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Finite Groups and Their Properties

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Dedication

All praise is due to Allah Almighty, whose guidance, mercy, and blessing have given me the strength and determination to accomplish this work. Without his support, this achievement would not have been possible.

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إهْدَاء

«اَلْحَمْدُ لِلّٰهِ رَبِّ الْعَالَمِين، الذَّي بِنِعْمَتِهِ تَتِمُّ الصَّالِحَاتُ، وَبِهِ تُسْتَفْتَحُ الْبِدَايَاتُ وَتُخْتَمُ الْغَايَاتُ»

بِاسْمِ اللهِ، وَعَلى هَدْيِهِ أَمْضِي، وَبِهِ أَسْتَعِينْ، أُهْدِي ثَمَرَةَ هَذَا الْجُهْدِ الْمُتَوَاضِع:

إِلَى وَالِدَيَّ الْكَرِيمَين، مَنْ سَقَيَانِي حُبًّا، وَغَمَرَانِي لُطْفًا، فَكُنْتُ بِفَضْلِهِمَا بَعْدَ اَللّٰهِ مَا أَنَا عَلَيهِ، وَإِنْ جَفَّ مِدَادِي فَلَنْ يَفِي بِحَقِّهِمَا، وَإِنْ طَالَ دُعَائِي فَلَنْ يَبْلُغَ بَعْضَ جَمِيلَهُمَا.
إِلَى إِخْوَتِي وَأَخَوَاتِي، عِقْدِ اَلْمَوَدَّةِ وَنَبْضِ الصُّحْبَةِ، الذِّينَ شَدُّوا أَزْرِي إِذَا فَتَرَ، وَكَانُوا لِي ظِلًّا إِذَا اَشْتَدَّ الْحَرُّ، وَنُورًا إِذَا ادْلَهَمَّ الدَّرْبُ.

إِلَى أَصْدِقَائِي اَلْأَوْفِيَاء زَادُ الطَّرِيقِ وَبَهْجَةِ الْمَسِيرِ، الذِّينَ طَيَّبُوا الرِّحْلَةَ بِصِدْقِهِمْ، وَزَيَّنُوهَا بِوَفَائِهِمْ.

إِلَى أُسْتَاذِي الْمُشْرِفْ **اَلدُّكْتُور يُومْبَعِي رِيَاض،** مَنَارَ اَلْعِلْمِ، الذِّي أَفَاضَ مِنْ عِلْمِهِ، وَأَجَادَ فِي تَوْجِيهِهِ، فَلَهُ مِنِّي شُكْرٌ لاَ يَنْقَضِي، جَزَاهُ اللهُ عَنِّي خَيْرَ الْجَزَاءِ.

«رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَى وَالِدَيَّ، وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ»

شَرْفِي أَحْمَد -- بِلْفَرْدِي أَحْمَد

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Notations

Symbol	**Meaning**
$\mathcal{G}$	A group
$\mathbb{Z}_n$	The set of integers mod n
$GL_n(\mathbb{K})$	Group of invertible $n \times n$ matrices over $\mathbb{K}$
$GL(V)$	General linear group of invertible operators on V
$SL_n(\mathbb{K})$	Special linear group of $n \times n$ matrices over $\mathbb{K}$ with determinant 1
$\lvert\mathcal{G}\rvert$	The order of a group
$\lvert x\rvert$	The degree an element x
$\mathcal{S} \leq \mathcal{G}$	$\mathcal{S}$ is a subgroup of $\mathcal{G}$
$\langle x \rangle$	Generator of a group
$[\mathcal{G} : \mathcal{S}]$	The index of $\mathcal{S}$ in $\mathcal{G}$
$\mathcal{N} \trianglelefteq \mathcal{G}$	$\mathcal{N}$ is a normal subgroup of $\mathcal{G}$
$\ker \varphi$	Kernel of φ
$\mathrm{Im}\ \varphi$	Image of φ
$\mathrm{Cl}(x)$	Set of conjugacy classes of x
$x \sim y$	x conjugates to y
$\mathrm{C}_{\mathcal{G}}(x)$	The centralizer of x in $\mathcal{G}$
$\mathrm{Z}(\mathcal{G})$	The center of $\mathcal{G}$
$\mathrm{N}_{\mathcal{G}}(\mathcal{S})$	Normalizer of $\mathcal{S}$ in $\mathcal{G}$
$\mathfrak{C}_n$	Cyclic group of order n
$\mathfrak{S}_n$	Symmetric group of n elements
$\mathfrak{A}_n$	Alternating group of n elements
$\mathfrak{D}_n$	Dihedral group of order $2n$
$\mathrm{sgn}(\sigma)$	Sign of permutation σ
$(V, \mathscr{L})$	A Representation of $\mathcal{G}$
$\mathbb{I}d_V$	The identity operator on a vector space V
$\mathbb{I}_n$	The identity matrix of size $n \times n$
A^{-1}	The inverse of a matrix A
$\mathscr{L}_1 \oplus \mathscr{L}_2$	Direct sum of representations $\mathscr{L}_1$ and $\mathscr{L}_2$
$\mathscr{L}_1 \otimes \mathscr{L}_2$	Tensor product of representations $\mathscr{L}_1$ and $\mathscr{L}_2$
$\mathrm{Hom}_{\mathcal{G}}(V, W)$	The set of $\mathcal{G}-$homomorphisms from V to W
$\mathrm{Tr}(\mathscr{L}(x))$	Trace of a representation $\mathscr{L}$
$\chi(x)$	Character of a representation
$\langle \cdot, \cdot \rangle$	Inner product on class functions
χ_i	Irreducible character
$\mathbb{C}[\mathcal{G}]$	Group algebra of $\mathcal{G}$ over $\mathbb{C}$
χ_{reg}	Character of regular representation
δ_{ij}	Kronecker delta

Introduction

The idea of a group is really important in mathematics. It helps us understand symmetry and how things change. We can look at how objects move and change shape like a solid object rotating or how the roots of an equation can be rearranged. Groups show us what stays the same and how things are connected.

This text is about groups, which are groups with a limited number of elements and how we can use linear representations to understand them better. Linear representations are a way to take the ideas of group theory and make them more concrete using linear algebra. The theory of groups and their linear representations is a powerful tool that helps us understand groups, in a more tangible way. Our journey begins with:

Chapter 1 covers the basics of group theory. It starts with what a group's goes through subgroups, cosets, normal subgroups and quotient groups. The isomorphism theorems, conjugacy classes, the class equation and Sylow theorems give us tools to work with groups. The chapter ends with examples of dihedral and symmetric groups that we'll use throughout the book.

Chapter 2 is about representations of finite groups. We start with what a representation's how it looks in matrix form. Then we look at subspaces, subrepresentations and irreducible representations. Two important results are Schurs lemma and Maschkes theorem. They show that every representation of a finite group can be broken down into parts. The chapter ends with examples for groups.

Chapter 3 focuses on characters which are the traces of representation matrices. Characters make representations easier to work with. We look at two rules that show how irreducible characters work. We also examine the representation and how it breaks down. This helps us link the groups order to the sums of squares of character degrees.

Chapter 4 puts all this information into the character table. This is a way to encode a groups representation theory. We discuss properties of character tables. How to construct them. We use groups to illustrate this.

Chapter 5 shows the power of characters in applications. We prove Burnsides $p^a q^b$ -theorem, which is a classic result, in number theory. Characters help us test if a group is simple or not. We also use characters to study the groups center via linear characters.

Chapter 1

Basic Group Theory Background

In this chapter, we review the basic notions of group theory, beginning with the definitions of groups and ending with Sylow theorems. These concepts will be used throughout the study of group representation. These results provide the algebraic tools needed for the study of representation theory.

1.1 Groups

✍**Definition 1.1.1.** *[3] A **group** is a pair $(\mathcal{G}, *)$, where $\mathcal{G}$ is a nonempty set equipped with a binary operation. These satisfy the following conditions:*

(a) ***Associativity :*** *For all $x, y, z \in \mathcal{G}$, we have:*

$$(x * y) * z = x * (y * z).$$

(b) ***Existence of an identity :*** *There is an element $e \in \mathcal{G}$ satisfying:*

$$e * x = x * e = x \quad \text{for all } x \in \mathcal{G}.$$

(c) ***The existence of the inverse :*** *For each element $x \in \mathcal{G}$, there is an element $x^{-1} \in \mathcal{G}$ satisfying:*

$$x * x^{-1} = x^{-1} * x = e$$

where e is the identity element.

✯**Remarks 1.1.1.** *[3]*

1. *If $x * y = y * x$, for all $x, y \in \mathcal{G}$. We say that the group $(\mathcal{G}, *)$ is **abelian** or **commutative**.*
2. *The identity element e is unique in $\mathcal{G}$.*
3. *The inverse $x^{-1} \in \mathcal{G}$ is unique.*

✎**Example 1.1.1.** ✏ *Some classic examples of groups:*

$$(\mathbb{C}, +), \quad (\mathbb{Q}^{\times}, \cdot), \quad (\mathbb{Z}_n, +), \quad (GL_n(\mathbb{C}), \cdot), \quad (SL_n(\mathbb{R}), \cdot).$$

✍**Definition 1.1.2.** *[14] **The order of a finite group** $\mathcal{G}$ is the number of elements in $\mathcal{G}$, denoted by $|\mathcal{G}|$.*
*For an element $x \in \mathcal{G}$, **the order** of x is the smallest positive integer n such that:*

$$x^n = e \quad (e \text{ is the identity element}).$$

1.2 Subgroups and Cyclic Groups

✍**Definition 1.2.1.** *[2] A subset $\mathcal{S}$ of a group $\mathcal{G}$ is said to be a* ***subgroup*** *if $\mathcal{S}$ is itself a group under the operation of $\mathcal{G}$.*
Equivalently, $\mathcal{S}$ is a subgroup if the following conditions hold:

(a) $e \in \mathcal{S}$.

(b) *For all $x, y \in \mathcal{S}$ then $x \cdot y \in \mathcal{S}$.*

(c) *For all $x \in \mathcal{S}$ we have $x^{-1} \in \mathcal{S}$.*

✎**Example 1.2.1.** ✏ *Some examples of subgroups:*

$$n\mathbb{Z} \leq \mathbb{Z} \quad \mathbb{R}_+ \leq \mathbb{R}^* \quad SL_2(\mathbb{R}) \leq GL_2(\mathbb{R}).$$

➢**Notation 1.** *We use the notation $\mathcal{S} \leq \mathcal{G}$ to mean that $\mathcal{S}$ is a subgroup of $\mathcal{G}$.*

▼**Theorem 1.2.1.** *[2] Let $\mathcal{G}$ be a group and $\mathcal{S}$ a nonempty subset of $\mathcal{G}$. If $xy^{-1} \in \mathcal{S}$ for all $x, y \in \mathcal{S}$ then $\mathcal{S}$ is a subgroup.*

Proof. Let $t \in \mathcal{S}$, and $x = t, y = t$, we see that:

$$xy^{-1} = tt^{-1} = e \in \mathcal{S}.$$

Let $x = e, y = t$, then:

$$xy^{-1} = et^{-1} = t^{-1} \in \mathcal{S}.$$

For all $z, t \in \mathcal{S}$, we want to show that $zt \in \mathcal{S}$. Choose $x = z, y = t^{-1}$, then:

$$xy^{-1} = z(t^{-1})^{-1} = zt \in \mathcal{S}.$$

Hence $\mathcal{S} \leq \mathcal{G}$. □

✍**Definition 1.2.2.** *[14] A group $\mathcal{G}$ is called* ***cyclic*** *if there is an element $x \in \mathcal{G}$ such that $\mathcal{G} = \{x^n \mid n \in \mathbb{Z}\}$. The element x is called* ***the generator*** *of $\mathcal{G}$.*

➢**Notation 2.** *We use the notation $\mathcal{G} = \langle x \rangle$ to denote that $\mathcal{G}$ is generated by x.*

