الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي والبحث العلمي
جامعة الشهيد حمه لخضر بالوادي

كلية التكنولوجيا

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مستخرج من محضر اجتماع اللجنة العلمية لقسم: الهندسة الميكانيكية

في يوم الاثنين 02 جوان 2025 وفي حدود الساعة 10:00 بمكتب رئيس القسم بمبنى كلية التكنولوجيا وبدعوة من رئيس اللجنة العلمية لقسم الهندسة الميكانيكية تم عقد اجتماع ضم أعضاء اللجنة العلمية الآتية أسمائهم:

- الاسم واللقب: منصوري خالد (رئيس اللجنة العلمية)
- الاسم واللقب: منصر رضا (رئيس القسم)
- الاسم واللقب: غربي محمد الطاهر (عضو)
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- الاسم واللقب: لعويني عبد الجليل (عضو)
- الاسم واللقب: زين علي (عضو)
- الاسم واللقب: نسيب الجيلاني (عضو)
- الاسم واللقب: مقدود سفيان (عضو)

بعد إطلاع اللجنة العلمية على تقارير الخبراء المتعلقة بالمطبوعات البيداغوجية المقدمة من طرف أساتذة القسم. وقد تضمنت هذه المراجعة تقييمًا دقيقًا للمحتوى العلمي ومدى ملاءمته للبرامج التعليمية، وبناءً على ذلك، تم اعتماد هذه المطبوعات كمرجع أكاديمي رسمي ضمن البرامج الدراسية. بغرض تقديمه ونشره على مستوى موقع التعليم عن بعد وذلك بناء على التقارير الإيجابية للخبراء كما هو مبين في الجدول التالي:

الأستاذ	المادة	المستوى	التخصص	لجنة الخبراء
محمودي عبد القادر	Modeling and Simulation of Electromechanical Systems (باللغة الإنجليزية)	ثانية ماستر	كهروميكانيك	منصر رضا – جامعة الوادي بتاريخ : 2025/03/29
				يحي خالد – جامعة بسكرة بتاريخ : 2025/04/07

رفعت الجلسة في حدود الساعة 12:00 زوالا من نفس التاريخ واليوم المشار إليه أعلاه

رئيس اللجنة العلمية

رئيس اللجنة العلمية
لقسم الهندسة الميكانيكية
د. منصوري خالد

وزارة التعليم العالي والبحث العلمي جامعة الوادي كلية التكنولوجيا
اللجنة العلمية لقسم الهندسة الميكانيكية

REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR
ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITE MOHAMED KHIDER - BISKRA
Faculté des Sciences et de la Technologie
Département de Génie Electrique

الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي والبحث العلمي
جامعة محمد خيضر بسكرة
كلية **:العلوم و التكنولوجيا**
قسم **: الهندسة الكهربائية**

Rapport d'Expertise

Années Universitaire **: 2024/2025**

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Etablissement universitaire **: Université de Biskra, FST, Département de Génie électrique**

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Etablissement universitaire **: Université d'Eloued, FT, Département de Génie électrique**

Intitulé du polycopié de cours : **Modeling and Simulation of Electromechanical Systems**

Mr. MAHMOUDI Abdelkader, *Maitre de conférences classe B à l'université d'Eloued*, a effectué un travail pédagogique intéressant dans le domaine de modélisation et simulation des machines electriques. Son polycopié de cours destiné aux étudiants de Master 1 et 2 toutes spécialités et plus particulièrement Master 2 Electromécanique est conforme aux canevas du ministère de l'enseignement supérieur et de la recherche scientifique.

Sur le plan méthodologique, il y'a un effort considérable fourni par Mr. MAHMOUDI. En plus de l'introduction et de la conclusion, Le polycopié est structuré en cinq chapitres. Le document contient également plusieurs figures, tableaux et photos. Le cours est bien structuré et la méthodologie est en adéquation avec le sujet traité. La partie bibliographie comporte un grand nombre des références qui s'accordent bien au contenu du cours.

Dans le premier chapitre, l'auteur présente une modélisation dynamique de la machine à courant continu passant par la description des concepts généraux des systèmes dynamiques et les différentes méthodes de modélisation.

Une étude détaillée de la modélisation dynamique des machines synchrones a été présentée dans le deuxième chapitre. Il examine le principe de fonctionnement de la machine, ses caractéristiques et ses modèles mathématiques.

Le troisième chapitre traite l'application de la transformation de Park à la machine synchrone dans le cas d'une distribution non sinusoïdale du champ. L'étude est validée par des graphes et allures des flux et des forces électromotrices.

Dans le quatrième chapitre, l'association convertisseurs-machines est développée à travers une modélisation adéquate et des simulations des comportements de l'onduleur commandé par des techniques de modulation à largeur d'impulsion MLI sinus-triangle et du système global.

La modélisation du moteur asynchrone triphasé a été présentée dans le cinquième chapitre. L'auteur élabore une étude méthodologique complète pour l'analyse dynamique de ces machines en donnant les équations fondamentales jusqu'aux représentations sophistiquées destinées aux applications de commande.

En conclusion, vu l'excellente qualité de ce support de cours en terme de rédaction, structure et présentation et en tant qu'expert, j'exprime un avis favorable à sa publication.

Date : 07/04/2025

Pr. YAHIA Khaled

Rapport d'Expertise du Polycopié de Cours

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Nom et Prénom du candidat : Dr. Abdelkader MAHMOUDI
Titre du polycopié: Modeling and Simulation of Electromechanical Systems
Spécialité : 2^{émé} master électromécanique**.**

Ce polycopié, intitulé « Modeling and Simulation of Electromechanical Systems », a été rédigé par Dr. Abdelkader MAHMOUDI, maître de conférences de classe B à l'Université d'El Oued. Il est conforme au programme recommandé et établi par le ministère de l'enseignement supérieur et de la recherche scientifique. Ce document constitue un excellent support pédagogique pour les étudiants en deuxième année master électromécanique.

Le contenu de ce polycopié est structuré en cinq chapitres :

Dans le premier chapitre, l'auteur présente les principes fondamentaux des systèmes dynamiques et leur application à la modélisation des machines à courant continu. Il aborde d'abord les concepts généraux des systèmes dynamiques, incluant leur représentation mathématique et les différentes méthodes de modélisation. La représentation par équation d'état est particulièrement développée, avec les techniques de transition vers la fonction de transfert.

Le deuxième chapitre présente une analyse approfondie de la modélisation dynamique des machines synchrones. Il examine dans un premier temps leurs caractéristiques constructives, en détaillant leur architecture ainsi que leur principe de fonctionnement, lequel repose sur la synchronisation entre le champ rotorique et le champ tournant statorique. La partie centrale du chapitre développe les modèles mathématiques des machines synchrones. Enfin, l'étude présente deux modèles d'état distincts : l'un applicable aux machines synchrones classiques, l'autre spécifiquement adapté aux machines synchrones à aimants permanents.

Dans le troisième chapitre, un cadre théorique et pratique pour l'analyse des machines synchrones présentant des distributions de champ complexes, élargissant significativement le domaine d'application de la transformation classique de Park.

Dans le chapitre quatre, cette partie développe une méthodologie complète pour modéliser et simuler les systèmes convertisseur-machine. Essentielle pour analyser leurs interactions et concevoir des entraînements performants, cette approche permet notamment de comprendre les couplages énergétiques et d'optimiser les architectures de conversion.

Dans ce chapitre final, l'auteur élabore un cadre méthodologique complet pour l'analyse dynamique des machines asynchrones. Le développement progressif va des équations fondamentales jusqu'aux représentations sophistiquées destinées aux applications de commande. Cette approche systématique éclaire particulièrement les interactions électromécaniques complexes qui caractérisent ces machines omniprésentes dans les applications industrielles.

En conclusion, l'auteur a élaboré un polycopié de grande qualité, à la fois bien rédigé, structuré et présenté, qui mérite pleinement d'être publié. Ce document constitue un excellent outil pédagogique et peut être utilisé directement par les étudiants en deuxième année master, spécialité électromécanique.

En tant qu'expert, j'exprime **un avis favorable** à sa publication.

El-Oued le 29/03/2025

Signature de l'expert

Course Title

Modeling and Simulation of Electromechanical Systems

Prepared by:

Abdelkader MAHMOUDI

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Chapter I. Dynamic Properties of the Direct Current Machine

I.1.1. General Introduction

Dynamic system modeling is essential for studying such systems. It involves understanding the mathematical model that describes how the system's dynamics evolve over time. This model provides insights into the system's behavior under the influence of controls, disturbances, and over time, and how these changes are reflected in the system’s outputs. Altogether, these insights form what is known as the system model. Numerous resources are available on dynamic system modeling

The model can be either linear or nonlinear, and either stationary or non-stationary. Modeling plays a significant role in identifying system parameters, designing control strategies, and detecting faults.

I.1.2. Dynamic System

A dynamic system is characterized by the following three concepts:

Functionality: These are the objectives for which the system was created, defining its purpose for existence.

Structure: This represents the means implemented to fulfill the system’s functions. Generally, a system’s structure refers to the organization of hardware and/or software resources, which may involve various technologies. Traditionally, different system structures are categorized according to their technological nature. Dynamic systems can thus be classified as follows:

- Electrical or electronic systems, whether logical or analog,
- Thermoelectric systems,
- Mechanical systems,
- Computer systems,
- Biological, chemical systems, etc.

Behavior: This defines how the system performs one or more functions.

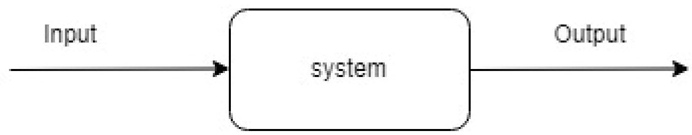

Figure 1: General Diagram of a Dynamic System

The behavior of a dynamic system reflects the evolution of its outputs over time in response to applied inputs. In other words, a dynamic system expresses the causal relationship between inputs (causes) and outputs (effects). This relationship between inputs and outputs forms the model or process of the system.

From a system perspective, inputs and outputs are measurable physical quantities. From a model perspective, inputs and outputs correspond to quantified external variables. Thus, the inputs and outputs of a system provide quantitative information about the system's behavior due to the analytical nature of the model. For the system, this information is obtained through known inputs or by measuring outputs (using sensors, for example).The known inputs of a system are called control inputs. These inputs are known or given because they correspond to outputs from another system, allowing them to be defined. Control inputs can either be directly measured at the output of the generating system or be calculated. Inputs are converted into action quantities through other systems known as actuators. The diagram shown in Figure (2) represents the structure of an open-loop dynamic system.

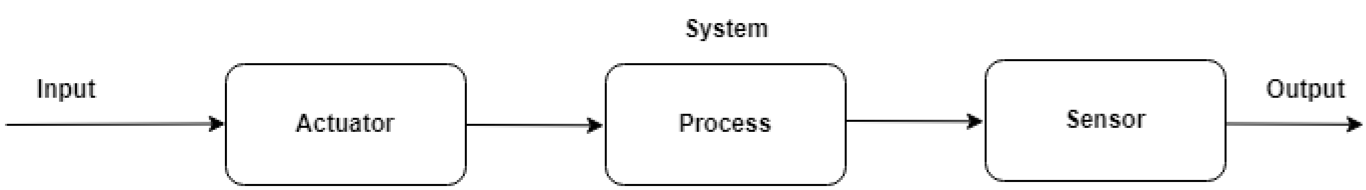

Figure 2: Structure of an Open-Loop System

Finally, a dynamic system can be characterized by:

1. **The Mathematical Representation**: The mathematical model of the system, which describes its behavior.
2. **General Organization:** Its overall structure, encompassing three subsystems: actuators, processes, and sensors.

3. **Interaction with a Control System:** Its interaction with a control system that generates the inputs needed to achieve the desired output function.

I.1.3. Representation of Systems

When studying systems, they can be represented by linear models. There are three types of representation for these models:

a. Input-Output Differential Form:

The mathematical description of the operation of a system involves writing all the algebraic and differential relationships that connect the m input variables, represented by a vector denoted $u(t)$, the n output variables represented by a vector denoted $y(t)$, and the internal variables of the system.

Using the flow graph method, it is possible to eliminate defined mathematical relationships and the internal variables, thereby obtaining algebraic and differential relationships that link the outputs and inputs. Ultimately, we arrive at a relationship that connects the following input and output variables:

$$\sum_{i=c}^{n} a_i \frac{d^i y}{dt^i} = \sum_{i=0}^{m} b_i \frac{d^i u}{dt^i}$$

where:

— the coefficients a_i and b_i are real constants, with a_c, a_n, b_0, and b_m being non-zero;

— n and m are positive integers such that $m \leq n$ for the system to be causal; n is the order of the system;

— $c \leq n$ is a non-negative integer called the class of the system.

The solution $y(t)$ of this equation, referred to as the time response of the system,

b. Input-Output Transfer function:

The input-output transfer matrix, denoted $F(s)$, where s represents the differentiation operator (with $s = d/dt$; Laplace operator), describes the evolution of the output $y(t)$ in relation to the input $u(t)$ in the Laplace domain:

$$y(s) = F(s) \cdot u(s)$$

This relationship highlights how the output is influenced by the input through the transfer function ***F(s),*** facilitating analysis and design in the frequency domain.

Laplace Transform:

Let *f(t)* be a causal continuous-time signal. The Laplace transform of this signal is defined by the following integral:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

where:

- *F(s)* is the Laplace transform of *f(t)*,
- *s* is a complex number $s = \sigma + j\omega$
- $e^{-st}$is the exponential decay factor,
- The integral is evaluated from *0* to ∞

This transform is particularly useful for analyzing linear time-invariant (LTI) systems and solving differential equations, as it converts differential equations into algebraic equations in the ***s*** -domain.

c. State-Space Form:

It is used for modeling systems. When designing control methods, it is necessary to use either the input-output transfer matrix or the state-space form.

I.1.4. State Equation

I.1.4.1. Mathematical Representation

The state of a model represents the set of parameters required to understand the behavior of a system's output based on the inputs it receives. Modeling a system using state equations involves constructing a model that only uses first-order differential equations. The structure of state representation is common to all linear systems, and a continuous linear system with constant coefficients is described by the following state equation, known as the canonical form (LTI = Linear Time-Invariant Systems):

Continuous Case:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The state space is R^n; we assume there are p outputs and m inputs. The state, output, and input variables are grouped into the state vector x, the output vector y, and the input vector u, respectively. The relationships are linear and can thus be expressed in matrix form.

In both the continuous and discrete cases, this is referred to as the (A,B,C,D) system, with $x \in R^n$.

$y \in R^p, u \in R^m$,

where :
$A[n \times n]$:the state matrix
$B[n \times m]$: the input matrix
$C[p \times n]$: the output matrix
$D[p \quad x_m]$: the direct input-output transfer matrix.

Equations (1) and (2) are represented by the diagrams in Figure 3.

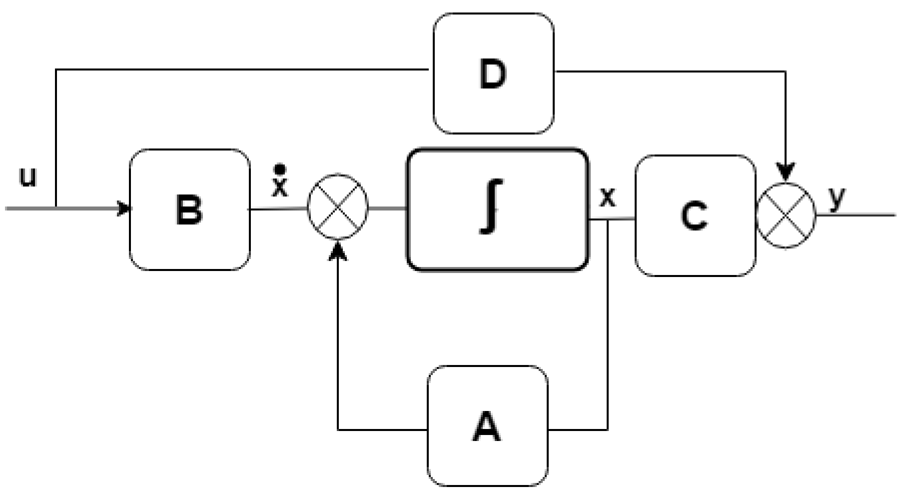

Figure 3: Standard diagram for the canonical form (continuous).

I.1.4.2. State definitions

- **State:** The state of a dynamic system is the smallest set of variables or quantities such that knowing this set at time $t=t_0$, , along with the input signal for $t \geq t_0$, is sufficient to fully determine the system's behaviour for $t \geq t_0$.

- **State Variables:** These are the variables or quantities that constitute the system's state.

- **State Vector:** Mathematically, the state is represented by concatenating all state variables into a real vector of dimension *n*, denoted as $x=[x_1,\ldots,x_n]^T$.

- **State Space:** This is the vector space in which the state vector *x* can evolve, with each instance of *x* corresponding to a point in this space. Therefore, this space is R^n .

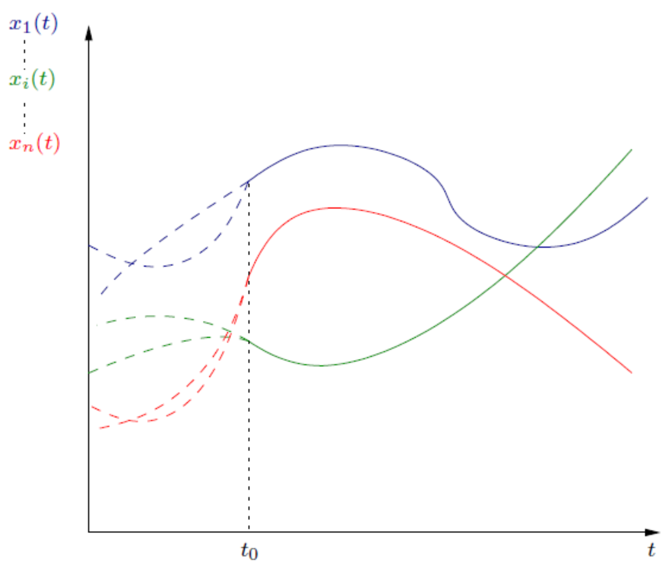

Figure 4: The evolution of the components of $x(t)$ after t_0 (solid line) is independent of their evolution before t_0 (dotted line).

I.1.4.3. Properties of Continuous-Time Systems

Continuous-time systems have specific properties that characterize their behavior over time. These include:

- **Linearity:** A system is linear if it satisfies the principles of superposition and homogeneity. Specifically, let y_1 and y_2 be the responses of a system Σ excited separately by the inputs u_1 and u_2. If α is any real number, the system is linear if its output equals αy_1+y_2 in response to the input αu_1+u_2 (see Figure 5).
- **Time-Invariance:** A system is time-invariant if its behavior and characteristics do not change over time. Shifting the input signal in time results in a corresponding shift in the output signal.

- **Causality:** A continuous-time system is causal if its output at any time t depends only on the values of the input at the current or past times $t \leq t_0$, and not on future values.

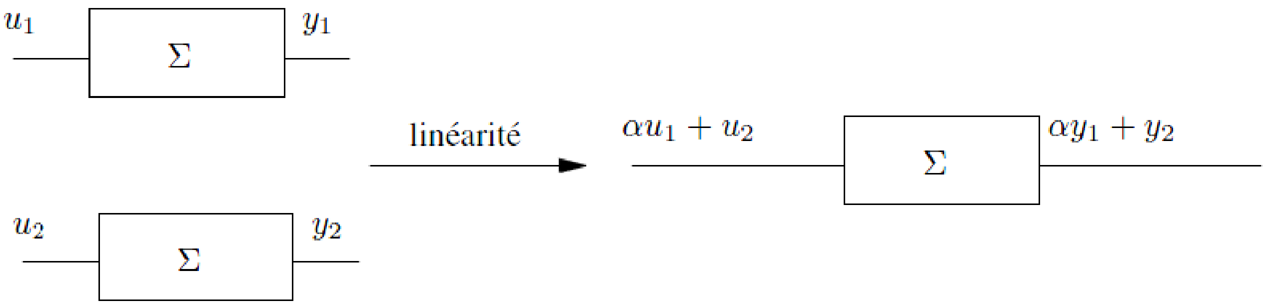

Figure 5: Linearity of a System

Example:

Let's consider an RLC circuit as shown in Figure 6:

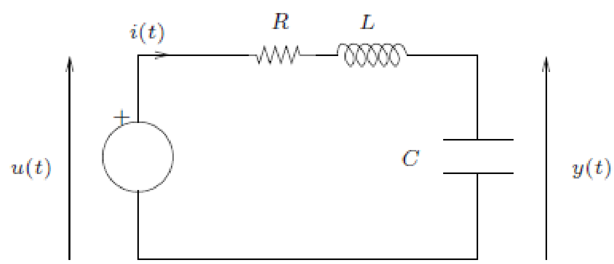

Figure 6: RLC Circuit

$$\begin{cases} u(t) = Ri(t) + L\dfrac{di}{dt} + y(t) \\ y(t) = \dfrac{1}{C}\displaystyle\int_0^t i(\tau)d\tau \Leftrightarrow \dfrac{dy(t)}{dt} = \dfrac{1}{C}i(t) \end{cases}$$

Let: $x_1 = i$ et $x_2 = y$

The equations can be expressed as:

$$\begin{cases} u(t) = Rx_1 + L\dfrac{dx_1}{dt} + x_2 \\ \dot{x}_2 = y = \dfrac{1}{c}x_1 \end{cases}$$

This leads to the following system of equations:

$$\Rightarrow \begin{cases} \dot{x}_1 = -\dfrac{R}{L}x_1 - \dfrac{1}{L}x_2 + \dfrac{1}{L}u \\ \dot{x}_2 = \dfrac{1}{c}x_1 \end{cases}$$

In matrix form, this can be represented as::

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This gives:

$$A = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{c} & 0 \end{bmatrix},\ B = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathrm{D} = 0$$

I.4.4 Transition from State Space Representation to Transfer Function

Continuous Case: Let

$$\dot{x} = Ax + Bu(*)$$
$$y = Cx + Du(**)$$

Taking the Laplace transform gives: $sX(s) - X(0) = AX(s) + BU(s)$

Assuming $\mathrm{X}(0) = 0$, We have:

$$sX(s) - AX(s) = BU(s) \Rightarrow (sI - A)X(s) = BU(s) \Rightarrow$$
$$X(s) = (sI - A)^{-1}.BU(s)$$

Substituting into equation (**) yields:

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s) = (C(sI - A)^{-1}B + D)U(s)$$

Thus, the transfer function is given by:

$$F(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

For Single Input Single Output (SISO) systems, this is a function.

