Orthogonal Polynomials Linear Combination of the Chebyshev Polynomials of First and Second Kind

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Abstract

In this paper we derive useful results regarding the analysis of differential properties and resultants. Moreover, we study a three-term recursive relation of associated Chebyshev polynomial-type families which are orthogonal polynomials linear combination of the Chebyshev polynomials of first and second kind,orthogonal with respect to some appropriate weight function.

Keywords: *Chebychev Polynomials, Three-Term Recursive Relation, Quadratic transformation*

2000 MSC: 42C05, 33C45.

Gaussian quadrature formula for Chebyshev type,

1 Mathematical basis

The main aim of this contribution is to study a new set of associated Chebychev polynomial-type which are orthogonal polynomials with respect to a given weight function $w(x) = \frac{\sqrt{4pq-x^2}}{1-x^2}$ with $-2\sqrt{pq} \leq x \leq 2\sqrt{pq}$. The paper is structured as follows: in Section 1 we present some useful terminology as well as some necessary definitions regarding Chebyshev polynomials used later on. In Section 2, we focus attention on the properties of this associated Chebychev polynomial-type. In Section 3, we evaluate the discriminants and resultants of such families of associated Chebyshev polynomial-type. Then, we briefly present the basic notions as well as useful properties of Chebyshev polynomials which are orthogonal polynomials with respect to a given weight function such as $w\left(x\right)=\frac{2}{\pi}\frac{1}{\sqrt{1-x^{2}}}$, $-1\leq x\leq 1$. They satisfy, see ([1] and [2])

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{T_{n}\left(x\right)T_{m}\left(x\right)}{\sqrt{1-x^{2}}}dx=\frac{1}{2}\delta_{mn},\quad n>1 \tag{1}$$

and

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{T_{0}^{2}\left(x\right)}{\sqrt{1-x^{2}}}dx=1$$

The Chebyshev polynomials of second kind are orthogonal polynomials with respect to the weight function $w\left(x\right)=\frac{2}{\pi}\sqrt{1-x^{2}}$, $-1\leq x\leq 1$, because

$$\frac{2}{\pi}\int\limits_{-1}^{1}U_{n}\left(x\right)U_{m}\left(x\right)\sqrt{1-x^{2}}dx=\delta_{mn}. \tag{2}$$

The Chebyshev polynomials of first kind $T_{n}\left(x\right)$ may be defined by the following recursive relation: $T_{0}\left(x\right)=1$, $T_{1}\left(x\right)=x$, and

$$T_{n}\left(x\right)=2xT_{n-1}\left(x\right)-T_{n-2}\left(x\right). \tag{3}$$

Alternatively, they may be defined as

$$T_{n}\left(x\right)=\cos\left(n\arccos x\right)$$

and

$$T_{n}\left(x\right)=2^{n-1}x^{n}+...$$

The n^{th} orthogonal Chebyshev polynomials of the second kind $U_{n}(x)$ satisfy, (see [1])

$$U_{n}=\frac{1}{n+1}T_{n+1}',\quad\forall n\in\mathbb{N}^{*} \tag{4}$$

and

$$U_{n}\left(x\right)=\frac{\sin\left(\left(n+1\right)\arccos x\right)}{\sin\left(\arccos x\right)},\quad\forall n\in\mathbb{N}^{*}$$

We remind that they also satisfy the well known TTRR, (see [1]),

$$U_{n}\left(x\right)=2xU_{n-1}\left(x\right)-U_{n-2}\left(x\right) \tag{5}$$

with initial values $U_{0}\left(x\right)=1,U_{1}\left(x\right)=2x$ and

$$U_{n+1}\left(x\right)=2^{n}x^{n}+...$$

The monic Chebyshev polynomials of the second kind satisfy the recursive relation

$$\frac{1}{4}U_{n}\left(x\right)=xU_{n-1}\left(x\right)-U_{n-2}\left(x\right). \tag{6}$$