✎**Example 1.2.2.**
- ✏ *The group $\mathbb{Z}_n = \{0, 1, ..., n-1\}$ under addition modulo n is a cyclic group generated by $\overline{1}$.*
- ✏ *The trivial group $(\{e\}, \cdot)$ is cyclic, and its generator is e.*

☛**Proposition 1.2.1.** *[2]*

1. *Every cyclic group is abelian.*
2. *Every subgroup of a cyclic group is cyclic.*

1.3 Cosets and Lagrange's Theorem

✍**Definition 1.3.1.** *[2] Let $\mathcal{S}$ be a subgroup of $\mathcal{G}$ and let $x \in \mathcal{G}$.* ***The left coset*** *of $\mathcal{S}$ determined by x is the set:*

$$x\mathcal{S} = \{xt \mid t \in \mathcal{S}\}.$$

We call ***the right coset*** *of $\mathcal{S}$ the set:*

$$\mathcal{S}x = \{tx \mid t \in \mathcal{S}\}.$$

The element x is called a ***representative*** *of the coset $x\mathcal{S}$ or $\mathcal{S}x$.*

◆**Lemma 1.3.1.** *[2] Let $\mathcal{S}$ be a subgroup of $\mathcal{G}$, and let $x, y \in \mathcal{G}$. Then:*

1. $x \in x\mathcal{S}$.
2. $x\mathcal{S} = \mathcal{S} \Longleftrightarrow x \in \mathcal{S}$.
3. $(xy)\mathcal{S} = x(y\mathcal{S})$ *and* $\mathcal{S}(xy) = (\mathcal{S}x)y$.
4. $x\mathcal{S} = y\mathcal{S} \Longleftrightarrow x \in y\mathcal{S}$.
5. *Either* $x\mathcal{S} = y\mathcal{S}$ *or* $x\mathcal{S} \cap y\mathcal{S} = \emptyset$.
6. $x\mathcal{S} = y\mathcal{S} \Longleftrightarrow x^{-1}y \in \mathcal{S}$.
7. $|x\mathcal{S}| = |y\mathcal{S}|$.
8. $x\mathcal{S} = \mathcal{S}x \Longleftrightarrow \mathcal{S} = x\mathcal{S}x^{-1}$.
9. $x\mathcal{S} \leq \mathcal{G} \Longleftrightarrow x \in \mathcal{S}$.

Proof.

1. $x = xe \in x\mathcal{S}$.
2. Suppose that $x\mathcal{S} = \mathcal{S}$. Then $x = xe \in x\mathcal{S} = \mathcal{S}$.
   Now, assume that $x \in \mathcal{S}$ and show that $x\mathcal{S} \subseteq \mathcal{S}$ and $\mathcal{S} \subseteq x\mathcal{S}$. The first inclusion comes directly from the closure of $\mathcal{S}$.
   Let $y \in \mathcal{S}$, since $x \in \mathcal{S}$ and $y \in \mathcal{S}$, we know that $x^{-1}y \in \mathcal{S}$. Thus, $y = ey = (xx^{-1})y = x(x^{-1}y) \in x\mathcal{S}$.
3. This follows directly from $(xy)t = x(yt)$ and $t(xy) = (tx)y$.
4. If $x\mathcal{S} = y\mathcal{S}$, then $x = xe \in x\mathcal{S} = y\mathcal{S}$. Conversely, if $x \in y\mathcal{S}$ we have $x = yt$ where $t \in \mathcal{S}$, therefore $x\mathcal{S} = (yt)\mathcal{S} = y(t\mathcal{S}) = y\mathcal{S}$.
5. This property comes directly from property 4, for if there is an element z in $x\mathcal{S} \cap y\mathcal{S}$, then $z\mathcal{S} = x\mathcal{S}$ and $z\mathcal{S} = y\mathcal{S}$.
6. Notice that $x\mathcal{S} = y\mathcal{S}$ if and only if $\mathcal{S} = x^{-1}y\mathcal{S}$. The result now follows from property 2.
7. It suffices to define a one-to-one mapping from $x\mathcal{S}$ onto $y\mathcal{S}$. Obviously, the correspondence $xt \to yt$ maps $x\mathcal{S}$ onto $y\mathcal{S}$. That is one-to-one follows directly from the cancellation property.

8. Notice that $x\mathcal{S} = x\mathcal{S}$ if and only if $(x\mathcal{S})x^{-1} = (\mathcal{S}x)x^{-1} = \mathcal{S}(xx^{-1}) = \mathcal{S}$ that is, if and only if $x\mathcal{S}x^{-1} = \mathcal{S}$.

9. If $x\mathcal{S}$ is a subgroup, then it contains the identity e. Thus, $x\mathcal{S} \cap e\mathcal{S} \neq \emptyset$; and by property 5, we have $x\mathcal{S} = e\mathcal{S} = \mathcal{S}$. Thus, from property 2, we have $x \in \mathcal{S}$. Conversely, if $x \in \mathcal{S}$, then $x\mathcal{S} = \mathcal{S}$.

□

✍**Definition 1.3.2.** *[14] Let $\mathcal{S} \leq \mathcal{G}$, then **the index** of $\mathcal{S}$ in $\mathcal{G}$ is the number of right cosetes of $\mathcal{S}$ in $\mathcal{G}$.*

➢**Notation 3.** *We denote by $[\mathcal{G} : \mathcal{S}]$ the index of $\mathcal{S}$ in $\mathcal{G}$.*

▼**Theorem 1.3.1.** *[3](Lagrange's Theorem)*
Let $\mathcal{S}$ be a subgroup of a finite group $\mathcal{G}$. Then $|\mathcal{S}|$ divides $|\mathcal{G}|$.

Proof. Let $x_1\mathcal{S}, x_2\mathcal{S}, ...x_k\mathcal{S}$ denote the distinct left cosets of $\mathcal{S}$ in $\mathcal{G}$. Then for each $x \in \mathcal{G}$ we have

$$x\mathcal{S} = a_i\mathcal{S} \quad \text{for some } i.$$

Also, using property 1 of Lemma(1.3.1), $w \in x\mathcal{S}$. So each member of $\mathcal{G}$ belongs to one of the cosets $a_i\mathcal{S}$;

$$\mathcal{G} = x_1\mathcal{G} \cup x_2\mathcal{S} \cup ... \cup x_k\mathcal{S}.$$

by property 5 of Lemma(1.3.1), these cosets are pairwise disjoint, so

$$|\mathcal{G}| = |x_1\mathcal{S}| + |x_2\mathcal{S}| + ... + |x_k\mathcal{S}|.$$

Therefore, since $|x_i\mathcal{S}| = |\mathcal{S}|$ for each i, we have $\mathcal{G} = k|\mathcal{S}|$. □

➤**Corollary 1.3.1.** *[2]*

1. *If $\mathcal{S}$ is a subgroup of a finite group $\mathcal{G}$, then $[\mathcal{G} : \mathcal{S}] = |\mathcal{G}|/|\mathcal{S}|$.*

2. *In a finite group the order of each element of the group divides the order of the group.*

3. *A group $\mathcal{G}$ of prime order is cyclic.*

1.4 Normal Subgroups

✍**Definition 1.4.1.** *[14] A subgroup $\mathcal{N} \leq \mathcal{G}$ is called **a normal subgroup** of $\mathcal{G}$ if*

$$x\mathcal{N} = \mathcal{N}x \quad \textit{for all } x \in \mathcal{G}.$$

➢**Notation 4.** *We use the notation $\mathcal{N} \trianglelefteq \mathcal{G}$ to mean $\mathcal{N}$ is normal subgroup of a group $\mathcal{G}$.*

✎**Example 1.4.1.** ✏ *The trivial subgroup $\{e\}$ and $\mathcal{G}$ are normal subgroups*

$$\{e\} \trianglelefteq \mathcal{G}, \quad \mathcal{G} \trianglelefteq \mathcal{G}.$$

✏ *Every group is normal in itself.*

✏ *If $\mathcal{G}$ is abelian, then every subgroup of $\mathcal{G}$ is normal.*

✏ *Let $GL_n(\mathbb{K})$ be the group of invertible $n \times n$ matrices over $\mathbb{K}$ and $SL_n(\mathbb{K})$ the subgroup of matrices with determinant 1. Then:*

$$SL_n(\mathbb{K}) \trianglelefteq GL_n(\mathbb{K}).$$

▼**Theorem 1.4.1.** *[2] A subgroup $\mathcal{N}$ of $\mathcal{G}$ is normal if and only if $x\mathcal{N}x^{-1} \subseteq \mathcal{N}$ for all $x \in \mathcal{G}$.*

Proof. ($\Longrightarrow$) Suppose $\mathcal{N} \trianglelefteq \mathcal{G}$. By definition, $x\mathcal{N}x^{-1} = \mathcal{N}$. Hence $x\mathcal{N}x^{-1} \subseteq \mathcal{N}$.
($\Longleftarrow$) Suppose $x\mathcal{N}x^{-1} \subseteq \mathcal{N}$ for all $x \in \mathcal{G}$. Take $y \in \mathcal{N}$, then we have:

$$x^{-1}\mathcal{N}x \subseteq \mathcal{N}$$

In particular $x^{-1}yx \in \mathcal{N}$. Let $t = x^{-1}yx$. Then $t \in \mathcal{N}$;

$$y = xtx^{-1} \in x\mathcal{N}x^{-1} \Longrightarrow \mathcal{N} \subseteq x\mathcal{N}x^{-1}$$

Thus

$$x\mathcal{N}x^{-1} = \mathcal{N}.$$

□

1.5 Quotient Groups

▼**Theorem 1.5.1.** *[1] Let $\mathcal{N}$ be a normal subgroup of $\mathcal{G}$. then the set of left cosets of $\mathcal{N}$ in $\mathcal{G}$ forms a group under the operation*

$$(x\mathcal{N})(y\mathcal{N}) = xy\mathcal{N}.$$

The resulting group is called ***the quotient group*** *of $\mathcal{G}$ by $\mathcal{N}$*

$$\{x\mathcal{N} \mid x \in \mathcal{G}\}.$$

➢**Notation 5.** *The notation $\mathcal{G}/\mathcal{N}$ denoted the quotient group.*

❖**Properties 1.5.1.** *[1]*

(i) *If $\mathcal{G}$ is a finite group and $\mathcal{N} \trianglelefteq \mathcal{G}$ the order of the quotient group is $|\mathcal{G}/\mathcal{N}| = \dfrac{|\mathcal{G}|}{|\mathcal{N}|}$.*

(ii) *If $\mathcal{G}$ is an abelian group then $\mathcal{G}/\mathcal{N}$ is abelian.*

(iii) *If $\mathcal{G}$ is cyclic then $\mathcal{G}/\mathcal{N}$ is cyclic.*

1.6 Group Homomorphism and Isomorphism Theorem

✍**Definition 1.6.1.** *[1] Let $(\mathcal{G}, \cdot)$ and $(\mathcal{H}, *)$ be two groups. A map $\varphi : \mathcal{G} \to \mathcal{H}$ is called a* ***homomorphism*** *if:*

$$\varphi(x \cdot y) = \varphi(x) * \varphi(y).$$

✍**Definition 1.6.2.** *[5] Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a homomorphism:*

- ***The kernel*** *of φ is the set of all elements of $\mathcal{G}$ that map to the identity of $\mathcal{H}$:*

$$\ker \varphi = \{x \in \mathcal{G} \mid \varphi(x) = e_{\mathcal{H}}\}.$$

- ***The image*** *of φ is the set of all elements of $\mathcal{H}$ obtained as $\varphi(x)$ for some $x \in \mathcal{G}$:*

$$\text{Im } \varphi = \{y \in \mathcal{H} \mid \exists x \in \mathcal{G} : \varphi(x) = y\}.$$

A homomorphism φ can have the following additional properties:

(a) *If φ is* ***injective*** *(one-to-one), it is called a* ***monomorphism****.*

(b) *If φ is* ***surjective*** *(onto), it is called an* ***epimorphism****.*

(c) *If φ is* ***bijective*** *(injective and surjective), it is called* ***isomorphism****.*

☛**Proposition 1.6.1.** *[7] Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a homomorphism:*

1. *$\varphi(e_{\mathcal{G}}) = e_{\mathcal{H}}$ (e is the identity element).*
2. *For all $x \in \mathcal{G} : \varphi(x^{-1}) = (\varphi(x))^{-1}$.*

Proof. 1. We have

$$\begin{aligned} e_{\mathcal{H}} \cdot \varphi(e_{\mathcal{G}}) &= \varphi(e_{\mathcal{G}}) \\ &= \varphi(e_{\mathcal{G}} \cdot e_{\mathcal{G}}) \\ &= \varphi(e_{\mathcal{G}}) \cdot \varphi(e_{\mathcal{G}}) \end{aligned}$$

By the cancellation law, we obtain $\varphi(e_{\mathcal{G}}) = e_{\mathcal{H}}$.

