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Theme

(FPTs) involving tow metrics and applications to differential and integral systems with improved results

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Abstract

In this work, we highlight some versions of the fixed point theory (FPT) involving two metrics. This is done in two setting: scalar metric spaces and vector-valued metric spaces. We provide also some of their applications in the solvability of differential equations, which illustrate that improved results can be acquired in comparison with the classical Banach's principle.

Key words:
Maia's fixed point theorem, Perov's fixed point theorem, existence and uniqueness, boundary value problems, integral equation systems.

.

Résumé

Dans ce travail, nous mettons en évidence certaines versions de la théorie du point fixe (FPT) impliquant deux métriques. Cela se fait dans deux contextes : les espaces muni d'une distance scalaire et les espaces muni d'une distance vectorielle. Nous fournissons également certaines de ses applications dans la solvabilité des équations différentielles, qui illustrent que des résultats améliorés peuvent être acquis, par rapport au principe de Banach classique.

Mots clés :
Théorème du point fixe de Maia, théorème du point fixe de Perov, existence et unicité, problèmes aux limites, systèmes d'équations intégrales.

ملخص

في هذا العمل، نسلط الضوء على بعض الإصدارات لنظرية النقطة الثابتة التي تنطوي على متريين. يتم ذلك في إعدادين: فضاءات المزودة بالمسافة السلميةوفضاءات المزودة بالمسافة الشعاعية. نقدم أيضاًا بعض تطبيقاته في قابلية حل المعادلات التفاضلية، التي توضح أنه يمكن الحصول على نتائج محسنة مقارنة بمبدأ باناخ التقليدي.

الكلمات المفتاحية:
نظام المعادلة غير نظرية مايا للنقطة الثابتة، نظرية بيروف للنقطة الثابتة، الوجود والوحدنية، الجمل الحدية، نظم المعادلات التكاملية.

Dedicace

We dedicate our graduation to our parents, may Allah protect and preserve them, and may they always be a source of pride for us. They have waited for years to see their sons fulfill their dreams. We dedicate our graduation to all our friends and companions throughout the various stages of our educational journey. We dedicate our graduation to our teachers, from primary school to college, as well as our professors and mentors at the university, who have taught us so much, not only in terms of knowledge but also in terms of moral values, love, tolerance, dedication, and seriousness. Thanks to them, we have been able to see life from a different perspective. They have been a source of light, guiding us on many paths in our lives.

Contents

Notations and Abbreviations

$\mathbb{R}$	:	Set of real numbers.
$\mathbb{R}_+$	:	Set of positive numbers.
$\mathbb{R}^n$	:	Vector space of dimension n constrained over the field of reals.
$\mathbb{N}$	:	Set of natural numbers.
$(X, d), (X, \delta)$	:	Metric spaces.
$(X, D), (X, \Delta)$	:	Generalized metric spaces.
$\mathcal{M}_n$	:	The set of real square matrices of order n.
$d(.,.), \delta(.,.)$	:	scalar metrics
$D(.,.), \Delta(.,.)$	:	Vector-valued metrics
$B(x_0, r)$	:	Open ball.
$B_G(x_0, r)$	:	Generalized open ball.
$(X, \to)$	:	L-space.
$(X, \to^d)$	:	L-space with a convergence structure induced by a metric
$\mathcal{C}([a, b])$	:	Space of real continuous functions on $[a, b]$
(FPT)	:	Fixed point theorem
(BVP)	:	Boundary value problem
F_T	:	Fixed point set of the operator T
(PO)	:	Picard operator
(WPO)	:	Weakly Picard operator

Introduction

Banach's contraction principle is one of the most useful tools in nonlinear functional analysis, that ensures the existence and uniqueness of fixed points on complete normed or metric spaces. In the setting of a metric space, this principle is stated as follows

Theorem :

Let $(X.d)$ be a complete metric space and $T : X \longrightarrow X$ a contraction, that is a map satisfying: $\exists\, k \in [0,1[: d(Tx,Ty) \leqslant kd(x,y), \;\; \forall x,y \in X$

Then

(i) T has a unique fixed point $p \in X$

(ii) the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, converges to P, for every $x_0 \in X$

Up to now, this principle is widely applied in various branches of pure and applied mathematics (see for example [18, 19]) and the references cited therein). Many generalizations of this principle have been given in recent years, for details on some of such kind of extensions, we refer to the monographs [20, 21].

In this line of research, we can distinguish tow main directions: one is by finding more general contraction inequalities and the other is by modifying the structure of the space. In connection with the second line, in 1968, Maia [1] extended previous theorem by distributing the conditions on two metrics d and δ on the set X, satisfying the following assumptions

- $d(x,y) \leq \delta(x,y), \ \forall x,y \in X$
- (X,d) is a complete metric space
- $\exists\, \alpha \in [0,1[: \delta(Tx,Ty) \leqslant \alpha\delta(x,y), \ \forall x,y \in X$
- T is continuous with respect to d

Later, many other generalizations on this direction have been investigated [22, 23, 24]. It is remarkable that these extensions have very limited applications, given that the first ones were published more that 40 years ago, [13].

An other extension of previous theorem in spaces endowed with vector valued metrics, was done by Perov [13]. The use of the so called " generalized metric spaces " allowed him to define a new class of maps which satisfy a type of contraction similar to that of Banach, but with a matrix $M \in \mathcal{M}_n$ with positive entries, converging to zero instead of a scalar $0 \leq k < 1$.

Motivated by the previous discussion, we highlight in the present thesis, some versions of the fixed point theory with tow metrics. This will be done in the following tow setting: scalar metric spaces and vector valued metric spaces. We provide also some of their applications in the solvability of differential equation systems, which illustrate that improved results can be obtained in comparison with Banach's principle.

The content of this work is composed of three chapters distributed as follows.

The first chapter is devoted to recalling some notions that will be useful to us later: recall of some notions of functional analysis, L-spaces, Picard operators and matrices convergent to zero.

In the second chapter, we begin by presenting the first version in the fixed point theory involving tow metrics, which comes back to Maia [1]. We give next some extensions of this theorem. Finally, we provide an application to the following (BVP) considered in [13]:

$$\begin{cases} x'''(t) + f(t, x(t)) = 0, \ t \in [a, b] \\ x(a) = 0, \, x'(a) = 0, \ x(b) = kx(\eta), \end{cases}$$

The comparison of the result with the one obtained by Banach's approach, illustrate that improved results can be acquired using (FPT) with tow metrics.

In the first part of the third chapter, we give some version of Maia's (FPT) in the setting of vector valued metric spaces. An application to Fredholm integral equations systems, will be the focus of the second part of this chapter. More precisely, inspired by [13], we consider the following system

$$\begin{cases} x_1(t) = g_1(t) + \displaystyle\int_a^b F_1(t, s, x_1(s), x_2(s))ds \\ x_2(t) = g_2(t) + \displaystyle\int_a^b F_2(t, s, x_1(s), x_2(s))ds \end{cases}$$

where $g_1, g_2 \in \mathcal{C}([a, b]), F_1, F_2 \in \mathcal{C}([a, b] \times [a, b] \times \mathbb{R}^2, \mathbb{R})$ are given functions.

It should be noted that the above system is more general then those considered in [13]. After converting the considered system into a fixed point problem, we give sufficient conditions for the existence and uniqueness of the solution using the (FPT) given in the first part. We provide furthermore the following example to illustrate the validity of our theoretical result:

$$\begin{cases} x_1(t) = 2 + \dfrac{1}{4} \displaystyle\int_0^1 (t+1) + \dfrac{1}{1 + x_1(s) + x_2(s)} \, ds \\ x_2(t) = 2 + \dfrac{1}{2} \displaystyle\int_0^1 t^2 + sin(x_1(s) + x_2(s)) \, ds \end{cases}$$

We conclude with a discussion and some remarks.

Chapter 1

Preliminaries

In this chapter, we recall some necessary definitions and basic mathematical notions that will be used in the rest of the work.

1.1 Recall of some notions of functional analysis

1.1.1 Measure space

Let X be a set endowed with a $\sigma-$algebra $\mathcal{F}$. That is $(X, \mathcal{F})$ is a measurable space.

Definition 1.1.1. Let $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ be two measurable spaces.

A function $f : X_1 \longrightarrow X_2$ is said measurable if

$$\forall B \in \mathcal{F}_2 : \ f^{-1}(B) \in \mathcal{F}_1$$

Definition 1.1.2. [25]

A positive measure on $(X, \mathcal{F})$ is a function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ that satisfies the following axioms:

(i) $\mu(\varnothing) = 0$;

(ii) if $\{E_i : i \in I\}$ is a countable, pairwise disjoint collection sets in $\mathcal{F}$,then

$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$$

The triple $(X, \mathcal{F}, \mu)$ is then called a measure space and the members of $\mathcal{F}$ are called measurable sets.

