\alpha_3)\int_0^\pi \varphi u_x dx &\le \frac{5}{4}\frac{(\sigma\alpha_1+\omega\alpha_3)^2}{(b\sigma+d\omega)}\int_0^\pi \varphi^2 dx+\frac{1}{5}\mu(b\sigma+d\omega)\int_0^\pi u_x^2 dx,\\ &\le \frac{5c}{4}\frac{(\sigma\alpha_1+\omega\alpha_3)^2}{(b\sigma+d\omega)}\int_0^\pi \varphi_x^2 dx+\frac{1}{5}\mu(b\sigma+d\omega)\int_0^\pi u_x^2 dx,\end{aligned}$$

$$\begin{aligned}-\mu(\sigma\alpha_3+\omega\alpha_2)\int_0^\pi \psi u_x dx &\le \frac{5}{4}\frac{(\sigma\alpha_3+\omega\alpha_2)^2}{(b\sigma+d\omega)}\int_0^\pi \psi^2 dx+\frac{1}{5}\mu(b\sigma+d\omega)\int_0^\pi u_x^2 dx,\\ &\le \frac{5c}{4}\frac{(\sigma\alpha_3+\omega\alpha_2)^2}{(b\sigma+d\omega)}\int_0^\pi \psi_x^2 dx+\frac{1}{5}\mu(b\sigma+d\omega)\int_0^\pi u_x^2 dx,\end{aligned}$$

$$F_2'(t)\le -\frac{1}{5}\mu(b\sigma+d\omega)\int_0^\pi u_x^2 dx$$

$$+\left[\frac{1}{2}(\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma))+b(\sigma\alpha+\omega\beta)+\frac{5c}{4}\frac{(\sigma\alpha_1+\omega\alpha_3)^2}{(b\sigma+d\omega)}\right]\int_0^\pi \varphi_x^2 dx$$

$$+\left[\frac{1}{2}(\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma))+d(\sigma\beta+\omega\gamma)+\frac{5c}{4}\frac{(\sigma\alpha_3+\omega\alpha_2)^2}{(b\sigma+d\omega)}\right]\int_0^\pi \psi_x^2 dx$$

$$+\frac{1}{2}\mu(\sigma\tau_1+\omega\tau_3)^2\int_0^\pi \varphi_t^2 dx+\frac{1}{2}\mu(\sigma\tau_2+\omega\tau_4)^2\int_0^\pi \psi_t^2 dx,$$

we put

$$a_1=\left[\frac{1}{2}(\sigma(d\alpha+b\beta)+\omega(d\beta+b\gamma))+b(\sigma\alpha+\omega\beta)+\frac{5c}{4}\frac{(\sigma\alpha_1+\omega\alpha_3)^2}{(b\sigma+d\omega)}\right],$$

$$a_2 = \left[\frac{1}{2}\left(\sigma\left(d\alpha + b\beta\right) + \omega\left(d\beta + b\gamma\right)\right) + d\left(\sigma\beta + \omega\gamma\right) + \frac{5c}{4}\frac{\left(\sigma\alpha_3 + \omega\alpha_2\right)^2}{\left(b\sigma + d\omega\right)}\right],$$

then we choose $\frac{m}{(b\sigma+d\omega)} = \max\left(a_1, a_2, \frac{1}{2}\mu\left(\sigma\tau_1 + \omega\tau_3\right)^2, \frac{1}{2}\mu\left(\sigma\tau_2 + \omega\tau_4\right)^2\right)$ ,new by substituting in (3.1), we get

$$\begin{aligned}\left(b\sigma + d\omega\right) F_2'(t) \leq & -\frac{\mu}{5}\left(b\sigma + d\omega\right)^2 \int_0^\pi u_x^2 dx \\ & + m\left(\int_0^1 \varphi_x^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_t^2 dx\right).\end{aligned} \tag{3.34}$$

□

**Lemme 3.3.2.** *Along the solution* $(u, \varphi, \psi)$ *of (3.1), the functional*

$$F_3(t) = -\rho\int_0^1 u_t u dx$$

*satisfies*

$$F_3'(t) \leq -\rho\int_0^1 u_t^2 dx + 2\mu\int_0^1 u_x^2 dx + \frac{b^2}{2\mu}\int_0^1 \varphi^2 dx + \frac{d^2}{2\mu}\int_0^1 \psi^2 dx. \tag{3.35}$$

*Proof.* Differentiating $F_3(t)$, we get

$$F_3(t) = -\rho\int_0^1 u_t^2 dx - \rho\int_0^1 u_{tt} u dx.$$

By exploiting the equations of (3.1) and using integration by parts, we get

$$F_3'(t) = -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\int_0^1 \varphi u_x dx + d\int_0^1 \psi u_x dx,$$

using Young's inequality,

$$b\int_0^1 \varphi u_x dx \leq \frac{b^2}{2\mu}\int_0^1 \varphi^2 dx + \frac{\mu}{2}\int_0^1 u_x^2 dx,$$

and

$$d\int_0^1 \psi u_x dx \leq \frac{d^2}{2\mu}\int_0^1 \psi^2 dx + \frac{\mu}{2}\int_0^1 u_x^2 dx,$$

estimate (**??**) follows immediately.

$$F_3'(t) \leq -\rho\int_0^1 u_t^2 dx + 2\mu\int_0^1 u_x^2 dx + \frac{b^2}{2\mu}\int_0^1 \varphi^2 dx + \frac{d^2}{2\mu}\int_0^1 \psi^2 dx,$$

□

**End of the proof of Theorem 3.3.1**

At this point we define the Lyapunov fonctional $\mathcal{L}(t)$ as follows

$$\mathcal{L}(t) = NE(t) + N_1 F_1(t) + N_2 (b\sigma + d\omega) F_2(t) + F_3(t),$$

where $N, N_1$ and $N_2$ are positive constants to be properly chosen later.

First, we have

$$|\mathcal{L}(t) - NE(t)| = |N_1 F_1(t) + N_2 (b\sigma + d\omega) F_2(t) + F_3(t)|,$$

$$\begin{aligned}|\mathcal{L}(t) - NE(t)| \leq{} & N_1 \int_0^{\pi} \left(\kappa_1 |\varphi_t \varphi| + \kappa_2 |\psi_t \psi| + \frac{\tau_1}{2} |\varphi|^2 + \frac{\tau_4}{2} |\psi|^2\right) dx \\ & -\frac{\rho}{\mu} \int_0^{\pi} \left| u_t \left( \int_0^x (b\varphi + d\psi)(y)\, dy \right) \right| dx + \rho \int_0^1 |u_t u|\, dx \\ & + N_2 |b\sigma + d\omega| \int_0^{\pi} (\rho |\sigma\alpha + \omega\beta| |u_t \varphi_x| + \rho |\sigma\beta + \omega\gamma| |u_t \psi_x|)\, dx \\ & + N_2 \int_0^{\pi} (\mu\kappa_1 |b\varphi_t u_x| + \mu\kappa_2 |d\psi_t u_x|)\, dx.\end{aligned}$$

Using Young's, Cauchy Schwarz and Poincaré's inequalities, we obtain

$$\begin{aligned}|\mathcal{L}(t) - NE(t)| &\leq c_0 \int_0^1 \left(u_t^2 + \varphi_t^2 + \psi_t^2 + (\varphi_x + \psi_x)^2 + \psi_x^2 + (u_x + \varphi + \psi)^2\right) dx \\ &\leq cE(t).\end{aligned}$$

Thus,

$$(N - c) E(t) \leq \mathcal{L}(t) \leq (N + c) E(t).$$

Secondly, substituting (3.12),(3.24),(**??**) and (3.35) in the expression of $\mathcal{L}'(t)$ we get

$$\begin{aligned}\mathcal{L}'(t) \leq{} & -\left[\frac{1}{2}\left(\tau_1 - \frac{(\tau_2 + \tau_3)^2}{4\tau_4}\right) N - m_\delta N_1 - m N_2\right] \int_0^1 \varphi_t^2 dx \\ & -\left[\frac{1}{2}\left(\tau_4 - \frac{(\tau_2 + \tau_3)^2}{4\tau_1}\right) N - m_\delta N_1 - m N_2\right] \int_0^1 \psi_t^2 dx \\ & -\mu \left(\frac{(\sigma b + \omega d)^2}{2} N_2 - 2\right) \int_0^1 u_x^2 dx - (\rho - \delta N_1) \int_0^1 u_t^2 dx \\ & -(\hat{\alpha} N_1 - m N_2) \int_0^1 \varphi_x^2 dx - (\hat{\gamma} N_1 - m N_2) \int_0^1 \psi_x^2 dx \\ & -\frac{1}{2}\left(\hat{\alpha}_1 N_1 - \frac{b^2}{\mu}\right) \int_0^1 \varphi^2 dx - \frac{1}{2}\left(\hat{\alpha}_2 N_1 - \frac{d^2}{\mu}\right) \int_0^1 \psi^2 dx.\end{aligned}$$

New, we have to choose the coefficients carefully. First, we take

$$\delta = \frac{\rho}{2N_1}.$$

Secondly, We choose $N_2$ large enough such that

$$\frac{(\sigma b + \omega d)^2}{2} N_2 - 2 > 0.$$

Next, we pick $N_1$ large enough such that

$$\begin{aligned} &\hat{\alpha} N_1 - m_\delta N_2 > 0, \ \hat{\gamma} N_1 - m_\delta N_2 > 0, \\ &\hat{\alpha}_1 N_1 - \frac{b^2}{\mu} > 0, \text{ and } \ \hat{\alpha}_2 N_1 - \frac{d^2}{\mu} > 0. \end{aligned}$$

Finally, we take $N$ large enough such that $\mathcal{L}(t) \sim E(t)$ (i.e. $N - c > 0$) and

$$\begin{aligned} &\frac{1}{2}\left(\tau_1 - \frac{(\tau_2 + \tau_3)^2}{4\tau_4}\right) N - mN_1 - m_\delta N_2 > 0, \\ &\frac{1}{2}\left(\tau_4 - \frac{(\tau_2 + \tau_3)^2}{4\tau_1}\right) N - mN_1 - m_\delta N_2 > 0. \end{aligned}$$

Therefore, there exist $\sigma$ and $\widetilde{\sigma}$ positives constants such that

$$\begin{aligned} \mathcal{L}'(t) &\leq -\sigma \int_0^\pi \left(\varphi_t^2 + \psi_t^2 + u_t^2 + u_x^2 + \varphi_x^2 + \psi_x^2 + \psi^2 + \varphi^2\right) dx, \\ &\leq -\widetilde{\sigma} E(t), \qquad \forall t \geq 0. \end{aligned}$$

Since $E(t)$ is equivalent to $\mathcal{L}(t)$, we infer that

$$\mathcal{L}'(t) \leq -\omega \mathcal{L}(t), \quad \forall t \geq 0,$$

for some positive constant $\omega$. Thus

$$\mathcal{L}(t) \leq \lambda_1 \mathcal{L}(0) e^{-\omega t}, \quad \forall t \geq 0.$$

Using again the equivalence between $\mathcal{L}(t)$ and $E(t)$ we conclude that

$$E(t) \leq \lambda e^{-\omega t}, \quad \forall t \geq 0,$$

which completes the proof of Theorem 3.3.1.

**Remark 3.3.2.** *The same proof is valid for the following boundary conditions*

$$\begin{aligned} &u_x(t,\pi) = \varphi(t,\pi) = \psi(t,\pi) = 0, \\ &u(t,0) = \varphi_x(t,0) = \psi_x(t,0) = 0, \end{aligned} \qquad t \geq 0.$$

### 3.3.1 Lack of exponential decay

In this section we suppose that (3.21) does not hold, and prove that the solution $(u, \varphi, \psi)$ of the system (3.1) lacks exponentially stability. The proof is based on the following theorem due to Gerhart-Prüss-Huang [22, 34, 24].

**Theorem 3.3.2.** *Let $S(t) = e^{\mathcal{A}t}$ be a $C_0-$semigroup of contractions on a Hilbert space $\mathcal{H}$, with infinitesimal generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if:*

- $i\mathbb{R} \subset \rho(\mathcal{A})$,
- $\overline{\lim}_{|\lambda| \longrightarrow \infty} \left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty.$

Our result of non exponential stablity reads as follow.

