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Thème

On The Semilinear Parabolic Equations

Présenté par: REDOUNI Farouk

Soutenu devant le jury composé de

Abdelfeteh FAREH	MCB	Rapporteur	Univ. d'El Oued
Mansour Abdelwaheb	MCB	Examinateur	Univ. d'El Oued
Nisse Khadidja	MCB	Examinatrice	Univ. d'El Oued

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Acknowledgement

I would like to dedicate this humble work to my Mother

.

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I am also very thankful to my family and friends for their endless understanding and favour.

Notations

$C_1, C_2 ...$		are arbitrary constants;
grad a		gradient of a scalar a, also denoted by ∇a;
t		time $(t \geq 0)$;
x, y, z		space (Cartesian coordinate);
Δ		Laplace operator ;
Δ	$=$	$\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$, in n-dimensional case ;
w_x	$=$	$\frac{\partial w}{\partial x}$;
w_{xx}	$=$	$\frac{\partial^2 w}{\partial x^2}$;
∂_x	$=$	$\frac{\partial}{\partial x}$;
H^1, H^2, H_0^1		Sobolev spaces;
$C(\Omega)$		continuous function space;

-The ODE and PDE are conventional abbreviations for ordinary differential equation and partial differential equation, respectively.

Table des matières

Introduction

Semilinear parabolic partial differential equations are encountered in various fields on mathematics, physics and numerous applications in many fields.

Exact solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and process in various areas of natural science.

This work has three major parts : at first we introduced the Burgers equation, then we give some other models and at last we discuss the semigroup method.

In chapter 1 we discuss the burgers equation problem and the problem of the appearance of shock waves. We give the traffic flows as an example of this problem and finally we discuss the mathematical analysis of this equation.

In the second chapter, we treat some interesting models of semilinear equations in one dimensional space variable such as the unnormalized Burgers equation and the the Burgers-Huxley equation. We detail next the two dimensional space variables as the Boussinesq equation and the Kolmogorov type equation.

Chapter 3 is concerned with the semigroup method. This method was developed from the end of the 1940s to the 1960s, and it is still useful nowadays. We briefly give some definitions and properties and we gradually switch to the abstract case. We describe some nice theorems with Lipschitz conditions.

Based on these theorems, we state and prove existence and uniqueness theorems for a quite large class of semilinear parabolic partial differential equations.

We sincerely hope that this work will be helpful for the readers engaged in the fields of mathematics and other sciences.

Chapitre 1

BURGERS EQUATION

Résumé

In this chapter we present the Burgers equation in its viscous and non-viscous version. After submitting as a motivation, some applications of this equations, we continue with the mathematical analysis of them. this will lead us to confront one of the main problems linked to nonlinear partial differential equations : the appearance of shock waves.

Definition 1.1 *A semilinear partial differential equation with two independent variables has the form*

$$a(x,y)\frac{\partial^2 w}{\partial x^2} + b(x,y)\frac{\partial^2 w}{\partial x \partial y} + c(x,y)\frac{\partial^2 w}{\partial y^2} = F(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}), \tag{1.1}$$

where $w(x,y)$ is the unknown function and F is a given function. The functions $a(x,y)$, $b(x,y)$ and $c(x,y)$ are called the coefficients of Equation 1.1.

This Definition can be extended to more than two variables.

Definition 1.2 *One calls exact solution of a semilinear PDE any solution defined in the whole domain of definition of the PDE and which is given in closed form, i.e.as a finite expression*

The opposite of an exact solution is of course not a wrong solution, but what **Painlevé** calls "une solution illusoire ". For instance, the Taylor series and Laurent series outside their disk of definition.

Note that a multivalued expression is not excluded. From this definition, an exact solution is global, as opposed to local. This generally excludes series or infinite product, unless their domain of validity can be made the full domain of definition, like for linear ODEs.

1.1 Introduction

In this chapter we will consider the viscid Burgers equation.That is the semilinear parabolic partial differential equation

$$u_t + uu_x = \varepsilon u_{xx}, \tag{1.2}$$

where $\varepsilon > 0$ is the constant of viscosity. This is the simplest partial differential equation including both nonlinear propagation's effects and diffusive effects. When the right-hand side term is removed from (1.2), we then obtain the hyperbolic partial differential equation

$$u_t + uu_x = 0. \tag{1.3}$$

We will refer to (1.3) as the inviscid Burgers equation.

There is a crucial connection between the above equations : Equation 1.3 is the limit as $\varepsilon \to 0$ of 1.2.

Motivation

We have probably heard of some planes breaking the sound barrier. What does it mean ? This means that a shock wave in the atmosphere is created as in Figure 1.1

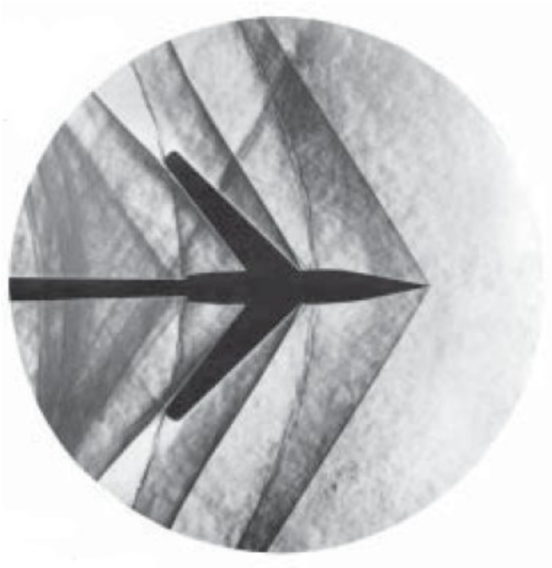

FIGURE 1.1 –

But what is a shock wave ? Imagine heavy traffic on highway as a wave. A shock wave corresponds to collisions. To explain this, we develop our intuition with one dimensional model, a traffic on one line road at different speeds. We know that collisions could occur if drivers do not adjust their speed. The atmosphere is a fluid, and traffic is a rough model of 1D fluid which is convenient to develop our intuition. Under which conditions do shock waves or other singularities occur in fluid ? This is what we are going to explain you.

1.2 Burgers Model

Burgers equations appear often as a simplification of a more complex and sophisticated model. Hence it is usually thought as a *toy model*, namely, a tool that is used to understand some of the inside behaviour of the general problem. Here we will present some examples.

1.2.1 The One Dimensional Model (Traffic Flow)

Let us construct a simple traffic model. We have cars at each position x along a line representing a road. At time $t = 0$, the car at position x_0 has a speed $V_0(x_0)$. The car starting at x_0 will be at position $x(t, x_0)$ at time t. Let $V(x(t, x_0), t)$ be the speed of the car at position $x(t, x_0)$ and time t. The hypothesis is that cars move at constant speed along a straight road. Hence, we have $V(x(t, x_0), t) = V_0(x_0)$, which means that this function is constant in t. Hence, its derivative should be identically zero ! By the chain rule for functions of several variables we have

$$\begin{aligned}\frac{d}{dt}V(x(t,x_0),t) &= \frac{\partial V}{\partial x}(x(t,x_0),t)\frac{\partial x}{\partial t}(t,x_0) + \frac{\partial V}{\partial t}(x(t,x_0),t)\\ &= \frac{\partial V}{\partial x}(x(t,x_0),t)V(x_0) + \frac{\partial V}{\partial t}(x(t,x_0),t)\end{aligned}$$

Simply writing that the position on the road is given by x, and using that $V(x(t, x_0), t) = V_0(x_0)$, the traffic equation becomes what is called *Burgers equation* as in (1.3)

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}.V = 0$$

This equation says that car maintains a constant speed as it moves long the road.

Another way of saying this is that if a car at position x_0 starts off with speed $V_0 = V(x_0, 0)$, then at time t it will be at position $x_0 + tV_0$ still with the same speed : $V(x_0 + tV_0, t) = V_0$ for all t.