2. Using (1) we have

$$\varphi(x^{-1})\varphi(x) = \varphi(x^{-1}x) = \varphi(e_{\mathcal{G}}) = e_{\mathcal{H}}$$

and

$$\varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(e_{\mathcal{G}}) = e_{\mathcal{H}}.$$

Hence $\varphi(x^{-1})$ is the inverse of $\varphi(x)$.

□

◆**Lemma 1.6.1.** *[1] Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a homomorphism:*

1. *$\ker \varphi \trianglelefteq \mathcal{G}$.*
2. *$\text{Im } \varphi \leq \mathcal{H}$.*

➤**Corollary 1.6.1.** *[2] Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a homomorphism:*

(i) *If $\mathcal{S} \leq \mathcal{G} \Longrightarrow \varphi(\mathcal{S}) \leq \mathcal{G}$.*

(ii) *If $\mathcal{S} \leq \mathcal{H} \Longrightarrow \varphi^{-1}(\mathcal{S}) \leq \mathcal{G}$.*

(iii) *If $\mathcal{N} \trianglelefteq \mathcal{H} \Longrightarrow \varphi^{-1}(\mathcal{N}) \leq \mathcal{G}$.*

1.6.1 The Isomorphism Theorems

Let $\mathcal{N}$ be a normal subgroup of $\mathcal{G}$ and consider the homomorphism

$$\varphi : \mathcal{G} \to \mathcal{G}/\mathcal{N}, \quad x \longmapsto x\mathcal{N}.$$

Let

$$\mathbf{P}_{\mathcal{N}}(\mathcal{G}) = \{\mathcal{H} : \mathcal{N} \leq \mathcal{H} \leq \mathcal{G}\}$$

and

$$\mathbf{P}(\mathcal{G}/\mathcal{N}) = \{\mathcal{S} : \mathcal{S} \leq \mathcal{G}/\mathcal{N}\}.$$

We define the map

$$\Phi : \mathbf{P}_{\mathcal{N}} \to \mathbf{P}(\mathcal{G}/\mathcal{N})$$
$$\Phi(\mathcal{H}) = \varphi(\mathcal{H}) = \mathcal{H}/\mathcal{N}.$$

▼**Theorem 1.6.1.** *[3] Φ is a bijection and $\mathcal{H} \trianglelefteq \mathcal{G}$ if and only if $\Phi(\mathcal{H}) \trianglelefteq \mathcal{G}/\mathcal{N}$.*

Proof. Let $\mathcal{N} \leq \mathcal{H}, \mathcal{B} \leq \mathcal{G}$ and suppose $\Phi(\mathcal{H}) = \mathcal{H}/\mathcal{N}$ is equal to $\Phi(\mathcal{B}) = \mathcal{B}/\mathcal{N}$. Then

$$\mathcal{H} = \bigcup_{x\mathcal{N}\in\mathcal{H}/\mathcal{N}} x\mathcal{N} = \bigcup_{x\mathcal{N}\in\mathcal{B}/\mathcal{N}} = \mathcal{B}.$$

For a subgroup $\mathcal{S}$ of $\mathcal{G}/\mathcal{N}$. Then by Corollary 1.6.1, $\varphi^{-1}(\mathcal{S})$ is subgroup of $\mathcal{G}$ and as φ is surjective, we have

$$\Phi(\varphi^{-1}(\mathcal{S})) = \varphi(\varphi^{-1}(\mathcal{S})) = \mathcal{S}.$$

□

▼**Theorem 1.6.2.** *[3](The First Isomorphism Theorem)*
Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a group homomorphism. Then $\mathrm{Im}\ \varphi$ is a subgroup of $\mathcal{H}$, $\ker\varphi$ is a normal subgroup of $\mathcal{G}$ and there is an isomorphism from the quotient group $\mathcal{G}/\ker\varphi$ to $\mathrm{Im}\ \varphi$.

Proof. Since $\mathrm{Im}\ \varphi \leq \mathcal{H}$ and $\ker\varphi \trianglelefteq \mathcal{G}$, we define the map:

$$\Psi : \mathcal{G}/\ker\varphi \to \mathrm{Im}\ \varphi, \quad \Psi(x\ker\varphi) = \varphi(x).$$

Suppose that $x\ker\varphi = y\ker\varphi$, then

$$x^{-1}y \in \ker\varphi \Longrightarrow \varphi(x^{-1}y) = 1.$$

By the homomorphism property we get:

$$\varphi(x^{-1})\varphi(y) = 1$$
$$\Longrightarrow \varphi(x)^{-1}\varphi(y) = 1$$
$$\Longrightarrow \varphi(x) = \varphi(y).$$

So, the map Ψ is well defined.
If we take $y \in \mathrm{Im}\ \varphi \Longrightarrow y = \varphi(x), x \in \mathcal{G}$, then:

$$\Psi(x\ker\varphi) = \varphi(x) = y$$

Hence, Ψ is surjective.
Now, let $\Psi(x\ker\varphi)=\varphi(y\ker\varphi)$, this is equivalent to

$$\varphi(x)=\varphi(y)$$
$$\Longleftrightarrow \varphi(x^{-1}y)=1$$
$$\Longleftrightarrow x^{-1}y\in\ker\varphi$$
$$\Longleftrightarrow x\ker\varphi=y\ker\varphi.$$

Therefore Ψ is injective. Since the quotient group $x\ker\varphi y\ker\varphi=xy\ker\varphi$, we obtain:

$$\Psi(xy\ker\varphi)=\varphi(xy)=\varphi(x)\varphi(y)=\Psi(x\ker\varphi)\Psi(y\ker\varphi).$$

Consequently, Ψ is a group homomorphism. □

➢**Notation 6.** *We denote by $\mathcal{G}/\ker\varphi\cong \mathrm{Im}\ \varphi$ the isomorphic between $\mathcal{G}/\ker\varphi$ and $\mathrm{Im}\ \varphi$.*

▼**Theorem 1.6.3.** *[3](The Second Isomorphism Theorem)*
Let $\mathcal{S}$ be a subgroup of $\mathcal{G}$ and $\mathcal{N}$ be a normal subgroup of $\mathcal{G}$. Then

$$\mathcal{S}\mathcal{N}\leq\mathcal{G},\quad \mathcal{S}\cap\mathcal{N}\trianglelefteq\mathcal{S}\quad \textit{and}\quad \mathcal{S}/(\mathcal{S}\cap\mathcal{N})\cong\mathcal{S}\mathcal{N}/\mathcal{N}.$$

Proof. Consider the homomorphism

$$\varphi:\mathcal{G}\to\mathcal{G}/\mathcal{N}$$
$$\varphi(x)=x\mathcal{N}.$$

Let ψ be the restriction of φ on $\mathcal{S}$:

$$\psi:\mathcal{S}\to\mathcal{G}/\mathcal{N}$$
$$\psi(t)=t\mathcal{N}\quad\text{(which is a homomorphism).}$$

We have:

$$\mathrm{Im}\ \psi=\{t\mathcal{N}\mid t\in\mathcal{S}\}$$

since every coset $t\mathcal{N}$ corresponds to an element of $\mathcal{S}\mathcal{N}$, then

$$\mathrm{Im}\ \psi=\mathcal{S}\mathcal{N}/\mathcal{N}.$$

And:

$$\ker\psi=\{t\in\mathcal{S}\mid\psi(t)=\mathcal{N}\}$$

with

$$\psi(t)=\mathcal{N}\Longleftrightarrow t\mathcal{N}=\mathcal{N},\ \text{for } t\in\mathcal{N}$$

we obtain:

$$\ker\psi=\mathcal{S}\cap\mathcal{N}.$$

By The First Isomorphism Theorem we found:

$$\mathcal{S}/(\mathcal{S}\cap\mathcal{N})\cong\mathcal{S}\mathcal{N}/\mathcal{N}.$$

□

▼**Theorem 1.6.4.** *[3](The Third Isomorphism Theorem)*
Let $\mathcal{N}$ and $\mathcal{N}'$ be two normal subgroups of $\mathcal{G}$ with $' \leq \mathcal{N}$. Then:

$$\mathcal{N}/\mathcal{N}' \trianglelefteq \mathcal{G}/\mathcal{N}' \quad \text{and} \quad (\mathcal{G}/\mathcal{N}')/(\mathcal{N}/\mathcal{N}') \cong \mathcal{G}/\mathcal{N}.$$

Proof. Similarly, we define the same homomorphism

$$\varphi : \mathcal{G}/\mathcal{N}' \to \mathcal{G}/\mathcal{N}$$

$$\varphi(x\mathcal{N}') = x\mathcal{N}, \quad x \in \mathcal{G}.$$

φ is well defined.
φ is homomorphism since;

$$\varphi((x\mathcal{N}')(y\mathcal{N}')) = \varphi(xy\mathcal{N}') = xy\mathcal{N} = (x\mathcal{N})(y\mathcal{N}) = \varphi(x\mathcal{N}')\varphi(y\mathcal{N}').$$

For $x\mathcal{N} \in \mathcal{G}/\mathcal{N}$,

$$x\mathcal{N} = \text{Im } \varphi(x\mathcal{N}') \Longrightarrow \text{Im } \varphi = \mathcal{G}/\mathcal{N}.$$

And with:

$$\begin{aligned} \ker\varphi &= \{x\mathcal{N}' \in \mathcal{G}/\mathcal{N}' : \varphi(x\mathcal{N}') = \mathcal{N}\} \\ &\Longleftrightarrow \{x\mathcal{N}' \in \mathcal{G}/\mathcal{N}' : x\mathcal{N} = \mathcal{N}\} \\ &\Longleftrightarrow x \in \mathcal{N}. \end{aligned}$$

Therefore

$$\ker \varphi = \{x\mathcal{N}' : x \in \mathcal{N}\} = \mathcal{N}/\mathcal{N}'.$$

Using The First Isomorphism Theorem we get:

$$(\mathcal{G}/\mathcal{N}')/(\mathcal{N}/\mathcal{N}') \cong \mathcal{G}/\mathcal{N}.$$

□

1.7 Conjugacy Classes

✍**Definition 1.7.1.** *[7] Let $\mathcal{G}$ be a group and let $xy \in \mathcal{G}$. We say that y is **conjugate** to x in $\mathcal{G}$ if there exists $z \in \mathcal{G}$ such that:*

$$y = zxz^{-1}.$$

✯**Remark 1.7.1.** *[5] The relation of conjugacy on $\mathcal{G}$ is an equivalence relation since it satisfies the reflexivity, symmetricity and the transitivity.*

✍**Definition 1.7.2.** *[8] Let $\mathcal{G}$ be a group and $x \in \mathcal{G}$. **The conjugacy classes** of x in $\mathcal{G}$, denoted by $\text{Cl}(x)$ is the set of all elements in $\mathcal{G}$ conjugate to x:*

$$\text{Cl}(x) = \{yxy^{-1} \mid y \in \mathcal{G}\}$$

☛**Proposition 1.7.1.** *[14] A subgroup $\mathcal{N}$ of $\mathcal{G}$ is normal if and only if $\mathcal{N}$ is an union of conjugacy classes of $\mathcal{G}$.*