1.1.2 L^p spaces

In what follows, Ω is an open set of $\mathbb{R}^n$ endowed with the Lebesgue measure dx

Definition 1.1.3. [25] Let $p \in \mathbb{R}$ with $1 \leqslant p < \infty$. The space $L^p(\Omega)$ is defined by:

$$L^p(\Omega) = \left\{ f : \Omega \longleftrightarrow \mathbb{R}; f \;\text{ is measurable and }\; \int_{\Omega} \mid f(x) \mid^p dx < \infty \right\}$$

we notice

$$\|f\|_p = \left[\int_{\Omega} \mid f(x) \mid^p dx \right]^{\frac{1}{p}}$$

Definition 1.1.4. $L^{\infty}(\Omega)$ is the vector space of measurable functions satisfying:

$$\exists C > 0 : \; |f| \leq C \; a.e.$$

The lower bound of all C satisfying this property is denoted by $||f||_{\infty}$:

$$\|f\|_{\infty} = \inf\{C > 0 : |f(x)| \leq C \quad \text{a.e. on } \; \Omega\}$$

1.1.3 Holder's inequality

[25] Let p and q be two conjugate numbers, that is: $\dfrac{1}{p} + \dfrac{1}{q} = 1$

Theorem 1.1.1. Let p and q be two conjugate numbers, with $1 \leq p \leq \infty$. If $f \in L^p$ and $g \in L^q$, then $f.g \in L^1$ and

$$\int | f(x)g(x) | \, dx \leq f \|_p\| \, g \, \|_q \tag{1.1}$$

Proof. The conclusion is obvious if $p = 1$ and $q = \infty$. Suppose therefore that $1 < p < \infty$. Recall Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \ \ \forall \, a \geq 0, \forall \, b \geq 0 \tag{1.2}$$

The inequality (1.2) can be obtained using the fact that the log function is concave on $]0, \infty[$, which leads to

$$\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) = \log(ab)$$

So

$$| \, f(x) \, || \, g(x) \, |\leq \frac{1}{p} \, | \, f(x) \, |^p \, \frac{1}{q} \, | \, g(x) \, |^q, \text{ a.e. in } \Omega$$

Consequently:

$$\int | \, f(x)g(x) \, | \, dx \leq \frac{1}{p}\|f\|_p^P + \frac{1}{q}\|g\|_q^q \tag{1.3}$$

Thus, $fg \in L^1$. Now, replacing f in (1.3) by $\lambda f (\lambda > 0)$, we get

$$\int | \, f(x)g(x) \, | \, dx \leq \frac{\lambda^{p-1}}{p}\|f\|_p^p + \frac{1}{\lambda q}\|g\|_q^q \tag{1.4}$$

we choose $\lambda = \|f\|_p^{-1}\|g\|_q^{\frac{q}{p}}$ (so as to minimize the right hand side in (1.4)), we then obtain (1.1). $\square$

1.1.4 Banach's fixed point theorem

Theorem 1.1.2. Let X be a nonempty set and let d be a metric on X such that (X, d) forms a complete metric space. If the mapping $T : X \longrightarrow X$ satisfies

$$d(Tx, Ty) \leq \alpha\, d(x, y), \text{ for some } 0 < \alpha < 1 \text{ and all } x, y \in X; \tag{1.5}$$

then there is a unique $z \in X$ such that $Tz = z$.

1.2 L-space and Picard operators

1.2.1 L-space

An L-space is a nonempty set endowed with a structure implying a notion of convergence for sequences, which will be defined below. To this end, let us precise some notations.

Let X be a nonempty set. Denote $s(X) := \{\{x_n\}_{n \in N\mathbb{N}} | x_n \in X,\, n \in \mathbb{N}\}$. Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \longrightarrow X$ is an operator.

Definition 1.2.1. [7, 8]

The triple $(X, c(X), Lim)$ is called an L-space [6] and denoted by $(X, \rightarrow)$, if the following conditions are satisfied:

1. If $x_n = x$ for all,$n \in \mathbb{N}$, then $\{x_n\}_{n \in N} \in c(X)$ and $Lim\{x_n\}_{n \in N} = x$.
2. If $\{x_n\}_{n \in N\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n \in \mathbb{N}} = x$, then for all sub-sequences $\{x_{n_i}\}_{i \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$, we have that $\{x_{n_i}\}_{i \in \mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n \in \mathbb{N}} = x$.

Example 1.2.1.

(i) If (X, d) is a metric space, then $(X, \rightarrow^d)$ is an L-space, where $\rightarrow^d$ is the convergence structure induced by the metric d on X.

(ii) Any Hausdorff topological space, generalized metric space (see Definition 3.1.1) and gauge space is an L-space.

1.2.2 Picard operators

Let $(X, \to)$ be an L-space and $T : X \to X$ an operator. We shall use the following notations

$$F_A = \{x \in X \mid T(x) = x\}$$

$$T^0 = I_X, \;\; T^1 = T, \;\; T^{n+1} = T \circ T^n, \;\; n \in \mathbb{N}$$

Definition 1.2.2. [7, 8]

An operator $T : X \to X$ is said to be weakly Picard operator (WPO), if the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of T.

Definition 1.2.3. [7, 8]

If an operator T is (WPO) and $F_T = \{x^*\}$, then by definition the operator T is a Picard operator (PO).

Example 1.2.2.

(i) Let (X, d) be a complete metric space and $T : X \to X$ a contraction. Then, it follows from Theorem 1.1.2 that T is (PO).

(ii) Let (X, d) be a complete generalized metric space and $T : X \to X$ a contraction in the sens of Definition 3.1.3. Then, it follows from Theorem 3.1.1 that T is (PO).

1.3 Matrices which converge to zero

We denote by$M_m(R_+)$the set of all $m \times m$ square matrices with positive real elements, by I_m the identity $m \times m$ matrix and by O_m the zero $m \times m$ matrix.
$A \in M_m(R_+)$ is said to be convergent to zero if $A^n \longrightarrow 0_m$as $n \longrightarrow \infty$.

Definition 1.3.1. [4] A square matrix $\mathcal{M}_{n\times n}(\mathbb{R}_+)$ with non-negative elements is said to be converges to zero if

$$M^k \longrightarrow 0, \text{comme } k \longrightarrow \infty.$$

Property 1.3.1. [4]

Let$M \in \mathcal{M}_n(\mathbb{R}_+)$ with non-negative elements. Then the following assertions are equivalent:

(i) M is convergent to zero.

(ii) $(I - M)$ is not singular and : $(I - M)^{-1} = I + M + M^2 + ... + M^k + ... + ...$ such as I represents the unitary matrix of the same order as M.)

(iii) $|\lambda| < 1$ for everything $\lambda \in \mathbb{C}$ such that $det(M - \lambda I) = 0$.

(iv) $(I - M)$ is not singular and $(I - M)^{-1}$ has not negative elements.

Lemma 1.3.1. [5] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, at the matrix M is convergent to zero if and only if $I - M$ is not singular and

$$(I - M)^{-1} = I + M + M^2 + ... = \sum_{k=0}^{\infty} M^k, \tag{1.6}$$

Proof. Let's first suppose that M converges to zero. To show that $I - M$ is not singular, it suffices to show that the linear system $(I - M)Z = 0$ only has the null solution.suppose that $Z \in \mathbb{R}^n$ be a solution of this system, then:

$$Z = MZ = M^2Z = M^3Z = ... = M^kZ = ...$$

and leaving $k \longrightarrow \infty$ we can deduce $Z = 0$. SO, $I - M$ is not singular .

moreover, (1.6) follows from the identity

$$I - (I - M)(I + M + M^2 + ... + M^k) = M^{k+1}.$$

conversely,if, $I - M$ is not singular and that (1.6) is verified, then from the convergence of the series $\sum_{k=0}^{\infty} M^k$ we conclude that

$$M^k \longrightarrow 0 \text{ as } \ k \longrightarrow \infty.$$

□

Definition 1.3.2. We say a not singular matrix

$$A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$$

the property of absolute value if $A^{-1} \mid A \mid \leq I$ as

$$\mid A \mid = (\mid a_{ij} \mid)_{1 \leq i,j \leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$$

Lemma 1.3.2. Let $M = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a triangular matrix with $\max\{\mid a_{ii} \mid, \quad i = 1, ..., n\} < \dfrac{1}{2}$.

Then the matrix $A = (I - M)^{-1}$ is convergent to zero.

Proof. Suppose

$$M = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \\ 0 \cdots a_{nn} \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{R}_+),$$

then the eigenvalues of M are $\lambda_i = \dfrac{a_{ii}}{1-a_{ii}}$ for $i = 1, ..., n$. Since all the eigenvalues of M are in open unit ball, the conclusion follows from the Proposition. □

Here are some examples of matrices convergent to zero.

Example 1.3.1.

$$A = \begin{pmatrix} a & a \\ \\ b & b \end{pmatrix}, \in M_2(\mathbb{R}_+)$$

where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

Example 1.3.2.

$$A = \begin{pmatrix} a & b \\ \\ a & b \end{pmatrix}, \in M_2(\mathbb{R}_+)$$

where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

Example 1.3.3.

$$A = \begin{pmatrix} a & a \\ \\ 0 & b \end{pmatrix}, \in M_2(\mathbb{R}_+)$$

where $a, b, c \in \mathbb{R}_+$ and $a + b < 1$;

Chapter 2

On (FPTs) with two metrics and application to (BVPs)

We begin by presenting the first version in the fixed point theory involving tow metrics, which comes back to Maia [1]. We give next some extensions of this theorem. Finally, we provide an application to some (BVPs).

2.1 Maia's Fixed Point Theorem

Theorem 2.1.1.

Let X be a space with two metrics d and δ, such that:

1. $d(x, y) \leq \delta(x, y)$ for all points x, y of X.
2. X is complete with respect to d.
3. $T : X \longrightarrow X$ is a continuous map with respect to d.
4. T is a contraction with respect to δ.

Then there exists a unique fixed point for T in X.

Proof.