For Multiple Input Multiple Output (MIMO) systems, this is a matrix.

Example :

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

To find the transfer function $F(s) = \frac{Y(s)}{U(s)}$, we start with the equation:

$$F(s) = C \cdot (sI - A)^{-1} \cdot B$$

Given that $D = 0$, we can simplify it to:

$$F(s) = C \cdot (sI - A)^{-1} \cdot B$$

Step 1: Formulate $sI - A$
Given:

$$A = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$$

Then:

$$sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

So:

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} s+2 & -1 \\ 3 & s-2 \end{bmatrix}$$

Step 2: Calculate the Determinant
The determinant Δ of $sI - A$ is given by:

$$\Delta = (s+2)(s-2) - (-1)(3) = (s+2)(s-2) + 3$$

Expanding this:

$$\Delta = s^2 - 4 + 3 = s^2 + 5s + 4$$

Step 3: Calculate $(sI - A)^{-1}$
The inverse is calculated using the formula:

$$(sI-A)^{-1}=\frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $a=s+2, b=-1, c=3, d=s-2$:

$$(sI-A)^{-1}=\frac{1}{s^2+5s+4}\begin{bmatrix} s-2 & 1 \\ -3 & s+2 \end{bmatrix}$$

Step 4: Multiply by B
Let:

$$B=\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (assuming } L=1 \text{ for simplicity)}$$

So:

$$(sI-A)^{-1}B=\frac{1}{s^2+5s+4}\begin{bmatrix} s-2 \\ -3 \end{bmatrix}$$

Step 5: Calculate $F(s)$
Now, we need C :

$$C=\begin{bmatrix} 1 & 2 \end{bmatrix}$$

Then:

$$F(s)=C\cdot(sI-A)^{-1}\cdot B=\begin{bmatrix} 1 & 2 \end{bmatrix}\cdot\frac{1}{s^2+5s+4}\begin{bmatrix} s-2 \\ -3 \end{bmatrix}$$

Calculating this gives:

$$\begin{aligned} F(s) = & \frac{1}{s^2+5s+4}(1\cdot(s-2)+2\cdot(-3)) \\ & =\frac{s-2-6}{s^2+5s+4}=\frac{s-8}{s^2+5s+4} \end{aligned}$$

Thus, the transfer function is:

$$F(s)=\frac{s-8}{s^2+5s+4}$$

I.2. Modeling of the Direct Current (DC) Motor

I.2.1 Formulating Equations for a Physical System

The equations describing the behavior of a dynamic system are derived by applying the laws of physics. However, the resulting model may only approximate real phenomena, as it is generally challenging to account for all the physical effects involved.

I.2.2 Description of the Direct Current (DC) Motor

A direct current motor (DCM), with its schematic shown in Figure (7), is an electromechanical device that converts electrical input energy into mechanical energy. The electrical energy is supplied by a power converter that feeds the winding located on the rotating armature (rotor). This winding is placed in a magnetic field, either permanent or variable, generated by the stator (field coil). For simplicity, we assume that this excitation is separate and constant, as is the case when the field coil consists of permanent magnets. The current flowing through the motor's armature windings generates electromagnetic forces, and through a suitable mechanism (brushes and commutator), these forces combine to drive rotation. Thus, the motor can be considered a system where the armature voltage is the input, and the output is a parameter related to the rotor's angular position. Initially, the rotor's rotational speed is chosen as the output parameter.

I.2.3 Modeling

As a DC motor is an electromechanical system, its dynamic equations result from combining the mechanical and electrical models of the motor, as schematically illustrated in Figure (7).

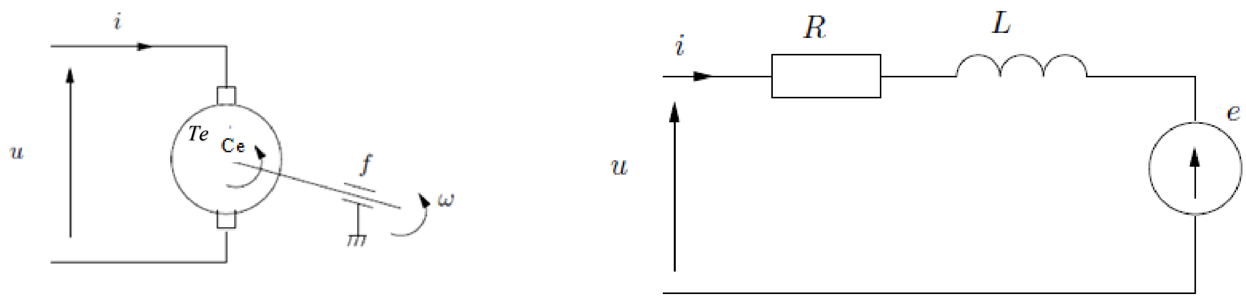

Figure 7: Schematic of a Direct Current Motor

For the electrical part, we calculate the voltage across the armature. The electrical equation, relating the armature voltage u to the armature current i, is expressed as:

$$u(t) = \mathrm{R}i(t) + L\frac{di(t)}{dt} + e(t) \quad (1)$$

R: Resistance of the motor's armature.

L: Its inductance.

The electromotive force, which is proportional to the rotor's rotational speed:

$$e(t) = k_e\omega \quad (2)$$

For the mechanical part, the fundamental principle of dynamics is applied around the axis of rotation. The mechanical equation describing the torques acting on the rotor is written as:

$$T_e - T_L = J\frac{d\omega}{dt} + f\omega \quad (3)$$

where

T_e: is the motor torque,

f: the viscous friction coefficient.

J: Moment of inertia of the rotor.

T_L: Load torque or resisting torque.

The torque T_e is proportional to the armature current, such that:

$$T_e = k_m \cdot i \quad (4)$$

In general, the coefficients k_e and k_m are close enough that they can reasonably be considered equal. By setting $K=k_e=k_m$,, equations (3) and (4) yield (assuming T_L=0):

$$ki = J\frac{d\omega}{dt} + f\omega \tag{5}$$

By differentiating (5), we obtain:

$$k\frac{di}{dt} = J\frac{d^2\omega}{dt^2} + f\frac{d\omega}{dt} \tag{6}$$

Combining equations (5), (6), (1), and (2), we get:

$$\frac{R}{k}\left(f\omega + J\frac{d\omega}{dt}\right) + \frac{L}{k}\left(J\frac{d^2\omega}{dt^2} + f\frac{d\omega}{dt}\right) + K\omega = u \tag{7}$$

Rearranging gives:

$$\frac{d^2\omega}{dt^2} + \frac{RJ+Lf}{LJ}\frac{d\omega}{dt} + \frac{Rf+K^2}{LJ}\omega = \frac{K}{LJ}u \tag{8}$$

This differential equation links ω and u through time-invariant parameters. It is a second-order linear differential equation with constant coefficients. According to equation (8), the transfer function of the DC motor can be written as:

$$G(p) = \frac{\Omega(s)}{U(s)} = \frac{K}{LJp^2 + \frac{RJ+Lf}{LJ}p + \frac{Rf+K^2}{LJ}} \tag{9}$$

where:
Numerator: $N(s) = K =$ constant
Denominator: $D(s) = p^2 + \frac{RJ+Lf}{LJ}p + \frac{Rf+K^2}{LJ}$, which determines the system poles.
The transfer function can also be written as:

$$G(p) = \frac{\Omega(s)}{U(s)} = \frac{K_G}{\tau_{el}\tau_{em}\left(p+\frac{1}{\tau_{el}}\right)\left(p+\frac{1}{\tau_{em}}\right)} \tag{10}$$

or:

$$G(p) = \frac{\Omega(s)}{U(s)} = \frac{K_G}{(1+\tau_{el}\cdot p)(1+\tau_{em}\cdot p)} \tag{11}$$

with:

$$\tau_{el} = \frac{L}{R}$$
$$\tau_{em} = \frac{RJ}{Rf + K^2}$$
$$K_G = \frac{K}{Rf + K^2}$$

Thus, the DC motor has two poles:

$$p_1 = -\frac{1}{\tau_{el}} \text{ and } p_2 = -\frac{1}{\tau_{em}}$$

Associated with two time constants, where p_2 is the dominant pole (slow response):
τ_{el} : electrical time constant.
τ_{em} : electromechanical time constant (larger when the inertia J is high).
K_G is the static gain of the DC motor.

In general, the electromechanical part responds more slowly than the electrical part, so we have:

$$\tau_{em} \gg \tau_{el}$$

In expanded form, we have:

$$G(p) = \frac{\Omega(s)}{U(s)} = \frac{K_G}{1+(\tau_{el}+\tau_{em})p+\tau_{el}\tau_{em}p^2} \tag{12}$$

I.2.4.State Model of the DC Motor

A state model of the DC motor can be readily determined. The system input is the armature voltage u, and its output is the rotor's rotational speed. Two independent state variables are chosen: the rotational speed $x_1 = \omega$ and the armature current $x_2 = i$. The electrical equation (1) then becomes:

$$Rx_2 + L\frac{dx_2}{dt} + Kx_1 = u$$

The mechanical equation (3) provides:

$$Kx_2 - fx_1 = J\frac{dx_1}{dt}$$

Therefore:

$$J\frac{dx_1}{dt} = Kx_2 - fx_1 \Rightarrow \dot{x}_1 = \frac{dx_1}{dt} = -\frac{f}{J}x_1 + \frac{K}{J}x_2$$
$$L\frac{dx_2}{dt} = -Rx_2 - Kx_1 + u \underset{\downarrow\downarrow}{\Rightarrow} \dot{x}_2 = \frac{dx_2}{dt} = -\frac{K}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u$$

Hence, the state representation is:

$$\frac{d}{dt}\binom{x_1}{x_2} = \begin{pmatrix} -\frac{f}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{pmatrix}\binom{x_1}{x_2} + \binom{0}{\frac{1}{L}}u,$$
$$y = (1 \quad 0)\binom{x_1}{x_2}.$$

The chosen configuration is not unique. Therefore, a block diagram of this system can be implemented in Simulink, as shown in Figure (8).

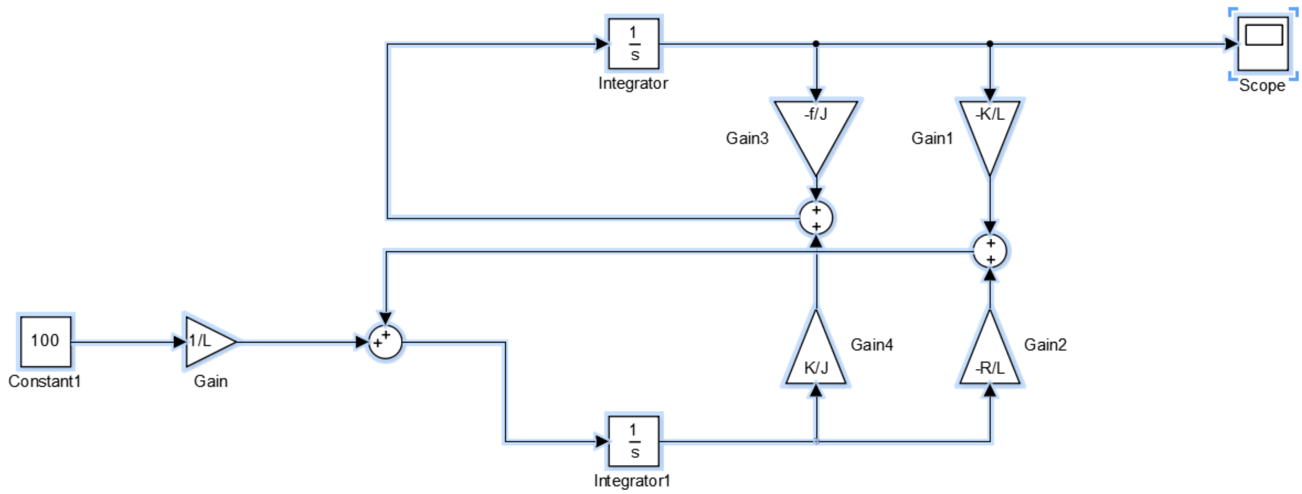

Figure 8: Simulation Diagram of the DC Motor (ME)

I.2.5. Transition from State Model to Transfer Equation

$$A = \begin{pmatrix} -\frac{f}{J} & -\frac{K}{J} \\ 0 & -\frac{k}{L} \end{pmatrix},\ B = \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix},\ C = (1 \quad 0),\ D = 0.$$

By applying the relation:

$$F(P) = Y(P)U(P) = C \cdot (PI - A)^{-1}B + D$$

we get:

$$\Omega(P) = (1 \quad 0) \cdot \left(PI + \frac{f}{J}K - \frac{K}{L}\right)^{-1} \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} U(P)$$

which simplifies to:

$$\Omega(P) = (1 \quad 0) \cdot \frac{1}{\left(P + \frac{f}{J}\right)\left(P + \frac{R}{L}\right) + \frac{K^2}{LJ}} \cdot \begin{pmatrix} P + \frac{R}{L} & +\frac{K}{J} \\ -\frac{K}{L} & \mathrm{P} + \frac{f}{J} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} U(P)$$

After performing all calculations, we obtain:

$$G(P) = \frac{\Omega(P)}{U(P)} = \frac{\frac{K}{LJ}}{P^2 + \frac{RJ + Lf}{LJ}P + \frac{Rf + K^2}{LJ}}$$

This expression is identical to that in (8).
Figure (9) shows the block diagram of the DC motor simulation in Simulink.

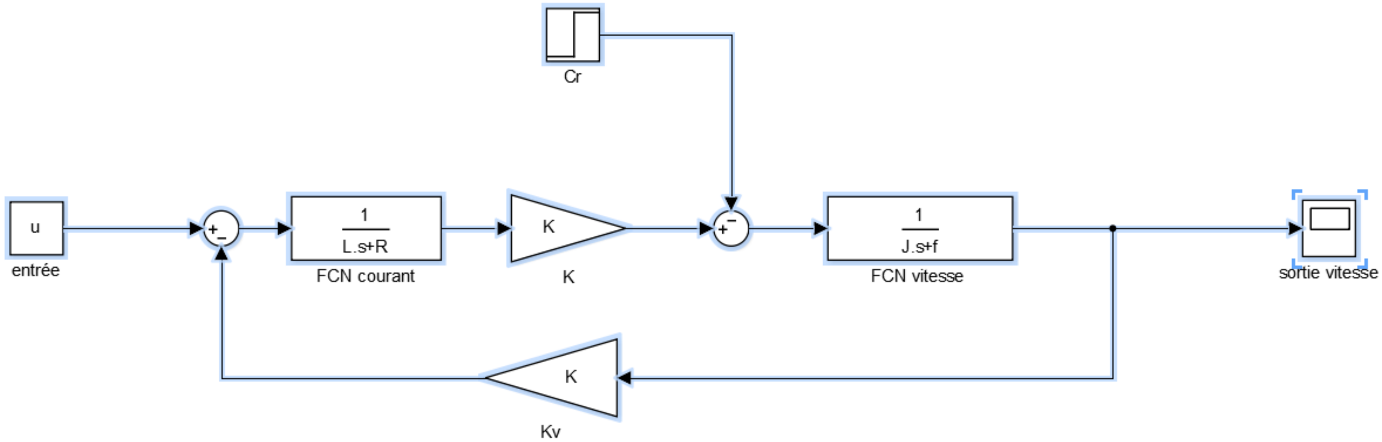

Figure 9: Simulation Diagram of the DC Motor (FT)

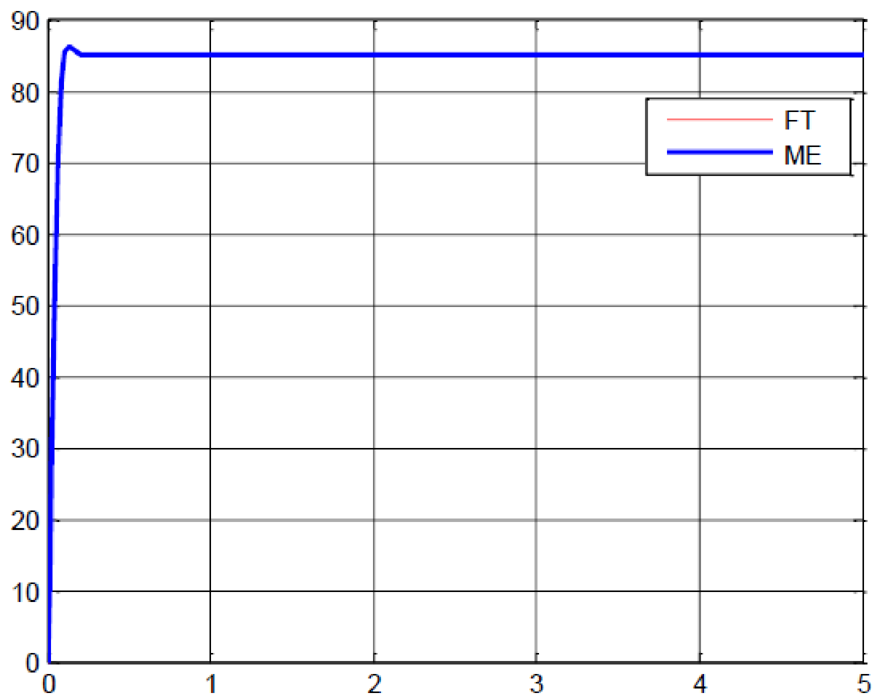

Figure 10: Simulation Graphs for the Speed of the DC Motor by FT and ME

The parameters of the simulated machine are R=42.31R = 42.31R=42.31, L=0.63L = 0.63L=0.63, J=0.0012J = 0.0012J=0.0012, f=0.001f = 0.001f=0.001, K=1.137K = 1.137K=1.137.
In both simulations, the curves are identical.

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Chapter II

Dynamic Model of Synchronous Motor

II.1. Introduction

This lecture presents a dynamic model of the synchronous machine. We demonstrate how to use this model in power system simulations, and explain the relations between the machine's dq0 model and time-varying phasor model. Synchronous machines are often operated as generators, and are a major source of energy in electric power systems. In several applications synchronous machines are also operated as motors. A basic diagram of the machine is shown in Fig. 1. The key mechanical components of the machine are the rotor and stator. There are also two key electric parts: the field winding on the rotor, and the three-phase winding on the stator. The field winding is typically connected to a DC source, which creates a magnetic field with alternating north and south polarities, as illustrated in Fig. 1. As the rotor rotates AC voltages are induced in the stator windings. In addition, the interaction between magnetic fields and currents in the machine produces a torque that acts to decelerate the rotor. These two processes result in energy conversion, from mechanical energy to electromagnetic energy as a generator, or vice-versa as a motor.

The machine is named "synchronous" since at steady-state the rotor speed is proportional to the frequency of currents and voltages in the stator. This is not necessarily the case in other machines. For instance, in induction motors the frequency of the rotor must be slightly lower than the frequency of the AC current.

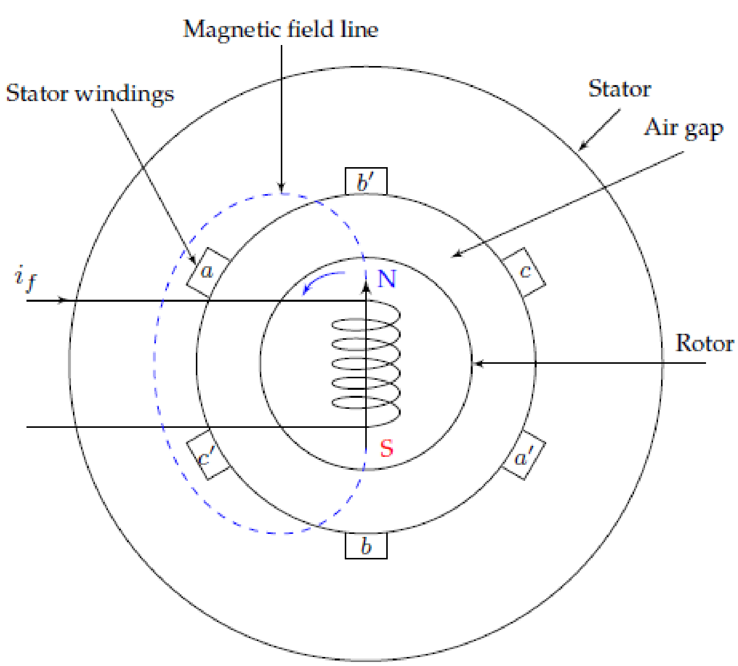

Figure 1: Schematic diagram of a three-phase synchronous machine with two poles.

II.2. Construction of synchronous machines

Synchronous machines are AC machines that have a field circuit supplied by an external DC source.
In a synchronous generator, a DC current is applied to the rotor winding producing a rotor magnetic field. The rotor is then turned by external means producing a rotating magnetic field, which induces a 3-phase voltage within the stator winding.
In a synchronous motor, a 3-phase set of stator currents produces a rotating magnetic field causing the rotor magnetic field to align with it. The rotor magnetic field is produced by a DC current applied to the rotor winding.
Field windings are the windings producing the main magnetic field (rotor windings for synchronous machines).
Armature windings are the windings where the main voltage is induced (stator windings for synchronous machines).