✍**Definition 1.7.3.** *[14] Let $\mathcal{G}$ be a group.* ***The centralizer*** *of x in $\mathcal{G}$, denoted by $\mathrm{C}_{\mathcal{G}}(x)$ is the set*

$$\mathrm{C}_{\mathcal{G}}(x) = \{y \in \mathcal{G} \mid yx = xy\}.$$

✯**Remark 1.7.2.** *[1] For every $x \in \mathcal{G}$, the centralizer $\mathrm{C}_{\mathcal{G}}(x)$ is a subgroup of $\mathcal{G}$.*

✍**Definition 1.7.4.** *[7]* ***The center*** *of a group $\mathcal{G}$ is the set of elements that commute with every element in $\mathcal{G}$:*

$$\mathrm{Z}(\mathcal{G}) = \{z \in \mathcal{G} \mid zx = xz \quad \textit{for all } x \in \mathcal{G}\}.$$

▼**Theorem 1.7.1.** *[8] Let $\mathcal{G}$ be a finite group let $x \in \mathcal{G}$. The number of elements in the conjugacy class of x is equal to the index of the centralizer of x in $\mathcal{G}$.*

$$|\mathrm{Cl}(x)| = [\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x)].$$

1.8 Class Equation of Finite Group

▼**Theorem 1.8.1.** *[7] Let $\mathcal{G}$ be a finite group. Let $\mathrm{Z}(\mathcal{G})$ be the center of $\mathcal{G}$ and $x_1, x_2, ..., x_n$ be a set of representatives of distinct conjugacy classes of $\mathcal{G}$, that are not contained in $\mathrm{Z}(\mathcal{G})$, then:*

$$|\mathcal{G}| = |\mathrm{Z}(\mathcal{G})| + \sum_{k=1}^{n} [\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_k)].$$

Proof. Since conjugacy class is an equivalence relation on $\mathcal{G}$, then:

$$\mathcal{G} = \bigcup_{i=1}^{n} \mathrm{Cl}(x_i).$$

An element x is in $\mathrm{Z}(\mathcal{G})$ if and only if its conjugacy class has size 1. Each element of $\mathrm{Z}(\mathcal{G})$ forms its own singleton conjugacy class:

$$|\mathcal{G}| = |\mathrm{Z}(\mathcal{G})| + \sum_{|\mathrm{Cl}(x_k)|>1} |\mathrm{Cl}(x_k)|.$$

By the Orbit-Stabilizer theorem:

$$|\mathrm{Cl}(x_k)| = [\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_k)].$$

Therefore

$$|\mathcal{G}| = |\mathrm{Z}(\mathcal{G})| + \sum_{k=1}^{n} [\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_k)].$$

□

▼**Theorem 1.8.2.** *[5] If $\mathcal{S}$ is a subgroup of $\mathcal{G}$ and $x \in \mathcal{G}$, then the conjugate subgroup $x\mathcal{S}x^{-1}$ has the same order as $\mathcal{S}$ and is isomorphic to $\mathcal{S}$.*

1.9 $p-$Groups and Sylow's Theorems

✍**Definition 1.9.1.** *[6] Let p be a prime number. A group $\mathcal{G}$ is said to be a $p-$**group** if every element in $\mathcal{G}$ has order a power of p.*

$$\text{For all } x \in \mathcal{G}, \ \exists k \geq 0 : ord(x) = p^k.$$

✍**Definition 1.9.2.** *[8] Let $\mathcal{G}$ be a finite group of order $p^n m$, where p is prime and $gcd(p, m) = 1$. A subgroup $\mathcal{S}$ of $\mathcal{G}$ is said to be a **Sylow** $p-$**group** if $|\mathcal{S}| = p^n$.*

▼**Theorem 1.9.1.** *[8](Cauchy Theorem)*
If $\mathcal{G}$ is a finite group and p is prime such that $p \mid |\mathcal{G}|$, then $\mathcal{G}$ contains an element of order p.

Proof. Suppose that $|\mathcal{G}| \geq 2$, consider the class equation

$$|\mathcal{G}| = |\mathrm{Z}(\mathcal{G})| + \sum_{i=1}^{n} \underbrace{[\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_i)]}_{\text{each } \geq 2}.$$

Assume that some indices $|\mathrm{C}_{\mathcal{G}}(x_i)|$ are divisible by p, then, since $|\mathrm{C}_{\mathcal{G}}(x)| < |\mathcal{G}|$, using the induction hypothesis, we conclude that $\mathrm{C}_{\mathcal{G}}(x_i)$ contains an element of order p.
Assume that none of $|\mathrm{C}_{\mathcal{G}}(x_i)|$ are divisible by p, but then as $|\mathcal{G}| = [\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_i)] \cdot |\mathrm{C}_{\mathcal{G}}(x_i)|$, all the indices $[\mathrm{C}_{\mathcal{G}}(x_i)]$ are divisible by p and the class equation implies that $\mathrm{Z}(\mathcal{G})|$ is divisible by p.
But $\mathrm{Z}(\mathcal{G})$ is abelian, so it contains an element of order p. □

✯**Remark 1.9.1.** *[6] If $\mathcal{G}$ is a finite abelian group and p is prime such that $p \mid |\mathcal{G}|$, then there exists $x \in \mathcal{G} : |x| = p$.*

✍**Definition 1.9.3.** *[2] Let $\mathcal{S}$ be a subgroup of $\mathcal{G}$. **The normalizer** of $\mathcal{S}$ in $\mathcal{G}$, denoted by $\mathrm{N}_{\mathcal{G}}(\mathcal{S})$ is the set*

$$\mathrm{N}_{\mathcal{G}}(\mathcal{S}) = \{x \in \mathcal{G} \mid x\mathcal{S}x^{-1} = \mathcal{S}\}.$$

▼**Theorem 1.9.2.** *[6](Sylow's First Theorem)*
Let $\mathcal{G}$ be a finite group of order $|\mathcal{G}| = p^n m$ with $gcd(p, m) = 1$. Then $\mathcal{G}$ contains at least one Sylow $p-$subgroup of order p^n.

Proof. If $|\mathcal{G}| = 1$ then $\{1\}$ is the Sylow $p-$subgroup for any prime p thus the Sylow $p-$subgroups exist.
If $|\mathcal{G}| \geq 2$, suppose that $|\mathcal{G}| = p^n m$. Assume that $n \geq 1$, we use the class equation

$$|\mathcal{G}| = |\mathrm{Z}(\mathcal{G})| + \sum_{i=1}^{n} \underbrace{[\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_i)]}_{\text{each } \geq 2}.$$

Assume that some of $[\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_i)]$ are divisible by p. Notice that $|\mathcal{G}| = [\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_i)] \cdot |\mathrm{C}_{\mathcal{G}}(x_i)|$ and as p does not divide $[\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x_i)]$, whereas p^n divides $|\mathcal{G}|$, so p^n divides $|\mathrm{C}_{\mathcal{G}}(x_i)|$.
Since $|\mathrm{C}_{\mathcal{G}}(x_i)| < |\mathcal{G}|$, and by induction hypothesis, $\mathrm{C}_{\mathcal{G}}(x_i)$ contains a Sylow $p-$subgroup that is of order p^n.
Using Cauchy' Theorem(1.9.1) we know that $\mathrm{Z}(\mathcal{G})$ has a normal subgroup $\mathcal{N}$ of order p in $\mathcal{G}$, since $\mathcal{N}x = x\mathcal{N}$. By induction hypothesis, $\mathcal{G}/\mathcal{N}$ contains a Sylow $p-$subgroup that is a subgroup of order p^{n-1}.
By Theorem(1.6.1), this subgroup is of the form $\mathcal{P}/\mathcal{N}$ where $\mathcal{N} \leq \mathcal{P} \leq \mathcal{G}$. Notice that $|\mathcal{P}| = |\mathcal{N}| \cdot |\mathcal{P}/\mathcal{N}| = p \cdot p^{n-1} = p^n$, so $\mathcal{P}$ is a Sylow $p-$subgroup of $\mathcal{G}$. □

◆**Lemma 1.9.1.** *[3] Let $\mathcal{S}$ be a $p-$power order subgroup of $\mathcal{G}$, and $\mathcal{P}$ be a Sylow $p-$subgroup of $\mathcal{G}$. Then:*

$$\mathcal{S} \leq \mathcal{P}^x, \quad \textit{for some } x \in \mathcal{G}.$$

▼**Theorem 1.9.3.** *[8](Sylow's Second Theorem)*
Any two Sylow's $p-$subgroups of a finite group $\mathcal{G}$ are conjugate.

Proof. Let $\mathcal{P}$ and $\mathcal{R}$ be Sylow subgroups of $\mathcal{G}$. By Lemma(1.9.1) we know that

$$\mathcal{R} \subseteq \mathcal{P}^x, \quad \text{for some } x \in \mathcal{G}.$$

But those two subgroups have the same order. Hence we have $\mathcal{R} = \mathcal{P}^x$. □

▼**Theorem 1.9.4.** *[3](Sylow's Third Theorem)*
Let $\mathcal{G}$ be a finite group, we denote by $\mathfrak{n}(p)$ the number of Sylow $p-$subgroups of $\mathcal{G}$. Then:

- *$\mathfrak{n}(p)$ divides $|\mathcal{G}|$.*
- *$\mathfrak{n}(p) = 1 + pr$.*

Proof. Let $\mathcal{P}$ be a Sylow $p-$subgroup of $\mathcal{G}$. Since the Sylow $p-$subgroups form a single conjugacy class

$$\{\mathcal{P}^x : x \in \mathcal{G}\},$$

so from remark we get

$$\mathfrak{n}(p) = [\mathcal{G} : \mathrm{N}_{\mathcal{G}}(\mathcal{P})].$$

In particular, $\mathfrak{n}(p)$ divides $|\mathcal{G}|$.
Let $E = \mathrm{N}_{\mathcal{G}}(\mathcal{P})$ and let Ω be the collection of all the right cosets of $\mathcal{G}$ that we consider as a $\mathcal{P}-$set. We write Ω as a disjoint union of $\mathcal{P}-$orbits;

$$\Omega = E\alpha_1 * \mathcal{P} \cup E\alpha_2 * \mathcal{P} \cup \cdots \cup E\alpha_m * \mathcal{P},$$

where the first orbit $E\alpha_1 * \mathcal{P}$ is containing the coset $E \cdot 1 = E$ then $\alpha_1 = 1$. From this we get that

$$\begin{aligned} \mathfrak{n}(p) &= |E\alpha_1 * \mathcal{P}| + |E\alpha_2 * \mathcal{P}| + \cdots + |E\alpha_m * \mathcal{P}| \\ &= [\mathcal{P} : \mathcal{P} \cap E^{\alpha_1}] + [\mathcal{P} : \mathcal{P} \cap E^{\alpha_2}] + \cdots + [\mathcal{P} : \mathcal{P} \cap E^{\alpha_m}]. \end{aligned}$$

Notice that $\mathcal{P} \cap E^{\alpha_i} = \mathcal{P}$ if and only if $\mathcal{P} \leq E^{\alpha_i}$ then $\mathcal{P}^{\alpha_i^{-1}} \leq E$. However, this happens if and only if $\mathcal{P}^{\alpha_i^{-1}} = \mathcal{P}$ and that happens if and only if $\alpha_i \in \mathrm{N}_{\mathcal{G}}(\mathcal{P})$. But then $E\alpha = E\alpha_i \in E\alpha_i * \mathcal{P}$ and as the only orbit containing E is $E\alpha_1 * \mathcal{P}$, it follows that $i = 1$. It follows that $[\mathcal{P} : \mathcal{P} \cap E^{\alpha_i}]$ is divisible by p for $i = 2, ..., m$ and that $[\mathcal{P} : \mathcal{P} \cap E^{\alpha_1}] = 1$. Hence $\mathfrak{n}(p) = 1 + pr$. □

1.10 Examples of Finite Groups

1.10.1 Cyclic Groups

✍**Definition 1.10.1.** *[14] A group $\mathcal{G}$ is generated by an element $x \in \mathcal{G}$ if for all $y \in \mathcal{G}, y = x^n$ for some $n \in \mathbb{Z}$. We denote $\mathcal{G} := \langle x \rangle$ and say that $\mathcal{G}$ is* ***the cyclic group*** *generated by x.*