Existence:

Let x_0 be a point of X, let us consider the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$. This sequence is Cauchy with respect to δ, indeed:

Since by (4), T is a contraction with respect to δ, so $\exists K < 1$ such that:

$\delta(T(x), T(y)) \leq K\delta(x, y)$, for all points x, y of X

Let n, m two numbers such that $m < n$, then:

$$\delta(T^n(x), T^m(y)) \leq K^m \delta(T^{n-m}(x_0), x_0)$$

Since $k < 1$, so for each $\varepsilon > 0$, $\exists l \in \mathbb{N}$ such that for every $m > l$, We have $K^m < \varepsilon$. Then

$$\delta(T^n(x), T^m(y)) \leq \varepsilon\delta(T^{n-m}(x_0), x_0) \text{ if } l < m < n$$

So $\{T^n(x_0)\}_{n\in N}$ is a Cauchy sequence with respect to δ.

Now, from (1) we have

$$d(T^n(x_0), T^m(x_0)) \leqslant \delta(T^n(x_0), T^m(x_0))$$

Then

$$d(T^n(x_0), T^m(x_0) \leqslant \varepsilon\delta(T^{n-m}(x_0), x_0)$$

So $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d.

According to (2), that is, X is complete with respect to d, so $\{T^n(x_0)\}_{n\in N}$ converges in (X, d). Let x be the limit point. Using (3), one gets

$$x = \lim_{n\longrightarrow\infty} T^n(x_0) = T(\lim_{n\longmapsto\infty} T^{n-1}(x_0)) = T(x)$$

Which means that x is a fixed point for T in X.

Uniqueness:

Suppose that there is $x, y \in X$. $(x \neq y)$ such that:

$$T(x) = x \text{ and } T(y) = y$$

f is $k-$Lipschitz then we can write

$$\delta(T(y), T(x)) \leq k\delta(x, y)$$

$$\Leftrightarrow k \geqslant \frac{\delta(T(x), T(y))}{\delta(x, y)} = \frac{\delta(x, y)}{\delta(x, y)} = 1$$

which is a contradiction with the hypothesis $(k < 1)$ □

Remark 2.1.1.

The continuity hypothesis of T with respect to d in Theorem 2.1.1 is not superfluous. In fact, if X is the set of natural numbers (including zero), consider the following metrics on X:

$$\delta(p, q) = \left| \tfrac{1}{2^p} - \tfrac{1}{2^q} \right|;$$

and

$$d(p, q) = \begin{cases} \left| \tfrac{1}{2^{p+1}} - \tfrac{1}{2^{q+1}} \right| & \text{if } p > 0, q > 0; \\ \dfrac{1}{2^{p+1}} & \text{if } p > 0, q = 0; \\ \dfrac{1}{2^{q+1}} & \text{if } p = 0, q > 0; \\ 0 & \text{if } p = 0, q = 0; \end{cases}$$

Then, $d(p, q) \leq \delta(p, q)$ for all points p, q of X; furthermore X is complete with respect to δ but not with respect to d.

Let $T : X \longmapsto X$ such that : $T(p) = p + 1$;T is a contraction with respect to δ, but has no fixed points in X.

Remark 2.1.2.

As a particular case of Theorem 2.1.1 when $d = \delta$, is the Theorem 1.1.2

2.2 Construction of majorant metrics

Theorem 2.2.1.

Let (X, d) be a metric space, $T : X \longmapsto X$ be an application. Consider the power series:

$$\sum_{n=0}^{\infty} \lambda^n d(T^n(x), T^n(y)) \tag{2.1}$$

Suppose that, for some $\lambda > 1$; the series (2.1) converges for every two points x, y in X. Then, set for such a λ :

$$\delta(x, y) = \sum_{n=0}^{\infty} \lambda^n d(T^n(x)), T^n(y))$$

Then, we have:

(i) δ is a metric on X, that bounds the metric d

(ii) T is a contraction with respect to δ

Proof.

a) δ is a metric on X:

1. $\delta(x, y) = 0 \Leftrightarrow x = y$. Indeed:

$* \; x = y \Longrightarrow \delta(x, y) = 0$ is clear.

$* \; \delta(x, y) = 0 \Longrightarrow \sum_0^\infty \lambda^n d(T^n(x), T^n(y)) = 0$

we have $\lambda > 1 \Longrightarrow d(T^n(x), T^n(y)) = 0$

$T^n(x) = T^n(y) = 0$

$\forall n \in N : T^0(x) = T^0(y) \Longrightarrow x = y$

2. $\delta(x,y)=\delta(y,x)$

$$\delta(x,y)=\sum_{0}^{\infty}\lambda^n d(T^n(x),T^n(y)=\sum_{0}^{\infty}\lambda^n d(T^n(y),T^n(x))=\delta(x,y)$$

3. $\delta(x,y)\leq\delta(x,z)+\delta(z+y)$

$$\delta(x,y)=\sum_{0}^{\infty}\lambda^n d(T^n(x),T^n(y))\leq\sum_{0}^{\infty}\lambda^n d(T^n(x),T^n(z))+\sum_{0}^{\infty}\lambda^n d(T^n(z),T^n(y))$$

b) We shall prove now that T is a contraction with respect to δ:

$\exists k\in[0,1[$:

$\delta(x,y)=\sum_0^\infty\lambda^n d(T^n(x),T^n(y))$

$$=d(x,y)+\lambda(d(T(x),T(y))+\lambda(d(T^2(x),T^2(y))+.....+\lambda^{n-1}(d(T^n(x),T^n(y))$$

$$\delta(x,y)=d(x,y)+\lambda\sum_{0}^{\infty}\lambda^n d(T^{n+1}(x),T^{n+1}(y))$$

Since $\sum_0^\infty\lambda^n d(T^{n+1}(x),T^{n+1}(y))$ is convergent, Then:

$$\delta(x,y)=d(x,y)+\lambda\delta(T(x),T(y))$$

$$\delta(T(x),T(y))=\frac{1}{\lambda}(\delta(x,y)-d(x,y))$$

$$\delta(T(x),T(y))=\sum_{0}^{\infty}\lambda^n d(T^{n+1}(x),T^{n+1}(y))\leq\frac{1}{\lambda}\delta(x,y)$$

□

Theorem 2.2.2.

Let $T:X\longrightarrow X$ be an application of the metric space X itself. If a certain iterate of T is a contraction, then the series (2.1) of Theorem (2.2.1) converges for an $\lambda>1$, for any two points x, y in X.

Proof.

By hypothesis we have that there exist $k < 1$ and an integer v such that:

$$d(T^v(x), T^v(y)) \leq kd(x, y)$$

from which, for every integer n:

$$d(T^{vn}(x), T^{vn}(y)) \leq k^n d(x, y)$$

Then:

$$\sum_0^\infty \lambda^n d(T^n(x), T^n(y)) = \sum_0^\infty \lambda^{vn} d(T^n(x), T^n(y)) +$$

$$\sum_0^\infty \lambda^{vn+1} d(T^{vn+1}(x), T^{vn+1}(y)) + \text{.........} +$$

$$\sum_0^\infty \lambda^{vn+n-1} d(T^{vn+n-1}(x), T^{vn+n-1}(y)) \leq \sum_0^\infty \lambda^{vn} k^n d(T(x), T(y)) +$$

$$\lambda \sum_0^\infty \lambda^{vn} K^n d(T(x), T(y)) + \text{.....} +$$

$$\lambda^{n-1} \sum_0^\infty \lambda^{vn} K^n d(T^{n-1}(x), T^{n-1}(y))$$

Then it is enough to take a λ such that : $1 < \lambda^n < \frac{1}{k}$, so that the series converges regardless of the points x, y in X. □

2.3 Some extensions of Maia's (FPT)

In 1968, Maia [1] was the first author who extended the Banach's (FPT) in the metric space setting by using two comparable metrics. Then, there were appeared a series of similar extensions of this result by many authors. In 1977, I.N.Rus [3], stated and proved the following (FPT) of Maia type by replacing one of the conditions in Maia's Theorem 2.1.1.

Theorem 2.3.1. [[3]]

LetX be a nonempty set, d and δ two metrics on X such that (X, d) forms a complete metric space. If the mapping $T : X \to X$ is continuous with respect to d and furthermore:

$$d(Tx, Ty) \leq c\delta(x, y), \text{ for some } c > 0 \text{ and all } x, y \in X; \tag{2.2}$$

$$\delta(Tx, Ty) \leq \alpha\delta(x, y), \text{ for some } 0 < \alpha < 1 \text{ and all } x, y \in X; \tag{2.3}$$

then there is a unique $z \in X$ such that $Tz = z$.

More recently, in 2007, Anton S. Muresan [14] proved the following fixed point theorems for expansion mappings $T : X \to X$ on metric space endowed with tow metrics d and δ and satisfying the following two conditions:

(1) There exists a constant $C > 0$ such that $d(T(x), T(y)) \leq C\delta(x, y), \ \forall x, y \in X$

(2) (X, d) is a complete e metric space.

Theorem 2.3.2. [14]

Suppose that the above conditions (1) and (2) hold. If

(3) f is surjective;

(4) there exists $a, b, c \in \mathbb{R}$, $a < 1, a + b + c > 1$ such that

$$\delta^2(T(x), T(y)) \geq a\delta^2(x, T(x)) + b\delta^2(y, T(y)) + c\delta(x, y), \ \forall x, y \in X, x \neq y$$

(5) There exists $c_1 > 0$ such that $\delta(x, y) \leq c_1 d(x, y)$, for all $x, y \in X$

Then $T:(X,d)\to(X,d)$ is a (WPO). Moreover, if $c>1$ then $T:(X,d)\to(X,d)$ is a (PO).

