Poeple's democratic republic of algeria
ministry of higher education and
scientific research

ECHAHID HAMMA LAKHDAR UNIVERSITY ELOUED

FACULTY OF EXACT SCIENCES

Final Master’s Thesis

MASTER ACADEMIQUE

Domain: Mathematics and Informatics
Option: Mathematics
Specialty: fundamental and applied Mathematics

Theme

Existence results for functional differential equations with infinite delay

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Academic year : 2023-2024

شكر وتقدير

بعد رحلة بحث وجهد واجتهاد تكللنا بإنجاز هذه المذكرة القيمة، نحمد الله عز وجل على النعمة التي منَ بها علينا فهو العلي القدير، كما لا يسعنا إلا أن نخص بأسمى عبارات الشكر والتقدير للدكتور ***"قابسي حسين "*** لما قدمه لنا من جهد ونصح ومعرفة طيلة انجاز هذه المذكرة فلولا الله ثم وجوده لما أحسسنا بمتعة العمل وحلاوة البحث، ولا وصلنا إلى ما وصلنا إليه فله منا كل الشكر والتقدير.

إهداء

يا رب لك الحمد كما ينبغي لجلال وجهك وعظيم سلطانك تباركت يا رب وتعاليت، "سبحانك لا علم لنا إلا ما علمتنا إنك أنت العليم الحكيم "ونصلي ونسلم على خير نبي أرسل للعالمين سيدنا محمد عليه أزكى الصلاة وأفضل التسليم وعلى آله وصحبه الطاهرين ".إلى أعز ما أملك في الوجود إلى من منحتني الحنان، الحب والقوة بدعواتها **" أمي العزيزة والغالية "** . إلى الذي علمني حروف الحياة ومهد كل شيء جميل ومواجهة الصعوبات إلى رمز التفاؤل والقناعة **"أبي الحبيب رحمة الله عليه "**. إلى من حبهم يجري في عروقي ويلهج بذكرهم فؤادي **"إخوتي وأخواتي "**. إلى رفيقة هذا العمل والصديقة **"درداخ بسمة "**. إلى كل من تصفح هذه المذكرة وانتفع بها وتذكرنا بدعائه، إلى كل هؤلاء أهدي ثمرة جهدي .

"بن بردي حنان "

إهداء

الحمد لله على منه وامتنانه و الشكر له على نعمه و انعامه حمدا كثيرا طيبا، الذي أنعم عليَّ بنعمة العلم و سهل لي طريقا ابغي فيه علما ووفقني في إنهاء عملي المتواضع هذا، إلى من بلغ الرسالة وأدى الأمانة ونصح الأمة، إلى نبي الرحمة ونور العالمين سيدنا محمد عليه الصلاة وأزكى التسليم، إلى التي بالأماني حملتني، وبالتهاني استقبلتني، وبالحنان رعتني، إلى من لم يعرف دعاؤها حدودا ولا عطاؤها قيودا، إلى بسمة الحياة وسرٌّ الوجود، إلى ينبوع الصبر والتفاؤل والأمل **" أمي الغالية "**.إلى من جرع الكأس فارغا ليسقيني قطرة حب، إلى من كنت أتأمله ليقدم لنا لحظة سعادة، إلى من حصد الأشواك عن دربي ليمهد لي طريق العلم، إلى القلب الكبير الذي أحمل اسمه بكل فخر وأعتز به في كل مكان **"أبي العزيز "**.إلى سندي وقوتي وملاذي بعد الله، إلى من آثروني على أنفسهم، إلى من علموني علم الحياة، إلى القلوب الطاهرة **"إخوتي وأخواتي "**. إلى من جمعني بهم دم واحد وقلب واحد وبيت واحد **"كل الأهل والأقارب "**. إلى رفيقة هذا العمل ورفيقة الدرب **"بن بردي حنان "**. إلى كل من تصفح هذه المذكرة وانتفع بها وتذكرنا بدعائه، إلى كل هؤلاء أهدي ثمرة جهدي .

"درداخ بسمة "

Contents

Introduction

Many real-life phenomena in physics, engineering, biology, medicine, economics, etc, can be modeled by an initial value problem (IVP), or Cauchy problem, for ordinary differential equations (ODEs) of the type

$$\begin{cases} x'(t) = f(t, x(t)), & t \geq t_0 \\ x(t_0) = x_0, \end{cases},$$

where the function $x(t)$, called the state variable, represents some physical quantity that evolves over time. The functional differential equations are not only an extension of ordinary differential equations but also provide good models in many fields including Biology, Mechan-ics, Economics and biomathematics (see [7], [16], [27]). For example (see [16]), in population dynamics, since a growing population consumes more (or less) food than a matured one, depending on individual species, this leads to neutral functional equations. Positive periodic solutions of differential equations have been studied extensively in recent times. We refer to the references in this article and references therein for a wealth of information on this subject.

In first chapter of this memoir include some concepts, definition and theoremes related to the fixed point results about differential equation with a delay. For more details, we refer to the following references(see [24], [4], [11]).

The objective in the second chapter is to investigate the existence of positive ω-periodic solutions for the first order delay differential equations of the form

$$\dot{x}(t) + p(t)x(t) + q(t)x(\tau(t)) = 0, \ t \geq t_0.$$

With respect to this equation throughout we will assume the following conditions:

i) $p, q \in C([t_0, \infty), \mathbb{R})$.

ii) $\tau \in C([t_0, \infty), [0, \infty)), \tau(t) < t$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

We will obtain the existence criteria for the positive ω-periodic solutions of Eq(II.0.1). For equations of the type above, the existence results in the literature are largely based on the assumption that the functions $p(t)$, $q(t)$ are ω-periodic and satisfy the conditions

$$\int_0^\omega p(t)dt > 0, \ \int_0^\omega q(t)dt > 0.$$

In the last chapter, we obtain various sufficient conditions for the existence of positive periodic solutions for problems the following two types of differential equations with several delays of the logistic type

$$\frac{dx}{dt} = \pm x(t)\left[G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t))) - \varphi(t)x'(t-r(t))\right], \quad (.0.1)$$

$$\frac{du}{dt} = a(t)u(t) - \lambda b(t)f(t, u(t-\delta(t))) - c(t)x(t-\sigma(t)), \quad (.0.2)$$

by employing two available operators and by applying the coincidence degree theorem and the fixed point theorem. where $G(t, x_1, ..., x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R})$, $G(t + \omega, x_1, ..., x_n) = G(t, x_1, ..., x_n)$ and $f : \mathbb{R}\times \mathbb{R}^+ \to \mathbb{R}, \in C(\mathbb{R}, \mathbb{R}^+)$ $\varphi, a, b, c \in C(\mathbb{R}, \mathbb{R}^+)$ r, σ, δ, $\tau_i \in C(\mathbb{R}, \mathbb{R})$ for $i = 1, ..., n$. All of the above functions are continuous, ω-periodic functions and $\omega > 0$ is a constant. Here $\lambda > 0$ is a parameter. In [16], Yongkun Li studied existence and global attractivity of a positive periodic solution of a class of delay differential equations having the form

$$x'(t) = x(t)F(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t))).$$

The above equation was extensively investigated in literature as biomathematics models. It contains many biomathematics models of delay differential equations, such as the single species growth logistic equation with delay

$$x'(t) = r(t)x(t)\left[1 - \frac{x(t-\tau(t))}{R(t)}\right]. \quad (.0.3)$$

It was proved in reference [27] that if $\tau(t) = m\omega$, then (III.0.1) has a positive ω-periodic solution, where m is a nonnegative integer. Also, the next so called "food—limited" population growth model (see [27]) was studied in reference [7], and it was shown that if $\tau(t) = m\omega$ holds, then the equation

$$x'(t) = r(t)x(t)\left[\frac{R(t) - x(t-\tau(t))}{R(t) + r(t)c(t)x(t-\tau(t))}\right], \quad (.0.4)$$

has a positive ω-periodic solution. In references [13], [27], the following single-species population model exhibiting the so-called Allee effect was considered

$$x'(t) = x(t)\left[a(t) + b(t)x(t-\tau(t)) - c(t)x^2(t-\tau(t))\right]. \quad (.0.5)$$

It was proved that if $\tau(t) = m\omega$ holds, then (III.1.1) has a positive ω-periodic solution (see [16]). Special cases of (III.1.3)-(III.1.4) below have been considered and investigated by many other authors. For example, very recently, in [9] Hai-Feng Huo and Wan-Tong Li discussed the existence of a positive periodic solutions of the delay system

$$\begin{cases} \frac{dx}{dt} = \pm x(t)G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t)), u(t-\delta(t))) \\ \frac{du}{dt} = -a(t)u(t) + b(t)x(t-\sigma(t)), \end{cases}, \quad (.0.6)$$

where all of the above functions are ω-periodic functions where $\omega > 0$ is a constant and G satisfies some conditions.

Chapter I

Preliminaires

In this section we discus compact sets. The following elements have been gathered from several analysis and some specialized books on the functional analysis and the most is in the following bibliography [24], [1], [4].

I.1 Norm and metric

Definition I.1.1 *Let X be a nonempty set and $\rho : X \times X \to [0, \infty)$ a function. Then ρ is called a metric on X if the following properties hold*

i) $\rho(x, y) = 0$ if and only if $x = y$, for all x, $y \in X$.

ii) $\rho(x, y) = \rho(y, x)$ for all x, $y \in X$.

iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all x, y, $z \in X$.