The roots of $U_{n}\left(x\right)$ are all real, distinct, symmetric with respect to the line $x=0$ and are given by the expression

$$\eta_{k}=\cos\frac{k\pi}{n+1},\quad k=1,2,...,n. \tag{7}$$

2 Example of associated orthogonal Chebyshev polynomial-type

Let $(Q_n(x))_{n\geq 0}$ denote the sequence of associated Chebyshev polynomials. These polynomials are orthogonal with respect to the weight function (see [5]),

$$w_{p,q}(x) = \frac{\sqrt{4pq - x^2}}{1 - x^2}, \text{ for } \quad -2\sqrt{pq} \leq x \leq 2\sqrt{pq}, \tag{8}$$

with $0 \leq p \leq 1, q = 1 - p$ and

$$Q_{-1}(x) = 0, \quad Q_0(x) = 1.$$

If $p \geq \frac{1}{2}$, then this system of polynomials satisfies ([5])

$$\left(\frac{p}{q}\right)^n \int\limits_{-2\sqrt{pq}}^{2\sqrt{pq}} Q_n(x) Q_m(x) \frac{\sqrt{4pq - x^2}}{1 - x^2} dx = \delta_{mn} 2p(1-p)\pi, \text{ for } n \geq 1 \tag{9}$$

and

$$\left(\frac{p}{q}\right)^n \int\limits_{-2\sqrt{pq}}^{2\sqrt{pq}} Q_0(x) Q_m(x) \frac{\sqrt{4pq - x^2}}{1 - x^2} dx = \delta_{0m} 2(1-p)\pi, \text{ for } m \geq 1 \tag{10}$$

The explicit expression of these polynomials is given by (see [5]),

$$Q_n(x) = \left(\frac{q}{p}\right)^n \left((2-2p) T_n\left(\frac{x}{2\sqrt{pq}}\right) + (2p-1) U_n\left(\frac{x}{2\sqrt{pq}}\right)\right) \tag{11}$$

where T_n and U_n are respectively the Chebyshev polynomials of first and second kind.

Now we are in position to state and prove the following main theorem.

Theorem 1 *If the first moment corresponding to $w_{p,q}(x)$ is given by*

$$C_0 = \int\limits_{-2\sqrt{pq}}^{2\sqrt{pq}} \frac{\sqrt{4pq - x^2}}{1 - x^2} dx \tag{12}$$

then,

$$C_0 = 1. \tag{13}$$

Moreover, the moment corresponding to $w_{p,q}(x)$ *with* $\xi = \frac{1}{2\sqrt{pq}}$ *is as follows*

$$\begin{aligned} C_n &= \int_{-2\sqrt{pq}}^{2\sqrt{pq}} x^n \frac{\sqrt{4pq - x^2}}{1 - x^2} dx \qquad (14) \\ &= (-1)^n 2^{2n+3} (pq)^{\frac{n+1}{2}} \xi \\ &\quad \times \int_{-1}^{1} \left(1 + \frac{(1 - \xi^2)(1 + m^2)^2}{(\xi m^2 + 2m + \xi)(\xi m^2 - 2m + \xi)}\right) \frac{m^n}{(1 + m^2)^{n+1}} dm. \end{aligned}$$

Proof. Setting $x = 2\sqrt{pq}t$ in (12) yields

$$C_0 = 4pq \int_{-1}^{1} \frac{\sqrt{1 - t^2}}{1 - 4pqt^2} dt.$$

It follows that

$$C_0 = 2pq \int_{-1}^{1} \left(\frac{1}{1 - 2\sqrt{pq}t} + \frac{1}{1 + 2\sqrt{pq}t}\right) \sqrt{1 - t^2} dt,$$

this moment can be expressed as

$$C_0 = \sqrt{pq} \int_{-1}^{1} \frac{1}{\frac{1}{2\sqrt{pq}} - t} \sqrt{1 - t^2} dt - \sqrt{pq} \int_{-1}^{1} \frac{1}{-\frac{1}{2\sqrt{pq}} - t} \sqrt{1 - t^2} dt.$$