✍**Definition 1.10.2.** *[1] If $x^n = 1$ for some $n \geq 1 \in \mathbb{Z}$, then $\langle x \rangle$ is finite. If $k \in \mathbb{Z}$ is the least possible integer ≥ 1 such that $x^k = 1$, then $\langle x \rangle = \{1, x, ..., x^{k-1}\}$, and k is the order of $\langle x \rangle$.*

➢**Notation 7.** *We denote the cyclic group of order n as $\mathfrak{C}_n$.*

☛**Proposition 1.10.1.** *[1] Every cyclic group is abelian.*

Proof. Let $\mathcal{G} = \langle x \rangle$. Then for any $x^m, x^n \in \mathcal{G}$:

$$x^m x^n = x^{m+n} = x^{n+m} = x^n x^m.$$

Hence $\mathcal{G}$ is abelian. □

✎**Example 1.10.1.**

- ✏ *Let $\mathcal{G} = \mathbb{Z}$ under $+$, then $\mathbb{Z} = \langle 1 \rangle$. Thus $\mathbb{Z}$ is a cyclic group.*
- ✏ *$\mathcal{G} = \mathbb{Z}_n$ under $+$ is a cyclic group with $\mathbb{Z}_n = \langle 1 \rangle$. We denote by $\mathbb{Z}/n\mathbb{Z}$ the integers mod n.*

❖**Properties 1.10.1.** *[2]*

(i) *If $\mathcal{G}$ is finite of order n, then:*

$$\mathcal{G} = \{e, x, x^2, ..., x^{n-1}\}, \quad x^n = e.$$

(ii) *Every cyclic group is abelian.*

(iii) *Let $\mathcal{G} = \langle x \rangle$ and $\mathcal{S} \leq \mathcal{G}$. For each $t \in \mathcal{G}$, the left coset of $\mathcal{S}$ is:*

$$t\mathcal{S} = \{ts \mid s \in \mathcal{S}\}.$$

(iv) *Every subgroup $\mathcal{S} \leq \mathcal{G}$ is cyclic:*

$$\mathcal{S} = \langle x^m \rangle \quad \text{for } m \mid n.$$

(v) *The number of cosets:*

$$[\mathcal{G} : \mathcal{S}] = \frac{|\mathcal{G}|}{|\mathcal{S}|}.$$

(vi) *Every subgroup of a cyclic group is normal.*

(vii) *Let $\mathcal{G} = \langle x \rangle$, and $\mathcal{N} = \langle x^m \rangle$. The quotient group:*

$$\mathcal{G}/\mathcal{N} = \{x^k\mathcal{N} \mid k = 0, 1, 2, ...m - 1\}$$

is a cyclic.

(viii) *Since a cyclic group is abelian:*

$$xyx^{-1} = y \Longrightarrow \mathrm{Cl}(x) = \{x\}.$$

(ix) *In cyclic groups $\mathrm{Z}(\mathcal{G}) = \mathcal{G}$ and every element is central, which gives all conjugacy classes are trivial.*

(x) *Every cyclic group of order p^n is a $p-$group.*

(xi) *Subgroups of cyclic $p-$groups are unique for each divisor.*

(xii) *For a cyclic group $\mathcal{G}$, there is exactly one subgroup of order p^k for $k \leq n$ so $\mathfrak{n}(\mathfrak{p}) = 1$.*

➢**Notation 8.** *We use the notation $\mathfrak{C}_n$ to denote the cyclic groups.*

1.10.2 Dihedral Groups

✍**Definition 1.10.3.** *[1] **The dihedral groups** denoted by $\mathfrak{D}_{2n}$ is the group of symmetries of a regular $n-$gon, it consists of $n-$rotations and $n-$reflections. Then $|\mathfrak{D}_{2n}| = 2n$.*

$$\mathfrak{D}_{2n} = \langle r, s \mid r^n = e, s^2 = e, srs = r^{-1} \rangle,$$

where r is the rotation and s is the reflection.

❖**Properties 1.10.2.** *[5]*

(i) *The elements of $\mathfrak{D}_{2n}$ are of the form:*

$$\mathfrak{D}_{2n} = \{e, r, r^2, ..., r^{n-1}, s, sr, sr^2, ..., sr^{n-1}\}.$$

(ii) *Let $\mathcal{S} \leq \mathfrak{D}_{2n}$, then the rotation $\langle r \rangle$ is a subgroup.*

(iii) *Since $[\mathfrak{D}_{2n} : \langle r \rangle] = 2$, then the cosets are*

$$\langle r \rangle \quad \text{and} \quad s\langle r \rangle.$$

(iv) *The conjugacy classes in $\mathfrak{D}_{2n}$ are as follows:*

(a) *If n is odd,*
- *The identity element is $\{1\}$.*
- *$(n-1)/2$ conjugacy classes of size 2: $\{r^k, r^{-k}\}, k = 1, 2, ..., (n-1)/2$.*
- *all the reflections: $\{r^i s : 0 \leq i \leq n-1\}$.*

(b) *If n is even,*
- *Two conjugacy classes of size 1: $\{1\}, \{r^{\frac{n}{2}}\}$.*
- *$n/2 - 1$ conjugacy classes of size 2: $\{r^k, r^{-k}\}, k = 1, 2, ..., (n/2) - 1$.*
- *The reflections splits into two conjugacy classes:*

$$\{r^{2i}s : 0 \leq i \leq (n/2) - 1\} \quad \text{and} \quad \{r^{2i+1}s : 0 \leq i \leq (n/2) - 1\}.$$

(v) *$\langle r \rangle$ is always normal, and any subgroups of $\langle r \rangle$ are also normal in $\mathfrak{D}_{2n}$.*

(vi) *If n is odd, the only normal subgroups are $\{e\}, \langle r^d \rangle$ and $\mathfrak{D}_{2n}$.*

(vii) *If n is even, there are other normal subgroups.*

(viii) *The group $\mathfrak{D}_{2n}$ is not abelian group for $n \geq 3$, so the conjugacy classes are nontrivial.*

(ix) *Since $|\mathfrak{D}_{2n}| = 2n$. Then $\mathfrak{D}_{2n}$ is a $p-$group only if*

$$2n = p^k \Longrightarrow \begin{cases} p = 2 \\ n = 2^{k-1} \end{cases}$$

(x) *If n is odd, Sylow $2-$subgroups have order 2 and generated by reflections.*

(xi) *If n is even, there exist larger $2-$subgroups.*

✎**Example 1.10.2.** ✏ *For $n = 3$. The dihedral group is:*

$$\mathfrak{D}_3 = \langle r, s \mid r^3 = 1, s^2 = 1, rsr = s \rangle.$$

The identity forms its class:

$$\{1\}.$$

Conjugacy class of size 2:

$$\{r, r^2\}.$$

Reflections:

$$\{s, rs, r^2 s\}.$$

Thus, we get three conjugacy classes.

✏ *For $n = 6$. Conjugacy classes of size 1:*

$$\{1\}, \quad \{r^3\}.$$

Conjugacy classes of size 2:

$$\{r, r^5\}, \quad \{r^2, r^4\}.$$

Reflections:

$$\{s, r^2 s, r^4 s\} \quad \text{and} \quad \{rs, r^3 s, r^5 s\}.$$

1.10.3 Symmetric Groups

✍**Definition 1.10.4.** *[7]* ***The symmetric group*** *denoted by $\mathfrak{S}_n$ is the group of permutations on a set with n elements*

$$\{1, 2, ..., n\}.$$

There are $n!$ possible permutations, $|\mathfrak{S}_n| = n!$.

✍**Definition 1.10.5.** *[8]* ***A permutation*** *of a set with n elements is a bijection from the set onto itself:*

$$\alpha : \begin{matrix} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{matrix} \qquad \beta : \begin{matrix} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{matrix}$$

$$\alpha \circ \beta : \begin{matrix} 1 \to 2 \to 1 \\ 2 \to 3 \to 3 \\ 3 \to 1 \to 2 \end{matrix}$$

the operation between permutations is composition.

✍**Definition 1.10.6.** *[3] We can represent a permutation $\sigma \in \mathfrak{S}_n$ by a matrix:*

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

✎**Example 1.10.3.** ✏ *Consider the symmetric group* $\mathfrak{S}_3$. *We have* $|\mathfrak{S}_3| = 3! = 6$ *with the following permutations:*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \kappa = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

$$\beta \circ \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \epsilon$$

☛**Proposition 1.10.2.** *[5] Every permutation* $\sigma \in \mathfrak{S}_n$ *can be written as a product of disjoint cycles.*

✎**Example 1.10.4.** ✏ *Consider the symmetric group* $\mathfrak{S}_5$*:*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix} = (134)(25).$$

The permutations (134) *and* (25) *are* ***cycles****.*

✮**Remarks 1.10.1.** *[5]*

1. *A cycle of length* 3 *(*(abc)*) is called "*$3-$*cycle".*
2. *A cycle of length* 2 *(*(ab)*) is called "*$2-$*cycle" or a "transposition".*
3. *The order of cycles does not matter.*

❖**Properties 1.10.3.** *[8]*

(i) *Let* $\mathcal{S} \leq \mathfrak{S}_n$*, for* $\sigma \in \mathfrak{S}_n$*:*

$$\sigma\mathcal{S} = \{\sigma s \mid s \in \mathcal{S}\}.$$

Let $\mathfrak{A}_n =\{$ *even permutation*$\}$*, then* $\mathfrak{A}_n \leq \mathfrak{S}_n$*.*

(ii) *Two permutations in* $\mathfrak{S}_n$ *are conjugate if and only if they have the same cycle type.*

(iii) $\mathfrak{S}_n$ *is a* $p-$*group if and only if* $n \leq 2$

1.10.4 Quaternion Groups

✍**Definition 1.10.7.** *[1]* ***The quaternion group*** *is a finite non-abelian group defined by:*

$$\mathfrak{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

with multiplication, satisfying:

$$i^2 = j^2 = k^2 = -1$$

with $|\mathfrak{Q}_8| = 8$*.*

✍**Definition 1.10.8.** *[2] We can present* $\mathfrak{Q}_8$ *by the following presentation:*

$$\mathfrak{Q}_8 = \langle i, j \mid i^4 = 1, i^2 = j^2, jij^{-1} = i^{-1} \rangle.$$

❖**Properties 1.10.4.** *[5]*

(i) *The elements of* $\mathfrak{Q}_8$ *are:*

$$\mathfrak{Q}_8 = \{1, -1, i, -i, j, -j, k, -k\}.$$

(ii) *Let* $\mathcal{S} \leq \mathfrak{Q}_8$*, for every* $x \in \mathfrak{Q}_8$*. The left coset is:*

$$x\mathcal{S} = \{xs \mid s \in \mathcal{S}\}.$$

(iii) *Every subgroup of* $\mathfrak{Q}_8$ *is normal.*

(iv) *The conjugacy classes of* $\mathfrak{Q}_8$ *are:*

$$\{1}, \quad \{-1\}, \quad \{i, -i\}, \quad \{j, -j\}, \quad \{k, -k\}.$$

(v) *Since* $|\mathfrak{Q}_8| = 8 = 2^3$*, then* $\mathfrak{Q}_8$ *is a* $2-$*group and because* 2 *is the only prime number dividing the order,* $\mathfrak{Q}_8$ *is the unique Sylow* $2-$*subgroup.*

Chapter 2

Linear Representations of Finite Groups

We introduce linear representations of finite groups and study their actions on vector spaces over a field (like $\mathbb{C}$). We also study irreducible and completely reducible representations, together with Schur's Lemma and Maschke's Theorem.

2.1 Representation

✍**Definition 2.1.1.** *[10] Let $\mathcal{G}$ be a group and V be a vector space over field $\mathbb{K}$.* ***A linear representation*** *of $\mathcal{G}$ on V is a homomorphism:*

$$\mathscr{L} : \mathcal{G} \to GL(V)$$

where $GL(V)$ is the general linear group of invertible linear operators of V.