However, collisions will occur. Figure 1.2 shows one such resulting chaos.

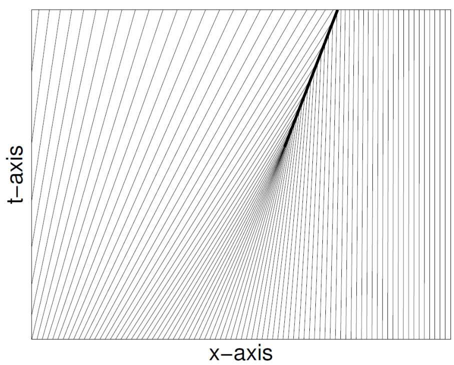

FIGURE 1.2 – A shock wave in Burgers equation for the initial speed given by V(x)=exp(-x2). The heavy black line is where the collisions happens.

Describing the Shocks

What happens when we approach the shock ? Suppose that at the beginning ($t = 0$) two points $x_0 < y_0$ had velocities $V(x_0, 0) > V(y_0, 0)$. Let us denote the position of x_0 at time t by x_t. Similarly, y_t denotes the position of y_0 at time t. Then x_t gets closer and closer to y_t and just before the shock x_t is very close to y_t. Yet, the difference of their velocities $V(x_t, t) - v(y_t, t) = V(x_0, 0) - V(y_0, 0)$ has remained the same. This means that the mean slope of the function $V(x, t)$ as a function of x for fixed t is becoming very large (see Figure 1.3). Hence we see that $\frac{\partial V}{\partial x}$ is becoming infinite when we approach the shock. This means that our differential equation 1.3 has a *singularity* at the shock.

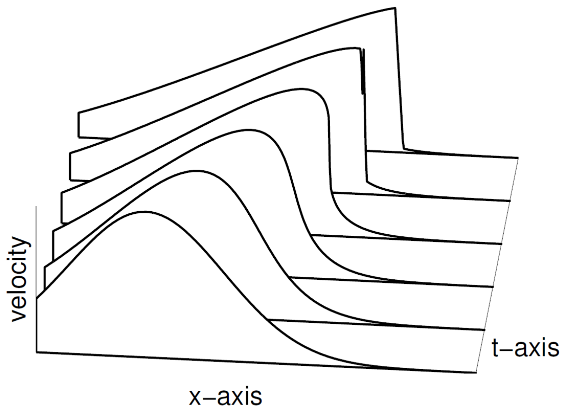

FIGURE 1.3 – The graph of V as a function of x for several values of t

1.2.2 Avoiding the Shock

We have modelled a situation in which all cars travelling along the road never change their speed. If the function $V(x)$ is increasing with x, then the cars in the front are faster that the ones behind, and there is no problem. But if some slower cars are ahead of the faster ones, there will be a crash if no one modifies their speed. There are two ways to develop the model further. Coming back to our initial motivation, the supersonic shock wave, we can refine our equation to include collisions, a one dimensional version of shocks. The underlying cause is the same for supersonic planes, namely the air in front of the planes cannot travel fast enough to escape the plane coming up from behind! Or we can refine our equation to allow for more intelligent driver behaviour, having the faster car slow down and slower car speed up.

1.2.3 Behaviour of Smart Drivers

When a collision is imminent, the conscious driver would surely react responsively, tending to compromise the speed with that of the other drivers. Logically, thinking of a *viscous fluid.*

If you have noticed honey flowing, you may have remarked that it behaves as a whole, as if all particles were tied together, no drops are splashing around, they remain linked to the main stream. Water has a viscosity about $10,000$ times less and air 50 less than that (but not zero). So one possible adjustment to Burgers equation in order to avoid collisions will involve adding *viscosity term.* Such a term is just multiple of the second derivative $\frac{\partial^2 V}{\partial x^2}$. Hence, a modified equation takes the form which is known as the *viscous Burgers equation* :

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}.V = \nu \frac{\partial^2 V}{\partial x^2} \tag{1.4}$$

with $\nu > 0$ being called the viscosity. The modified world lines can be seen in figure 1.4 for the same function $V(x)$ describing the initial speed as in figure 1.2.

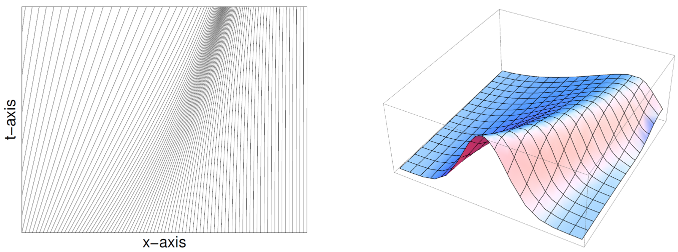

FIGURE 1.4 – In the same situation as in figure 1.2, we add some viscosity. Note how the cars bunch up but do not collide any more. On the right the graph of V(x,t)

But why adding a second derivative ?
We know that the second derivative is the limit of

$$\frac{V(x+\delta x,t) - 2V(x,t) + V(x-\delta x,t)}{(\delta x)^2}$$

as $\delta x \longrightarrow 0$.

It will be close to the first derivative of the first derivative as

$$\frac{V'(x+\delta x/2,t) - V'(x-\delta x/2,t)}{\delta x} \approx \frac{1}{\delta x}\left(\frac{V(x+\delta x,t) - V(x,t)}{\delta x} - \frac{V(x+\delta x,t) - V(x,t)}{\delta x}\right).$$

So, we can write this expression as

$$\frac{2}{\delta x^2}\frac{(V(x+\delta x,t) - V(x,t)) + (V(x-\delta x,t) - V(x,t))}{2}. \tag{1.5}$$

What does mean the second factor ? It had two meaning terms, the first is the difference of speed in front and the middle and the second is the difference of speed in the back and in the middle. The risk of a collision is when $V(x-\delta x,t) > V(x+\delta x,t)$. In this case, the first term is negative and the second is positive. If the first term wins i.e. if $|V(x - - + \delta x, t) - V(x,t)| > |V(x - \delta x, t) - V(x,t)|$, then expression 1.5 is negative. Thus, it tells you that you should break. On the other hand, If the second term wins, expression 1.5 is positive and you should accelerate.

Note that for x_0 fixed, the left-hand side of 1.2 represents the acceleration because it is the derivative of the speed function $V(x(t,x_0),t)$ with respect to t.

1.3 Mathematical Analysis

From the mathematical point of view Burgers equations are a very interesting and suggestive topic : It turns out that a study of them leads to most of the ideas that arise in the field of semilinear parabolic equations.

1.3.1 Viscid Case

We will focus first on equation (1.2). Specifically we will deal with the initial value problem

$$\begin{cases} u_t + uu_x = \epsilon u_{xx}, & x \in \mathbb{R}, \quad t > 0, \quad \epsilon > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.6}$$

The Cole-Hopf Transformation.

Hopf (1950) and Cole (1951) independently discovered a transformation that reduces the Burgers equation to a linear heat equation.

The Cole-Holpf transformation is defined by

$$u = -2\epsilon \frac{\varphi_x}{\varphi} \tag{1.7}$$

Operating in 1.7 we find that

$$\begin{aligned} u_t &= \frac{2\epsilon(\varphi_t\varphi_x - \varphi\varphi_{xt})}{\varphi^2}, \\ uu_x &= \frac{4\epsilon^2\varphi_x(\varphi\varphi_{xx} - \varphi_x^2)}{\varphi^3}, \\ \epsilon u_{xx} &= -\frac{2\epsilon^2(2\varphi_x^3 - 3\varphi\varphi_{xx}\varphi_x + \varphi^2\varphi_{xxx})}{\varphi^3}. \end{aligned}$$

Substituting this expression into 1.2,

$$\frac{2\epsilon(-\varphi\varphi_{xt} + \varphi_x(\varphi_t - \epsilon\varphi_{xx}) + \epsilon\varphi\varphi_{xxx})}{\varphi^2} = 0$$

$$\iff -\varphi\varphi_{xt} + \varphi_x(\varphi_t - \epsilon\varphi_{xx}) + \epsilon\varphi\varphi_{xxx} = 0$$

$$\iff \varphi_x(\varphi_t - \epsilon\varphi_{xx}) = \varphi(\varphi_{xt} - \epsilon\varphi_{xxx}) = \varphi(\varphi_t - \epsilon\varphi_{xx})_x.$$

Therefore, if φ solves the heat equation $\varphi_t - \epsilon\varphi_{xx} = 0 \quad x \in \mathbb{R}$, then $u(x,t)$ given by transformation (1.7) solves the viscid Burgers equation (1.2).