The rotor of a synchronous machine is a large electromagnet. The magnetic poles can be either salient (sticking out of rotor surface) or non-salient construction.

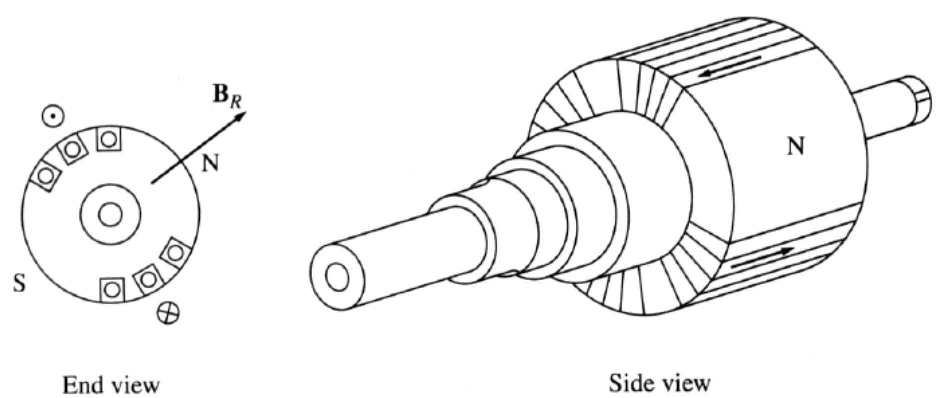

Figure 2: Non-salient-pole rotor: usually two- and four-pole rotors.

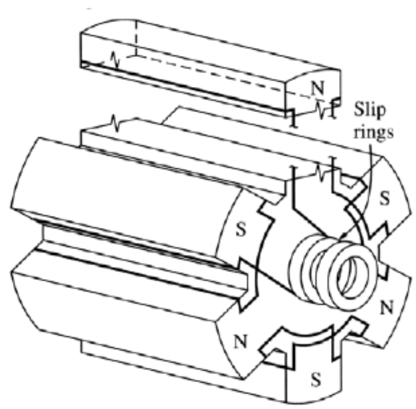

Figure 3: Salient-pole rotor: four and more poles.

II.3. Principle of Operation of a Synchronous Motor

A synchronous motor operates based on the interaction between a rotating magnetic field produced by the stator and a steady magnetic field generated by the rotor. The motor runs at a constant speed, synchronized with the frequency of the supply current. Here is the step-by-step principle:

- **Generation of Rotor Magnetic Field (BRB_RBR):**
  - The rotor contains an electromagnet or permanent magnets.
  - A **field current (I_F)** is applied to the rotor windings, creating a steady magnetic field (B_R).
- **Creation of Stator Magnetic Field (B_S):**
  - The stator is equipped with three-phase windings.
  - A **three-phase AC voltage** is applied to these windings, producing three-phase currents.
  - These currents generate a **rotating magnetic field (B_S)** that rotates at the synchronous speed (N_s).

$$N_s = \frac{60 \cdot f}{P}$$

Where:

N_s = synchronous speed (in RPM),

f = supply frequency (in Hz),

P = number of pair poles.

- **Magnetic Field Interaction :**
  - Two magnetic fields are present in the motor: the rotor field (B_R) and the stator field (B_S).
  - The rotor's magnetic field attempts to align with the stator's rotating magnetic field.

- **Synchronization:**
  - When the rotor achieves synchronous speed, its magnetic field locks with the stator's magnetic field.
  - The rotor spins at the same speed as the stator field (N_s), maintaining synchronization.
- **Torque Production:**
  - Torque is generated as a result of the angle (δ) between the rotor and stator magnetic fields.
  - As the angle increases (up to a certain limit), the torque also increases.
  - Once steady-state operation is reached, this angle remains constant, and the rotor continues to rotate synchronously with the stator field.

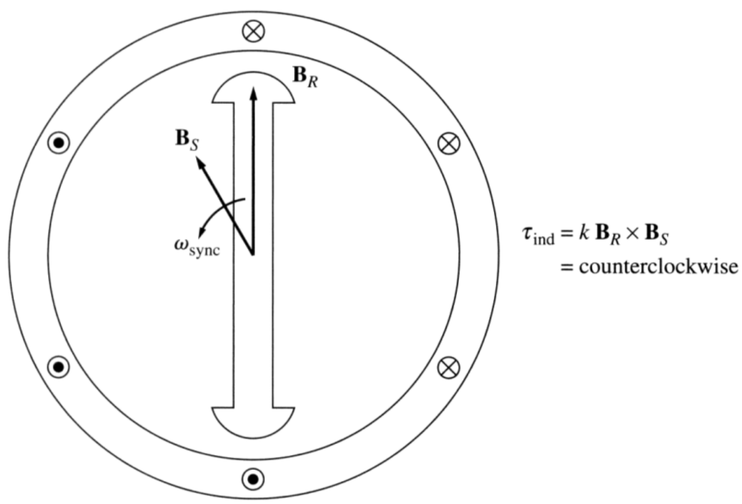

Figure 4: two pole Synchronous Motor

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II.4. Dynamic Models of Synchronous Machines

II .4.1 Simplifying Assumptions:

- Constant air gap;
- Negligible slotting effect;
- Negligible saturation as well as hysteresis and eddy currents;
- Winding resistance does not vary with temperature;
- Negligible skin effect.

II.4.2 Mathematical Modeling of the Machine:

Electrical Equations:

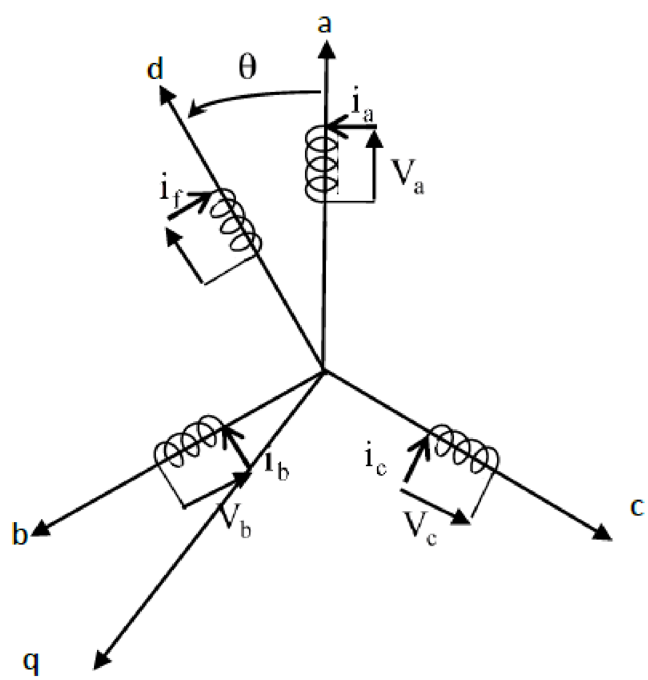


Figure 5: Representation of the windings of a synchronous machine without dampers

a) at the stator:

$$\begin{cases} V_{sa} = R_s I_{sa} + \frac{d\varphi_{as}}{dt} \\ V_{sb} = R_s I_{sb} + \frac{d\varphi_{bs}}{dt} \\ V_{sc} = R_s I_{sc} + \frac{d\varphi_{cs}}{dt} \end{cases} \tag{1}$$

The variables are at the stator, and we define $V, I,$ and φ as the voltages, currents, and fluxes of the machine's stator.
b) at the rotor:

$$V_f = R_f I_f + \frac{d\varphi_f}{dt} \tag{2}$$

V_f, I_f, φ_f : as the voltages, currents, and fluxes of the machine's stator.

Magnetic Equations:

a) Stator Flux:

$$\begin{cases} \varphi_{sa} = L_s I_{sa} + M_{ss} I_{sb} + M_{ss} I_{sc} + M_{sf} I_f \cos(\theta) \\ \varphi_{sb} = L_s I_{sb} + M_{ss} I_{sc} + M_{ss} I_{sa} + M_{sf} I_f \cos(\theta - 2 * \pi/3) \\ \varphi_{sc} = L_s I_{sc} + M_{ss} I_{sa} + M_{ss} I_{sb} + M_{sf} I_f \cos(\theta + 2 * \pi/3) \end{cases} \tag{3}$$

b) Rotor Flux:

$$\varphi_f = L_f I_f + M_{fs} I_{sa} \cos(\theta) + M_{fs} I_{sb} \cos(\theta - 2 * \pi/3) + M_{fs} I_{sc} \cos(\theta + 2 * \pi/3) \tag{4}$$

The model can be expressed in matrix form as follows:

$$[V] = [R][I] + [L]\frac{d}{dt}[I] + [M]\frac{d}{dt}[I]$$

By substituting flux equations (3) and (4) into the stator and rotor models given by equations (1) and (2), the matrix representation of the system becomes:

$$\begin{bmatrix} V_{sa} \\ V_{sb} \\ V_{sc} \\ V_f \end{bmatrix} = \begin{bmatrix} R_s & 0 & 0 & 0 \\ 0 & R_s & 0 & 0 \\ 0 & 0 & R_s & 0 \\ 0 & 0 & 0 & R_f \end{bmatrix} \begin{bmatrix} I_{sa} \\ I_{sb} \\ I_{sc} \\ I_f \end{bmatrix} + \begin{bmatrix} L_s & 0 & 0 & 0 \\ 0 & L_s & 0 & 0 \\ 0 & 0 & L_s & 0 \\ 0 & 0 & 0 & L_f \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_{sa} \\ I_{sb} \\ I_{sc} \\ I_f \end{bmatrix} +$$

$$\begin{bmatrix} 0 & M_{ss} & M_{sf}\cos(\theta) \\ M_{ss} & M_{ss} & M_{sf}\cos(\theta - 2 * \pi/3) \\ M_{ss} & M_{ss} & 0 \\ M_{fs}\cos(\theta) & M_{fs}\cos(\theta - 2 * \pi/3) & M_{fs}\cos(\theta + 2 * \pi/3) \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_{sa} \\ I_{sb} \\ I_{sc} \\ I_f \end{bmatrix} \qquad (5)$$

II.4.3. Three-Phase to Two-Phase Transformation:

The purpose of this transformation is to convert a three-phase abc system into a two-phase $\alpha\beta$ system. There are two main transformations commonly used: Clarke and Concordia.

The Clarke transformation preserves the amplitude of the quantities but does not conserve power or torque (a scaling factor of $\sqrt{3/2}$ must be applied).

The Concordia transformation, being normalized, preserves power but does not maintain the amplitudes of the original signals.

II.4.3.1.Clarke-Concordia Transformation

The Clarke transformation, proposed by Edith Clarke in 1951, is used to convert a three-phase reference system (a, b, c) into a two-phase reference system (α, β, o). Assuming that $\theta_0 = 0$, the transformation matrix simplifies as follows:

$$X_{\alpha\beta} = \begin{bmatrix} \cos(\theta) & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin(\theta) & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

For a more practical application, the transformation can be expressed as:

$$X_{\alpha\beta} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

The initial system (a, b, c) can be described using a sinusoidal vector rotating at a speed ωt :

$$X = \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix} = \begin{bmatrix} X_{\max}\sin(\omega t) \\ X_{\max}\sin\left(\omega t - \frac{2\pi}{3}\right) \\ X_{\max}\sin\left(\omega t + \frac{2\pi}{3}\right) \end{bmatrix}$$

To understand this transformation for a three-phase system, let us assume a vector X rotating at a speed ωt. This vector can be represented in the stationary a, b, c reference frame as X_a, X_b, and X_c, as defined previously.
The Clarke transformation projects this vector $X = [X_a, X_b, X_c]^T$ from the fixed a, b, c reference frame onto the α, β axes. This is expressed as:

$$\begin{bmatrix} X_\alpha \\ X_\beta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin(\theta) & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

Assuming $\theta_0 = 0$, the transformation matrix simplifies to:

$$\begin{bmatrix} X_\alpha \\ X_\beta \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

This simplified matrix transforms the three-phase components (X_a, X_b, X_c) into the two-phase components (X_α, X_β) in the α, β plane.

The transformation can be expressed as follows:

$$\begin{bmatrix} X_0 \\ X_\alpha \\ X_\beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

Thanks to the Concordia transformation, the Clarke matrix can be normalized into a square matrix with a determinant of 1 . This normalization ensures that the inverse of the matrix is equal to its transpose, making it convenient for reversing the transformation. The normalized Concordia matrix is given by:

$$\begin{bmatrix} X_0 \\ X_\alpha \\ X_\beta \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

This Concordia transformation ensures power preservation in the transformation, making it particularly useful in energy-based systems.

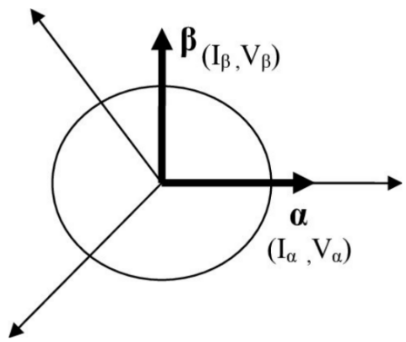


Figure 6: a-b-c to α-β Transformation Instantaneous voltage and currents on the a-b-c coordinates can be transformed into the quadrature α, β coordinates by Clarke transformation

II.4.3.2. Park Transformation

The difference between the Park transformation and the Clarke-Concordia transformation lies in the reference frames they use. In the Park transformation, the axes are rotating (as shown in Figure 6), while the Clarke-Concordia transformation uses a stationary reference frame. The Park transformation, often confused with the $dq0$ transformation, is a mathematical tool widely used in electrical engineering, particularly in vector control. It enables modeling a three-phase system using a two-phase model through a change of reference frame.In the Park transformation, the two main axes in the new reference frame are traditionally labeled d and q. The transformed quantities typically represent physical variables such as currents, voltages, or fluxes. This rotating reference frame in the Park transformation provides an advantage in simplifying the analysis and control of AC machines by aligning the d-axis with the rotor flux, thus making the equations time-invariant under steady-state conditions.

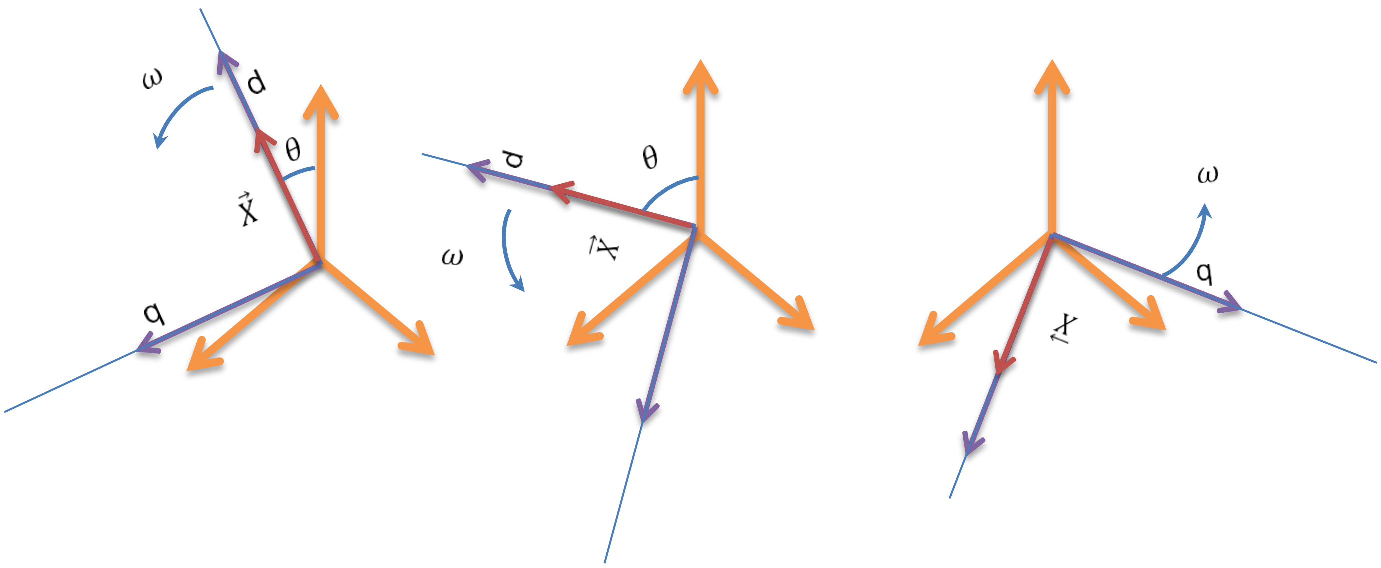


Figure 7: The representation of vector x on the rotating d-q axes at different time instances.

According to the Park transformation, we can express the transformation as follows:

$$\begin{bmatrix} X_d \\ X_q \\ X_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\theta) & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin(\theta) & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

Park Matrix
The Park transformation matrix is therefore:

$$P = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\theta) & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin(\theta) & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

This matrix transforms the three-phase stationary reference frame (X_a, X_b, X_c) into the rotating $d\ q$ reference frame (X_d, X_q), along with a zero-sequence component X_0.

II.5. Park Equations for Synchronous Machines

The Park transformation is employed to eliminate the angular dependence (θ) from the stator inductance matrix $[L_{ss}]$, enabling control algorithms to work with DC electrical quantities. This transformation simplifies the analysis and control of synchronous machines by converting the threephase stator quantities (a, b, c) into two orthogonal components (d, q) in the rotor reference frame.

In this framework, the stator windings (a, b, c) are replaced by two windings (d, q), which are in quadrature (mutually perpendicular). The transformation ensures the conversion of the magnetomotive force (MMF) and instantaneous power from the stator reference frame to the $d - q$ reference frame.

Transformation Details

The transformation is given by:

$$\begin{bmatrix} X_d \\ X_q \\ X_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\theta) & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin(\theta) & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \downarrow\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

This allows the electrical quantities of the stator (currents, voltages, or flux linkages) to be expressed in the dq frame:

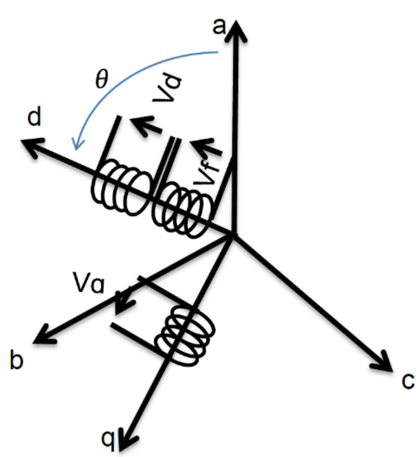


Figure 8: Equivalent Diagram of a Synchronous Machine in the Park Reference Frame

The equivalent machine shown in Figure 8 resembles a direct current (DC) machine, with the following structure:

- The winding f serves as the field winding (inductor).
- The system includes two armature windings:
- d -axis winding aligned with the inductor's axis (f).

- q -axis winding in quadrature with the f-axis.

Transformation Matrices
The forward transformation matrix $[P(\theta)]$ is given in Equation (16). Its inverse, denoted as $[P^{-1}(\theta)]$, is expressed as:

$$P^{-1}(\theta) = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta) & \sin(\theta) & \frac{1}{\sqrt{2}} \\ \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta - \frac{2\pi}{3}\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Transformation to the d - q Rotor-Linked System
The transformation from the three-phase system to the $d - q$ system is given by:

$$\begin{bmatrix} \phi_d \\ \phi_q \\ \phi_0 \end{bmatrix} = P(\theta)\begin{bmatrix} \phi_a \\ \phi_b \\ \phi_c \end{bmatrix}$$

By applying this transformation to the system equations:

$$\frac{d}{dt}\begin{bmatrix} \phi_d \\ \phi_q \\ \phi_0 \end{bmatrix} = [L]\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix}$$

After multiplying by $[P^{-1}(\theta)]$, the new system becomes:

$$\begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix} = [L]\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \begin{bmatrix} \phi_d \\ \phi_q \\ \phi_0 \end{bmatrix} + [P^{-1}(\theta)]([L] - I)[\phi]$$

Model Demonstration
For synchronous motors, the flux generated b: ↓: field winding (f) is constant. By assuming I_f = constant, the following magnetic and electrical equations can be derived:

a) Magnetic Equations

- On the d-axis:

$$\phi_d = L_d I_d + M_f I_f$$

- On the f-axis:

$$\phi_f = L_f I_f + M_f I_d$$

- On the q-axis:

$$\phi_q = L_q I_q$$

b) Electrical Equations

$$\begin{cases} V_d = RI_d + \dfrac{d\varphi_d}{dt} - \left(\dfrac{d\theta}{dt}\right)\varphi_q \\ V_q = RI_q + \dfrac{d\varphi_q}{dt} + \left(\dfrac{d\theta}{dt}\right)\varphi_d \\ V_f = R_f I_f + \dfrac{d\varphi_f}{dt} \end{cases}$$

It becomes

$$\begin{cases} V_d = RI_d + L_d \dfrac{dI_d}{dt} + M_f \dfrac{dI_f}{dt} - \omega L_q I_q \\ V_q = RI_q + L_q \dfrac{dI_q}{dt} + \omega\left(L_d I_d + M_f I_f\right) \\ V_f = R_f I_f + L_f \dfrac{di_f}{dt} + L_d \dfrac{dI_d}{dt} \end{cases}$$

Then, the model in matrix form will be

$$\begin{bmatrix} \varphi_d \\ \varphi_q \\ \varphi_f \end{bmatrix} = \begin{bmatrix} L_d & 0 & M_f \\ 0 & L_q & 0 \\ M_f & 0 & L_f \end{bmatrix} \begin{bmatrix} I_d \\ I_q \\ I_f \end{bmatrix}$$

$$\begin{bmatrix} V_d \\ V_q \\ V_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & 0 & 0 \\ 0 & \boldsymbol{R} & 0 \\ 0 & 0 & \boldsymbol{R}_f \end{bmatrix} \begin{bmatrix} I_d \\ I_q \\ I_f \end{bmatrix} + \begin{bmatrix} L_d & 0 & M_f \\ 0 & L_q & 0 \\ M_f & 0 & L_f \end{bmatrix} \frac{\boldsymbol{d}}{\boldsymbol{dt}} \begin{bmatrix} I_d \\ I_q \\ I_f \end{bmatrix} + \boldsymbol{\omega} \begin{bmatrix} 0 & -L_q & 0 \\ L_d & 0 & M_f \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_d \\ I_q \\ I_f \end{bmatrix}$$

II.6. Expression of Electromagnetic Power and Torque:
The expression for the electromagnetic torque is given by:

$$T_e = \frac{\partial W_e}{\partial \theta_{geo}} = \frac{\partial}{\partial \theta_{geo}}(W_e)$$

In electrical machine field theory, the electromagnetic torque involved in the equation is expressed as the partial derivative of the stored electromagnetic energy with respect to the geometric rotational angle of the rotor.