✯**Remarks 2.1.1.** *[10]*

1. *$\mathscr{L}(x)$ is linear operator on V.*
2. *For all $x, y \in \mathcal{G} : \mathscr{L}(xy) = \mathscr{L}(x)\mathscr{L}(y)$.*

✎**Example 2.1.1.** ✏ *Let $\mathcal{G}$ be any group and V any vector space. Define:*

$$\mathscr{L}(x) = \mathbb{I}d_V, \quad \text{for all } x \in \mathcal{G}.$$

Hence $\mathscr{L}$ defines a linear representation of $\mathcal{G}$.

✏ *Let $\mathcal{G} = \mathbb{Z}, V = \mathbb{C}$. Define:*

$$\mathscr{L}(n) = \mathscr{L}(1)^n.$$

✍**Definition 2.1.2.** *[4] The vector space V called* ***representation space****, and the pair $(V, \mathscr{L})$ is called* ***a representation*** *of $\mathcal{G}$.*

✎**Example 2.1.2.** ✏ *Let $\mathcal{G} = \mathbb{Z}_2 = \{0, 1\}, V = \mathbb{R}^2$. Define:*

$$\mathscr{L}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathscr{L}(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$V = \mathbb{R}^2$ is representation space.
$\mathscr{L}$ is the homomorphism.
$(\mathbb{R}^2, \mathscr{L})$ is the representation.

✍**Definition 2.1.3.** *[13] Let $\mathscr{L}_1, \mathscr{L}_2$ be two representations, such that:*

$$\mathscr{L}_1 : \mathcal{G} \to GL(V), \quad \mathscr{L}_2 : \mathcal{G} \to GL(W)$$

*$\mathscr{L}_1, \mathscr{L}_2$ are said to be **equivalent** if there exists an isomorphism $\boldsymbol{T} : V \to W$ such that:*

$$\mathbf{T}\mathscr{L}_1(x) = \mathscr{L}_2(x)\mathbf{T}, \quad \textit{for all } x \in \mathcal{G}.$$

☛**Proposition 2.1.1.** *[1] Let $\mathscr{L}$ be a representation, then:*

- ***The kernel** of $\mathscr{L}$ is normal subgroup of $\mathcal{G}$, such that*

$$\ker(\mathscr{L}) = \{x \in \mathcal{G} \mid \mathscr{L}(x) = \mathbb{I}d_V\}.$$

- ***The image** $\mathscr{L}(\mathcal{G})$ is a subgroup of $GL(V)$.*

2.2 Matrix Representation

✍**Definition 2.2.1.** *[10] Let $\mathcal{G}$ be a group and $\mathbb{K}$ be a field. **A matrix representation** of $\mathcal{G}$ over $\mathbb{K}$ is a homomorphism:*

$$\mathscr{L} : \mathcal{G} \to GL_n(\mathbb{K})$$

where $GL_n(\mathbb{K})$ is the group of invertible $n \times n$ matrices.

☛**Proposition 2.2.1.** *[4] Every finite-dimensional representation*

$$\mathscr{L} : \mathcal{G} \to GL(V)$$

where $\dim(V) = n$ can be expressed as a matrix representation after choosing a basis of V.

$$\mathscr{L} : \mathcal{G} \to GL_n(\mathbb{K}).$$

✎**Example 2.2.1.** ✏ *For any group $\mathcal{G}$. define:*

$$\mathscr{L}(x) = \mathbb{I}_n, \quad \textit{for all } x \in \mathcal{G}.$$

This is the trivial matrix representation.

✍**Definition 2.2.2.** *[12] Let $\mathscr{L}_1, \mathscr{L}_2$ be two matrix representations. $\mathscr{L}_1$ and $\mathscr{L}_2$ are said to be **equivalent** if there exists an invertible matrix A such that*

$$\mathscr{L}_2(x) = A^{-1}\mathscr{L}_1(x)A, \quad \textit{for all } x \in \mathcal{G}.$$

✎**Example 2.2.2.** ✏ *Let $\mathscr{L}_1(x) = \mathbb{I}_n$ and $\mathscr{L}_2(x) = \mathbb{I}_n$ for all $x \in \mathcal{G}$. Take $A = \mathbb{I}_n$. Then:*

$$A^{-1}\mathscr{L}_1(x)A = \mathbb{I}_n = \mathscr{L}_2(x).$$

Consequently, the two trivial representations are equivalent.

✏ *Let* $\mathcal{G} = \mathbb{Z}_2 = \{0,1\}$. *Define:*

$$\mathscr{L}_1(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathscr{L}_2(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. *Then:*

$$A^{-1}\mathscr{L}_1(x)A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathscr{L}_2(x).$$

Thus $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *are equivalent.*

2.3 Degree of a Representation

✍**Definition 2.3.1.** *[13] Let* $\mathcal{G}$ *be a group and* $\mathscr{L} : \mathcal{G} \to GL(V)$ *be a representation of* $\mathcal{G}$ *over* $\mathbb{K}$. ***The degree*** *of the representation is defined as*

$$\deg(\mathscr{L}) = \dim(V).$$

✯**Remark 2.3.1.** *[4] If the representation can be written as a matrix representation*

$$\mathscr{L} : \mathcal{G} \to GL_n(\mathbb{K}).$$

The degree of the representation is defined as $\deg(\mathscr{L}) = \dim(V) = n$.

✍**Definition 2.3.2.** *[12] A representation of degree* 1 *is called* ***a linear character***

$$\mathscr{L} : \mathcal{G} \to GL_1(\mathbb{K}) = \mathbb{K}^{\times}$$

$\mathbb{K}^{\times}$ *denotes the set of all non-zero elements of* $\mathbb{K}$.

☛**Proposition 2.3.1.** *[10] The degree of a representation does not depend on the choice of basis.*

◆**Lemma 2.3.1.** *[1] If*

$$\mathscr{L} : \mathcal{G} :\to GL_n(\mathbb{K})$$

is a representation of a finite group $\mathcal{G}$, *then:*

$$\mathscr{L}(\mathcal{G}) \leq GL_n(\mathbb{K}).$$

2.4 Invariant Subspaces

✍**Definition 2.4.1.** *[13] Let*

$$\mathscr{L} : \mathcal{G} \to GL(V)$$

be a representation of group $\mathcal{G}$ *on a vector space* V. *A subspace* $\mathcal{I} \subset V$ *is called* $\mathcal{G}-$*invariant if*

$$\mathscr{L}(x)(w) \in \mathcal{I}, \quad \textit{for all } x \in \mathcal{G}, w \in \mathcal{I}.$$

☛**Proposition 2.4.1.** *[4] Let $\mathscr{L} : \mathcal{G} \to GL(V)$ be a representation, then:*

$$\{0\} \quad \text{and} \quad V \quad \text{are invariant subspaces.}$$

Proof. Let $w \in \mathcal{I}$. Then $w = 0$. For any $x \in \mathcal{G}$:

$$\mathscr{L}(x)(w) = \mathscr{L}(x)(0) = 0 \in \{0\}.$$

Thus $\mathscr{L}(x)(\{0\}) \subseteq \{0\}$.
Let $w \in V$. For any $x \in \mathcal{G}$:

$$\mathscr{L}(x)(w) \in V$$

Since $\mathscr{L}(x)$ is a linear operator on V. we have $\mathscr{L}(x)(V) \subseteq V$. □

✎**Example 2.4.1.** ✏ *Let $\mathscr{L}(x) = \mathbb{I}d_V$ for all $x \in \mathcal{G}$. Since $\mathscr{L}(x)(w) = w \in \mathcal{I}$ for any $w \in \mathcal{I}$ then every subspace $\mathcal{I} \subset V$ is $\mathcal{G}-$invariant.*

✏ *Let $V = \mathbb{C}^3$ and $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.*
Since $Ae_1 = 2e_1$, thus the subspace generated by e_1 is invariant. The subspaces generated by e_2 and e_3 are also invariant.

➤**Corollary 2.4.1.** *[9] Let $\mathscr{L} : \mathcal{G} \to GL_n(\mathbb{K})$ be a representation, and let $\mathcal{I} = span(e_1, e_2, ..., e_k)$. Then $\mathcal{I}$ is invariant if and only if every matrix $\mathscr{L}(x)$ has the form*

$$\mathscr{L}(x) = \begin{pmatrix} a_x & b_x \\ 0 & c_x \end{pmatrix}.$$

◆**Lemma 2.4.1.** *[1] Let $(V, \mathscr{L})$ be a representation of a group $\mathcal{G}$, and let $\mathcal{I}_1, \mathcal{I}_2 \subseteq V$ be two $\mathcal{G}-$invariant subspaces, then:*

1. *$\mathcal{I}_1 + \mathcal{I}_2$ is $\mathcal{G}-$invariant.*
2. *$\mathcal{I}_1 \cap \mathcal{I}_2$ is $\mathcal{G}-$invariant.*

Proof. 1. Let $w \in \mathcal{I}_1 + \mathcal{I}_2$. Then $w = v_1 + v_2$ for $v_1 \in \mathcal{I}_1, v_2 \in \mathcal{I}_2$. For any $x \in \mathcal{G}$:

$$\mathscr{L}(x)(w) = \mathscr{L}(x)(v_1 + v_2) = \mathscr{L}(x)(v_1) + \mathscr{L}(x)(v_2) \in \mathcal{I}_1 + \mathcal{I}_2.$$

Therefore, $\mathcal{I}_1 + \mathcal{I}_2$ is $\mathcal{G}-$ invariant.

2. Let $w \in \mathcal{I}_1 \cap \mathcal{I}_2$. Then $w \in \mathcal{I}_1$ and $w \in \mathcal{I}_2$. For any $x \in \mathcal{G}$ we have:

$$\mathscr{L}(x)(w) \in \mathcal{I}_1 \quad \text{and} \quad \mathscr{L}(x)(w) \in \mathcal{I}_2.$$

Therefore, $\mathscr{L}(x)(w) \in \mathcal{I}_1 \cap \mathcal{I}_2$, hence $\mathcal{I}_1 \cap \mathcal{I}_2$ is $\mathcal{G}-$invariant.

□

2.5 Subrepresentation

✍**Definition 2.5.1.** *[9] Let $\mathscr{L} : \mathcal{G} \to GL(V)$ be a representation of a group $\mathcal{G}$. Let $\mathcal{I} \subseteq V$ be $\mathcal{G}-$invariant subspaces. The restriction of the action of $\mathcal{G}$ to $\mathcal{I}$:*

$$\mathscr{L}_{\mathcal{I}}(x) = \mathscr{L}|_{\mathcal{I}}(x).$$

*This defines a representation $\mathscr{L}_{\mathcal{I}} : \mathcal{G} \to GL(V)$, called **the subrepresentation** associated with $\mathcal{I}$.*

✎**Example 2.5.1.** ✏ *Let $\mathcal{G} = \mathbb{Z}, V = \mathbb{C}^3$ with basis $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. Define the representation $\mathscr{L} : \mathbb{Z} \to GL(V)$*

$$A = \mathscr{L}(1) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then $\mathscr{L}(n) = A^n$.
Let $\mathcal{I}_1 = span(e_1) = \{\lambda e_1 : \lambda \in \mathbb{C}\}$. Then:

$$Ae_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2e_1.$$

Thus $Ae_1 = 2e_1 \in \mathcal{I}_1$, then $A(\lambda e_1) = 2\lambda e_1 \in \mathcal{I}_1$.
Therefore, $\mathcal{I}_1$ is a $1-$dimensional subrepresentation.

☛**Proposition 2.5.1.** *[11] If $\mathcal{I}$ is an invariant subspace of V, then the subrepresentation*

$$\mathscr{L}_{\mathcal{I}}(x) = \mathscr{L}|_{\mathcal{I}}(x)$$

is a representation of $\mathcal{G}$ in $\mathcal{I}$.