Theorem 2.3.3. [14]

We suppose that conditions $(1)-(3)$ and (5) in Theorem 2.3.2 hold. If there exists $K\in\left[\frac{2}{3},1\right]$ such that:

$$\delta^2(T(x),T(y))\geqslant K\frac{\delta^2(x,T(x))+\delta(x,T(x))\delta(y,T(y))+\delta^2(y,T(y))}{\delta(x,T(x))+\delta(y,T(y))},$$

for any $x,y\in X, x\neq y$ for which $\delta(x,T(x))+\delta(y,T(y))\neq 0$, then $T:(X,d)\to(X,d)$ is a (WPO).

Further extensions of Maia's (FPT) can be found in [15, 16, 17].

2.4 Application to (BVPs)

2.4.1 Setting of the problem

In this part , we give an application of Theorem 2.3.1 to the following (BVP) considered in [12].

$$x'''(t)+f(t,x(t))=0,\ \ t\in[a,b]. \tag{2.4}$$

$$x(a)=0,x'(a)=0,x(b)=kx(\eta), \tag{2.5}$$

where $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a given function, $a<\eta<b$ and $k\in\mathbb{R}$.

(2.4)-(2.5) was considered firstly in [2], where an existence-uniqueness result has been proved using Banach's approach. The result was stated as follows.

Theorem 2.4.1. [2]

Suppose that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to x on $[a,b]$, namely there is a constant L such that, for every $(t,x,y) \in [a,b] \times \mathbb{R}^2$, we have:

$$\mid f(t,x) - f(t,y) \mid \leqslant L \mid x - y \mid .$$

If $(b-a)^2 \neq k(a-\eta)^2$, $(a-\eta)$ and $(b-a)$ is so small that

$$\frac{(b-a)^3}{3} + \frac{\mid k \mid}{3} \frac{(b-a)^5}{(b-a)^2 - k(\eta-a)^2 \mid} < \frac{1}{L}, \tag{2.6}$$

then there exists a unique solution of (2.4)-(2.5)

2.4.2 Equivalent integral form

We start by constructing the Green's function for (2.4)−(2.5) in the case $k = 0$, namely:

$$\begin{cases} u'''(t) + f(t) = 0, \quad t \in [a,b]. \\ u(a) = u'(a) = u(b) = 0. \end{cases} \tag{2.7}$$

Lemma 2.4.1.

let $f \in \mathcal{C}([a;b])$, (2.7) has a solution given by :

$$u(t) = \int_a^b R(t,s) f(s) ds,$$

where

$$R(t,s) = \begin{cases} \dfrac{(a-t)^2(s-b)^2}{2(a-b)^2} - \dfrac{(s-t)^2}{2}, \ a \leqslant s \leqslant t \leqslant b, \\ \\ \dfrac{(a-t)^2(s-b)^2}{2(a-b)^2}, a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{2.8}$$

Proof.

To prove the desired result, we use the variation of parameters formula. That is, we put:

$$u(t) = c_1 + c_2 t + c_3 t^2 - \frac{1}{2}\int_a^t (s-t)^2 f(s)ds$$

Using boundary conditions in (2.7), we obtain:

$$c_1 = \frac{a^2}{2(a-b)^2}\int_a^b (s-b)^2 f(s)ds$$

$$c_2 = \frac{a}{(a-b)^2}\int_a^b (s-b)^2 f(s)ds$$

$$c_3 = \frac{1}{2(a-b)^2}\int_a^b (s-b)^2 f(s)ds$$

Thus, we get

$$u(t) = \int_a^b \frac{(a-t)^2(s-b)^2}{2(a-b)^2} f(s)ds - \int_a^t \frac{(s-t)^2}{2} f(s)ds$$

$$= \int_a^t \frac{(a-t)^2(s-b)^2}{2(a-b)^2} f(s)ds + \int_t^b \frac{(a-t)^2(s-b)^2}{2(a-b)^2} f(s)ds - \int_a^t \frac{(s-t)^2}{2} f(s)ds$$

$$\int_a^t \left[\frac{(a-t)^2(s-b)^2}{2(a-b)^2} - \frac{(s-t)^2}{2}\right] f(s)ds + \int_t^b \frac{(a-t)^2(s-b)^2}{2(a-b)^2} f(s)ds$$

The uniqueness follows from the fact, that the corresponding homogeneous problem has only the trivial solution. Hence the proof is complete. □

Lemma 2.4.2.

let $f \in \mathcal{C}([a;b])$ and $k(a-\eta)^2 \neq (a-b)^2, (a \neq \eta)$ then (2.4)-(2.5) has a unique solution given by:

$$x(t) = u(t) + \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2} u(\eta),$$

that we can rewrite as

$$x(t) = \int_a^b G(t,s) f(s)ds$$

where

$$G(t,s) = R(t,s) + \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta, s) \tag{2.9}$$

Proof.

Let $x(t) = u(t) + (\lambda_0 + \lambda_1 t + \lambda_2 t^2)u(\eta)$, where $\lambda_0, \lambda_1, \lambda_2$ are constants that will be determined and $u(t) = \int_a^b R(t,s)f(s)ds$. So,

$$x(a) = u(a) + (\lambda_0 + \lambda_1 a + \lambda_2 a^2)u(\eta) = (\lambda_0 + \lambda_1 a + \lambda_2 a^2)u(\eta),$$

$$x'(a) = u'(a) + (\lambda_1 + 2\lambda_2 a)u(\eta) = (\lambda_1 + 2\lambda_2 a)u(\eta)$$

$$x(b) = u(b) + (\lambda_0 + \lambda_1 b + \lambda_2 b^2)u(\eta) = (\lambda_0 + \lambda_1 b + \lambda_2 b^2)u(\eta),$$

$$x(\eta) = u(\eta) + (\lambda_0 + \lambda_1 \eta + \lambda_2 \eta^2)u(\eta) = (\lambda_0 + \lambda_1 \eta + \lambda_2 \eta^2 + 1)u(\eta),$$

We get

$(\lambda_0 + \lambda_1 a + 2\lambda_2 a^2)u(\eta) = 0$

$(\lambda_1 + 2\lambda_2 a)u(\eta) = 0$

$(\lambda_0 + \lambda_1 b + \lambda_2 b^2)u(\eta) = (\lambda_0 + \lambda_1 \eta + \lambda_2 \eta^2 + 1)ku(\eta)$

or, equivalently:

$$\begin{cases} \lambda_0 + \lambda_1 a + 2\lambda_2 a^2 = 0 \\ \lambda_1 + 2\lambda_2 a) = 0 \\ (1-k)\lambda_0 + (b - k\eta)\lambda_1 + (b^2 - k\eta^2)\lambda_2 = k \end{cases}$$

Solving the above system, one gets:

$$\lambda_0 = \frac{a^2 k}{(a-b)^2 - k(a-\eta)^2}$$

$$\lambda_1 = \frac{-2ak}{(a-b)^2 - k(a-\eta)^2}$$

$$\lambda_2 = \frac{k}{(a-b)^2 - k(a-\eta)^2}$$

Therefore

$$x(t) = u(t) + \left(\frac{a^2k}{(a-b)^2 - k(a-\eta)^2} - \frac{2akt}{(a-b)^2 - k(a-\eta)^2} + \frac{kt^2}{(a-b)^2 - k(a-\eta)^2} \right) u(\eta)$$

$$= u(t) + \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2} u(\eta)$$

Let us prove the uniqueness. Assume that $y(t)$ is also a solution of (2.4)-(2.5), that is

$$y'''(t) + f(t) = 0, t \in [a,b]$$

$$y(a) = y'(a) = 0, y(b) = ky(\eta)$$

Let $z(t) = y(t) - x(t)$; $t \in [a,b]$. Thus we have

$$z'''(t) = y'''(t) - x'''(t) = f(t) - f(t) = 0, t \in [a,b]$$

Therefore $z(t) = c_1t^2 + c_2t + c_3$, where$c_1, c_2$ and c_3 are constants that we will determine. We have

$$\begin{aligned} z(a) &= y(a) - x(a) = 0 \\ z'(a) &= y'(a) - x'(a) = 0 \\ z(b) &= y(b) - x(b) = ky(\eta) - kx(\eta) = k(y(\eta) - x(\eta)) = kz(\eta) \end{aligned}$$

or

$$z(a) = c_1a^2 + c_2a + c_3$$

$$z'(a) = 2c_1a + c_2 = 0$$

$z(b) = c_1b^2 + c_2b + c_3 = k(z(t) = c_1\eta^2 + c_2\eta + c_3) = kz(\eta)$

We get homogeneous system

$$\begin{cases} c_1a^2 + c_2a + c_3 = 0 \\ 2c_1a + c_2 = 0 \\ (b^2 - k\eta^2)c_1 + (b - k\eta)c_2 + (1-k)c_3 = 0 \end{cases}$$

with determinant

$$\begin{vmatrix} a^2 & a & 1 \\ 2a & 1 & 0 \\ (b^2 - k\eta^2) & (b - k\eta) & (1-k) \end{vmatrix} = k(a-\eta)^2 - (a-b)^2 \neq 0$$

So the homogeneous system has only the trivial solution and hence $z(t) = 0, t \in [a,b]$, or $x(t) = y(t), t \in [a,b]$. The proof is complete □

2.4.3 Existence and uniqueness result

Let us now state and prove the result of existence and uniqueness of solutions to (2.4)-(2.5), where we employ Theorem 2.3.1.

Theorem 2.4.2.

Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous, let $f(t,0) \neq 0$ for all $t \in [a,b]$ and let L be a non-negative constant such that

$$\mid f(t,v) - f(t,v) \mid \leq L \mid u - v \mid, \text{ for all } (t,u),(t,v) \in [a,b] \times \mathbb{R} \tag{2.10}$$

If $k(\eta - a)^2 \neq (b-a)^2$ with $a < \eta < b$ and there are constants $p > 1$ and $q > 1$ such that

$1/p + 1/q = 1$ with

$$L\left(\int_a^b \left(\int_a^b \mid G(t,s) \mid^q ds\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} < 1 \tag{2.11}$$

then the (BVP) (2.4)-(2.5) has a unique (non trivial) solution in $C^3([a,b])$.

Proof.

Consider the operator $T : C([a,b]) \to C([a,b])$ defined by

$$(Tx)(t) = \int_a^b G(t,s)f(s,x(s))ds, \;\; t \in [a,b]$$

Where $G(t,s)$ is given by (2.9).

To establish the existence and uniqueness to $Tx = x$, we show that the conditions of Theorem 2.3.1 are satisfied.

Consider the pair $(X,d) = (\mathcal{C}([a,b]), d)$ which forms a complete metric space. In addition, consider the metric δ on X with $p > 1$, where:

$$d(x,y) = \sup_{t\in[a,b]} \mid x(t) - y(t) \mid \tag{2.12}$$

$$\delta(x,y) = \left(\int_a^b \mid x(t) - y(t) \mid^p dt\right)^{\frac{1}{p}} \;\; , p > 1 \tag{2.13}$$

For $x, y \in C([a,b])$ and $t \in [a,b]$, we have:

$$\mid (Tx)(t) - (Ty)(t) \mid \leq \int_a^b G(t,s) \mid f(s,x(s)) - f(s,y(s)) \mid ds$$

,

$$\leq \int_a^b \mid G(t,s) \mid L \mid x(s) - y(s) \mid ds \leq \left(\int_a^b \mid G(t,s) \mid^q ds\right)^{\frac{1}{q}}$$

$$\times L\left(\int_a^b \mid x(s) - y(s) \mid^p ds\right)^{1/p} \leq L \sup_{t\in[a,b]} \left(\int_a^b \mid G(t,s) \mid^q ds\right)^{1/p} \delta(x,y) \tag{2.14}$$

Where, we have used the hypothesis (2.10) together with Holders inequality.

Thus, defining:

$$c = L \sup_{t\in[a,b]} \left(\int_a^b \mid G(t,s) \mid^q ds \right)^{1/p}$$

we see that the inequality 2.14, can be rewritten as:

$$d(Tx,Ty) \leq c\delta(x,y), \quad \text{for some } \; c > 0 \; \text{ and all } \; x, y \in \mathcal{C}([a,b]) \tag{2.15}$$

Which means that (2.2) of Theorem 2.3.1 holds.

Let us check now the continuity of the operator T:

Let $x, y \in \mathcal{C}([a,b]$. For every $\varepsilon > 0$, we can choose $\eta = \varepsilon / c(b-a)^{\frac{1}{p}}$ so that, if $d(x,y) < \eta$, then from (2.15) together with the definition of δ given by (2.13), we have:

$$d(Tx,Ty) \leq c\delta(x,y) = c \left(\int_a^b \Big| x(t) - y(t) \Big|^p dt \right)^{\frac{1}{p}} \leq c\,(b-a)^{\frac{1}{p}} d(x,y) < \varepsilon$$

Hence T is continuous on $C([a,b])$ with respect to the metric d.

Finally, we show that T is a contraction on $C([a,b])$ with respect to the metric δ, that is, the inequality (2.3) in Theorem 2.3.1 holds. Indeed:

From (2.14), for each $x, y \in C([a,b])$ we have

$$\left(\int_a^b \mid (Tx)(t) - (Ty)(t) \mid^p dt \right)^{\frac{1}{p}} \leq L \left(\int_a^b \left(\int_a^b \mid G(t,s) \mid^q ds \right)^{p/q} dt \right)^{\frac{1}{p}} \delta(x,y)$$

So we obtain:

$$\delta(Tx,Ty) \leq L \left(\int_a^b \left(\int_a^b \mid G(t,s) \mid^q ds \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} \delta(x,y)$$

From the assumption (2.11), we thus get

$$\delta(Tx,Ty) \leq \alpha\delta(x,y) \text{ for some } \alpha < 1 \; \text{ and all } x, y \in C([a,b])$$

. $\square$

The following corollary results immediately from Theorem 2.4.2, by taking $p = 2$ and $q = 2$.

Corollary 2.4.1.

Let $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, let $f(t, 0) \neq 0$ for all $t \in [a, b]$ and let L be a non-negative constant such that

$$\mid f(t,u) - f(t,v) \mid \leq L \mid u - v \mid \quad \text{for all } (t,u),(t,v) \in [a,b] \times \mathbb{R} \tag{2.16}$$

If $k(\eta, a)^2 \neq (b-a)^2$ with $a < \eta < b$ and

$$L \left(\int_a^b \left(\int_a^b \mid G(t,s) \mid^2 ds \right) ds \right)^{\frac{1}{2}} < 1 \tag{2.17}$$

then the (BVP) (2.4)-(2.5) has a unique (nontrivial) solution in $\mathcal{C}^3([a, b])$

2.4.4 Comparison with Banach's approach

In this part, we discuss and compare the existence-uniqueness results for (2.4)-(2.5), acquired in Theorem 2.4.1 (using Banach's approach) and Theorem 2.4.2 (using Maia's type (FPT)).

Let us consider the case $[a, b] = [0, 1]$.

Condition (2.6) in Theorem 2.4.1, becomes in this case:

$$\frac{L}{3} \left(1 + \frac{\mid k \mid}{\mid 1 - k\eta^2 \mid} \right) < 1 \tag{2.18}$$

Let us focus on the limit on the size of the Lipschitz constant L governed by (2.18).

Given k and η, for sufficiently small L, the inequality (2.18) will hold. Let us illustrate how condition (2.11) is sharpened through Theorem 2.4.1 by discussing an example. Consider $k = 1, \eta = \frac{1}{2}$. In this situation, Smirnovs condition (2.18) becomes

$$L < \frac{9}{7} \tag{2.19}$$

Whereas the left hand side of (2.17) can be evaluated with the given particular values of k, a, b and η, which leads to

$$\int_0^1 G(t,s)^2 ds = \frac{4}{27}t^7 - \frac{5}{18}t^6 + \frac{2}{15}t^5 - \frac{4}{45}t^4(t-1)^5$$

and so

$$\int_0^1 \int_0^1 G(t,s)^2 dsdt = \frac{16}{14175}$$

Thus, in this special case, (2.17) takes the form

$$L\frac{4\sqrt{7}}{315} < 1 \tag{2.20}$$

Condition (2.20) will be satisfied, for example, if $L \leqslant 29$: For an $f$such as

$$f(t,x) = 25\sin \text{x},\, + (t+1)^2$$

Let us check (2.10) :

$$\Big|f(t,x) - f(t,y)\Big| = \Big|25\sin x - 25\sin y\Big| \leqslant \Big|25(\sin x - \sin y)\Big| \leqslant 25\Big|x - y\Big|$$

The smallest constant L that can be chosen so that f satisfies (2.10) on $[0,1]\times\mathbb{R}$ is $L = 25$. The value $L = 25$ does not satisfy Smirnovs condition (2.19), but it does satisfy (2.20).

The above discussion illustrate that Theorem 2.4.2 and Corollary 2.4.1 cover a wider class of problems than Theorem 2.4.1.

Chapter 3

Vectorial forms of (FPT) with tow metrics and application

In the first part of this chapter, we give some version of Maia's (FPT) in the setting of vector valued metric spaces. An application to Fredholm integral equations systems, will be the focus of the second part of this chapter.

3.1 Perov's fixed point theorem

We start by stating the first version of fixed point theorem in spaces endowed with vector-valued metrics, which comes back to Perov [13].

3.1.1 Generalized metric space

In this section, we will define the generalized metric spaces (that is, the spaces endowed with vector-valued metrics) and and we precise some related concepts.

Let $x, y \in \mathbb{R}^n : x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, in what follows, we use the following notations and concepts:

$x \leq y$ if and only if $x_i \leq y_i$ for every $i = 1, ..., n$.

$| x |= (| x_1 |, ..., | x_n |)$

$\max(x, y) = \max\left(\max(x_1, y_1), ..., \max(x_n, y_n)\right)$.

If $c \in \mathbb{R}$, then $x \leq c$ means that $x_i \leq c$ for each $i = 1, ..., n$, for $x \in \mathbb{R}^n$, $(x)_i = x_i$, $i = 1, ..., n$.

Definition 3.1.1. [11]

Let X be a nonempty set. A generalized metric on X is an application $D : X \times X \to \mathbb{R}^n$ such that:

a) $D(x, y) = D(y, x), \ \forall x, y \in \mathbb{R}^n$

b) $D(x, y) \geq O_n$ and $D(x, y) = O_n$ if and only if $x = y$, where $O_n = (0, ..., 0) \in \mathbb{R}^n$.

c) $D(x, y) \leq D(x, z) + D(z, y), \ \forall x, y, z \in X$.

Note that for all $i \in \{1, ..., n\}$ $(D(x, y))_i = D_i(x, y)$ is a metric space in X.

We call the pair (X, D) a generalized metric space.

For $r = (r_1, r_2, ..., r_n) \in \mathbb{R}^n_+$ We will note by

$$B(x_0, r) = \{x \in X, \ : D(x_0, x) < r\},$$

the open ball with center x_0 and a radius r.

$$\bar{B}(x_0, r) = \{x \in X, \ : D(x_0, x) \leq r\}.$$

The closed ball with center x_0 and radius r, as $r = (r_1, r_2, ..., r_n) > 0$, $r_i > 0$, $i = 1, ..., n$.