The value of metric $\rho(x, y)$ is called distance between x and y, and the ordered pair (X, ρ) is called metric space.

Definition I.1.2 *Let $(X, +, .)$ be a linear space over a field ($\mathbb{R}$ or $\mathbb{C}$) and $N : X \to [0, +\infty)$ a function. Then, N is said to be a norm if the following properties hold*

i) $N(x) = 0$ if and only if $x = 0$.

ii) $N(\lambda x) = |\lambda|\, N(x)$ for all $x \in X$ and $\lambda \in \mathbb{k}$.

iii) $N(x + y) \leq N(x) + N(y)$ for all x, $y \in X$.

The ordered pair (X, N) is called a normed space. We use the notation $\|.\|$ for norm. Then every normed space $(X, \|.\|)$ is a vector space and it is a metric space (X, ρ) with induced metric $\rho(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Example I.1.1 *The real line $\mathbb{R}$ with $(x, y) = |x - y|$ is a metric space. The metric is called the usual metric for $\mathbb{R}$. Let X be a linear space over field ($\mathbb{R}$ or $\mathbb{C}$) and $N : X \to \mathbb{R}^+$a function. Then, N is said to be a norm if the following properties hold*

(N1) $N(x) = 0$ if and only if $x = 0$.

(N2) $N'(x) = |\lambda|\, N(x)$ for all $x \in X$ and $\lambda \in \mathbb{k}$.

(N3) $N(x + y) \leq N(x) + N(y)$ for all x, $y \in X$.

Example I.1.2 *The ordered pair* (X, N) *is called a normed space. We use the notation* $\|.\|$ *for norm. Then every normed space* $(X, \|.\|)$ *is a metric space* (X, ρ) *with induced metric* $\rho(x, y) = \|x - y\|$ *.*

Example I.1.3 *Let* $X = \mathbb{R}^n$*,* $n > 1$ *be a linear space. Then* $\mathbb{R}^n$ *is a normed space with the following norms :*

1) $\|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$ *for all* $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

2) $\|x\|_p = (\sum\limits_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ *for all* $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ *and* $p \in (1, \infty)$.

3) $\|x\|_\infty = \max\limits_{1 \le i \le n} |x_i|$ *for all* $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Definition I.1.3 *Let* X *be a nonempty set and is a metric on* X*, a sequence* $\{x_n\} \subset X$ *is a Cauchy sequence if for each* $\epsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $\rho(x_n, x_m) < \epsilon$ *for all* n*,* $m > n_0$*, i.e* $\lim\limits_{n,m \to +\infty} \rho(x_n, x_m) = 0$.

Definition I.1.4 *The metric space* (X, ρ) *is complete if every Cauchy sequence in* (X, ρ) *has a limit in that space.*

Definition I.1.5 *A Banach space is a complete normed space. We often say a complete normed vector space.*

Theorem I.1.1 *A closed subspace of a Banach space is a Banach space.*

Example I.1.4 *The linear space* $C([a, b])$ *of continuous functions on the closed and bounded interval* $[a, b]$ *is a Banach space with the uniform convergence norm* $\|f\|_\infty = \sup\limits_{t \in [a,b]} |f(t)|$*. Let* (X, ρ) *be a metric space. Recall that a subset* Ω *of* X *is called compact if every open cover of* Ω *has a finite subcover. Equivalently, a subset* Ω *of* X *is compact if every sequence in* Ω *contains a convergent subsequence with a limit in* Ω*.*

Example I.1.5 *Let* $\varphi : [a, b] \to \mathbb{R}^n$ *be continuous and let* X *the set of continuous functions* $f : [a, c] \to \mathbb{R}^n$ *with* $c > b$ *and* $f(t) = \varphi(t)$ *for* $a \le t \le b$*. Define* $\rho(f, g) = \sup\limits_{a \le t \le c} |f(t) - g(t)|$ *for* f*,* $g \in X$*. Then* (X, ρ) *is complete metric space but not a Banach space because* $f + g$ *is not in* X*.*

Definition I.1.6 *Let* (X, ρ) *be a metric space. A subset* Ω *of* X *is said to be totally bounded if for each* $\epsilon > 0$*, there exists a finite number of elements* x_1*,* x_2*, ...,* x_n *in* X *such that* $\Omega \subseteq \cup_{i=1}^{n} B_\epsilon(x_i)$*.*

Proposition I.1.1 *Let* (X, ρ) *be a metric space. Then the following are equivalent*

i) X *is compact.*

ii) Every sequence in X *has aconvergent subsequence.*

iii) X *is complete and totally bounded.*

Definition I.1.7 *A set* Ω *in a metric space* (X, ρ) *is relatively compact if its closure is compact, i.e,* $\bar{\Omega}$ *is compact.*

Proposition I.1.2 *Let Ω be a closed subset of a complete metric space (X, d). Then Ω is compact if and only if it is relatively compact.*

Definition I.1.8 *Let X be a compact metric space Ω and be a subset of $C(X)$.*

1) Ω is uniformly bounded if there exists $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$.

2) Ω is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that t_1, $t_2 \in X$ and $\rho(t_1, t_2) < \delta$ imply $|u(t_1) - u(t_2)| < \epsilon$ for all $u \in \Omega$.

The following result gives the main method of proving compactness in the spaces in which we are interested.

Definition I.1.9 *Let (X, ρ) be a complete metric space and $S : X \to X$ be a contraction operator if there exists $\alpha \in (0, 1)$tel que $x, y \in X$ implique*

$\rho(Sx, Sy) \leq \alpha d(x, y), \quad \alpha \in (0, 1)$

Theorem I.1.2 *Let X be a compact metric space. If Ω is an equicontinuous, uniformly bounded subset of $C(X)$, then is relatively compact.*

Definition I.1.10 *Let S be a mapping from a metric space (X, ρ) into another metric space (Y, ρ'). Then S is said to satisfy Lipschitz condition on X if there existe a constant $L > 0$ such that $\rho'(Sx, Sy) \leq L\rho(x, y)$ for all x, $y \in X$.*

The following proposition guarantees the existence of Lipschitzian mappings.

Proposition I.1.3 *Let $S : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a, b). Suppose S is continuous on $[a, b]$. Then, S is a Lipschitz continuous function (and hence is uniformly continuous).*

Definition I.1.11 *Let X and Y be two Banach spaces and let the mapping $S : X \to X$. Then*

1) S is said to be bounded if Ω is bounded in X implies $S(\Omega)$ is bounded.

2) S is said to be closed if $x_n \to x$ in X and $Sx_n \to y$ in Y imply $Sx = y$.

3) S is said to be compact if Ω is bounded in X implies $S(\Omega)$ is relatively compact ($S(\Omega)$ is compact), i.e for every bounded sequence $\{x_n\}$ in X, $\{Sx_n\}$ has convergent subsequence in Y.

4) S is said to be completely continuous if it is continuous and compact. In the case of linear mappings, the concepts of continuity and boundedness are equivalent, but it is not true in general

I.2 Fixed point theorems and coincidence degree theorem

In this section we state some fixed point theorems that we employ to help us in proving existence and stability of solutions(see [21], [14], [3]).

Definition I.2.1 *Let S be a mapping in the set Ω. we call fixed point of S any point x satisfying $S(x) = x$. If there exists such x, we say that S has a fixed point, which is equivalent to saying that the equation $S(x) - x = 0$ has a null solution. Fixed point theorems guarantee the existence of a fixed point under appropriate conditions on the map S and the set Ω.*

I.2.1 Banach fixed point theorem

A fixed point of S is a solution of the problem $Sx^* = x^*$. The idea had two outstanding features. First, it had application to problems in every area of mathematics which used complete metric spaces. And it was clean. For example, the standard muddy and shaky proofs of implicit function theorems became clear and solid using the fixed-point theory. We will use it here to prove existence of solutions of various kinds of differential equations.

Theorem I.2.1 *Let (X, ρ) be a complete metric space and $S : X \to X$ a contraction operator. Then there is a unique point $x \in X$ with $Sx = x$.*

I.2.2 Schauder's theorem

Definition I.2.2 *Let X be a normed vector space and $F = \{x_1, x_2, ..., x_n\}$ a finite subset of X. Then $conv(F)$, the convex hull of F, is defined by*

$$Conv(F) = \left\{ \sum_{j=1}^{n} t_j x_j = 1 : \sum_{j=1}^{n} t_j = 1, \quad t_j \geq 0 \right\}.$$

For future applications, we will need a more general definition to handle the case in which F is infinite.

Theorem I.2.2 *(Schauder's fixed point) Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x \in \Omega$ such that $Sx = x$.*

I.2.3 Krasnoselskii's fixed point theorem

In 1955 Krasnoselskii's observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, $(S\varphi := A\varphi + B\varphi)$ a contraction and a compact application. It declares then Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator. Krasnoselskii's found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result .

Theorem I.2.3 *(Krasnoselskii's fixed point) Let Ω be a closed bounded convex nonempty subset of a Banach space $(X, \|.\|)$. Suppose that A and B map into X such that*

(i) A is a contraction mapping.

(ii) B is compact and continuous.

(iii) x, $y \in \Omega$, implies $Ax + By \in \Omega$, Then there exists $z \in \Omega$ with $z = Az + Bz$.

