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Study of approximate solutions with time scales

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Study of approximate solutions With time scales

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Thanks

First of all, we thank ALLAH, almighty, for giving us the strength to survive, as well as the daring to overcome all difficulties..

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In order not to forget anyone, our sincere thanks go to all those who helped us to realize our memory.

General Notations

$\delta_n(\epsilon)$ order functions.

$E(x)$ the approximation error.

$F(y, \dot{y})$ a numerical function with respect to y et $\dot{y}$.

$\dot{y} = \frac{dy}{d\theta}$ the prime derivative of y with respect to θ.

$\ddot{y} = \frac{d^2y}{d\theta^2}$ the second derivative of y with respect to θ.

$F_y = \frac{\partial F}{\partial y}$ the first derivative of F with respect to y.

$F_{yy} = \frac{\partial^2 F}{\partial y^2}$ the second derivative of F with respect to y.

$F_{\dot{y}} = \frac{\partial F}{\partial y}$ the first derivative of F with respect to $\dot{y}$.

H Hamiltonian system.

S^1 the circle or 1 torus.

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General Introduction

The development of the theory of the influence of small perturbations on solutions of differential equations started in the eighteenth century. Since Poincaré (around 1900) perturbation theory is flourishing again with important new fundamental insights and applications. One of the theories which can be considered old as well as new and which is now gaining considerable popularity among researchers and mathematicians in science such as mechanics is the theory of perturbation.

In mathematics and physics, multiple-scale analysis (also called the method of multiple scales) comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, both for small as well as large values of the independent variables. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent. In the solution process of the perturbation problem thereafter, the resulting additional freedom introduced by the new independent variables is used to remove secular terms.
This work is divided into three chapters.

- The first chapter is devoted to basic notions and functional tools concerning dynamical systems and the theory of perturbation used in this work.
- In the second chapter, we introduce the initial concepts and principle of some perturbation methods for calculating analytic approximations with basic examples.
- In the last chapter, we introduced some definitions of analytical approximations with time scales using previous perturbation methods with basic examples.

Chapter 1

Preliminaries

In this chapter, we introduce some fundamental notions about dynamical systems, perturbation theory and time scales that we will use in chapters 2 and 3, (see [2, 3, 10]).

1.1 Background and elementary examples

In the eighteenth century one of the most important fields of application of perturbation theory is the theory describing the motion of celestial bodies. The motion of the earth around the sun according to Newton's laws envisaged as the dynamics of two bodies, point masses, bound by gravity, leads to periodic revolution in an elliptic orbit. However, with the accuracy of measurement acquired in those times one could establish considerable deviations of this motion in an elliptic orbit. Looking for causes of these deviations one considered the perturbational influence of the moon and the large planets like Jupiter and Saturn. Such models are leading to equations of motion which, apart from terms corresponding with the mutual attraction of the earth and the sun, contain small perturbation terms. The study of these new equations turns out to be very difficult, in fact the fundamental questions of this problem have not been solved up till now, but it started off the development of perturbation theory, (see [1, 4, 6]).

Example 1.1. *A simple example of a perturbation arising in a natural way is the following* *problem. Consider a harmonic oscillation, described by the equation* (1.1).

$$\ddot{x} + x = 0 \tag{1.1}$$

In deriving this equation the effect of friction has been neglected; in practice however, friction will always be present. If the oscillator is such that the friction is small, an improved model for the oscillations is given by the equation (1.2).

$$\ddot{x} + \varepsilon\dot{x} + x = 0 \tag{1.2}$$

The term $\varepsilon\dot{x}$ is called "friction term" or "damping term" and this particular simple form of the friction term has been based on certain assumptions concerning the mechanics of friction.

The parameter ε is small:

$$0 \leq \varepsilon << 1$$

as will always be the case, in this chapter and in subsequent chapters. If one puts $\varepsilon = 0$ in equation 1.2 one recovers the original equation 1.1; we call equation 1.1 the "unperturbed problem".

One of the interesting conclusions which we shall draw is, that in a great many problems, the introduction of small perturbations triggers off qualitatively and quantitatively behaviour of the solutions which diverges very much from the behaviour of the solutions of the unperturbed problem. One can observe this already in the simple example of the damped oscillator described by equation 1.2 We shall consider some other examples.

Example 1.2. *Consider the initial value problem*

$$\dot{x} = -x + \varepsilon, x(0) = 1 \tag{1.3}$$

The solution is $x(t) = \varepsilon + (1 - \varepsilon)e^{-t}$. The unperturbed problem is

$$\dot{y} = -y, y(0) = 1.$$

The solution is $y(t) = e^{-t}$. It is clear that

$$|x(t) - y(t)| = \varepsilon - \varepsilon e^{-t} \leq \varepsilon, t \geq 0$$

The error, arising in approximating $x(t)$ by $y(t)$ is never bigger that ε.

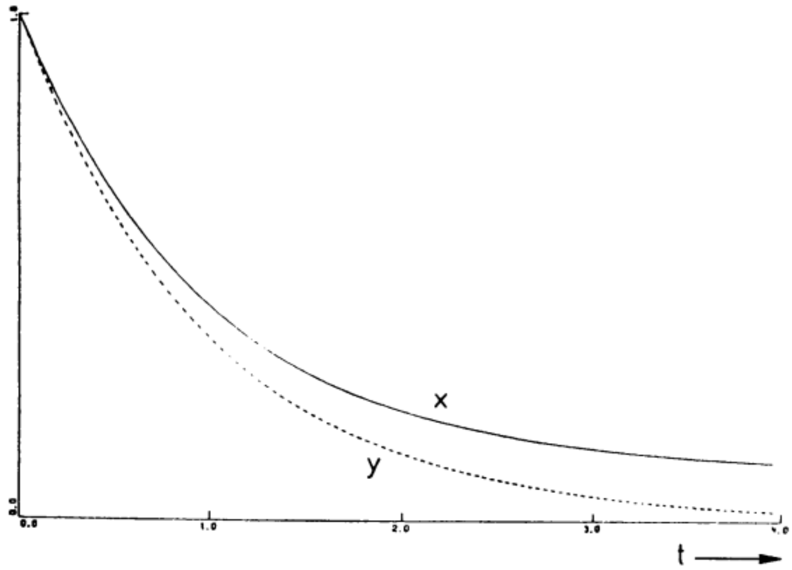

Figure 1.1: solution of the equation 1.3; $0 \leq x \leq 1$

Example 1.3. *The situation is very different for the problem*

$$\dot{x} = +x + \varepsilon, x(0) = 1.$$

The solution is $x(t) = -\varepsilon + (1+\varepsilon)e^t$. *The unperturbed problem is*

$$\dot{y} = +y, y(0) = 1$$

with solution $y(t) = e^t$. *We find*

$$|x(t) - y(t)| = \varepsilon \left|1 - e^t\right|.$$

On the interval $0 \leq t \leq 1$, *the error caused by approximating* $x(t)$ *by* $y(t)$ *is of the order of magnitude* ε. *This is not true anymore for* $t \geq 0$, *where the difference increases without bounds.*

In example 1.3 the solutions are not bounded for $t \geq 0$. We shall show now that also in the case of bounded solutions the difference between the solutions of the perturbed and the unperturbed problem can be considerable. Consider the initial value problem.

Example 1.4.

$$\ddot{x} + (1+\varepsilon)^2 x = 0, x(0) = 0, \dot{x}(0) = 1. \tag{1.4}$$

The solution is

$$x(t) = \frac{1}{1+\varepsilon}\sin(1+\varepsilon)t$$

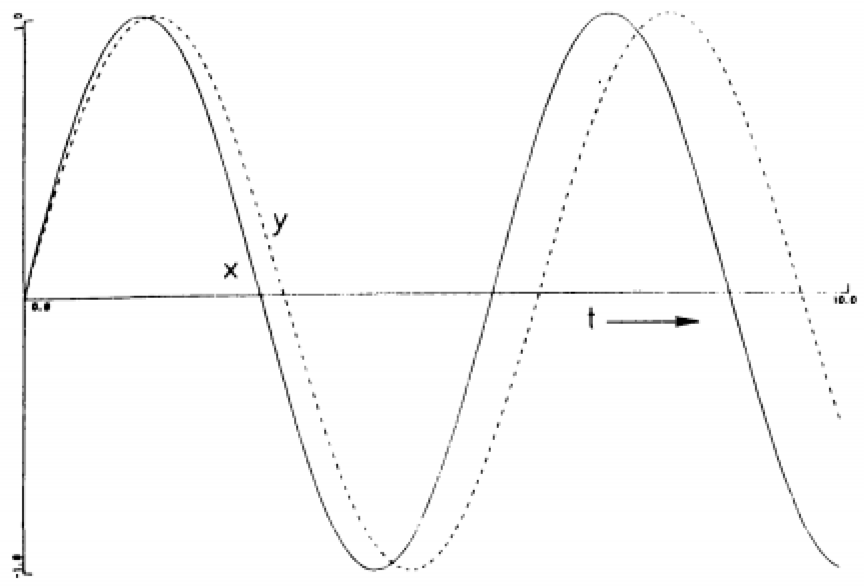

Figure 1.2: solution of the equation 1.4; $0 \leq x \leq 20$

The unperturbed problem is

$$\ddot{y} + y = 0, y(0) = 0, \dot{y}(0) = 1$$

with solution $y(t) = \sin t$. *we should analyse the difference* $x(t) - y(t)$. *The solutions are close for a long time, but if one waits long enough, take for example* $t = \pi/(2\varepsilon)$, *the difference becomes considerable.*

These examples do illustrate that in constructing approximations of solutions of initial value problems, one has to indicate on what interval of time one is looking for an approximation. In many cases we would like this interval of time to be as large as possible. We shall introduce now a number of concepts which enable us to estimate vector functions in terms of a small parameter ε.

1.2 Basic material

Consider the vector function $f : R \times R^n \times R \to R^n$; the function $f(t, x, \varepsilon)$ is continuous in the variables $t \in \mathbb{R}$ and $x \in D \subset R^n, \varepsilon$ is a small parameter. The function f has to be expanded with respect to the small parameter ε. In the simple case that f has a Taylor expansion with respect to ε near $\varepsilon = 0$ we have

$$f(t, x, \varepsilon) = f(t, x, 0) + \varepsilon f_1(t, x) + \varepsilon^2 f_2(t, x) + \cdots \varepsilon^n f_n(t, x) + \cdots$$

with coefficients $f_1, f_2, \cdots$ which depend on t and x. The expressions $\varepsilon, \varepsilon^2, \cdots, \varepsilon^n, \cdots$ are called order functions. We shall look for an expansion of the form

$$f(t, x, \varepsilon) = \sum_{n=0}^{N} \delta_n(\varepsilon) f_n(t, x) + \cdots$$

in which $\delta_n(\varepsilon), n = 0, 1, 2, \cdots$ are order functions, (see [2, 7]).

Definition 1.2.1. *The function $\delta(\varepsilon)$ is continuous and positive in $(0, \varepsilon_0]$ and it has the property that $\lim_{\varepsilon \to 0} \delta(\varepsilon)$ exists whereas $\delta(\varepsilon)$ is monotonically decreasing as ε tends to zero; $\delta(\varepsilon)$ is called an order function.*

In the case that f has a Taylor expansion, the order functions which have been used are

$$\{\varepsilon^n\}_{n=0}^{\infty}$$

Other examples of order functions on $(0, 1]$ are

$$\varepsilon |\ln \varepsilon|, \sin \varepsilon, 2, e^{-1/e}$$

Very often we shall compare the magnitude of order functions with a characterisation like ${}^n\varepsilon^4$ goes faster to zero that ε^{2n}. What we mean is that we compare the behaviour of these two order functions as ε tends to zero; for this comparison we shall use the Landau O-symbols.

Definition 1.2.2. *a. $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ as $\varepsilon \to 0$ if there exists a constant k such that $\delta_1(\varepsilon) \leq k\delta_2(\varepsilon)$ as $\varepsilon \to 0$ b. $\delta_1(\varepsilon) = o(\delta_2(\varepsilon))$ as $\varepsilon \to 0$ if*

$$\lim_{\varepsilon \to 0} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} = 0$$

Remark 1.2.1. *In a number of cases the following limit exists*

$$\lim_{\varepsilon \to 0} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} = k$$

In this case we find with the definition $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$. So if this limit exists, we have in a simple way the O -estimate for order functions.

Example 1.5. *1.* $\varepsilon^4 = o(\varepsilon^2)$ *as* $\varepsilon \to 0$.

2. $\sin \varepsilon = O(\varepsilon)$ *as* $\varepsilon \to 0$.

3. $\varepsilon |\ln \varepsilon| = o(1)$ *as* $\varepsilon \to 0$.

In perturbation theory we usually omit ^{n} as $\varepsilon \to 0^n$ as this is always the case to be considered. We are now able to compare functions of ε, but this does not apply to vector functions $f(t, x, \varepsilon)$. Consider for instance the function

$$\varepsilon t \sin x, 0 \leq t \leq 1, x \in \mathrm{R}$$

Intuitively we would estimate $\varepsilon t \sin x = O(\varepsilon)$ on the domain $[0, 1] \times R$; to do this we have to extend our definitions.

Definition 1.2.3. *Consider the vector function $f(t, x, \varepsilon), t \in I \subset \mathbf{R}, x \in D \subset \mathbf{R}^n, 0 \leq \varepsilon \leq \varepsilon_0$*

a. $f(t, x, \varepsilon)$ is $O(\delta(\varepsilon))$ if there exists a constant k such that $\|f\| \leq k\delta(\varepsilon)$ as $\varepsilon \to 0$ with $\delta(\varepsilon)$ an order function ($\|.\|$ is the sup norm on $I \times D$).

b. $f(t, x, \varepsilon)$ is $o(\delta(\varepsilon))$ as $\varepsilon \to 0$ if $\lim_{\varepsilon \to 0} \frac{\|f\|}{\delta(\varepsilon)} = 0$.

In the example given above we have now

$$\varepsilon t \sin x = O(\varepsilon), 0 \leq t \leq 1, x \in \mathbf{R}$$

Note however that the $O(\varepsilon)$ estimate does not hold for the functions

$$\begin{aligned} &\varepsilon t \sin x \quad , t \geq 0 \quad , x \in \mathrm{R} \\ &\varepsilon^2 t \sin x \quad , t \geq 0, x \in \mathrm{R}. \end{aligned}$$

As we are especially interested in initial value problems for differential equations, the variable t plays a special part. Often we shall study a solution or its approximation on an interval which we would like to take as large as possible. So in these cases we shall not fix the interval of time apriori. It turns out to be useful to characterise the size of the interval of time in terms of the small parameter ε.

1.2.1 Time scale

Definition 1.2.4. *Consider the vector function $f(t,x,\varepsilon), t \geq 0, x \in D \subset \mathbf{R}^n$, and the order functions $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$; $f(t,x,\varepsilon) = O\left(\delta_1(\varepsilon)\right)$ as $\varepsilon \to 0$ on the time scale $1/\delta_2(\varepsilon)$ if the estimate is valid for $x \in D, 0 \leq \delta_2(\varepsilon)t \leq C$ with C a constant independent of ε,(see [3, 5]).*

Example 1.6. $f_1(t,x,\varepsilon) = \varepsilon t \sin x, t \geq 0, x \in \mathbf{R}$.
$f_1(t,x,\varepsilon) = O(\varepsilon)$ *on the time scale* 1, $f_1(t,x,\varepsilon) = O(1)$ *on the time scale* $1/\varepsilon$.
$f_2(t,x,\varepsilon) = \varepsilon^2 t \sin x, t \geq 0, x \in \mathbf{R}$.
$f_2(t,x,\varepsilon) = O(\varepsilon)$ *on the time scale* $1/\varepsilon$.
$f_2(t,x,\varepsilon) = O\left(\varepsilon^{1/2}\right)$ *on the time scale* $1/\varepsilon^{3/2}$.
We are now able to introduce the concept of an asymptotic approximation of a function.