Proof. Since $\mathcal{I}$ is $\mathcal{G}-$invariant, by definition:

$$\mathscr{L}(x)(w) \in \mathcal{I}, \quad \text{for all } x \in \mathcal{G} \quad \text{and all } w \in \mathcal{I}.$$

So the restriction $\mathscr{L}|_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ is a well-defined linear map.
For any $x \in \mathcal{G}$, we have $\mathscr{L}(x^{-1})|_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ as well. Moreover, for any $w \in \mathcal{I}$:

$$\mathscr{L}_{\mathcal{I}}(x) \circ \mathscr{L}_{\mathcal{I}}(x^{-1})(w) = \mathscr{L}_{\mathcal{I}}(x)(\mathscr{L}(x^{-1})(w)) = \mathscr{L}_{\mathcal{I}}(xx^{-1})(w) = \mathscr{L}(e)(w) = w$$

and similarly $\mathscr{L}_{\mathcal{I}}(x^{-1}) \circ \mathscr{L}_{\mathcal{I}}(x) = w$, so $\mathscr{L}_{\mathcal{I}}(x)$ is invertible with inverse $\mathscr{L}_{\mathcal{I}}(x^{-1})$. Hence $\mathscr{L}_{\mathcal{I}}(x) \in GL(\mathcal{I})$.
For any $x, y \in \mathcal{G}$ and any $w \in \mathcal{I}$:

$$\begin{aligned} \mathscr{L}_{\mathcal{I}}(xy)(w) = \mathscr{L}(xy)(w) &= \mathscr{L}(x)(\mathscr{L}(y)(w)) \\ &= \mathscr{L}(x)|_{\mathcal{I}}(\mathscr{L}(y)|_{\mathcal{I}}(w)) \\ &= (\mathscr{L}_{\mathcal{I}}(x) \circ \mathscr{L}_{\mathcal{I}}(y))(w). \end{aligned}$$

Thus $\mathscr{L}_{\mathcal{I}}(xy) = \mathscr{L}_{\mathcal{I}}(x) \circ \mathscr{L}_{\mathcal{I}}(y)$. □

☛**Proposition 2.5.2.** *[8] Every representation has two trivial subrepresentation $\{0\}$ and V.*

✎**Example 2.5.2.** ✏ *Any 1−dimensional representation has two subrepresentation $\{0\}$ and V.*

✍**Definition 2.5.2.** *[11] Let $\mathcal{G}$ be a group and $\mathscr{L} : \mathcal{G} \to GL(V)$ a representation of $\mathcal{G}$ on V. Let $\mathcal{I} \subseteq V$ be a subrepresentation. Then **the quotient space** $V/\mathcal{I}$ admits a **quotient representation** defined by:*

$$x \cdot (v + w) = \mathscr{L}(x)(v) + w, \quad \text{for all } x \in \mathcal{G}, v \in V.$$

2.6 Irreducible Representation

✍**Definition 2.6.1.** *[10] Let $(V, \mathscr{L})$ be a representation of a group $\mathcal{G}$. The representation is said to be **irreducible** if the only $\mathcal{G}$−invariant subspaces of V are $\{0\}$ and V.*

✎**Example 2.6.1.** ✏ *Let $\mathcal{G}$ be any group, any one-dimensional trivial representation is irreducible.*

✏ *Take $\mathcal{G} = GL_n(\mathbb{C}), V = \mathbb{C}^n$ and define:*

$$\mathscr{L} : GL_n(\mathbb{C}) \to GL(V)$$
$$\mathscr{L}(T)(v) = Tv.$$

Let $v \in \mathcal{I}$ be a nonzero vector, for any nonzero vector $w \in \mathbb{C}^n$ we have $Tv = w$. Since $v \in \mathcal{I}$ and $\mathcal{I}$ is invariant we have:

$$w = Tv \in T(\mathcal{I}) \subseteq \mathcal{I}.$$

Therefore the only invariant subspaces are $\{0\}$ and $\mathbb{C}^n$. Hence the representation is irreducible.

✍**Definition 2.6.2.** *[9] A representation $(V, \mathscr{L})$ that is not irreducible is called **reducible**, that is, it admits a non-trivial invariant subspace.*

✎**Example 2.6.2.** ✏ *Let $\mathscr{L}(x) = \mathbb{I}d_V$, with $\dim V = n \geq 2$. Then any 1−dimensional subspace is invariant, which means there exist non-trivial invariant subspaces. Hence the representation is reducible.*

✏ *Let $\mathcal{G} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C}^\times \right\}, V = \mathbb{C}^2$. Define:*

$$\mathscr{L} : G \to GL(V)$$
$$\mathscr{L}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}.$$

Let $\mathcal{I}_1 = span\{(1, 0)\} = \{(\lambda, 0) \mid \lambda \in \mathbb{C}\}$. For any $x \in \mathcal{G}$ and any vector $v = (\lambda, 0) \in \mathcal{I}_1$ we have:

$$\mathscr{L}(x)(v) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} \lambda \\ 0 \end{pmatrix} = \begin{pmatrix} a\lambda \\ 0 \end{pmatrix} \in \mathcal{I}_1.$$

Thus, $\mathscr{L}(x)(\mathcal{I}_1) \subseteq \mathcal{I}_1$, therefore $\mathcal{I}_1$ is a subrepresentation. Similarly, $\mathcal{I}_2 = span\{(0, 1)\} = \{(0, \lambda) \mid \lambda \in \mathbb{C}\}$ is invariant.

☛**Proposition 2.6.1.** *[10] Every representation of degree 1 is irreducible.*

2.7 Completely Reducible Representation

✍**Definition 2.7.1.** *[13] Let*

$$\mathscr{L}_1 : \mathcal{G} \to GL(V_1) \quad \text{and} \quad \mathscr{L}_2 : \mathcal{G} \to GL(V_2)$$

*be two representation of $\mathcal{G}$. **The direct sum** representation*

$$\mathscr{L}_1 \oplus \mathscr{L}_2 : \mathcal{G} \to GL(V_1 \oplus V_2)$$

is defined by

$$(\mathscr{L}_1 \oplus \mathscr{L}_2)(x)(v_1, v_2) = (\mathscr{L}_1(x)v_1, \mathscr{L}_2(x)v_2), \quad \text{for all } x \in \mathcal{G}.$$

✍**Definition 2.7.2.** *[9] A representation $(V, \mathscr{L})$ of group $\mathcal{G}$ is called **completely reducible** if it can be written as a direct sum of irreducible subrepresentations*

$$V = V_1 \oplus V_2 \oplus \cdots V_k$$

where each V_i is an irreducible representation.

✎**Example 2.7.1.** ✏ *Let $\mathcal{G}$ be any group and V any $1-$dimensional representation. Therefore, it is completely reducible.*

◆**Lemma 2.7.1.** *[11] If V is completely reducible and $\mathcal{I} \subset V$ is invariant, then:*

$$V = \mathcal{I} \oplus \mathcal{I}'$$

where $\mathcal{I}'$ is invariant subspace.

2.8 Schur's Lemma

✍**Definition 2.8.1.** *[10] Let*

$$\mathscr{L}_1 : \mathcal{G} \to GL(V), \quad \mathscr{L}_2 : \mathcal{G} \to GL(W)$$

*be two representations of a group $\mathcal{G}$. A linear map $\mathbf{T} : V \to W$ is called a $\mathcal{G}-$**homomorphism** if it commutes with the group action:*

$$\mathbf{T}(\mathscr{L}_1(x)(v)) = \mathscr{L}_2(x)(\mathbf{T}(v)), \quad \text{for all } x \in \mathcal{G}, v \in V.$$

▼**Theorem 2.8.1.** *[9](First Form of Schur's Lemma)*
Let $(V, \mathscr{L}_1)$ and $(W, \mathscr{L}_2)$ be two irreducible representations of $\mathcal{G}$, and $\mathbf{T} : V \to W$ be a $\mathcal{G}-$homomorphism. Then either:

$$\mathbf{T} = 0 \quad \text{or} \quad \mathbf{T} \text{ is an isomorphism.}$$

Proof. Consider $\ker \mathbf{T} \subset V$. For any $x \in \mathcal{G}$ and $v \in \ker \mathbf{T}$:

$$\mathbf{T}(\mathscr{L}_1(x)(v)) = \mathscr{L}_2(x)(\mathbf{T}(v)) = \mathscr{L}_2(x)(0) = 0.$$

Hence, $\ker \mathbf{T}$ is stable under the action of $\mathcal{G}$. Therefore $\ker \mathbf{T}$ is an invariant subspace of V.
Since V is irreducible, then $\ker \mathbf{T}$ is either $\{0\}$ or V.
Similarly with $\mathrm{Im}\mathbf{T} \subset W$:

$$\mathbf{T}(\mathscr{L}_1(x)(v)) = \mathscr{L}_2(x)(\mathbf{T}(v)).$$

Then $\mathrm{Im}\mathbf{T}$ is $\mathcal{G}-$invariant and since W is irreducible, $\mathrm{Im}\mathbf{T}$ is either $\{0\}$ or W.
If $\ker \mathbf{T} = V$, then $\mathbf{T} = 0$.
If $\ker \mathbf{T} = \{0\}$, then $\mathbf{T}$ is injective, so $\mathrm{Im}\mathbf{T} \neq \{0\}$. Therefore $\mathrm{Im}\mathbf{T} = W$ so $\mathbf{T}$ is bijective and consequently $\mathbf{T}$ is an isomorphism. □

▼**Theorem 2.8.2.** *[12](Second Form of Schur's Lemma)*
Let $(V, \mathscr{L})$ be an irreducible representation of $\mathcal{G}$ over an algebraically closed field $\mathbb{K}$. If $\mathbf{T} : V \to V$ is a $\mathcal{G}-$endomorphism, then $\mathbf{T}$ is a scalar multiple of the identity
$\mathbf{T} = \lambda \cdot \mathbb{I}d_V, \quad \lambda \in \mathbb{K}$.

Proof. Since $\mathbb{K}$ is algebraically closed, the operator $\mathbf{T}$ admits an eigenvalue $\lambda \in \mathbb{K}$. Let

$$\mathbf{A} = \mathbf{T} - \lambda \mathbb{I}d_V.$$

$\mathbf{A}$ is $\mathcal{G}-$endomorphism. Since λ is an eigenvalue, then $\ker \mathbf{A} \neq \{0\}$ by The First Form of Schur's Lemma(2.8.1) $\mathbf{A} = 0$. Thus

$$\mathbf{T} - \lambda \mathbb{I}d_V = 0$$

so

$$\mathbf{T} = \lambda \mathbb{I}d_V.$$

□

✯**Remark 2.8.1.** *[9] If V and W are irreducible representations and $V \ncong W$. Then:*

$$\mathrm{Hom}_{\mathcal{G}}(V, W) = \{0\}.$$

➢**Notation 9.** *We write $\mathrm{Hom}_{\mathcal{G}}(V, W)$ to denote the set of all $\mathcal{G}-$homomorphisms from V to W.*

2.9 Maschke's Theorem

Let $\mathcal{G}$ be a finite group and V be a finite dimensional representation of $\mathcal{G}$ over $\mathbb{K}$. Let

$$\mathscr{L} : \mathcal{G} \to GL(V)$$

be a representation.

▼**Theorem 2.9.1.** *[10](Maschke's Theorem)*
If $\mathrm{char}(\mathbb{K}) \nmid |\mathcal{G}|$, then every representation of $\mathcal{G}$ is completely reducible. In other words, V can be decomposed as a direct sum of irreducible representations.