**Theorem 3.3.3.** *Suppose that (3.21) does not hold, then the energy associated with the solution $(u, \varphi, \psi)$ of the system (3.1) is not exponentially stable.*

*Proof.* It suffices to prove that there exists a sequence $(F_n) \subset \mathcal{H}$ with bounded norm $\|F_n\| < 1$, such that

$$\overline{\lim}_{|\lambda| \longrightarrow \infty} \left\| (\lambda I - \mathcal{A})^{-1} F_n \right\|_{\mathcal{H}} = \overline{\lim}_{|\lambda| \longrightarrow \infty} \|U_n\|_{\mathcal{H}} = \infty.$$

Let $(U_n)_{n \in \mathbb{N}} \subset D(\mathcal{A})$ be the solution of $(\lambda I - \mathcal{A}) U_n = F_n$, then, omitting $n$ we have

$$\begin{aligned}
i\lambda u + v &= f_1 \\
i\lambda \rho v + \mu u_{xx} + b\varphi_x + d\psi_x &= \rho f_2 \\
i\lambda \varphi + \phi &= f_3 \\
i\lambda \kappa_1 \phi + \alpha \varphi_{xx} + \beta \psi_{xx} - bu_x - \alpha_1 \varphi - \alpha_3 \psi - \tau_1 \phi - \tau_2 \chi &= \kappa_1 f_4 \\
i\lambda \psi + \chi &= f_5 \\
i\lambda \kappa_2 \chi + \beta \varphi_{xx} + \gamma \psi_{xx} - du_x - \alpha_3 \varphi - \alpha_2 \psi - \tau_3 \phi - \tau_4 \chi &= \kappa_2 f_6.
\end{aligned}$$

Taking $f_1 = f_3 = f_4 = f_5 = f_6 = 0$ and $f_2 = \dfrac{1}{\rho} \sin(n\pi x)$, then eliminating $v, \phi$ and $\chi$ we obtain

$$\begin{aligned} v &= -i\lambda u \\ i\lambda\rho v + \mu u_{xx} + b\varphi_x + d\psi_x &= \sin(n\pi x) \\ \phi &= -i\lambda\varphi \\ i\lambda\kappa_1\phi + \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\phi - \tau_2\chi &= 0 \\ \chi &= -i\lambda\psi \\ i\lambda\kappa_2\chi + \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_3\phi - \tau_4\chi &= 0. \end{aligned}$$

thus

$$\begin{aligned} \lambda^2\rho u + \mu u_{xx} + b\varphi_x + d\psi_x &= \sin(n\pi x) \\ \lambda^2\kappa_1\varphi + \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - (\alpha_1 - i\lambda\tau_1)\varphi - (\alpha_3 - i\lambda\tau_2)\psi &= 0 \\ \lambda^2\kappa_2\psi + \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - (\alpha_3 - i\lambda\tau_3)\varphi - (\alpha_2 - i\lambda\tau_4)\psi &= 0. \end{aligned}$$

Taking into accound the boundary conditions (4.6), we are looking for $(u, \varphi, \psi)$ of the form

$$u = A\sin(n\pi x), \; \varphi = B\cos(n\pi x), \; \psi = C\cos(n\pi x).$$

That is

$$\begin{aligned} \lambda^2\rho A - \mu A(n\pi)^2 - bBn\pi - dCn\pi &= 1 \\ \lambda^2\kappa_1 B - \alpha B(n\pi)^2 - \beta C(n\pi)^2 - bAn\pi - (\alpha_1 - i\lambda\tau_1)B - (\alpha_3 - i\lambda\tau_2)C &= 0 \\ \lambda^2\kappa_2 C - \beta B(n\pi)^2 - \gamma C(n\pi)^2 - dAn\pi - (\alpha_3 - i\lambda\tau_3)B - (\alpha_2 - i\lambda\tau_4)C &= 0. \end{aligned}$$

$$\begin{cases} (\rho\lambda^2 - \mu\pi^2 n^2)A - bn\pi B - dn\pi C = 1 \\ -b(n\pi)A + \left[\kappa_1\lambda^2 - (n\pi)^2\alpha - (\alpha_1 - i\lambda\tau_1)\right]B - \left[\beta(n\pi)^2 + (\alpha_3 - i\lambda\tau_2)\right]C = 0 \\ -d(n\pi)A - \left[\beta(n\pi)^2 + (\alpha_3 - i\lambda\tau_3)\right]B + \left[\kappa_2\lambda^2 - (n\pi)^2\gamma - (\alpha_2 - i\lambda\tau_4)\right]C = 0, \end{cases}$$

which can be written

$$\begin{pmatrix} p_1(\lambda) & -bn\pi & -dn\pi \\ -bn\pi & p_2(\lambda) & p_4(\lambda) \\ -dn\pi & p_5(\lambda) & p_3(\lambda) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.36}$$

Then

$$\begin{vmatrix} p_1(\lambda) & -bn\pi & -dn\pi \\ -bn\pi & p_2(\lambda) & p_4(\lambda) \\ -dn\pi & p_5(\lambda) & p_3(\lambda) \end{vmatrix} = \lambda^3 p_1p_2p_3 - \lambda^3 p_1p_4p_5 - \pi^2b^2n^2\lambda p_3 - \pi^2d^2n^2\lambda p_2 + \pi^2bdn^2\lambda p_4 + \pi^2bdn^2\lambda p_5,$$

$$\begin{vmatrix} 1 & -bn\pi & -dn\pi \\ 0 & p_2(\lambda) & p_4(\lambda) \\ 0 & p_5(\lambda) & p_3(\lambda) \end{vmatrix} = \lambda^2 p_2 p_3 - \lambda^2 p_4 p_5,$$

$$A = \frac{\lambda p_2 p_3 - \lambda p_4 p_5}{p_1 \lambda (\lambda p_2 p_3 - \lambda p_4 p_5) - \pi^2 b^2 n^2 p_3 - \pi^2 d^2 n^2 p_2 + \pi^2 bdn^2 p_4 + \pi^2 bdn^2 p_5}$$

where

$$p_1(\lambda) := \rho\lambda^2 - \mu(\pi n)^2, \; p_2(\lambda) := \kappa_1 \lambda^2 - (n\pi)^2 \alpha - (\alpha_1 - i\lambda\tau_1),$$

$$p_3(\lambda) := \kappa_2 \lambda^2 - (n\pi)^2 \gamma - (\alpha_2 - i\lambda\tau_4), \; p_4(\lambda) := -\beta(n\pi)^2 - (\alpha_3 - i\lambda\tau_2),$$

$$p_5(\lambda) := -\beta(n\pi)^2 - (\alpha_3 - i\lambda\tau_3).$$

Solving (3.36) we obtain

$$\begin{vmatrix} p_1(\lambda) & -bn\pi & -dn\pi \\ -bn\pi & p_2(\lambda) & p_4(\lambda) \\ -dn\pi & p_5(\lambda) & p_3(\lambda) \end{vmatrix} = p_1(p_2 p_3 - p_4 p_5) + (n\pi)^2 b(dp_4 - bp_3) + (n\pi)^2 d(bp_5 - dp_2),$$

$$\begin{vmatrix} 1 & -bn\pi & -dn\pi \\ 0 & p_2(\lambda) & p_4(\lambda) \\ 0 & p_5(\lambda) & p_3(\lambda) \end{vmatrix} = p_2 p_3 - p_4 p_5,$$

$$A = \frac{p_2 p_3 - p_4 p_5}{p_1(p_2 p_3 - p_4 p_5) + (n\pi)^2 b(dp_4 - bp_3) + (n\pi)^2 d(bp_5 - dp_2)}$$

$$A = \frac{K_1}{p_1 K_1 + K_2},$$

where,

$$K_1 := p_2 p_3 - p_4 p_5, \; K_2 := b(n\pi)^2 (dp_4 - bp_3) - d(n\pi)^2 (dp_2 - bp_5).$$

Let $\lambda$ be such that $p_1(\lambda) = 0$, then $(n\pi)^2 = \dfrac{\rho\lambda^2}{\mu}$ and

$$p_1(\lambda) := \rho\lambda^2 - \mu(\pi n)^2, \; p_2(\lambda) := \kappa_1 \lambda^2 - (n\pi)^2 \alpha - (\alpha_1 - i\lambda\tau_1),$$

$$p_3(\lambda) := \kappa_2 \lambda^2 - (n\pi)^2 \gamma - (\alpha_2 - i\lambda\tau_4), \; p_4(\lambda) := -\beta(n\pi)^2 - (\alpha_3 - i\lambda\tau_2),$$

$$p_5(\lambda) := -\beta(n\pi)^2 - (\alpha_3 - i\lambda\tau_3).$$

$$K_1 = \frac{\rho}{\mu}\left[\left(\frac{\mu\kappa_1}{\rho} - \alpha\right)\left(\frac{\mu\kappa_2}{\rho} - \gamma\right) - \beta^2\right]\lambda^4$$
$$+i\frac{\rho}{\mu}\left[\left(\frac{\mu\kappa_1}{\rho} - \alpha\right)\tau_4 + \left(\frac{\mu\kappa_2}{\rho} - \gamma\right)\tau_1 + \beta\left(\tau_2 + \tau_3\right)\right]\lambda^3 + K_3$$

and

$$K_2 = -\frac{\rho^2}{\mu^2}\left[b^2\left(\frac{\mu\kappa_2}{\rho} - \gamma\right) + d^2\left(\frac{\mu\kappa_1}{\rho} - \alpha\right) + 2bd\beta\right]\lambda^4$$
$$+i\frac{\rho}{\mu}\left[bd\left(\tau_2 + \tau_3\right) - b^2\tau_4 - d^2\tau_1\right]\lambda^3 + K_4,$$

where $K_3, K_4$ are polynomials of degree 2 in $\lambda$.

At this point we discuss three cases:

**1)** Suppose that $\chi_0 \neq 0$ and $\chi_1 \neq 0$,then

$$A = \frac{K_1}{K_2} \approx \frac{\mu\left[\left(\frac{\mu\kappa_1}{\rho} - \alpha\right)\left(\frac{\mu\kappa_2}{\rho} - \gamma\right) - \beta^2\right]}{-\rho\left[b^2\left(\frac{\mu\kappa_2}{\rho} - \gamma\right) + d^2\left(\frac{\mu\kappa_1}{\rho} - \alpha\right) + 2bd\beta\right]} \equiv c,$$

for some constant $c \neq 0$.

Therefore

$$\|U\|^2 \geq \rho\|v\|^2 = \rho c^2|\lambda|^2\int_0^1 \sin^2(n\pi x)\,dx = \frac{\rho c^2|\lambda|^2}{2}.$$

Therefore,

$$\lim_{|\lambda|\longrightarrow\infty}\|U\|^2 = \infty.$$

**2)** Suppose that $\chi_0 = \chi_1 = 0$, then

$$\frac{\mu\kappa_1}{\rho} - \alpha = -\frac{b\beta}{d},\ \frac{\mu\kappa_2}{\rho} - \gamma = -\frac{d\beta}{d}.$$

Consequently

$$\left(\frac{\mu\kappa_1}{\rho} - \alpha\right)\tau_4 + \left(\frac{\mu\kappa_2}{\rho} - \gamma\right)\tau_1 + \beta\left(\tau_2 + \tau_3\right) \neq 0,$$
$$bd\left(\tau_2 + \tau_3\right) - b^2\tau_4 - d^2\tau_1 \neq 0,$$

by virtue of (3.10) and

$$A = \frac{K_1}{K_2} \approx \frac{\left[\left(\frac{\mu\kappa_1}{\rho} - \alpha\right)\tau_4 + \left(\frac{\mu\kappa_2}{\rho} - \gamma\right)\tau_1 + \beta\left(\tau_2 + \tau_3\right)\right]}{\left[bd\left(\tau_2 + \tau_3\right) - b^2\tau_4 - d^2\tau_1\right]} \equiv c.$$

Therefore

$$\|U\|^2 \geq \rho \|v\|^2 = \rho c^2 |\lambda|^2 \int_0^1 \sin^2(n\pi x)\, dx = \frac{\rho c^2 |\lambda|^2}{2}.$$

Therefore,

$$\lim_{|\lambda| \longrightarrow \infty} \|U\|^2 = \infty.$$

**3)** Suppose that $\chi_0 \neq 0$ and $\chi_1 = 0$, then

$$A = \frac{K_1}{K_2} \approx \frac{\left[\left(\frac{\mu\kappa_1}{\rho} - \alpha\right)\left(\frac{\mu\kappa_2}{\rho} - \gamma\right) - \beta^2\right]\lambda}{-i\left[bd\left(\tau_2 + \tau_3\right) - b^2\tau_4 - d^2\tau_1\right]} \approx c\lambda,$$

$$\|U\|^2 \geq \|u_x\|^2 = A^2 (n\pi)^2 \int_0^1 \cos^2(n\pi x)\, dx = \frac{c^2 |\lambda|^4 \mu}{2\rho}$$

and

$$\lim_{|\lambda| \longrightarrow \infty} \|U\|^2 = \infty.$$

which completes the proof of Theorem 3.3.3. □

Chapter 4

# A double porous elastic system with memory

## 4.1 Introduction

In the present paper we are concerned by the following double porous elastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x & \text{in } (0,\pi)\times(0,\infty), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi & \\ \qquad\qquad - \int_0^t g(t-s)\,\varphi_{xx}(s)\,ds & \text{in } (0,\pi)\times(0,\infty), \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi & \\ \qquad\qquad - \int_0^t h(t-s)\,\psi_{xx}(s)\,ds & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{4.1}$$

with boundary conditions

$$u_x(0,t) = u_x(\pi,t) = \varphi(0,t) = \varphi(\pi,t) = \psi(0,t) = \psi(\pi,t) = 0 \tag{4.2}$$

and initial data

$$\begin{gathered} u(x,0) = u_0(x)\,,\ u_t(x,0) = u_1(x)\,,\ \varphi(x,0) = \varphi_0(x)\,,\ \varphi_t(x,0) = \varphi_1(x)\,, \\ \psi(x,0) = \psi_0(x)\,,\ \psi_t(x,0) = \psi_1(x)\,. \end{gathered} \tag{4.3}$$

Here, $u$ is the transversal displacement of a one-dimensional elastic solid of length $\pi$, $\varphi$ and $\psi$ are the unknown porous functions, the coefficients $\rho, \kappa_1, \kappa_2, \mu, \alpha,\ \gamma, \alpha_1, \alpha_2$ are positive.