Definition 1.2.5. *Consider the functions $f(t,x,\varepsilon), g(t,x,\varepsilon), t \geq 0, x \in D \subset \mathbf{R}^n$; $g(t,x,\varepsilon)$ is an asymptotic approximation of $f(t,x,\varepsilon)$ on the time scale $1/\delta(\varepsilon)$ if $f(t,x,\varepsilon) - g(t,x,\varepsilon) = o(1)$ as $\varepsilon \to 0$ on the time scale $1/\delta(\varepsilon)$ Approximation of functioris will take the form of asymptotic expansions like*

$$f(t,x,\varepsilon) = \sum_{n=0}^{N} \delta_n(\varepsilon) f_n(t,x,\varepsilon) + o\left(\delta_N(\varepsilon)\right)$$

with $\delta_n(\varepsilon), n = 0, \cdots, N$, order functions such that $\delta_{n+1}(\varepsilon) = o\left(\delta_n(\varepsilon)\right), n = 0, \cdots, N-1$. The expansion coefficients $f_n(t,x,\varepsilon)$ are bounded and $O(1)$ as $\varepsilon \to 0$ on the given t, x-domain.

Example 1.7. $f_1(t,x,\varepsilon) = x \sin(t + \varepsilon t), t \geq 0, 0 \leq x \leq 1$ *The function $x \sin t$ is an asymptotic approximation of $f_1(t,x,\varepsilon)$ on the time scale 1.*

$f_2(t,x,\varepsilon) = \sin\varepsilon + \varepsilon^3 te^{-\varepsilon t}, t \geq 0$ *The function* ε *is an asymptotic approximation of* $f_2(t,x,\varepsilon)$ *on the time scale* $1/\varepsilon$.

1.2.2 Naive expansion

Consider the initial value problem

$$\dot{x} = f(t,x,\varepsilon),\ x(0) \text{ given}$$

$t \geq 0, x \in D \subset \mathrm{R}^n$. If we can expend $f(t,x,\varepsilon)$ in a Taylor series with respect to ε

$$f(t,x,\varepsilon) = f_0(t,x) + \varepsilon f_1(t,x) + \varepsilon^2 \cdots$$

then one might suppose that a similar expansion exists for the solution

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 \cdots$$

Substitution of the expansion in the equation, equating coefficients of corresponding powers of ε and subsequently imposing the initial values produces a so-called formal expansion of the solution. It seems natural to expect that the formal expansion represents an asymptotic approximation of the solution. We shall show that this is correct, but only in a very restricted sense; first a simple example.

Example 1.8. *Consider the oscillator with damping*

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0$$

with initial values $x(0) = a, \dot{x}(0) = 0$. *The solution of the initial value problem is*

$$x(t) = ae^{-et}\cos\left(\sqrt{1-\varepsilon^2}t\right) + \varepsilon\frac{a}{\sqrt{1-\varepsilon^2}}e^{-\varepsilon t}\sin\left(\sqrt{1-\varepsilon^2}t\right)$$

Ignoring the explicit solution we substitute

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 \cdots$$

Equating coefficients of equal powers of ε, *we find*

$$\begin{aligned} \ddot{x}_0 + x_0 &= 0 \\ \ddot{x}_n + x_n &= -2\dot{x}_{n-1}, n = 1, 2, \cdots \end{aligned}$$

The initial values do not depend on ε so it is natural to put

$$x_0(0) = a \quad , \quad \dot{x}_0(0) = 0$$

$$x_n(0) = 0 \quad , \quad \dot{x}_n(0) = 0, n = 1, 2, \cdots$$

Solving the equations and applying the initial conditions we obtain

$$\begin{aligned} x_0(t) &= a \cos t \\ x_1(t) &= a \sin t - at \cos t, \ \textit{etc.} \end{aligned}$$

The formal expansion is

$$x(t) = a \cos t + a\varepsilon(\sin t - t \cos t) + \varepsilon^2 \cdots$$

This expansion corresponds with an oscillation with increasing amplitude.

The formal expansion can be characterised as follows.

Theorem 1.1. *Consider the initial value problem*

$$\dot{x} = f_0(t, x) + \varepsilon f_1(t, x) + \cdots + \varepsilon^m f_m(t, x) + \varepsilon^{m+1} R(t, x, \varepsilon)$$

with $x(t_0) = \eta$ and $|t - t_0| \leq h, x \in D \subset R^n, 0 \leq \varepsilon \leq \varepsilon_0$. Assume that in this domain we have

a. $f_i(t, x), i = 0, \cdots, m$ continuous in t and $x, (m + 1 - i)$ times continuously differentiable in x;

b. $R(t, x, \varepsilon)$ continuous in t, x and ε, Lipschitz-continuous in x.

Substitution in the equation for x the formal expansion

$$x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^m x_m(t)$$

Taylor expansion with respect to powers of ε, equating corresponding coefficients and applying the initial values $x_0(t_0) = \eta, x_i(t) = 0, i = 1, \cdots, m$ produces an approximation of $x(t)$:

$$\|x(t) - (x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^m x_m(t))\| = O\left(\varepsilon^{m+1}\right)$$

on the time scale 1 .

Proof: See [7]. A natural question is whether, on taking the limit $m \to \infty$, while assuming that the righthand side of the equation can be represented by a convergent power series with respect to ε, we obtain in this way a convergent series for $x(t)$. This question has been answered in a positive way by Poincaré.

1.2.3 The Poincare expansion theorem

Consider the initial value problem

$$\dot{x} = f(t, x, \varepsilon), x\left(t_0\right) = \eta$$

where we assume that $f(t, x, \varepsilon)$ can be expanded in a convergent Taylor series with respect to ε and x in a certain domain. The unperturbed problem is

$$\dot{x}_0 = f\left(t, x_0, 0\right)$$

In many applications, for instance if we are looking for periodic solutions, we do not know precise initial conditions; we allow for this by admitting deviations of order μ

$$x\left(t_0\right) \equiv x_0\left(t_0\right) + \mu$$

with μ a constant, at this point independent of ε. We translate

$$x = y + x_0(t)$$

to find for y

$$\dot{y} = F(t, y, \varepsilon), y\left(t_0\right) = \mu$$

with $F(t, y, \varepsilon) = f\left(t, y + x_0(t), \varepsilon\right) - f\left(t, x_0(t), 0\right)$.
The expansion properties of $f(t, x, \varepsilon)$ imply that $F(t, y, \varepsilon)$ possesses a convergent power expansion with respect to y and ε in a neighbourhood of $y = 0, \varepsilon = 0$. The proof of theorem 1.1, in making use of the Lipschitz-continuity of the derivatives to order m, can be viewed as contraction in a C^m -space. In the subsequent proof we use contraction in a Banach space of analytic functions.

Theorem 1.2. ***(Poincaré)***
Consider the initial value problem 1.6 $\dot{y} = F(t, y, \varepsilon), y\left(t_0\right) = \mu$ *with* $|t - t_0| \leq h, y \in D \subset \mathbb{R}^n, 0 \leq \varepsilon \leq \varepsilon_0, 0 \leq \mu \leq \mu_0$. *If* $F(t, y, \varepsilon)$ *is continuous with respect to* t, y *and* ε *and can be expanded in a convergent power series with respect to* y *and* ε *for* $\|y\| \leq \rho, 0 \leq \varepsilon \leq \varepsilon_0$, *then* $y(t)$ *can be expanded in a convergent power series with respect to* ε *and* μ *in a neighbourhood of* $\varepsilon = \mu = 0$, *convergent on the time scale* 1.

Proof: See [7].

Chapter 2

Approximate solutions

In this chapter we shall show how to find convergent series approximations of periodic solutions by using the expansion theorem and the periodicity of the solution. This method is called Poincaré-Lindstedt, it is also called the continuation method, (see [3, 10]).

2.1 Periodic solutions of autonomous second-order equations

In this section we shall consider equations of the form

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, \varepsilon), (x, \dot{x}) \in D \subset \mathrm{R}^2 \tag{2.1}$$

If $\varepsilon = 0$, all nontrivial solutions are $2\pi-$periodic. To start with we assume that periodic solutions of equation (2.1) exist for small, positive values of ε. Furthermore we shall assume that in D and for $0 \leq \varepsilon \leq \varepsilon_0$, the requirements of the Poincaré expansion theorem have been satisfied. Both the period T of the solution and the initial values (and so the location of the solution in the phase-plane) will depend on the small parameter ε. This is the reason to put

$$T = T(\varepsilon), x(0) = a(\varepsilon), \dot{x}(0) = 0$$

Note that putting $\dot{x} = 0$ for periodic solutions of equation 2.1 is no restriction. The approximations which we shall construct concern solutions which exist for all time. The time scale of validity of the approximations will therefore play a part in the discussion. The expansion theorem tells us that, on the time scale 1 ,

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = a(0)\cos t$$

with $a(0)$ at this point unknown; moreover we have clearly $T(0) = 2\pi$. In the sequel we shall expand both the initial value and the period with respect to the small parameter ε. It is convenient to transform $t \to \theta$ such that, in the new time-like variable θ, the periodic solution is 2π-periodic. We transform

$$wt = \theta, w^{-2} = 1 - \varepsilon\eta(\varepsilon)$$

Equation 2.1 becomes after transforming and with notation $\dot{x} = dx/d\theta$

$$\ddot{x} + x = \varepsilon\left[\eta x + (1-\varepsilon\eta) f\left(x, (1-\varepsilon\eta)^{-\frac{1}{2}}\dot{x}, \varepsilon\right)\right] \tag{2.2}$$

initial values $x(0) = a(\varepsilon), \dot{x}(0) = 0$.

We shall abbreviate equation 2.2 by

$$\ddot{x} + x = \varepsilon g\left(x, \dot{x}, \varepsilon, \eta\right)$$

The initial value a and the parameter η, which determines the unknown period, have to be chosen such that we obtain a 2π -periodic solution in θ of equation 2.2. By variation of constants we find that the initial value problem for equation 2.2 is equivalent with the following integral equation

$$x(\theta) = a\cos\theta + \varepsilon\int_0^\theta \sin(\theta-\tau) g\left(x(\tau), \dot{x}(\tau), \varepsilon, \eta\right) d\tau$$

For the periodic solution we have $x(\theta) = x(\theta + 2\pi)$ which yields the periodicity condition

$$\int_\theta^{\theta+2\pi} \sin(\theta-\tau) g\left(x(\tau), \dot{x}(\tau), \varepsilon, \eta\right) d\tau = 0$$

Expanding $\sin(\theta-\tau)$ we find two independent conditions

$$\begin{aligned} &\textstyle\int_0^{2\pi} \sin\tau g\left(x(\tau), \dot{x}(\tau), \varepsilon, \eta\right) d\tau = 0 \\ &\textstyle\int_0^{2\pi} \cos\tau g\left(x(\tau), \dot{x}(\tau), \varepsilon, \eta\right) d\tau = 0 \end{aligned} \tag{2.3}$$

The periodic solution depends on ε but also on a and η, so system 2.3 can be viewed as a system of two equations with two unknowns, a and η. According to the implicit function theorem this system of equations 2.3 is uniquely solvable in a neighbourhood of $\varepsilon = 0$ if the corresponding Jacobian does not vanish. Writing system 2.3 in the form $F_1(a,\eta) = 0, F_2(a,\eta) = 0$ this means that in a neighbourhood of $\varepsilon = 0$

$$\frac{\partial\,(F_1, F_2)}{\partial(a,\eta)} \neq 0 \tag{2.4}$$

If condition 2.4 has been satisfied, we have with the assumptions on the right-hander of equation 2.1, that $a(\varepsilon)$ and $\eta(\varepsilon)$ can be expanded in a Taylor series with respect to ε. From equation 2.2 we find for system 2.3 with $\varepsilon = 0$

$$\int_0^{2\pi} \sin\tau f(a(0)\cos\tau, -a(0)\sin\tau, 0)d\tau = 0 \tag{2.5}$$

$$\pi\eta(0)a(0) + \int_0^{2\pi} \cos\tau f(a(0)\cos\tau, -a(0)\sin\tau, 0)d\tau = 0$$

Applying 2.4 to system 2.5 we find the condition (notation : $f = f(x,y,\varepsilon)$)

$$a(0)\int_0^{2\pi} \left[\frac{1}{2}\sin 2\tau\frac{\partial f}{\partial x}(a(0)\cos\tau, -a(0)\sin\tau, 0) + \; -\sin^2\tau\frac{\partial f}{\partial y}(a(0)\cos\tau, -a(0)\sin\tau, 0)\right] d\tau \neq 0 \tag{2.6}$$

Remark 2.1.1. *If $\varepsilon = 0$, all solutions of equation* 2.1 *are periodic. Condition* 2.4 *and consequently condition* 2.6*, is a condition for the existence of an isolated periodic solution which branches off for $\varepsilon > 0$. If, however, there exists for $\varepsilon > 0$ a continuous family of periodic solutions, a one-parameter family depending on $a(\varepsilon)$, then condition* 2.4 *or* 2.6 *will not be satisfied. If we know apriori that this family of periodic solutions exists, then we can of course still apply the Poincaré-Lindstedt method. In this respect we should study the equation*

$$\ddot{x} + x = \varepsilon f(x)\ .$$

We conclude that if condition 2.4 has been satisfied, the corresponding periodic solution of equation 2.2 can be represented by the convergent series

$$x(\theta) = a(0)\cos\theta + \sum_{n=1}^{\infty} \varepsilon^n \gamma_n(\theta) \tag{2.7}$$

in which $\theta = \omega t = (1 - \varepsilon\eta)^{-\frac{1}{2}} t$ while using the convergent series

$$a = \sum_{n=0}^{\infty} \varepsilon^n a_n, \eta = \sum_{n=0}^{\infty} \varepsilon^n \eta_n, x(0) = a(0) + \sum_{n=1}^{\infty} \varepsilon^n \gamma_n(0), 0 < \varepsilon \leq \varepsilon_0$$

The expansions of $T(\varepsilon)$ and $a(\varepsilon)$ which we used in the beginning of the construction have been validated in this way. We shall now demonstrate the construction of the Poincaré-Lindstedt method for an important example.