Using the Cauchy integral principal value ([6])

$$\int_{-1}^{1} \frac{1}{\delta - t} \sqrt{1 - t^2} dt = \left(\delta + \sqrt{\delta^2 - 1}\right) \pi, \quad \delta \in \mathbb{R} \backslash [-1 \quad 1] \qquad (16)$$

we get, $C_0 = 1$.Thus, property (13) is established. Let us now prove identity (14). Setting $x = 2\sqrt{pq}t$ in (14) we obtain

$$C_n = 2^{n+2} (pq)^{\frac{n+2}{2}} \int_{-1}^{1} t^n \frac{\sqrt{1 - t^2}}{1 - 4pqt^2} dt.$$

We may notice that,

$$C_n = 2^{n+2} (pq)^{\frac{n+2}{2}} \int_{-1}^{1} t^n \left(\frac{1}{1 - 2\sqrt{pq}t} + \frac{1}{1 + 2\sqrt{pq}t}\right) \sqrt{1 - t^2} dt.$$

Putting $\xi = \frac{1}{2\sqrt{pq}}$ we get at once

$$\begin{aligned} C_{n,1} &= 2^{n+2}\left(pq\right)^{\frac{n+2}{2}}\int\limits_{-1}^{1}\frac{t^n}{1-2\sqrt{pq}t}\sqrt{1-t^2}dt \\ &= 2^{n+1}\left(pq\right)^{\frac{n+1}{2}}\int\limits_{-1}^{1}\frac{t^n}{\xi-t}\sqrt{1-t^2}dt. \end{aligned}$$

Making the change of variable : $\sqrt{1-t^2} = 1+mt$,(second Euler's substitution)

$$t = -\frac{2m}{1+m^2}, dt = \frac{2\left(m^2-1\right)}{\left(1+m^2\right)^2}dm$$

we arrive at

$$\int\limits_{-1}^{1}\frac{t^n}{\xi-t}\sqrt{1-t^2}dt = \left(-1\right)^{n-1}2^n\int\limits_{-1}^{1}\frac{m^n}{\left(1+m^2\right)^n}\frac{1+m^2}{\xi m^2+2m+\xi}\frac{1-m^2}{1+m^2}\frac{2\left(m^2-1\right)}{\left(1+m^2\right)^2}dm$$

so that the equality

$$\int\limits_{-1}^{1}\frac{t^n}{\xi-t}\sqrt{1-t^2}dt = \left(-1\right)^n 2^{n+1}\int\limits_{-1}^{1}\frac{m^n}{\left(1+m^2\right)^{n+2}}\frac{\left(m-1\right)^2\left(m+1\right)^2}{\xi m^2+2m+\xi}dm$$

follows. Noticing that

$$\frac{\left(m-1\right)^2\left(m+1\right)^2}{\left(\xi m^2+2m+\xi\right)\left(1+m^2\right)^2} = \frac{\xi}{1+m^2} - \frac{2m}{\left(1+m^2\right)^2} + \frac{1-\xi^2}{\xi m^2+2m+\xi}$$

we get

$$\int\limits_{-1}^{1}\frac{t^n}{\xi-t}\sqrt{1-t^2}dt =$$

$$\begin{aligned} &\left(-1\right)^n 2^{n+1}\left(\xi\int\limits_{-1}^{1}\frac{m^n}{\left(1+m^2\right)^{n+1}}dm - 2\int\limits_{-1}^{1}\frac{m^{n+1}}{\left(1+m^2\right)^{n+2}}dm\right. \\ &\left. + \left(1-\xi^2\right)\int\limits_{-1}^{1}\frac{m^n}{\left(\xi m^2+2m+\xi\right)\left(1+m^2\right)^n}dm\right) \end{aligned}$$

Let us now compute the integral

$$C_{n,2} = -2^{n+1}\left(pq\right)^{\frac{n+1}{2}}\int\limits_{-1}^{1}t^n\frac{1}{-\xi-t}\sqrt{1-t^2}dt.$$