As coupling is considered, the constants $b, d$ and $\beta$ must be different from zero. It is also assumed that the internal energy density associated with the system (4.1) is a positive definite quadratic form, which may be satisfied by requiring that the matrix

$$A = \begin{pmatrix} \mu & b & d \\ b & \alpha_1 & \alpha_3 \\ d & \alpha_3 & \alpha_2 \end{pmatrix}$$

is positive definite. The functions $g, h$ are relaxation functions that assumed to satisfy some hypotheses that will be specified later.

The system considered here, represents a thermoelastic solid with double porosity structure in the framework of the theory of elastic materials with voids developed by Nunziato and Cowin [16]. This approach has been used by Ieşan and Quintanilla [36] to derive a new theory of thermoelastic solids which have a double porosity structure. In contrast to the classical theory the new one is not based on Darcy's law, and the porosity structure in the case of equilibrium is influenced by the displacement field.

In the beginning of the second decade of this century, Svanadze [41, 42] established the basic properties of the plane waves in the dynamic theory of elastic solids with double porosity.

The viscoelastic damping represented by a memory term was introduced first by Dafermos [17, 18]. He proved that the solution of the viscoelastic equation

$$\rho u_{tt} = c u_{xx} - \int_0^t g\left(t-\tau\right) u_{xx}\left(x,\tau\right) d\tau,\ \left(0,1\right) \times \left(0,\infty\right),$$

with Dirichlet boundary conditions is asymptotically stable, but he does not specify the rate of decay of the solution [18].

Soufyane [40] examined the porous thermoelastic system of memory type

$$\begin{cases} \rho_1 u_{tt} = k\left(u_x + \varphi\right)_x - \theta_x, \\ \rho_2 \varphi_{tt} = \alpha \varphi_{xx} - k\left(u_x + \varphi\right) + \theta - \displaystyle\int_0^t g(t-s)\varphi_{xx}(x,t-s)ds, \\ \theta_t = \kappa \theta_{xx} - u_{tx} - \varphi_t, \end{cases} \tag{4.4}$$

where all the coefficients are equal to 1 and the kernel $g$ satisfies

$$l = \alpha - \int_0^\infty g(s)\, ds > 0, \tag{4.5}$$

$$g'(t) \le -\xi g^p(t), \quad t \ge 0. \tag{4.6}$$

He used the multiplier method and established an exponential decay rate for $p = 1$, and a polynomial decay rate for $1 < p < \frac{3}{2}$.

Messaoudi and Fareh [19, 20] considered (4.4) and assumed the relaxation function $g$ satisfies (4.5) and $g'(t) \le -\xi(t)g(t)$ for a non increasing function $\xi$. They obtained a general decay result for which the exponential and the polynomial rates of decay are only special cases.

Recently, Apalara [2] considered the porous thermoelastic system of memory type

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 & in\ (0,1)\times(0,+\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\,\phi_{xx}(s)\, ds = 0 & in\ (0,1)\times(0,+\infty), \end{cases}$$

with Neumann-Dirichlet boundary conditions. He studied the case of equal wave speeds $\dfrac{\mu}{\rho} = \dfrac{\delta}{J}$ and proved, in contrary to [28], that the unique dissipation given by the memory term leads to a general decay.

The basic evolution equations for one-dimensional theories of double porous materials with memory effect are given by

$$\rho u_{tt} = \mathbb{T}_x,$$
$$\kappa_1 \varphi_{tt} = \sigma_x + \xi,$$
$$\kappa_2 \psi_{tt} = \chi_x + \zeta,$$

where $u$ is the displacement, $\varphi$ and $\psi$ are the porous variables, $\rho, \kappa_1$ and $\kappa_2$ are positive constants. $\mathbb{T}$ is the first Piola-Kirchhoff stress tensor, $\sigma$, $\chi$ are equilibrated stress vectors, $\xi$ and $\zeta$ are the intrinsic equilibrated body forces that they must be given by constitutives assumptions. We assume

$$\mathbb{T} = \mu u_x + b\varphi + d\psi,$$
$$\sigma = \alpha\varphi_x + \beta\psi_x,\ \chi = \beta\varphi_x + \gamma\psi_x,$$
$$\xi = -bu_x - \alpha_1\varphi - \alpha_3\psi - \int_0^t g(t-s)\,\varphi_{xx}(s)\, ds,$$
$$\zeta = -du_x - \alpha_3\varphi - \alpha_2\psi - \int_0^t h(t-s)\,\psi_{xx}(s)\, ds.$$

Here $\mu, b, d, \lambda, \alpha, \alpha_1, \alpha_2, \alpha_3, \beta$,and $\gamma$ are constants.
If we introduce the constitutive equations into the evolution equations we obtain the system (4.1).

In the present chapter we consider the isothermal case of (**??**) ($\beta = \gamma_1 = \gamma_2 = 0$), with weaker dissipations of memory type instead of the strong dampings $\varepsilon_i \varphi_t, \varepsilon_i \psi_t$. We use the multiplier method and establish a general decay result depends on the coefficients of the system and the rates of decay of the relaxation functions $g$ and $h$.

To the best of our knowledge, the longtime behavior of thermoelastic systems with double porosity structures has been investigated only in [7, 30], and none of them have considered the dissipation of memory type. Our result improves the previous results, in the sense that we have weakened and minimized the dissipations and have generalized the rate of decay.

The rest of the chapter is organized as follows: in Section 2, we introduce some notations and preliminaries needed for our work and prove several technical lemmas. In Section 3, we state and prove the general decay of the energy associated to the solution of system (4.1)-(4.3).

## 4.2 Preliminaries

We begin this section by requiring the following assumptions on the relaxation functions $g$ and $h$ :

(H1) $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are $C^1$ non increasing functions satisfying

$$g(0) > 0,\ l = \alpha - \int_0^{+\infty} g(s)\, ds > 0,$$
$$h(0) > 0,\ k = \gamma - \int_0^{+\infty} h(s)\, ds > 0,$$

with

$$lk > \beta^2. \tag{4.7}$$

(H2) There exist two non-increasing $C^1$ functions $\eta, \xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$g'(t) \leq -\xi(t)\, g(t),\ h'(t) \leq -\eta(t)\, h(t),\ t \geq 0.$$

We recall the notation

$$(f \circ \phi)(t) := \int_0^\pi \int_0^t f(t-s)(\phi(t) - \phi(s))^2 \, dsdx.$$

**Remark 4.2.1.** *From (4.1)$_1$ and (4.3) we easily show that*

$$\frac{d^2}{dt^2} \int_0^\pi u(x,t)\, dx = 0. \tag{4.8}$$

*By solving (4.8) and using the initial data of $u$, we obtain*

$$\int_0^\pi u(x,t)\, dx = t \int_0^\pi u_1(x)\, dx + \int_0^\pi u_0(x)\, dx.$$

*Consequently, if we set*

$$\overline{u}(x,t) = u(x,t) - t \int_0^\pi u_1(x)\, dx - \int_0^\pi u_0(x)\, dx,$$

*then $(\overline{u}, \varphi, \psi)$ satisfies (4.1)–(4.2) and the initial data*

$$\overline{u}(x,0) = u_0(x) - \int_0^\pi u_0(x)\, dx,\ \overline{u}_t(x,0) = u_1(x) - \int_0^\pi u_1(x)\, dx.$$

*Moreover, we have*

$$\int_0^\pi \overline{u}(x,t)\, dx = 0,$$

*which allows to apply Poincaré's inequality. In the sequel, we work with $(\overline{u}, \varphi, \psi)$ but for convenience we write $(u, \varphi, \psi)$.*

**Remark 4.2.2.** *We notice that the matrix $A$ of coefficients is positive definite if and only if*

$$\left(\alpha_1 - \frac{b^2}{\mu}\right)\left(\alpha_2 - \frac{d^2}{\mu}\right) > \left(\alpha_3 - \frac{bd}{\mu}\right)^2. \tag{4.9}$$

*Moreover, since any principal submatrix of a positive definite matrix is also positive definite, we have*

$$\alpha_1 - \frac{b^2}{\mu} > 0 \text{ and } \alpha_2 - \frac{d^2}{\mu} > 0.$$

Now, let us prove some useful lemmas.

**Lemme 4.2.1.** *Let*

$$E(t) = \frac{1}{2} \int_0^\pi \left[\rho u_t^2 + \kappa_1 \varphi_t^2 + \kappa_2 \psi_t^2 + \mu u_x^2 + \alpha_1 \varphi^2 + \alpha_2 \psi^2 + 2\alpha_3 \varphi\psi\right] dx$$

$$+\frac{1}{2}\left(\alpha-\int_0^t g(s)\,ds\right)\int_0^\pi \varphi_x^2 dx+\frac{1}{2}\left(\gamma-\int_0^t h(s)\,ds\right)\int_0^\pi \psi_x^2 dx$$
$$+b\int_0^\pi u_x\varphi dx+d\int_0^\pi u_x\psi dx+\beta\int_0^\pi \varphi_x\psi_x dx+\frac{1}{2}\left[g\circ\varphi_x+h\circ\psi_x\right],$$

*be the energy associated with the solution of (4.1)–(4.2). Assume that (H1) holds and that $A$ is positive definite , then*

$$E(t)\geq 0,\ \forall t\geq 0.$$

*Proof.* Let $\widetilde{\alpha}=\alpha-\int_0^t g(s)\,ds$ and $\widetilde{\gamma}=\gamma-\int_0^t h(s)\,ds$, clearly, $\widetilde{\alpha}\geq l$ and $\widetilde{\gamma}\geq k$, then using (4.7) and the fact that the matrix $A$ is positive definite, we infer that

$$E(t)=\langle A(u_x,\varphi,\psi),(u_x,\varphi,\psi)\rangle+\frac{1}{2}\left[g\circ\varphi_x+h\circ\psi_x\right]$$
$$+\frac{1}{4}\left(\widetilde{\alpha}-\frac{\beta^2}{\widetilde{\gamma}}\right)\int_0^\pi \varphi_x^2 dx+\frac{1}{4}\left(\widetilde{\gamma}-\frac{\beta^2}{\widetilde{\alpha}}\right)\int_0^\pi \psi_x^2 dx$$
$$+\frac{\widetilde{\alpha}}{2}\int_0^\pi\left(\varphi_x+\frac{\beta}{\widetilde{\alpha}}\psi_x\right)^2 dx+\frac{\widetilde{\gamma}}{2}\int_0^\pi\left(\psi_x+\frac{\beta}{\widetilde{\gamma}}\varphi_x\right)^2 dx\geq 0.$$

□

**Lemme 4.2.2.** *Under the assumptions (H1)-(H2), we have*

$$\int_0^\pi \varphi_t(t)\int_0^t g(t-s)\varphi_{xx}(s)\,dsdx$$
$$=\frac{1}{2}\frac{d}{dt}\left[(g\circ\varphi_x)(t)-\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2(t)\,dx\right]+g(t)\int_0^\pi \varphi_x^2(t)\,dx-\frac{1}{2}(g'\circ\varphi_x).$$