Remark I.2.1 *Note that if $B = 0$, the theorem becomes the theorem of Banach. If $A = 0$, then the theorem is not other than the theorem of Schauder. We first state the well-known fixed point theorem in cones [17], [2]. For the proof, we refer to the classical works [23], [25].*

Lemma I.2.1 *(Deimling [12], Guo and Lakshmikantham [2] and Krasnoselskii [17]) Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$ Assume that $T : \bar{K_r} \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$*
(i) $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.
(ii) $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Lemma I.2.2 *Let X be a Banach space, and K be a cone of X. Assume Ω_1, Ω_2, are open subsets of X with $0 \in \Omega_1$, $\bar{\Omega_1} \subseteq \Omega_2$ and let $A : K \cap (\Omega_2 - \Omega_1) \to K$ be a completely continuous operator such that*
(i)$\|Ax\| \leq \|x\|$ for every $x \in K \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|$ for every $x \in K \cap \partial\Omega_2$ or
(ii) $\|Ax\| \geq \|x\|$ for every $x \in K \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|$ for every $x \in K \cap \partial\Omega_2$.
Then A has at least one fixed point in $K \cap (\bar{\Omega_2} - \Omega_1)$.

I.3 Coincidence degree theorem

We state the coincidence degree theorem which enables us to prove the problem of existence periodic solutions. For details on Mawhin techniques, we refer the reader to (Gaines and Mawhin [6], p. 40)

Definition I.3.1 *A linear mapping $L : X \cap DomL \to Z$ with $\ker L = L^{-1}(0)$ and $\operatorname{Im} L = L(DomL)$, will be called a Fredholm mapping if the following two conditions hold:*
(i) $\ker L$ has a finite dimension.
(ii) $\operatorname{Im} L$ is closed and has a finite codimension.

Recall also that the codimension of $\operatorname{Im} L$ is the dimension of $Z/\operatorname{Im} L$, i.e the dimension of the cokernel $co \ker L$ of L. When L is a Fredholm mapping, its index is the integer

$$Ind(L) = \dim\ker L - co\dim \operatorname{Im} L$$

Let X and Z be two Banach spaces. Consider the operator equation

$$Lx = \eta Nx,\ \ \eta \in (0, 1),$$

where $L : X \cap DomL \to Z$ is a linear operator andη is a parameter. Let P and Q denote two projectors such that

$$P : X \cap DomL \to \ker L \ \text{ and } \ Q : Z \to Z/\operatorname{Im} L$$

Definition I.3.2 *We say that a mapping N is L-compact on Ω if the mapping $QN : \bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded, and $K_p(I - Q)N : \bar{\Omega} \to X$ is compact, i.e, it is continuous and $K_p(I - Q)N(\bar{\Omega})$ is relatively compact, where*

$$K_p : \operatorname{Im} L \to DomL \cap \ker P$$

is a inverse of the restriction L_p of L to $DomL \cap \ker P$, so that

$$LK_p = I \ \ \text{and } K_pL = I - P.$$

Lemma I.3.1 *Let X and Z be two Banach spaces and L a Fredholm mapping of index zero. Assume that $N : \bar{\Omega} \to Z$ is L-compact on $\bar{\Omega}$ with open bounded in X. In addition, we assume that*

(a) For each $\eta \in (0, 1)$, $x \in \partial\Omega \cap DomL$

$$Lx \neq \eta Nx.$$

(b) For each $x \in \partial\Omega \cap \ker L$

$$QNx \neq 0.$$

and

$$\deg\{QNx, \partial\Omega \cap \ker L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap DomL$.

I.4 Delay differential equations

Delayed problems are innumerable in literature. Some of these problems have been of particular interest. We have chosen in this thesis some models intervening in dilerent real world domains (see [19], [26]).

I.4.1 Definition of delay differential equations

For $\tau \geq 0$ is a given real number $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C([a, b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping in the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence.

If $[a, b] = [-\tau, 0]$ we let $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element ψ in C by $\|\psi\| = \sup\limits_{-\tau<s<0} \|\psi(s)\|$. Even though single bars are used for norms in different spaces, no confusion should arise. If $t_0 \in \mathbb{R}$, $A \geq 0$ and $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$.

Definition I.4.1 *If Ω is a subset of $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation*

$$x'(t) = f(t, x_t), \tag{I.4.1}$$

where

$$x_t(s) = x(t + s), -\tau \leq s \leq 0.$$

is a delay differential equation on Ω and will denote this equation by DDE.

Definition I.4.2 *A function x is said to be a solution of Equation (I.4.1) on $[t_0 - \tau, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C([t_0 - \tau, t + A], \mathbb{R}^n)$, $(t, x_t) \in \Omega$ and $x(t)$ satisfies Equation (I.4.1) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}$, $\psi \in C$.*

Definition I.4.3 *we say $x(t, t_0, \psi)$ is a solution of Equation (I.4.1) with initial value ψ at t_0 or simply a solution through (t_0, ψ) if there is an $A > 0$ such that $x(t, t_0, \psi)$ is a solution of Equation (I.4.1) on $[t_0 - \tau, t_0 + A)$ and $xt_0(t, t_0, \psi) = \psi$.*

Remark I.4.1 *If $\tau = 0$, Equation(I.4.1) is a very general type of equation and includes ordinary differential equations. If $f(t, \psi) = L(t, \psi) + h(t)$ where $L(t, \psi)$ is linear in ψ.*

Remark I.4.2 *we say Equation (I.4.1) is linear and is homogeneous if $h \equiv 0$ and nonhomogeneous if $h \neq 0$. If $f(t, \psi) = g(\psi)$ where g does not depend on t, we claim Equation (I.4.1)is autonomous.*

Example I.4.1 *The following equations are delay differential equations*

$$x'(t) = 3x(t) + 2x(t - 5), \tag{I.4.2}$$

$$x'(t) = a(t)x(t) + b(t)x'(t - \tau(t)) + h(t), \tag{I.4.3}$$

$$x'(t) = \int_{-\tau}^{0} x(t + s)ds, \tag{I.4.4}$$

where a, b, τ are continuous functions. Equation (I.4.2) is an linear autonomous delay differential equation with constant $\tau = 5$. Equation (I.4.3) is nonhomogeneous, linear nonautonomous delay functional differential equations and Equation (I.4.4) is a delay linear integro-differential equation. If $t_0 \in \mathbb{R}$, $\psi \in C$ are given and $f(t, \psi)$ is continuous, then finding a solution of Equation (I.4.1) through (t_0, ψ) is equivalent to solving the integral equation

$$x(t_0) = \psi,$$

$$x(t) = \psi(0) + \int_{t_0}^{t} f(s, x_s)ds, \ t \geq t_0.$$

We define Sx by

$$(Sx)(t) = \psi(0) + \int_{t_0}^{t} f(s, x_s)ds, t \geq t_0, \ x = \psi \text{ on } [t_0 - \tau, t_0].$$

To prove the existence of the solution through a point $(t_0, \psi) \in \mathbb{R} \times C$, we consider an $\eta > 0$ and all functions x on $[t_0 - \tau, t_0 + A]$ which are continuous and coincide with ψ on $[t_0 - \tau, t_0]$, that is, $xt_0 = \psi$. The values of these functions on $[t_0, t_0 + \eta]$ are restricted to the class of x such that $|x(t) - \psi(0)| < \delta$ or $t \in [t_0, t_0 + \eta]$. The usual mapping S obtained from the corresponding integral equation is defined and it is then shown that and can be so chosen that S maps this class into itself and is completely continuous. Thus, Schauder's fixed-point theorem implies existence.(for exemple details see the books [11], [26].

I.5 Existence and Uniqueness of the solutions

Theorem I.5.1 *suppose Ω is an open subset in $\mathbb{R} \times C$ and f is continuous on Ω. If $(t_0, \psi) \in \Omega$, then there is a solution of (I.4.1) passing through (t_0, ψ). We say $f(t, \psi)$ is Lipschitz in ψ in a compact set X of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \psi_i) \in K$, $i = 1, 2$, $|f(t, \psi_1) - f(t, \psi_2)| \leq k\,|\psi_1 - \psi_2|$*

Theorem I.5.2 *Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \psi)$ is Lipschitz in ψ in each compact set in Ω. If $(t_0, \psi) \in \Omega$, then there is a unique solution of (I.4.1) through (t_0, ψ).*

I.6 Neutral functional differential equations

In order to define a general class of neutral delay differential equations NDDEs (or neutral functional differential equations NFDEs), we need the definition of atomic.

Definition I.6.1 *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, ψ). A function $\Psi : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if Ψ is continuous together with its first and second Fréchet derivatives with respect to ψ and $\Psi\psi$, the derivative with respect to ψ, is atomic at β on Ω.*

Definition I.6.2 *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $\Psi : \Omega \to \mathbb{R}^n$ are given continuous functions with Ψ atomic at zero.The equation*

$$\frac{d}{dt}\Psi(t, x_t) = f(t, x_t), \tag{I.6.1}$$

is called the neutral delay differential equation NDDE (Ψ, f).