Example 2.1. *(van der Pol equation)*
We showed that the equation

$$\ddot{x} + x = \varepsilon \left(1 - x^2\right) \dot{x}$$

has one periodic solution for all positive values of ε. We take ε small and we put again

$$\omega t = \theta, \omega^{-2} = 1 - \varepsilon\eta(\varepsilon)$$

so that the van der Pol equation transforms to

$$\ddot{x} + x = \varepsilon \left[\eta x + (1 - \varepsilon\eta)^{1/2} \left(1 - x^2\right) \dot{x}\right] \tag{2.8}$$

with $x(0) = a(\varepsilon), \dot{x}(0) = 0$. We can determine $a(0)$ and $\eta(0)$ with the system of equations 2.5. *We find*

$$\begin{aligned} \tfrac{1}{2}a(0)\left(1 - \tfrac{1}{4}a(0)^2\right) &= 0 \\ \eta(0)a(0) &= 0 \end{aligned}$$

with non-trivial solutions $a(0) = 2, \eta(0) = 0$ (putting $a(0) = -2$ doesnot yield a new periodic solution). Condition 2.4 *has been satisfied. To determine the terms of the series* 2.7 *systematically, we substitute the series with its derivatives into equation* 2.8. *Collecting terms which are coefficients of equal powers of ε produces equations for the coefficients $\gamma_n(\theta)$. Then we apply the periodicity condition; this is aequivalent to using system* 2.3 *where g has been expanded with respect to powers of ε. We write*

$$\begin{aligned} x(\theta) &= a_0 \cos\theta + \varepsilon\gamma_1(\theta) + \varepsilon^2\gamma_2(\theta) + \cdots \\ \dot{x}(\theta) &= -a_0 \sin\theta + \varepsilon\dot{\gamma}_1(\theta) + \varepsilon^2\dot{\gamma}_2(\partial) + \cdots \ \textit{etc.} \end{aligned}$$

to find after substitution

$$\ddot{\gamma}_1 + \gamma_1 = a_0\eta_0 \cos\theta + a_0 \left(\frac{1}{4}a_0^2 - 1\right) \sin\theta + \tag{2.9}$$

$$\frac{1}{4}a_0^3 \sin 3\theta$$

The solutions of equation 2.9 *are periodic if* $a_0 = 2, \eta_0 = 0$. *The general solution of the equation is*

$$\gamma_1(\theta) = A_1 \cos\theta + B_1 \sin\theta - \frac{a_0^3}{32} \sin 3\theta$$

From the initial conditions we have $B_1 = \frac{3}{32}a_0^3, A_1 = a_1$; *in the next step* a_1 *will be determined. In the literature this process of applying the periodicity condition is sometimes called " elimination of the secular terms". The term "secular" originates from celestial mechanics and refers to terms which are unbounded with time; we have met with such terms in section* 1.3. *Putting* $a_0 = 2$ *we find*

$$\gamma_1(\theta) = a_1 \cos\theta + \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

The equation for γ_2 *is*

$$\begin{aligned}\ddot{\gamma_2} + \gamma_2 = \big(2\eta_1 + \tfrac{1}{4}\big)\cos\theta + 2a_1 \sin\theta + 3a_1 \sin 3\theta + \\ -\tfrac{3}{2}\cos 3\theta + \tfrac{5}{4}\cos 5\theta\big)\end{aligned} \tag{2.10}$$

The periodicity condition yields $\big(2\eta_1 + \frac{1}{4}\big) = 0, a_1 = 0$, *which determines* $\gamma_1(\theta)$ *completely. The general solution of equation* 2.10 *is*

$$\gamma_2(\theta) = A_2 \cos\theta + B_2 \sin\theta + \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta$$

From the initial conditions we have $B_2 = 0, A_2 + \frac{13}{96} = a_2$; *in the next step* a_2 *will be determined. The equation for* γ_3 *is*

$$\ddot{\gamma_3} + \gamma_3 = 2\eta_2 \cos\theta + \left(2A_2 + \frac{1}{4}\right)\sin\theta + \left(3A_2 - \frac{9}{32}\right)\sin 3\theta + \frac{35}{24}\sin 5\theta - \frac{7}{12}\sin 7\theta \tag{2.11}$$

Note that the equations for γ_n *are all linear. This is in agreement with remark 1.2. The periodicity condition produces* $\eta_2 = 0, 2A_2 + \frac{1}{4} = 0$ *which determines* a_2. *Using the relation* $\omega^{-2} = 1 - \varepsilon\eta_0 - \varepsilon^2\eta_1 - \varepsilon^3\eta_2 - \cdots$ *we find*

$$\omega = 1 + \frac{1}{2}\varepsilon\eta_0 + \varepsilon^2\left(\frac{1}{2}\eta_1 + \frac{3}{8}\eta_0^2\right) + \varepsilon^3\left(\frac{1}{2}\eta_2 + \frac{3}{4}\eta_0\eta_1 + \frac{15}{48}\eta_0^3\right) + \cdots$$

For the period we have

$$\begin{aligned}T = \tfrac{2\pi}{\omega} = 2\pi\big(1 - \tfrac{1}{2}\varepsilon\eta_0 - \varepsilon^2\big(\tfrac{1}{2}\eta_1 + \tfrac{1}{8}\eta_0^2\big) - \varepsilon^3\big(\tfrac{1}{2}\eta_2 + \tfrac{1}{4}\eta_0\eta_1 + \\ \tfrac{3}{48}\eta_0^3\big) + \cdots\big)\end{aligned}$$

For the van der Pol equation we find with $\eta_0 = 0, \eta_1 = -\frac{1}{8}, \eta_2 = 0$ *:*

$$\omega = 1 - \tfrac{1}{16}\varepsilon^2 + O\left(\varepsilon^4\right)$$
$$T = 2\pi\left(1 + \tfrac{1}{16}\varepsilon^2\right) + O\left(\varepsilon^4\right)$$

The approximation of the periodic solution of the van der Pol equation to accuracy $O\left(\varepsilon^4\right)$

$$\begin{gathered} x(\theta) = 2\cos\theta + \varepsilon\left\{\tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta\right\} + \\ \varepsilon^{\frac{2}{2}}\left\{-\tfrac{1}{8}\cos\theta + \tfrac{3}{16}\cos 3\theta\right. \\ \left.-\tfrac{5}{96}\cos 5\theta\right\} + \varepsilon^3\left\{A_3\cos\theta - \tfrac{7}{256}\sin\theta + \tfrac{21}{256}\sin 3\theta\right. \\ \left.-\tfrac{35}{576}\sin 5\theta + \tfrac{7}{576}\sin 7\theta\right\} + O\left(\varepsilon^4\right) \end{gathered} \tag{2.12}$$

in which $\theta = \left(1 - \frac{1}{10}\varepsilon^2 t\right)$ *; the* $O\left(\varepsilon^4\right)$ *estimate is valid for* $0 \leq \theta \leq 2\pi$ *and* $0 < \varepsilon \leq \varepsilon_0$ *We note that* ε_0 *is the radius of convergence of the power series with respect to* ε *of the periodic solution. It is not so easy to compute this radius of convergence as we need for this the explicit form of the coefficients for arbitrary* n *(or the ratio of subsequent coefficients). In expression* 2.12 *there is a still unknown coefficient* A_3*, which can be computed when applying the periodicity condition to the solution of the equation for* γ_4*. Andersen and Geer (1982) used a computer program which formally manipulated the expansions; they calculated the expansion of* ω *to* $O\left(\varepsilon^{164}\right)$*. Their result to* $O\left(\varepsilon^{10}\right)$ *is*

$$\begin{gathered} \omega = 1 - \tfrac{1}{16}\varepsilon^2 + \tfrac{17}{3072}\varepsilon^4 + \tfrac{35}{884736}\varepsilon^6 - \tfrac{678899}{5096079360}\varepsilon^8 + \\ \tfrac{28160413}{2293235712000}\varepsilon^{10} + \cdots \end{gathered} \tag{2.13}$$

The computations suggest a radius of convergence $\varepsilon_0 = \sqrt{3.42}$ *which is rather large.*

In the next section we shall improve the validity of the approximations for periodic solutions strongly.

2.2 Poincaré-Lindstedt method

2.2.1 Introduction

This section presents the simplest and perhaps one of the most useful of all approximation methods: the expansion of a solution to a differential equation in a series in a small parameter. This technique is known as the perturbation method, (see [8, 9]). It will be used to construct uniformly valid periodic solutions to second-order nonlinear differential equations of the form

$$\frac{d^2y}{dt^2} + y = \epsilon F\left(y, \frac{dy}{dt}\right), \quad \epsilon > 0 \tag{2.14}$$

where ϵ is a small positive parameter and, in general, F is assumed to be an analytic function of y and dy/dt. The starting point for the perturbation method is the assumption that a periodic solution to Eq. (2.14) can be written as

$$y(t, \epsilon) = \sum_{m=0}^{n} \epsilon^m y_m(t) + O\left(\epsilon^{n+1}\right) \tag{2.15}$$

In general, this series is an asymptotic expansion. The original justification for this expansion was given by Poincaré [3].

The general procedure is to substitute Eq. (2.15) into Eq. (2.14), expand in powers of ϵ, and set the various coefficients of the powers of ϵ equal to zero. This leads to a set of linear inhomogeneous differential equations that can be solved recursively.

To illustrate the method, consider the following nonlinear differential equation:

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0 \tag{2.16}$$

Examination of the first-integral shows that all its solutions are periodic. The substitution of Eq. (2.15) into Eq. (2.16) gives

$$\left(\frac{d^2y_0}{dt^2} + \epsilon\frac{d^2y_1}{dt^2} + \epsilon^2\frac{d^2y_2}{dt^2} + \cdots\right) + \left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) + \epsilon\left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right)^3 = 0,$$

and

$$\left(\frac{d^2y_0}{dt^2} + y_0\right) + \epsilon\left(\frac{d^2y_1}{dt^2} + y_1 + y_0^3\right) + \epsilon^2\left(\frac{d^2y_2}{dt^2} + y_2 + 3y_0^2y_1\right) + \cdots = 0$$

Setting the coefficients of the various powers of ϵ to zero leads to the following system of linear differential equations;

$$\frac{d^2y_0}{dt^2} + y_0 = 0 \tag{2.17}$$

$$\frac{d^2y_1}{dt^2} + y_1 = -y_0^3 \tag{2.18}$$

$$\frac{d^2y_2}{dt^2} + y_2 = -3y_0^2 y_1$$
$$\vdots \quad \vdots\vdots$$
$$\frac{d^2y_n}{dt^2} + y_n = F_n\left(y_0, y_1, \ldots, y_{n-1}\right)$$

where F_n is a cubic polynomial function of its arguments. Note that this system of equations can be solved recursively, i.e., the determination of y_k involves only knowledge of the functions y_m for $0 \leq m \leq k-1$

To solve these equations, initial conditions have to be selected. The choice

$$y(0) = A, \quad \frac{dy(0)}{dt} = 0$$

gives

$$\begin{cases} y_0(0) = A, & \text{for } i \geq 1 \\ \frac{dy_k(0)}{dt} = 0 & \text{for } k \geq 0 \end{cases} \tag{2.19}$$

The solution to Eq. (2.17) is

$$y_0(t) = A\cos t$$

Substituting this result into the right-side of Eq. (2.18) gives

$$\frac{d^2y_1}{dt^2} + y_1 = -y_0^3 = -\left(\frac{3A^3}{4}\right)\cos t - \left(\frac{A^3}{4}\right)\cos 3t.$$

The particular solution to this equation is

$$y_1^P(t) = \left(\frac{A^3}{32}\right)\cos 3t - \left(\frac{3A^3}{8}\right)t\sin t.$$

Consequently, the complete solution is

$$y_1(t) = C_1\cos t + C_2\sin t + \left(\frac{A^3}{32}\right)\cos 3t - \left(\frac{3A^3}{8}\right)t\sin t$$

The initial conditions, as specified by Eqs. (2.19), allow the arbitrary constants C_1 and C_2 to be determined; they are $C_1 = (-A^3/32)$ and $C_2 = 0$, and $y_1(t)$ is

$$y_1(t) = \left(\frac{A^3}{32}\right)(\cos 3t - \cos t) - \left(\frac{3A^3}{8}\right)t\sin t$$

Thus, to order ϵ, the "solution" to Eq. (2.16) is

$$y(t,\epsilon) = A\cos t + \epsilon\left(\frac{A^3}{32}\right)[(\cos 3t - \cos t) - 12t\sin t] \tag{2.20}$$

However, inspection of Eq. (2.20) shows that $y_1(t)$, the supposedly small correction term to the periodic function $y_0(t)$, is not only nonperiodic, but, in addition, is unbounded as $t \to \infty$. Thus a direct, naive application of Eq. (2.15) leads to serious difficulties if the goal is to calculate analytic periodic approximations to the solutions of nonlinear differential equations of the form given by Eq. (2.14).

2.2.2 Secular Terms

The calculations of the previous section show that using a solution of the form given by Eq. (2.15) does not give an analytic periodic approximation to the actual periodic solution of Eq. (2.21) if only a finite number of terms are retained. The resulting approximation is aperiodic and becomes unbounded as $t \to \infty$. This lack of periodicity comes about because even if $y(t)$ is periodic in t, the retention of only a finite number of terms, in its expansion, can give a function that is not periodic. For example, consider the function $\sin(1+\epsilon)t$. Its expansion in terms of ϵ is

$$\sin(1+\epsilon)t = \sin t + \epsilon t\cos t - \left(\frac{\epsilon^2 t^2}{2}\right)\sin t + \cdots \tag{2.21}$$

It is seen that the retention of a finite number of terms on the right-side of Eq. (2.14) gives a function that is not only aperiodic, but, is unbounded as $t \to \infty$.

Terms such as $t^m \cos t$ or $t^m \sin t$ are called secular terms. These expressions arise because the expansion of Eq. (2.15) is nonuniformly valid.

In actual applications, a variety of time and/or calculational considerations force the use of only a small number of terms for inclusion in the perturbation expansion. Therefore, to construct a uniformly valid solution, an approximation is needed that eliminates secular terms.

A technique to avoid the presence of secular terms was developed by Lindstedt . Later, Poincaré proved that the expansions obtained by Lindstedt's technique are both asymptotic and uniformly valid,(see [2, 7, 8, 10]).

2.2.3 Poincaré-Lindstedt Method

The essence of this method consists in the introduction of a new independent variable that is linearly related to the old independent variable [1, 2]. This transformation allows the secular terms to be eliminated systematically. The fun- damental idea came from the astronomer Lindstedt and is based on the observation that one of the consequences of the nonlinear terms in Eq. (2.14) is to change the frequency of the linear system from $\omega_0 = \omega(0) = 1$ to $\omega(\epsilon) \neq 1$. To account for this change in frequency, a new variable $\theta = \omega t$ is introduced, and both y and ω are expanded in powers of ϵ as follows:

$$\begin{aligned} y(\theta, \epsilon) &= y_0(\theta) + \epsilon y_1(\theta) + \cdots + \epsilon^n y_n(\theta) + \cdots \\ \omega(\epsilon) &= 1 + \epsilon\omega_1 + \cdots + \epsilon^n \omega_n + \cdots \end{aligned} \tag{2.22}$$

where, at this point, the ω_i are unknown constants. Introduce the following notation:

$$\dot{y} \equiv \frac{dy}{d\theta}, \quad \ddot{y} \equiv \frac{d^2y}{d\theta^2}$$

$$F_y(y, \dot{y}) \equiv \frac{\partial F(y, \dot{y})}{\partial y}, \quad F_{\dot{y}} \equiv \frac{\partial F(y, \dot{y})}{\partial \dot{y}}.$$

If Eqs. (2.22) are substituted into Eq. (2.14) and the coefficients of the various powers of ϵ are set equal to zero, then the following equations are obtained for the y_n :

$$\begin{aligned} &\ddot{y}_0 + y_0 = 0 \\ &\ddot{y}_1 + y_1 = -2\omega_1\ddot{y}_0 + F\left(y_0, \dot{y}_0\right) \\ &\ddot{y}_2 + y_2 = -2\omega_1\ddot{y}_1 - \left(\omega_1^2 + 2\omega_2\right)\ddot{y}_0 + F_y\left(y_0, \dot{y}_0\right)y_1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad + F_{\dot{y}}\left(y_0, \dot{y}_0\right)\left(\omega_1\dot{y}_0 + \dot{y}_1\right), \\ &\vdots : \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ &\ddot{y}_n + y_n = G_n\left(\dot{y}_0, \dot{y}_1, \ldots, \dot{y}_{n-1}; \dot{y}_0, \dot{y}_1, \ldots \dot{y}_{n-1}\right) \end{aligned} \tag{2.23}$$

If F is a polynomial function of y and dy/dt, then G_n is also a polynomial function of its arguments.

The periodicity condition for the new variable θ can be expressed as

$$y(\theta) = y(\theta + 2\pi) \tag{2.24}$$

The corresponding conditions for the $y_n(\theta)$ are

$$y_n(\theta) = y_n(\theta + 2\pi) \tag{2.25}$$

If Eq. (2.22) is to be a periodic solution of Eq. (2.14), then the right-sides of Eq .(2.23) , must not contain constant multiples of either $\sin\theta$ or $\cos\theta$; otherwise, secular terms would exist. Therefore, if $y_n(\theta)$ is to be periodic, then two conditions must be satisfied at each step of the calculation. This means that two free parameters are needed. Detailed examination of Eqs. (2.23) shows that one of the constants is ω_n. The only other place where a second constant can be introduced is from the initial condition on y_{n-1}. This implies that the initial conditions take the form:

$$\begin{aligned} y(0) &= A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots \\ \frac{dy(0)}{d\theta} &= 0 \end{aligned} \tag{2.26}$$

where the A_n are, a priori, unknown constants. Thus, the periodicity requirement forces the right-side of Eq. (2.23) to have a term linear in ω_1 and, generally, another term nonlinear in A_0. These two terms are expressions involving $\sin\theta$ and $\cos\theta$. Consequently, by setting the coefficients equal to zero for the elimination of secular terms, both ω_1 and A_0 can be determined. Likewise, for $n \geq 2$, the periodicity condition on $y_n(\theta)$ gives a pair of equations for ω_n and A_{n-1}. Again, the elimination of secular terms allows both ω_n and A_{n-1} to be determined. At any given step in

the calculation ω_n, A_{n-1}, and $y_n(\theta)$ can be simultaneously determined. In this way, a uniformly valid periodic approximation to the periodic solutions of Eq. (2.14) can be found.