Definition 3.1.2. [11]

Let (X, D) be a generalized metric space, a subset $A \subseteq X$ is said to be open if for all $x_0 \in A$, there exists $r \in \mathbb{R}^n_+$ with $r > 0$ such that $B(x_0, r) \subseteq A$.

Any open ball is an open set and the collection of all open balls of X generates the generalized metric topology

Remark 3.1.1. [11]

For the generalized metric space (X, D) the following properties hold true:

1. Any convergent sequence is a Cauchy sequence.
2. Any Cauchy sequence is bounded.
3. If a Cauchy sequence (x_p) has a sub-sequence (x_{p_k})

$$x_{p_k} \to x, \text{when } p_k \to \infty,$$

then

$$x_p \to x, \text{when } p \to \infty.$$

3.1.2 Perov's theorem in generalized metric spaces

The Russian mathematician A.I Perov defined the generalized metric space by introducing a vector valued metric in $\mathbb{R}^n$. Then, this concept of metric spaces allowed him to define a new class of maps called "Perov's contractions" which satisfy a contraction condition similar to that of Banach, but with a matrix $M \in \mathcal{M}_n(\mathbb{R}^+)$ (with non-negative inputs) instead of a constant q [9].

Definition 3.1.3. [10]

Let (X, D) be a generalized metric space. An application $T : X \to X$ is said to be contraction if there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ which converges to zero, such that:

$$D(T(u), T(v)) \leqslant MD(u, v), \forall u, v \in X. \tag{3.1}$$

Theorem 3.1.1. (Perov) [10]

Let (X, D) be a complete generalized metric space and $T : X \longrightarrow X$ a contraction application with the Lipschitz matrix M. Then T has a unique fixed point u^* , and for all $u_0 \in X$ we have:

$$D(T^k(u_0), u^*) \leq M^k(I - M)^{-1}D(u_0, T(u_0)), \tag{3.2}$$

for all $k \in \mathbb{N}$.

3.2 Perov's type (FPT) with tow metrics

Theorem 3.2.1.

Let X be a nonempty set, endowed with two vector-valued metrics, D, $\Delta : X \times X \longrightarrow \mathbb{R}^m_+$. Let $T : X \longrightarrow X$ be an operator. We assume that:

1. There exists a matrix $C \in \mathcal{M}_m(\mathbb{R}_+)$ such that: $D(T(x), T(y)) \leq C\,\Delta(x, y), \forall x, y \in X$;
2. (X, D) is a complete generalized metric space .
3. $T : (X, D) \longrightarrow (X, D)$ is continuous;
4. $T : (X, \Delta) \longrightarrow (X, \Delta)$ is an A-contraction, i.e. there exists a matrix $A \in \mathcal{M}_m(\mathbb{R}_+)$ converging to zero, such that:

$$\Delta(T(x), T(y)) \leq \mathrm{A}\,\Delta(x, y) \quad , \ ,\forall x, y \in X.$$

Then T is (PO) in $(X, \to^D)$ and $(X, \to^\Delta)$.

Proof.

Let $x_0 \in X$. By (4), the sequence of successive approximations $T^n(x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, Δ). Indeed, $\forall n, p \in \mathbb{N}$ we have:

$$\Delta(T^n(x_0), T^{n+p}(x_0) \leq \sum_{k=n}^{n+p-1} \Delta(T^k(x_0), T^{k+1}(x_0)) \leq \sum_{k=n}^{n+p-1} \Delta(x_0, T(x_0))$$

$$\leq A^n(I_m - A)^{-1}\Delta(x_0, T(x_0)) \longrightarrow 0 \, \text{as} \, n, p \longrightarrow \infty$$

By (1), we get that $T^n(x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, D).

By (2), there exists $x^* \in X$, such that $T^n(x_0) \longrightarrow^D x^*$ as $n \to \infty$.

By (3), it follows that $x^* \in F_T$, since

$$D(x^*, T(x^*)) \leq D(x^*, T(x_0)) + D(T^n(x_0), T(x^*))$$

$$= D(x^*, T^n(x_0)) + D(T(T^{n-1}(x_0), T(x^*)) \longrightarrow D(x^*, x^*) + D(T(x^*), T(x^*)), \, \text{as} \, n \longrightarrow \infty$$

By (4), we obtain the uniqueness of the fixed point x^*. Hence T is (PO) in $(X, \to^D)$.

Next we show that T is (PO) in $(X, \to^\Delta)$.

For any $x_0 \in X$, since $x^* \in F_T$, by (4) we have

$$\Delta(x^*, T^n(x_0)) = \Delta(T^n(x^*), T^n(x^*)) \leq A^n\Delta(x^*, x_0) \to 0 \, \text{as} \, n \to \infty$$

which implies that Since $T^n(x_0) \longrightarrow^\Delta x^*$ is the unique fixed point, we get that T is (PO) in $(X, \to^\Delta)$. □

Notice that, in the proof of the above result, Perov's Theorem can not be applied for $T : (X, \delta) \to (X, \Delta)$, because the lack of completeness of the generalized metric space (X, Δ).

The following Corollary results from Theorem 3.2.1.

Corollary 3.2.1.

Let (X, Δ) be a generalized metric space, where $\Delta : X \times X \to \mathbb{R}^m_+$.

Let $T : X \to X$ be an operator. We assume that:

1. $F_T \neq \emptyset$.

2. There exists a matrix $A \in \mathcal{M}_m(\mathbb{R}_+)$ which converges to zero, such that:

$$D(T(x), T(y)) \leq AD(x, y),\, \forall x, y \in X.$$

Then T is (PO) in the L-space $(X, \to^{\Delta})$.

Proof.

In view of 1, let x^* be a fixed point of T.

Now, using 2, we have for every $x_0 \in X$ and $n \in \mathbb{N}$:

$$\Delta(x^*, (T^n(x_0))) = \Delta(T^n(x^*), T^n(x_0)) \leq A^n \Delta(x^*, x_0) \longrightarrow 0 \,\text{as}\, n \longrightarrow \infty.$$

Which means that T is (WPO) in $(X, \longrightarrow^{\Delta})$.

Again, using 2, we prove the uniqueness of the fixed point of T. Which implies that T is (PO) in $(X, \longrightarrow^{\Delta})$ and completes the proof. □

Another fixed point result of Maia type in vectorial form is the following.

Theorem 3.2.2.

Let X be a nonempty set, endowed with two vector-valued metrics $D, \Delta : X \times X \longrightarrow \mathbb{R}^m_+$. Let $T : X \to X$ be an operator. We assume that:

1. $F_T \neq \emptyset$.

2. There exists a matrix $C \in \mathcal{M}_m(\mathbb{R}_+)$ such that $D(T(x), T(y)) \leq C\Delta(x, y), \forall x, y \in X$

3. $T : (X, \Delta) \longrightarrow (X, \Delta)$ is an A-contraction, i.e. there exists a matrix $A \in \mathcal{M}_m(\mathbb{R}_+)$ convergent to zero, such that

$$\Delta(T(x), T(y)) \leq A\, \Delta(x, y), \;\; \forall x, y \in X$$

Then T is (PO) in the L-spaces $(X, \xrightarrow{D})$ and $(X, \xrightarrow{\Delta})$.

Proof.

We deduce immediately from Corollary 3.2.1, that T is (PO) in $(X, \xrightarrow{\Delta})$, so it remains to prove that T is (PO) in $(X, \xrightarrow{D})$. Indeed:

Let $F_T = \{x^*\}$. For any $x_0 \in X$ and $n \in \mathbb{N}$, we have, using conditions 2. and 3.:

$$D(x^*, T^{n+1}(x_0)) = D(T^{n+1}(x^*), T^{n+1}(x_0))$$

$$\leq C\Delta(T^n(x^*), T^n(x_0))$$

$$\leq CA^n\Delta(x^*), x_0) \longrightarrow 0 \,\text{as}\, n \longrightarrow \infty$$

Hence T is (PO) in $(X, \xrightarrow{D})$ and the proof is complete. □

3.3 Applications to integral equations systems

Inspired by [13] we consider the following coupled system of Fredholm integral equations

$$\begin{cases} x_1(t) = g_1(t) + \displaystyle\int_a^b F_1(t, s, x_1(s), x_2(s))ds \\ x_2(t) = g_2(t) + \displaystyle\int_a^b F_2(t, s, x_1(s), x_2(s))ds \end{cases} \tag{3.3}$$

where $g_1, g_2 \in \mathcal{C}([a, b]), F_1, F_2 \in \mathcal{C}([a, b] \times [a, b] \times \mathbb{R}^2, \mathbb{R})$ are give functions

Remark 3.3.1. The systems (1.1) and (1.2) in [13] are special cases of (3.3) let $X = C([a,b]) \times C([a,b])$ and :

$$T_i : X \longrightarrow C([a,b]),\ \ (i = 1, 2)$$

defined by :

$$T_i(x_1(t), x_2(t)) = g_i(t) + \int_a^b F_i(t, s, x_1(s), x_2(s))ds$$

Then, (3.3) is equivalent to:

$$(x_1(t), x_2(t)) = T(x_1(t), x_2(t))$$

where

$$T : X \to X$$

$$(x_1, x_2) = (T_1(x_1, x_2), T_2(x_1, x_2)) \tag{3.4}$$

Thus, the solutions of (3.3) are the fixed points of T.

In what follows, we give sufficient conditions for the existence and uniqueness of the solution for (3.3) using Theorem 2.1.1.