Using the same change of variable as above, we get

$$\int_{-1}^{1}\frac{t^n}{-\xi-t}\sqrt{1-t^2}dt=(-1)^n\,2^{n+1}\int_{-1}^{1}\frac{m^n}{\left(1+m^2\right)^{n+2}}\frac{\left(m^2-1\right)^2}{-\xi m^2+2m-\xi}dm.$$

A similar result can be readily obtained, by a similar technique, we have

$$\frac{(m-1)^2\,(m+1)^2}{\left(-\xi m^2+2m-\xi\right)\left(1+m^2\right)^2}=\frac{-\xi}{1+m^2}-\frac{2m}{\left(1+m^2\right)^2}+\frac{1-\xi^2}{-\xi m^2+2m-\xi}.$$

Since

$$\frac{1}{\xi m^2+2m+\xi}+\frac{1}{\xi m^2-2m+\xi}=\frac{2\xi\left(m^2+1\right)}{\left(\xi m^2+2m+\xi\right)\left(\xi m^2-2m+\xi\right)}$$

then

$$\int_{-1}^{1}\frac{t^n}{-\xi-t}\sqrt{1-t^2}dt=$$

$$\begin{aligned}&(-1)^n\,2^{n+1}\left(-\xi\int_{-1}^{1}\frac{m^n}{\left(1+m^2\right)^{n+1}}dm-2\int_{-1}^{1}\frac{m^{n+1}}{\left(1+m^2\right)^{n+2}}dm\right.\\&\left.-\left(1-\xi^2\right)\int_{-1}^{1}\frac{m^n}{\left(\xi m^2-2m+\xi\right)\left(1+m^2\right)^n}dm\right)\end{aligned}$$

Taking into account the above two steps we find

$$\begin{aligned}&2^{n+1}\,(pq)^{\frac{n+1}{2}}\left(\int_{-1}^{1}\frac{t^n}{\xi-t}\sqrt{1-t^2}dt-\int_{-1}^{1}t^n\frac{1}{-\xi-t}\sqrt{1-t^2}dt\right)\\=\;&(-1)^n\,2^{2n+3}\,(pq)^{\frac{n+1}{2}}\,\xi\left(\int_{-1}^{1}\frac{m^n}{\left(1+m^2\right)^{n+1}}dm\right.\\&\left.+\left(1-\xi^2\right)\int_{-1}^{1}\frac{1}{\left(\xi m^2+2m+\xi\right)\left(\xi m^2-2m+\xi\right)}\frac{m^n}{\left(1+m^2\right)^{n-1}}dm\right)\end{aligned}$$

The proof of formula (15) is now completed. ■

We obtain the following recursive formulas.

Proposition 2 *The orthogonal polynomials $Q_n(x)$ satisfy the three-term recursive relation*

$$Q_n(x)=\frac{q}{p\sqrt{pq}}xQ_{n-1}(x)-\left(\frac{q}{p}\right)^2Q_{n-2}(x)\tag{17}$$

with initial conditions

$$Q_{-1}(x)=0,\quad Q_0(x)=1,Q_1(x)=\frac{q}{p\sqrt{pq}}x,\tag{18}$$

and starting values

$$Q_2(x)=\left(\frac{q}{p\sqrt{pq}}\right)^2x^2-\left(\frac{q}{p}\right)^2,Q_3(x)=\left(\frac{q}{p\sqrt{pq}}\right)^3x^3-2\frac{q}{p\sqrt{pq}}\left(\frac{q}{p}\right)^2x\tag{19}$$

Proof. Thanks to (11), we have

$$xQ_n(x)=2\sqrt{pq}\left(\frac{q}{p}\right)^n\left((2p-2)\left(\frac{x}{2\sqrt{pq}}T_n\left(\frac{x}{2\sqrt{pq}}\right)+(2p-1)\frac{x}{2\sqrt{pq}}U_n\left(\frac{x}{2\sqrt{pq}}\right)\right)\right)$$