Definition I.6.3 *A function x is said to be a solution of the NDDE (Ψ, f) or equation (I.6.1), if there are $t_0 \in \mathbb{R}$, $A > 0$, such that $x \in C([t_0 - \tau, t_0 + A), \mathbb{R}^n)$, $(t, x_t) \in \Omega$, $t \in [t_0, t_0 + A)$, $\Psi(t, x_t)$ is continuously differentiable and satisfies (I.6.1) on $[t_0, t_0 + A)$. For a given $t_0 \in \mathbb{R}$, $\psi \in C$, and $(t_0, \psi) \in \Omega$, we say $x(t_0, \psi)$ is a solution of (I.6.1) with initial value ψ at t_0, or simply a solution through (t_0, ψ), if there is an $A > 0$ such that $x(t_0, \psi)$, is a solution of (I.6.1) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \psi) = \psi$.*

Theorem I.6.1 *If Ω is an open set in $\mathbb{R} \times C$ and $(t_0, \psi) \in \Omega$, then there exists a solution of the NDDE (Ψ, f) through (t_0, ψ).*

Theorem I.6.2 *If $\Omega \subseteq \mathbb{R} \times C$ is open and $f : \Omega \to \mathbb{R}^n$ as Lipschitz in ψ on compact sets of Ω, then, for any $(t_0, \psi) \in \Omega$, there exists a unique solution of the NDDE (Ψ, f) through (t_0, ψ).*

Example I.6.1 *The equations*

$$x'(t) = 3x'(t - 5),$$
$$x'(t) = x(t) + [x'(t - 4) + 1]^2,$$
$$x'(t) = x(t) + x'(t - 2) - x'(t - 7).$$

are neutral differential equations.

I.7 Stability of delay differential equations

The theory of stability was created by the Russian mathematician Lyapunov (1857-1918). This theory has found wide application in various fields of physics and mathematical sciences. From a mathematical point of view, the theory of stability presents a particular case of the qualitative theory of differential equations. Lyapunov's method was the ultimate object for studying stability for differential equations and partial differential equations. Nevertheless, this method has encountered serious obstacles and there are still a lot of problems that resist this technique. The simplest notion of stability is the one related to stability of equilibrium points.

Definition I.7.1 *A point $x(t) = x_e$ in the state space is said to be an equilibrium point of the autonomous system $x' = f(x)$ if and only if it has the property that when ever the state of the system starts at x_e, it remains at xe for all future time.Accordin $\delta_1 g$ to the definition, the equilibrium points are the real roots of the equation $f(x_e) = 0$. This is made clear by noting that if $x_e^{'} = f(x_e) = 0$, then it follows that x_e is constant and, by definition, an equilibrium point. Without loss of generality, we assume that 0 is an equilibrium point of the system. If the equilibrium point under study, x_e, is not at zero we may define a new (shifted) coordinate system $x_s(t) = x(t) - x_e$ and note that*

$$x_s^{'}(t) = x'(t) = f(x(t)) = f(x_s(t) + x_e) =: f_s(x_s(t)), x_s(0) = x(0) - x_e.$$

The claim follows by noting that $f_s(0) = f(x_e) = 0$.

In summary, the study of the zeroequilibrium point of $x_s'(t) = f_s(x_s(t))$ is equivalent to the study of the nonzero equilibrium point x_e of $x'(t) = f(x(t))$. We consider the system

$$x' = f(t, x\,(t)) \text{ for all } t \geq t_0, \tag{I.7.1}$$

$$x(t) = \psi(t) \text{ for all } t_0 - \tau \leq t \leq t_0. \tag{I.7.2}$$

where $f : (-\infty, +\infty) \times C \to \mathbb{R}^n$, with $C = C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions $\psi : [t_0 - \tau, t_0] \to \mathbb{R}^n$, $\tau > 0$ equipped with the supremum norm $\|\psi\| = \sup\limits_{-\tau \leq t \leq 0} |\psi(t)|$. We suppose that f is continuous and is supposed to satisfy all the conditions which guarantee a solution $x(t, t_0, \psi)$ through of the problem (I.7.1)-(I.7.2) and to be continuous in (t, t_0, ψ) of the definition domain of f [8].

Definition I.7.2 *Suppose that $f(t, 0) = 0$ for all $t \in \mathbb{R}$.*

1) The trivial solution $x(t) = 0$ of (I.7.1) is Stable in $t_0 (t_0 \in \mathbb{R})$, if for every $\epsilon > 0$ there exists a$\delta = \delta(\epsilon, t_0) > 0$ such that if $\|\psi\| < \delta$, the solution of (I.7.1)-(I.7.2) exists on $[t_0 - \tau, \infty)$ and $|x(t; t_0; \psi)| < \epsilon$ for all $t \geq t_0$. Otherwise we will say that the solution is unstable in t_0.

2) The solution $x = 0$ of (I.7.1) is said to be uniformly stable if the number $\delta = \delta(\epsilon)$ is independent of t_0.

3) The trivial solution $x(t) = 0$ of (I.7.1) is Asymptotically stable in t_0 if it is stable in t_0 and if there exists $\delta_1 = \delta_1(t_0) > 0$ such that whenever all $\|\psi\| < \delta_1$, the solution of the problem (I.7.1)-(I.7.2) satisfies

$$|x(t, t_1, \psi)| \to 0 \text{ as } t \to \infty$$

Definition I.7.3 *The trivial solution $x(t) = 0$ of (I.7.1) is Asymptotically uniformly stable in t_0 if it is uniformly stable and if there exist $\delta_1 > 0$ (independent of t_0) as for all t_0 and $\|\psi\| < \delta_1$ the solution x of the problem (I.7.1)-(I.7.2) satisfies the condition $x(t, t_0, \psi) \to 0$ when $t \to \infty$, in the following way: for all $\eta > 0$ there is a $T = T(\eta) > 0$ such that $|x(t, t_0, \psi)| < \eta$ for all $t \geq t_0 + T$.*

Remark I.7.1 *If all the solutions tend to zero, then $x = 0$ is globally asymptotically stable.*

Example I.7.1 *Consider, for $t \geq 1$, the differential delay equation*

$$x' = \frac{1}{t}x(t) - \frac{27t}{(t+2)^3}x^3(\frac{t+2}{3}) = \frac{1}{t}x(t) - \frac{27t}{(t+2)^3}x^3(t-\tau(t)),$$

with $\tau(t) = \frac{3}{2}(t-1)$, with the initial condition $x(1) = x_0$. We easily check that the unique solution of this problem is $x(t) = x_0 t\ \exp(-x_0^2(t-1))$, for all $t \geq 1$. Thus, $\lim\limits_{t\to\infty} x(t) = 0$, for all x_0. Suppose that $|x_0| = \delta$, then

$$\left| x(1 + \frac{1}{\delta^2}) \right| = \frac{1}{e}(\delta + \frac{1}{\delta}) \geq \frac{2}{e}.$$

Therefore, for all $\delta$the solution is outside the bali $|x| < \frac{2}{e}$, in times $t = 1 + \frac{1}{\delta^2}$, and the solution is unstable.

I.8 Stability by method of the fixed point

When one wants to study the stability of the trivial solution of a differential equation with delay by the method of fixed point one will have to proceed as follows

(1) A delay differential equation requires primarily a an initial function defined on an appropriate initial interval I_{t_0} i.e $\psi : I_{t_0} \to \mathbb{R}^n$. We must fall immediately after a suitable space C of functions $\varphi : I_{t_0} \cup [t_0, +\infty) \to \mathbb{R}^n$ which coincide on I_{t_0} with ψ .According to the case of needs we can always add other restrictions to the functions φ of C such as the boundary or the condition $\varphi(t) \to \infty$ when $t \to \infty$. This last condition is necessary if we wish to study asymptotic stability.

(2) Then we have to invert the differential equation to define what we call a fixed point application i.e, a mapping $S : C \to C$ whose fixed point is the solution of the given delay equation (the original equation). Nevertheless, this inversion can be a delicate task in many cases. For example if the equation does not have a linear term in its structure we will not be able to use the variation of the parameters. It is therefore essential to act differently and to try if a transformation of this equation is possible.

(3) A fixed point theorem must be chosen allowing the equation $S(x) = x$ to have a solution. Specially if S is a contraction we can apply the Banach fixed point theorem, if S is compact then we will apply the theorem of Schauder or Schaeffer and if S is puts in the form of a sum of a contraction and a compact application then the Krasnoselski ihybrid theorem can give satisfaction. It thus becomes clear that the stability method by the fixed point method relies on three essential things,

the variation of the parameters,a complete space and a fixed point theorem.n one stage we can conclude the existence (or even uniqueness) and stability.n addition, it will be seen that this method always requires conditions on average however the conditions of the Lyapunov method are always punctual.

Chapter II

Positive periodic solutions of delay differential equations

In this work we investigate the existence of positive ω-periodic solutions for the first-order delay differential equations of the form

$$\dot{x}(t) + p(t)x(t) + q(t)x((t)) = 0, \ t \geq t_0. \tag{II.0.1}$$

With respect to Eq (II.0.1) throughout we will assume the following conditions:

i) $p, q \in C([t_0, \infty), \mathbb{R})$,

ii) $\tau \in C([t_0, \infty); [0, \infty)), \tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

we will obtain the existence criteria for the positive ω-periodic solutions of Eq (II.0.1). For equations of the type above, the existence results in the literature are largely based on the assumption that the functions $p(t)$, $q(t)$ are ω-periodic and satisfy the conditions

$$\int_0^\omega p(t)dt > 0 \text{ and } \int_0^\omega q(t)dt > 0. \tag{II.0.2}$$

It would be interesting to know whether there is a positive periodic solution of Eq (II.0.1) when the conditions above are not satisfied. In the second section we will give several examples of equations to which none of the results in [22], [10] can be applied. Schauder's Fixed Point Theorem will be used to prove the main results in the next section.