3.3.1 Hypotheses and important Lemmas

Let us consider the following assumptions

(H_1) : F_1 is a continuous function and $\exists L_1, L_2 > 0$ such that:

$$\mid F_1(t, s, \xi_1, \eta_1) - F_1(t, s, \xi_2, \eta_2) \mid \leqslant L_1 \mid \xi_1 - \xi_2 \mid + L_2 \mid \eta_1 - \eta_2 \mid$$

Whenever the left hand side is defined.

(H_2) : F_2 is a continuous function and $\exists M_1, M_2 > 0$ such that:

$$\mid F_2(t,s,\xi_1,\eta_1) - F_2(t,s,\xi_2,\eta_2) \mid \leqslant M_1 \mid \xi_1 - \xi_2 \mid + M_2 \mid \eta_1 - \eta_2 \mid$$

Let us consider the scalar metrics d and δ defined on $\mathcal{C}([a,b])$ by (2.12) and (2.13) respectively. Let D and Δ be the generalized metrics defined on X by:

$$D((x_1,x_2),(y_1,y_2)) = \begin{pmatrix} d(x_1,y_1) \\ d(x_2,y_2) \end{pmatrix} \tag{3.5}$$

and

$$\Delta((x_1,x_2),(y_1,y_2)) = \begin{pmatrix} \delta(x_1,y_1) \\ \delta(x_2,y_2) \end{pmatrix} \tag{3.6}$$

Lemma 3.3.1.

Let the hypotheses (H_1) and (H_2) hold true. Then, for every $t \in [a,b]$ and $(x_1,x_2),(y_1,y_2) \in X$ we have the following inequalities:

$$\left| T_1(x_1,x_2)(t) - T_1(y_1,y_2)(t) \right| \leqslant (b-a)^{\frac{1}{q}} \Big[L_1\delta(x_1,y_1) + L_2\delta(x_2,y_2) \Big]$$

$$\left| T_2(x_1,x_2)(t) - T_2(y_1,y_2)(t) \right| \leqslant (b-a)^{\frac{1}{q}} \Big[M_1\delta(x_1,x_1) + M_2\delta(x_2,x_2) \Big]$$

Proof.

We consider the operator $T : X \longrightarrow X$,defined by

$$T(x)(t) = \begin{pmatrix} T_1(x)(t) \\ T_2(x)(t) \end{pmatrix}$$

$$\begin{pmatrix} T_1(x_1(t),x_2(t)) = g_1(t) + \displaystyle\int_a^b F_1(t,s,x_1(s),x_2(s))ds \\ T_2(x_1(t),x_2(t)) = g_2(t) + \displaystyle\int_a^b F_2(t,s,x_1(s),x_2(s))ds \end{pmatrix}$$

for all $x=(x_1;x_2)\in X, y=(y_1;y_2)\in X$ we have :

(1)

$$\Big|T_1(x_1,x_2)(t)-T_1(y_1,y_2)(t)\Big| = \Big|\int_a^b F_1(t,s,x_1(s),x_2(s))ds - \int_a^b F_1(t,s,y_1(s),y_2(s))ds\Big|$$

using (H_1) we get

$$\Big|T_1(x_1,x_2)(t)-T_1(y_1,y_2)(t)\Big| \leqslant \int_a^b \Big(L_1\Big|x_1(s)-y_1(s)\Big| + L_2\Big|x_2(s)-y_2(s)\Big|\Big)\,ds$$

holder's inequality gives :

$$\Big|T_1(x_1,x_2)(t)-T_1(y_1,y_2)(t)\Big|$$

$$\leqslant \Big(\int_a^b L_1^q ds\Big)^{\frac{1}{q}}\Big(\int_a^b \Big|x_1(s)-y_1(s)\Big|^p ds\Big)^{\frac{1}{p}} + \Big(\int_a^b L_2^q ds\Big)^{\frac{1}{q}}\Big(\int_a^b \Big|x_2(s)-y_2(s)\Big|^p ds\Big)^{\frac{1}{p}}$$

$$= L_1(b-a)^{\frac{1}{q}}\delta(x_1,y_1) + L_2(b-a)^{\frac{1}{q}}\delta(x_2,y_2)$$

(2)

$$\Big|T_2(x_1,x_2)(t)-T_2(y_1,y_2)(t)\Big| = \Big|\int_a^b F_2(t,s,x_1(s),x_2(s))ds - \int_a^b F_2(t,s,y_1(s),y_2(s))ds\Big|$$

using (H_2) we get

$$\Big|T_2(x_1,x_2)(t)-T_2(y_1,y_2)(t)\Big| \leqslant \int_a^b \Big(M_1\Big|x_1(s)-y_1(s)\Big| + M_2\Big|x_2(s)-y_2(s)\Big|\Big)\,ds$$

holder's inequality gives :

$$\left|T_2(x_1,x_2)(t)-T_2(y_1,y_2)(t)\right|$$

$$\leqslant \left(\int_a^b M_1^q ds\right)^{\frac{1}{q}} \left(\int_a^b \left|x_1(s)-y_1(s)\right|^p ds\right)^{\frac{1}{p}} + \left(\int_a^b M_2^q ds\right)^{\frac{1}{q}} \left(\int_a^b \left|x_2(s)-y_2(s)\right|^p ds\right)^{\frac{1}{p}}$$

$$= M_1(b-a)^{\frac{1}{q}}\delta(x_1,y_1) + M_2(b-a)^{\frac{1}{q}}\delta(x_2,y_2)$$

□

Lemma 3.3.2.

The generalized metrics D and Δ satisfy the following inequalities:

$$\Delta\Big(T(x_1,x_2),T(y_1,y_2)\Big) \leq A\,\Delta\Big((x_1,y_1),(x_2,y_2)\Big),\quad \forall (x_1,x_2),\,(y_1,y_2)\in X \tag{3.7}$$

$$D\Big(T(x_1,x_2),T(y_1,y_2)\Big) \leq C\,\Delta\Big((x_1,y_1),(x_2,y_2)\Big),\quad \forall (x_1,x_2),\,(y_1,y_2)\in X \tag{3.8}$$

where, C and A are the square matrices given by

$$A = \begin{pmatrix} (b-a)\,L_1 & (b-a)\,L_2 \\ (b-a)\,M_1 & (b-a)\,M_2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} (b-a)^{\frac{1}{q}}\,L_1 & (b-a)^{\frac{1}{q}}\,L_2 \\ (b-a)^{\frac{1}{q}}\,M_1 & (b-a)^{\frac{1}{q}}\,M_2 \end{pmatrix}$$

Proof.

For all $x=(x_1,x_2), y=(y_1,y_2)\in X$, we have:

$$\left|T(x_1,x_2)-T(y_1,y_2)\right| = \Big(\left|T_1(x_1,x_2)-T_1(y_1,y_2)\right| ; \left|T_2(x_1,x_2)-T_2(y_1,y_2)\right|\Big)$$

From the Lemme 3.3.1, we have for every $s \in [a, b]$:

$$\Big|T_1(x_1, x_2)(s) - T_1(y_1, y_2)(s)\Big| \leqslant \Big|L_1(b-a)^{\frac{1}{q}}\delta(x_1, y_2) + L_2(b-a)^{\frac{1}{q}}\delta(x_1, y_1)\Big|$$

Now, integrating the p^{th} power of the previous inequality between a and b, and then raising it to the $\frac{1}{p}$ power, we get

$$\left(\int_a^b \Big|T_1(x_1, x_2)(s) - T_1(y_1, y_2)(s)\Big|^p ds\right)^{\frac{1}{p}} \leqslant \left(\int_a^b \Big|L_1(b-a)^{\frac{1}{q}}\delta(x_1, y_2)\Big|^p ds\right)^{\frac{1}{p}}$$

$$+\left(\int_a^b \Big|L_2(b-a)^{\frac{1}{q}}\delta(x_1, y_2)\Big|^p ds\right)^{\frac{1}{p}}$$

Recall that $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\left(\int_a^b \Big|T_1(x_1, x_2)(s) - T_1(y_1, y_2)(s)\Big|^p ds\right)^{\frac{1}{p}} \leq L_1(b-a)\delta(x_1, y_1) + L_2(b-a)\delta(x_2, y_2) \quad (3.9)$$

Using the same arguments, we prove that:

$$\left(\int_a^b \Big|T_2(x_1, x_2)(s) - T_2(y_1, y_2)(s)\Big|^p ds\right)^{\frac{1}{p}} \leq M_1(b-a)\delta(x_1, y_1) + M_2(b-a)\delta(x_2, y_2) \quad (3.10)$$

Now, taking into account the definition of the generalized metric Δ given by (3.6), it can be easily seen that (3.7), follows immediately from (3.9) and (3.10).

Again, from the Lemme 3.3.1, we have for every $t \in [a, b]$:

$$\Big|T_1(x_1, x_2)(t) - T_1(y_1, y_2)(t)\Big| \leqslant \Big|L_1(b-a)^{\frac{1}{q}}\delta(x_1, y_1) + L_2(b-a)^{\frac{1}{q}}\delta(x_2, y_2)\Big|$$

Now, taking the supremum on $[a,b]$ in the above inequality, we get:

$$\sup_{t\in[a,b]}\left|T_1(x_1,x_2)(t)-T_1(y_1,y_2)(t)\right|\leqslant L_1(b-a)^{\frac{1}{q}}\delta(x_1,y_1)+L_2(b-a)^{\frac{1}{q}}\delta(x_2,y_2)\qquad(3.11)$$

Similarly, we easily obtain:

$$\sup_{t\in[a,b]}\left|T_2(x_1,x_2)(t)-T_2(y_1,y_2)(t)\right|\leqslant M_1(b-a)^{\frac{1}{q}}\delta(x_1,y_1)+M_2(b-a)^{\frac{1}{q}}\delta(x_2,y_2)\qquad(3.12)$$

Taking into account the definition of the generalized metrics D and Δ given by (3.5) and (3.6) respectively, (3.8), follows immediately from (3.11) and (3.12).