From the two three-term recursivee relations (3), (5) (by replacing x by $\frac{x}{2\sqrt{pq}}$) we get

$$xQ_n(x)=\sqrt{pq}\left(\frac{q}{p}\right)^n\left((2p-2)\left(T_{n+1}\left(\frac{x}{2\sqrt{pq}}\right)+T_{n-1}\left(\frac{x}{2\sqrt{pq}}\right)\right)\right)$$

$$+\sqrt{pq}\left(\frac{q}{p}\right)^n\left((2p-1)\left(U_{n+1}\left(\frac{x}{2\sqrt{pq}}\right)+U_{n-1}\left(\frac{x}{2\sqrt{pq}}\right)\right)\right)$$

It follows immediately that

$$xQ_n(x)=\frac{p}{q}\sqrt{pq}\left(\frac{q}{p}\right)^{n+1}\left((2p-2)T_{n+1}\left(\frac{x}{2\sqrt{pq}}\right)+(2p-1)U_{n+1}\left(\frac{x}{2\sqrt{pq}}\right)\right)$$

$$+\frac{q}{p}\sqrt{pq}\left(\frac{q}{p}\right)^{n-1}\left((2p-2)T_{n-1}\left(\frac{x}{2\sqrt{pq}}\right)+(2p-1)U_{n-1}\left(\frac{x}{2\sqrt{pq}}\right)\right).$$

This relation yields

$$xQ_n(x)=\frac{p}{q}\sqrt{pq}Q_{n+1}(x)+\frac{q}{p}\sqrt{pq}Q_{n-1}(x).$$

Therefore, the recursive formula (17) is proved. Equations (17) and (18) yield (19). ■

Corollary 3 *Let* $B_n^{p,q}(x)$ *be a sequence of orthogonal polynomials with respect to the weight function* $w_{p,q}(x)=(x-c)\frac{\sqrt{4pq-x^2}}{1-x^2}$ *, for* $-2\sqrt{pq}\le x\le 2\sqrt{pq}$*, such that*

$$\int_{-2\sqrt{pq}}^{2\sqrt{pq}}B_n^{p,q}(x)B_m^{p,q}(x)(x-c)\frac{\sqrt{4pq-x^2}}{1-x^2}dx=0,\quad n\neq m,\quad c\ge 2\sqrt{pq},\tag{20}$$

then

$$(x-c)B_n^{p,q}(x)\ =c_{n+1}(p,q)Q_{n+1}(x)+c_n(p,q)Q_n(x),\tag{21}$$

where

$$c_{n+1}(p,q) = -\frac{Q_n(c)\, B_n^{p,q\prime}(c)}{Q_{n+1}(c)\, Q_n'(c) - Q_n(c)\, Q_{n+1}'(c)}, \tag{22}$$

and

$$c_n(p,q) = \frac{Q_{n+1}(c)\, B_n^{p,q\prime}(c)}{Q_{n+1}(c)\, Q_n'(c) - Q_n(c)\, Q_{n+1}'(c)}. \tag{23}$$

Here

$$Q_n(c) = \left(\frac{q}{p}\right)^n \left((2-2p)\, T_n\left(\frac{c}{2\sqrt{pq}}\right) + (2p-1)\, U_n\left(\frac{c}{2\sqrt{pq}}\right)\right)$$

$$Q_n'(c) = \left(\frac{q}{p}\right)^n \left(n\,(2-2p)\, U_{n-1}\left(\frac{c}{2\sqrt{pq}}\right) + (2p-1)\, U_n'\left(\frac{x}{2\sqrt{pq}}\right)\right)$$

$$Q_{n+1}(c) = \left(\frac{q}{p}\right)^{n+1} \left((2-2p)\, T_{n+1}\left(\frac{c}{2\sqrt{pq}}\right) + (2p-1)\, U_{n+1}\left(\frac{c}{2\sqrt{pq}}\right)\right)$$

$$Q_{n+1}'(c) = \left(\frac{q}{p}\right)^{n+1} \left((n+1)\,(2-2p)\, U_n\left(\frac{c}{2\sqrt{pq}}\right) + (2p-1)\, U_{n+1}'\left(\frac{c}{2\sqrt{pq}}\right)\right)$$