To completely transform the problem (1.6) we still have to work with the initial condition function. To do this, note that (1.7) can be written as

$$u = -2\epsilon(\log\varphi)_x.$$

Hence

$$\varphi(x,t) = e^{-\int \frac{u(x,t)}{2\epsilon}dx}.$$

It is clear from (1.7) that multiplying φ by a constant does not affect u, so we can write the last equation as

$$\varphi(x,t) = e^{-\int_0^x \frac{u(y,t)}{2\epsilon}dy}. \tag{1.8}$$

The initial condition on (1.6) must therefore be transformed by using (1.8) into

$$\varphi(x,0) = \varphi_0(x) = e^{-\int_0^x \frac{u_0(y)}{2\epsilon}dy}.$$

Summing up, we have reduced the problem (1.6) to the heat equation problem

$$\begin{cases} \varphi_t - \epsilon\varphi_{xx} = 0 & x\in\mathbb{R} \quad t>0 \quad \epsilon>0, \\ \varphi(x,0) = \varphi_0(x) = e^{-\int_0^x \frac{u_0(y)}{2\epsilon}dy}, & x\in\mathbb{R}. \end{cases} \tag{1.9}$$

Heat Equation.

The general solution of the initial value problem for the heat equation is well known and can be handled by a variety of methods. Taking the Fourier transform with respect to x for both heat equation and the initial condition $\varphi_0(x)$ in problem (1.9) we obtain the first order ordinary differential equations

$$\begin{cases} \widehat{\varphi}_t = \xi^2\epsilon\widehat{\varphi}, & \xi\in\mathbb{R} \quad t>0 \quad \epsilon>0 \\ \widehat{\varphi}(\xi,0) = \widehat{\varphi}_0(\xi) & \xi\in\mathbb{R}, \end{cases} \tag{1.10}$$

where $\widehat{\varphi}(\xi,t) = \int_{-\infty}^{+\infty}\varphi(x,t)e^{i\xi x}dx$. The solution for this problem is

$$\widehat{\varphi}(\xi,t) = \widehat{\varphi}_0(\xi)e^{\xi^2\epsilon t}.$$

To recover $\varphi(x,t)$ we have to use the inverse Fourier transform F^{-1}, namely,

$$\varphi(x,t) = F^{-1}(\widehat{\varphi}(\xi,t)) = F^{-1}(\widehat{\varphi}_0 e^{\xi^2\epsilon t}) = \varphi_0(x) * F^{-1}(e^{\xi^2\epsilon t}),$$

where $*$ denote the convolution product.
On the other hand

$$F^{-1}(e^{\xi^2\epsilon t}) = \frac{1}{2\sqrt{\pi\epsilon t}}e^{-\frac{x^2}{4\epsilon t}},$$

so the initial value problem (1.9) has the analytic solution

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi\epsilon t}} \int_{-\infty}^{+\infty} \varphi_0(\xi) e^{-\frac{(x-\xi)^2}{4\epsilon t}} d\xi.$$

Finally, from (1.7), we obtain the analytic solution for the problem (1.6)

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-\xi}{t} \varphi_0(\xi) e^{-\frac{(x-\xi)^2}{4\epsilon t}} d\xi}{\int_{-\infty}^{+\infty} \varphi_0(\xi) e^{-\frac{(x-\xi)^2}{4\epsilon t}} d\xi}. \tag{1.11}$$

Remark 1 *The heat equation*

$$\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = 0$$

has infinitely particular solutions. In particular, it admits solutions

$$\begin{aligned}
w(x,t) &= A(x^2 + 2t) + B, \\
w(x,t) &= Ae^{(\mu^2 t \pm \mu x)} + B, \\
w(x,t) &= A\frac{1}{\sqrt{t}} e^{-\dfrac{x^2}{4t}} + B, \\
w(x,t) &= Ae^{-\mu^2 t} \cos(\mu x + B) + C, \\
w(x,t) &= Ae^{-\mu x} \cos(\mu x - 2\mu^2 t + B) + C,
\end{aligned}$$

where A, B, C, and μ are arbitrary constants.

Note 1 *1-Burgers equation 1.2 admits many particular solutions in which we state :*

$$\begin{aligned}
w(x,t) &= \frac{A-x}{B-t}, \\
w(x,t) &= \lambda + \frac{2}{x + \lambda t + A}, \\
w(x,t) &= \frac{4x + 2A}{x^2 + Ax + 2t + B}, \\
w(x,t) &= \frac{2\lambda}{1 + Ae^{-\lambda^2 t - \lambda x}}, \\
w(x,t) &= -\lambda + A\frac{e^{A(x-\lambda t)} - B}{e^{A(x-\lambda t)} + B},
\end{aligned}$$

where A,B,C and λ are arbitrary constants

2- If $w(x,t)$ is a solution of Burgers equation. Then the function

$$u(x,t) = C_1 w(C_1 x + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where C_1, C_2, C_3 and C_4 are arbitrary constants, is also a solution of the equation.

Chapitre 2

Resolution of Some Interesting Models

Since we get a solution for Burgers equation , we have achieved a quite large class solutions of many semilinear equation related with it directly and indirectly way. We will provide some details on some different semilinear equations arose in many problems.

2.1 Exact Solutions of equations with two independent variables

2.1.1 Unnormalized Burgers Equation

The form of this equation is

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\frac{\partial w}{\partial x}. \tag{2.1}$$

By the change of variables

$$x = \frac{a}{b}z, \quad t = \frac{a}{b^2}\tau$$

we get

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial x}.$$

Since $\dfrac{\partial \tau}{\partial x} = 0$, then

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{b^2}{a}\frac{\partial w}{\partial \tau}, \tag{2.2}$$

also

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}. \tag{2.3}$$

As $\dfrac{\partial z}{\partial t} = 0$ we obtain

$$\frac{\partial w}{\partial x} = \frac{b}{a}\frac{\partial w}{\partial z}.$$

The second derivative with respect to x is

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{b}{a}\frac{\partial w}{\partial z}(z,\tau)\right) = \frac{b}{a}\frac{\partial^2 w}{\partial z^2}\frac{\partial z}{\partial x} = \frac{b^2}{a^2}\frac{\partial^2 w}{\partial z^2}. \tag{2.4}$$

Coming back to the original equation, it results

$$\frac{b^2}{a}\frac{\partial w}{\partial \tau} = a\left(\frac{b^2}{a^2}\frac{\partial^2 w}{\partial z^2}\right) + bw\left(\frac{b}{a}\frac{\partial w}{\partial z}\right).$$

Dividing all members by $\frac{b^2}{a}$ we get

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2} + w\frac{\partial w}{\partial z},$$

which is the Burgers equation.