There does exist a situation for which the preceding analysis can be simplified. This occurs when the function $F(y, dy/dt)$ is an even function of dy/dt. For this case, $y(t)$ can be chosen to be an even function of t by using the initial conditions

$$y(0) = A, \quad \frac{dy(0)}{dt} = 0 \tag{2.27}$$

Consequently, both $y(\theta)$ and $y_n(\theta)$ are even functions of θ. Thus, the right-sides of Eq. (2.23) contain no term in$\sin\theta$. For this situation, only one free parameter ω_n is needed to ensure that there is no term in $\cos\theta$. This result implies that Eqs. (2.26) become

$$y(\theta=0)=A=A_0, \quad \frac{dy(0)}{d\theta}=0$$

where $A_k=0$ for $k\geq 1$. In summary, the $(n+1)$ th approximation to the solution of Eq. (2.14) according to the Poincaré-Lindstedt perturbation method is given by

$$y(\theta,\epsilon)=\sum_{m=0}^{n}\epsilon^m y_m(\theta)+O\left(\epsilon^{n+1}\right) \tag{2.28}$$

where

$$\theta=\omega t=\left[1+\epsilon\omega_1+\cdots+\epsilon^n\omega_n+O\left(\epsilon^{n+1}\right)\right]t$$

The expansion given by Eq. (2.28) is an example of a so-called generalized asymptotic series.

2.2.4 Examples

Example 2.2. *The van der Pol equation is*

$$\frac{d^2y}{dt^2}+y=\epsilon\left(1-y^2\right)\frac{dy}{dt} \tag{2.29}$$

The initial conditions to be used are given in Eq. (2.26). For this equation, $F=(1-y^2)\,dy/dt$. *Substitution of this function into Eqs. (2.23), and (2.22) gives*

$$\ddot{y}_0+y_0=0, \quad y_0(0)=A_0, \quad \dot{y}_0(0)=0, \tag{2.30}$$

$$\ddot{y}_1+y_1=-2\omega_1\ddot{y}_0+\left(1-y_0^2\right)\dot{y}_0, \quad y_1(0)=A_1, \quad \dot{y}_1(0)=0, \tag{2.31}$$

$$\begin{aligned}\ddot{y}_2+y_2=&-2\omega_1\ddot{y}_1-\left(\omega_1^2+2\omega_2\right)\ddot{y}_0-2y_0y_1\dot{y}_0\\&+\left(1-y_0^2\right)\left(\dot{y}_1+\omega_1\dot{y}_0\right), \quad y_2(0)=A_2, \quad \dot{y}_2(0)=0\end{aligned} \tag{2.32}$$

The solution to Eq. (2.30) *is*

$$y_0(\theta)=A_0\cos\theta. \tag{2.33}$$

Substituting Eq. (2.33) into Eq. (2.31) and simplifying the resulting expression gives

$$\ddot{y}_1+y_1=2\omega_1A_0\cos\theta-A_0\left(1-\frac{A_0^2}{4}\right)\sin\theta+\left(\frac{A_0^3}{4}\right)\sin 3\theta$$

Elimination of the secular terms gives

$$A_0=2, \quad \omega_1=0 \tag{2.34}$$

and

$$\ddot{y}_1 + y_1 = \left(\frac{A_0^3}{3}\right)\sin 3\theta, \quad y_1(0) = A_1, \quad \dot{y}_1(0) = 0.$$

The solution to this equation is

$$y_1(\theta) = A_1\cos\theta + \left(\frac{1}{4}\right)(3\sin\theta - \sin 3\theta) \tag{2.35}$$

Substitution of Eqs. (2.33), (2.34), *and* (2.35) *into Eq.* (2.32) *gives*

$$\ddot{y}_2 + y_2 = \left(4\omega_2 + \frac{1}{4}\right)\cos\theta + 2A_1\sin\theta - \left(\frac{3}{2}\right)\cos 3\theta + 3A_1\sin 3\theta + \left(\frac{5}{4}\right)\cos 5\theta.$$

The requirement of no secular terms gives the following results:

$$A_1 = 0, \quad \omega_2 = -\frac{1}{16}$$

$$\ddot{y}_2 + y_2 = -\left(\frac{3}{2}\right)\cos 3\theta + \left(\frac{5}{4}\right)\cos 5\theta, \quad y_2(0) = A_2, \tag{2.36}$$

$$\dot{y}_2(0) = 0$$

The solution to Eq. (2.36) *is*

$$y_2(\theta) = \left(A_2 - \frac{13}{96}\right)\cos\theta + \left(\frac{1}{96}\right)(18\cos 3\theta - 5\cos 5\theta)$$

Continuing this procedure, it can be shown that to terms of order ϵ^2, Eq. (2.29) has the periodic solution

$$\begin{aligned} y(\theta,\epsilon) = 2\cos\theta &+ \left(\frac{\epsilon}{4}\right)(3\sin\theta - \sin 3\theta) \\ &+ \left(\frac{\epsilon^2}{96}\right)(-13\cos\theta + 18\cos 3\theta - 5\cos 5\theta) + O\left(\epsilon^3\right) \end{aligned}$$

where $\theta = \omega t$ and

$$\omega(\epsilon) = 1 - \frac{\epsilon^2}{16} + O\left(\epsilon^3\right)$$

This isolated periodic solution is a limit cycle.

Example 2.3. *The mixed parity differential equation*

$$\frac{d^2y}{dt^2} + y + \alpha y^2 + \beta y^3 = 0 \tag{2.37}$$

Let the initial conditions be

$$y(0)=A, \quad dy(0)/dt=0,$$

and introduce, for small amplitude oscillations, the scaling variable ϵ, i.e.

$$y=\epsilon x \tag{2.38}$$

where

$$0<\epsilon \ll 1, \quad x=O(1)$$

Writing the first of the initial conditions as

$$y(0)=A=\epsilon a$$

the following results are obtained for $x(0)$ and $dx(0)/dt$:

$$x(0)=a, \quad \frac{dx(0)}{dt}=0$$

by Substitution we have

$$\frac{d^2x}{dt^2}+x+\epsilon x^2+\epsilon^2\beta x^3=0 \tag{2.39}$$

Transforming from the independent variable t to the new variable θ, where

$$\theta=\omega(\epsilon)t=1+\epsilon\omega_1+\epsilon^2\omega_2+O\left(\epsilon^3\right) \tag{2.40}$$

allows $x(t)$ to be expressed as

$$x(\theta,\epsilon)=x_0(\theta)+\epsilon x_1(\theta)+\epsilon^2 x_2(\theta)+O\left(\epsilon^3\right) \tag{2.41}$$

Substituting Eqs. (2.40) and (2.41) into Eq. (2.39), and using the notation, $\dot{x}(\theta)=dx/d\theta$ and $\ddot{x}(\theta)=d^2x/d\theta^2$, the following equations are obtained for $x_0(\theta), x_1(\theta)$, and $x_2(\theta)$:

$$\ddot{x}_0+x_0=0,$$

$$\ddot{x}_1+x_1=-2\omega_1\ddot{x}_0-\alpha x_0^2$$

$$\ddot{x}_2+x_2=-2\omega_1\ddot{x}_1-\left(2\omega_2+\omega_1^2\right)\ddot{x}_0-2\alpha x_0x_1-\beta x_0^3,$$

where these equations should satisfy the initial conditions:

$$x_0(0) = a, \quad \dot{x}_0(0) = 0$$

$$x_1(0) = \dot{x}_1(0) = 0,$$

$$x_2(0) = \dot{x}_2(0) = 0$$

The solutions to these differential equations are

$$x_0(\theta) = a\cos\theta$$

$$x_1(\theta) = \left(\frac{\alpha a^2}{6}\right)(-3 + 2\cos\theta + \cos 2\theta)$$

$$x_2(\theta) = \left(\frac{a^3}{3}\right)\left[-\alpha^2\left(\frac{174\alpha^2 - 27\beta}{288}\right)\cos\theta + \left(\frac{\alpha^2}{3}\right)\cos 2\theta + \left(\frac{2\alpha^2 + 3\beta}{32}\right)\cos 3\theta\right]$$

with

$$\omega_1 = 0, \quad \omega_2 = \left(\frac{9\beta - 10\alpha^2}{24}\right)a^2.$$

Putting all of these results together gives

$$\begin{aligned} x(\theta, \epsilon) = a\cos\theta + \epsilon\left(\frac{\alpha a^2}{6}\right)[-3 + 2\cos\theta + \cos 2\theta] \\ + \epsilon^2\left(\frac{a^3}{3}\right)\left[-\alpha^2 + \left(\frac{174\alpha^2 - 27\beta}{288}\right)\cos\theta\right. \\ \left. + \left(\frac{\alpha^2}{3}\right)\cos 2\theta + \left(\frac{2\alpha^2 + 3\beta}{32}\right)\cos 3\theta\right] + O\left(\epsilon^3\right) \end{aligned}$$

where $\theta = \omega t$ *and*

$$\omega(\epsilon) = 1 + \epsilon^2\left(\frac{9\beta - 10\alpha^2}{24}\right)a^2 + O\left(\epsilon^3\right)$$

In terms of the original variable $y = \epsilon x$*, the last two expressions can be rewrit- ten, using* $A = \epsilon a$*, to the forms:*

$$\begin{aligned} y(\theta, A) = A\cos\theta + \left(\frac{\alpha A^2}{6}\right)[-3 + 2\cos\theta + \cos 2\theta] \\ + \left(\frac{A^3}{3}\right)\left[-\alpha^2 + \left(\frac{174\alpha^2 - 27\beta}{288}\right)\cos\theta\right. \\ \left. + \left(\frac{\alpha^2}{3}\right)\cos 2\theta + \left(\frac{2\alpha^2 + 3\beta}{32}\right)\cos 3\theta\right] + O\left(A^3\right), \end{aligned}$$

$$\omega(A) = 1 + A^2 \left(\frac{9\beta - 10\alpha^2}{24} \right) + O\left(A^3\right)$$

where

$$0 < A \ll 1$$

2.3 Averaging method

In this section we consider the equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x}) \tag{2.42}$$

If $\varepsilon = 0$, the solutions are known: a linear combination of $\cos t$ and $\sin t$. We can also write this linear combination as

$$r_0 \cos\left(t + \psi_0\right)$$

The amplitude r_0 and the phase ψ_0 are constants and they are determined by the initial values. To study the behaviour of the solutions for $\varepsilon \neq 0$, Lagrange introduces "variation of constants". One assumes that for $\varepsilon \neq 0$, the solution can still be written in this form where the amplitude r and the phase ψ are now functions of time. So we put for the solution of equation (2.42)

$$x(t) = r(t)\cos(t + \psi(t)) \tag{2.43}$$

$$\dot{x}(t) = -r(t)\sin(t + \psi(t)) \tag{2.44}$$

Substitution of these expressions for x and $\dot{x}$ into equation (2.42) produces an equation for r and ψ

$$\begin{gathered} -\dot{r}\sin(t+\psi) - r\cos(t+\psi)(1+\dot{\psi}) + r\cos(t+\psi) = \\ = \varepsilon f(r\cos(t+\psi), -r\sin(t+\psi)) \end{gathered}$$

or

$$-\dot{r}\sin(t+\psi) - r\cos(t+\psi)\dot{\psi} = \varepsilon f(r\cos(t+\psi), -r\sin(t+\psi)) \tag{2.45}$$

Another requirement is that differentiation of the righthandside of equation (2.43) must produce an expression which equals the righthand side of equation (2;44). So we find

$$\dot{r}\cos(t+\psi) - r\sin(t+\psi)(1+\dot{\psi}) = -r\sin(t+\psi)$$

or

$$\dot{r}\cos(t+\psi) - \dot{\psi} r \sin(t+\psi) = 0 \tag{2.46}$$

Equations (2.45) and (2.46) can be considered as two algebraic equations in $\dot{r}$ and $\dot{\psi}$; the solutions are

$$\begin{aligned} \dot{r} &= -\varepsilon \sin(t+\psi) f(r\cos(t+\psi), -r\sin(t+\psi)) \\ \dot{\psi} &= -\frac{\varepsilon}{r}\cos(t+\psi) f(r\cos(t+\psi), -r\sin(t+\psi)) \end{aligned} \tag{2.47}$$

The transformation $2.43-44$ x , $\dot{x} \to r, \psi$ presupposes of course that $r \neq 0$; in problems where $r\,(t_0) = 0$ we have to use a different transformation. The system of differential equations (2.47) is supplied with initial conditions, for instance at $t_0 = 0$ determined by

$$\begin{aligned} x(0) &= r(0)\cos\psi(0) \\ \dot{x}(0) &= -r(0)\sin\psi(0). \end{aligned}$$

The reasoning of Lagrange is now as follows. The righthand sides of system (2.65) can be expanded in the form

$$\varepsilon\left[g_0(r,\psi) + g_1(r,\psi)\sin t + h_1(r,\psi)\cos t + \ldots\right]$$

where the dots represent the higher order harmonics $g_n(r,\psi)\sin nt$ and $h_n(r,\psi)\cos nt$ $n = 2, 3, \ldots$ So this is what we are calling nowadays a Fourier expansion of the righthand sides. According to equations (2.65) the variables r and ψ change slowly with time; the contribution of these changes is, averaged over the period 2π, zero, except for the term $g_0(r,\psi)$. So we omit all the terms which have average zero and we solve the simplified system of differential equations.
The idea of averaging is so natural, that for a long time the method has been used in many fields of application without people bothering about proofs of validity.

A survey of the theory of averaging and many new results can be found in [2]. We conclude this introduction with an explicit calculation.

Example 2.4. *Consider the equation for a nonlinear oscillator with damping*

$$\ddot{x} + x = \varepsilon\left(-\dot{x} + x^2\right)\ . \tag{2.48}$$

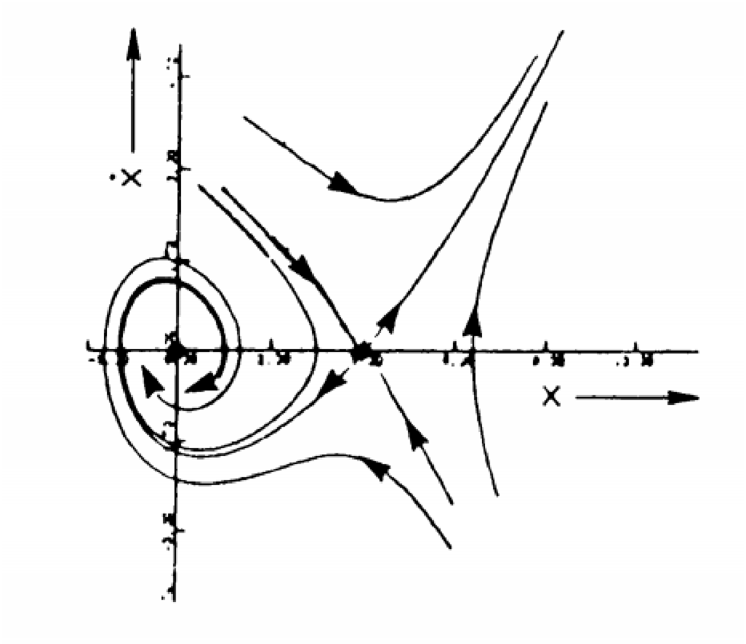

Figure 2.1: solution of the equation 2.48; $0 \leq x \leq 20$

The trivial solution is asymptotically stable. We shall apply the method of Lagrange. Transformation 2.43 − 44 *yields*

$$\dot{r} = -\varepsilon r \sin^2(t+\psi) - \varepsilon r^2 \sin(t+\psi)\cos^2(t+\psi)$$
$$\dot{\psi} = -\varepsilon \cos(t+\psi)\sin(t+\psi) - \varepsilon r \cos^3(t+\psi)$$