Proof. Since $(x-a)\, B_n^{p,q}(x)$ is a polynomial of degree $n+1$, it can be written into the form,

$$(x-c)\, B_n^{p,q}(x) = \sum_{k=0}^{n+1} c_k Q_k(x),$$

where

$$c_k = \int_{-2\sqrt{pq}}^{2\sqrt{pq}} B_n^{p,q}(x)\, L_k(x)\, (x-c)\, \frac{\sqrt{4pq - x^2}}{1-x^2} dx \qquad k = 0, 1, ..., n+1,$$

and $B_n^{p,q}(x)$ is orthogonal to every polynomial of lower degree with respect to the weight function $w_{p,q}(x)$. We have in each case

$$(x-c)\, B_n^{p,q}(x) = c_{n+1}(p,q)\, Q_{n+1}(x) + c_n(p,q)\, Q_n(x).$$

Setting $x = c$, we obtain

$$c_{n+1}(p,q)\, Q_{n+1}(c) + c_n(p,q)\, Q_n(c) = 0.$$

Differentiating, and setting $x = c$, gives

$$c_{n+1} Q_{n+1}'(c) + c_n Q_n'(c) = B_n^{p,q\prime}(c),$$

so that

$$c_{n+1} = -\frac{Q_n(c)\, B_n^{p,q\prime}(c)}{Q_{n+1}(c)\, Q_n'(c) - Q_n(c)\, Q_{n+1}'(c)}$$

and

$$c_n = \frac{Q_{n+1}(c)\, B_n^{p,q'}(c)}{Q_{n+1}(c)\, Q_n'(c) - Q_n(c)\, Q_{n+1}'(c)}.$$

Thanks to (11) we find the expansions of $Q_n(c), Q_{n+1}(c), Q_n'(c), Q_{n+1}'(c)$ respectively. Thus properties (21),(22),(23) are proved. ■

The resultant of a two-term linear combination of Chebyshev polynomials of second kind is given in the recent work of Dilcher and Stolarsky ([1]). Different types of resultants related to Chebyshev polynomials of first and second kind are computed in order to calculate the resultants of this special form of linear combinations of Chebyshev polynomials.

3 Resultants and discriminants

With the above motivation about Chebyshev polynomials of second kind we are in position to give the connection between these polynomials as follows ([1])

$$T_n(x) = \frac{1}{2}\left(U_n(x) - U_{n-2}(x)\right) = xT_{n-1}(x) - \left(1 - x^2\right) U_{n-2}(x). \tag{24}$$

Since

$$\left(1 - x^2\right) T_n'' - xT_n' + n^2 T_n = 0, \tag{25}$$

then

$$U_n'(x) = \frac{(n+1)\, T_{n+1}(x) - xU_n(x)}{x^2 - 1} \tag{26}$$

where $\{U_n(x)\}_{n\geq 0}$ satisfies the recursive relation of the differential equation

$$U_{n+1}'(x) = \frac{2x(n+1)\, U_{n+1}(x) - (n+2)\, U_n(x)}{x^2 - 1}. \tag{27}$$

It is well known that $T_n\left(\cos\frac{(2k-1)\pi}{2n}\right) = 0,\ k = 1, 2, ..., n$. The resultant of two polynomials $T_n(z)$ and $Q_n(z)$ is given by ([1])

$$\operatorname{Re} s\left(T_n, Q_n\right) = 2^{n(n-1)} \prod_{k=1}^{n} Q_n\left(\cos\frac{(2k-1)\pi}{2n}\right). \tag{28}$$

The discriminant of $Q_n(z)$ of degree n and its leading coefficient is γ, it can be computed in terms of the resultants between this polynomial and its derivatives. It is given by

$$Disc\ Q_n(x) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{\gamma} \operatorname{Re} s\left(Q_n, Q_n'\right) = (-1)^{\frac{n(n-1)}{2}} \gamma^{n-2} \prod_{k=1}^{n} Q_n'\left(r_{n,k}\right) \tag{29}$$

where $Q_n(r_{n,k}) = 0,\ k = 1, 2, ..., n$. Next, we state and prove a lemma which is important in the sequel.