*Proof.* We have

$$\int_0^\pi \varphi_t(t)\int_0^t g(t-s)\varphi_{xx}(s)\,dsdx=-\int_0^\pi \varphi_{xt}(t)\int_0^t g(t-s)\varphi_x(s)\,dsdx$$
$$=\int_0^\pi \varphi_{xt}(t)\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx-\int_0^\pi \varphi_{xt}(t)\varphi_x(t)\int_0^t g(t-s)\,dsdx,$$
$$=\frac{1}{2}\int_0^\pi\int_0^t g(t-s)\frac{d}{dt}(\varphi_x(t)-\varphi_x(s))^2\,dsdx-\frac{1}{2}\left(\int_0^t g(s)\,ds\right)\frac{d}{dt}\int_0^\pi \varphi_x^2(t)\,dx,$$
$$=\frac{1}{2}\frac{d}{dt}\int_0^\pi\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx-\frac{1}{2}\int_0^\pi\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx$$
$$-\frac{1}{2}\frac{d}{dt}\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2(t)\,dx+g(t)\int_0^\pi \varphi_x^2(t)\,dx,$$
$$=\frac{1}{2}\frac{d}{dt}\left[(g\circ\varphi_x)(t)-\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2(t)\,dx\right]+g(t)\int_0^\pi \varphi_x^2(t)\,dx-\frac{1}{2}(g'\circ\varphi_x).$$

□

**Lemme 4.2.3.** *Suppose that (H1) holds, then along the solution of the system (4.1)–(4.3) the energy $E(t)$ satisfies the property*

$$E'(t) \leq \frac{1}{2}(g' \circ \varphi_x) + \frac{1}{2}(h' \circ \psi_x) \leq 0. \tag{4.10}$$

*Proof.* Multiplying the equations of (4.1) by $u_t, \varphi_t, \psi_t$ respectively and integrating with respect to $x$ over $(0, \pi)$ we obtain

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi u_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^\pi u_x^2 dx + b\int_0^\pi \varphi u_{tx} dx + d\int_0^\pi \psi u_{tx} dx = 0, \tag{4.11}$$

$$\begin{aligned}&\frac{\kappa_1}{2}\frac{d}{dt}\int_0^\pi \varphi_t^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^\pi \varphi_x^2 dx + \frac{\alpha_1}{2}\frac{d}{dt}\int_0^\pi \varphi^2 dx + \beta\int_0^\pi \psi_x\varphi_{xt} dx + b\int_0^\pi u_x\varphi_t dx + \alpha_3\int_0^\pi \psi\varphi_t dx \\ &+ \frac{1}{2}\frac{d}{dt}\left[(g \circ \varphi_x)(t) - \int_0^t g(s)\, ds \int_0^\pi \varphi_x^2(t)\, dx\right] = -g(t)\int_0^\pi \varphi_x^2(t)\, dx + \frac{1}{2}(g' \circ \varphi_x)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\frac{\kappa_2}{2}\frac{d}{dt}\int_0^\pi \psi_t^2 dx + \frac{\gamma}{2}\frac{d}{dt}\int_0^\pi \psi_x^2 dx + \frac{\alpha_2}{2}\frac{d}{dt}\int_0^\pi \psi^2 dx + \beta\int_0^\pi \psi_{xt}\varphi_x dx + d\int_0^\pi u_x\psi_t dx + \alpha_3\int_0^\pi \psi_t\varphi dx \\ &+ \frac{1}{2}\frac{d}{dt}\left[(h \circ \psi_x)(t) - \int_0^t h(s)\, ds \int_0^\pi \psi_x^2(t)\, dx\right] = -h(t)\int_0^\pi \psi_x^2(t)\, dx + \frac{1}{2}(h' \circ \psi_x).\end{aligned} \tag{4.13}$$

The addition of (4.11)-(4.13) gives

$$\begin{aligned}E'(t) &= -g(t)\int_0^\pi \varphi_x^2(t)\, dx + \frac{1}{2}(g' \circ \varphi_x) - h(t)\int_0^\pi \psi_x^2(t)\, dx + \frac{1}{2}(h' \circ \psi_x), \\ &\leq \frac{1}{2}(g' \circ \varphi_x) + \frac{1}{2}(h' \circ \psi_x) \leq 0.\end{aligned}$$

□

**Lemme 4.2.4.** *Let $v \in L^2(0, \pi)$ and $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ an integrable function, then for any $\epsilon > 0$, we have*

$$\begin{aligned}\int_0^\pi v(x,t)\int_0^t f(t-s)\, v(x,s)\, dsdx \leq & \left(\epsilon + \int_0^t f(s)\, ds\right)\int_0^\pi v^2(x,t)\, dx \\ & + \frac{1}{4\epsilon}\left(\int_0^t f(s)\, ds\right)(g \circ v).\end{aligned} \tag{4.14}$$

*Proof.* We have

$$\int_0^\pi v(x,t)\int_0^t f(t-s)\, v(x,s)\, dsdx$$

$$= -\int_0^\pi v(x,t)\int_0^t f(t-s)(v(x,t)-v(x,s))\,dsdx + \left(\int_0^t f(s)\,ds\right)\int_0^\pi v^2dx. \quad (4.15)$$

Using Young's and Cauchy Schwarz' inequalities we obtain

$$-\int_0^\pi v(x,t)\int_0^t f(t-s)(v(x,t)-v(x,s))\,dsdx$$

$$\leq \epsilon\int_0^\pi v^2(x,t)\,dx + \frac{1}{4\epsilon}\int_0^\pi\left(\int_0^t f(t-s)(v(x,t)-v(x,s))\,ds\right)^2dx,$$

$$\leq \epsilon\int_0^\pi v^2(t)\,dx + \frac{1}{4\epsilon}\left(\int_0^t f(s)\,ds\right)\int_0^\pi\int_0^t f(t-s)(v(t)-v(s))^2\,dsdx. \quad (4.16)$$

Substituting (4.16) into (4.15), inequality (4.14) follows immediately. □

**Remark 4.2.3.** *Applying Lemma 4.2.4 and the fact that* $\int_0^t g(s)\,ds \leq \alpha - l$,

$\int_0^t h(s)\,ds \leq \gamma - k$, *we have,*

$$\int_0^\pi \varphi_x(t)\int_0^t g(t-s)\varphi_x(s)\,dsdx \leq \left(\epsilon+\int_0^t g(s)\,ds\right)\int_0^\pi \varphi_x^2dx + \frac{1}{4\epsilon}\left(\int_0^t g(s)\,ds\right)(g\circ\varphi_x)$$
$$\leq (\alpha+\epsilon)\int_0^\pi \varphi_x^2dx + \frac{\alpha-l}{4\epsilon}(g\circ\varphi_x)$$

*and*

$$\int_0^\pi \psi_x(t)\int_0^t h(t-s)\psi_x(s)\,dsdx \leq \left(\epsilon+\int_0^t h(s)\,ds\right)\int_0^\pi \psi_x^2dx + \frac{1}{4\epsilon}\left(\int_0^t h(s)\,ds\right)(h\circ\psi_x)$$
$$\leq (\gamma+\epsilon)\int_0^\pi \psi_x^2dx + \frac{\gamma-k}{4\epsilon}(h\circ\psi_x).$$

*On the other hand, Cauchy Schwarz inequality yields*

$$\left(\int_0^x u_t(y)\,dy\right)^2 \leq \left(\int_0^\pi dx\right)\int_0^\pi u_t^2(x)\,dx = \pi\int_0^\pi u_t^2(x)\,dx.$$

For the sake of completeness, we end this section by stating the following theorem, where the proof can by done by the use of Faedo-Galerkin method.

**Theorem 4.2.1.** *Assume that the assumptions (H1), (H2) and (4.9) hold, then for every* $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in H_0^1(0,\pi)\times L^2(0,\pi)$ *and* $(u_0,u_1) \in H_*^1(0,\pi)\times L_*^2(0,\pi)$, *the problem (4.1)–(4.11) has a unique weak solution* $(u,\varphi,\psi)$ *satisfies*

$$u \in C\left((0,+\infty), H_*^1(0,\pi)\right)\cap C^1\left((0,+\infty), L_*^2(0,\pi)\right),$$
$$\varphi,\psi \in C\left((0,+\infty), H_0^1(0,\pi)\right)\cap C^1\left((0,+\infty), L^2(0,\pi)\right),$$

*where*

$$L_*^2(0,\pi) := \left\{ v \in L^2(0,\pi) \, ; \int_0^\pi v(x)\, dx = 0 \right\} \text{ and } H_*^1(0,\pi) = H^1(0,\pi) \cap L_*^2(0,\pi).$$

## 4.3 General decay

Now we are able to state and prove our main result, which reads as follows

**Theorem 4.3.1.** *Let* $(u_0, u_1) \in H_*^1(0,\pi) \times L_*^2(0,\pi)$ *and* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ *and assume that (H1)–(H2) hold and*

$$\frac{\mu}{\rho} = \frac{\alpha}{\kappa_1} + \frac{d\beta}{b\kappa_2} = \frac{\gamma}{\kappa_2} + \frac{b\beta}{d\kappa_1}. \tag{4.17}$$

*Then for any* $t_0 > 0$, *there exist two positive constants* $\sigma, \omega$ *such that the energy* $E(t)$ *satisfies*

$$E(t) \leq \sigma e^{-\omega \int_{t_0}^t \chi(s) ds}, \text{ for a.e. } t \geq t_0, \tag{4.18}$$

*where* $\chi(t) = \min\{\xi(t), \eta(t)\}$.

**Remark 4.3.1.** *Our result is optimal if:*

1. *for all* $t \geq 0$, $\xi(t) \neq \eta(t)$, *or,*
2. *there exists* $t_1 > 0$; $\xi(t_1) = \eta(t_1)$ *and* $\lim\limits_{t \longrightarrow t_1} \dfrac{\xi(t) - \eta(t)}{t - t_1} = 0.$

*In both cases* $\chi$ *is differentiable for all* $t \geq 0$.

The proof of Theorem 4.3.1 will be established through several lemmas.

**Lemme 4.3.1.** *For any* $\varepsilon > 0$, *the functional*

$$F_1(t) = \kappa_1 \int_0^\pi \varphi\varphi_t dx + \kappa_2 \int_0^\pi \psi_t \psi dx + \frac{\rho}{\mu} \int_0^\pi (b\varphi + d\psi) \int_0^x u_t(y)\, dy dx$$

*satisfies along the solution of (4.1)–(4.3), the estimate*

$$\begin{aligned} F_1'(t) \leq & (\kappa_1 + C_\varepsilon) \int_0^\pi \varphi_t^2 dx + (\kappa_2 + C_\varepsilon) \int_0^\pi \psi_t^2 dx - \frac{\widehat{l}}{2} \int_0^\pi \varphi_x^2 dx - \frac{\widehat{k}}{2} \int_0^\pi \psi_x^2 dx \\ & - \widehat{\alpha}_1 \int_0^\pi \varphi^2 dx - \widehat{\alpha}_2 \int_0^\pi \psi^2 dx + c(g \circ \varphi_x) + c(h \circ \psi_x) + \varepsilon \int_0^\pi u_t^2 dx, \end{aligned} \tag{4.19}$$

*where,* $\widehat{l}$ *and* $\widehat{k}$ *are positive constants depend on* $l, k, \beta$ *and* $\epsilon$, *and* $\widehat{\alpha}_1, \widehat{\alpha}_2$ *are positive constants depend on* $b, d, \mu, \alpha_1, \alpha_2$ *and* $\alpha_3$.