Where:

- W_e is the energy stored in the magnetic circuit.
- θ_{geo} is the angular displacement of the moving part (rotor relative to the stator).

With:

$$\theta_{geo} = p \cdot \theta$$

According to Park, the expression for the power is written as:

$$P = \frac{3}{2}\left(V_d I_d + V_q I_q\right)$$

By replacing the V_d and V_q expressions, we obtain the following:

$$p(t) = \frac{3}{2}\left[\left(R_s\left(I_d^2 - I_q^2\right)\right)\right] + \frac{3}{2}\left[I_d \frac{d\phi_d}{dt} + I_q \frac{d\phi_q}{dt}\right] + \frac{3}{2}\left[\phi_d I_d - \phi_q I_q\right]\omega$$

Where:

- The 1st term represents the Ohmic losses (Joule losses),
- The 2 nd term represents the change in the stored magnetic energy,
- The 3rd term represents the power transferred from the stator to the rotor through the air gap (electromagnetic power).

Knowing that:

$$P_e = T_e W_e$$

We can also write:

$$C_e = \phi_d I_q - \phi_q I_d$$

$$C_e = \frac{3}{2} p\left(L_d I_d - L_q I_q\right) + M_f I_f$$

9 Equation of Motion:

The equation governing the motion of the machine is given by:

$$J \frac{dW_r}{dt} = C_e - C_r - f W_r$$

Where:

- J: Moment of inertia of the rotating masses;
- C_r : Load torque applied to the machine's shaft;
- C_e : Electromagnetic torque;
- W_r : Mechanical angular velocity ($\omega_r = pW_r$);
- f :Coefficent of viscous friction

II.7. State-Space Model of the Synchronous Machine (SM):

Let us consider the voltages V_d and V_q, along with the excitation flux ϕ_f, as control variables, the stator currents I_d and I_q as state variables, and the torque C_r as a disturbance. Based on equations (1.17), the following system of equations can be written:

With:

$$\mathbf{x} = \begin{bmatrix} I_d \\ I_q \end{bmatrix}, \ \mathbf{U} = \begin{bmatrix} V_d \\ V_q \\ \phi_f \end{bmatrix}$$

And:

$$\begin{bmatrix} \dot{I}_d \\ \dot{I}_q \end{bmatrix} = \begin{bmatrix} -\frac{R_s}{L_f} & \omega\frac{L_q}{L_d} \\ -\omega\frac{L_d}{L_q} & -\frac{R_s}{L_q} \end{bmatrix} \begin{bmatrix} I_d \\ I_q \end{bmatrix} + \begin{bmatrix} \frac{1}{L_d} & 0 \\ 0 & \frac{1}{L_q} \end{bmatrix} \begin{bmatrix} V_d \\ V_q \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\phi_f}{L_q} \end{bmatrix} \omega$$

Where we define:

$$\mathbf{A} = \begin{bmatrix} -\frac{R_s}{L_\mathrm{d}} & \omega\frac{L_q}{L_d} \\ -\omega\frac{L_d}{L_q} & -\frac{R_s}{L_q} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \frac{1}{L_d} & 0 \\ 0 & \frac{1}{L_q} \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 0 \\ -\frac{\phi_f}{L_q} \end{bmatrix}$$

The matrix $[A]$ can also be expressed as:

$$\mathbf{A} = \begin{bmatrix} -\frac{R_s}{L_{d_d}} & \omega\frac{L_q}{L_{L_d}} \\ -\omega\frac{L_d}{L_q} & -\frac{R_s}{L_q} \end{bmatrix}$$

The matrix $[A]$ can be expressed as:

$$\mathbf{A} = \begin{bmatrix} -\frac{R_s}{L_d} & \omega\frac{L_q}{L_d} \\ -\omega\frac{L_d}{L_q} & -\frac{R_s}{L_q} \end{bmatrix}$$

It can also be decomposed as:

$$\mathbf{A} = \mathbf{A}_{01} + \omega\mathbf{A}_{02}$$

Where:

$$\mathbf{A}_{01} = \begin{bmatrix} -\frac{R_s}{L_d} & 0 \\ 0 & -\frac{R_s}{L_q} \end{bmatrix}, \ \mathbf{A}_{02} = \begin{bmatrix} 0 & \omega\frac{L_q}{L_d} \\ -\omega\frac{L_d}{L_q} & 0 \end{bmatrix}$$

To achieve the same system, it is necessary to include the matrices $\mathbf{C}$ and $\mathbf{D}$:

$$\mathbf{C} = [1 \quad 1], \ \mathbf{D} = 0$$

This ensures the complete representation of the system in its state-space form.

The system is thus represented in the state-space form:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{BU}$$

$$\mathrm{Y} = \mathbf{Cx} + \mathbf{DU}$$

II.8. State-Space Model of the Permanent Magnet Synchronous Machine (PMSM):

The model of the Permanent Magnet Synchronous Machine (PMSM) is simpler than the model previously presented. Alternatively, it can be considered a specific case of the wound-rotor synchronous machine. Thus, the electrical model of the PMSM will be:

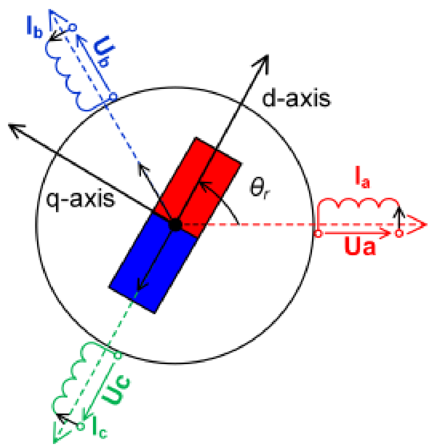


Figure 9: Equivalent Diagram of a Permanent Magnet Synchronous Machines in the Park Reference Frame

$$\begin{cases} V_d = RI_d + \dfrac{d\varphi_d}{dt} - \left(\dfrac{d\theta}{dt}\right)\varphi_q \\ V_q = I_q + \dfrac{d\varphi_q}{dt} + \left(\dfrac{d\theta}{dt}\right)\varphi_d \end{cases}$$

$$\begin{cases} \varphi_d = L_d I_d + \varphi_f \\ \varphi_q = L_q I_q \end{cases}$$

$$\begin{bmatrix} V_d \\ V_q \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \end{bmatrix} \begin{bmatrix} I_d \\ I_q \\ \varphi_f \end{bmatrix} + \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_d \\ I_q \\ \varphi_f \end{bmatrix} + \omega \begin{bmatrix} 0 & -L_q & 0 \\ L_d & 0 & 1 \end{bmatrix} \begin{bmatrix} I_d \\ I_q \\ \varphi_f \end{bmatrix}$$

The mechanical model remains the same as that of the wound-rotor synchronous machine.

$$\begin{cases} T_e = \dfrac{3}{2} P \left(\varphi_d I_q - \varphi_q I_d\right) \\ T_e - T_L - f\omega_m = J\dfrac{d\omega_m}{dt} \\ \omega = \omega_m p \end{cases}$$

Thus, where we consider the voltages V_d, V_q, and the excitation flux ϕ as control variables, and the stator currents I_d, I_q as state variables, we formulate the model as follows:

$$\begin{cases} \dfrac{dI_d}{dt} = \dfrac{V_d}{L_d} - \dfrac{R}{L_d} I_d + \omega \dfrac{L_q}{L_d} I_q \\ \dfrac{dI_q}{dt} = \dfrac{V_q}{L_q} - \dfrac{R}{L_q} I_q - \omega \dfrac{L_d}{L_q} I_d - \dfrac{\omega}{L_q} \varphi_f \end{cases}$$

Finally, the matrix form is written as:

$$\frac{d}{dt}\begin{bmatrix} I_d \\ I_q \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L_d} & \omega \dfrac{L_q}{L_d} \\ -\omega \dfrac{L_d}{L_q} & -\dfrac{R}{L_q} \end{bmatrix} \begin{bmatrix} I_d \\ I_q \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_d} & 0 & 0 \\ 0 & \dfrac{1}{L_q} & -\dfrac{\omega}{L_q} \end{bmatrix} \begin{bmatrix} V_d \\ V_q \\ \varphi_f \end{bmatrix}$$

$$[A] = \begin{bmatrix} -\dfrac{R}{L_d} & \omega \dfrac{L_q}{L_d} \\ -\omega \dfrac{L_d}{L_q} & -\dfrac{R}{L_q} \end{bmatrix} \quad [A] = [A_1] + \omega [A_2] = \begin{bmatrix} -\dfrac{R}{L_d} & 0 \\ 0 & -\dfrac{R}{L_q} \end{bmatrix} + \omega \begin{bmatrix} 0 & \dfrac{L_q}{L_d} \\ -\dfrac{L_d}{L_q} & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} \dfrac{1}{L_d} & 0 & 0 \\ 0 & \dfrac{1}{L_q} & -\dfrac{\omega}{L_q} \end{bmatrix} \quad [B] = [B_1] + \omega [B_2] = \begin{bmatrix} \dfrac{1}{L_d} & 0 & 0 \\ 0 & \dfrac{1}{L_q} & 0 \end{bmatrix} + \omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{L_q} \end{bmatrix}$$

The model can be summarized as::

$$[\dot{I}] = ([A_1] + \omega [A_2])[I] + ([B_1] + \omega [B_2])[V] \ldots$$

On the other hand, considering the voltages V_d, V_q, the excitation flux ϕ_f, the resistive torque T_L, and the electromagnetic torque T_e as control variables, and the stator currents I_d, I_q, and the mechanical speed ω_m as state variables, we formulate the model as follows:

:

$$\begin{cases} \dfrac{dI_d}{dt} = \dfrac{V_d}{L_d} - \dfrac{R}{L_d} I_d + \omega \dfrac{L_q}{L_d} I_q \\ \dfrac{dI_q}{dt} = \dfrac{V_q}{L_q} - \dfrac{R}{L_q} I_q - \omega \dfrac{L_d}{L_q} I_d - \dfrac{\omega}{L_q} \varphi_f \\ \dfrac{d\omega_m}{dt} = \dfrac{1}{J}(C_e - C_r - f\omega_m) \end{cases}$$

$$T_e = \frac{3}{2}P\left(\varphi_f I_q + L_d I_d I_q - L_q I_q I_d\right)$$

The matrix form of this statement is written as:

$$\frac{d}{dt}\begin{bmatrix} I_d \\ I_q \\ \omega_m \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_d} & \omega\frac{L_q}{L_d} & 0 \\ -\omega\frac{L_d}{L_q} & -\frac{R}{L_q} & 0 \\ 0 & 0 & \frac{f}{J} \end{bmatrix}\begin{bmatrix} I_d \\ I_q \\ \omega_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L_d} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{L_q} & -\frac{\omega}{L_q} & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{J} & \frac{1}{J} \end{bmatrix}\begin{bmatrix} V_d \\ V_q \\ \varphi_f \\ Cr \\ Ce \end{bmatrix}$$

Chapter III

Extension of the Park transformation to synchronous motors with non-sinusoidal field distribution

III.1. Introduction

In the simplified hypothesis for modeling electrical machines, it is almost always assumed that the flux linkage is distributed sinusoidally. Since this course focuses solely on the synchronous machine with a non-sinusoidal flux distribution, an accurate derivation of its transient model is necessary. Therefore, in this course, the machine model is based on the following four assumptions:

- The stator winding is sinusoidally distributed around the air gap.
- The effect of stator slots on the inductances is neglected.
- Linear magnetic circuit (no saturation).
- The effect of temperature or frequency on resistances and inductances is negligible.

The vector control of the PMSM (Permanent Magnet Synchronous Machine) based on the Park transformation will be covered in this course for better understanding The use of a pulse-width modulation (PWM) voltage inverter is of great interest for the control of electrical machines. It allows the shifting of harmonics to higher orders and facilitates the adjustment of both the amplitude and frequency of the supply voltage, which is particularly well-suited for controlling asynchronous machines. Before proceeding with the simulation of the combined converter-machine system, the power converter component is modeled first..

III.2. Extensions to the case of machines with non-sinusoidal field distribution

III.2. 1. Case of machines with trapezoidal field distribution

Machines with sinusoidal distribution are referred to as "well-constructed in the Park sense." These machines are designed to be naturally powered by sinusoidal currents. However, for legitimate technological reasons, the field distribution may be non-sinusoidal (e.g., to maximize the torque-toweight ratio). The most commonly described case is the "trapezoidal" distribution (see the curves below in Figure 1: a series of "plateaus" and slopes).

The most useful quantity is the derivative of the flux (equal to the back electromotive force divided by the angular velocity, $\phi_{af} = e_{af}/\Omega$). Over a quarter period, its expression is:

$$\dot{\varphi}_{af} = \frac{e_{af}}{\Omega} = \begin{cases} \dfrac{-E_m}{\Omega.\,\delta} \cdot p \cdot \theta \\ \dfrac{-E_m}{\Omega} \end{cases} \text{for} \begin{cases} 0 \leq p \cdot \theta \leq \delta \\ 0 \leq P \cdot \theta \leq \dfrac{\pi}{2} \end{cases} \ldots..$$

$\frac{E_m}{\Omega} = \frac{\pi \cdot \delta \cdot P_1 \cdot M_{f0} \cdot i_f}{4 \cdot \sin(\delta)}$ represents the amplitude of the plateau φ_{af}

E_m is the back electromotive force during the plateau.

The Fourier series expansions of the magnetic flux and its derivative with respect to the first phase are expressed as follows:

$$\varphi_{af}(\theta) = M_{f0} \cdot i_f \sum_{k=0}^{\infty} \frac{\sin[(2 \cdot k + 1) \cdot \delta]}{(2 \cdot k + 1)^3 \cdot \sin(\delta)} \cdot \cos[(2 \cdot k + 1) \cdot p \cdot \theta]$$

$$\dot{\varphi}_{af}(\theta) = -p \cdot M_{f0} \cdot i_f \cdot \sum_{k=0}^{\infty} \frac{\sin[(2 \cdot k + 1) \cdot \delta]}{(2 \cdot k + 1)^2 \cdot \sin(\delta)} \cdot \sin[(2 \cdot k + 1) \cdot p \cdot \theta]$$

Thus, the back electromotive force can also be expressed as:

$$e_{af}(\theta) = \frac{-4 \cdot p \cdot -E_m \cdot \sin(\delta)}{\pi \cdot \delta} \cdot \sum_{k=0}^{\infty} \frac{\sin[(2 \cdot k + 1) \cdot \delta]}{(2 \cdot k + 1)^2 \cdot \sin(\delta)} \sin[(2 \cdot k + 1) \cdot p \cdot \theta] \ldots$$

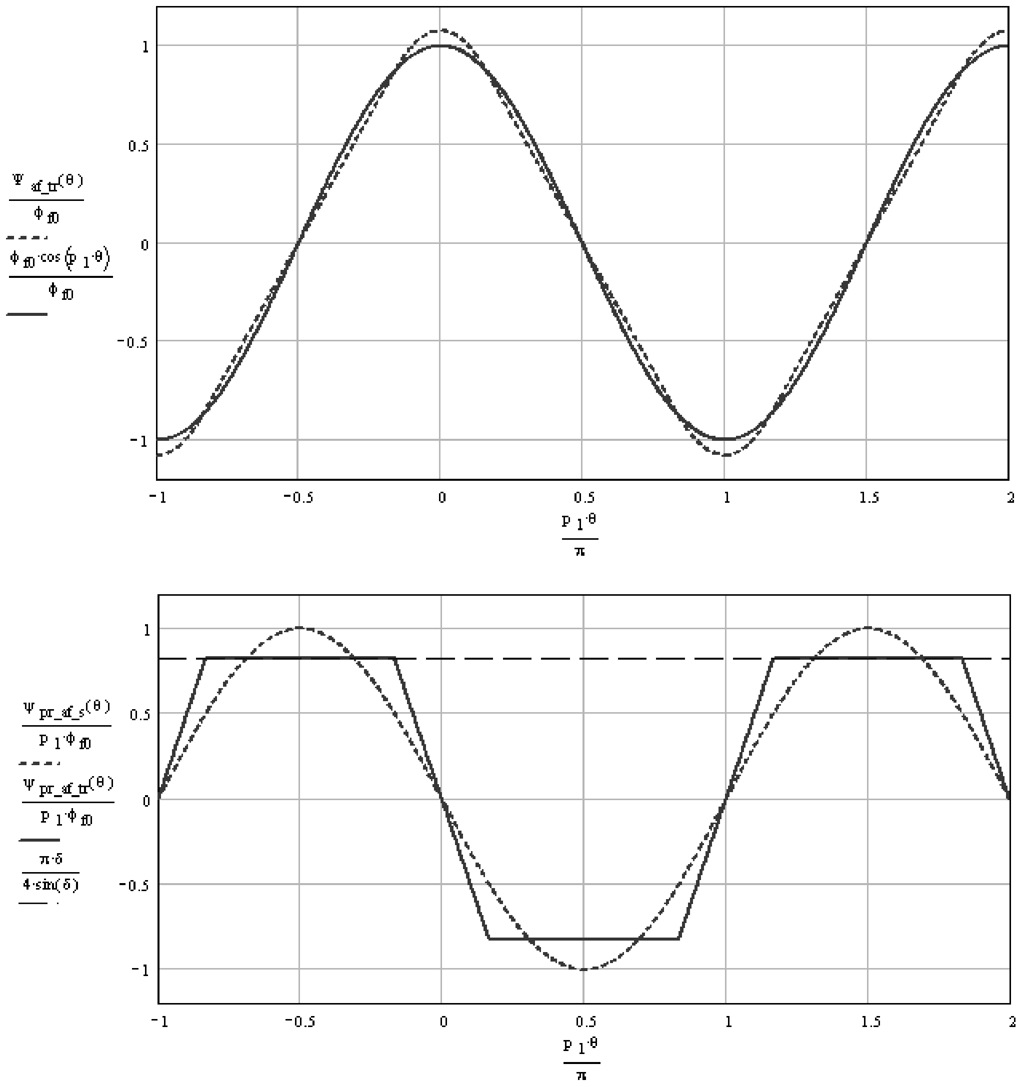


Figure 1: Flux and Flux Derivatives of a Trapezoidal Distribution Machine Compared to Their Fundamentals.

The example in Figure 1 corresponds to a classic case where $\delta = \pi/6$. We will see later that this machine can be easily supplied by a system of three-phase square wave currents. It is observed that the flux of the trapezoidal distribution machine consists of straight-line segments (when the derivative plateau, and consequently the back electromotive force, is constant), forming sawtooth patterns. These segments are connected by parabolic arcs (when the derivative varies linearly).

III.2.2. Case of Non-Sinusoidal Field Distribution Machines

The most effective cases are often those illustrated in Figure 2

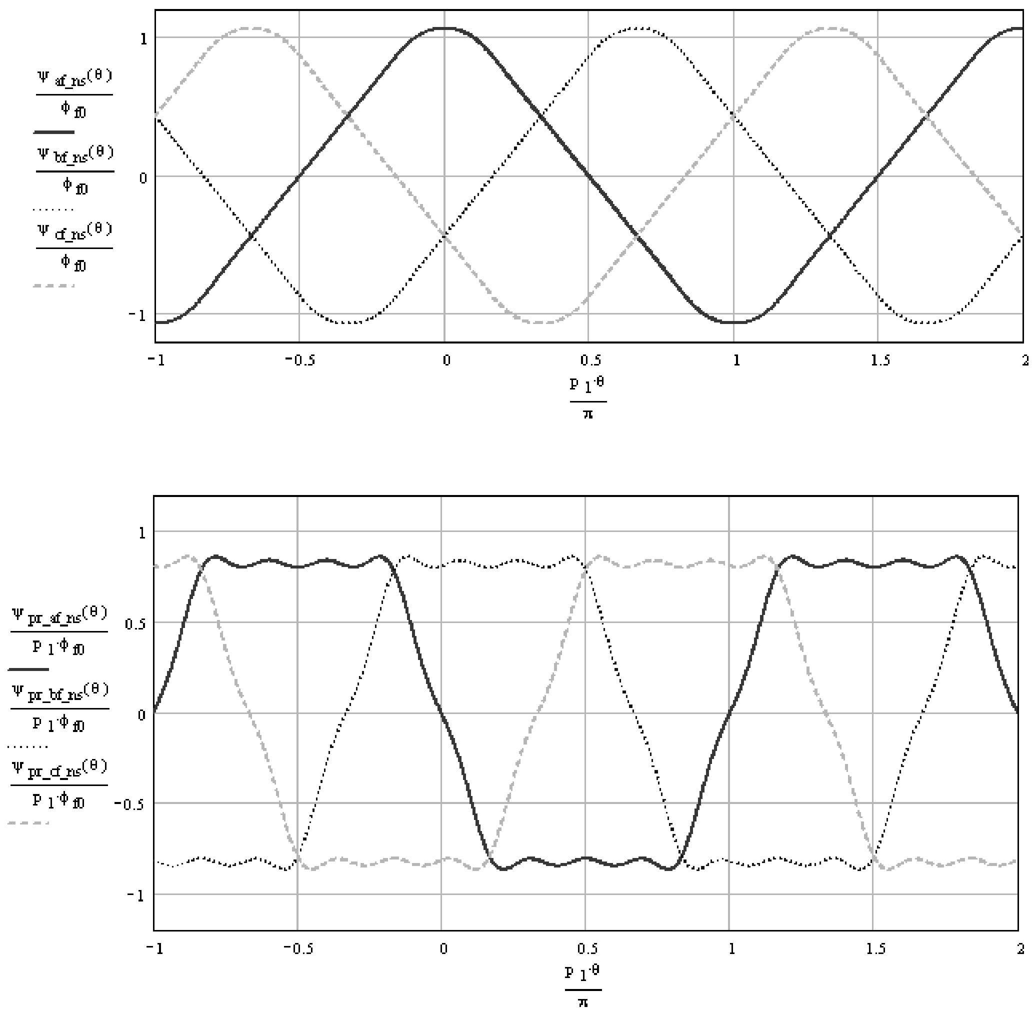


Figure 2: Flux and Flux Derivatives with Non-Sinusoidal Distribution.