Lemma 4 *([1]) Let $p_n(x)$ be a sequence of polynomials satisfying the recursive formula*

$$p_n(x) = (a_n x + b_n) p_{n-1}(x) - c_n p_{n-2}(x), \; n = 2, 3, ...$$

with initial conditions

$$p_0(x) = 1, p_1(x) = a_1 x + b_1.$$

Let $a_1 a_n c_n \neq 0$ *and* $(x_{n,k}), k = 1, 2, , ..., n$ *be the zeros of* $p_n(x)$. *Then*

$$\operatorname{Re} s(p_n, p_{n-1}) = \prod_{k=1}^{n} p_{n-1}(x_{n,k}) = (-1)^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} a_k^{n-2k+1} c_k^{k-1} \tag{30}$$

In ([1]) , if we can construct $A_n(x)$ *and* $B_n(x)$ *so that,*

$$p_n'(x) = A_n(x) p_{n-1}(x) + B_n(x) p_n(x).$$

Then

$$Disc\ p_n(x) = \gamma^{n-2} \prod_{k=1}^{n} a_k^{n-2k+1} c_k^{k-1} \prod_{k=1}^{n} A_n(x_{n,k}) \tag{31}$$

Proof. Details and proof of the above Lemma can be found in ([1]). ■

Theorem 5 *If* $\triangle_n = \prod_{k=1}^{n} Q_{n-1}(r_{n,k})$, *then*

$$\triangle_n = (-1)^{\frac{n(n-1)}{2}} \left(\frac{q}{p}\right)^{n(n-1)} \tag{32}$$

and

$$\operatorname{Re} s(Q_n, Q_{n-1}) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{q}{p\sqrt{pq}}\right)^{n(n-1)} \left(\frac{q}{p}\right)^{n(n-1)}. \tag{33}$$

Analogously, if $R_n(x) = \prod_{k=1}^{n} \left(x - 2\sqrt{pq}\cos\frac{k\pi}{n+1}\right)$ *(monic polynomial), then*

$$\frac{\operatorname{Re} s(R_n, Q_{n+1}')}{\operatorname{Re} s(U_n, U_{n-1})} = 2^{-(n-1)^2} (n+1)^n (pq)^{-\frac{n}{2}} \left(\frac{q}{p}\right)^{n(n+1)} (2p-1)^n \frac{\prod_{k=1}^{n} \cos\frac{k\pi}{n+1}}{\left(\prod_{k=1}^{n} \sin\frac{k\pi}{n+1}\right)^2} \tag{34}$$

Proof. We have by virtue of TTRR (17) $Q_n(r_{n,k}) = 0$, $k = 1, 2, ...$ Putting

$$Q_n(x) = \left(\frac{q}{p\sqrt{pq}}\right)^n \prod_{k=1}^{n} (x - r_{n,k})$$

we get

$$\triangle_n = \left(\frac{q}{p\sqrt{pq}}\right)^{n(n-1)} \prod_{k=1}^{n} \left(r_{n,k} - r_{n-1,1}\right)\left(r_{n,k} - r_{n-1,2}\right) .... \left(r_{n,k} - r_{n-1,n-1}\right)$$

or else

$$\triangle_n = \frac{\left(\frac{q}{p\sqrt{pq}}\right)^{n(n-1)}}{\left(\frac{q}{p\sqrt{pq}}\right)^{(n-1)n}} \prod_{k=1}^{n-1} Q_n\left(r_{n-1,k}\right) = (-1)^{n-1}\left(\frac{q}{p}\right)^{2(n-1)} \triangle_{n-1}$$