The proof is complete. ☐

3.3.2 Existence-uniqueness result

We are now ready to state and prove our existence-uniqueness result for (3.3).

Theorem 3.3.1.

Let $(H_1)-(H_2)$ be satisfied, then (3.3) has a unique solution in X, provided that

$$\lambda_1=\frac{(b-a)}{2}\left((L_1+M_2)-\sqrt{(L_1+M_2)^2-4(L_1M_2-L_2M_1)}\right)<1\qquad(3.13)$$

$$\lambda_2=\frac{(b-a)}{2}\left((L_1+M_2)+\sqrt{(L_1+M_2)^2-4(L_1M_2-L_2M_1)}\right)<1\qquad(3.14)$$

Proof.

The proof is based on Theorem 3.2.1.

Let us recall that (X,D) is a complete generalized metric space. So, condition 2 in Theorem 3.2.1 is satisfied.

According to Lemma 3.3.2, it suffices to show that the matrix A in (3.7) is convergent to zero, to conclude that conditions 1 and 4 in Theorem 3.2.1 are fulfilled.

It is not hard to see that the matix A in (3.7) has the following eigenvalues:

$$\lambda_{1,2} = \frac{(b-a)}{2}\left(\left|(L_1+M_2) \pm \sqrt{(L_1+M_2)^2 - 4(L_1M_2 - L_2M_1)}\right|\right)$$

Now, it follows from Property 1.3.1, that conditions (3.13)-(3.14) lead to the fact that A converges to zero.

It remains now to check the continuity of the operator T with respect to the generalized metric D. Indeed:

For every $(x_1,x_2),(y_1,y_2) \in X$, we have from (3.11):

$$d\left(T_1(x_1,x_2),T_1(y_1,y_2)\right) \leqslant L_1(b-a)^{\frac{1}{q}}\delta(x_1,y_1) + L_2(b-a)^{\frac{1}{q}}\delta(x_2,y_2)$$

Given that:

$$\delta(x,y) = \left(\int_a^b \left|x(t)-y(t)\right|^p\, dt\right)^{\frac{1}{p}} \leq (b-a)^{\frac{1}{p}} d(x,y), \quad \forall x,y \in X,$$

it follows that:

$$d\left(T_1(x_1,x_2),T_1(y_1,y_2)\right) \leqslant L_1(b-a)\, d(x_1,y_1) + L_2(b-a)\, d(x_2,y_2)$$

Similarly, we can obtain:

$$d\left(T_2(x_1,x_2),T_2(y_1,y_2)\right) \leqslant M_1(b-a)\, d(x_1,y_1) + M_2(b-a)\, d(x_2,y_2)$$

So, for every $\epsilon > 0$, we can choose $\eta < \min\left\{\frac{\epsilon}{(b-a)(L_1+L_2)}, \frac{\epsilon}{(b-a)(M_1+M_2)}\right\}$, so that, if

$D((x_1,x_2),(y_1,y_2)) < \eta$, that is $d((x_1,y_1)) < \eta$ and $d((x_2,y_2)) < \eta$,

then, it follows that:

$d\left(T_1(x_1,x_2),T_1(y_1,y_2)\right) < \epsilon$ and $d\left(T_2(x_1,x_2),T_2(y_1,y_2)\right) < \epsilon$

or, equivalently:

$D\left(T(x_1,x_2),T(y_1,y_2)\right) < \epsilon$

Hence, T is continuous with respect to D and the proof is complete. □

3.3.3 Discussion, example and remarks

We begin this part by an example illustrating the validity of our theoretical result.

Example 3.3.1.

Let us consider the following system of integral equations:

$$\begin{cases} x_1(t) = 2 + \dfrac{1}{4}\displaystyle\int_0^1 (t+1) + \dfrac{1}{1+x_1(s)+x_2(s)}\,ds \\ \\ x_2(t) = 2 + \dfrac{1}{2}\displaystyle\int_0^1 t^2 + sin(x_1(s)+x_2(s))\,ds \end{cases} \tag{3.15}$$

(3.15) is identified to (3.3) with:

$$[a,b] = [0,1], \;\; g_1(t) = g_2(t) = 2,$$

$$F_1(t,s,\xi_1,\eta_1) = \frac{1}{4}\left(t+1+\frac{1}{(1+\xi_1+\eta_1)}\right)$$

and

$$F_2(t,s,\xi_2,\eta_2) = \frac{1}{2}\left(t^2+\sin(\xi_2+\eta_2)\right)$$

We have:

$$
\begin{aligned}
\Big|F_1(t,s,\xi_1,\eta_1) - F_1(t,s,\xi_2,\eta_2)\Big| &= \Big|\frac{1}{4}\frac{1}{(1+\xi_1+\eta_1)} - \frac{1}{4}\frac{1}{(1+\xi_2+\eta_2)}\Big| \\
&= \Big|\frac{1}{4}\frac{1+\xi_2+\eta_2-1-\xi_1-\eta_1}{(1+\xi_1+\eta_1)(1+\xi_2+\eta_2)}\Big| \\
&\leq \Big|\frac{1}{4}\frac{1}{(1+\xi_1+\eta_1)(1+\xi_2+\eta_2)}(\xi_1-\xi_2)\Big| \\
&+ \Big|\frac{1}{4}\frac{1}{(1+\xi_1+\eta_1)(1+\xi_2+\eta_2)}(\eta_1-\eta_2)\Big| \\
&\leq \frac{1}{4}\Big|(\xi_1-\xi_2)\Big| + \frac{1}{4}\Big|(\eta_1-\eta_2)\Big|
\end{aligned}
$$

So, hypothesis (H_1) is satisfied with $L_1 = L_2 = \dfrac{1}{4}$.

On the other hand, we have:

$$
\begin{aligned}
|F_2(t,s,\xi_1,\eta_1) - F_2(t,s,\xi_2,\eta_2)| &= \Big|\frac{1}{2}\sin(\xi_1+\eta_1) - \frac{1}{2}\sin(\xi_2+\eta_2)\Big| \\
&\leq \Big|\sin\Big(\frac{(\xi_1+\eta_1)-(\xi_2+\eta_2)}{2}\Big)\Big| \\
&\leq \Big|\frac{(\xi_1+\eta_1)-(\xi_2+\eta_2)}{2}\Big| \\
&\leq \frac{1}{2}\Big|\xi_1-\xi_2\Big| + \frac{1}{2}\Big|\eta_1-\eta_2)\Big|
\end{aligned}
$$

Hence, hypothesis (H_2) is satisfied with $M_1 = M_2 = \dfrac{1}{2}$.

Consequently, the matrix A in (3.7) becomes in this case:

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

which admits the following eigenvalues:

$$\lambda_1 = \frac{1}{2}\left|\frac{1}{4}+\frac{1}{2}-\sqrt{(\frac{1}{4}+\frac{1}{2})^2-4(\frac{1}{8}-\frac{1}{8})}\right| = 0 < 1$$

and

$$\lambda_2 = \frac{1}{2}\left|\frac{1}{4}+\frac{1}{2}+\sqrt{(\frac{1}{4}+\frac{1}{2})^2-4(\frac{1}{8}-\frac{1}{8})}\right| = \frac{3}{4} < 1$$

Then, conditions (3.13)-(3.14) are fulfilled.

So, the existence and uniqueness of solutions for (3.15), follows from Theorem 3.3.1.

Remark 3.3.2.

Because of the form of the functions F_1 and F_2 in (3.15), it is clear that [Theorem 4.1,[13]] is not applicable to (3.15). So Theorem 3.3.1 cover more general systems than those given by (1.2) in [13].

Let us now, focus on the following Particular case of (3.3):

$$\begin{cases} F_1(t,s,\xi_1,\eta_1) \text{ is independent of } \eta \\ \\ F_1(t,s,\xi_1,\eta_1) \text{ is independent of } \xi \end{cases}$$

Hence, (H_1) and (H_2) are reduced to:

$$\begin{cases} \mid F_1(t,s,\xi_1,\eta_1) - F_1(t,s,\xi_2,\eta_2) \mid \leqslant L_1 \mid \xi_1 - \xi_2 \mid \\ \\ \mid F_1(t,s,\xi_1,\eta_1) - F_1(t,s,\xi_2,\eta_2) \mid \leqslant M_2 \mid \eta_1 - \eta_2 \mid \end{cases}$$

That is $L_2 = M_1 = 0$

So condition (3.13) is reduced to (ii) of [Theorem 4.1,[13]] with

$L_{K_1} + L_{H_1} = L_1$

and

$L_{K_2} + L_{H_2} = M_2.$

Which means that Theorem 3.3.1 is reduced to [Theorem 4.1,[13]] in this particular case.

We conclude by the following Remark.

Remark 3.3.3.

It should be noted that, in addition of the generalization of the considered problem (see Remark 3.3.1 and the discussion above), this work provide a slight completeness to the proofs made in [13], regarding the check of continuity of the considered operators in (X, D).

General conclusion

In this memory, We are interested in the fixed point theory with two metric. We started with some definitions that we used and presented some versions of fixed point theorems of the Maia's and perov's type. We provide also some of their applications in the solutions of differential and integral systems.

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