We turn now to more complicated equations. As we will see next, we will add a continuous function depending on the second variable t. Then, the previous equation (unnormalized Burgers equation) becomes

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\frac{\partial w}{\partial x} + f(t). \tag{2.5}$$

using the transformation

$$w = u(z,t) + \int_0^t f(\tau)d\tau$$

where

$$z = x + b\int_0^t (t-\tau)f(\tau)d\tau$$

we obtain $dz = dx$ and

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial u}{\partial t} + f(t). \\
\frac{\partial w}{\partial x} &= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}. \\
&= \frac{\partial u}{\partial z} \\
\frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) \\
&= \frac{\partial^2 u}{\partial z^2}.
\end{aligned}$$

Substituting in 2.5 we obtain

$$\frac{\partial u}{\partial t} + f(t) = a\frac{\partial u}{\partial x^2} + bu\frac{\partial u}{\partial x} + f(t).$$

So

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x^2} + bu\frac{\partial u}{\partial x}$$

which is the unnormalized Burgers equation solved above.

2.1.2 The Burgers-Huxley equation

Let us consider the equation

$$\frac{\partial w}{\partial t} + \sigma w\frac{\partial w}{\partial x} = a\frac{\partial^2 w}{\partial x^2} + \mu + bw + cw^2 + dw^3. \tag{2.6}$$

For $\mu = 0$, it is called the Burgers-Huxley equation.
There is an exact solution of the equation 2.6 given by

$$w(x,t) = \frac{\beta}{z}\frac{\partial z}{\partial x} + \lambda. \tag{2.7}$$

Here, β is any of the roots of the quadratic equation

$$d\beta^2 + \sigma\beta + 2a = 0, \tag{2.8}$$

λ is any of the roots of the cubic equation

$$d\lambda^3 + c\lambda^2 + b\lambda + \mu = 0 \tag{2.9}$$

and the specific form of $z = z(x,t)$ depends on the equation coefficients.

In the following we will give an example of Burgers-Huxley equation.

Example 2.1 *Consider the Burgers-Huxley equation*

$$\frac{\partial w}{\partial t} + 4w\frac{\partial w}{\partial x} = 2\frac{\partial^2 w}{\partial x^2} + w + w^2 + w^3. \tag{2.10}$$

By the transformation

$$w(x,t) = \frac{\beta}{z}\frac{\partial z}{\partial x} + \lambda$$

where β is the solution of the equation

$$\beta^2 + 4\beta + 4 = 0$$

That is

$$\beta = -2$$

Moreover, the cubic equation is

$$\lambda^3 + \lambda^2 + \lambda = 0.$$

Clearly $\lambda = 0$ *is a solution for the last equation. Then, we take*

$$w(x,t) = \frac{-2}{z}\frac{\partial z}{\partial x}$$

For more similar equation's forms, see Polyanin and Zaitsev [2].

2.2 Second order Parabolic equations with two space variables

2.2.1 Boussinesq Equation

Let us consider Boussinesq equation with two space variables of the form

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(w\frac{\partial w}{\partial y}\right). \tag{2.11}$$

It arises in nonlinear heat conduction theory and the theory of unsteady flows through porous media with a free surface (see Polubarinova-Kochina, 1962)[3].

We suppose that the solution has a first order polynomial form of three variables t, x, y like :

$$w(x, y, t) = Ax + By + Ct + D,$$

where A, B, C, D are constant .
We will investigate under which conditions between the coefficients of w made it a solution. In other word, we will try to indicate the relation between its coefficients. We have

$$\frac{\partial w}{\partial t} = C,$$

$$\frac{\partial w}{\partial x} = A,$$

and

$$\frac{\partial w}{\partial y} = B.$$

Substituting in 2.11 we found

$$C = A^2 + B^2.$$

Finally, under this condition we get a linear solution in all its independent variables

$$w(x, y, t) = Ax + By + (A^2 + B^2)t + C$$

where A, B, and C are arbitrary constants.

2.2.2 Kolmogorov Type Equation

This equation has the semilinear form

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\frac{\partial w}{\partial y}. \tag{2.12}$$

It arises in mathematical finance when studying agents decisions under risk (see Citti, Pascucci and Polidoro (2001)[4]).

There are many ideas for solving a such problem in which we state :

1- Two Dimensional Solution

One could think the solution of 2.12 has the form

$$w(x,y,t) = Au(z,t) + B,$$

where

$$z = x + \lambda y + bB\lambda t.$$

A, B, and λ are arbitrary constants and the function $u = u(z,t)$ is described by the unnormalized Burgers equation of the form 2.1

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial z^2} + Ab\lambda u\frac{\partial u}{\partial z}.$$

Indeed, we put

$$z = x + \lambda y + bB\lambda t.$$

This implies that

$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = \lambda \quad \text{and } \frac{\partial z}{\partial t} = bB\lambda.$$

As we said before $w(x,y,t) = Au(z,t) + B$, then

$$\begin{aligned}
\frac{\partial w}{\partial x} &= A\frac{\partial\left(u(z,t)\right)}{\partial x} = A\frac{\partial u}{\partial z}(z,t)\frac{\partial z}{\partial x} = A\frac{\partial u}{\partial z}.\\
\frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial z}(z,t)\right) = A\frac{\partial^2 u}{\partial z^2}\frac{\partial z^2}{\partial x^2} = A\frac{\partial^2 u}{\partial z^2}.\\
\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y}\left(Au(z,t)\right) = A\frac{\partial u}{\partial z}(z,t)\frac{\partial z}{\partial y} = A\lambda\frac{\partial u}{\partial z}(z,t).\\
\frac{\partial w}{\partial t} &= \frac{\partial}{\partial t}\left(Au(z,t)\right) = A\left[\frac{\partial u}{\partial z}(z,t)\frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}(z,t)\right]\\
&= A\left[bB\lambda\frac{\partial u}{\partial z}(z,t) + \frac{\partial u}{\partial t}(z,t)\right].
\end{aligned}$$

Substituting in 2.12 we obtain

$$AbB\lambda\frac{\partial u}{\partial z}+A\frac{\partial u}{\partial t}=a\left(A\frac{\partial^2 u}{\partial z^2}\right)+b\left(Au(z,t)+B\right)A\lambda\frac{\partial u}{\partial z}.$$

After disposing of $AbB\lambda\frac{\partial u}{\partial z}$ term from both sides of the equation and dividing all members by A, finally it results

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial z^2}+bA\lambda u\frac{\partial u}{\partial z},$$

which is the unnormalized Burgers equation stated before.

We will give next another solution formula.

2- Generalized Separable Solution

By taking the transformation

$$w=\frac{u(x,t)-y}{b(t+C)},$$

where C is any constant. The function $u=u(x,t)$ is given by the linear heat equation

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}.$$

Actually, since we had

$$w=\frac{u(x,t)-y}{b(t+C)},$$

then

$$\begin{aligned}\frac{\partial w}{\partial x}&=\frac{\partial u}{\partial x}(x,t)\frac{1}{b(t+C)}.\\ \frac{\partial^2 w}{\partial x^2}&=\frac{1}{b(t+C)}\frac{\partial^2 u}{\partial x^2}(x,t),\\ \frac{\partial w}{\partial y}&=\frac{-1}{b(t+C)},\end{aligned}$$

and

$$\frac{\partial w}{\partial t}=\frac{b(t+C)\frac{\partial u}{\partial t}-b(u(x,t)-y)}{b^2(t+C)^2}.$$

Thus, we get

$$\frac{1}{b(t+C)}\frac{\partial u}{\partial t} - \frac{u(x,t)-y}{b(t+c)^2} = a\frac{1}{b(t+C)}\frac{\partial^2 u}{\partial x^2} + b\frac{u(x,t)-y}{b(t+C)}\left(\frac{-1}{b(t+C)}\right)$$

which is equivalent to

$$\frac{\partial u}{\partial t} - \frac{u(x,t)-y}{b(t+C)} = a\frac{\partial^2 u}{\partial x^2} - \frac{u(x,t)-y}{b(t+C)} \qquad .$$

So

$$\frac{\partial u}{\partial t} = a\frac{\partial^2}{\partial x^2}.$$

Chapitre 3

Semigroup Method

The semigroup method is a powerful tool for solving semilinear parabolic equation. It can be used to deal with many initial value problems or initial boundary value problems for both linear and nonlinear equations this is why we will present it in this chapter