It follows

$$\triangle_{n-1} = \prod_{k=1}^{n-1} Q_{n-2}\left(r_{n-1,k}\right) =$$

$$\left(\frac{q}{p\sqrt{pq}}\right)^{(n-1)(n-2)} \prod_{k=1}^{n-1} \left(r_{n-1,k} - r_{n-2,1}\right)\left(r_{n-1,k} - r_{n-2,2}\right) ... \left(r_{n-1,k} - r_{n-2,n-2}\right),$$

that is

$$\triangle_{n-1} = \frac{\left(\frac{q}{p\sqrt{pq}}\right)^{(n-1)(n-2)}}{\left(\frac{q}{p\sqrt{pq}}\right)^{(n-2)(n-1)}} \prod_{k=1}^{n-2} Q_{n-1}\left(r_{n-2,k}\right) = (-1)^{n-2}\left(\frac{q}{p}\right)^{2(n-2)} \prod_{k=1}^{n-2} Q_{n-3}\left(r_{n-2,k}\right).$$

Thus

$$\triangle_n = (-1)^{n-1}\left(\frac{q}{p}\right)^{2(n-1)} (-1)^{n-2}\left(\frac{q}{p}\right)^{2(n-2)} \triangle_{n-2}$$

The formula (32) follows by induction. From (32) we get (33). Differentiating both sides of identity (11) and replacing x by $2\sqrt{pq}x$ give

$$Q'_{n+1}\left(2\sqrt{pq}x\right) = \frac{1}{2\sqrt{pq}}\left(\frac{q}{p}\right)^{n+1} \left((n+1)(2-2p)U_n(x) + (2p-1)U'_{n+1}(x)\right).$$

We obtain at once from identities (26) and (27) the following

$$U'_{n+1}(x) = \frac{(n+2)\left(\frac{1}{2}\left(U_{n+2}(x) - U_n(x)\right)\right) - xU_{n+1}(x)}{x^2-1}.$$

We have by (5),$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$, it follows that

$$U'_{n+1}(x) = \frac{(n+2)\left(xU_{n+1}(x) - U_n(x)\right) - x\left(2xU_n(x) - U_{n-1}(x)\right)}{x^2-1}.$$

Once again from the relation $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$,we get

$$U'_{n+1}(x) = \frac{(n+2)\left(2x^2U_n(x) - xU_{n-1}(x)\right) - (n+2)U_n(x) - 2x^2U_n(x) + xU_{n-1}(x)}{x^2-1}.$$

Thus

$$Q'_{n+1}\left(2\sqrt{pq}x\right) = B_n\left(x,p\right)U_n\left(x\right) + A_n\left(x,p\right)U_{n-1}\left(x\right),$$

where

$$B_n\left(x,p\right) = \frac{1}{2\sqrt{pq}}\left(\frac{q}{p}\right)^{n+1}\frac{\left(2n+2\right)x^2 + \left(n+1\right)\left(2-2p\right)\left(x^2-1\right) - n - 2}{x^2-1}$$

and

$$A_n\left(x,p\right) = \frac{1}{2\sqrt{pq}}\left(\frac{q}{p}\right)^{n+1}\frac{x\left(n+1\right)\left(2p-1\right)}{1-x^2}.$$

It follows that,

$$\frac{\operatorname{Re}s\left(R_n, Q'_{n+1}\right)}{\prod\limits_{k=1}^{n} U_{n-1}\left(\cos\frac{k\pi}{n+1}\right)} = 2^{-n}\left(n+1\right)^n\left(pq\right)^{-\frac{n}{2}}\left(\frac{q}{p}\right)^{n(n+1)}\left(2p-1\right)^n\prod_{k=1}^{n}\frac{\cos\frac{k\pi}{n+1}}{\sin^2\frac{k\pi}{n+1}}$$

with $U_n\left(x\right) = 2^{n-1}x^{n+1} + ...$ We know according to ([1]) that

$$\operatorname{Re}s\left(U_n, U_{n-1}\right) = 2^{(n-1)^2}\prod_{k=1}^{n}U_{n-1}\left(\cos\frac{k\pi}{n+1}\right).$$

This proves property (34). ■

Acknowledgement 6 *I warmly express my deep gratitude to the referees for their criticism of the manuscript and for their helpful suggestions.*

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