The righthand sides are 2π *-periodic in* t*. We average now over* t*, keeping* r *and* ψ *fixed. The result is*

$$\dot{r}_a = -\frac{1}{2}\varepsilon r_a, \dot{\psi}_a = 0$$

with index a *to indicate that we approximate* r *and* ψ *by* r_a *and* ψ_a*. The averagec equations are very easy to solve:*

$$r_a(t) = r(0)e^{-\frac{1}{2}\epsilon t}, \psi_a(t) = \psi(0)$$

As an approximation of $x(t)$ *we propose*

$$x_a(t) = r(0)e^{-\frac{1}{2}\epsilon t}\cos(t + \psi(0)) \tag{2.49}$$

For which initial values does expression 2.49 *appear to be a reasonable approxi mation of the solution (see figure* 2.1 *for the phase-plane of equation 2.48)?*

2.3.1 Lagrange standard form

In proving the asymptotic validity of the approximations one usually starts off with a slowly varying system like equation 2.47. Here we show how to obtain such equations by the method of "variation of parameters". Consider for $x \in \mathbf{R}^n, t \geq 0$ the initial value problem

$$\dot{x} = A(t)x + \varepsilon g(t,x), x(0) = x_0 \tag{2.50}$$

$A(t)$ is a continuous $n \times n$ -matrix, $g(t,x)$ is a sufficiently smooth function of t and x. The unperturbed ($\varepsilon = 0$) equation is linear and it has n independent solutions which are used to compose a fundamental matrix $\phi(t)$. We put

$$x = \Phi(t)y \quad \text{(Lagrange)}$$

In mechanics this is called sometimes "introducing comoving coordinates". Substitution in equation (2.50) produces

$$\dot{\Phi}y + \Phi\dot{y} = A(t)\Phi y + \varepsilon g(t, \Phi y)$$

and as $\dot{\Phi} = A(t)\Phi$ we have

$$\Phi\dot{y} = \varepsilon g(t, \Phi y)$$

Equations (2.45-46) are a special case of this vector equation. We solve for $\dot{y}$ by inverting the fundamental matrix:

$$\dot{y} = \varepsilon\Phi^{-1}(t)g(t, \Phi(t)y) \tag{2.51}$$

The initial values follow from

$$y(0) = \Phi^{-1}(0)x_0$$

Writing down the equations for the components of y while using Cramer's rule we have

$$\dot{y}_i = \varepsilon\frac{W_i(t,y)}{W(t)}, i = 1, \ldots, n \tag{2.52}$$

with $W(t) = |\Phi(t)|$, the wronskian (determinant of $\Phi(t)$); $W_i(t,y)$ is the determinant of the matrix which one obtains by replacing the i^{th} column of $\Phi(t)$ by g. standard form is

$$\dot{y} = \varepsilon f(t, y)$$

Remark 2.3.1. *The unperturbed equation in the case of equation* (2.50) *is linear; this facilitates the variation of parameters procedure. If the unperturbed equation is nonlinear the variation of parameters technique still applies. In practice however there are usually many technical obstructions while carrying out the procedure.*

Example 2.5. *Consider the equation*

$$\ddot{x} + \omega^2 x = \varepsilon g(t, x, \dot{x}) \tag{2.53}$$

with constant $\omega > 0$. *We have some freedom in choosing the transformation to the standard form as there are several representations of the solutions of the unperturbed problem. We start with an amplitude-phase transformation*

$$\begin{aligned} x(t) &= r(t)\cos(\omega t + \psi(t)) \\ \dot{x}(t) &= -r(t)\omega\sin(\omega t + \psi(t)) \end{aligned} \tag{2.54}$$

We find the equations

$$\begin{aligned} \dot{r} &= -\frac{\varepsilon}{\omega}\sin(\omega t + \psi)g(t, r\cos(\omega t + \psi), -r\omega\sin(\omega t + \psi)) \\ \dot{\psi} &= -\frac{\epsilon}{\omega r}\cos(\omega t + \psi)g(t, r\cos(\omega t + \psi), -r\omega\sin(\omega t + \psi)) \end{aligned} \tag{2.55}$$

Another possibility is to use the transformation

$$\begin{aligned} x(t) &= y_1(t)\cos\omega t + \frac{y_2(t)}{\omega}\sin\omega t \\ \dot{x}(t) &= -\omega y_1(t)\sin\omega t + y_2(t)\cos\omega t \end{aligned} \tag{2.56}$$

We find the equations

$$\begin{aligned} \dot{y}_1 &= -\tfrac{\varepsilon}{\omega}\sin\omega t g(t, x(t), \dot{x}(t)) \\ \dot{y}_2 &= \varepsilon\cos\omega t g(t, x(t), \dot{x}(t)) \end{aligned} \tag{2.57}$$

In g *the expressions for* $x(t)$ *and* $\dot{x}(t)$ *still have to be substituted. Equation* (2.53) *can be treated both with the standard form* (2.55) *and the standard form* (2.57). *It turns out that the explicit form of the righthand side* $\varepsilon g(t, x, \dot{x})$ *determines which choice is the best one.*

2.3.2 Averaging in the periodic case

We shall now consider the asymptotic validity of the averaging method. Consider the initial value problem

$$\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon), x(0) = x_0 \tag{2.58}$$

We assume that $f(t,x)$ is T -periodic in t and we introduce the average

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt$$

In performing the integration y has been kept constant. Consider now the initial value problem for the averaged equation

$$\dot{y} = \varepsilon f^0(y), y(0) = x_0 \tag{2.59}$$

The vector function $y(t)$ represents an approximation of $x(t)$ in the following way :

Theorem 2.1. *Consider the initial value problems* 2.58 *and* 2.59 *with* $x, y, x_0 \in D \subset R^n, t \geq 0$ *Suppose that*
a. the vector functions f, g *and* $\partial f/\partial x$ *are defined, continuous and bounded by a constant* M *(independent of* ε *) in* $[0,\infty) \times D$*;*
b. g *is Lipschitz-continuous in* x *for* $x \in D$*;*
c. $f(t,x)$ *is* T *-periodic in* t *with average* $f^0(x); T$ *is a constant which is independent of* ε
d. $y(t)$ *is contained in an internal subset of* D*.*
Then we have $x(t) - y(t) = O(\varepsilon)$ *on the time scale* $1/\varepsilon$*.*

Proof: See [7].

Example 2.6. *Consider the autonomous equation*

$$\ddot{x} + x = \varepsilon f(x, \dot{x}) \tag{2.60}$$

with given initial values. Transforming $x, \dot{x} \to r, \psi$ *with* $(2.43-44)$ *we obtain the equations*

$$\begin{aligned} \dot{r} &= -\varepsilon \sin(t+\psi) f(r\cos(t+\psi), -r\sin(t+\psi)) \\ \dot{\psi} &= -\tfrac{\varepsilon}{r}\cos(t+\psi) f(r\cos(t+\psi), -r\sin(t+\psi)) \end{aligned} \tag{2.61}$$

The righthand side is 2π *-periodic in* t*; while averaging* ψ *is kept constant, so for the averaging process the system can also be considered to be* 2π *-periodic in* $t+\psi$*. With* $s = t+\psi$ *we find*

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(t+\psi) f(r\cos(t+\psi), -r\sin(t+\psi))dt =$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(t+\psi)f(r\cos(t+\psi),-r\sin(t+\psi))dt =$$
$$\frac{1}{2\pi}\int_\psi^{2\pi+\psi}\sin s f(r\cos s,-r\sin s)ds = f_1(r)$$

Because of the 2π *-periodicity of the integrand,* $f_1(r)$ *doesnot depend on* ψ*. In the same way we find*

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(t+\psi)f(r\cos(t+\psi),-r\sin(t+\psi))dt = f_2(r)$$

An asymptotic approximation r_a, ψ_a *of* r, ψ *can be found as solutions of*

$$\begin{aligned} \dot{r}_a &= -\varepsilon f_1\left(r_a\right), r_a(0) = r(0) \\ \dot{\psi}_a &= -\frac{\varepsilon}{r_a} f_2\left(r_a\right), \psi_a(0) = \psi(0) \end{aligned} \tag{2.62}$$

As ψ_a *is not present in the righthand sides, the order of the differential equations to be solved has been reduced to one. We apply this to a standard example, the van der Pol-equation*

$$\ddot{x} + x = \varepsilon\left(1 - x^2\right)\dot{x}.$$

Using the formulas in appendix 3 we find

$$\dot{r}_a = \frac{1}{2}\varepsilon r_a\left(1 - \frac{1}{4}r_a^2\right), \dot{\psi}_a = 0 \tag{2.63}$$

Taking $r(0) = 2$ *we have* $r_a(t) = 2, t \geq 0$ *(critical point of the first equation in 2.63). Using averaging we find with* $r(0) = 2, \psi(0) = 0$

$$x(t) = 2\cos t + O(\varepsilon) \text{ on the time scale } 1/\varepsilon$$

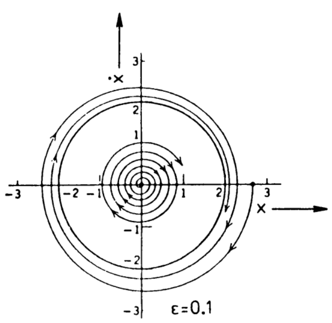

Figure 2.2: solutions of the equation 2.60

By the averaging method we find moreover an approximation for the other solutions by solving equations (2.63) *:*

$$r_a(t) = \frac{r(0)e^{\frac{1}{2}et}}{\left[1 + \frac{1}{4}r_0^2\left(e^{et} - 1\right)\right]^{\frac{1}{2}}}, \quad \psi_a(t) = \psi(0)$$

As $x_a(t) = r_a(t)\cos\left(t + \psi_a(t)\right)$ we find that with increasing time t, the solutions approach the periodic solution. In figure 2.2 *we have presented the phase-plane in the case $\varepsilon = .1$.*

Example 2.7. *Consider the Mathieu-equation*

$$\ddot{x} + (1 + 2\varepsilon\cos 2t)x = 0 \tag{2.64}$$

with initial values $x(0) = x_0, \dot{x}(0) = 0$. Equation (2.64) *models linear oscillators in engineering which are subjected to frequency modulation. We apply the transformations from example* 2.5 *with $g = -2\varepsilon\cos(2t)x$ and $\omega = 1$. Introducing the*

amplitude r and phase ψ, the averaged equations become rather complicated. In equations like this it is more convenient to use transformation $(2;56)$

$$x(t) = y_1(t)\cos t + y_2(t)\sin t$$
$$\dot{x}(t) = -y_1(t)\sin t + y_2(t)\cos t$$

which leads to

$$\dot{y}_1 = 2\varepsilon\sin t\cos 2t\left(y_1\cos t + y_2\sin t\right), \quad y_1(0) = x_0$$

$$\dot{y}_2 = -2\varepsilon\cos t\cos 2t\left(y_1\cos t + y_2\sin t\right), \quad y_2(0) = 0$$

The righthand side is 2π -periodic in t. We find for the averaged equations :

$$\dot{y}_{1a} = -\frac{1}{2}\varepsilon y_{2a}, \dot{y}_{2a} = -\frac{1}{2}\varepsilon y_{1a}$$

We solve these linear equations while imposing the initial values with the result

$$y_{1a}(t) = \frac{1}{2}x_0e^{-\frac{1}{2}et} + \frac{1}{2}x_0e^{\frac{1}{2}et}, y_{2a}(t) = \frac{1}{2}x_0e^{-\frac{1}{2}\epsilon t} - \frac{1}{2}x_0e^{\frac{1}{2}et}$$

An $O(\varepsilon)$ approximation of $x(t)$ on the time scale $1/\varepsilon$ is

$$x_a(t) = \frac{1}{2}x_0e^{-\frac{1}{2}et}(\cos t + \sin t) + \frac{1}{2}x_0e^{\frac{1}{2}et}(\cos t - \sin t)$$

If $\varepsilon = 0$, the solutions of equation 2.64 are all stable.

Chapter 3

Approximate solutions with time scales

A standard time is a specification for measuring time: either the rate at which time passes, points in time, or both. An example of a standard type of time could be a time scale, which defines a way to measure divisions of time. The civil time standard can specify time and intervals.

3.1 Approximation of periodic solutions on arbitrary long timescales

When analysing a periodic solution of an equation with the Poincaré- Lindstedt method, the period and the other characteristic quantities (amplitude, phase) can be approximated with arbitrary good precision. One of the consequences is that we can find approximations which are valid on an interval of time which is much longer than the period. We shall illustrate the ideas again for equation 2.1 which has been discussed in the preceding section. In other problems where the Poincaré Lindstedt method is being used, we can follow an analogous reasoning. In the $x, \dot{x}$ -phase plane of equation 2.1, the T -periodic solution corresponds with a closed orbit; this also holds for the approximation to order ε^N which we have constructed. During an interval of time of length T, the phase points corresponding with the periodic solution and the approximation are ε^N -close, we have almost synchronisation. After a longer period of time however, the phase points of the periodic solution and the approximation will separate more and more. We shall make this precise. It follows from $\omega^{-2} = 1 - \varepsilon\eta(\varepsilon)$

that ω can be expanded in a power series, which is convergent for $0 < \varepsilon \leq \varepsilon_0$:

$$\omega = 1 + \sum_{n=1}^{\infty} \varepsilon^n \omega_n$$

We now introduce the quantities

$$\theta_N = \left(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots + \varepsilon^N \omega_N\right) t$$

and

$$x_M(\theta) = a_0 \cos\theta + \varepsilon\gamma_1(\theta) + \varepsilon^2\gamma_2(\theta) + \ldots + \varepsilon^M \gamma_M(\theta).$$

Note that in the Poincaré-Lindstedt method one has $M = N$ and the algorithm produces $x_N(\theta_N)$ which is an $O\left(\varepsilon^{N+1}\right)$ approximation of the periodic solution on the time scale 1. We shall demonstrate that the problem of "almost synchronisation" can be discussed on a longer time scale by considering $M \neq N$. We shall use

$$x(\theta) - x_M(\theta_N) = x(\theta) - x(\theta_N) + x(\theta_N) - x_M(\theta_N)$$

and by application of the triangle inequality

$$\|x(\theta) - x_M(\theta_N)\| \leq \|x(\theta) - x(\theta_N)\| + \|x(\theta_N) - x_M(\theta_N)\| . \tag{3.1}$$

The function $x(\theta)$ is continuously differentiable, so, for ε sufficiently small, $x(\theta)$ is Lipschitz-continuous in θ with Lipschitz-constant L. So we have

$$\|x(\theta) - x(\theta_N)\| \leq L\|\theta - \theta_N\| = L\left|\sum_{n=N+1}^{\infty} \varepsilon^n \omega_n\right| t$$

Furthermore it follows from the expansion theorem that

$$x(\theta_N) - x_M(\theta_N) = \sum_{n=M+1}^{\infty} \varepsilon^n \gamma_n(\theta_N)$$

with bounded coefficients γ_n. Using these results, the estimate 3.1 becomes

$$\begin{aligned} \|x(\theta) - x_M(\theta_N)\| &\leq L\left|\sum_{n=N+1}^{\infty} \varepsilon^n \omega_n\right| t + \left|\sum_{n=M+1}^{\infty} \varepsilon^n \gamma_n(\theta_N)\right| \\ &= O\left(\varepsilon^{N+1} t\right) + O\left(\varepsilon^{M+1}\right) \end{aligned} \tag{3.2}$$

For $M = N$ and $0 \leq t \leq T$ we recover the results of section 2.1 from 3.2. However with $M = N - 1$ we find for $0 \leq \varepsilon t \leq T$ $\|x(\theta) - x_{N-1}(\theta_N)\| = O\left(\varepsilon^N\right)$ on the time scale $1/\varepsilon$.

We also find $\|x(\theta) - x_{N-2}(\theta_N)\| = O\left(\varepsilon^{N-1}\right)$ on the time scale $1/\varepsilon^2$.