In this report, the non-sinusoidal examples have been selected such that they all share the same first harmonic (or fundamental component). This component is identical to the sinusoidal model and can be considered as the first harmonic approximation of the non-sinusoidal machine models in Figures 1 and 2. It is common for these non-sinusoidal machines to be analyzed in terms of their first harmonic. However, much more accurate studies can be conducted by taking into account all the harmonic components.

In the example shown in Figure 2, the flux and its derivative are modeled using Fourier series:

$$\varphi_{af\_ns} = \phi_{f1} \cdot \cos(p \cdot \theta_1) + \phi_{f3} \cdot \cos(3 \cdot p \cdot \theta_1) + \phi_{f5} \cdot \cos(5 \cdot p \cdot \theta_1) + \phi_{f7} \cdot \cos(7 \cdot p \cdot \theta_1) + \phi_{f9} \cdot \cos(9 \cdot p \cdot \theta_1) \ldots (5)$$

$$\dot{\varphi}_{af\_ns} = -\phi_{f1} \cdot \sin(p \cdot \theta_1) - 3 \cdot \phi_{f3} \cdot \sin(3 \cdot p \cdot \theta_1) - 5 \cdot \phi_{f5} \cdot \sin(5 \cdot p \cdot \theta_1) - 7\phi_{f7} \cdot \sin(7 \cdot p \cdot \theta_1) - 9.\ \phi_{f9} \cdot \sin(9 \cdot p \cdot \theta_1) \ldots (6)$$

Figure 3 compares the flux derivatives for three cases: sinusoidal, non-sinusoidal, and trapezoidal ($\delta = \pi 6$). If we consider the "non-sinusoidal case" as the general case, we can view the trapezoidal and sinusoidal cases as two levels of approximation and idealization.

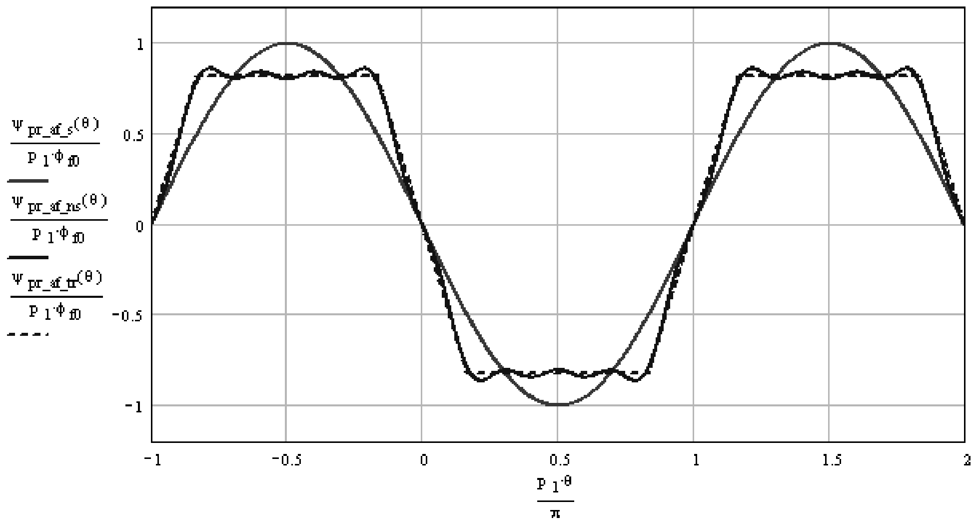


Figure 3: Flux Derivative for a Non-Sinusoidal Distribution Compared to the Sinusoidal and Trapezoidal Cases

III.3. Application of the Park Transformation to Machines with Non-Sinusoidal Flux Distribution

The Park transformation has excellent properties. It leads to the simplest possible form of the machine's equations under dynamic conditions and enables the determination of efficient control laws. However, can it be extended to machines that do not satisfy the three classical assumptions (linearity, circularity, sinusoidal field distribution)? Strictly speaking, no. However, if we are willing to abandon certain properties, we can define transformations (referred to as 'extended Park transformations') that address significant practical problems, such as directly determining the currents required to impose a desired torque without ripple and with minimized Joule losses.

In this section, we will limit our analysis to motors with constant excitation, no saturation, and no cogging torque, but with non-sinusoidal flux distribution. The flux density distribution $B_r(\theta)$ is assumed to be as illustrated in Figure 4.

In the fundamental electrical equation, the expression for flux is no longer given by the first harmonic theory but is represented by a more general expression:

$$[\varphi_{abc}] = [L_{ss}(\theta)][i_{abc}] + [\varphi_{abcf}(\theta)] \ldots . (7)$$

In this model, the flux density distribution $B_r(\theta)$ is assumed to follow the pattern shown in Figure 4.

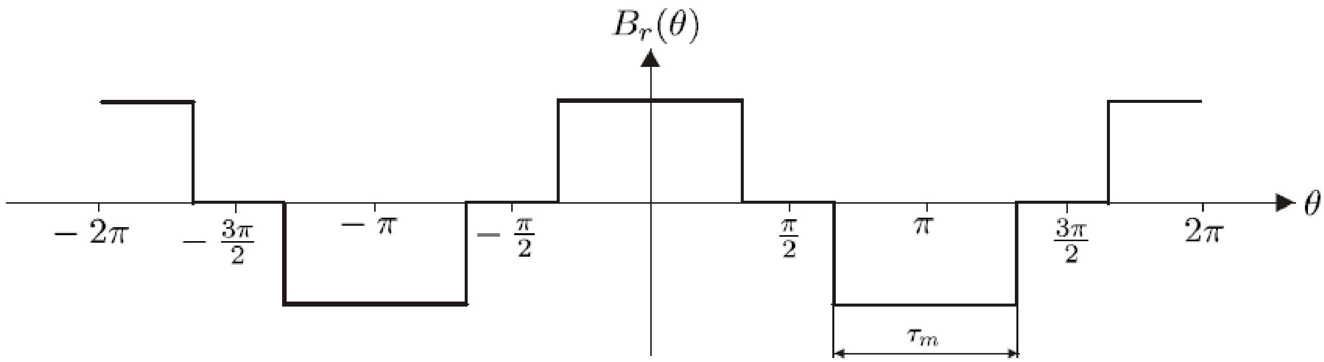


Figure 4: The Assumed Square Flux Density in the Air Gap

The flux density $B_r(\theta)$ can be expressed as a Fourier series:

$$B_r(\theta) = \sum_{i=1}^{\infty} B_{2i-1} \cdot \cos\left[(2i-1)\theta\right] = B_1 \cdot \cos(\theta) + B_3 \cdot \cos(3\theta) + B_5 \cdot \cos(5\theta) + \cdots$$

The analytical calculation of the Fourier coefficients gives:

$$B_i = \frac{4 \cdot \hat{B}}{2 \cdot i^2} \cdot \sin\left(\frac{i \cdot \tau_m}{2}\right)$$

where

$$\hat{B} = B_m(0).$$

To calculate the flux induced in a phase, we assume that the stator winding has a partially uniform distribution. This means that each phase winding is distributed over an angle with a constant turn density, as illustrated in Figure 5 below (where $\beta = \frac{2\pi}{3}$).

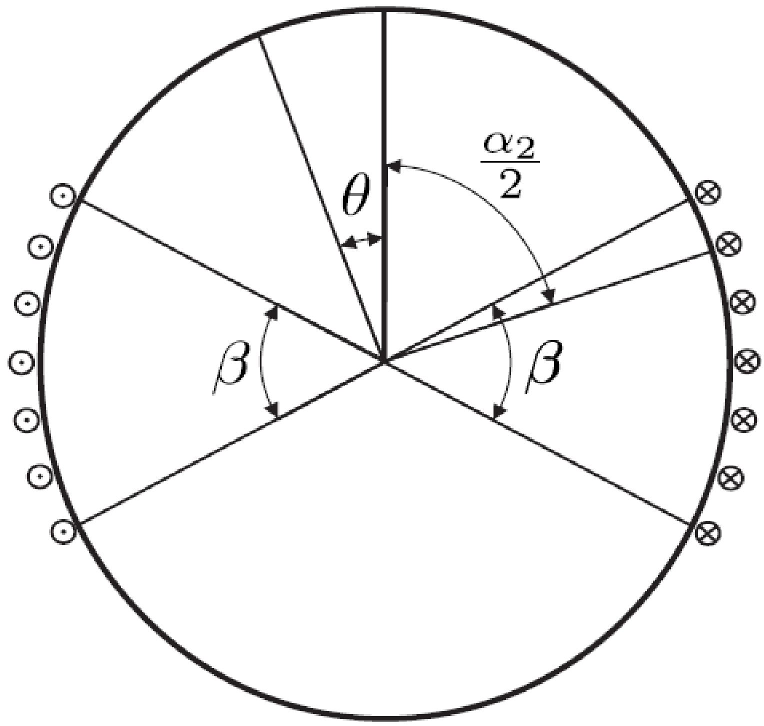


Figure 5: Simplified Model of Stator Winding for Phase a

To calculate the induced flux in a phase, the standard form $\phi = \int_S B \cdot ds$ is used. Thus, the flux can be calculated as:

$$\phi_{m,a}(\theta) = k_f \cdot \sum_{j-1}^{N_c} \left[\int_{\theta-\frac{\alpha_j}{2}}^{\theta+\frac{\alpha_j}{2}} B_r(\theta') \cdot r_s \cdot l_s d\theta' \right]$$

$$= \quad k_f \cdot \sum_{j-1}^{N_c} \left[\int_{\theta-\frac{\alpha_j}{2}}^{\theta+\frac{\alpha_j}{2}} r_s \cdot l_s \left(\sum_{i-1}^{\infty} B_{2k-1} \cdot \cos\left[(2k-1)\theta'\right] \right) d\theta' \right]$$

$$= \quad k_f \cdot r_s \cdot l_s \sum_{j-1}^{N_c} \sum_{i=1}^{\infty} \left[\int_{\theta-\frac{\alpha_j^2}{2}}^{\theta+\frac{\alpha_j}{2}} B_{2k-1} \cdot \cos\left[(2k-1)\theta'\right] d\theta' \right] \ldots$$

Where N_c is the number of stator coils for phase a, α_j is defined in Figure 5, r_s is the inner radius of the stator, l_s is the length of the stator, and k_f is a constant that depends on the specific stator design.

$$\phi_{m,a}(\theta) = 2 \cdot k_f \cdot r_s \cdot l_s \sum_{j=1}^{N_c} \sum_{i=1}^{\infty} \left[\int_{\theta-\frac{\alpha_j}{2}}^{\theta+\frac{\alpha_j}{2}} \sin \left[\frac{(2i-1)\alpha_j}{2}\right] \frac{B_{2k-1} \cdot \cos[(2k-1)\theta]}{(2k-1)} \right] \dots$$

The important conclusion that can be drawn from the last Equation is as follows: The total flux induced by the rotor in the stator winding can also be expressed as a sum of odd cosine terms, where the coefficients decrease rapidly.

$$\phi_{m,a}(\theta) = \sum_{i=1}^{\infty} \phi_{(2i-1)} \cdot \cos[(2k-1)\theta] = \phi_1 \cdot \cos(\theta) + \phi_3 \cdot \cos(3\theta) + \phi_5 \cdot \cos(5\theta) + \cdots$$

From the geometry of Figure 5, we can see that the flux induced in each phase by the constant rotor flux (as in the example of PMSM) can be written as:

$$\begin{bmatrix} \varphi_{m,a}(\theta) \\ \varphi_{m,b}(\theta) \\ \varphi_{m,c}(\theta) \end{bmatrix} = \begin{bmatrix} \varphi_{m,a}(\theta) \\ \varphi_{m,a}\left(\theta - 2\frac{\pi}{3}\right) \\ \varphi_{m,a}\left(\theta + 2\frac{\pi}{3}\right) \end{bmatrix} = \varphi_{m,ph..}$$

By applying the Park transformation, the flux $\phi_{m,\mathrm{ph}}$ is as follows:

$$\phi_{m,dq0} = P(\theta)\phi_{m,\mathrm{ph}} = \begin{bmatrix} \phi_1 + (\phi_5 + \phi_7)\cos(6\theta) + (\phi_{11} + \phi_{13})\cos(12\theta) + \cdots \\ (-\phi_5 + \phi_7)\sin(6\theta) + (-\phi_{11} + \phi_{13})\sin(12\theta) + \cdots \\ \phi_3\cos(3\theta) + \phi_9\cos(9\theta) + \phi_{15}\cos(15\theta) + \cdots \end{bmatrix}$$
$$= \begin{bmatrix} \phi_m + \phi_{d6}\cos(6\theta) + \phi_{d12}\cos(12\theta) + \cdots \\ \phi_{q6}\sin(6\theta) + \phi_{q12}\sin(12\theta) + \cdots \\ \phi_{03}\cos(3\theta) + \phi_{09}\cos(9\theta) + \phi_{015}\cos(15\theta) + \cdots \end{bmatrix}$$

We now assume that the stator inductances in the d and q directions (L_d and L_q) are constant and independent of the rotor position. This means that we can write the total flux in the stator as:

$$\begin{bmatrix} \phi_d \\ \phi_q \\ \phi_0 \end{bmatrix} = \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \\ 0 & 0 & L_0 \end{bmatrix} \begin{bmatrix} I_d \\ I_q \\ I_0 \end{bmatrix} + \phi_{m,dq0}$$

In matrix form, the stator phase voltages can now be written as:

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \phi_a \\ \phi_b \\ \phi_c \end{bmatrix}$$

By applying the transformation to the system:

$$[V] = [R][I] + \frac{d}{dt}[\phi]$$

$$P^{-1}[V_{dq0}] = [R]P^{-1}[I_{dq0}] + \frac{d}{dt}\left(P^{-1}[\phi_{dq0}]\right)$$

Then, multiplying by $P(\theta)^{-1}$:

$$[V_{dq0}] = [R][I_{dq0}] + \frac{d}{dt}[\phi_{dq0}] + P\left(\frac{d}{dt}P^{-1}\right)[\phi_{dq0}]$$

For the demonstration of the model, we have:

$$P\left(\frac{d}{dt}P^{-1}\right) = \frac{d\theta}{dt}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From equations 18 and 19, we have:

$$V_{dq0} = RI_{dq0} + L\frac{d}{dt}I_{dq0} + \frac{d}{dt}\phi_{m,dq0} + \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(LI_{dq0} + \phi_{m,dq0}\right)$$

Following equation (23), this can be written as:

$$\begin{cases} V_d = Ri_d + L_d\dfrac{di_d}{dt} - \omega \cdot L_q i_q - \omega\left(\phi_{q6}\sin(6\theta) + \phi_{q12}\sin(12\theta) + \cdots\right) \\ V_q = Ri_q + L_q\dfrac{di_q}{dt} + \omega \cdot L_d i_d + \omega(\phi_m + \phi_{d6}\cos(6\theta) + \phi_{d12}\cos(12\theta) + \cdots) \end{cases}$$

$$\begin{cases} V_d = Ri_d + L_d\dfrac{di_d}{dt} - \omega \cdot L_q i_q + \left(-\varphi_{q6} - 6 \cdot \varphi_{d6}\right)\omega \cdot \sin(6.\theta) + \left(-\varphi_{q12} - \right. \\ \left. 12.\varphi_{q12}\right)\omega \cdot \sin(12.\theta) + \cdots \\ V_q = Ri_q + L_q\dfrac{di_q}{dt} + \omega \cdot L_d i_d + \left(\varphi_{d6} + 6 \cdot \varphi_{q6}\right)\omega \cdot \cos(6 \cdot \theta) + \left(\varphi_{d12} + 12 \cdot \varphi_{q12}\right)\omega \cdot \cos(12 \cdot \theta) \\ \quad + \omega \cdot \varphi_m + \cdots \end{cases}$$

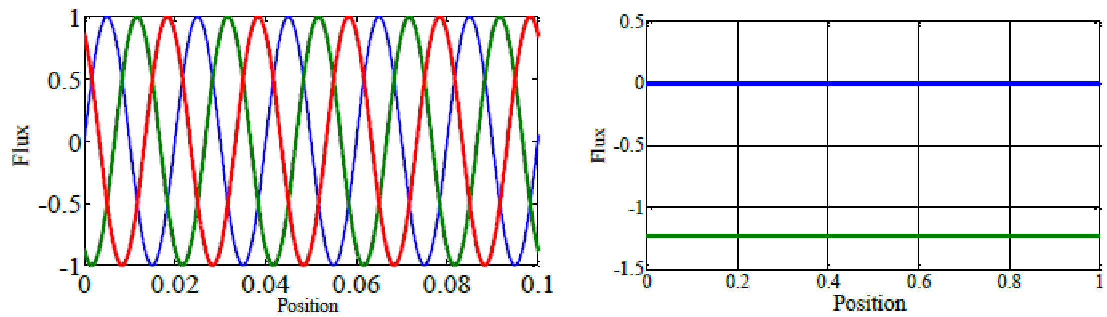


Figure 6: Example of the application of the Park transformation for a three-phase sinusoidal system

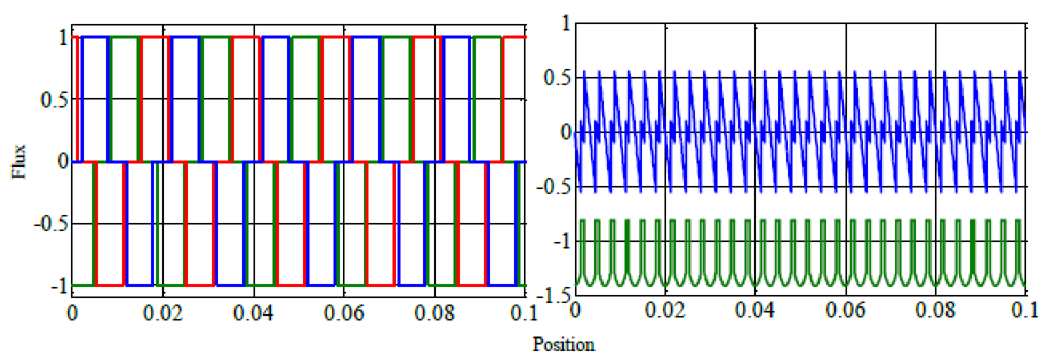


Figure 7: Example of the application of the Park transformation for a three-phase non-sinusoidal system

Chapter Iv

Converter-Machine System Modeling

IV.1. Introduction

The control of a machine cannot be discussed without considering the converter used for its operation and the associated power supply. For example, the control of an asynchronous machine (MAS) is commonly achieved by adjusting the frequency of its power supply through a voltage inverter.

This chapter covers the operating principles of a three-phase voltage inverter, the different types of PWM control, vector modeling of PWM control, as well as classical vector control and sinusoidal-triangle control. The chapter concludes with a summary.

IV.2. Modeling of the Three-Phase Voltage Inverter

The two-level three-phase voltage inverter consists of two switches per leg, characterized by their bidirectional nature and full controllability during both opening and closing. Depending on the application, these switches can be MOSFETs for low power and very high frequencies, IGBTs for high power and elevated frequencies, or GTOs for very high power and low frequencies. To enable bidirectional current flow, each switch is connected in anti-parallel with a diode.

The general structure of a two-level voltage inverter is illustrated in Figure (1). To develop a functional model of the inverter, it is assumed that each arm consists of two switches T_i and T_i', which are considered ideal (i.e., switching phenomena are neglected).