II.1 Existence of positive periodic solutions

In this section we will study the existence of positive ω-periodic solutions of Eq (II.0.1). In the next lemma and theorem we choose T sufficiently large that $\tau(t) \geq t_0$ for $t \geq T$.

Lemma II.1.1 *Suppose that there exists a function $k \in C([T, \infty), (0, \infty))$such that*

$$\int_t^{t+\omega} [p(s) + q(s)k(s)]ds = 0, \ t \geq T \tag{II.1.1}$$

Then the function

$$f(t) = \exp(-\int_T^t [p(s) + q(s)k(s)]ds), \ t \geq T$$

is ω-periodic.

Proof: For $t \geq T$ we obtain

$$\begin{aligned} f(t+\omega) &= \exp(-\int_T^{t+\omega}[p(s)+q(s)k(s)]ds) \\ &= \exp(-\int_T^{t}[p(s)+q(s)k(s)]ds)\exp(-\int_t^{t+\omega}[p(s)+q(s)k(s)]ds) \\ &= f(t). \end{aligned}$$

Thus the function $f(t)$ is ω-periodic. ■

Theorem II.1.1 *Suppose that there exists a function $k \in C([T,\infty),\ (0,\infty))$ such that (II.0.2) holds and*

$$\int_{\tau(t)}^{t}[p(s)+q(s)k(s)]ds = \ln k(t), \tau(t) \geq T. \tag{II.1.2}$$

Then Eq. (II.0.1) has a positive ω-periodic solution.

Proof: Let $X = \{x \in C([t_0,\infty),\mathbb{R})\}$ be the Banach space with the norm

$$\|x(t)\| = \sup|x(t)|$$

With regard to Lemma I.3.1 we define

$$\begin{aligned} M &= \max_{t\in[T,\infty)}\{\exp(-\int_T^t[p(s)+q(s)k(s)]ds)\} \\ m &= \min_{t\in[T,\infty)}\{\exp(-\int_T^t[p(s)+q(s)k(s)]ds)\} \end{aligned}$$

We now define a closed, bounded and convex subset of X as follows:

$$\Omega = \{x \in X : x(t+\omega) = x(t),\, m \leq x(t) \leq M,\ k(t)x(t) = x(\tau(t)),\, t \geq T\ /\ x(t) = 1, t_0 \leq t \leq T\}$$

Define the operator $S : \Omega \to X$ as follows:

$$(Sx)(t) = \begin{cases} \exp(-\int_T^t[p(s)+q(s)\frac{x(\tau(s))}{x(s)}]ds) \\ 1, t_0 \leq t \leq T, \end{cases}$$

We will show that for any $x \in \Omega$ we have $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq T$we get

$$\begin{aligned} (Sx)(t) &= \exp(-\int_T^t[p(s)+q(s)\frac{x(\tau(s))}{x(s)}]ds) \\ &= \exp(-\int_T^t[p(s)+q(s)k(s)]ds) \leq M. \end{aligned}$$

Furthermore for $t \geq T$ and $x \in \Omega$ we obtain

$$(Sx)(t) = \exp(-\int_T^t[p(s)+q(s)k(s)]ds) \geq m.$$

For $t \in [t_0, T]$ we have$(Sx)(t) = 1$, that is $(Sx)(t) \in \Omega$. Further for every $x \in \Omega$ and $\tau(t) \geq T$ we get

$$\begin{aligned}(Sx)(\tau(t)) &= \exp(-\int_{\tau(t)}^{t}[p(s)+q(s)\frac{x(\tau(s))}{x(s)}]ds)\\ &= (Sx)(t)\exp(\int_{\tau(t)}^{t}[p(s)+q(s)\frac{x(\tau(s))}{x(s)}]ds).\end{aligned}$$

With regard to (III.1.1) and (III.1.3) for $\tau(t) \geq T$ it follows that

$$(Sx)(\tau(t)) = (Sx)(t)\exp(\int_{\tau(t)}^{t}[p(s)+q(s)k(s)]ds) = k(t)(Sx)(t).$$

Finally we will show that for $x \in \Omega$, $t \geq T$ the function $(Sx)(t)$ is ω-periodic. For $x \in \Omega$, $t \geq T$ and according to (II.0.2) we have

$$\begin{aligned}(Sx)(t+\omega) &= \exp(-\int_{T}^{t+\omega}[p(s)+q(s)k(s)]ds)\\ &= \exp(-\int_{T}^{t}[p(s)+q(s)k(s)]ds)\exp[(-\int_{t}^{t+\omega}p(s)+q(s)k(s)]ds) = (Sx)(t).\end{aligned}$$

So $(Sx)(t)$ is ω-periodic on $[T, \infty)$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$. We now show that S is completely continuous. First we will show that S is continuous. Let $x_i = x_i(t) \in \Omega$ be such that $x_i(t) \to x(t) \in \Omega$ as $i \to \infty$. For$t \geq T$ we have

$$|(Sx_i)(t)-(Sx)(t)| = \left|\exp(-\int_{T}^{t}[p(s)+q(s)\frac{x_i(\tau(s))}{x_i(s)}]ds) - \exp(-\int_{T}^{t}p(s)+q(s)\frac{x(\tau(s))}{x(s)}]ds)\right|$$

By applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{i\to\infty}\|(Sx_i)(t)-(Sx)(t)\| = 0.$$

For $t \in [t_0, T]$ the relation above is also valid. This means that S is continuous. We now show that $S\Omega$ is relatively compact. It is sufficient to show by the Arzela–Ascoli theorem that the family of functions$\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, 1)$. The uniform boundedness follows from the definition of Ω. With regard to (III.1.5) for $t \geq T$, $x \in \Omega$ we get

$$\begin{aligned}\left|\frac{d}{dt}(Sx)(t)\right| &= \left|p(t)+q(t)\frac{x(\tau(t))}{x(t)}\right|\exp(-\int_{T}^{t}[p(s)+q(s)\frac{x(\tau(t))}{x(t)}]ds)\\ &= \mid p(t)+q(t)k(t)\mid \exp(-\int_{T}^{t}[p(s)+q(s)k(s)]ds) \leq M1,\ M1>0.\end{aligned}$$

For $t \in [t_0, T]$, $x \in \Omega$ we have

$$\left|\frac{d}{dt}(Sx)(t)\right| = 0$$

This shows the equicontinuity of the family $S\Omega$, (cf. [15], p. 265, [20]). Hence S is relatively compact and therefore S is completely continuous. By Theorem III.3.1 there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that $x_0(t)$ is a positive ω-periodic solution of Eq (II.0.1). The proof is complete. ■

II.2 Examples

In the following examples none of the results in [22], [10] can be applied.

Example II.2.1 *Consider the differential equation*

$$\dot{x}(t) + 0.5x(t) + (\sin t - 0.5)x(t - 4\pi) = 0, \ t \geq 0. \tag{II.2.1}$$

We choose $k(t) = 1$. Then for conditions (II.1.1), (II.1.2) and $\omega = 2\pi$ we have

$$\begin{aligned} \int_t^{t+\omega} [p(s) + q(s)k(s)]ds &= \int_t^{t+2\pi} \sin s ds = 0, \\ \int_{\tau(t)}^t [p(s) + q(s)k(s)]ds &= \int_{t-4\pi}^t \sin s ds = 0, \ t \geq 0. \end{aligned}$$

All conditions of Theorem III.3.1 are satisfied. Thus Eq (II.2.1) has a positive $\omega = 2\pi$-periodic solution

$$x(t) = \exp(-\int_T^t \sin s ds) = e^{\cos t - \cos T}, \ t \geq T.$$

Example II.2.2 *Consider the delay differential equation*

$$\dot{x}(t) + (0.5\cos t - e^{-t})x(t) + e^{-t-\sin t} - x(t - \pi) = 0, \ t \geq 0. \tag{II.2.2}$$

We take $k(t) = e^{\sin t}$. Then for conditions (II.1.1), (II.1.2) and $\omega = 2\pi$ we obtain

$$\int_t^{t+\omega} [p(s) + q(s)k(s)]ds = 0.5 \int_t^{t+2\pi} \cos s ds = 0,$$

and

$$\int_{\tau(t)}^t [p(s) + q(s)k(s)]ds = 0.5 \int_{t-\pi}^t \cos s ds = \sin t, \ t \geq 0.$$

All conditions of Theorem III.3.1are satisfied. Then Eq (II.2.2) has a positive $\omega = 2\pi$-periodic solution

$$x(t) = \exp(-0.5 \int_T^t \cos s ds) = e^{0.5(\sin T - \sin t)}, \ t \geq T.$$

Example II.2.3 *Consider the delay differential equation*

$$\dot{x}(t) - e^{-t}x(t) + (e^{-t} + 2\cos 2t)e^{\sin t \sin 2t}x(\frac{1}{2}t) = 0, \ t \geq 0. \tag{II.2.3}$$