3.1 Semigroups of Linear Operators

3.1.1 Semigroups

To introduce the concept of semigroups of linear operators, we will firstly look at a simple example.
Consider the following initial boundary value problem for the heat equation

$$\begin{cases} u_t - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \\ u(0,x) = u_0(x) \in L^2(0,\pi). \end{cases} \tag{3.1}$$

Applying the method of separate variables, it is easy to see that the solution of the problem can be expressed as a Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k e^{-k^2 t} \sin(kt)$$

where α_k is the Fourier coefficient

$$\alpha_k = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin(kx) dx.$$

Since it is assumed that $u_0 \in L^2$ using the Parseval equality (see Songmu Zheng[9])

$$\sum_{k=1}^{\infty} \alpha_k^2 = \frac{2}{\pi} \int_0^{\pi} u_0^2(x) dx.$$

The series $(\alpha_k e^{-k^2t}\sin(kx))$ converges rapidly because it had a fast decay factor e^{-k^2t} for $t > 0$. Then, we can easily prove that $u(x,t)$ given by this series is infinitely time differentiable with respect to x and t for $t > 0$. On the other hand, it satisfies the equation and the boundary conditions of (3.1). Then we have

$$\begin{aligned}\int_0^{\pi} [u(x,t) - u_0(x)]^2\,dx &= \int_0^{\pi} \left|\sum_{k=1}^{\infty} \alpha_k (e^{-k^2t} - 1)\sin(kx)\right|^2 dx \\ &\leq \frac{\pi}{2}\sum_{K=1}^{\infty} \alpha_k^2 (e^{-k^2t} - 1)^2 \longrightarrow 0 \text{ as } t \longrightarrow 0\end{aligned}$$

In other word, as $t \longrightarrow 0$, $u(x,t)$ converges to $u_0(x)$ in $L^2(0,\pi)$ sense. It can be easily seen that the solution $u(x,t)$ is uniquely determined by u_0. Thus we can view $u(x,t)$ as the image of $u_0(x)$ by an abstract mapping $T(t)$, i.e

$$u(x,t) = T(t)u_0(x) = \sum_{k=1}^{\infty} \alpha_k e^{-k^2t}\sin(kx). \tag{3.2}$$

We deduce from this definition that for any $t \in [0,\infty[$, $T(t)$ is a linear operator from $L^2(0,\pi)$ to $L^2(0,\pi)$. Now, we investigate the properties of $T(t)$. By definition 3.2 we have

i) $T(0) = I$

Moreover, for any $t, s \geq 0$

$$T(t).T(s)u_0(x) = \sum_{k=1}^{\infty} \left(a_k e^{-k^2t}\right) e^{-k^2s}\sin(kx) \tag{3.3}$$

$$= \sum_{k=1}^{\infty} a_k e^{-k^2(t+s)}\sin(kx) = T(t+s)u_0(x). \tag{3.4}$$

Since u_0 is any function in $L^2(0,\pi)$, we have

ii)
$$T(t+s) = T(t).T(s) = T(s).T(t).$$

As in (3.3) we can show that

iii) For any $u_0 \in L^2(0,\pi)$, $T(t)u_0 \in C\left([0,\infty[, L^2(0,\pi)\right)$.

In other word, for $t \in [0,\infty[$, $u(x,t)$ is an abstract continuous function valued in $L^2(0,\pi)$

We usually call one parameter operator $\{T(t); t \geq 0\}$ satisfying the three properties in the above as a one parameter, strongly continuous semigroup or simply a C_0-semigroup in a Banach space. It follows from the previous argument that solving problem (3.1) alternatively is to find a linear semigroup of operators $T(t)$. In general, in the definition of a semigroup of operators $T(t)$, L^2 should be replaced by a general Banach space B. Now, we turn to the important question on how to find $T(t)$ for a given problem. In the next section we will see that actually $T(t)$ can be generated from an infinitesimal generator A which can be easily identified for a given problem.

3.2 The infinitesimal generator

Let $\{T(t); t \geq 0\}$ be a C_0-semigroup defined on a Banach space B. We denote by $D(A)$ a subset of B such that for $x \in D(A)$, $T(t)$ is differentiable at $t = 0$ from the right, i.e

$$D(A) = \left\{ x \in B, \lim_{h \to 0^+} \frac{T(h)x - x}{h} \quad \text{exists} \right\}. \tag{3.5}$$

For $x \in D(A)$ we denote

$$-Ax = \lim_{h \to 0^+} \frac{T(h)x - x}{h}. \tag{3.6}$$

It is clear that A is a linear operator defined on $D(A)$. ($A = DT_0(x)$)
The operator A is usually called the infinitesimal generator of $T(t)$.

We will admit that once $T(t)$ is known, then A is derived from (3.6). On the other hand, once A is given, then $T(t)$ can be generated by A. For more information see Engel and Naguel [7].

Remark 2 *In some references in the literature, instead of A, the operator $-A$ is called the infinitesimal generator of $T(t)$*

First, we will prove some important lemmas.

Lemma 3.1 *For any $x \in D(A)$, $x \longrightarrow T(t)x \in C^1([0, \infty[, B)$. Moreover for $t \geq 0$,*

$$x - T(t)x = \int_0^t AT(s)x ds = \int_0^t T(s)Ax ds. \tag{3.7}$$

and

$$\frac{d(T(t)x)}{dt} + A(T(t)x) = 0 \tag{3.8}$$

Proof. For $x \in D(A)$, and $h > 0$ we have

$$\frac{T(t+h)x - T(t)x}{h} = \frac{(T(h) - I)T(t)x}{h} \tag{3.9}$$

$$= T(t)\frac{T(h) - I}{h}x. \tag{3.10}$$

as $h \longrightarrow 0^+$, the right hand side in the above equalities exists (see relation (3.6)). It turns out that the right-hand side of 3.9 in the above has a limit. So, it implies $T(t)x \in D(A)$ and

$$-AT(t)x = -T(t)Ax.$$

On the other hand, when $t > 0$ and $h \longrightarrow 0$, from

$$\frac{T(t)x - T(t-h)x}{h} = T(t-h)\frac{T(h) - I}{h}x$$

and the fact that $T(t)x \in C\left([0, \infty[, B\right)$, we can deduce that as $h \longrightarrow 0$, equation (3.3) has a limit, which should be $T(t)Ax$.

The previous discussion yields that $x \longrightarrow T(t)x \in C^1\left([0, \infty[, B\right)$ and

$$\frac{d}{dt}\left(T(t)x\right) = -T(t)Ax = -AT(t)x. \tag{3.11}$$

Integrating with respect to t, we get

$$[T(s)x]_0^t = -\int_0^t AT(s)xds = \int_0^t -T(s)Axds.$$

Thus the proof is complete. ■

Remark 3 *(3.8) says that : for $x \in D(A)$, $u = T(t)x$ is a classical solution to the following initial value problem for the abstract homogeneous parabolic equation*

$$\begin{cases} \frac{du}{dt} + Au = 0 \\ u(0) = x. \end{cases} \tag{3.12}$$

Lemma 3.2 *If A is the infinitesimal generator of a strongly continuous semigroup, then the domain $D(A)$ is dense in B and A is a closed linear operator.*

Proof. See Engel and Naguel [7]. ■

For $x \in D(A)$, we introduce the graph norm

$$\|x\|_{D(A)} = \left(\|x\|^2 + \|Ax\|^2\right)^{\frac{1}{2}} \tag{3.13}$$

Throughout this chapter we simply denote by $\|.\|$ the norm in B. It is easy to verify that $D(A)$ equipped with the graph norm is a Banach space, continuously embedded in B.