*Proof.* The differentiation of $F_1$ gives

$$F_1'(t) = \kappa_1 \int_0^\pi \varphi_t^2 dx + \kappa_2 \int_0^\pi \psi_t^2 dx + \frac{\rho}{\mu} \int_0^\pi (b\varphi_t + d\psi_t) \int_0^x u_t(y)\, dydx$$

$$+ \int_0^\pi \varphi \left( \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \int_0^t g(t-s)\, \varphi_{xx}(s)\, ds \right) dx$$

$$+ \int_0^\pi \psi \left( \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \int_0^t h(t-s)\, \psi_{xx}(s)\, ds \right) dx$$

$$+\frac{1}{\mu} \int_0^\pi (b\varphi + d\psi)(\mu u_x + b\varphi + d\psi)\, dx,$$

$$F_1'(t) = \kappa_1 \int_0^\pi \varphi_t^2 dx - \alpha \int_0^\pi \varphi_x^2 dx + \left( \frac{b^2}{\mu} - \alpha_1 \right) \int_0^\pi \varphi^2 dx + \kappa_2 \int_0^\pi \psi_t^2 dx$$

$$-\gamma \int_0^\pi \psi_x^2 dx + \left( \frac{d^2}{\mu} - \alpha_2 \right) \int_0^\pi \psi^2 dx - 2\beta \int_0^\pi \psi_x \varphi_x dx + \underbrace{\int_0^\pi \varphi_x \int_0^t g(t-s)\, \varphi_x(s)\, dsdx}_{I_4}$$

$$+\underbrace{\int_0^\pi \psi_x \int_0^t h(t-s)\, \psi_x(s)\, dsdx}_{I_5} + \underbrace{2\left( \frac{db}{\mu} - \alpha_3 \right) \int_0^\pi \varphi\psi dx}_{I_3}$$

$$+\underbrace{\frac{\rho b}{\mu} \int_0^\pi \varphi_t \int_0^x u_t(y)\, dydx}_{I_1} + \underbrace{\frac{\rho d}{\mu} \int_0^\pi \psi_t \int_0^x u_t(y)\, dydx}_{I_2}.$$

Now we estimate the terms $I_1$ by using Young inequality

$$\begin{aligned} I_1 &= \frac{\rho b}{\mu} \int_0^\pi \varphi_t \int_0^x u_t(y)\, dydx \\ &\leq \frac{\pi^2}{2\varepsilon} \left( \frac{\rho b}{\mu} \right)^2 \int_0^\pi \varphi_t^2 dx + \frac{\varepsilon}{2} \int_0^\pi \left( \int_0^x u_t(y)\, dy \right)^2 dx \\ &\leq C_\varepsilon \int_0^\pi \varphi_t^2 dx + \frac{\varepsilon}{2} \int_0^\pi u_t^2 dx, \end{aligned}$$

then, we estimate the terms $I_2$

$$\begin{aligned} I_2 &= \frac{\rho d}{\mu} \int_0^\pi \psi_t \int_0^x u_t(y)\, dydx \\ &\leq \frac{\pi^2}{2\varepsilon} \left( \frac{\rho d}{\mu} \right)^2 \int_0^\pi \psi_t^2 dx + \frac{\varepsilon}{2} \int_0^\pi \left( \int_0^x u_t(y)\, dy \right)^2 dx \\ &\leq C_\varepsilon \int_0^\pi \psi_t^2 dx + \frac{\varepsilon}{2} \int_0^\pi u_t^2 dx, \end{aligned}$$

then,

$$I_3 = 2\left(\frac{bd}{\mu} - \alpha_3\right)\int_0^\pi \varphi\psi dx \le \varepsilon_2\left(\frac{bd}{\mu} - \alpha_3\right)\int_0^\pi \varphi^2 dx + \frac{1}{\varepsilon_2}\left(\frac{bd}{\mu} - \alpha_3\right)\int_0^\pi \psi^2 dx,$$

estimation of the $I_4$ and $I_5$ in Remark 4.2.3, then we get

$$\begin{aligned} F_1'(t) &= \kappa_1 \int_0^\pi \varphi_t^2 dx - \alpha \int_0^\pi \varphi_x^2 dx + \left(\frac{b^2}{\mu} - \alpha_1\right)\int_0^\pi \varphi^2 dx + \kappa_2 \int_0^\pi \psi_t^2 dx \\ &- \gamma \int_0^\pi \psi_x^2 dx + \left(\frac{d^2}{\mu} - \alpha_2\right)\int_0^\pi \psi^2 dx - 2\beta \int_0^\pi \psi_x \varphi_x dx + (\alpha + \epsilon)\int \varphi_x^2 dx \\ &+ \frac{\alpha - l}{4\epsilon}(g \circ \varphi_x) + (\gamma + \epsilon)\int_0^\pi \psi_x^2 dx + \frac{\gamma - k}{4\epsilon}(h \circ \psi_x) + \varepsilon_2\left(\frac{bd}{\mu} - \alpha_3\right)\int_0^\pi \varphi^2 dx \\ &+ \frac{1}{\varepsilon_2}\left(\frac{bd}{\mu} - \alpha_3\right)\int_0^\pi \psi^2 dx + C_\varepsilon \int_0^\pi \varphi_t^2 dx + \frac{\varepsilon}{2}\int_0^\pi u_t^2 dx + C_\varepsilon \int_0^\pi \psi_t^2 dx + \frac{\varepsilon}{2}\int_0^\pi u_t^2 dx, \end{aligned}$$

then, we infer that for any $\epsilon, \varepsilon, \varepsilon_1, \varepsilon_2 > 0$ we have

$$\begin{aligned} F_1'(t) &= (\kappa_1 + C_\varepsilon)\int_0^\pi \varphi_t^2 dx + (\kappa_2 + C_\varepsilon)\int_0^\pi \psi_t^2 dx - (l - \epsilon + -\varepsilon_1\beta)\int_0^\pi \varphi_x^2 dx \\ &- \left(k - \epsilon - \frac{\beta}{\varepsilon_1}\right)\int_0^\pi \psi_x^2 dx + \frac{\alpha - l}{4\epsilon}(g \circ \varphi_x) + \frac{\gamma - k}{4\epsilon}(h \circ \psi_x) + \varepsilon \int_0^\pi u_t^2 dx \\ &- \left(\alpha_1 - \frac{b^2}{\mu} - \varepsilon_2\left(\frac{bd}{\mu} - \alpha_3\right)\right)\int_0^\pi \varphi^2 dx - \left(\alpha_2 - \frac{d^2}{\mu} - \frac{1}{\varepsilon_2}\left(\frac{bd}{\mu} - \alpha_3\right)\right)\int_0^\pi \psi^2 dx. \end{aligned}$$

Inequality (4.7) and (4.9) allow us to choose $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\widehat{l} = l - \beta\varepsilon_1 > 0 \text{ and } \widehat{k} = k - \frac{\beta}{\varepsilon_1} > 0$$

and

$$\widehat{\alpha}_1 = \alpha_1 - \frac{b^2}{\mu} - \left(\frac{bd}{\mu} - \alpha_3\right)\varepsilon_2 > 0 \text{ and } \widehat{\alpha}_2 = \alpha_2 - \frac{d^2}{\mu} - \left(\frac{bd}{\mu} - \alpha_3\right)\frac{1}{\varepsilon_2} > 0.$$

Next we choose $\epsilon = \min\left\{\frac{\widehat{l}}{2}, \frac{\widehat{k}}{2}\right\}$ to get (4.19). □

**Lemme 4.3.2.** *For any $t_0 > 0$ and any $\varepsilon_1, \delta > 0$, the functional*

$$F_2(t) := -\kappa_1 \int_0^\pi \varphi_t(t)\int_0^t g(t-s)(\varphi(t) - \varphi(s))\, ds dx$$

*satisfies along the solution of (4.1)–(4.3) the estimate*

$$\begin{aligned} F_2'(t) \le -\kappa_1\left(\int_0^{t_0} g(s)\,ds - \delta\right)\int_0^\pi \varphi_t^2(t)\,dx + 3\varepsilon_1 \int_0^\pi \varphi_x^2 dx + \varepsilon_1 \int_0^\pi \psi_x^2 dx \\ + \frac{c}{\varepsilon_1} g \circ \varphi_x + \varepsilon_1 \int_0^\pi u_x^2 dx - \frac{c}{\delta} g' \circ \varphi_x. \end{aligned} \tag{4.20}$$

*Proof.* The differentiation of $F_2$ and integration by parts give

$$F_2'(t) = \alpha \int_0^\pi \varphi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx$$
$$+\beta \int_0^\pi \psi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx$$
$$+b \int_0^\pi u_x \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx + \alpha_1 \int_0^\pi \varphi \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx$$
$$+\alpha_3 \int_0^\pi \psi \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx$$
$$-\int_0^\pi \left(\int_0^t g(t-s)\varphi_x(s)\, ds\right)\left(\int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, ds\right) dx$$
$$-\kappa_1 \int_0^\pi \varphi_t(t) \int_0^t g'(t-s)(\varphi(t) - \varphi(s))\, dsdx - \kappa_1 \left(\int_0^t g(s)\, ds\right) \int_0^\pi \varphi_t^2(t)\, dx.$$

Now we estimate the terms in the right hand side of $F_2'(t)$ term by term recalling that

$$\int_0^t g(s)\, ds < \alpha \text{ and } \int_0^t h(s)\, ds < \gamma.$$

First, using Young's and Cauchy Schwarz's inequalities, we have for any $\varepsilon_1 > 0$,

$$I_1 = \alpha \int_0^\pi \varphi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx$$

$$\leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{\alpha^2}{4\varepsilon_1} \int_0^\pi \left(\int_0^t \left(\sqrt{g(t-s)}\right)\left(\sqrt{g(t-s)}(\varphi_x(t) - \varphi_x(s))\, ds\right)\right)^2 dx,$$
$$\leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{\alpha^2}{4\varepsilon_1}\left(\int_0^t g(t-s)\, ds\right) \int_0^\pi \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))^2\, dsdx,$$
$$I_1 \leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{\alpha^3}{4\varepsilon_1} g \circ \varphi_x. \tag{4.21}$$

Similarly, for any $\varepsilon_2 > 0$, we have

$$I_2 = \beta \int_0^\pi \psi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx \leq \varepsilon_1 \int_0^\pi \psi_x^2 dx + \frac{\beta^2\alpha}{4\varepsilon_1} g \circ \varphi_x. \tag{4.22}$$

$$\begin{aligned} I_3 &= \alpha_3 \int_0^\pi \psi \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx \leq \varepsilon_1 \int_0^\pi \psi^2 dx + \frac{\beta^2\alpha}{4\varepsilon_1} g \circ \varphi \\ &\leq C\varepsilon_1 \int_0^\pi \psi_x^2 dx + \frac{C\beta^2\alpha}{4\varepsilon_1} g \circ \varphi_x, \end{aligned} \tag{4.23}$$

Next, we use Young's, Poincaré's and Cauchy Schwarz inequalities to obtain for any $\varepsilon_1 > 0$ and $\delta > 0$,

$$\begin{aligned} I_4 &= b\int_0^\pi u_x \int_0^t g(t-s)(\varphi(t)-\varphi(s))\,dsdx \\ &\leq \varepsilon_1 \int_0^\pi u_x^2 dx + \frac{b^2}{4\varepsilon_1}\int_0^\pi \left(\int_0^t g(t-s)(\varphi(t)-\varphi(s))\,ds\right)^2 dx \\ &\leq \varepsilon_1 \int_0^\pi u_x^2 dx + \frac{b^2}{4\varepsilon_1}\left(\int_0^t g(t-s)\,ds\right)\int_0^\pi\int_0^t g(t-s)(\varphi(t)-\varphi(s))^2\,dsdx \\ &\leq \varepsilon_1 \int_0^\pi u_x^2 dx + \frac{Cb^2\alpha}{4\varepsilon_1} g\circ\varphi_x. \end{aligned} \tag{4.24}$$

Similarly, for any $\varepsilon_1 > 0$, we have

$$I_5 = \int_0^\pi \varphi \int_0^t g(t-s)(\varphi(t)-\varphi(s))\,dsdx \leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{C\alpha}{4\varepsilon_1} g\circ\varphi_x, \tag{4.25}$$

$$I_6 = -\kappa_1 \int_0^\pi \varphi_t(t) \int_0^t g'(t-s)(\varphi(t)-\varphi(s))\,dsdx$$

$$\leq \kappa_1\delta \int_0^\pi \varphi_t^2 dx + \frac{\kappa_1}{4\delta}\int_0^\pi \left(\int_0^t g'(t-s)(\varphi(t)-\varphi(s))\,ds\right)^2 dx$$

$$\leq \kappa_1\delta \int_0^\pi \varphi_t^2 dx + \frac{\kappa_1}{4\delta}\left(-\int_0^t g'(s)\,ds\right)\int_0^\pi\int_0^t -g'(t-s)(\varphi(t)-\varphi(s))^2\,dsdx$$

$$\leq \kappa_1\delta \int_0^\pi \varphi_t^2 dx - \frac{\kappa_1 C g(0)}{4\delta}\int_0^\pi\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx,$$

then

$$I_6 \leq \kappa_1\delta \int_0^\pi \varphi_t^2 dx - \frac{c}{\delta} g'\circ\varphi_x, \tag{4.26}$$

$$I_7 = -\int_0^\pi \left(\int_0^t g(t-s)\varphi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,ds\right)dx$$

$$= \int_0^\pi \left(\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,ds\right)^2 dx$$

$$-\left(\int_0^t g(s)\,ds\right)\int_0^\pi \varphi_x(t)\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx$$

$$\leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \left[1 + \frac{\left(\int_0^t g(s)\,ds\right)^2}{4\varepsilon_1}\right]\int_0^\pi \left(\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\right)^2 dsdx$$

$$\leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \left[1 + \frac{\left(\int_0^t g(s)\,ds\right)^2}{4\varepsilon_1}\right] \left(\int_0^t g(t-s)\,ds\right) \int_0^\pi \int_0^t g(t-s)\left(\varphi_x(t) - \varphi_x(s)\right)^2 ds dx$$

$$I_7 \leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{c}{\varepsilon_1} g \circ \varphi_x. \tag{4.27}$$