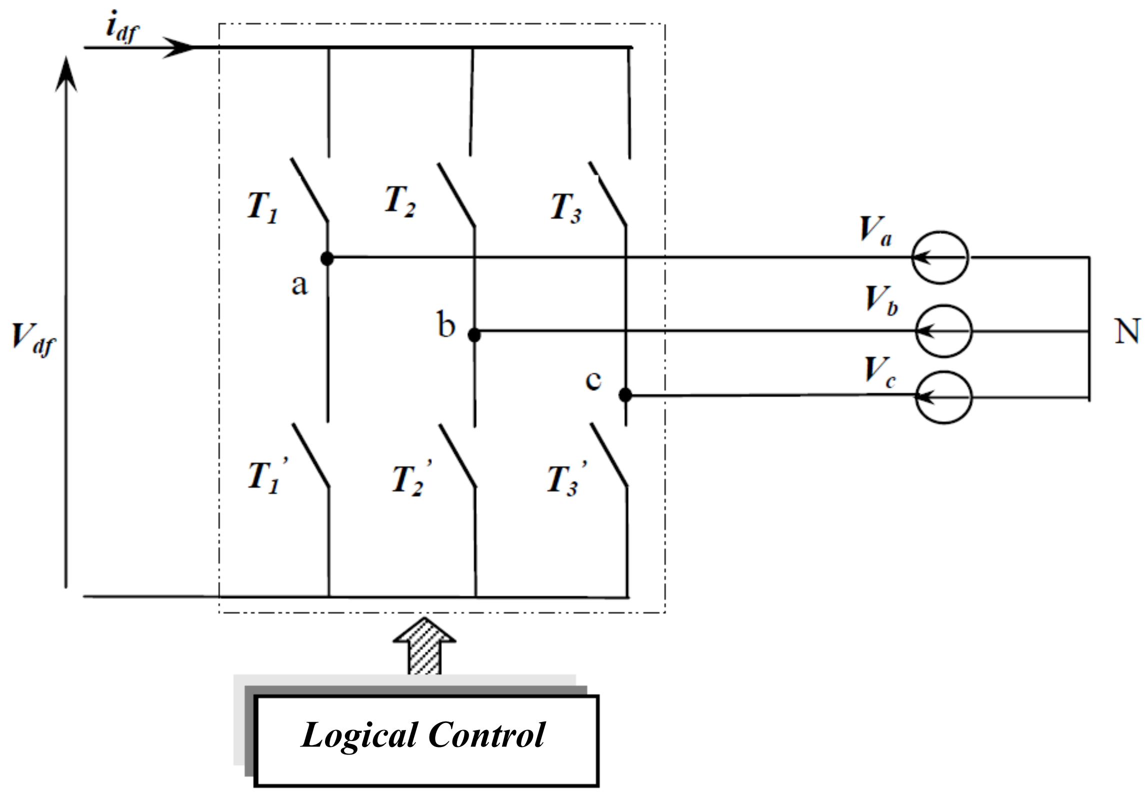


Figure 1: Simplified Diagram of a Voltage Inverter

For each arm of the inverter, a logical connection function F_i is defined to describe the state of each switch. Its value is:

$$F_i = \begin{cases} 1 & \text{if } T_i \text{ is closed and } T_i' \text{ is open} \\ 0 & \text{if } T_i \text{ is open and } T_i' \text{ is closed} \end{cases}$$

The potentials at the nodes a, b, c of the inverter relative to point N are given by the following relationships:

$$V_{aN} = F_1 V_{df},\ V_{bN} = F_2 V_{df},\ V_{cN} = F_3 V_{df}$$

The line-to-line voltages of the inverter can be determined using the connection functions as follows:

$$\begin{aligned} U_{ab} &= V_{aN} - V_{bN} = V_{df}(F_1 - F_2) \\ U_{bc} &= V_{bN} - V_{cN} = V_{df}(F_2 - F_3) \\ U_{ca} &= V_{cN} - V_{aN} = V_{df}(F_3 - F_1) \end{aligned}$$

The phase voltages can also be expressed in terms of the line-to-line voltages as follows:

$$V_a = V_{aN} = \frac{U_{ab} - U_{ca}}{3},\ V_b = V_{bN} = \frac{U_{bc} - U_{ab}}{3},\ V_c = V_{cN} = \frac{U_{ca} - U_{bc}}{3}$$

The matrix form of the phase voltages of the inverter, expressed using the logical connection functions, can be derived as following:

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \frac{V_{df}}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Alternatively:

$$[V_{abc}] = V_{df}[T_c][F]$$

With :

$$[T_c] = \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

$[T_c]$: continuous-to-alternating transfer matrix
The input current to the inverter i_{df} can be expressed in terms of the load currents as follows:

$$i_{df} = F_1 i_a + F_2 i_b + F_3 i_c$$

where:

$$i_a + i_b + i_c = 0,\ V_a + V_b + V_c = 0$$

when the neutral of the load is isolated.

b. Inverter Control Strategy

b.1. Sinusoidal-Triangle Control

Various control laws have been implemented to generate pulse-width modulated (PWM) waveforms. Historically, the sinusoidal-triangle method-also known as the "sub-oscillation method"—was the first to be used. Despite its age, it remains the most frequently utilized technique due to its simplicity.

The fundamental principle of sinusoidal-triangle pulse-width modulation (PWM) is achieved by comparing a low-frequency modulating signal (reference voltage) with a high-frequency triangular carrier wave. Switching instants are determined by the intersection points between the carrier and the modulating signals. The switching frequency of the inverter switches is governed by the carrier frequency.

In a three-phase system, three sinusoidal reference signals are used, phase-shifted by $2\pi/3$, and all have the same frequency f_s. Since the output voltage of the inverter is not purely sinusoidal, the resulting current is also non-sinusoidal, containing harmonics. These harmonics are the primary source of disturbances, such as electromagnetic torque pulsations, which cause additional losses. The sinusoidal PWM technique is employed to address these issues and offers several advantages:

- **Adjustable output voltage frequency**, allowing control flexibility.
- **Harmonics are shifted to higher frequencies**, making them easier to filter.

The benefits of these advantages include:

- **Reduced current distortion**, enhancing system performance.
- **Lower cost for the output filter**, making the system more economical.

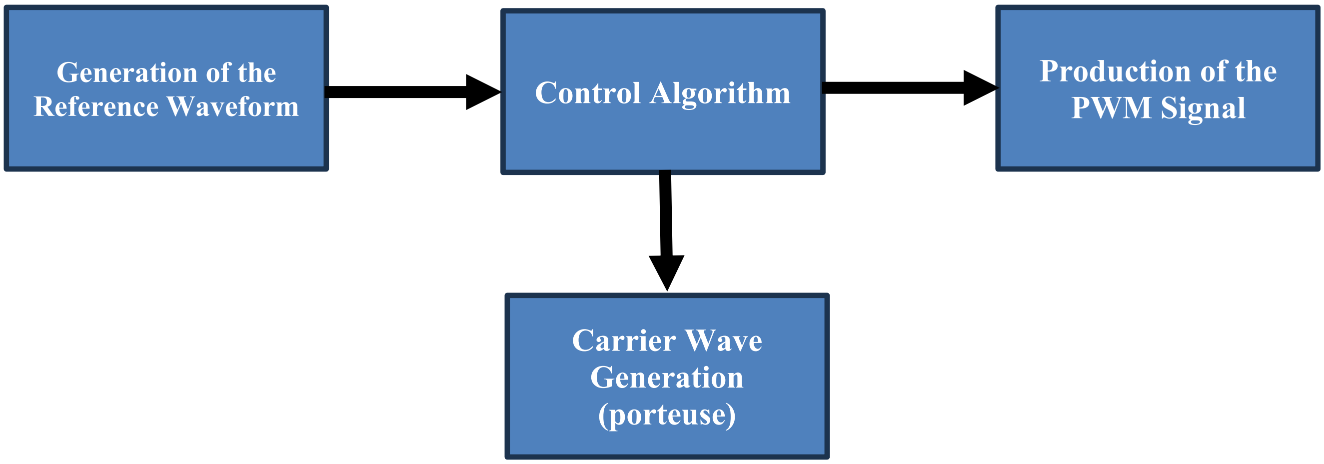


Figure 2: Principle of Sinusoidal Control (Sinusoidal PWM)

The principle of this strategy can be summarized by the following algorithm:

$$\text{If } U_r(t) \geq U_p(t) \Rightarrow S(t) = 1, \text{ else } S(t) = 0$$

Where:

- $U_r(t)$ is the reference voltage,
- $U_p(t)$ is the carrier voltage,
- $S(t)$ is the resulting PWM signal.

If the reference is sinusoidal, two parameters characterizing the PWM are used:

- The modulation index m_r which represents the ratio of the carrier frequency f_p to the reference frequency $f_{\text{ref:}}$:

$$m = \frac{f_p}{f_{\text{ref}}} = \frac{T_p}{T_{\text{ref}}}$$

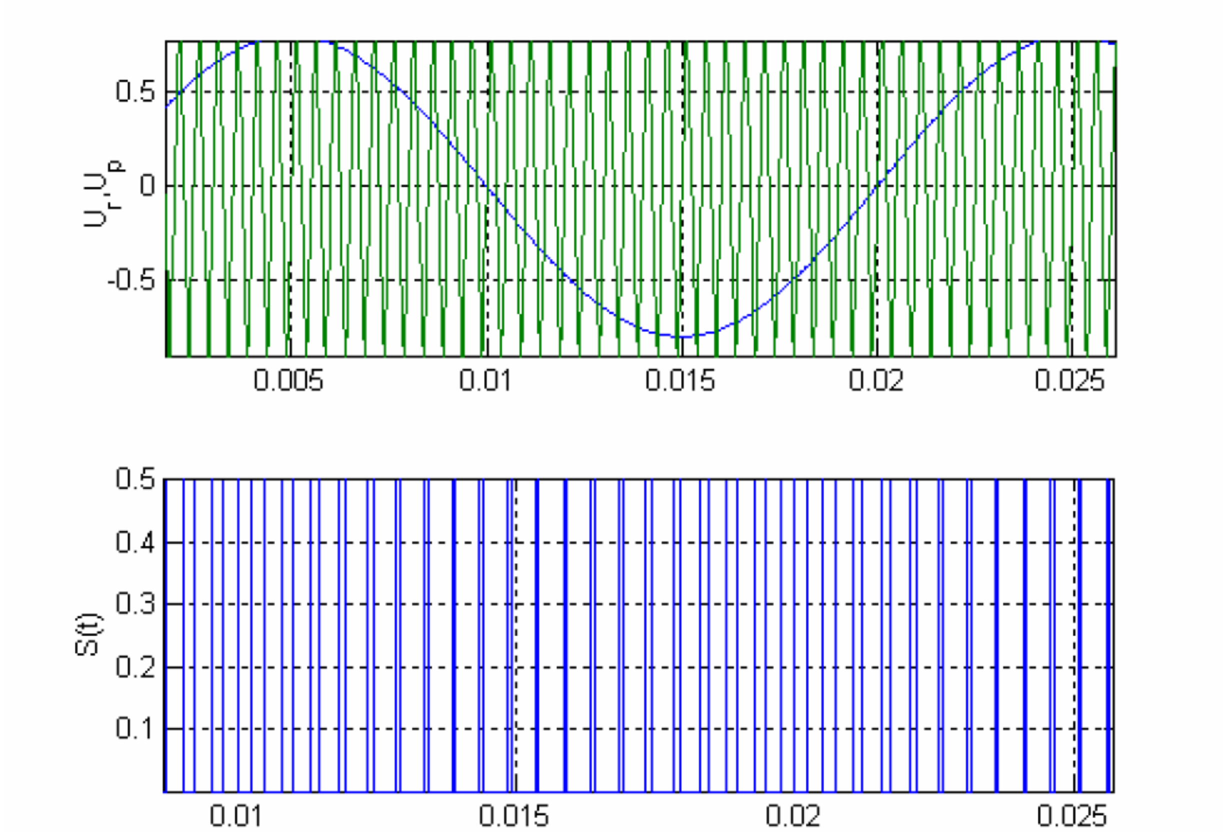


Figure 3: Sinusoidal Triangle PWM (Sinus-Carrier PWM)

The modulation index (voltage control coefficient) " r ", which represents the ratio of the amplitudes of the reference voltage $V_{\mathrm{m\ ref}}$ to the carrier voltage V_{mp}, is given by:

$$r = \frac{V_{\mathrm{m\ ref}}}{V_{\mathrm{mp}}}$$

The reference voltages for the three-phase inverter are designed to generate a balanced three-phase system. These are:

$$V_1 = E_{\mathrm{ref}}\sin(wt)$$
$$V_2 = E_{\mathrm{ref}}\sin\left(wt - \frac{2\pi}{3}\right)$$
$$V_3 = E_{\mathrm{ref}}\sin\left(wt - \frac{4\pi}{3}\right)$$

Block diagram of the Sinusoidal Triangle Control (MLIST)

The functional diagram in **Figure 4** illustrates the principle of the sinusoidal triangle control. The output voltage and current waveforms of the three-phase inverter for an RLC load are shown in the following figures:

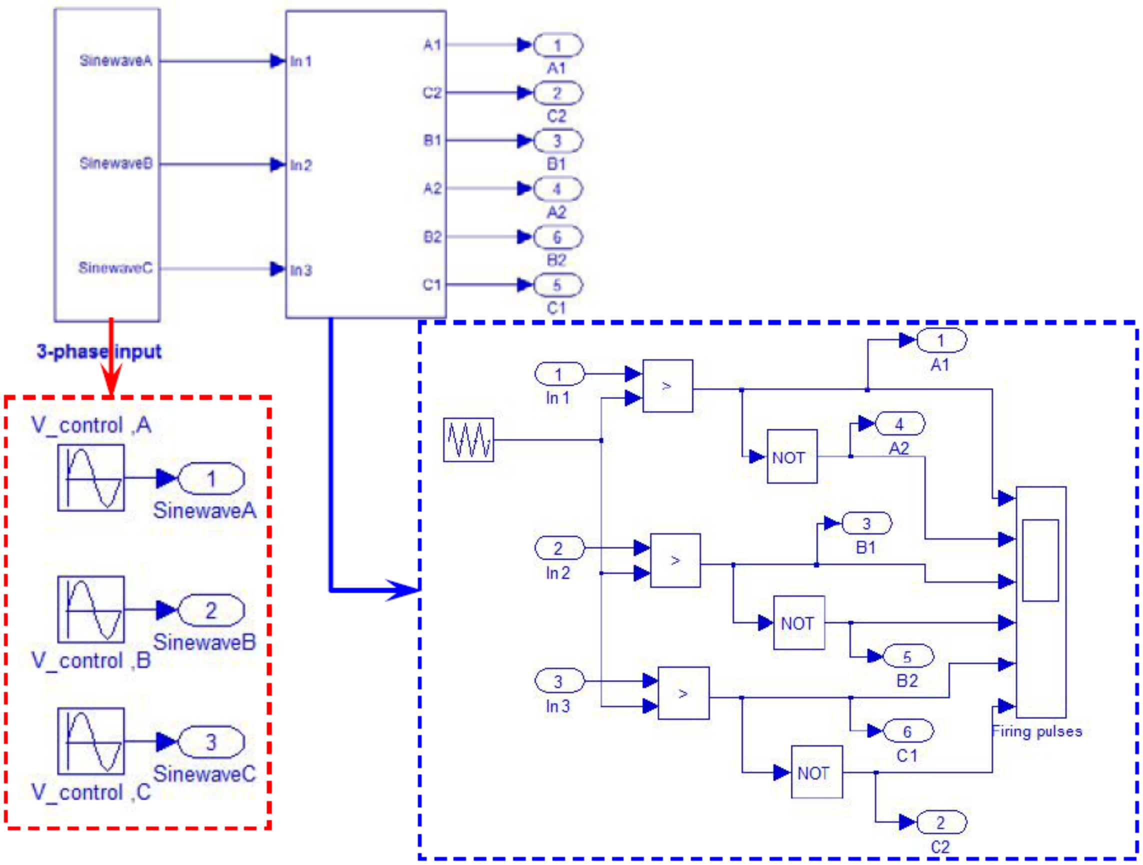


Figure 4: Functional Diagram of the Sinusoidal Triangle Control

B.2. Sinusoidal Sawtooth PWM Control

The sinusoidal sawtooth modulation technique, often referred to as **sinusoidal-triangle** or **sinusoidal-sawtooth PWM**, is an alternative to the sinusoidal triangle method for generating Pulse Width Modulation (PWM) signals. It uses a sawtooth waveform as the carrier signal instead of a triangular waveform, and its principle is similar to the sinusoidal-triangle method, where the sinusoidal reference signal is compared with the sawtooth carrier to generate the switching signals for the inverter. **Figure 5** shows the waveform of the so-called sinusoidal sawtooth PWM.

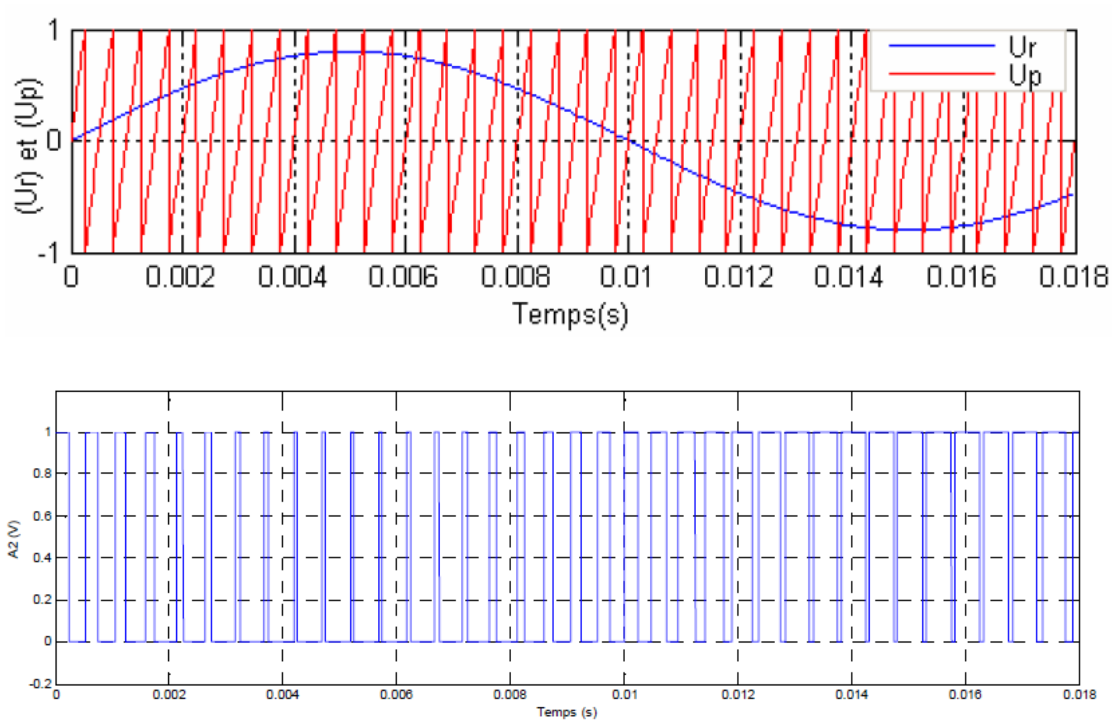


Figure 5: Illustration of Sawtooth PWM.

The block diagram of the sinusoidal sawtooth control (MLISD)

The functional diagram in **Figure 6** shows the principle of sinusoidal sawtooth control.

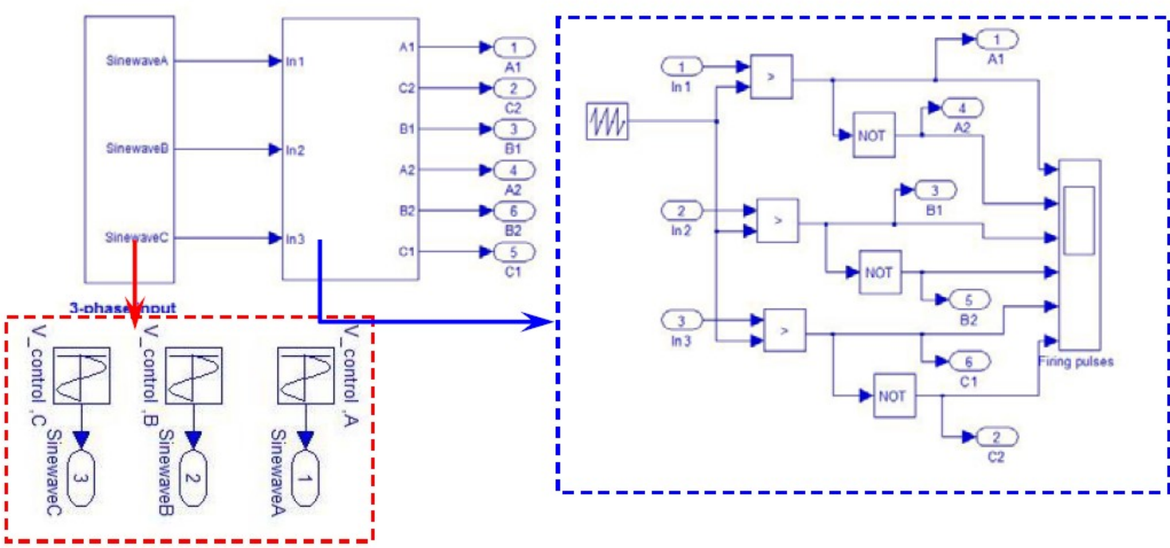


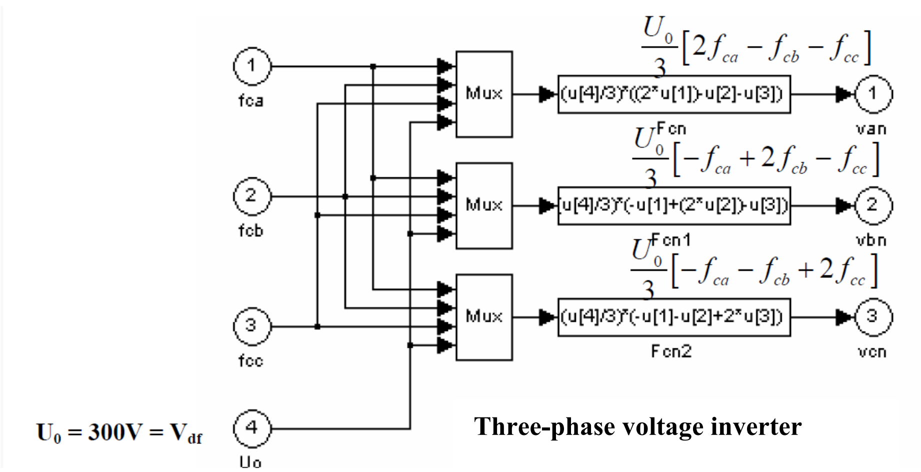
Figure 6: Functional diagram of the sinusoidal sawtooth control (MLISD).