We choose $k(t) = e^{\sin 2t \sin t}$. Then for conditions (II.1.1), (II.1.2) and $\omega = \pi$ we get

$$\int_t^{t+\omega} \cos 2s ds = 0 \ \textit{and} \ \int_{0.5t}^t 2\cos 2s ds = \sin 2t - \sin t, \ t \geq 0.$$

All conditions of Theorem III.3.1 are satisfied and Eq (II.2.3) has a positive π-periodic solution

$$x(t) = \exp(-2 \int_T^t \cos 2s ds) = e^{\sin 2T - \sin 2t}, \ t \geq T.$$

Chapter III

Study the existence of positive periodic solutions of Nonlinear neutral differential systems

In this chapter, we study the existence of a positive periodic solution of systems of equations. The study on functional differential equations is more intricate than ordinary differential equations, that is why there are plenty of results on the existence of positive periodic solutions for various types of first-order or second-order ordinary delay differential equations, while studies on positive periodic solutions for delay differential equations by using the Mawhin coincidence degree theory are relatively less, and most of them are confined to first-order delay differential. For that purpose we recall the problems (.0.1)-(.0.2)

$$\frac{dx}{dt} = \pm x(t)\left[G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t))) - \varphi(t)x'(t-r(t))\right], \quad \text{(III.0.1)}$$

$$\frac{du}{dt} = a(t)u(t) - \lambda b(t)f(t, u(t-\delta(t))) - c(t)x(t-\sigma(t)). \quad \text{(III.0.2)}$$

Where $G(t, x_1, ..., x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R})$, $G(t+\omega, x_1, ..., x_n) = G(t, x_1, ..., x_n)$ and $f : \mathbb{R} \times \ \mathbb{R}^+ \to \mathbb{R}, \in C(\mathbb{R}, \mathbb{R}^+)$, φ, a, b, $c \in C(\mathbb{R}, \mathbb{R}^+)$, r, σ, δ, $\tau_i \in C(\mathbb{R}, \mathbb{R})$ for $i = 1, ..., n$. All of the above functions are continuous, ω-periodic functions and $\omega > 0$ is a constant. Here $\lambda > 0$ is a parameter. The object here, we obtain various sufficient conditions for the existence of positive periodic solutions for problems (III.0.1)–(III.0.2).

III.1 Periodic solutions for equation (III.0.1)

The main tool employed in their study is based on some techniques of the Mawhin coincidence degree. Indeed, first of all, let us give some notions and notations used in the theory of coincidence degree theorem (see [5], [6], [18]) which we will apply in this part. To state existence of ω-periodic solution criteria for (III.0.1) some preparations and notations are needed. For $\omega > 0$, let X be the set of all continuous scalar functions x, periodic in t of period ω. Clearly, $(X, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,\omega]} |x(t)| .$$

For convenience, we shall introduce the notation

$$\bar{u} = \frac{1}{\omega}\int_0^\omega u(t)dt,$$

where u is a periodic continuous function with period ω.As a first case, we consider the following nonlinear differential equation with delay

$$\frac{dx}{dt} = -x(t)[G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t))) - \varphi(t)x'(t-r(t))], \tag{III.1.1}$$

where $G(t, x_1, ..., x_{n+1}) \in C(\mathbb{R}^{n+1}, \mathbb{R})$, $G(t+\omega, x_1, ..., x_n) = F(t, x_1, ..., x_n)$, r' ,$\varphi' \in C(\mathbb{R}, \mathbb{R})$ and r, φ, $\tau_i \in C(\mathbb{R}, \mathbb{R}^+)$ for $i = 1, ..., n$. All of the above functions are supposed to be ω-periodic functions with $\omega > 0$ is a constant. We can introduce a change of variables by the formula

$$y(t) = \ln(x(t)), \tag{III.1.2}$$

which implies that

$$x(t)y'(t) = x'(t),$$

and

$$x(t-r(t))y'(t-r(t)) = x'(t-r(t)).$$

So, y satisfies the delay equation

$$\begin{aligned} \frac{dy}{dt} &= -[G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t))) - \varphi(t)x\prime(t-r(t))] \\ &= -[G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) - \varphi(t)y\prime(t-r(t))e^{y(t-r(t))}], \end{aligned} \tag{III.1.3}$$

where

$$r'(t) \neq 1 \text{ for } t \in [0, \omega]. \tag{III.1.4}$$

It is obvious that the existence problem of ω-periodic solution of (III.1.1) is equivalent to that of ω-periodic solutions of the Eq (III.1.3).

Lemma III.1.1 *Assume the condition (III.1.4) holds. Suppose that in (III.1.1) the following conditions hold.*

(i) There exists a constant $C > 0$ such that if x is continuous ω-periodic function and satisfies

$$\int_0^\omega [G(t, e^{x(t-\tau_1(t))}, ..., e^{x(t-\tau_n(t))}) + \psi(t)e^{x(t-r(t))}]dt = 0,$$

then we have

$$\int_0^\omega \left|G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) + \psi(t)e^{y(t-r(t))}\right| dt \leq C.$$

(ii) There exists a constant $H > 0$ such that, when $v_i \geq H$, $i = 1, ..., n+1$,$G(t, e^{v_1}, ..., e^{v_n}) + \psi(t)e^{v_{n+1}} > 0$, $G(t, -e^{v_1}, ..., -e^{v_n}) - \psi(t)e^{v_n+1} < 0$,uniformly hold for $[0, \infty)$ with

$$\psi(t) = \frac{\varphi'(t)(1-r'(t) + \varphi(t)r''(t)}{[1-r'(t)]^2},$$

is positive continuous ω-periodic function.Then, the Eq. (III.1.1) has at least one positive ω-periodic solution.

Proof: In order to apply Lemma I.3.1, we take $X = Z := \{y \in C(\mathbb{R},\mathbb{R}) : y(t+\omega) = y(t)\}$, and use

$$\|y\| = \sup_{t\in\mathbb{R}} |y(t)| = \sup_{t\in[0,\omega]} |y(t)|.$$

Then, X and $Z$are Banach spaces when they are endowed with the norm $\|.\|$. Set

$$Ny = -[G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) - \varphi(t)y\prime(t-r(t))e^{y(t-r(t))}],$$

and

$$Ly = y', Py = Qy = \frac{1}{\omega}\int_0^\omega y(t)dt, y \in X.$$

Obviously,

$$\ker L = \{y|y \in X, y = \xi, \xi \in R\}, \operatorname{Im} L = y|y \in X, \int_0^\omega y(t)dt = 0\},$$

are closed in X and

$$\dim\ker L = co\dim\operatorname{Im} L.$$

Hence, L is Fredholm mapping of index zero. Furthermore, the generalized inverse (to L)

$$K_p : \operatorname{Im} L \to \ker P \cap DomL,$$

has the form

$$K_p(y) = \int_0^t y(s)ds - \frac{1}{\omega}\int_0^\omega\int_0^t y(s)dsdt.$$

Thus,

$$(QN)(y) = -\frac{1}{\omega}\int_0^\omega [G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) - \varphi(t)y'(t-r(t))e^{y(t-r(t))}]dt,$$

but

$$\begin{aligned}
&-\int_0^\omega \varphi(t)y'(t-r(t))e^{y(t-r(t))}dt \\
&= -\int_0^\omega [\frac{\varphi(t)}{1-r'(t)}\frac{d}{dt}e^{y(t-r(t)}]dt + \int_0^\omega [\frac{\varphi'(t)1-r'(t)+\varphi(t)(r''(t))}{[1-r'(t)]^2}e^{y(t-r(t))}]dt \\
&= \int_0^\omega \psi(t)e^{y(t-r(t))}dt,
\end{aligned}$$

one has

$$(QN)(y) = -\frac{1}{\omega}\int_0^\omega [G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) + \psi(t)e^{y(t-r(t))}]dt,$$

and

$$\begin{aligned}
K_p(I-Q)N(y) = &-\frac{1}{\omega}\int_0^\omega G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) + \psi(t)e^{y(t-r(t))}dt \\
&+ \frac{1}{\omega}\int_0^\omega\int_0^t [G(s, e^{y(s-\tau_1(s))}, ..., e^{y(s-\tau_n(s))} + \psi(s)e^{y(s-r(s))}dsdt \\
&+ (\frac{t}{\omega} - \frac{1}{2})\int_0^\omega [G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) + \psi(t)e^{y(t-r(t))}]dt.
\end{aligned}$$

Clearly, QN and $K_p(I-Q)N$are continuous and, moreover, $QN(\bar{\Omega})$and $K_p(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L—compact on $\bar{\Omega}$. Here Ω is any open bounded set in X. Now we reach the position to search for an appropriate open bounded subset X for the application of Lemma I.3.1. Corresponding to equation $Ly = \eta Ny$, $\eta \in (0,1)$ we have

$$y'(t) = -\eta[G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) - \varphi(t)y\prime(t-r(t))e^{y(t-r(t))}]. \tag{III.1.5}$$

Suppose that $y(t) \in X$ is a solution of system (III.1.5) for a certain $\eta \in (0,1)$. By integrating (III.1.5) over the interval $[0,\omega]$, we obtain

$$\int_0^\omega [G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) + \psi(t)e^{y(t-r(t))}]dt = 0, \tag{III.1.6}$$

by this, (III.1.5) and (i) we find

$$\int_0^\omega |x'(t)|\, dt \leq \eta \int_0^\omega \left|G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) + \psi(t)e^{y(t-r(t))}\right| dt \leq C. \tag{III.1.7}$$