Substituting $I_1 - I_7$,in the expression of $F_2'(t)$ we arrive at

$$F_2'(t) \leq -\kappa_1 \left(\int_0^t g(s)\,ds\right) \int_0^\pi \varphi_t^2(t)\,dx + 3\varepsilon_1 \int_0^\pi \varphi_x^2 dx + \varepsilon_1 (1+C) \int_0^\pi \psi_x^2 dx$$
$$+ \frac{c}{\varepsilon_1} g \circ \varphi_x + \varepsilon_2 \int_0^\pi u_x^2 dx + \delta\kappa_1 \int_0^\pi \varphi_t^2 dx - \frac{c}{\delta} g' \circ \varphi_x,$$

then for any $t \geq t_0 > 0$, we have

$$F_2'(t) \leq -\kappa_1 \left(\int_0^{t_0} g(s)\,ds - \delta\right) \int_0^\pi \varphi_t^2(t)\,dx + 3\varepsilon_1 \int_0^\pi \varphi_x^2 dx + \varepsilon_1 (1+C) \int_0^\pi \psi_x^2 dx$$
$$+ \frac{c}{\varepsilon_1} g \circ \varphi_x + \varepsilon_1 \int_0^\pi u_x^2 dx - \frac{c}{\delta} g' \circ \varphi_x.$$

□

**Remark 4.3.2.** *Similarly, the functional*

$$F_3(t) := -\kappa_2 \int_0^\pi \psi_t(t) \int_0^t h(t-s)\left(\psi(t) - \psi(s)\right) ds dx,$$

*satisfies for any $\varepsilon_1, \delta > 0$, the estimate*

$$F_3'(t) \leq -\kappa_2 \left(\int_0^{t_0} h(s)\,ds - \delta\right) \int_0^\pi \psi_t^2(t)\,dx + 3\varepsilon_1 \int_0^\pi \psi_x^2 dx + \varepsilon_1 \int_0^\pi \varphi_x^2 dx$$
$$+ \frac{c}{\varepsilon_1} h \circ \psi_x + \varepsilon_1 \int_0^\pi u_x^2 dx - \frac{c}{\delta} h' \circ \psi_x. \tag{4.28}$$

**Lemme 4.3.3.** *Suppose that (4.17) holds, then for any $\varepsilon_3 > 0$, the functional*

$$F_4(t) := b \int_0^\pi \varphi_x u_t dx + b \int_0^\pi u_x \varphi_t dx - \frac{b\rho}{\mu\kappa_1} \int_0^\pi u_t \int_0^t g(t-s)\varphi_x(s)\,ds dx$$
$$+ d \int_0^\pi \psi_x u_t dx + d \int_0^\pi u_x \psi_t dx - \frac{d\rho}{\mu\kappa_2} \int_0^\pi u_t \int_0^t h(t-s)\psi_x(s)\,ds dx,$$

*satisfies along the solution of (4.1)-(4.3), the estimate*

$$F_4'(t) \leq -\frac{1}{2}\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2}\right) \int_0^\pi u_x^2 dx + \frac{c}{\varepsilon_3} \int_0^\pi \varphi_x^2 dx + \left(c + \frac{c}{\varepsilon_3}\right) \int_0^\pi \psi_x^2 dx$$
$$+ \frac{bd\alpha}{2\mu\kappa_1} g \circ \varphi_x + \frac{b^2}{4\mu\kappa_1} g \circ \varphi_x + c\varepsilon_3 \int_0^\pi u_t^2 - \frac{c}{\varepsilon_3} g' \circ \varphi_x \tag{4.29}$$
$$+ \frac{bd\gamma}{2\mu\kappa_2} h \circ \psi_x + \frac{d^2}{4\mu\kappa_2} h \circ \psi_x - \frac{c}{\varepsilon_3} g' \circ \varphi_x - \frac{c}{\varepsilon_3} h' \circ \psi_x.$$

*Proof.* Differentiating $F_4(t)$ we obtain

$$F_4(t) := b\int_0^\pi \varphi_{xt}u_t dx + b\int_0^\pi \varphi_x u_{tt}dx + b\int_0^\pi u_{xt}\varphi_t dx + b\int_0^\pi u_x\varphi_{tt}dx$$

$$-\frac{b\rho}{\mu\kappa_1}\int_0^\pi u_{tt}\int_0^t g(t-s)\varphi_x(s)\,dsdx - \frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\left(\int_0^t g(t-s)\varphi_x(s)\,ds\right)_t dx$$

$$+d\int_0^\pi \psi_{xt}u_t dx + d\int_0^\pi \psi_x u_{tt}dx + d\int_0^\pi u_{xt}\psi_t dx + d\int_0^\pi u_x\psi_{tr}dx$$

$$-\frac{d\rho}{\mu\kappa_2}\int_0^\pi u_{tt}\int_0^t h(t-s)\psi_x(s)\,dsdx - \frac{d\rho}{\mu\kappa_2}\int_0^\pi u_t\left(\int_0^t h(t-s)\psi_x(s)\,ds\right)_t dx.$$

Taking into account (4.1)-(4.3) and integrating by parts, we have

$$F_4'(t) = \frac{b}{\rho}\int_0^\pi \varphi_x(\mu u_{xx} + b\varphi_x + d\psi_x)\,dx$$

$$+\frac{b}{\kappa_1}\int_0^\pi u_x\left(\alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \int_0^t g(t-s)\varphi_{xx}(s)\,ds\right)dx$$

$$-\frac{b}{\mu\kappa_1}\int_0^\pi (\mu u_{xx} + b\varphi_x + d\psi_x)\int_0^t g(t-s)\varphi_x(s)\,dsdx$$

$$-\frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x(t)\,dsdx - \frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\int_0^t g'(t-s)\varphi_x(s)\,dsdx$$

$$+\frac{d}{\rho}\int_0^\pi \psi_x(\mu u_{xx} + b\varphi_x + d\psi_x)\,dx$$

$$+\frac{d}{\kappa_2}\int_0^\pi u_x\left(\beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \int_0^t h(t-s)\psi_{xx}(s)\,ds\right)dx$$

$$-\frac{d}{\mu\kappa_2}\int_0^\pi (\mu u_{xx} + b\varphi_x + d\psi_x)\int_0^t h(t-s)\psi_x(s)\,dsdx$$

$$-\frac{d\rho h(0)}{\mu\kappa_2}\int_0^\pi u_t\psi_x(t)\,dsdx - \frac{d\rho}{\mu\kappa_2}\int_0^\pi u_t\int_0^t h'(t-s)\psi_x(s)\,dsdx,$$

then

$$F_4'(t) = \left(\frac{b\mu}{\rho} - \frac{b\alpha}{\kappa_1} - \frac{d\beta}{\kappa_2}\right)\int_0^\pi \varphi_x u_{xx}dx + \left(\frac{d\mu}{\rho} - \frac{b\beta}{\kappa_1} - \frac{d\gamma}{\kappa_2}\right)\int_0^\pi \psi_x u_{xx}dx$$

$$-\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2}\right)\int_0^\pi u_x^2dx + \frac{b^2}{\rho}\int_0^\pi \varphi_x^2dx + \frac{d^2}{\rho}\int_0^\pi \psi_x^2dx + 2\frac{bd}{\rho}\int_0^\pi \varphi_x\psi_x dx$$

$$-\left(\frac{b\alpha_1}{\kappa_1} + \frac{d\alpha_3}{\kappa_2}\right)\int_0^\pi u_x\varphi dx - \left(\frac{b\alpha_3}{\kappa_1} + \frac{d\alpha_2}{\kappa_2}\right)\int_0^\pi u_x\psi dx$$

$$-\frac{b^2}{\mu\kappa_1}\int_0^\pi \varphi_x\int_0^t g(t-s)\varphi_x(s)\,dsdx - \frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x\int_0^t g(t-s)\varphi_x(s)\,dsdx$$

$$-\frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x(t)\,dsdx-\frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\int_0^t g'(t-s)\varphi_x(s)\,dsdx$$
$$-\frac{bd}{\mu\kappa_2}\int_0^\pi \varphi_x\int_0^t h(t-s)\psi_x(s)\,dsdx-\frac{d^2}{\mu\kappa_2}\int_0^\pi \psi_x\int_0^t h(t-s)\psi_x(s)\,dsdx$$
$$-\frac{d\rho h(0)}{\mu\kappa_2}\int_0^\pi u_t\psi_x(t)\,dsdx-\frac{d\rho}{\mu\kappa_2}\int_0^\pi u_t\int_0^t h'(t-s)\psi_x(s)\,dsdx.$$

At this point we estimate the terms of the right hand side of $F_4'(t)$ term by term, taking into account (4.17). First we have

$$\begin{aligned}J_1 &:= \frac{b^2}{\rho}\int_0^\pi \varphi_x^2dx+\frac{d^2}{\rho}\int_0^\pi \psi_x^2dx+2\frac{bd}{\rho}\int_0^\pi \varphi_x\psi_xdx,\\ &= \frac{1}{\rho}\int_0^\pi (b\varphi_x+d\psi_x)^2\,dx,\\ &\le 2\frac{b^2}{\rho}\int_0^\pi \varphi_x^2dx+2\frac{d^2}{\rho}\int_0^\pi \psi_x^2dx.\end{aligned}$$

Secondly, we have for any $\xi>0$,

$$J_2 := -\left(\frac{b\alpha_1}{\kappa_1}+\frac{d\alpha_3}{\kappa_2}\right)\int_0^\pi u_x\varphi dx-\left(\frac{b\alpha_3}{\kappa_1}+\frac{d\alpha_2}{\kappa_2}\right)\int_0^\pi u_x\psi dx,$$

$$\le \frac{\xi}{2}\int_0^\pi u_x^2dx+\frac{C\left(\frac{b\alpha_1}{\kappa_1}+\frac{d\alpha_3}{\kappa_2}\right)^2}{2\xi}\int_0^\pi \varphi_x^2dx+\frac{\xi}{2}\int_0^\pi u_x^2dx+\frac{C\left(\frac{b\alpha_3}{\kappa_1}+\frac{d\alpha_2}{\kappa_2}\right)^2}{2\xi}\int_0^\pi \psi_x^2dx,$$
$$\le \xi\int_0^\pi u_x^2dx+\frac{c}{\xi}\int_0^\pi \varphi_x^2dx+\frac{c}{\xi}\int_0^\pi \psi_x^2dx.$$

Thirdly,

$$J_3 := -\frac{b^2}{\mu\kappa_1}\int_0^\pi \varphi_x\int_0^t g(t-s)\varphi_x(s)\,dsdx=-\frac{b^2}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2dx$$
$$+\frac{b^2}{\mu\kappa_1}\int_0^\pi \varphi_x\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx,$$
$$\le \frac{\delta b^2}{\mu\kappa_1}\int_0^\pi\int_0^t g(t-s)\varphi_x^2dsdx+\frac{b^2}{4\delta\mu\kappa_1}\int_0^\pi\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx$$
$$-\frac{b^2}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2dx,$$
$$\le (\delta-1)\frac{b^2}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2dx+\frac{b^2}{4\delta\mu\kappa_1}\int_0^\pi\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx.$$