Remark 4 $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ *: The Laplacian with respect to the space variable with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ ($\Omega \subset \mathbb{R}^n$). It is a very important linear operator because it arises in many second order semilinear equation as their principal part. It had many important properties, one of them for example, it generates an analytic semigroup(see Pazy [8]) and then a C_0-semigroup.*

3.3 The Abstract Cauchy Problem

3.3.1 The Homogeneous Initial value Problem

Let X be a Banach space and let A be a linear operator from $D(A) \subset X$ into X. Given $x \in X$, the abstract Cauchy problem for A for initial data x consists of finding a solution $u(t)$ to the initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t), & t > 0 \\ u(0) = x \end{cases} \tag{3.14}$$

where by a solution we mean an X valued function $u(t)$ ($u(t) : [0, \infty[\to X$) such that $u(t)$ is continuous for $t \geq 0$, continuously differentiable and $u(t) \in D(A)$ for $t > 0$ and (3.14) is satisfied. Note that since $u(t) \in D(A)$ for $t > 0$ and u is continuous at $t = 0$, (3.14) cannot have a solution for $x \notin \overline{D(A)}$(the closure of $D(A)$).

From the results of the last section it is clear that if A is the infinitesimal generator of C_0 semigroup $T(t)$, the abstract Cauchy problem for A has a solution, namely $u(t) = T(t)x$, for every $x \in D(A)$. It is not difficult to show that for $x \in D(A)$, $u(t) = T(t)x$ is the only solution of (3.14) for more information see Pazy [8].

Definition 3.1 *The linear operator A is positive if*

$$\langle Ax, x\rangle > 0 \qquad \forall x \in D(A). \tag{3.15}$$

Theorem 3.3 *Let A be a densely defined positive linear operator. The initial value problem (3.14) has a unique solution $u(t)$ which is continuously differentiable on $[0, \infty[$, for every initial value $x \in D(A)$ if and only if A is the infinitesimal generator of C_0 semigroup $T(t)$.*

Proof. See Pazy[8]. ∎

Theorem 3.4 *If A is the infinitesimal generator of differential semigroup (see Engel and Naguel [7]) then for every $x \in X$ the initial value problem (3.14) has a unique solution.*

Corollary 1 *If A is the infinitesimal generator of analytic semigroup then for every $x \in X$ the initial value problem (3.14) has a unique solution.*

If A is the infinitesimal generator of a C_0 semigroup which is not differentiable then, in general if $x \notin D(A)$, the initial value problem (3.14) does not have a solution. The function $t \to T(t)x$ is then a "generalized solution " of (3.14) which we will call a mild solution. There are many different ways to define generalized solutions of the problem (3.14), all lead eventually to $T(t)x$. One such way of defining a generalized solution of (3.14) is the following : A continuous function u on $[0, \infty[$ is a generalized solution of (3.14) if there are $x_n \in D(A)$ such that $x_n \to u(0)$ as $n \to \infty$ and $T(t)x_n \to u(t)$ uniformly on bounded intervals. It is obvious that the generalized solution thus defined is independent of the sequence (x_n), is unique and if $u(0) \in D(A)$ it gives the solution of (3.14). Clearly, with this definition of generalized solution , (3.14) has a generalized solution for every $x \in X$ and this generalized solution is $T(t)x$

3.3.2 Inhomogeneous Initial value Problem

In the following, we consider the inhomogeneous initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), & t > 0 \\ u(0) = x \end{cases} \tag{3.16}$$

where $f : [0, \infty[\to X$. We will assume from now on that A is the infinitesimal generator of a C_0-semigroup $T(t)$ so that the corresponding homogeneous equation, i.e., the equation with $f = 0$, has a unique solution for every initial value $x \in D(A)$.

Definition 3.2 *The function $u : [0, T) \to X$ is a classical solution of (3.16) on $[0, T)$ if $u \in C^1(0, T)$, $u(t) \in D(A)$ for all $0 < t < T$ and the problem is satisfied on $[0, T)$.*

Let $T(t)$ be the semigroup generated by A, and let u be a solution of the above problem. We define the function $g : \mathbb{R} \to X$ as $g(s) := T(t-s)u(s)($ here $t - s > 0)$ and

$$\begin{aligned} \dot{g}(s) &= AT(t-s)u(s) + T(t-s)\dot{u}(s) \\ &= T(t-s)u(s) - T(t-s)Au(s) + T(t-s)f(s) \\ &= T(t-s)f(s). \end{aligned}$$

If $f \in L^1([0, T); X)$ then, $T(t-s)f(s)$ is integrable and integrating from 0 to t yields

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds. \tag{3.17}$$

Consequently we have

Corollary 2 *If $f \in L^1([0, T); X)$ then for every $u_0 \in X$ the initial value problem (3.16) has at most one solution. If it has a solution, then it is given by 3.17.*

For every $f \in L^1(0, T : X)$ the right-hand side of (3.17) is a continuous function on $[0, T[$. It is natural to consider it as a generalized solution of (3.16) even it is not differentiable and does not strictly satisfy the equation in the sense of Definition (3.2). We therefore define

Definition 3.3 *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$. Let $x \in X$ and $f \in L^1(0,T : X)$, the function $u \in C([0,T] : X)$ given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \qquad 0 \le t \le T,$$

is the mild solution of the initial value Problem (3.16) on $[0,T]$.

The definition of the mild solution of the initial value Problem (3.16) coincides when $f = 0$ with the definition of $T(t)x$ as the mild solution of the corresponding homogeneous equation. It is therefore clear that not every mild solution of (3.16) is indeed a classical solution even in the case $f \equiv 0$.

For $f \in L^1(0,T : X)$ the initial value Problem (3.16) has by Definition (3.17) a unique mild solution.We will now be interested in posing further conditions on f so that for $x \in D(A)$, the mild solution becomes a classical solution and thus proving, under these conditions, the existence of solutions of (3.16) for $x \in D(A)$.

We start by showing that the continuity of f, in general, is not sufficient to ensure the existence of solutions of (3.16) for $x \in D(A)$. Indeed, let A be the infinitesimal generator of a C_0 semigroup $T(t)$ and let $x \in X$ be such that $T(t)x \notin D(A)$ for any $t \ge 0$. Let $f(s) = T(s)x$. Then $f(s)$ is continuous for $s \ge 0$. Consider the initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + T(t)x \\ u(0) = 0 \end{cases} \tag{3.18}$$

We claim that (3.18) has no solution even though $u(0) = 0 \in D(A)$. Indeed, the mild solution of (3.18)is

$$u(t) = \int_0^t T(t-s)T(s)xds = tT(t)x,$$

but $tT(t)x$ is not differentiable for $t > 0$ and therefore cannot be the solution of (3.18).

Thus, in order to prove the existence of solutions of (3.18) we have to require more than just the continuity of f. We star with a general criterion for the existence of solutions of the initial value Problem (3.18)

Theorem 3.5 *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$, let $f \in L^1(0,T : X)$ be continuous on* $]0,T]$*and let*

$$v(t) = \int_0^t T(t-s)f(s)ds, \qquad 0 \le t \le T. \tag{3.19}$$

The initial value problem (3.16) has a solution u on $[0,T[$ for every $x \in D(A)$if one of the following conditions is satisfied;

(i) $v(t)$ is continuously differentiable on $]0,T[$.

(ii) $v(t) \in D(A)$ for $0 < t < T$ and $Av(t)$is continuous on $]0,T[$.

If (3.16) has a solution on $[0,T[$ for some $x \in D(A)$ then v satisfies both (i) and (ii).