Moreover, in view of (III.1.6) and (ii), it is easy to see that there exist an $i_0 \in \{1, ..., n\}$, a point$t^* \in [0,\omega]$ and a constant $l_1 > 0$ such that

$$y(t^* - \tau_{i_0}(t^*)) < l_1 \ , \ y(t^* - r(t^*)) < l_1. \tag{III.1.8}$$

Otherwise, for any $l_1 > 0$ and any $t \in [0,\omega]$, one has

$$y(t^* - \tau_i(t^*)) \geq l_1, i = 1, ..., n,$$

and

$$y(t - r(t)) \geq l_1.$$

In view of (ii), we see that this contradicts (III.1.6). Hence (III.1.8) holds. Denote

$$t^* - \tau_{i_0}(t^*) = \xi_1 + k\omega, \xi_1 \in [0,\omega],$$

with k being an integer. Then

$$y(\xi_1) < l_1. \tag{III.1.9}$$

In a similar way, from (III.1.6) and (ii), it easy to see that there exist $i_1 \in \{1, ..., n\}$, a point $\xi_1 \in [0,\omega]$ and a constant $l_2 > 0$ such that

$$y(\xi_2) > -l_2. \tag{III.1.10}$$

Therefore, it follows from (III.1.7), (III.1.9) and (III.1.10) that

$$y(t) \leq y(\xi_1) + \int_0^\omega |y'(t)|\, dt < l_1 + C,$$

and

$$y(t) \geq y(\xi_2) - \int_0^\omega |y'(t)|\, dt > -(l_2 + C).$$

Thus

$$\|y\| < \max\{l_1 + C, l_2 + C\} := l.$$

Now, take

$$\Omega = \{y \in X | \, \|y\| < J\},$$

where$J = \max\{l, H\}$. Notice first that is a closed convex bounded subset of a Banach space so Ω satisfies the condition (a) of the Lemma I.3.1. When $y \in \partial\Omega \cap \ker L = \partial\Omega \cap R$, y is a constant in R with$\|y\| = J$. Then,

$$(QN)(y) = -\frac{1}{\omega}\int_0^\omega [G(t, e^{y(t-\tau_1(t))}, ..., e^{y(t-\tau_n(t))}) + \psi(t)e^{y(t-r(t))}]dt \neq 0.$$

Set $\psi(\theta, y) = \theta y + (1-\theta)QNy, 0 \leq \theta \leq 1$.Since for $y \in \partial\Omega \cap \mathbb{R}$, $\theta \in [0,1]$, $y\psi(\theta, y) > 0$ one has $\psi(\theta, y) \neq 0$. It follows from the property of invariance under a homotopy that

$$\deg\{QNy, \partial\Omega \cap R, 0\} \neq 0.$$

We know that Ω verifies all the requirements of Lemma I.3.1. Then (III.1.3) has at least one ω-periodic solution y. From (III.1.2) the equation (III.1.1) has also at least one positive ω-periodic solution given by $x(t) = e^{y(t)}$. The proof is complete. ■

As a first case consider the equation

$$\frac{dx}{dt} = x(t)[G(t, x(t-\tau_1(t)), ..., x(t-\tau_n(t))) - \varphi(t)x\prime(t-r(t))]. \tag{III.1.11}$$

Note that the Eqs. (III.1.1) and (III.1.11) differ only by a sign in the first equation. So the treatment is the same as the first case. So, we have the following lemma which can be proved by similar method.

Lemma III.1.2 *Suppose that the assumptions of Lemma III.1.1 hold. Then the Eq. (III.1.11) has at least one positive ω-periodic solution.*

III.2 Periodic solution for equation (III.0.2)

In fact the Eq. (III.0.1) has a positive ω-periodic solution so in this connection we offer existence criteria for the periodic solutions of the (III.0.2). Throughout this part we assume that for all $t \in [0, \omega]$ and a positive $x \in X$, whenever necessary, we shall consider

- (H1) $a, b, c \in C(\mathbb{R}, \mathbb{R}^+)$are ω-periodic functions with$\int_0^\omega a(t)dt > 0$ and $\int_0^\omega b(t)dt > 0$.

- (H2) δ, $\sigma \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic.

- (H3) $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and is ω-periodic about t.

- (H4) $\lim\limits_{x\to+\infty} \frac{f(t,x)}{x} = N \in (0, +\infty]$uniformly on $[0, \omega]$.

Next, by using the previous results, the problem (III.0.1)–(III.0.2) is expressed as follows

$$\frac{du}{dt} = a(t)u(t) - \lambda b(t)f(t, u(t - \delta(t))) - c(t)x(t - \sigma(t)). \tag{III.2.1}$$

In this part we use different method to establish existence periodic solution (III.2.1). These assumptions will be used in the main results.

In order to simplify notation, we define the function φ by

$$\varphi(t) := c(t)x(t - \sigma(t)). \tag{III.2.2}$$

So, (III.2.2) can be reduced to the equation

$$\frac{du}{dt} = a(t)u(t) - \lambda b(t)f(t, u(t - \delta(t))) - \varphi(t). \tag{III.2.3}$$

Suppose that $\varphi : \mathbb{R} \to \mathbb{R}^+$ is continuous ω-periodic function and u is a solution of Eq. (III.2.3). One gets, by applying the variation of parameters formula by multiplying both sides of the Eq (III.2.3) by the factor $\exp\left(-\int_0^t a(s)ds\right)$ and integrating from t to $t + \omega$. We see that this solution of Eq (III.2.3) is exactly

$$u(t) = \lambda \int_t^{t+\omega} b(s)F(t, s)f(s, u(s - \delta(s)))ds + \int_t^{t+\omega} F(t, s)\varphi(s)ds,$$

where

$$F(t, s) := \frac{\exp(-\int_t^s a(v)dv)}{1 - \exp(-\int_0^\omega a(v)dv)}.$$

To begin with, We will introduce more notation

- (N1) $R(t) := \int_t^{t+\omega} F(t, s)\varphi(s)ds$ and $R_0 = \|R(t)\|$,
- (N2) $\mu = \exp(-\int_0^\omega a(v)dv)$,
- (N3) $M = \max\{f(t, x) : (t, x) \in [0, \omega] \times [0, R_0(\mu + 1)/\mu]\}$.

It is easy to see that $R(t)$ satisfies

$$R'(s) = -a(t)R(s) + \varphi(s).$$

Hence, in (III.2.3) we make the change of the variables $u(t) = z(t) + R(t)$. Then with respect to z we obtain the next equation

$$z'(t) = a(t)z(t) - \lambda b(t)f(t, z(t - \delta(t)) + R(t - \delta(t))). \tag{III.2.4}$$

It is obvious that to prove (III.2.3) has a positive ω-periodic solution $u(t)$, it is enough to prove that $z(t) = u(t) - R(t)$is a ω-periodic solution of (III.2.4) for every $\varphi : \mathbb{R} \to \mathbb{R}^+$ continuous ω-periodic function and $z(t) + R(t) > 0$ for all $t \in [0, \omega]$. Furthermore the solution of (III.2.4) given by

$$z(t) = \lambda \int_t^{t+\omega} b(s)F(t, s)f(s, z(t - \delta(t)) + R(t - \delta(t)))ds.$$

Lemma III.2.1 *If assumptions (H1)–(H4) and (N3) are satisfied. Assume that*

$$A < \lambda < B. \tag{III.2.5}$$

Then (III.2.3) has at least one positive ω-periodic solution, where A and B are defined as follows

$$A = \frac{2}{\mu N}\frac{1-\mu}{\mu\ \int_0^\omega b(s)ds} \quad \textit{and}\ B = \frac{R_0}{M}\frac{1-\mu}{\mu\ \int_0^\omega b(s)ds}.$$

Proof: Recall $F(.,.)$, one has

$$F(t,s) \leq \frac{1}{1-\mu} \text{ and } F(t,s) \geq \frac{\mu}{1-\mu}, \tag{III.2.6}$$

for $t \leq s \leq t+\omega$, thus

$$z(t) \leq \|z\| \leq \frac{\lambda}{1-\mu}\int_t^{t+\omega} b(s)\ f\ (s, z(s-\delta(s)) + R(s-\delta(s)))ds,$$

and

$$z(t) \geq \mu\frac{\lambda}{1-\mu}\int_t^{t+\omega} b(s)f(s, z(s-\delta(s)) + R(s-\delta(s)))ds \geq \mu z.$$

Hence $z(t) \geq \mu \|z\|$ for $t \in R$.Taking X to be the set of all real continuous ω-periodic functions endowed with the usual linear structure as well as the norm