Taking $\delta=1$, we get
$$J_3\le \frac{b^2}{4\mu\kappa_1}g\circ\varphi_x.$$

Fourthly,

$$J_4 := -\frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x \int_0^t g(t-s)\varphi_x(s)\,dsdx = \frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x \int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx$$
$$-\frac{bd}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \psi_x\varphi_x dx,$$
$$J_4 \le \frac{bd}{2\mu\kappa_1}\int_0^\pi \psi_x^2 + \frac{bd}{2\mu\kappa_1}\int_0^\pi \left(\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,ds\right)^2 dx$$
$$-\frac{bd}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \psi_x\varphi_x dx,$$
$$\le \frac{bd}{2\mu\kappa_1}\int_0^\pi \psi_x^2 + \frac{bd}{2\mu\kappa_1}\left(\int_0^t g(t-s)\,ds\right)\int_0^\pi\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx$$
$$+\frac{bd}{2\mu\kappa_1}\left(\int_0^t g(s)\,ds\right)^2\int_0^\pi \varphi_x^2 dx + \frac{bd}{2\mu\kappa_1}\int_0^\pi \psi_x^2 dx,$$
$$J_4 \le \frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x^2 + \frac{bd}{2\mu\kappa_1}\left(\int_0^t g(s)\,ds\right)^2\int_0^\pi \varphi_x^2 dx + \frac{bd}{2\mu\kappa_1}\left(\int_0^t g(s)\,ds\right)(g\circ\varphi_x).$$

Therefore,

$$J_4 \le \frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x^2 + \frac{bd\alpha^2}{2\mu\kappa_1}\int_0^\pi \varphi_x^2 dx + \frac{bd\alpha}{2\mu\kappa_1}(g\circ\varphi_x).$$

Fifthly, for any $\varepsilon_3 > 0$, we have

$$J_5 := -\frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\int_0^t g'(t-s)\varphi_x(s)\,dsdx = -\frac{b\rho}{\mu\kappa_1}\int_0^t g'(s)\,ds\int_0^\pi u_t\varphi_x dx$$
$$+\frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx,$$
$$J_5 \le \frac{b\rho\varepsilon_3}{2\mu\kappa_1}\int_0^\pi u_t^2 + \frac{b\rho}{2\varepsilon_3\mu\kappa_1}\int_0^\pi\left(\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))\,ds\right)^2 dx$$
$$+\frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx - \frac{b\rho g(t)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx,$$
$$J_5 \le \frac{b\rho\varepsilon_3}{\mu\kappa_1}\int_0^\pi u_t^2 + \frac{b\rho}{2\varepsilon_3\mu\kappa_1}\int_0^t g'(s)\,ds\int_0^\pi\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx$$
$$+\frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx + \frac{b\rho(g(t))^2}{2\varepsilon_3\mu\kappa_1}\int_0^\pi \varphi_x^2 dx,$$

then,

$$J_5 \le \frac{b\rho\varepsilon_3}{\mu\kappa_1}\int_0^\pi u_t^2 + \frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx + \frac{b\rho(g(0))^2}{2\varepsilon_3\mu\kappa_1}\int_0^\pi \varphi_x^2 dx - \frac{c}{\varepsilon_3}g'\circ\varphi_x.$$

Similarly, for the rest terms

$$J_6 := -\frac{bd}{\mu\kappa_2}\int_0^\pi \varphi_x \int_0^t h(t-s)\,\psi_x(s)\,dsdx$$

$$J_6 \leq \frac{bd}{\mu\kappa_2}\int_0^\pi \varphi_x^2 + \frac{bd\gamma^2}{2\mu\kappa_2}\int_0^\pi \psi_x^2 dx + \frac{bd\gamma}{2\mu\kappa_2} h\circ\psi_x,$$

$$J_7 := -\frac{d^2}{\mu\kappa_2}\int_0^\pi \psi_x \int_0^t h(t-s)\,\psi_x(s)\,dsdx \leq \frac{d^2}{4\mu\kappa_2} h\circ\psi_x,$$

$$J_8 := -\frac{d\rho}{\mu\kappa_2}\int_0^\pi u_t \int_0^t h'(t-s)\,\psi_x(s)\,dsdx,$$

then

$$J_8 \leq \frac{d\rho\varepsilon_3}{\mu\kappa_2}\int_0^\pi u_t^2 dx + \frac{d\rho h(0)}{\mu\kappa_2}\int_0^\pi u_t\psi_x dx + \frac{d\rho\,(h(0))^2}{2\varepsilon_3\mu\kappa_2}\int_0^\pi \psi_x^2 dx - \frac{c}{\varepsilon_3} h'\circ\psi_x.$$

Therefore,

$$F_4'(t) = \left(\frac{b\mu}{\rho} - \frac{b\alpha}{\kappa_1} - \frac{d\beta}{\kappa_2}\right)\int_0^\pi \varphi_x u_{xx} dx + \left(\frac{d\mu}{\rho} - \frac{b\beta}{\kappa_1} - \frac{d\gamma}{\kappa_2}\right)\int_0^\pi \psi_x u_{xx} dx$$

$$-\left(\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2}\right) - \xi\right)\int_0^\pi u_x^2 dx + \left(\frac{bd\alpha^2}{2\mu\kappa_1} + \frac{b\rho\,(g(0))^2}{2\varepsilon_3\mu\kappa_1} + \frac{bd}{\mu\kappa_2} + \frac{c}{\xi} + 2\frac{b^2}{\rho}\right)\int_0^\pi \varphi_x^2 dx$$

$$+\left(\frac{bd\alpha}{2\mu\kappa_1} + \frac{b^2}{4\mu\kappa_1}\right) g\circ\varphi_x - \frac{c}{\varepsilon_3} g'\circ\varphi_x + \left(\frac{bd\gamma}{2\mu\kappa_2} + \frac{d^2}{4\mu\kappa_2}\right) h\circ\psi_x - \frac{c}{\varepsilon_3} h'\circ\psi_x$$

$$+\left(\frac{b\rho\varepsilon_3}{\mu\kappa_1} + \frac{d\rho\varepsilon_3}{\mu\kappa_2}\right)\int_0^\pi u_t^2 dx + \left(\frac{bd\gamma^2}{2\mu\kappa_2} + \frac{d\rho\,(h(0))^2}{2\varepsilon_3\mu\kappa_2} + \frac{bd}{\mu\kappa_1} + \frac{c}{\xi} + 2\frac{d^2}{\rho}\right)\int_0^\pi \psi_x^2 dx$$

$$F_4'(t) \leq -\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2} - \xi\right)\int_0^\pi u_x^2 dx + \left(\frac{2b^2}{\rho} + \frac{c}{\xi} + \frac{bd\alpha^2}{2\mu\kappa_1} + \frac{c}{\varepsilon_3} + \frac{bd}{\mu\kappa_2}\right)\int_0^\pi \varphi_x^2 dx$$

$$+\left(\frac{2d^2}{\rho} + \frac{c}{\xi} + \frac{bd}{\mu\kappa_1} + \frac{bd\gamma^2}{2\mu\kappa_2} + \frac{c}{\varepsilon_3}\right)\int_0^\pi \psi_x^2 dx$$

$$+\frac{bd\alpha}{2\mu\kappa_1} g\circ\varphi_x + \frac{b^2}{4\mu\kappa_1} g\circ\varphi_x + \left(\frac{b\rho\varepsilon_3}{\mu\kappa_1} + \frac{d\rho\varepsilon_3}{\mu\kappa_2}\right)\int_0^\pi u_t^2 - \frac{c}{\varepsilon_3} g'\circ\varphi_x$$

$$+\frac{bd\gamma}{2\mu\kappa_2} h\circ\psi_x + \frac{d^2}{4\mu\kappa_2} h\circ\psi_x - \frac{c}{\varepsilon_3} g'\circ\varphi_x - \frac{c}{\varepsilon_3} h'\circ\psi_x.$$

If we choose $\xi = \frac{1}{2}\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2}\right)$ we obtain

$$F_4'(t) \leq -\frac{1}{2}\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2}\right)\int_0^\pi u_x^2 dx + \left(c + \frac{c}{\varepsilon_3}\right)\int_0^\pi \varphi_x^2 dx + \left(c + \frac{c}{\varepsilon_3}\right)\int_0^\pi \psi_x^2 dx$$

$$+\frac{bd\alpha}{2\mu\kappa_1}g\circ\varphi_x+\frac{b^2}{4\mu\kappa_1}g\circ\varphi_x+c\varepsilon_3\int_0^\pi u_t^2-\frac{c}{\varepsilon_3}g'\circ\varphi_x$$
$$+\frac{bd\gamma}{2\mu\kappa_2}h\circ\psi_x+\frac{d^2}{4\mu\kappa_2}h\circ\psi_x-\frac{c}{\varepsilon_3}g'\circ\varphi_x-\frac{c}{\varepsilon_3}h'\circ\psi_x.$$

□

**Lemme 4.3.4.** *The functional*

$$F_5(t):=-\rho\int_0^\pi u_t u dx$$

*satisfies along the solution of (4.1)-(4.3) the estimate*

$$F_5'(t)\le-\rho\int_0^\pi u_t^2+2\mu\int_0^\pi u_x^2+c\int_0^\pi\varphi_x^2+c\int_0^\pi\psi_x^2. \tag{4.30}$$

*Proof.* The Differentiation of $F_5'(t)$

$$F_5'(t)=\mu\int_0^\pi u_x^2+b\int_0^\pi\varphi u_x+d\int_0^\pi\psi u_x dx-\rho\int_0^\pi u_t^2 dx.$$

Using Young and point care inequality

$$F_5'(t)=\mu\int_0^\pi u_x^2+\frac{bC}{4\varepsilon}\int_0^\pi\varphi_x^2dx+\varepsilon b\int_0^\pi u_x^2dx+\frac{Cd}{4\epsilon}\int_0^\pi\psi_x^2dx+\epsilon d\int_0^\pi u_x^2dx-\rho\int_0^\pi u_t^2dx,$$

on pose $\varepsilon=\frac{\mu}{2b},\epsilon=\frac{\mu}{2d}$

$$F_5'(t)=\mu\int_0^\pi u_x^2+\frac{b^2C}{2\mu}\int_0^\pi\varphi_x^2dx+\frac{\mu}{2}\int_0^\pi u_x^2dx+\frac{Cd^2}{2\mu}\int_0^\pi\psi_x^2dx+\frac{\mu}{2}\int_0^\pi u_x^2dx-\rho\int_0^\pi u_t^2dx,$$

then

$$F_5'(t)=-\rho\int_0^\pi u_t^2dx+2\mu\int_0^\pi u_x^2+c\int_0^\pi\varphi_x^2dx+c\int_0^\pi\psi_x^2dx.$$

□

### 4.3.1 Proof of the main result

Now we define the Lyapunov functional

$$\mathcal{L}(t)=NE(t)+N_1F_1(t)+N_2(F_2(t)+F_3(t))+N_4F_4(t)+F_5(t).$$

Substituting (3.5), (4.19), (4.20), (4.11), (4.29) and (4.30) in the expression of $\mathcal{L}'(t)$, we obtain

$$\mathcal{L}'(t)\le-\left[N_2\kappa_1\left(\int_0^{t_0}g(s)\,ds-\delta\right)-N_1(\kappa_1+C_\varepsilon)\right]\int_0^\pi\varphi_t^2dx$$

$$-\left[N_2\kappa_2\left(\int_0^{t_0} h(s)\,ds-\delta\right)-N_1(\kappa_2+C_\varepsilon)\right]\int_0^\pi \psi_t^2dx$$
$$-\left[\rho-\varepsilon N_1-c\varepsilon_2N_4\right]\int_0^\pi u_t^2dx-N_1\widehat{\alpha}_1\int_0^\pi \varphi^2dx-N_1\widehat{\alpha}_2\int_0^\pi \psi^2dx$$
$$-\left[\frac{N_4}{2}\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)-2\mu-2N_2\varepsilon_1\right]\int_0^\pi u_x^2dx$$
$$-\left[N_1\frac{\widehat{l}}{2}-4\varepsilon_1N_2-N_4\left(c+\frac{c}{\varepsilon_2}\right)\right]\int_0^\pi \varphi_x^2dx-\left[N_1\frac{\widehat{k}}{2}-4\varepsilon_1N_2-N_4\left(c+\frac{c}{\varepsilon_2}\right)\right]\int_0^\pi \psi_x^2dx$$
$$+\left[\frac{N}{2}-\frac{cN_4}{\varepsilon_2}-\frac{cN_2}{\delta}\right](g'\circ\varphi_x)+\left[\frac{N}{2}-\frac{cN_4}{\varepsilon_2}-\frac{cN_2}{\delta}\right](h'\circ\psi_x)$$
$$+\left[N_4\frac{2bd\alpha+b^2}{4\mu\kappa_1}+cN_1+\frac{c}{\varepsilon_1}N_2\right]g\circ\varphi_x$$
$$+\left[cN_1+N_4\frac{2bd\gamma+d^2}{4\mu\kappa_2}+\frac{c}{\varepsilon_1}N_2\right]h\circ\psi_x.$$

At this point, we choose the constants $N, N_1, N_2, N_3, N_4, \varepsilon, \varepsilon_1, \varepsilon_2$ and $\delta$ carefully.