III.5 Simulation

In this section, we address a real case considering the static converter. The voltage inverter controlled by PWM, which we use to power an RL load, is characterized by a switching frequency of 2.5 kHz.

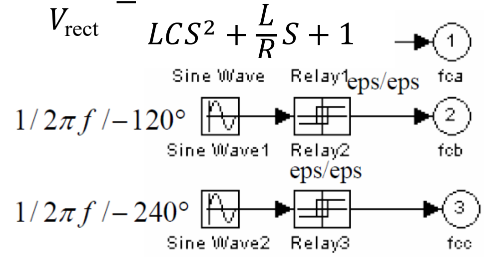
Function y=rect(x)
y=max(x)-min(x)

III.5.2. Diagram of the three-phase voltage inverter simulation:

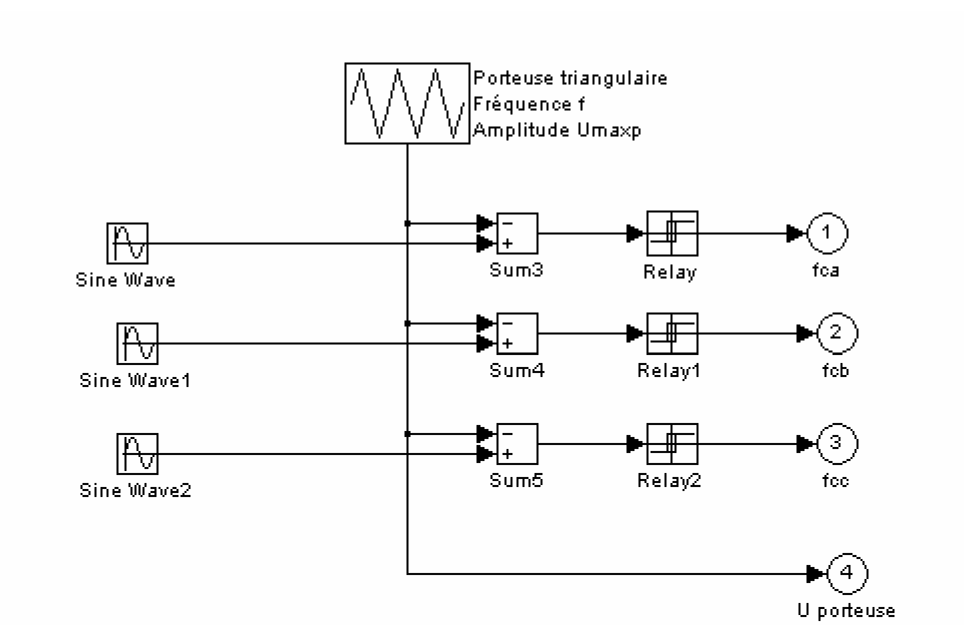


Three-phase voltage inverter

$$\frac{V_{df}}{V_{rect}} = \frac{1}{LCS^2 + \frac{L}{R}S + 1}$$



full-wave control



Pulse Width Modulation (PWM) Control

IV.6. Conclusion

This chapter explored the modeling and control of a three-phase voltage inverter, focusing on its structure, switching characteristics, and PWM techniques. Sinusoidal-triangle and sinusoidal-sawtooth control methods were examined for their role in improving waveform quality and system efficiency. Finally, a simulation demonstrated the practical application of a PWM-controlled inverter.

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Chapter v

Dynamic Modeling of Asynchronous Machines

V.1. Introduction

The asynchronous machine is widely used across various applications, surpassing synchronous and direct current (DC) machines. It exhibits a nonlinear dynamic system. Consequently, its control requires an accurate model that faithfully represents its behavior in terms of electrical, electromagnetic, and mechanical modes.

This chapter will introduce the linearized modeling of the asynchronous machine by expressing it in state-space form.

V.2. Structure and Modeling Assumptions

The asynchronous machine consists of two main components:
- The stator (primary component): A stationary structure that typically contains a three-phase winding, housed in evenly distributed slots on its inner surface and connected to the power supply.
- The rotor (secondary component): A rotating part around the machine's axis of symmetry, which can either be wound and short-circuited or designed as a squirrel-cage rotor.

For the purpose of simplifying analysis and mathematical modeling, the following assumptions are made:
- The winding distribution ensures a sinusoidal magnetomotive force (MMF) when supplied with sinusoidal currents.
- The machine operates in an unsaturated regime, where hysteresis, eddy currents, and skin effect are neglected.
- The homopolar component is considered negligible.

Under these conditions, assuming a three-phase configuration for both the stator and rotor (as illustrated in Figure 1), the machine's behavior is governed by three fundamental sets of equations:
- Electrical equations
- Magnetic equations
- Mechanical equation

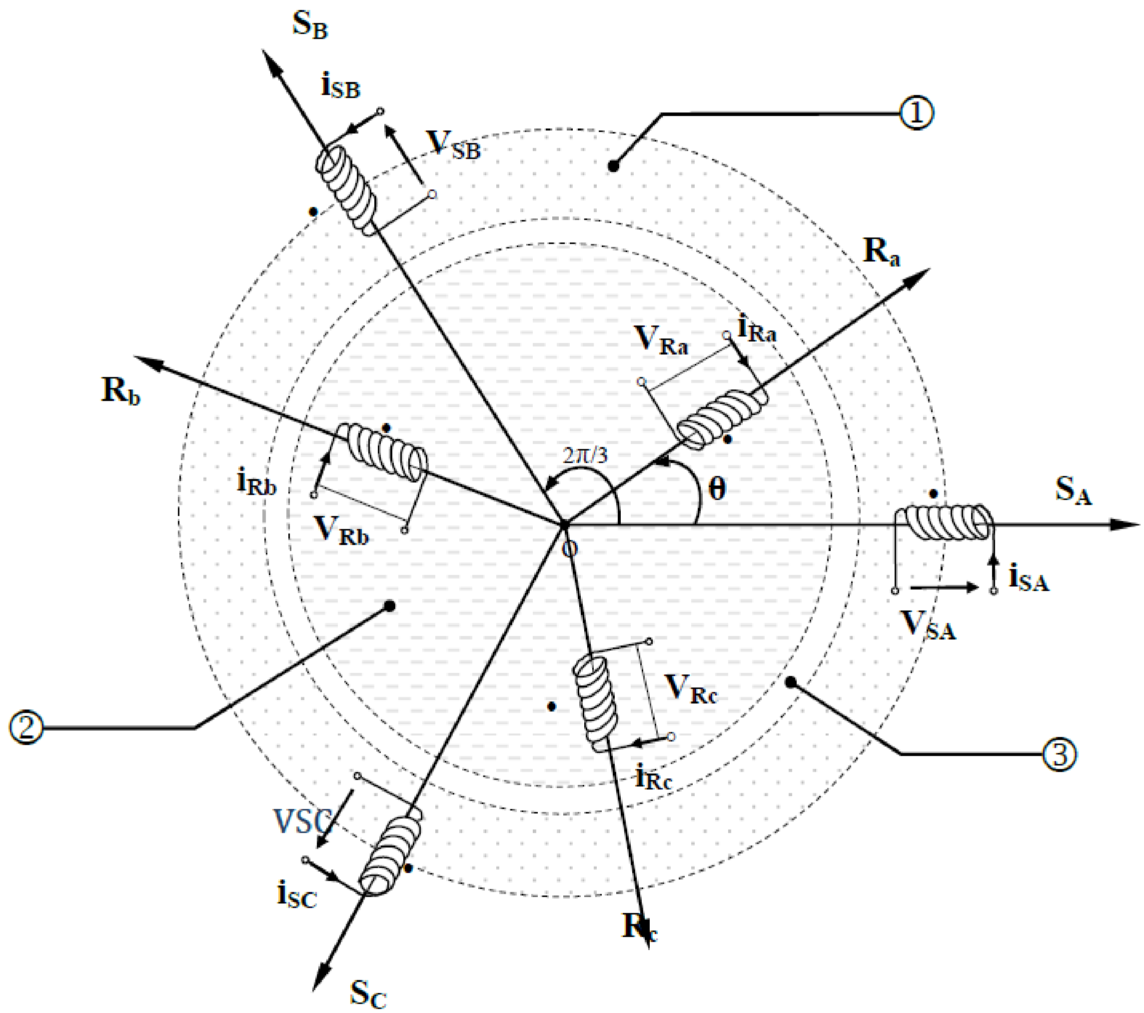

Figure 1: Three-Phase Model of the Asynchronous Machine**.**

V.3. Electrical,mechanical and magnetic equations

V.3. 1. Electrical equations

The windings of the three stator phases are spatially shifted by an angle of $2\pi/3$, as are those of the rotor, and can be represented as shown in Figure (1). The rotor phases are short-circuited. The angle θ represents the electrical angle between the axis of the stator phase (Sa) and the rotor phase (Ra). By applying the generalized Ohm's law to each phase of the stator (and similarly to the rotor), the following voltage equations are obtained.

For the stator phases:

$$\begin{bmatrix} V_{sa} \\ V_{sb} \\ V_{sc} \end{bmatrix} = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & R_s \end{bmatrix} \begin{bmatrix} I_{sa} \\ I_{sb} \\ I_{sc} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \varphi_{sa} \\ \varphi_{sb} \\ \varphi_{sc} \end{bmatrix}$$

Since the stator resistance is the same for all three phases, there is no need to explicitly write a resistance matrix. Alternatively, we can express it as:

$$[V_s] = [R_s][I_s] + \frac{d}{dt}[\varphi_s]$$

Similarly, for the rotor:

$$\begin{bmatrix} V_{ra} \\ V_{rb} \\ V_{rc} \end{bmatrix} = \begin{bmatrix} R_r & 0 & 0 \\ 0 & R_r & 0 \\ 0 & 0 & R_r \end{bmatrix} \begin{bmatrix} I_{ra} \\ I_{rb} \\ I_{rc} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \varphi_{ra} \\ \varphi_{rb} \\ \varphi_{rc} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or:

$$[V_r] = [R_r][I_r] + \frac{d}{dt}[\varphi_r] = [0]$$

Since the rotor is short-circuited, its phase voltages are zero.

Where:
V_s : is the stator phase voltage matrix.
V_r : is the rotor phase voltage matrix.
I_s: is the stator current matrix.
I_r : is the rotor current matrix.

V.3. 2. Magnetic equations

Each flux interacts with the currents of all phases, including its own (concept of flux/self-inductance):

$$\begin{bmatrix} [\varphi_s] \\ [\varphi_r] \end{bmatrix} = \begin{bmatrix} [L_s] & [M_{sr}] \\ [M_{rs}] & [L_r] \end{bmatrix} \begin{bmatrix} [I_s] \\ [I_r] \end{bmatrix}$$

[$\mathbf{L}_s$] is the stator inductance matrix, given by:

$$[\mathbf{L_s}] = \begin{bmatrix} L_s & M_s & M_s \\ M_s & L_s & M_s \\ M_s & M_s & L_s \end{bmatrix}$$

where:

- L_s is the self-inductance of a single winding.

- M_s is the mutual inductance between stator windings.
  $[\mathbf{L_r}]$ is the rotor inductance matrix, expressed as:

$$[\mathbf{L_r}] = \begin{bmatrix} L_r & M_r & M_r \\ M_r & L_r & M_r \\ M_r & M_r & L_r \end{bmatrix}$$

where:

- L_r is the self-inductance of a rotor winding.
- M_r is the mutual inductance between rotor windings.

$$[M_{sr}] = [M_{rs}]^t = M_{sr} \begin{bmatrix} \cos\ \theta & \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) \\ \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\ \theta & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\ \theta \end{bmatrix}$$

where:

- θ is the electrical angle, defining the instantaneous relative position between rotor and stator axes, with the stator axes chosen as the reference.

  Finally, we obtain the system of flux equations:

$$[V_{sabc}] = [R_s][I_{sabc}] + \frac{d}{dt}([L_s][I_{sabc}] + [M_{sr}][I_{rabc}]$$
$$[V_{rabc}] = [R_r][I_{rabc}] + \frac{d}{dt}([L_r][I_{rabc}] + [M_{rs}][I_{sabc}])$$

V.3.3. Mechanical equations

The study of the dynamic characteristics of the asynchronous machine involves variations not only in electrical parameters (voltage, current, flux, and EMF) but also in mechanical parameters (torque and speed). The equation of motion for the machine is given by:

$$J\frac{d\Omega}{dt} = T_e - T_r - f\Omega$$

where:

- T_e is the electromagnetic torque of the machine [Nm].

- T_r is the resistant (static) torque at the machine's shaft [Nm].
- J is the moment of inertia [$\text{kg} \cdot \text{m}^2$].
- Ω is the rotor angular velocity or the mechanical speed of the rotor.
- f is the friction coefficient [$Nm/rad/s$].

The electrical speed of the rotor is given by:

$$\omega_e = P \cdot \Omega$$

where:

- P is the number of pole pairs.
- ω_e is the electrical angular speed of the rotor.
- Ω is the mechanical angular speed of the rotor.

So:

$$T_e - T_r = \frac{J}{P}\frac{d\omega_r}{dt} + \frac{f\omega_r}{P}$$

V.4. Two-Phase Model

The transformation from a three-phase system to a two-phase system is based on the condition of generating a rotating electromagnetic field with equal magnetomotive forces (MMFs).

This three-phase to two-phase transformation results in a family of asynchronous machine models, where both stator and rotor quantities are projected onto two quadrature axes. The idea behind this transformation is that a rotating field created by a balanced three-phase system can be equivalently generated by a two-phase system with two coils:

- Spatially shifted by $\frac{\pi}{2}$.
- Supplied with currents phase-shifted by $\frac{\pi}{2}$ in time.

The objective of this transformation is to preserve both magnetomotive forces (MMFs) and instantaneous power.

For simplicity, our study first establishes a model in the stator reference frame.

- The equivalent winding for the three-phase stator consists of two windings along:
- The direct axis (α_s).
- The quadrature axis (β_s).

- The direct axis (α_s) is aligned with the axis of the first stator phase (a_s).
- Similarly, in the rotor, the equivalent three-phase windings are replaced by two windings, α_r and β_r.

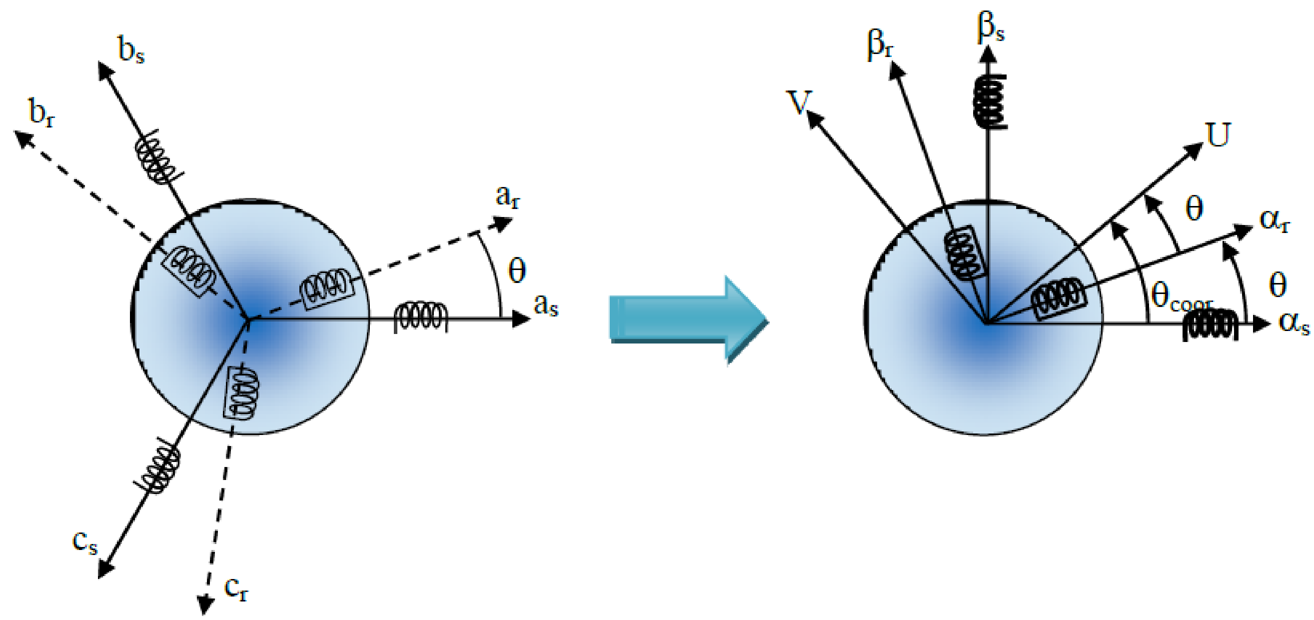

Figure 3: Representation of the Three-Phase to Two-Phase Transformation

V.4.1. Park Transformation

The Park transformation consists of a three-phase to two-phase transformation, followed by a rotation. It allows the transition from the (abc) reference frame to the $(\alpha\beta)$ reference frame, and then to the (dq) reference frame. It also enables direct transformation from the (abc) reference frame to the (dq) reference frame.

The $(\alpha\beta)$ reference frame remains fixed relative to the stator (abc) reference frame, whereas the (dq) reference frame is mobile.

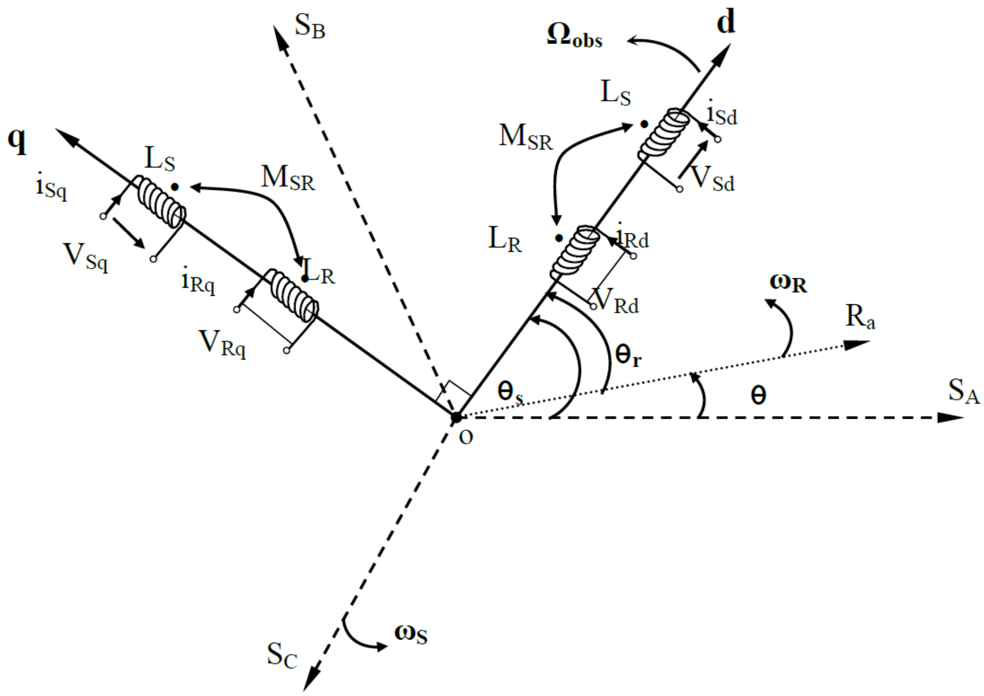

Figure 4: Angular Positioning of Axis Systems in Electrical Space

To simplify the equations, the reference frames of the Park transformation for both stator and rotor quantities must coincide, as illustrated in Figure 4. This is achieved by linking the angles θ_s and θ_r through the relation:

$$\theta_s = \theta + \theta_r$$

In this case, the normalized Park transformation is obtained using the transformation matrix:

$$[P]^{-1}[V_{dq0}] = [R][P]^{-1}[I_{dq0}] + \frac{d}{dt}\left([P]^{-1}[\phi_{dq0}]\right)$$

where the transformation matrix $[P]$ is given by:

$$[P] = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos\ \theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ -\sin\ \theta & -\sin\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The Park transformation matrix is orthogonal, meaning that its transpose is equal to its inverse:

$$[P(\theta)]^T = [P(\theta)]^{-1}$$

The Park transformation can be applied to voltages, currents, and fluxes. The variable transformation for currents, voltages, and fluxes is defined as:

$$\begin{bmatrix} x_d \\ x_q \\ x_0 \end{bmatrix} = P(\theta) \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}$$

where x represents voltage, current, or flux, and the indices denote:

- 0: Homopolar axis
- d : Direct axis
- q : Quadrature axis

Inverse of the Normalized Park Transformation Matrix

$$[P(\theta)]^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\,\theta & \sin\,\theta & \frac{1}{\sqrt{2}} \\ \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta - \frac{2\pi}{3}\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\theta - \frac{4\pi}{3}\right) & \sin\left(\theta - \frac{4\pi}{3}\right) & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The Park equations provide a dynamic electrical model for the equivalent two-phase winding of an asynchronous machine. The voltage equation in the $dq0$ reference frame is expressed as:

$$[V_{dq0}] = [R][I_{dq0}] + \frac{d}{dt}[\phi_{dq0}] + [P]\left(\frac{d}{dt}[P]^{-1}\right)[\phi_{dq0}]$$

It can be demonstrated that:

$$[P]\left(\frac{d}{dt}[P]^{-1}\right) = \frac{d\theta}{dt}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Final Park Equations

By substituting this into the voltage equation, we obtain the final system of Park equations, which serves as a dynamic electrical model for the equivalent two-phase winding:

$$\begin{cases} V_d = RI_d + \dfrac{d\phi_d}{dt} - \dfrac{d\theta}{dt}\phi_q \\ V_q = RI_q + \dfrac{d\phi_q}{dt} + \dfrac{d\theta}{dt}\phi_d \\ V_0 = RI_0 + \dfrac{d\phi_0}{dt} \end{cases}$$

Flux Transformations Between $dq0$ and abc Axes
To reduce the inductance matrix, the transformation equations establish the relationships between the flux components in the $dq0$ reference frame and the abc reference frame.

$$[\emptyset_{\text{sdqo}}] = [P(\theta_s)][\emptyset_{sabc}]$$
$$[\phi_{rdqo}] = [P(\theta_r)][\phi_{rabc}]$$

After calculation, we obtain:

$$\begin{bmatrix} \emptyset_{ds} \\ \emptyset_{qs} \\ \emptyset_{os} \\ \emptyset_{dr} \\ \emptyset_{qr} \\ \emptyset_{or} \end{bmatrix} = \begin{bmatrix} I_s - M_s & 0 & 0 & \frac{3}{2}M_{sr} & 0 & 0 \\ 0 & I_s - M_s & 0 & 0 & \frac{3}{2}M_{sr} & 0 \\ 0 & 0 & I_s + 2M_s & 0 & 0 & 0 \\ \frac{3}{2}M_{sr} & 0 & 0 & I_r - M_r & 0 & 0 \\ 0 & \frac{3}{2}M_{sr} & 0 & 0 & I_r - M_r & 0 \\ 0 & 0 & 0 & 0 & 0 & I_r + 2M_r \end{bmatrix} \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{os} \\ I_{dr} \\ I_{qr} \\ I_{or} \end{bmatrix}$$

where:

- $L_s = I_s - M_s$: Stator cyclic inductance.
- $L_r = I_r - M_r$: Rotor cyclic inductance.
- $M = \frac{3}{2}M_{sr}$: Cyclic mutual inductance between stator and rotor.