The approximations and their corresponding accuracy can be summarised in table 2.1.

time scale \ accuracy	$o(\varepsilon)$	$o(\varepsilon^2)$	$o(\varepsilon^3)$
$0 \leq t \leq T$	$a_0 \cos\theta_0$	$a_0 \cos\theta_1 + \varepsilon\gamma_1(\theta_1)$	$a_0 \cos\theta_2 + \varepsilon\gamma_1(\theta_2) + \varepsilon^2\gamma_2(\theta_2)$
$0 \leq \varepsilon t \leq T$	$a_0 \cos\theta_1$	$a_0 \cos\theta_2 + \varepsilon\gamma_1(\theta_2)$	$a_0 \cos\theta_3 + \varepsilon\gamma_1(\theta_3) + \varepsilon^2\gamma_2(\theta_3)$
$0 \leq \varepsilon^2 t \leq T$	$a_0 \cos\theta_2$	$a_0 \cos\theta_3 + \varepsilon\gamma_1(\theta_3)$	$a_0 \cos\theta_4 + \varepsilon\gamma_1(\theta_4) + \varepsilon^2\gamma_2(\theta_4)$

Table 2.1

In the table we have $\theta_0 = t, \theta_1 = (1 + \varepsilon\omega_1)\, t, \theta_2 = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2)\, t$, etc. The first row represents the convergent power series expansion which produces approximations, valid on the time scale 1. The subsequent rows in the table contain asymptotic approximations which are valid on a longer time scale.
We shall substitute the expressions in the table which apply to the periodic solution of the Van Der Pol equation $\ddot{x} + x = \varepsilon(1 - x^2)\,\dot{x}$:

Table 2.2 **Periodic solution Van Der Pol equation**

time scale \ accuracy	$o(\varepsilon)$	$o(\varepsilon^2)$	$o(\varepsilon^3)$
$0 \leq t \leq T$	$2\cos t$	$2\cos t + \varepsilon\gamma_1(t)$	$2\cos\theta_2 + \varepsilon\gamma_1(\theta_2) + \varepsilon^2\gamma_2(\theta_2)$
$0 \leq \varepsilon t \leq T$	$2\cos t$	$2\cos\theta_2 + \varepsilon\gamma_1(\theta_2)$	$2\cos\theta_2 + \varepsilon\gamma_1(\theta_2) + \varepsilon^2\gamma_2(\theta_2)$
$0 \leq \varepsilon^2 t \leq T$	$2\cos\theta_2$	$2\cos\theta_2 + \varepsilon\gamma_1(\theta_2)$	$2\cos\theta_4 + \varepsilon\gamma_1(\theta_4) + \varepsilon^2\gamma_2(\theta_4)$

in which

$$\begin{aligned}
\theta_2 &= \left(1 - \frac{1}{16}\varepsilon^2\right) t, \theta_4 = 1 - \frac{1}{16}\varepsilon^2 + \frac{17}{3072}\varepsilon^4 \\
\gamma_1(\theta) &= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \\
\gamma_2(\theta) &= -\frac{1}{8}\cos\theta + \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta \\
T &= 2\pi\left(1 + \frac{1}{16}\varepsilon^2 - \frac{5}{3072}\varepsilon^4\right).
\end{aligned}$$

Those approximations which are valid on the time scale $1/\varepsilon$.

Using the expansion, computed by Andersen and Geer (1982) which contains the expansion coefficients of ω to order ε^{104}, we can approximate the periodic solution on a very long time scale. Take for instance $N = 10$. The function $2\cos\theta_{10}$ with

$\theta_{10} = \left(1 - \frac{1}{16}\varepsilon^2 + \frac{17}{3072}\varepsilon^4 + \frac{35}{884736}\varepsilon^6 - \frac{678899}{5096079360}\varepsilon^8 + \frac{28160413}{2293235712000}\varepsilon^{10}\right) t$

approximates the periodic solution of the van der Pol equation with accuracy $O(\varepsilon)$ on the time scale ε^{-10}. If for instance $\varepsilon = 0.1$, the validity is on an interval of time of the order 10^{10} periods.

3.1.1 Periodic solutions of equations with forcing terms

The theory of nonlinear differential equations with inhomogeneous time-dependent terms, which represent oscillating systems with exciting forces, turns out to be very rich in phenomena. Here we consider only an important prototype problem, the forced Duffing-equation. This equation models many problems involving a nonlinear oscillator with damping and forcing. The forced Duffing-equation will be considered in the form

$$\ddot{x} + \varepsilon\mu\dot{x} + x - \varepsilon x^3 = \varepsilon h \cos\omega t \tag{3.3}$$

with constants $\mu \geq 0$ and $h > 0$; the forcing has a frequency which is near to 1 (a period which is near to 2π) so we may put

$$\omega^{-2} = 1 - \varepsilon\beta$$

with β a constant independent of ε. We shall look for solutions of equation 3.3 which are periodic with the period $2\pi/\omega$ of the forcing term. Note that when studying autonomous equations, the period is not determined apriori . Also, when studying autonomous equations, this enables us to put $\psi = 0$ for the phase ψ in those cases without loss of generality. In the case of equation 3.3 we have not the translation property so it is reasonable to introduce a phase ψ. We put

$$\omega t = \theta - \psi$$

which, with $dx/d\theta = \dot{x}$, transforms equation 3.3 to

$$\begin{aligned} \dot{x} + x =& \varepsilon g\left(x, \dot{x}, \psi, \theta, \varepsilon\right) \\ =& \varepsilon\beta x - \varepsilon\mu(1-\varepsilon\beta)^{\frac{1}{2}}\dot{x} + \varepsilon(1-\varepsilon\beta)x^3 + \\ &+ \varepsilon(1-\varepsilon\beta)h\cos(\theta-\psi) \end{aligned} \tag{3.4}$$

We shall look for 2π -periodic solutions of equation 3.4 with initial values $x(0) = a(\varepsilon), \dot{x}(0) = 0$; this problem is equivalent with looking for 2π -periodic solutions of the integral equations

$$\begin{aligned} x(\theta) &= a\cos\theta + \varepsilon\textstyle\int_0^\theta \sin(\theta-\tau)g\left(x(\tau), \dot{x}(\tau), \psi, \tau, \varepsilon\right)d\tau \\ \dot{x}(\theta) &= -a\sin\theta + \varepsilon\textstyle\int_0^\theta \cos(\theta-\tau)g\left(x(\tau), \dot{x}(\tau), \psi, \tau, \varepsilon\right)d\tau \end{aligned}$$

The periodicity condition becomes

$$\int_0^{2\pi} \sin\tau g\left(x(\tau), \dot{x}(\tau), \psi, \tau, \varepsilon\right)d\tau = 0, \int_0^{2\pi} \cos\tau g\left(x(\tau), \dot{x}(\tau), \psi, \tau, \varepsilon\right)d\tau = 0 \tag{3.5}$$

Apart from ε, the periodic solution depends on a and ψ. System 3.5 can be considered as a system of two equations with two unknowns, a and ψ. According to the implicit function theorem, this system is uniquely solvable in a neighbourhood of $\varepsilon = 0$, if the corresponding determinant of Jacobi does not vanish. Rewriting system 3.5 as $F_1(a,\psi) = 0, F_2(a,\psi) = 0$ this means that we have unique solvability if

$$\frac{\partial\left(F_1, F_2\right)}{\partial(a,\psi)} \neq 0 \tag{3.6}$$

with ε in a neighbourhood of 0.

According to the implicit function theorem we may expand

$$a(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n a_n, \psi(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \psi_n, x(\theta) = a_0\cos\theta + \sum_{n=1}^{\infty} \varepsilon^n \gamma_n(\theta)$$

Using these series, the expression for g in 3.4 yields the expansion

$$g\left(x, \dot{x}, \psi, \theta, \varepsilon\right) = \beta a_0\cos\theta + \mu a_0\sin\theta + a_0^3\cos^3\theta + h\cos\left(\theta - \psi_0\right) + \varepsilon\ldots$$

The periodicity condition 3.5 becomes at first order

$$\mu a_0 + h\sin\psi_0 \quad = 0; \beta a_0 + h\cos\psi_0 + \frac{3}{4}a_0^3 = 0 \tag{3.7}$$

System 3.7 is uniquely solvable if (3.6)

$$\mu \sin \psi_0 + \beta \cos \psi_0 + \frac{9}{4} a_0^2 \cos \psi_0 \neq 0 \tag{3.8}$$

The problem without damping, $\mu = 0$, is the simplest. We have in this case $\psi_0 = 0, \pi$; the possible values of a_0 follow from the equations $\beta a_0 \pm h + \frac{3}{4} a_0^3 = 0$ The requirement of uniqueness 3.8 becomes in this case $\beta + \frac{9}{4} a_0^2 \neq 0$ If, for given values of μ, h and β, we have determined the coefficients a_0 and ψ_0, we find for the periodic solution

$$x(t) = a_0 \cos(\omega t + \psi_0) + O(\varepsilon) \text{ on the time scale 1.}$$

From the discussion in the preceding section it is clear that, if we wish to construct an approximation which is valid on a longer time scale, we have to compute more terms of the expansion. Another remark is, that we have obtained an approximation of a periodic solution but until now there is no simple criterion to determine its stability at the same time.

3.1.2 Existence of periodic solutions

We studied expansion with respect to the small parameter ε for periodic solutions of the equation $\ddot{x} + x = \varepsilon f(x, \dot{x}, \varepsilon)$; and we considered a problem where the right hand side is time-dependent. In both cases we were able to apply the Poincaré expansion theorem. The periodicity condition has the form $F_1(a, \eta) = 0, F_2(a, \eta) = 0$ (equations 10.3 or 10.18) with a and η two parameters, which still have to be determined. The equations for a and η are uniquely solvable if

$$\frac{\partial (F_1, F_2)}{\partial (a, \eta)} \neq 0 \tag{3.9}$$

This is the most important condition for application of the implicit function theorem, the other conditions of this theorem have been satisfied by the assumptions of the Poincaré expansion theorem. We note that in principle, condition 3.9 is difficult to verify as the unknown periodic solution is contained in the expressions for F_1 and F_2. However, verification of condition 3.9 is possible by applying the expansion theorem to develop F_1 and F_2 with respect to the small parameter. We summarise this as follows.

Theorem 3.1. *Consider equation* 2.1 $\ddot{x}+x = \varepsilon f(x, \dot{x}, \varepsilon)$ *(or equation 2.16). If the conditions of the Poincaré expansion theorem have been satisfied and simultaneously the periodicity condition* 2.3 *(or 3.6) and the uniqueness condition* 3.9*, then there exists a periodic solution. This solution can be represented by a convergent power series in* ε *for* $0 \leq \varepsilon < \varepsilon_0$

The problem of the forced Duffing equation can be generalised as follows. Consider in R^n the equation

$$\dot{x} = f(t, x) + \varepsilon g(t, x, \varepsilon). \tag{3.10}$$

with (for instance) f and g T-periodic. The "unperturbed equation" is

$$\dot{y} = f(t, y) \tag{3.11}$$

and this equation is assumed to have a T -periodic solution $\phi(t)$. Does there exist a T -periodic solution $\psi(t, \varepsilon)$ which is close to $\phi(t)$ with $\psi(t, 0) = \phi(t)$? Consider equation 3.10 with initial condition

$$x|_{t=0} = \phi(0) + \mu. \tag{3.12}$$

The solution of this initial value problem will be $x(t, \varepsilon, \mu)$. We assume that the conditions of the expansion theorem have been satisfied so that

$$x(t, \varepsilon, \mu) = \phi(t) + \sum_{k_1+\dots k_n+k\geq 1} \gamma_{k_1\dots k_n k^{(t)}} \mu_1^{k_1} \dots \mu_n^{k_n} \varepsilon^k. \tag{3.13}$$

The series converges for $\|\mu\| \leq \rho, 0 \leq \varepsilon < \varepsilon_0$ and $t_0 \in [0, T]; \mu_1, \dots, \mu_n$ are the components of μ. The coefficients γ can be determined by substitution of the series 3.13 in equation 3.10 and by equating coefficients of equal powers of the parameters on both sides of the equation. This process produces a system of linear differential equations for the unknown expansion coefficients γ which presumably we can solve. The periodicity condition is

$$H(\mu_1, \dots, \mu_n, \varepsilon) = x(T, \varepsilon, \mu) - x(0, \varepsilon, \mu) = 0 \tag{3.14}$$

This system of n equations with n unknowns $\mu_1, \dots, \mu_n$ is uniquely solvable for ε near zero if for the Jacobian we have

$$\frac{\partial H}{\partial \mu} \neq 0$$

The corresponding solutions for the parameters $\mu_1, \ldots, \mu_n$ of 3.14 produce a T periodic solution 3.13 of equation 3.10

by applying the Poincaré-Lindstedt method all kinds of complications may arise. For instance in a number of problems, like in Hamiltonian systems, periodic solutions arise in continuous one-parameter families instead of isolated periodic solutions. In such a case condition 3.9 will not be satisfied and one has to take a closer look at the question whether the expansions which have been obtained represent periodic solutions. The verification of condition 3.9 can usually be carried out only by expansion with respect to the small parameter. condition 3.9 has been satisfied already at first order in the expansion. In practice it may happen that one needs higher order expansions to achieve this. We illustrate this as follows.

Example 3.1. *Consider the modified van der Pol equation for* $\varepsilon > 0$

$$\ddot{x} + x = \varepsilon x^3 + \varepsilon^2 \left(1 - x^2\right) \dot{x} \tag{3.15}$$

We shall look for periodic solutions of this equation, we put

$$\omega t = \theta, \omega^{-2} = 1 - \varepsilon \eta(\varepsilon).$$

Equation 3.15 *becomes with* $dx/d\theta = x'$

$$\ddot{x} + x = \varepsilon \left[\eta x + (1 - \varepsilon\eta) x^3 + \varepsilon (1 - \varepsilon\eta)^{\frac{1}{2}} \left(1 - x^2\right) \dot{x}\right] \tag{3.16}$$

The initial values will be again $x(0) = a(\varepsilon), \dot{x}(0) = 0$. *We expand*

$$\begin{gathered} a(\varepsilon) = \textstyle\sum_{n=0}^{\infty} \varepsilon^n a_n, \eta(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \eta_n \\ x(\theta) = a_0 \cos\theta + \varepsilon \gamma_1(\theta) + \varepsilon^2 \gamma_2(\theta) + \ldots \\ \dot{x}(\theta) = -a_0 \sin\theta + \varepsilon \dot{\gamma_1}(\theta) + \varepsilon^2 \dot{\gamma_2}(\theta) + \ldots \end{gathered}$$

Substitute in equation 3.16 *and equating coefficients of equal powers of* ε *yields*

$$\ddot{\gamma_1} + \gamma_1 = \eta_0 a_0 \cos\theta + a_0^3 \cos^3\theta \tag{3.17}$$

$$\ddot{\gamma_2} + \gamma_2 = \eta_1 a_0 \cos\theta + \eta_0 \gamma_1(\theta) - \eta_0 a_0^3 \cos^3\theta \tag{3.18}$$

$$+3a_0^2\cos^2\theta\gamma_1(\theta) - \left(1 - a_0^2\cos^2\theta\right)a_0\sin\theta$$

etc. The general solution of equation 3.17 *is*

$$\gamma_1(\theta) = A_1\cos\theta + B_1\sin\theta + \int_0^\theta \sin(\theta-\tau)\left[\eta_0 a_0\cos\tau + \right.$$
$$\left. + a_0^3\cos^3\tau\right]d\tau$$

From the initial values we have $A_1 = a_1, B_1 = 0; a_0$ *and* a_1 *are unknown. The periodicity condition produces*

$$\int_0^{2\pi}\sin\tau\left[\eta_0 a_0\cos\tau + a_0^3\cos^3\tau\right]d\tau = 0,$$
$$\int_0^{2\pi}\cos\tau\left[\eta_0 a_0\cos\tau + a_0^3\cos^3\tau\right]d\tau = 0 \tag{3.19}$$

The first equation of 3.19 *holds for all values of* η_0 *and* a_0*; from the second equation we find*

$$a_0\left(\eta_0 + \frac{3}{4}a_0^2\right) = 0$$

The Jacobian 3.9 *of system* 3.19 *is zcro. The parameters* a_0 *and* η_0 *have not been determined uniquely. We find with* $\eta_0 = -\frac{3}{4}a_0^2$

$$\gamma_1(\theta) = a_1\cos\theta + \int_0^\theta \sin(\theta-\tau)\frac{a_0^3}{4}\cos 3\tau d\tau$$
$$= a_1\cos\theta + \frac{a_0^3}{32}(\cos\theta - \cos 3\theta)$$

we should consider the consequences of the choice $a_0 = 0$*. by the same way we determine the solution of equation* 3.18 *The periodicity condition produces*

$$\begin{aligned}&\int_0^{2\pi}\sin\tau\left[\eta_1 a_0\cos\tau + \eta_0\gamma_1(\tau) - \eta_0 a_0^3\cos^3\tau + \right.\\ &\left. 3a_0^2\cos^2\tau\gamma_1(\tau) - \left(1 - a_0^2\cos^2\tau\right)a_0\sin\tau\right]d\tau = 0\\ &\int_0^{2\pi}\cos\tau\left[\eta_1 a_0\cos\tau + \eta_0\gamma_1(\tau) - \eta_0 a_0^3\cos^3\tau + \right.\\ &\left. 3a_0^2\cos^2\tau\gamma_1(\tau) - \left(1 - a_0^2\cos^2\tau\right)a_0\sin\tau\right]d\tau = 0\end{aligned} \tag{3.20}$$

Condition 3.20 leads to

$$a_0\left(1 - \frac{1}{4}a_0^2\right) = 0$$

$$\eta_1 a_0 + \eta_0 a_1 - \frac{23}{32}\eta_0 a_0^3 + \frac{9}{4}a_0^2 a_1 + \frac{a_0^3}{64} = 0$$

It follows that $a_0 = 2$ so that $\eta_0 = -3$. In this way the first term of the approximation of $x(\theta)$ has been determined uniquely. The second relation gives

$$2\eta_1 + 6a_1 + \frac{139}{8} = 0$$

The parameters η_1 and a_1 can be determined by analysing the periodicity condition for the solution of the equation for $\gamma_2(\theta)$. The periodic solution of equation 3.15 is approximated as $x(t) = 2\cos\left(1 - \frac{3}{2}\varepsilon\right)t + O(\varepsilon)$ on the time scale $1/\varepsilon$.