First we choose $\delta>0$ small such that

$$\int_0^{t_0} g(s)\,ds-\delta>0, \text{ and } \int_0^{t_0} h(s)\,ds-\delta>0,$$

then we choose $N_4$ such that $N_4\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)=6\mu$ we get

$$\frac{N_4}{2}\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)-2\mu=\mu.$$

Next we choose $\varepsilon_2>0$ such that

$$\rho-c\varepsilon_2N_4\geq\frac{\rho}{2}.$$

After that we choose $N_1$ large enough such that

$$\left[N_1\frac{\widehat{l}}{2}-N_4\left(c+\frac{c}{\varepsilon_2}\right)-c\right]\geq\frac{N_1\widehat{l}}{3}$$

and

$$\left[N_1\frac{\widehat{k}}{2}-N_4\left(c+\frac{c}{\varepsilon_2}\right)-c\right]\geq\frac{N_1\widehat{k}}{3}.$$

Now we choose $\varepsilon>0$ such that

$$\frac{\rho}{2}-\varepsilon N_1>0,$$

the next step is to choose $N_2$ large enough such that

$$N_2\kappa_1\left(\int_0^{t_0} g\left(s\right)ds-\delta\right)-N_1\left(\kappa_1+C_\varepsilon\right)>0$$

and

$$N_2\kappa_2\left(\int_0^{t_0} h\left(s\right)ds-\delta\right)-N_1\left(\kappa_2+C_\varepsilon\right)>0.$$

After that we choose $\varepsilon_1>0$ such that

$$\mu-2N_2\varepsilon_1>0,\ \frac{N_1\widehat{l}}{3}-4\varepsilon_1 N_2>0\ and\ \frac{N_1\widehat{k}}{3}-4\varepsilon_1 N_2>0.$$

Thus, there exists, $\varpi>0$ and $c_0>0$, such that

$$\mathcal{L}'\left(t\right)\leq-\varpi\int_0^\pi\left(\varphi_t^2+\psi_t^2+u_t^2+\varphi^2+\psi^2+u_x^2+\varphi_x^2+\psi_x^2\right)dx+c\left(g\circ\varphi_x+h\circ\psi_x\right)$$
$$+\left[\frac{N}{2}-c_0\right]\left(g'\circ\varphi_x+h'\circ\psi_x\right).$$

Now we let $\mathcal{L}\left(t\right)=N_1F_1\left(t\right)+N_2\left(F_2\left(t\right)+F_3\left(t\right)\right)+N_4F_4\left(t\right)+F_5\left(t\right)$, then

$$\left|\mathcal{L}\left(t\right)\right|\leq N_1\left(\kappa_1\int_0^\pi\left|\varphi\varphi_t\right|dx+\kappa_2\int_0^\pi\left|\psi_t\psi\right|dx+\frac{\rho}{\mu}\int_0^\pi\left|b\varphi+d\psi\right|\int_0^x\left|u_t\left(y\right)\right|dydx\right)$$
$$+N_2\left(\kappa_1\int_0^\pi\left|\varphi_t\left(t\right)\int_0^t g\left(t-s\right)\left(\varphi\left(t\right)-\varphi\left(s\right)\right)ds\right|dx\right)$$
$$+N_2\left(\kappa_2\int_0^\pi\left|\psi_t\left(t\right)\int_0^t h\left(t-s\right)\left(\psi\left(t\right)-\psi\left(s\right)\right)ds\right|dx\right)$$
$$+N_4\left(\left|b\right|\int_0^\pi\left|\varphi_x u_t\right|dx+b\int_0^\pi\left|u_x\varphi_t\right|dx+\frac{\left|b\right|\rho}{\mu\kappa_1}\int_0^\pi\left|u_t\int_0^t g\left(t-s\right)\varphi_x\left(s\right)ds\right|dx\right)$$
$$+N_4\left(\left|d\right|\int_0^\pi\left|\psi_x u_t\right|dx+d\int_0^\pi\left|u_x\psi_t\right|dx+\frac{\left|d\right|\rho}{\mu\kappa_2}\int_0^\pi\left|u_t\int_0^t h\left(t-s\right)\psi_x\left(s\right)ds\right|dx\right)$$
$$+\rho\int_0^\pi\left|u_t u\right|dx.$$

Exploiting, Young's, Cauchy Schwarz' and Poincaré's inequalities and using Remark 4.2.3, we arrive at

$$\left|\mathcal{L}\left(t\right)\right|\leq c\int_0^\pi\left[u_t^2+\varphi_t^2+\psi_t^2+u_x^2+\varphi^2+\psi^2+\varphi_x^2+\psi_x^2\right]dx+c\left[g\circ\varphi_x+h\circ\psi_x\right]\leq cE\left(t\right).$$

Therefore,

$$\left|\mathcal{L}\left(t\right)-NE\left(t\right)\right|\leq cE\left(t\right).$$

Consequently,

$$(N-c)\,E(t) \leq \mathcal{L}(t) \leq (N+c)\,E(t)\,.$$

Now, we choose $N$ large enough such that

$$N > c, \text{ and } \frac{N}{2} - c_0 > 0.$$

Thus, there exists $\lambda, c, c_1, c_2 > 0$, such that

$$\mathcal{L}'(t) \leq -\lambda E(t) + cg \circ \varphi_x + ch \circ \psi_x \tag{4.31}$$

and

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t)\,.$$

Let $\chi(t) = \min\{\xi(t), \eta(t)\}$ ,then, multiplying (4.31) by $\chi(t)$ we obtain

$$\begin{aligned}
\chi(t)\,\mathcal{L}'(t) &\leq -\lambda\chi(t)\,E(t) + c\chi(t)\,g \circ \varphi_x + c\chi(t)\,h \circ \psi_x, \\
&\leq -\lambda\chi(t)\,E(t) + c\xi(t)\,g \circ \varphi_x + c\eta(t)\,h \circ \psi_x, \\
&\leq -\lambda\chi(t)\,E(t) - cg' \circ \varphi_x - ch' \circ \psi_x, \\
&\leq -\lambda\chi(t)\,E(t) - cE'(t)\,.
\end{aligned}$$

That is

$$\chi(t)\,\mathcal{L}'(t) + cE'(t) \leq -\lambda\chi(t)\,E(t)\,.$$

Recalling that $\chi$ is non increasing and $\mathcal{L}(t) \geq 0$, we conclude that

$$\frac{d}{dt}\left(\chi(t)\,\mathcal{L}(t) + cE(t)\right) \leq -\lambda\chi(t)\,E(t)\,,\ a.e.\ t \geq t_0.$$

Let $\mathcal{F}(t) = \chi(t)\,\mathcal{L}(t) + cE(t)$, then $\mathcal{F}(t) \sim E(t)$ and there exists a positive constant $\omega$ such that

$$\mathcal{F}'(t) \leq -\omega\chi(t)\,\mathcal{F}(t) \quad \forall t \geq t_0.$$

An integration over $(t_0, t)$ gives

$$\mathcal{F}(t) \leq \mathcal{F}(t_0)\,e^{-\omega \int_{t_0}^{t} \chi(s)\,ds},\ \forall\, t \geq t_0.$$

Using again the fact that $\mathcal{F}(t) \sim E(t)$, we deduce that

$$E(t) \leq \sigma e^{-\omega \int_{t_0}^{t} \chi(s)\,ds},\ \forall t \geq t_0,$$

for a positive constant $\sigma$, which completes the proof of Theorem 4.3.1.

# Conclusion

Of concern in this work is a linear one-dimensional double porous-elastic system.

In the one porous elastic system Magaña and Quintanilla [28] considered:

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + \beta \phi_x + \lambda u_{txx}, \\ \rho \kappa \phi_{tt} = \alpha \phi_{xx} - \beta u_x - \xi \phi - \tau \phi_t, \end{cases}$$

they established an exponential rate of decay.

In the case $\lambda = 0$ Quintanilla [35] considered:

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + \beta \phi_x, \\ \rho \kappa \phi_{tt} = \alpha \phi_{xx} - \beta u_x - \xi \phi - \tau \phi_t, \end{cases} \tag{4.32}$$

with boundary conditions

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0,$$

He used Hurtwitz theorem and proved that the damping through porous-viscosity $(-\tau \phi_t)$ is not strong enough to obtain an exponential decay but only a slow (nonexponential) decay.

Apalara [1] considered the same system (4.32) with Dirichlet–Dirichlet, Neumann–Dirichlet and Dirichlet–Neumann boundary conditions, he prove that the porous dissipation is strong enough to exponentially stabilize the system, provided the wave speeds are equal.

In this work; chapter 2, we consider a one-dimensional double porous elastic system with two dissipative mechanisms : a viscoelastic dissipation in the displacement field and visco-porous dissipations.

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x + \lambda u_{txx}, & \text{in } (0,\infty)\times(0,L), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t & \text{in } (0,\infty)\times(0,L), \\ \kappa_2\psi_{tt} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t & \text{in } (0,\infty)\times(0,L). \end{cases}$$

We prove that the solution decays exponentially. without any restriction of the wave speeds.

However, in case $\lambda = 0$ the system with double porosity structure and frictional damping in both porous equations

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x, & \text{in } (0,\infty)\times(0,\pi), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t - \tau_2\psi_t, & \text{in } (0,\infty)\times(0,\pi), \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_3\varphi_t - \tau_4\psi_t, & \text{in } (0,\infty)\times(0,\pi). \end{cases}$$

We introduce two stability numbers $\chi_0$ and $\chi_1$ and prove that the solution of the system decays exponentially provided that $\chi_0 = 0$ and $\chi_1 \neq 0$. Otherwise, if $\chi_0 \neq 0$ we prove the lack of exponential decay.

Recently, In the one-dimensional porous-elastic system with memory effects. Apalara [2] considered

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 & in\ (0,1)\times(0,+\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\,\phi_{xx}(s)\,ds = 0 & in\ (0,1)\times(0,+\infty), \end{cases}$$

with Neumann-Dirichlet boundary conditions. He studied the case of equal wave speeds $\frac{\mu}{\rho} = \frac{\delta}{J}$ and proved, in contrary to [28], that the unique dissipation given by the memory term leads to a general decay.

In this work; chapter 3, we consider a one dimensional elastic system with two porous structures and memory effects in both porous equations.

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x & \text{in } (0,\pi)\times(0,\infty), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi & \\ \qquad\qquad - \int_0^t g(t-s)\,\varphi_{xx}(s)\,ds & \text{in } (0,\pi)\times(0,\infty), \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi & \\ \qquad\qquad - \int_0^t h(t-s)\,\psi_{xx}(s)\,ds & \text{in } (0,\pi)\times(0,\infty), \end{cases}$$

with boundary conditions

$$u_x(0,t) = u_x(\pi,t) = \varphi(0,t) = \varphi(\pi,t) = \psi(0,t) = \psi(\pi,t) = 0.$$

We prove that the weak dissipation generated by the memory terms produces a general rate of decay depending on the kernels of the memory terms and the coefficients of the system.

In other word, our results extend the previous results of single porous thermoelasticity to the case of double porous systems. The porosity equations are consolidated and can be viewed as a one equation of two coordinate in such away the resultant equation is damped if and only if the two porous equations are damped.

# Perspectives

It is interest to extend our result to thermoelastic system with two porous structures and only one porous dissipation, namely;

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x + \lambda u_{txx}, \\ \kappa_1 \varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi, \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t. \end{cases}$$

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x, \\ \kappa_1 \varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi, \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t. \end{cases}$$

Or with memory term

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x + \lambda u_{txx}, \\ \kappa_1 \varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi, \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \int_0^t h\left(t-s\right)\psi_{xx}\left(s\right)ds. \end{cases}$$

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x, \\ \kappa_1 \varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi, \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \int_0^t h\left(t-s\right)\psi_{xx}\left(s\right)ds. \end{cases}$$

Also, an interesting problem to is to the effect of the temperature is double porosity problems

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x + d\psi - \beta\theta_x, \\ \kappa_1 \varphi_{tt} = \alpha\varphi_{xx} + b_1\varphi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta - \displaystyle\int_0^t g(t-s)\varphi_{xx}(x,s)ds, \\ \kappa_2 \psi_{tt} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta - \displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)ds, \\ \theta_t = \kappa\theta_{xx} - \beta u_{tx} - \gamma_1\varphi_t - \gamma_2\psi_t. \end{cases}$$

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