$\|z\| = \max\{z(t) : t \in [0,\omega]\}$. Then $X$is a Banach space. Let $K = \{z \in X : z(t) \geq \mu z, t \in [0,\omega]\}$. Then K is a cone in the space X. Let the operator T be defined as follows

$$(Tz)(t) = \lambda \int_t^{t+\omega} b(s)F(t,s)f(s, z(s-\delta(s)) + R(s-\delta(s)))ds,$$

for $z \in X$. It is easy to see that T is completely continuous and $TK \subset K$ by the periodicity assumptions. Since $\lambda < B$, we choose $\alpha > 1$ such that $\lambda \leq B/\alpha$. Set

$$\Omega_1 := \{z \in X : \|z\| < R_0/\mu\}$$

Then one finds

$$u(t) = z(t) + R(t) \leq \|z + R\| \leq \|z\| + \|R\| = R_0/\mu + R_0 = R_0(\mu+1)/\mu,$$

and since $z(t) \geq \mu \|z\|$ this yields

$$u(t) = z(t) + R(t) \geq \mu \|z\| - \|R\| = \mu R_0/\mu - R_0 = 0$$

for each $z \in X \cap \partial\Omega_1$. Thus

$$\begin{aligned}
(Tz)(t) &\leq \frac{\lambda}{1-\mu}\max\{f(t,u) : (t,u) \in [0,\omega] \times\ [0, R_0(\mu+1)/\mu]\}\int_0^\omega b(s)ds \leq \frac{\lambda M}{1-\mu}\int_0^\omega b(s)ds \\
&\leq \frac{B}{\alpha}\frac{\lambda M}{1-\mu}\int_0^\omega b(s)ds \leq \frac{BM}{1-\mu}\int_0^\omega b(s)ds \\
&\leq \frac{R_0}{\mu} = \|z\|
\end{aligned}$$

that is $\|Tz\| \leq \|z\|$ for all $z \in X \cap \partial\Omega_1$. Now, choosing $\epsilon > 0$ so that

$$\lambda \frac{\mu^2(N-\epsilon)}{2(1-\mu)} \int_0^\omega b(s)ds > 1 \tag{III.2.7}$$

by $\lambda > A$. Again, one can choose $H > \frac{R_0}{\mu}$ such that

$$\lim_{x \to +\infty} \frac{f(t,x)}{x} = N \text{ for } t \in [0,\omega] \text{ and } x \geq H. \tag{III.2.8}$$

Set

$$\Omega_2 := \{z \in X : z < (H + R_0)/\mu\}$$

We find

$$u(t) = z(t) + R(t) \geq \mu \|z\| - R_0 \geq \mu(H + R_0)/\mu - R_0 = H. \tag{III.2.9}$$

From (III.2.8) and (III.2.9) we obtain

$$|f(t, z(t-\delta(t)) + R(t-\delta(t)))| \geq (N-\epsilon) \|z + R\| \geq (N-\epsilon)H \tag{III.2.10}$$

Finally, from, (III.2.5), (III.2.6), (III.2.7) and (III.2.10) it follows that

$$\begin{aligned}(Tz)(t) &\geq H\lambda \frac{\mu(N-\epsilon)}{1-\mu} \int_0^\omega b(s)ds \\ &\geq \frac{H+R_0}{\mu}(\lambda \frac{\mu^2(N-\epsilon)}{2(1-\mu)} \int_0^\omega b(s)ds \\ &\geq (H+R_0)/\mu = \|z\|\end{aligned}$$

We deduce that $\|Tz\| \geq \|z\|$ for each $z \in X \cap \partial\Omega_2$. Hence T has at least one fixed point z such that $\frac{R_0}{\mu} < \|z\| \leq \frac{H+R_0}{\mu}$ that is aω-periodic solution of (III.2.4). We claim that $R_0 < \|z\|$. Indeed, if there is t_0 such that $\frac{R_0}{\mu} = z(t_0) = (Tz)(t_0) < \frac{R_0}{\alpha\mu}$, we get a contradiction. On the other hand,

$$z(t) + R(t) \geq \mu \|z\| - R_0 = \mu \frac{R_0}{\mu} - R_0 = 0.$$

So, $u(t) = z(t) + R(t)$ is a positive ω-periodic solution of Eq (III.2.4). ■

III.3 Positive periodic solutions for the systems (III.0.1-III.0.2)

Theorem III.3.1 *Under the hypotheses of Lemmas III.1.1, III.1.2 and III.2.1. Then system (III.0.1)–(III.0.2) has at least one positive ω-periodic solution (x, u).*

Theorem III.3.2 *Under the hypotheses of Lemmas III.1.1, III.1.2 and III.2.1. And let the following conditions*

$$\lim_{x \to +\infty} \frac{f(t,x)}{x} = N = +\infty, \tag{III.3.1}$$

uniformly on $[0,\omega]$ and

$$0 < \lambda < B. \tag{III.3.2}$$

Then system (III.0.1)–(III.0.2) has at least one positive ω-periodic solution (x, u).

Proof: We deduce by the same reasoning using Lemmas III.1.1, III.1.2 and III.2.1 that Theorem III.3.1 is also valid if (H9) is replaced by (III.3.1) and (III.2.5) is replaced by (III.3.2). ■

III.4 Example

Let $r(t) \equiv r$ be a positive constant. Consider the following logistic model with several delays and feedback control

$$\begin{aligned}\frac{dx}{dt} &= x(t)[\eta(t) - \sum_{i=1}^{n} a_i(t)x(t-\tau_i(t)) - \varphi(t)x'(t-r(t))] \\ \frac{du}{dt} &= a_0(t)u(t) - \lambda b(t)u^2(t-\delta(t)) - c(t)x(t-\sigma(t)).\end{aligned} \tag{III.4.1}$$

where τ_i, σ, $\delta \in C(\mathbb{R},\mathbb{R})$, for all $i = 1,...,n$ and η, φ, a_0, c, b, $a_i \in C(\mathbb{R},\mathbb{R}^+)$ are ω-periodic functions. If

$$0 < \lambda < \frac{R_0(1-\mu)}{\mu M \int_0^\omega b(s)ds},$$

with R_0, μ and M are defined by (N1), (N2) and (N3) respectively. Then, the system (III.4.1) has at least one positive ω-periodic solution.
Proof: Let y be a continuous ω-periodic solution that satisfies

$$\int_0^\omega \left[\eta(t) - \left(\sum_{i=1}^{n} a_i(t)e^{y(t-\tau_i(t))} - \varphi(t)e^{y(t-r(t))}\right)\right] dt = 0,$$

where$\varphi(t) = \psi(t)$ is the same as in Lemma III.1.1. Then,

$$\int_0^\omega \eta(t)dt = \int_0^\omega \left[\sum_{i=1}^{n} a_i(t)e^{y(t-\tau_i(t))} - \varphi\prime(t)e^{y(t-r(t))}\right] dt = 0.$$

On the other hand

$$\begin{aligned}&\int_0^\omega \left|\eta(t) - \left(\sum_{i=1}^{n} a_i(t)e^{y(t-\tau_i(t))} - \varphi'(t)e^{y(t-r(t))}\right)\right| dt \\ &\leq \int_0^\omega |\eta(t)|\, dt + \int_0^\omega \left|\sum_{i=1}^{n} a_i(t)e^{y(t-\tau_i(t))} - \varphi'(t)e^{y(t-r(t))}\right| dt \\ &\leq 2\int_0^\omega |\eta(t)|\, dt = C > 0.\end{aligned}$$

Let v_i, $i = 1,...,n+1$ be continuous functions such that

$$\lim_{\{v_1,v_2,...,v_{n+1}\}\to+\infty} \left(\sum_{i=1}^{n} a_i(t)e^{v_i} - \varphi'(t)e^{v_{n+1}}\right) = +\infty,$$

with $v_i = v(t-\tau_i(t))$ for $i = 1,...,n$ and $v_{n+1} = v(t-r(t))$. Then, one has

$$\lim_{\{v_1,v_2,...,v_{n+1}\}\to+\infty} \left(\eta(t) - \left(\sum_{i=1}^{n} a_i(t)e^{v_i} - \varphi'(t)e^{v_{n+1}}\right)\right) = -\infty,$$

and the limit

$$\lim_{\{v_1,v_2,\ldots,v_{n+1}\}\to+\infty}\left(\eta(t)-\left(\sum_{i=1}^{n}a_i(t)e^{v_i}-\varphi'(t)e^{v_{n+1}}\right)\right)=\eta(t)>0,$$

holds uniformly in $t\in[0,\omega]$. Furthermore, the function $f:(t,u)\to u^2(t-\delta(t))$ is positive on$\mathbb{R}$ and satisfies the condition (III.3.1) of Theorem III.3.2. In fact

$$0<\lambda<B=\frac{R_0(1-\mu)}{\mu M\int_0^\omega b(s)ds}.$$

Consequently, under these hypotheses on the system (III.4.1) all the conditions of Theorem III.3.2 are satisfied. Thus, the system (III.4.1) has at least one positive ω-periodic solution. ■

Chapter IV

Conclusion

In this memoir we state some fixed point and concidence degree theoremes that we employ to help us in proving existence of solutions. Firstly, it includes some concept definitions and theorems about fixed-point results and differential equations with delays. Secondly, we investigate the existence of positiv periodic solutions for the first order delay differential equation. Finally, we study the existence of a positive periodic solution of system of delay equations.

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Abstract

Our purpose in this memoir is to establish the existence of positive periodic solutions for some nonlinear problems with functional delay by means of the fixed-point theory and technique of the Mawhin coincidence degree theory.

ملخص

هدفنا في هذه المذكرة هو إثبات وجود حلول دورية موجبة لبعض المسائل الغير الخطية ذات تأخر دالي، وهذا بتطبيق طريق النقطة الثابتة وتقنية تطابق المؤثرات المتساوية الدرجة علي مجموعة جزئية .