3.2 Averaging in the general case

We shall now consider the case in which the vector function $f(t,x)$ is not periodic but where the average exists in a more general sense. A simple example which arises often in applications is a vector function which consists of a finite sum of periodic vector functions with periods which are incommensurable. There exists no common period; for instance the function

$$\sin t + \sin 2\pi t$$

This function is not periodic but each of the two terms separately is periodic, periods 2π and 1. We can formulate the following result.

Theorem 3.2. *Consider the initial value problem*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), x(0) = x_0$$

with $x, x_0 \in D \subset R^n, t \geq 0$. *We assume that*

a. the vector functions f, g *and* $\partial f/dx$ *are defined, continuous and bounded by a constant (independent of* ε *) in* $[0,\infty) \times D$*;*

b. g *is Lipschitz-continuous in* x *for* $x \in D$;

c. $f(t,x) = \sum_{i=1}^{N} f_i(t,x)$ *with* N *fixed;* $f_i(t,x)$ *is* T_i *-periodic in* $t, i = 1,\ldots,n$ *with the* T_i *constants independent of* ε;

d. $y(t)$ *is the solution of the initial value problem*

$$\dot{y} = \varepsilon \sum_{i=1}^{N} \frac{1}{T_i} \int_0^{T_i} f_i(t,y)dt, y(0) = x_0$$

and $y(t)$ is contained in an interior subset of D. Then $x(t) - y(t) = O(\varepsilon)$ on the time scale $1/\varepsilon$.

Proof

In the proof of theorem 3.1 we replace $f^0(y), u(t,y)$ etc. by finite sums. Theorem 3.2 is a simple extension of theorem 3.1; the reason to formulate the theorem separately is that its conditions are often met in applications. In replacing the finite sum for $f(t,x)$ by a convergent series of periodic functions, we do encounter a problem of a different nature. One of the causes of this difference is that the expression $u(t,x)$, used in the near-identity transformation, need not be bounded anymore.

We now present a theorem for the case that the average of $f(t,x)$ exists in a more general sense.

Theorem 3.3. *Consider the initial value problem*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), x(0) = x_0$$

with $x, x_0 \in D \subset \mathbf{R}^n, t \geq 0$. We assume that

a. *the vector functions f, g and $\partial f/dx$ are defined, continuous and bounded by a constant (independent of ε) in $[0,\infty] \times D$;*

b. *g is Lipschitz-continuous in x for $x \in D$;*

c. *the average $f^\circ(x)$ of $f(t,x)$ exists where*

$$f^\circ(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,x)dt$$

d. *$y(t)$ is the solution of the initial value problem*

$$\dot{y} = \varepsilon f^\circ(y), y(0) = x_0$$

Then $x(t) - y(t) = O(\delta(\varepsilon))$ on the time scale $1/\varepsilon$ with

$$\delta(\varepsilon) = \sup_{x\in D} \sup_{0\leq \varepsilon t\leq C} \varepsilon \left\| \int_0^t \left[f(s,x) - f^\circ(x) \right] ds \right\|$$

.

Proof

See Sanders and Verhulst (1985).

If we replace assumption a in theorem 3.3 by the requirement that f is Lipschitzcontinuous in x, we have the weaker estimate $x(t) - y(t) = O\left(\delta^{1/2}(\varepsilon)\right)$ on the time scale $1/\varepsilon$ We shall now discuss a simple example in which the theorem and the part played by the error $\delta(\varepsilon)$ is demonstrated.

Example 3.2. ***(increasing friction)***

Consider the oscillator with damping described by

$$\ddot{x} + \varepsilon f(t)\dot{x} + x = 0 \ . \tag{3.21}$$

We assume that for $t \geq 0$*, the function* $f(t)$ *increases monotonically;* $f(0) = 1$, $\lim\limits_{t\to\infty} f(t) = 2$, $(0,0)$ *is asymptotically stable. As an illustration we consider two cases for* $f(t)$

$$f_1(t) = 2 - e^{-t}$$

and

$$f_2(t) = 2 - (1+t)^{-1}$$

In the first case the frictions increases more rapidly. With 2.55 *we have in amplitude-phase variables*

$$\dot{r} = -\varepsilon r \sin^2(t+\psi) f(t)$$
$$\dot{\psi} = -\varepsilon \sin(t+\psi)\cos(t+\psi) f(t).$$

We compute the averages of the right hand sides in the sense of ***theorem 3.3****. It is easy to see that both in the case of the choice* $f = f_1(t)$ *and* $f = f_2(t)$ *we find the averaged equations*

$$\dot{r}_a = -\varepsilon r_a, \dot{\psi}_a = 0$$

Solutions of equation 3.21 *with* $f(t) = f_1(t)$ *and* $f(t) = f_2(t)$*; the asymptotic approximation* $x_a(t)$ *has been indicated by ?, so in both cases we have the approximation*

$$x_a(t) = r_0 e^{-\varepsilon t}\cos(t + \psi(0))$$

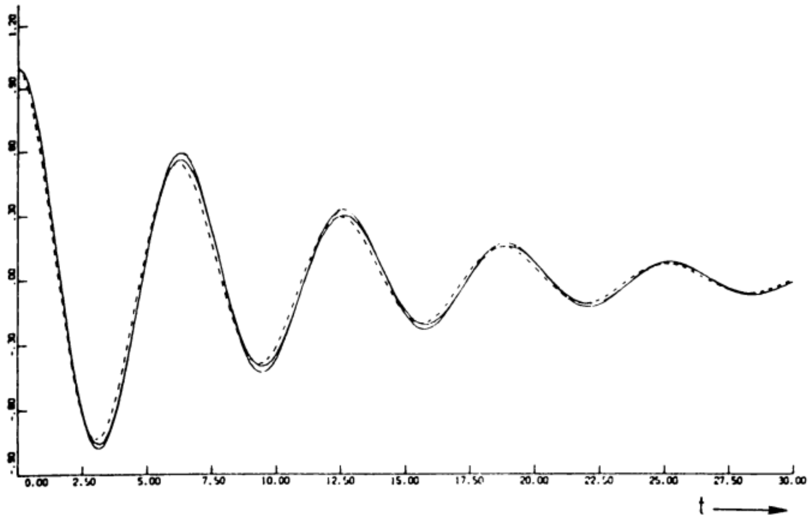

Figure 3.1: Solutions of the equation 3.21

with $x(t) - x_a(t) = O(\delta(\varepsilon))$ *on the time scale* $1/\varepsilon$ *This is a somewhat unexpected result as it means that right from the beginning the oscillator behaves approximately as if the friction has already reached its maximal value. We compute the error* $\delta(\varepsilon)$ *in both cases. By elementary estimates of the integrals we find:*
$f(t) = 2 - e^{-t}$ *yields* $\delta(\varepsilon) = O(\varepsilon)$
$f(t) = 2 - (1+t)^{-1}$ *yields* $\delta(\varepsilon) = O(\varepsilon|\ln \varepsilon|)$
In figure 3.1 *the solutions in the cases* $f = f_1(t)$ *and* $f = f_2(t)$ *have been obtained by numerical integration; they can be identified by realising that the solution corresponding with* $f = f_1(t)$ *experiences the strongest damping. In the case of* $f = f_1(t)$*, the approximation* $x_a(t)$ *is characterised by a better asymptotic error estimate than in the case* $f = f_2(t)$*. Also, it fits the numerical result slightly better.*

3.2.1 Averaging over one angle, resonance manifolds

In a number of examples, we have seen that systems of equations arise which contain two types of dependent variables : $x \in IR^n$ and ϕ which represents an angular variable, i.e. $0 \leq \phi \leq 2\pi$ or $\phi \in S^1$ (the circle or 1 torus). This is quite natural in mechanical systems like nonlinear oscillators with or without coefficients which are slowly varying with time, gyroscopic systems and Hamiltonian systems. The equations are of the form

$$\begin{aligned} \dot{x} &= \varepsilon X(\phi, x) + O\left(\varepsilon^2\right), x \in D \subset \mathbf{R}^n \\ \dot{\phi} &= \Omega(x) + O(\varepsilon), \phi \in S^1 \end{aligned} \tag{3.22}$$

with $X(\phi, x)$ periodic in ϕ and $t \geq 0$. We have seen the example 3.6 that if $\Omega(x)$ is bounded away from zero, we can replace time t by the time-like variable ϕ which yields

$$\frac{dx}{d\phi} = \varepsilon \frac{X(\phi, x)}{\Omega(x)} + O\left(\varepsilon^2\right)$$

after which we may apply theorem 3.1 while averaging over ϕ. We shall now present a theorem, the proof of which has the advantage that it can be generalized to the case of more than one angle.

Theorem 3.4. *Consider system* 3.22 *with initial values* $x(0) = x_0, \phi(0) = \phi_0$ *and suppose that:*

a. the right hand sides are C^1 *in* $D \times S^1$*;*

b. the solution of

$$\frac{dy}{dt} = \varepsilon X^0(y), y(0) = x_0$$

with

$$X^0(y) = \int_{S^1} X(\phi, x) d\phi$$

is contained in an interior subset of D *in which* $\Omega(x)$ *is bounded away from zero by a constant independent of* ε*; then*

$$x(t) - y(t) = O(\varepsilon)$$

on the time scale $1/\varepsilon$*.*

Proof

We introduce the near-identity-transformation

$$x = z + \varepsilon u(\phi, z)$$

with

$$u(\phi, z) = \frac{1}{\Omega(z)} \int^{\phi} \left(X(\theta, z) - X^0(z)\right) d\theta$$

System 3.22 becomes with this transformation

$$\dot{x} = \dot{z} + \varepsilon \tfrac{1}{\Omega(z)} \left(X(\phi, z) - X^0(z) \right) \dot{\phi} + \varepsilon \tfrac{\partial u}{\partial z} \dot{z} = \varepsilon X(\phi, z + \varepsilon u(\phi, z)) + O\left(\varepsilon^2\right)$$
$$\dot{\phi} = \Omega(z + \varepsilon u(\phi, z)) + O(\varepsilon)$$

Expanding as in the proof of theorem 3.1 while using the smoothness of the vector functions we find

$$\dot{z} = \varepsilon X^0(z) + O\left(\varepsilon^2\right)$$
$$\dot{\phi} = \Omega(z) + O(\varepsilon)$$

Note that $u(\phi, z)$ is uniformly bounded as long as assumption b holds. On adding the initial value $z(0) = y(0)$, theorem 1.1 tells us that $y(t) - z(t) = O(\varepsilon)$ on the time scale $1/\varepsilon$. Because of the near-identity-transformation we also have $x(t)-\ y(t) = O(\varepsilon)$ on the time scale $1/\varepsilon$

Let us start by examining an example.

Example 3.3. *Consider the system*

$$\dot{x} = \tfrac{1}{2}\varepsilon - \varepsilon \cos\phi \quad , x \in \mathbf{R}$$
$$\dot{\phi} = x - 1 \qquad\qquad , \phi \in S^1$$

In this problem $\Omega(x) = x - 1$ *so the resonance manifold is given by* $x = 1$. *First we apply theorem* 3.4 *with the assumption that* $x(0) = x_0$ *is not near the resonance manifold. Averaging over* ϕ *produces*

$$\dot{y} = \frac{1}{2}\varepsilon, y(0) = x_0$$

or $y(t) = x_0 + \frac{1}{2}\varepsilon t; x(t) - y(t) = O(\varepsilon)$ *on the time scale* $1/\varepsilon$ *as long as* $y(t)$ *doesnot enter a neighbourhood of the resonance manifold* $(x = 1)$. *Note that if* $x_0 > 1$, *the solution will stay away from the resonance manifold, if* $x_0 < 1$, *for instance* $x_0 = \frac{1}{2}$, *the solution will enter the neighbourhood of the resonance manifold and at this stage the approximation is no longer valid.*

It is instructive to analyse what is going on near the resonance manifold. First we note that there are two critical points : $x = 1, \phi = \pi/3$ *and* $x = 1, \phi = 5\pi/3$; *the first one is a saddle, the second one a centre point. Putting the initial values in one of these critical points we have clearly* $x(t) - y(t) = \frac{1}{2}\varepsilon t$. *Differentiating the equation for* ϕ *we derive easily*

$$\ddot{\phi} + \varepsilon \cos\phi = \frac{1}{2}\varepsilon$$

This equation has no attractors, so either a solution starts in or near the resonance manifold and remains there or it starts outside the resonance manifold and passes through this domain or it stays outside the resonance manifold altogether.

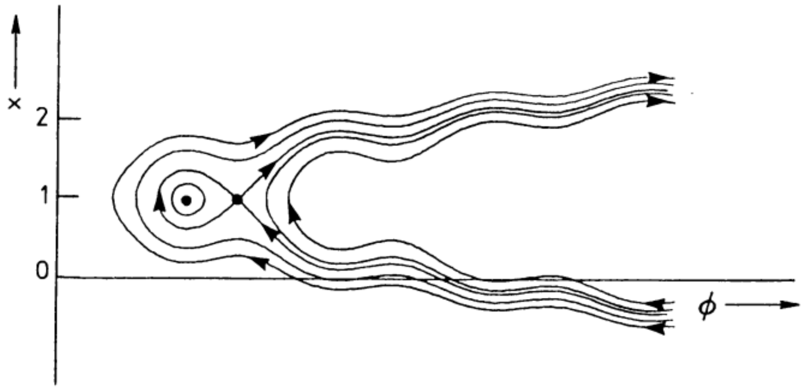

Figure 3.2: solution example 3.3

The x, ϕ -phase-plane with the resonance manifold at $x = 1, \varepsilon = .1$. Solutions stay near the resonance manifold or they are passing through it

In the second case there is the interesting phenomenon that solutions which start initially close can be dispersed when passing through resonance. In figure 3.2 the x, ϕ -phase-plane illustrates some of these phenomena. It is clear from the phaseplane that outside a neighbourhood of the resonance manifold the variable ϕ is time-like, while inside it is not.