Proof. See [8]. ■

Corollary 3 *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$. If $f(s)$ is continuously differentiable on $[0,T]$ then the initial value problem (3.16) has a solution u on $[0,T][$ for every $x \in D(A)$.*

Corollary 4 *Let A be the infinitesimal generator of C_0 semigroup$T(t)$. Let $f \in L^1(0,T:X)$ be continuous on $]0,T[$. If $f(s) \in D(A)$ for $0 < s < T$ and $Af(s) \in L^1(0,T:X)$ then for every $x \in D(A)$ the initial value Problem (3.16) has a solution on $[0,T[$*

Definition 3.4 *A function u which is differentiable almost everywhere on $[0,T]$ such that $u' \in L^1(0,T:X)$ is called a strong solution of (3.16) if $u(0) = x$ and $u'(t) = Au(t) + f(t)$ a.e. on $[0,T]$.*

We note that if $A = 0$ and $f \in L^1(0,T:X)$ the Cauchy Problem (3.16) has usually no solution unless f is continuous. However, it has always a strong solution given by $u(t) = u(0) + \int_0^t f(s)$. It is easy to show that if u is a strong solution of (3.16) and $f \in L^1(0,T:X)$ then u is is given by (3.17) and therefore is a mild solution of (3.16) and is the unique strong solution. A natural problem is to determine when a mild solution is a strong solution.

Theorem 3.6 *Let A be the infinitesimal generator of a C_o semigroup $T(t)$, let $f \in L^1(O,T:X)$ and let*

$$v(t) = \int_0^t T(t-s)f(s)ds, 0 \leq t \leq T.$$

The initial value Problem (3.16 has a strong solution u on $[0,T]$ for every $x \in D(A)$ if one of the following conditions is satisfied

(i) $v(t)$ is differentiable a.e. on $[0,T]$ and $v'(t) \in L^1(0,T:X)$.

(ii) $v(t) \in D(A)$ a.e. on $[0,T]$ and $Av(t) \in L^1(0,T:X)$.

If (3.16) has a strong solution u on [0, T] for some $x \in D(A)$ then v satisfies both (i) and (ii).

As a consequence of this theorem we have :

Corollary 5 *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$. If f is differentiable a.e. on $[0,T]$ and $f' \in L^(0,T:X)$ then for every $x \in D(A)$ the initial value problem (3.16) has a unique strong solution on $[0,T]$.*

In general, the Lipschitz continuity of f on $[0,T]$ is not sufficient to assure the existence of a strong solution of 3.16 for $x \in D(A)$. However, if X is reflexive and f is Lipschitz continuous on $[0,T]$ that is

$$\|f(t_1) - f(t_2)\| \leq C|t_1 - t_2| \qquad \text{for } t_1, t_2 \in [0,T]$$

then by a classical result f is differentiable a.e. and $f' \in L^1(0,T:X)$.
This corollary implies

Corollary 6 *Let X be a reflexive Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. If f is Lipschitz continuous on $[0,T]$ then for every $x \in D(A)$ the initial value problem (3.20) has a unique solution u on $[0,T]$ given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

3.4 Semilinear Parabolic Equations

In this section we will study the semilinear Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t, u(t)), & t > t_0 \\ u(t_0) = u_0 \end{cases} \tag{3.20}$$

where A is the infinitesimal generator of a C_0 semigroup $T(t)$,$t \geq t_0$ on a Banach space X and $f : [t_0, T] \times X \longrightarrow X$ is continuous in t and satisfies a Lipschitz condition in u, namely

$$\|f(t, u) - f(t, v)\| \leq L \|u - v\| \quad \forall t \geq t_0, \forall u, v \in X \tag{3.21}$$

The Cauchy problem (3.20) does not necessarily have a solution of any kind. However, if it has a classical or strong solution(see Definition 3.2), then the argument given at the beginning of Section 4.2 shows that this solution satisfies the integral equation

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)f(s, u(s))ds. \tag{3.22}$$

It is therefore natural to have the following definition

Definition 3.5 *A continuous solution u of the integral equation (3.22) will be called a mild solution of the Cauchy problem (3.20).*

We start with the following classical result which assures the existence and uniqueness of mild solution of (3.20) for Lipschitz continuous function f.

Theorem 3.7 *Let $f : [t_0, T] \times X \to X$ be continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X. If A is the infinitesimal generator of C_0 semigroup $T(t), t \geq 0$, on X then for every $u_0 \in X$ the problem (3.20) has a unique mild solution $u \in C([t_0, T] : X)$. Moreover, the mapping $u_0 \to u$ is Lipschitz continuous from X into $C([t_0, T] : X)$*

Proof. For a given $u_0 \in X$ we define

$$F : C([t_0, T]; X) \to C([t_0, T]; X)$$

by

$$(Fu)(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)f(s, u(s))ds \qquad \text{on}[t_0, T]. \tag{3.23}$$

Denoting by $\|u\|_\infty$ the norm of u as element of $C([t_0, T]; X)$. From the definition of F it clearly follows that

$$\|(Fu)(t) - (Fv)(t)\| \leq ML(t - t_0) \|u - v\|_\infty , \tag{3.24}$$

where M is a bound of $\|T(t)\|$ on $[t_0, T]$.

Using (3.23) and (3.24), by induction over n it easily follows that

$$\begin{aligned} \|(F^n u)(t) - (F^n v)(t)\| &\leq ML \int_{t_0}^{t} \frac{(MLs)^n}{n!} \|u - v\|_\infty \, ds & (3.25) \\ &\leq \frac{(ML(t - t_0))^n}{n!} \|u - v\|_\infty , & (3.26) \end{aligned}$$

this yields

$$\|F^n u - F^n v\| \leq \frac{(MLT)^n}{n!} \|u - v\|_\infty .$$

If n is large enough $\frac{(MLT)^n}{n!} < 1$, hence by a known extension of the contraction principle F has a unique fixed point $u \in C\left([t_0, T]; X\right)$. This fixed point is the solution of the integral equation and hence the mild solution of the problem.

The uniqueness of u and the Lipschitz continuity of the map $u_0 \mapsto u$ follows from the argument below.

Let v be a mild solution corresponding to the initial value v_0.Then

$$\|u(t) - v(t)\| \leq \|T(t - t_0)u_0 - T(t - t_0)v_0\| + \int_{t_0}^{t} \|T(t - s)\left(f(s, u(s)) - f(s, v(s))\right)\| \, ds$$

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|T(t)u_0 - T(t)v_0\| + \int_{t_0}^{t} \|T(t - s)\left(f(s, u(s)) - f(s, v(s))\right)\| \, ds \\ &\leq M\|u_0 - v_0\| + ML \int_{t_0}^{t} \|u(s) - v(s)\| \, ds, \end{aligned}$$

implying, by Gronwall's inequality[6],

$$\|u(t) - v(t)\| \leq Me^{MLT} \|u_0 - v_0\|$$

and finally

$$\|u - v\|_\infty \leq Me^{MLT} \|u_0 - v_0\| .$$

This means that the solution u is unique and the mapping is Lipschitz continuous. ■

It is not difficult to see that if $g \in C([t_0, T] : X)$ and if we modify in the proof of Theorem (3.7) the definition of F to

$$(Fu)(t) = g(t) + \int_{t_0}^{t} T(t - s) f(s, u(s)) ds,$$

we obtain the following slightly more general result.

Corollary 7 *If A and f satisfy the conditions of Theorem (3.7) then for every $g \in C([t_0, T] : X)$ the integral equation*

$$w(t) = g(t) + \int_{t_0}^{t} T(t-s) f(s, w(s)) ds \tag{3.27}$$

has a unique solution $w \in C([t_0, T] : X)$.

The uniform Lipschitz condition of the function f in Theorem (3.7) assures the existence of a global (i.e. defined on all of $[t_0, T]$) mild solution of (3.20). If we assume that f satisfies only a local Lipschitz condition in u, uniformly in t on bounded intervals, that is for every $t' \geq 0$ and constant $c \geq 0$ there is a constant $L(c, t')$ such that

$$\|f(t, u) - f(t, v)\| \leq L(c, t') \|u - v\| \tag{3.28}$$

holds for all $u, v \in X$ with $\|u\| \leq c$ and $t \in [0, t']$, then we have the following local version of Theorem (3.7).