Since the usual stator power supply mode and the rotor winding structure ensure that the sums of the stator and rotor currents are zero, the components with index (0) are null.

Under these non-degraded operating conditions, the fluxes along the d and q axes are simply defined by the three constant parameters L_s, L_r, M, and are related to the currents by the equation:

$$\begin{bmatrix} \phi_{ds} \\ \phi_{qs} \\ \phi_{dr} \\ \phi_{qr} \end{bmatrix} = \begin{bmatrix} L_s & 0 & M & 0 \\ 0 & L_s & 0 & M \\ M & 0 & L_r & 0 \\ 0 & M & 0 & L_r \end{bmatrix} \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr} \\ I_{qr} \end{bmatrix}$$

V.4.1.1. Electrical Equations

The Park equations for stator and rotor voltages are expressed as:

$$\begin{cases} V_{ds} = R_s I_{ds} + \dfrac{d\phi_{ds}}{dt} - \dfrac{d\theta_s}{dt}\phi_{qs} \\ V_{qs} = R_s I_{qs} + \dfrac{d\phi_{qs}}{dt} + \dfrac{d\theta_s}{dt}\phi_{ds} \\ V_{dr} = R_r I_{dr} + \dfrac{d\phi_{dr}}{dt} + \dfrac{d\theta_r}{dt}\phi_{qr} = 0 \\ V_{qr} = R_r I_{qr} + \dfrac{d\phi_{qr}}{dt} + \dfrac{d\theta_r}{dt}\phi_{dr} = 0 \end{cases}$$

In the Park reference frame (dq) rotating at the angular speed $\omega_s = \frac{d\theta_s}{dt}$, the equations become:

$$\begin{cases} V_{ds} = R_s I_{ds} + \dfrac{d\phi_{ds}}{dt} - \omega_s \phi_{qs} \\ V_{qs} = R_s I_{qs} + \dfrac{d\phi_{qs}}{dt} + \omega_s \phi_{ds} \end{cases}$$

$$\begin{cases} R_r I_{dr} + \dfrac{d\phi_{dr}}{dt} - (\omega_s - \omega)\phi_{qr} = 0 \\ R_r I_{qr} + \dfrac{d\phi_{qr}}{dt} + (\omega_s - \omega)\phi_{dr} = 0 \end{cases}$$

V.4.1.2. Magnetic Equations

Including the flux equations, the Park equations for the stator and rotor in the (dq) reference frame are expressed as:

$$\begin{cases} \phi_{ds} = L_s I_{ds} + M I_{dr} \\ \phi_{qs} = L_s I_{qs} + M I_{qr} \end{cases}$$
$$\begin{cases} \phi_{dr} = L_r I_{dr} + M I_{ds} \\ \phi_{qr} = L_r I_{qr} + M I_{qs} \end{cases}$$

where:

- L_s is the stator cyclic inductance
- L_r is the rotor cyclic inductance
- M is the mutual inductance between the stator and rotor

V.4.1.3. Mechanical Equations

The torque equation and the motion equation are written as follows:
Electromagnetic Torque Equation:

$$T_e = PM\left(I_{qs}I_{dr} - I_{ds}I_{qr}\right)$$

Equation of Motion:

$$J\frac{d\omega}{Pdt} = T_e - T_r - \frac{f}{P}\omega$$

where:

- J is the rotor moment of inertia.
- f is the viscous friction coefficient.
- P is the number of pole pairs.
- T_e is the electromagnetic torque.
- T_r is the load torque (resistant torque).
- ω is the rotor angular velocity.

V.4.2 Choice of the dq Reference Frame

So far, we have expressed the machine's equations and variables in a dq reference frame, which makes an electrical angle θ_s with the stator and an electrical angle θ_τ with the rotor. However, this reference frame has not been explicitly defined, meaning it remains free to choose.

There are three important choices for the dq reference frame:

1 Fixed to the stator

2 Fixed to the rotor

3 Fixed to the rotating field

Since the dq reference frame is mobile, we need to determine the transformation angles θ_S and θ_T to correctly apply Park's transformations.

V.4.2.1 Stator-Fixed Reference Frame

In this case, the conditions are:

$$\theta_s = 0,\ \theta_r = -\theta$$
$$\frac{d\theta_s}{dt} = 0,\ \frac{d\theta_r}{dt} = -\frac{d\theta}{dt}$$
$$\omega_s = 0,\ \omega_r = -\omega$$

With these conditions, the electrical equations take the following form:

Electrical Equations in the Stator-Fixed Reference Frame
The system of equations in this reference frame is:

$$\begin{cases} V_{ds} = R_s I_{ds} + \dfrac{d\Phi_{ds}}{dt} \\ V_{qs} = R_s I_{qs} + \dfrac{d\Phi_{qs}}{dt} \\ 0 = R_r I_{dr} + \dfrac{d\Phi_{d+}}{dt} + \omega\Phi_{qr} \\ 0 = R_r I_{qr} + \dfrac{d\Phi_{qr}}{dt} - \omega\Phi_{dr} \end{cases}$$

Rearranging the Equations
By expressing the system in terms of $I_{ds}, I_{qs}, I_{dr}, I_{qr}$, we obtain:

$$\begin{cases} \dfrac{dI_{ds}}{dt} = -\dfrac{1}{T_s\sigma} I_{ds} + \dfrac{M^2}{L_s L_r \sigma}\omega I_{qs} + \dfrac{M}{L_s T_r \sigma} I_{dr} + \dfrac{M}{L_s\sigma}\omega I_{qr} + \dfrac{V_{ds}}{L_{p_i}\sigma} \\ \dfrac{dI_{qs}}{dt} = -\dfrac{M}{L_s L_r \sigma}\omega I_{ds} - \dfrac{1}{T_s\sigma} I_{qs} - \dfrac{M}{L_s\sigma}\omega I_{dr} + \dfrac{M}{L_s T_r \sigma} I_{qr} + \dfrac{V_{qs}}{L_s\sigma} \\ \dfrac{dI_{dr}}{dt} = \dfrac{M}{L_r T_s \sigma} I_{ds} - \dfrac{M}{L_r\sigma}\omega I_{qs} - \dfrac{1}{T_r\sigma} I_{dr} - \dfrac{1}{\sigma}\omega I_{qr} - \dfrac{M}{L_s L_r \sigma} V_{ds} \\ \dfrac{dI_{qr}}{dt} = \dfrac{M}{L_r\sigma}\omega I_{ds} + \dfrac{M}{L_r T_s \sigma} I_{qs} + \dfrac{1}{\sigma}\omega I_{dr} - \dfrac{1}{T_r\sigma} I_{qr} - \dfrac{M}{L_s L_r \sigma} V_{qs} \end{cases}$$

Definitions of Parameters

- $\sigma = 1 - \frac{M^2}{I_s I_r}$ → Total leakage coefficient
- $T_s = \frac{I_s}{R_s}$ → Stator time constant
- $T_r = \frac{I_r}{R_r}$ → Rotor time constant

V.4.2.2 Rotor-Fixed Reference Frame

In this reference frame, the conditions are:

$$\begin{cases} \theta_s = 0, & \theta_r = 0 \\ \dfrac{d\theta_s}{dt} = \dfrac{d\theta}{dt}, & \dfrac{d\theta_r}{dt} = 0 \\ \omega_r = 0, & \omega_s = \omega \end{cases}$$

Thus, the electrical equations are:

$$\begin{cases} V_{ds} = R_s I_{ds} + \dfrac{d\Phi_{ds}}{dt} - \omega_s \Phi_{qs} \\ V_{qs} = R_s I_{qs} + \dfrac{d\Phi_{qs}}{dt} + \omega_s \Phi_{ds} \\ 0 = R_T I_{dr} + \dfrac{d\Phi_{dt}}{dt} \\ 0 = R_T I_{qr} + \dfrac{d\Phi_{qr}}{dt} \end{cases}$$

After rearranging the equations, we obtain:

$$\begin{cases} \dfrac{dI_{ds}}{dt} = -\dfrac{1}{T_s\sigma} I_{ds} + \dfrac{1}{\sigma}\omega I_{qs} + \dfrac{M}{L_s T_r \sigma} I_{dr} + \dfrac{M}{L_s\sigma}\omega I_{qr} + \dfrac{V_{ds}}{L_s\sigma} \\ \dfrac{dI_{qs}}{dt} = -\dfrac{1}{\sigma}\omega I_{ds} - \dfrac{1}{T_s\sigma} I_{qs} - \dfrac{M}{L_s\sigma}\omega I_{dr} + \dfrac{M}{L_s T_r\sigma} I_{qr} + \dfrac{V_{qs}}{L_s\sigma} \\ \dfrac{dI_{dr}}{dt} = \dfrac{M}{L_r T_s\sigma} I_{ds} - \dfrac{M}{L_r\sigma}\omega I_{qs} - \dfrac{1}{T_r\sigma} I_{dr} - \dfrac{M^2}{L_s L_r\sigma}\omega I_{qr} - \dfrac{M}{L_s L_r\sigma} V_{ds} \\ \dfrac{dI_{dr}}{dt} = \dfrac{M}{L_r\sigma}\omega I_{ds} + \dfrac{M}{L_r T_s\sigma} I_{qs} + \dfrac{M^2}{L_s L_r\sigma}\omega I_{dr} - \dfrac{1}{T_r\sigma} I_{qr} - \dfrac{M}{L_s L_r\sigma} V_{qs} \end{cases}$$

V.4.2.3 Rotating Field Reference Frame

This reference frame does not introduce simplifications in the formulation of the equations. However, it transforms sinusoidal quantities in steady-state operation into DC quantities, making it particularly useful for control applications.

The conditions for this reference frame are:

$$\begin{cases} \dfrac{d\theta_s}{dt} = \omega_s \\ \dfrac{d\theta_r}{dt} = \omega_s - \omega = \omega_r \end{cases}$$

Thus, the electrical equations take the form:

$$\begin{cases} V_{ds} = R_s I_{ds} + \dfrac{d\Phi_{ds}}{dt} - \omega_s \Phi_{qs} \\ V_{qs} = R_s I_{qs} + \dfrac{d\Phi_{qs}}{dt} + \omega_s \Phi_{ds} \\ 0 = R_r I_{dr} + \dfrac{d\Phi_{dr}}{dt} - \omega_r \Phi_{qr} \\ 0 = R_r I_{qr} + \dfrac{d\Phi_{qr}}{dt} + \omega_r \Phi_{dr} \end{cases}$$

After rearranging the equations, we obtain:

$$\begin{cases} \dfrac{dI_{ds}}{dt} = \dfrac{1}{T_s\sigma} I_{ds} + \left(\omega_r + \dfrac{1}{\sigma}\omega\right) I_{qs} + \dfrac{M}{L_s T_r \sigma} I_{dr} + \dfrac{M}{L_s\sigma}\omega I_{qr} + \dfrac{1}{L_s\sigma} V_{ds} \\ \dfrac{dI_{dr}}{dt} = -\left(\omega_r + \dfrac{1}{\sigma}\omega\right) I_{ds} - \dfrac{1}{T_s\sigma} I_{qs} - \dfrac{M}{L_s\sigma}\omega I_{dr} + \dfrac{M}{L_s T_r \sigma} I_{qr} + \dfrac{1}{L_s\sigma} V_{qs} \\ \dfrac{dI_{dr}}{dt} = \dfrac{M}{L_r T_s \sigma} I_{ds} - \dfrac{M}{L_r\sigma}\omega I_{qs} - \dfrac{1}{T_r\sigma} I_{dr} + \left(\omega_r - \dfrac{M^2}{L_s L_r \sigma}\omega\right) I_{qr} - \dfrac{M}{L_s L_r \sigma} V_{ds} \\ \dfrac{dI_{qr}}{dt} = \dfrac{M}{L_r\sigma}\omega I_{ds} + \dfrac{M}{L_r T_s \sigma} I_{qs} + \left(-\omega_r + \dfrac{M^2}{L_s L_r \sigma}\omega\right) I_{dr} - \dfrac{1}{T_r\sigma} I_{qr} - \dfrac{M}{L_s L_r \sigma} V_{qs} \end{cases}$$

V.5. State-Space Representation for Control

The generalized form of the system equations in state-space representation is given by:

$$\begin{cases} \dot{X}(t) = AX(t) + BU(t) \\ Y(t) = CX(t) + DU(t) \end{cases}$$

Where:

- $X(t)$ is the state vector of the system (dimension n).
- $U(t)$ is the input vector (control variables).
- $Y(t)$ is the output vector (measured variables).
- A is the state matrix (describes system dynamics).
- B is the input matrix (relates inputs to states).
- C is the output matrix (relates states to outputs).
- D is the direct transmission matrix (relates inputs directly to outputs, often zero in many cases).

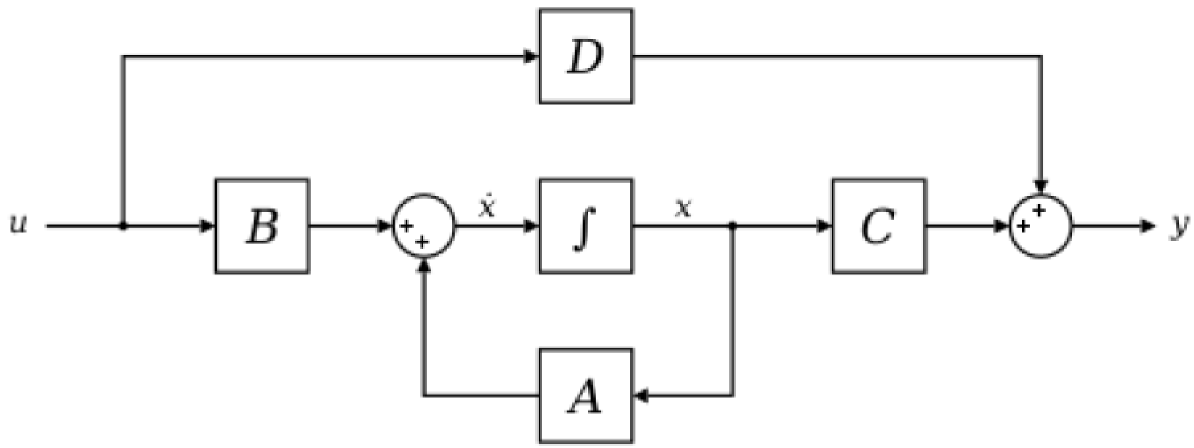

Figure.5: Representation of the Dynamic State Model in Functional Diagram Form

The state vector, input vector, or output vector is chosen based on control requirements or modeling needs. The models are as follows:

Considering the voltages V_{ds}, V_{qs}, V_{dr}, and V_{qr} as control variables and the stator and rotor currents I_{ds}, I_{qs}, I_{dr}, and I_{qr} as state variables, we can rewrite the system of equations based on the reference frame fixed to the rotating field.

By defining:

- $T_s = \frac{L_s}{R_s}$: Stator time constant.
- $T_r = \frac{L_r}{R_r}$: Rotor time constant.
- $\sigma = 1 - \frac{M^2}{L_s \cdot L_r}$: Blondel's leakage coefficient.

The state vector is:

$$[x] = \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr} \\ I_{qr} \end{bmatrix}$$

The state matrix A is written as:

$$[A] = \begin{bmatrix} -\frac{1}{T_s\sigma} & \left(\omega_r + \frac{\omega}{\sigma}\right) & \frac{M}{L_s T_i \sigma} & \frac{M}{L_s\sigma}\omega \\ -\left(\omega_r + \frac{\omega}{\sigma}\right) & -\frac{1}{T_s\sigma} & -\frac{M}{L_s\sigma}\omega & \frac{M}{L_s T_r\sigma} \\ \frac{M}{L_r T_s\sigma} & -\frac{M}{L_r\sigma}\omega & -\frac{1}{T_r\sigma} & \left(\omega_r - \frac{M}{L_s L_r\sigma}\omega\right) \\ \frac{M}{L_r\sigma}\omega & \frac{M}{L_r T_s\sigma} & \left(-\omega_r + \frac{M^2}{L_s L_r\sigma}\omega\right) & -\frac{1}{T_r\sigma} \end{bmatrix}$$

The matrix A can also be rewritten as:

$[A] = [A_1] + \omega[A_2] + \omega_r[A_3]$

Thus, the state matrix A can be expressed as:

$$[A] = \begin{bmatrix} -\frac{1}{T_s\sigma} & 0 & \frac{M}{L_s T_r\sigma} & 0 \\ 0 & -\frac{1}{T_s\sigma} & 0 & \frac{M}{L_s T_r\sigma} \\ \frac{M}{L_r T_s\sigma} & 0 & -\frac{1}{T_r\sigma} & 0 \\ 0 & \frac{M}{L_r T_s\sigma} & 0 & -\frac{1}{T_r\sigma} \end{bmatrix}$$

$$+\omega \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \omega_r \begin{bmatrix} 0 & \frac{1}{\sigma} & 0 & \frac{M}{L_s\sigma} \\ -\frac{1}{\sigma} & 0 & -\frac{M}{L_s\sigma} & 0 \\ 0 & -\frac{M}{L_r\sigma} & 0 & -\frac{M^2}{L_s L_r\sigma} \\ \frac{M}{L_r\sigma} & 0 & \frac{M^2}{L_s L_r\sigma} & 0 \end{bmatrix}$$

The system's control input vector is:

$$[U] = [V] = \begin{bmatrix} V_{ds} \\ V_{qs} \\ 0 \\ 0 \end{bmatrix}$$

The control matrix B is given by:

$$[B]=\begin{bmatrix}\frac{1}{L_s\sigma} & 0\\ 0 & \frac{1}{L_s\sigma}\\ -\frac{M}{L_sL_r\sigma} & 0\\ 0 & -\frac{M}{L_sL_r\sigma}\end{bmatrix}$$

When considering the state vector as the system's output vector Y, the matrices C and D are:

$$C=[1\quad 1\quad 1\quad 1],\ D=0$$

V.6. Power and Torque Equations

In the general case, the instantaneous electrical power P_e supplied to the stator and rotor windings is expressed as a function of the d - and q-axis components:

$$P_e=V_{sd}I_{sd}+V_{sq}I_{sq}+V_{rd}I_{rd}+V_{rq}I_{rq}$$

This power can be decomposed into three terms corresponding to the three columns of the voltage equations:

1 Joule Losses (Dissipated Power):

$$P_j=R_s\left(I_{sd}^2+I_{sq}^2\right)+R_r\left(I_{rd}^2+I_{rq}^2\right)$$

2 Electromagnetic Power Exchange with Sources:

$$P_{elm}=I_{sd}\frac{d\Phi_{sd}}{dt}+I_{sq}\frac{d\Phi_{sq}}{dt}+I_{rd}\frac{d\Phi_{rd}}{dt}+I_{rq}\frac{d\Phi_{rq}}{dt}$$

3 Mechanical Power P_m, related to angular position derivatives:

$$P_m=\left(\Phi_{sd}I_{sq}-\Phi_{sq}I_{sd}\right)\frac{d\theta_s}{dt}+\left(\Phi_{rd}I_{rq}-\Phi_{rq}I_{rd}\right)\frac{d\theta_r}{dt}$$

By considering the flux equations, we can further refine these expressions.

The mechanical power can also be written as:

$$P_m=\left(\Phi_{sd}I_{sq}-\Phi_{sq}I_{sd}\right)\frac{d(\theta_s-\theta_r)}{dt}$$

Since mechanical power is also expressed as:

$$P_m=T_e\Omega=\frac{T_e\omega}{P}$$

we derive the expression for the electromagnetic torque:

$$T_e = P\left(\Phi_{sd} I_{sq} - \Phi_{sq} I_{sd}\right)$$

Alternatively, using the rotor flux components, we can express the torque as:

$$T_e = \frac{PM}{L_r}\left(\Phi_{rd} I_{qs} - \Phi_{rq} I_{ds}\right)$$

V.7. Conclusion

The modeling of a three-phase and two-phase asynchronous machine has been demonstrated using the Park transformation. Then, the state-space formulation for control purposes and its representation in the form of a functional block diagram were also presented.

The mathematical models of the asynchronous machine allow us to simulate the machine to obtain a model that closely resembles or is equivalent to the real machine, as well as to integrate appropriate control techniques.

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