To study the behaviour of the solutions in the vicinity of the resonance manifold we introduce local variables near the zeros of $\Omega(x)$. So consider again equation 3.22

$$\begin{aligned} \dot{x} &= \varepsilon X(\phi, x) + O\left(\varepsilon^2\right) \quad &, x \in D \subset \mathrm{R}^n \\ \dot{\phi} &= \Omega(x) + O(\varepsilon) \quad &, \phi \in S^1 \end{aligned}$$

and suppose that $\Omega(r) = 0$ with $r \in R^n$. We introduce

$$x = r + \delta(\varepsilon)\xi$$

with $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Such a scaling is natural in boundary layer theory; for an extensive discussion of boundary layer variables see Eckhaus (1979). The order

function $\delta(\varepsilon)$ will be determined by a certain balancing principle. We introduce the local variable ξ in the equations and expand

$$\delta(\varepsilon)\dot{\xi} = \varepsilon X(\phi, r + \delta(\varepsilon)\xi) + O\left(\varepsilon^2\right) = \varepsilon X(\phi, r) + O(\varepsilon\delta) + O\left(\varepsilon^2\right)$$
$$\dot{\phi} = \Omega(r + \delta(\varepsilon)\xi) + O(\varepsilon) = \delta(\varepsilon)\frac{\partial\Omega}{\partial x}(r)\xi + O\left(\delta^2\right) + O(\varepsilon)$$

The terms in the first and the second equations balance, i.e. they are of the same order if

$$\delta(\varepsilon) = \sqrt{\varepsilon} \text{ and } \frac{\partial\Omega}{\partial x}(r) \neq 0$$

This determines the size of the boundary layer, i.e. the order of magnitude of the neighbourhood of the resonance manifold. The system becomes with this choice of $\delta(\varepsilon)$

$$\dot{\xi} = \sqrt{\varepsilon}X(\phi, r) + O(\varepsilon), \xi \in \mathrm{R}^n \tag{3.23}$$

$$\dot{\phi} = \sqrt{\varepsilon}\frac{\partial\Omega}{\partial x}(r)\xi + O(\varepsilon), \phi \in S^1 \tag{3.24}$$

The solutions of system $3.23-24$ can be approximated by omitting the $O(\varepsilon)$ terms and solving

$$\dot{\phi}_a = \sqrt{\varepsilon}\frac{\partial\Omega}{\partial x}(r)\xi_a \tag{3.25}$$

$$\dot{\xi}_a = \sqrt{\varepsilon}X\left(\phi_a, r\right) \tag{3.26}$$

Theorem 3.5. *tells us that, with appropriate initial values, the solutions of system* $3.23-24$ *and system* $3.25-26$ *are* $\sqrt{\varepsilon}$ *-close on the time scale* $1/\sqrt{\varepsilon}$. *System* $3.25-26$ *is* $(n+1)$ *-dimensional, the corresponding phase-flow is volume-preserving. This is surprising as the original system* 3.22 *may contain dissipation and forcing terms and has a quite general form. The phase-flow of system* $3.25-26$ *can be characterised by a two-dimensional system; differentiate the second equation of* $3.25-26$ *and we find after eliminating* ξ_a

$$\ddot{\phi}_a - \varepsilon\frac{\partial\Omega}{\partial x}(r)X\left(\phi_a, r\right) = 0 \tag{3.27}$$

Equation 3.27 *is a second-order equation; it is easy to write down a first integral (as for the mathematical pendulum) and critical points of the equation can only be saddles and centres. This means of course that to describe correctly what is going on in the resonance manifold, we must compute a second-order approximation.*

3.2.2 Periodic solutions

We have seen that the Poincaré-Lindstedt method is not only a quantitative method but also leads, by the implicit function theorem, to the existence of periodic solutions. A similar result holds for the averaging method where again the implicit function theorem with an appropriate periodicity condition plays a part. Consider again the equation

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon) \tag{3.28}$$

with $x \in D \subset R^n, t \geq 0$. Moreover we assume that both $f(t,x)$ and $g(t,x,\varepsilon)$ are T -periodic in t. Separately we consider in D the averaged equation

$$\dot{y} = \varepsilon f^0(y) \tag{3.29}$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with T -periodic solutions of equation 3.28.

Theorem 3.6. *Consider equation* 3.28 *and suppose that:*

a. the vector functions $f, g, \partial f/dx, \partial^2 f/\partial x^2$ and $\partial g/\partial x$ are defined, continous and bounded by a constant M (independent of ε) in $[0,\infty)\times\ D, 0 \leq \varepsilon \leq \varepsilon_0$

b. f and g are T -periodic in $t(T$ independent of $\varepsilon)$; If p is a critical point of the averaged equation 3.29 *whereas*

$$\left|\partial f^0(y)/\partial y\right|_{y=p} \neq 0 \tag{3.30}$$

then there exists a T -periodic solution $\phi(t,\varepsilon)$ of equation 3.28 *which is close to p such that*

$$\lim_{\varepsilon\to 0} \phi(t,\varepsilon) = p$$

Proof

First we shall impose the periodicity condition after which we can apply the implicit function theorem.

$$x(t) = z(t) + \varepsilon u(t, z(t))$$

The equation for z becomes

$$\dot{z} = \varepsilon f^0(z) + \varepsilon^2 R(t,z,\varepsilon) \tag{3.31}$$

Because of the choice of $u(t, z(t))$, a T -periodic solution $z(t)$ produces a T periodic solution $x(t)$. For R we have the expression

$$R(t, z, \varepsilon) = \frac{\partial f}{\partial z}(t, z)u(t, z) - \frac{\partial u}{\partial z}(t, z)f^0(z) + g(t, z, 0) + O(\varepsilon)$$

this expression is T -periodic in t and continuously differentiable with respect to z, Equation 3.31 is aequivalent with the integral equation

$$z(t) = z(0) + \varepsilon \int_0^t f^0(z(s))ds + \varepsilon^2 \int_0^t R(s, z(s), \varepsilon)ds$$

The solution $z(t)$ is T -periodic if $z(t + T) = z(t)$ for all $t \geq 0$ which leads to the equation

$$h(z(0), \varepsilon) = \int_0^T f^0(z(s))ds + \varepsilon \int_0^T R(s, z(s), \varepsilon)ds = 0 \tag{3.32}$$

It is clear that $h(p, 0) = 0$. With ε in a neighbourhood of $\varepsilon = 0$, equation 3.32 has a unique solution $z(0)$ because of the assumption on the Jacobi determinant 3.30. If $\varepsilon \to 0$, then $z(0) \to p$.

If we have concluded with theorem 3.5 that a periodic solution of equation 3.28 exists in a neighbourhood of $x = p$, we can often establish its stability in a simple way:

Theorem 3.7. *Consider equation* 3.28 *and suppose that the conditions of theorem* 3.5 *have been satisfied. If the eigenvalues of the critical point* $y = p$ *of the averaged equation* 3.29 *all have negative real parts, the corresponding periodic solution* $\phi(t, \varepsilon)$ *of equation* 3.28 *is asymptotically stable for* ε *sufficiently small. If one of the eigenvalues has positive real part,* $\phi(t, \varepsilon)$ *is unstable.*

Proof

We shall linearise equation 3.28 in a neighbourhood of the periodic solution $\phi(t, \varepsilon)$. After translating $x = z + \phi(t, \varepsilon)$

expanding with respect to z, omitting the nonlinear terms and renaming the dependent variable again x, we find a linear equation with T -periodic coefficients:

$$\dot{x} = \varepsilon A(t, \varepsilon)x \tag{3.33}$$

with $A(t, \varepsilon) = \frac{\partial}{\partial x}[f(t, x) + \varepsilon g(t, x, \varepsilon)]_{x=\phi(t,e)}$. We introduce the T -periodic matrix

$$B(t) = \frac{\partial f}{\partial x}(t, p)$$

From theorem 3.5 we have $\lim_{\varepsilon \to 0} A(t, \varepsilon) = B(t)$. We shall also use the matrices

$$B^0 = \frac{1}{T} \int_0^T B(t) dt$$

and

$$C(t) = \int_0^t \left[B(s) - B^0 \right] ds$$

Note that B^0 is the matrix of the linearised averaged equation. The matrix $C(t)$ is T -periodic and it has average zero. The near-identity transformation $x \to y$ with

$$y = (I - \varepsilon C(t))x$$

yields

$$\begin{aligned}
\dot{y} &= -\varepsilon \dot{C}(t)x + (I - \varepsilon C(t))\dot{x} \\
&= -\varepsilon B(t)x + \varepsilon B^0 x + (I - \varepsilon C(t))\varepsilon A(t, \varepsilon)x \\
&= \left[\varepsilon B^0 + \varepsilon (A(t, \varepsilon) - B(t)) - \varepsilon^2 C(t) A(t, \varepsilon) \right] (I - \varepsilon C(t))^{-1} y \\
&= \varepsilon B^0 y + \varepsilon (A(t, \varepsilon) - B(t))y + \varepsilon^2 R(t, \varepsilon)y
\end{aligned} \tag{3.34}$$

$R(t, \varepsilon)$ is T -periodic and bounded; we note that $(A(t, \varepsilon) - B(t)) \to 0$ as $\varepsilon \to 0$ and also that the characteristic exponents of equation 3.34 depend continuously on the small parameter ε. It follows that, for ε sufficiently small, the sign of the real parts of the characteristic exponents is equal to the sign of the real parts of the eigenvalues of the matrix B^0. We now apply theorem to conclude stability in the case of negative real parts. If at least one real part is positive, Flocquet transformation leads to instability. 0

Example 3.4. ***(autonomous equations)***
For equations of type $\vec{x} + x = \varepsilon f(x, \dot{x})$, and for autonomous equations in general, we shall always find one characteristic exponent with zero real part when linearising near a periodic solution.

In the case of second-order equations, it is then convenient to introduce polar coordinates and to consider the angle as a time-like variable. For instance in the case of the Van Der Pol equation

$$\ddot{x} + x = \varepsilon \left(1 - x^2 \right) \dot{x}$$

we have in phase-amplitude coordinates r, ψ

$$\dot{r} = \varepsilon \sin(t+\psi)\left(1 - r^2\cos^2(t+\psi)\right) r\sin(t+\psi)$$
$$\dot{\psi} = \varepsilon \cos(t+\psi)\left(1 - r^2\cos^2(t+\psi)\right)\sin(t+\psi)$$

We put $t + \psi = \theta$ to obtain

$$\frac{dr}{d\theta} = \frac{\dot{r}}{1+\dot{\psi}} = \varepsilon \sin\theta\left(1 - r^2\cos^2\theta\right) r\sin\theta + \varepsilon^2 \ldots$$

and after averaging over θ

$$\frac{dr_a}{d\theta} = \frac{1}{2}\varepsilon r_a\left(1 - \frac{1}{4}r_a^2\right)$$

This equation has a critical point $r_a = 2$, the corresponding eigenvalue of B^0 in theorem 3.6 is -1. Application of theorem 3.5 yields the existence of a 2π -periodic solution in θ, according to theorem 3.6 this solution is asymptotically stable. we should check that averaging over t of the equations for r and ψ does not lead to application of theorems 3.5-6. In using the Poincaré-Lindstedt method, where one also expands the period in the case of autonomous equations, this difficulty does not arise.

Example 3.5. ***(forced Duffing equation)***
Consider again the Duffing equation with forcing and damping from section 2.3

$$\ddot{x} + \varepsilon\mu\dot{x} + x - \varepsilon x^3 = \varepsilon h\cos\omega t$$

with constants $\mu \geq 0, h > 0$; *we put* $\omega^{-2} = 1 - \varepsilon\beta$. *We are looking for solutions which are periodic with period* $2\pi/\omega$ *so it makes sense to transform the variable* t. *Putting* $\omega t = s$, *the equations becomes*

$$\begin{aligned}\frac{d^2x}{ds^2} + x = \varepsilon\beta x - \varepsilon(1-\varepsilon\beta)^{\frac{1}{2}}\mu\frac{dx}{ds} + \varepsilon(1-\varepsilon\beta)x^3 + \\ +\varepsilon(1-\varepsilon\beta)h\cos s\end{aligned} \tag{3.35}$$

Introduction of amplitude-phase variables (section 3.2*) produces*

$$\begin{aligned}\frac{dr}{ds} =& -\varepsilon\sin(s+\psi)[\beta r\cos(s+\psi) + \mu r\sin(s+\psi) + \\ &+ r^3\cos^3(s+\psi) + h\cos s] + O\left(\varepsilon^2\right)\\ \frac{d\psi}{ds} =& -\varepsilon\cos(s+\psi)[\beta\cos(s+\psi) + \mu\sin(s+\psi) + \\ &+ r^2\cos^3(s+\psi) + \frac{h}{r}\cos s\Big] + O\left(\varepsilon^2\right)\end{aligned}$$

The righthand sides are 2π -periodic in s and averaging yields

$$\begin{aligned}\frac{dr_a}{ds} &= -\frac{1}{2}\varepsilon\mu r_a - \frac{1}{2}\varepsilon h \sin\psi_a \\ \frac{d\psi_a}{ds} &= -\frac{1}{2}\varepsilon\beta - \frac{3}{8}\varepsilon r_a^2 - \frac{1}{2}\varepsilon\frac{h}{r_a}\cos\psi_a\end{aligned} \tag{3.36}$$

Critical points of the averaged equations 3.36 satisfy the transcendental equations

$$\mu r_a = -h\sin\psi_a, \beta + \frac{3}{4}r_a^2 = -\frac{h}{r_a}\cos\psi_a$$

These critical points correspond with periodic solutions of equation 3.35 if

$$\mu\sin\psi_a + \beta\cos\psi_a + \frac{9}{4}r_a^2\cos\psi_a \neq 0$$

Finally we use theorem 3.6 to study the stability of the periodic solutions. Linearisation of system 3.36 in a critical point P produces the matrix

$$\begin{pmatrix} -\frac{1}{2}\varepsilon\mu & -\frac{1}{2}h\varepsilon\cos\psi_a \\ -\frac{3}{4}\varepsilon r_a + \frac{1}{2}\frac{h}{r_a^2}\varepsilon\cos\psi_a & \frac{1}{2}\frac{h}{r_a}\varepsilon\sin\psi_a \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{1}{2}\varepsilon\mu & \frac{1}{2}\varepsilon\left(\beta r_a + \frac{3}{4}r_a^3\right) \\ -\frac{9}{8}\varepsilon r_a - \frac{\epsilon}{2r_a}\beta & -\frac{1}{2}\varepsilon\mu \end{pmatrix}_P$$

The eigenvalues are

$$\lambda_{1,2} = -\frac{1}{2}\varepsilon\mu \pm \frac{1}{2}\varepsilon\left[-\left(\beta + \frac{9}{4}r_a^2\right)\left(\beta + \frac{3}{4}r_a^2\right)\right]^{\frac{1}{2}}$$

If for instance $\mu > 0, \beta \geq 0$ we have asymptotic stability.

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Absract

This work is devoted to the study of analytical approximations of approximate solutions of systems of nonlinear differential equations of the second order. We first introduced all the basic properties and concepts used in our study, then conducted the Lindestd Poincaré method and the averaging method. Finally, we have studied the approximate solutions with time scales according to the two previous methods and we have also made several applications in which we have shown these concepts.

Keywords: Approximation, Perturbation theory, Lindestd-Poincaré method, Averaging method, Periodic solutions, Time scales

Résumé

Ce travail est consacré à l'étude des approximations analytiques de solutions approchées de systèmes d'équations différentielles non linéaires du second ordre. Nous avons d'abord introduit toutes les propriétés et concepts de base utilisés dans notre étude, puis avons conduit la méthode Lindestd Poincaré et la méthode de moyenne. Enfin, nous avons étudié les solutions approchées avec des échelles de temps selon les deux méthodes précédentes et nous avons également fait plusieurs applications dans lesquelles nous avons montré ces concepts.

Mots clés : Approximation, Théorie de perturbation, Méthode de Lindestd-Poincaré, Méthode de moyenne, Solutions périodiques, Echelles de temps.

الملخص

هذا العمل مكرس لدراسة التقريب التحليلي للحلول التقريبيه لأنظمة المعادلات التفاضلية غير الخطية من الدرجة الثانية. قدمنا أولاً جميع الخصائص والمفاهيم الأساسية المستخدمة في دراستنا ، ثم قدنا طريقة لانداست بوانكاربيه وطريقة المتوسط. أخيرًا، درسنا الحلول التقريبية بالمقاييس الزمنية وفقا للطريقيتين السابقتين وقمنا أيضا بتطبيقات عديدة بيّنا فيها هاته المفاهيم.

كلمات مفتاحيه: التقريب، نظرية الذبذبه ، طريقة لانداست بوانكاربيه، طريقة المتوسط، مقاييس الزمن.