Theorem 3.8 *Let $f : [0, \infty[\times X \to X$ be continuous in t for $t \geq 0$ and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If A is the infinitesimal generator of a C_0-semigroup $T(t)$ on X then for every $u_0 \in X$ there is a $t_{max} \leq \infty$ such that the initial value Problem (3.20) has a unique mild solution u on $[0, t_{max}[$. Moreover, if $t_{max} < \infty$ then*

$$\lim_{t \to t_{max}} \|u(t)\| = \infty$$

It is well known that in general, if f just satisfies the conditions of Theorem (3.8) or Theorem (3.7) the mild solution of (3.20) does not need to be a classical solution or even a strong solution of (3.20). A sufficient condition for the mild solution to be a classical solution is given next

Theorem 3.9 (Regularity) . *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. If $f : [t_0, T] \times X \to X$ is continuously differentiable from $[t_0, T] \times X \to X$ into X, then the mild solution of (3.20) with $u_0 \in D(A)$ is a classical solution of the initial value Problem.*

In general, if $f : [t_0, T] \times X \to X$ is just Lipschitz continuous in both variables i.e.,

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq C \left(|t_1 - t_2| + \|x_1 - x_2\| \right), \quad t_1, t_2 \in [t_0, T]$$

the mild solution of (3.20) does not need to be a strong solution of the initial value problem. However, if X is reflexive, the Lipschitz continuity of f suffices to assure that the mild solution u with initial data $u_0 \in D(A)$ is a strong solution. Indeed we have

Theorem 3.10 *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on a reflexive Banach space X. if $f : [t_0, T] \times X \to X$ is Lipschitz continuous in both variables, $u_0 \in D(A)$ and u is the mild solution of the initial value problem (3.20), then u is the strong solution of this initial value problem.*

We conclude this section with an application of Theorem (3.7) which provides us with a classical solution of the initial value problem (3.20). Let A be the infinitesimal generator of the C_0 semigroup $T(t)$ on X. We endow the domain $D(A)$ of A with the graph norm It is not difficult to show that $D(A)$ with this norm is a Banach space which we denote by Y. The completeness of Y is a direct consequence of the closeness of A. Clearly $Y \subset X$, and since $T(t) : D(A) \to D(A)$, $T(t), t \geq 0$ is a semigroup on Y which is easily seen to be a C_0 semigroup on Y.

Theorem 3.11 *Let $f : [t_0, T] \times Y \to Y$ be uniformly Lipschitz in Y and for each $y \in Y$ let $f(t, y)$ be continuous from $[t_0, T]$ into Y. If $u_0 \in D(A)$, then the initial value problem (3.20) has a unique classical solution on $[t_0, T]$.*

3.5 Solutions of semilinear PDE problems

This section contains some of our main results. Using semigroup methods we conclude that a quite large class of semilinear parabolic partial differential equations has a unique solution in some sense.

Theorem 3.12 *Let q'_ξ be bounded and q be continuous in all its variables and let Ω be C^2 diffeomorphic to a convex set.Then the problem*

$$\left\{ \begin{array}{cl} \partial_t u(t,x) - \Delta u(t,x) + q(t,x,u(t,x)) = g(t,x) & ((t,x) \in ([0,T[\times\Omega) \\ u(t,x) = 0 & ((t,x) \in ([0,T[\times\partial\Omega) \\ u(0,x) = u_0(x) & \end{array} \right. \tag{3.29}$$

has a unique mild solution over some interval [0,T).

Proof. First we formulate our problem in the abstract form.

We will put X as the Hilbert space $L^2(\Omega)$, the domain of the operator $A = -\Delta$ is $H^2(\Omega) \cap H_0^1(\Omega)$, and by Remark (3.14) the operator A generates a strongly continuous semigroup.

The abstract problem is

$$\left\{ \begin{array}{l} \frac{du}{dt} + Au(t) = F(t,u(t)) \\ u(0) = u_0 \in L^2(\Omega), \end{array} \right. \tag{3.30}$$

where $F(t, u(t)) = -q(t, ., u(t, .)) + g(t, .)$ and q'_ξ is bounded, i.e. for some constant

$$|q'_\xi(t,x,\xi)| \leq K$$

Using Theorem (3.8) we have to prove that the mapping F is locally Lipschitz in u, uniformly in t on bounded intervals

$$\begin{aligned}\|F(t,u)-F(t,v)\|_{L^2}^2 &= \int_\Omega |-q(t,x,u(t,x))+g(t,x)-(-q(t,x,v(t,x))+g(t,x)|^2 \\ &= \int_\Omega |q'_\xi(t,x,\xi(t,x))|^2|u(t,x)-v(t,x)|^2 \leq K^2\,\|u-v\|_{L^2}^2 .\end{aligned}$$

So, by Theorem (3.8), equation 3.29 has a unique mild solution ■

Example 3.1 *Ω is a bounded domain in $\mathbb{R}^3$ with smooth lateral. Consider the problem*

$$\begin{cases} \frac{du}{dt}=\Delta u+f(t,x,u,\mathit{grad}\ u) \ \mathit{for}\ t>0, x\in\Omega \\ \frac{\partial u}{\partial n}+a(x)u=0 \ \mathit{on}\ \partial\Omega, \quad t>0 \\ u(x,0=u_0(x)) \qquad x\in\Omega \end{cases} \tag{3.31}$$

We assume that f(t,x,u,p) is locally Lipschitzian in all its arguments, and for some $k\in[1,3]$, there is a continuous function $B(t,r)$ that

$$\begin{aligned}|f(t,x,u,p)| &\leq B(t,|u|)(1+|p|^k), \\ |f(t,x,u,p)-f(t,x,u,q)| &\leq B(t,|u|)(1+|p|^{k-1}+|q|^{k-1})|p-q|, \\ |f(t,x,u,p)-f(t,x,v,p)| &\leq B(t,|u|+|v|)(1+|p|^k)|u-v|.\end{aligned}$$

Also, $a(x)>0$ is continuously differentiable on $\partial\Omega$. We take $A=-\Delta$ with $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$.

We have $F(t,u)(x)=f(t,x,u(x),\mathit{grad}\ u(x))$, $x\in\Omega$ where

$$\|F(t,u)\|_{L^2(\Omega)} \leq B(t,||u||_{L^\infty(\Omega)})(C+||u||_{W^{1,2k}(\Omega)}^k).$$

where

$$W^{1,2k}(\Omega)=\left\{u\in L^{2k}(\Omega), u'\in L^{2k}(\Omega)\right\}$$

So by Theorem (3.8) we have a solution of the abstract form of the original equation. After investigating his regularity(see Henry [6] page 76), the solution $u(t,x)$ is continuously differentiable in t, twice continuous differentiable in x, and we have a classical solution.

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ملخص

في هذا العمل قمنا بدراسة بعض المسائل المكافئية النصف الخطية **وإعطاء بعض طرق الحصول** على حلول دقيقه لهاته المعادلات ثم **إعطاء الوجود** والوحدانية الحلول لهاته المعادلات **باستعمل** نظرية **أنصاف الزمر.**

الكلمات المفتاحية :معادلات مكافئيه نصف خطية**، معادلة برغوس، أنصاف الزمر.**

Résumé

Dans ce travail nous avons étudie quelque équations paraboliques semi linéaires. Pour cela, nous avons donné quelque méthodes utiles pour obtenir des solutions exactes pour ces équations, et nous avons donné des théorèmes d'existence et d'unicité de solution des ces équations en utilisant la théorie de semi group.

Mots clés : équations parabolique semi linéaires, équations de Burgers , Semi groups.

Abstract

In this work we studied some semilinear parabolic equations and we give some useful methods which allow obtaining exact solutions for these equations, and we give some theorem of existence and uniqueness of these equation by semi group method

Key words: Semiliear parabolic equations, Burgers equation, semigroup method.