Proof. Let W' be the vector subspace of W such that $V = W \oplus W'$. Consider $\mathbf{p} : V \to W$ be the projection of V onto W along W', for $v = w + w'$ then $\mathbf{p}(v) = w$ for $w \in W, w' \in W'$. Define:

$$\tilde{\mathbf{p}} : v \longmapsto \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} \mathscr{L}(x)\mathbf{p}(\mathscr{L}(x^{-1})v).$$

The map $\tilde{\mathbf{p}}$ is called **the averaging operator**, and we can simplify it:

$$\tilde{\mathbf{p}} : v \longmapsto \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} x\mathbf{p}(x^{-1}v).$$

We first show that $\tilde{\mathbf{p}}$ has image in W, since for $v \in V : \mathbf{p}(x^{-1}v) \in W$ and $xW \leq W$.
For $w \in W$, we have $\tilde{\mathbf{p}}(w) = w$. This follows from the fact that $\mathbf{p}$ itself fixes W. Since W is $\mathcal{G}-$invariant we have $x^{-1}w \in W$, for all $w \in W$. So we get:

$$\begin{aligned}
\tilde{\mathbf{p}}(w) &= \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} x\mathbf{p}(x^{-1}w) \\
&= \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} xx^{-1}w \\
&= \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} w = w.
\end{aligned}$$

Thus $\tilde{\mathbf{p}}$ is a projection onto W.
For $y \in \mathcal{G}$, we have $y\tilde{\mathbf{p}}(v) = \tilde{\mathbf{p}}(yv)$, then it is invariant;

$$\begin{aligned}
y\tilde{\mathbf{p}}(v) &= y\frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} x\mathbf{p}(x^{-1}v) \\
&= \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} yx\mathbf{p}((yx)^{-1}yv) \\
&= \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} (yx)\mathbf{p}((yx)^{-1})yv.
\end{aligned}$$

Since y is invertible, we may write $z = yx$:

$$= \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} z\mathbf{p}((z)^{-1}(yv))$$

$$= \tilde{\mathbf{p}}(yv).$$

If $v \in \ker \tilde{\mathbf{p}}$ and $y \in \mathcal{G}$, then $\tilde{\mathbf{p}}(yv) = y\tilde{\mathbf{p}}(v) = 0$, so $y \in \ker \tilde{\mathbf{p}}$.
Thus

$$V = \mathrm{Im}\tilde{\mathbf{p}} \oplus \ker \tilde{\mathbf{p}}.$$

□

2.10 Examples of Representations of Small Groups

2.10.1 Cyclic Groups $\mathfrak{C}_n$

✍**Definition 2.10.1.** *[9] An* ***n-th root of unity*** *is a number $x \in \mathbb{C}$ such that $x^n = 1$. In general, n-th root of unity are given by $e^{\frac{2\pi i k}{n}}$ for $0 \leq k < n$.*

☆**Remark 2.10.1.** *[9] Every n-th root of unity is a power of* $\omega_n = e^{\frac{2\pi i}{n}}$.

✎**Example 2.10.1.** ✏ *We begin by examining the* $1-$*dimensional representations of* $\mathfrak{C}_4 := \langle x \mid x^4 = 1\rangle$.
Let $\mathscr{L} : \mathfrak{C}_4 \to GL_1(\mathbb{C})$ *be a representation. Since* $x^4 = 1$ *we have* $(\mathscr{L}(x))^4 = \mathbb{I}$. *Take* $\mathscr{L}(x) = [\alpha]$. *Since* $[\xi]$ *is a* 1×1 *matrix, we have* $[\alpha]^4 = [\alpha^4]$.
Then $[\alpha]^4 = [\alpha^4] = (\mathscr{L}(x))^4 = [1]$, *so* $\alpha^4 = 1$.
There are 4 such numbers in $\mathbb{C} : 1, -1, i, -i$. *So we have four distinct* $1-$*dimensional representations of* $\mathfrak{C}_4$*:*

- $\mathscr{L}_1 : \mathfrak{C}_4 \to GL_1(\mathbb{C})$
  $x \longmapsto 1.$
- $\mathscr{L}_2 : \mathfrak{C}_4 \to GL_1(\mathbb{C})$
  $x \longmapsto -1.$
- $\mathscr{L}_3 : \mathfrak{C}_4 \to GL_1(\mathbb{C})$
  $x \longmapsto i.$
- $\mathscr{L}_4 : \mathfrak{C}_4 \to GL_1(\mathbb{C})$
  $x \longmapsto -i.$

✏ *Consider now the following* $2-$*dimensional representation of* $\mathfrak{C}_4$. *Let* $\mathscr{L} : \mathfrak{C}_4 \to GL_2(\mathbb{C})$ *such that:*

$$\mathscr{L}(x) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Then

$$(\mathscr{L}(x))^4 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathscr{L}(1).$$

So $\mathscr{L}$ *is a representation.*

2.10.2 Symmetric Groups $\mathfrak{S}_n$

✍**Definition 2.10.2.** *[1] The group* $\mathfrak{S}_3$ *is the group of permutations of three elements.*

$$\mathfrak{S}_3 = \{e, (12), (13), (23), (123), (132)\}$$

with $|\mathfrak{S}_3| = 6$.

✎**Example 2.10.2.** ✏ *We first describe the* $1-$*dimensional representations.*
Let $\mathscr{L} : \mathfrak{S}_3 \to GL_1(\mathbb{C})$. *Define:*

$$\mathscr{L}(x) = 1, \quad \text{for all } x \in \mathfrak{S}_3.$$

This is a representation and it called ***the trivial representation***.
Define:

$$\mathscr{L}(x) = \begin{cases} 1, & \text{if } x \text{ is even,} \\ -1, & \text{if } x \text{ is odd.} \end{cases}$$

This representation is called ***the sign representation***.

✏ *Now, we construct a non-trivial representation:*

$$\mathscr{L} : \mathfrak{S}_3 \to GL_2(\mathbb{C}).$$

Let $\mathfrak{S}_3$ *act on* $\mathbb{R}^3$ *by permuting coordinates:*

$$\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

Consider the subspace: $\mathcal{I} = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$. *We choose a basis:*

$$u = (1, -1, 0), v = (1, 1, -2).$$

We apply it on the elements of $\mathfrak{S}_3$*:*

$$\text{Since } (12)(x_1, x_2, x_3) = (x_2, x_1, x_3) \Longrightarrow \begin{cases} (12)u = -u, \\ (12)v = v. \end{cases}$$

Thus:

$$\mathscr{L}((12)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By applying the same computation to the permutation (123) *we obtain:*

$$\mathscr{L}((123)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

2.10.3 Dihedral Group $\mathfrak{D}_{2n}$

✍**Definition 2.10.3.** *[1] The group* $\mathfrak{D}_{2n}$ *is the group of symmetries of a regular* $n-$*gon.*

$$\mathfrak{D}_{2n} = \langle r, s \mid r^n = e, s^2 = e, srs = r^{-1} \rangle.$$

with $|\mathfrak{D}_{2n}| = 2n$.

✎**Example 2.10.3.** ✏ *Let* $\mathscr{L} : \mathfrak{D}_{2n} \to GL_1(\mathbb{C})$. *So* $\mathscr{L}(r)$ *and* $\mathscr{L}(s) \in \mathbb{C}^\times$, *with:*

$$\mathscr{L}(r)^n = 1, \quad \mathscr{L}(s)^2 = 1, \quad \mathscr{L}(s)\mathscr{L}(r)\mathscr{L}(s) = \mathscr{L}(r)^{-1}.$$

- *Define:*

$$\mathscr{L}(r) = 1, \quad \mathscr{L}(s) = 1.$$

They are called ***the trivial representations****.*

- *Define:*

$$\mathscr{L}(r) = 1, \quad \mathscr{L}(s) = -1.$$

then:

$$\mathscr{L}(s)\mathscr{L}(r)\mathscr{L}(s) = 1 = \mathscr{L}(r)^{-1}.$$

These representation are called ***the reflection sign representations****.*

☆**Remark 2.10.2.** *[8] In dimension 1, all values commute:*

$$\mathscr{L}(s)\mathscr{L}(r)\mathscr{L}(s) = \mathscr{L}(r)$$

so:

$$\mathscr{L}(r) = \mathscr{L}(r)^{-1} \Longrightarrow \mathscr{L}(r)^2 = 1$$
$$\Longrightarrow \mathscr{L}(r) = \pm 1$$

Therefore, we have the following 1*−dimensional representations:*

$$\mathscr{L}(r) = \pm 1, \quad \mathscr{L}(s) = \pm 1.$$

✏ *Let*

$$\mathscr{L} : \mathfrak{D}_{2n} \to GL_2(\mathbb{C})$$

since $\mathfrak{D}_{2n}$ represents the rotation and reflection of a regular $n-$gon, we can represent them as matrices acting on $\mathbb{R}^2$.
Let:

$$\mathscr{L}(r) = \begin{pmatrix} \cos\dfrac{2\pi k}{n} & -\sin\dfrac{2\pi k}{n} \\ \sin\dfrac{2\pi k}{n} & \cos\dfrac{2\pi k}{n} \end{pmatrix} \quad \mathscr{L}(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad k = 1, 2, ..., n-1.$$

2.10.4 Quaternion Group $\mathfrak{Q}_8$

✍**Definition 2.10.4.** *[5] the group $\mathfrak{Q}_8$ is group whose elements:*

$$\mathfrak{Q}_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

satisfying $i^2 = j^2 = k^2 = ijk = -1$ with $|\mathfrak{Q}_8| = 8$.

✎**Example 2.10.4.** ✏ *Let $\mathscr{L} : \mathfrak{Q}_8 \to GL(V)$ be a homomorphism. Then we have the following representations*

$$\mathscr{L}(i), \quad \mathscr{L}(j)$$

These representations must satisfy:

$$\mathscr{L}(i)^2 = \mathscr{L}(j)^2 = -\mathbb{I}, \quad \mathscr{L}(i)\mathscr{L}(j) = \mathscr{L}(k), \quad \mathscr{L}(j)\mathscr{L}(i) = -\mathscr{L}(k).$$

✏ *We define the trivial representation:*

$$\mathscr{L} : \mathfrak{Q}_8 \to GL_1(\mathbb{C})$$

by

$$\mathscr{L}(x) = 1.$$

✏ *Because $\mathbb{C}^\times$ is abelian;*

$$\mathscr{L}(ij) = \mathscr{L}(i)\mathscr{L}(j) = \mathscr{L}(j)\mathscr{L}(i) = \mathscr{L}(ji)$$

and in $\mathfrak{Q}_8$ we have:

$$ij = k, \quad ji = -k$$

consequently:

$$\mathscr{L}(k) = \mathscr{L}(-k).$$

From this result;

$$\mathscr{L}(-k) = \mathscr{L}(-1)\mathscr{L}(k) \Longrightarrow \mathscr{L}(k) = \mathscr{L}(-1)\mathscr{L}(k) \Longrightarrow \mathscr{L}(-1) = 1.$$

Therefore:

$$\mathscr{L}(i) = \pm 1, \quad \mathscr{L}(j) = \pm 1, \quad \mathscr{L}(k) = \pm 1.$$

We summarize the $1-$*dimensional representations in the following table:*

representation	$\mathscr{L}(i)$	$\mathscr{L}(j)$	$\mathscr{L}(k)$
$\mathscr{L}_1$	1	1	1
$\mathscr{L}_2$	−1	1	−1
$\mathscr{L}_3$	1	−1	−1
$\mathscr{L}_4$	−1	−1	1

✏ *Consider the representation:*

$$\mathscr{L} : \mathfrak{Q}_8 \to GL_2(\mathbb{C})$$

defined by:

$$\mathscr{L}(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathscr{L}(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then:

$$\mathscr{L}(i)\mathscr{L}(j) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \mathscr{L}(k).$$

But:

$$\mathscr{L}(j)\mathscr{L}(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\mathscr{L}(k).$$

✯**Remarks 2.10.1.** *[5]*

1. *The quaternion group has five irreducible representations over* $\mathbb{C}$*; four of degree* 1 *and one of degree* 2.

2. $\mathfrak{Q}_8$ *is finite, then every representation of* $\mathfrak{Q}_8$ *is completely reducible.*

Chapter 3

Characters of Finite Groups

We introduce characters and show how representation theory can be studied through traces and class functions. We also develop the orthogonality relations and study the regular representation, which paly an important role in decomposing representations into irreducible components.

3.1 Character of Representation

✍**Definition 3.1.1.** *[10] Let V be a finite-dimensional vector space and $\mathbf{T} : V \to V$ a linear operator represented by a matrix A.* ***The trace*** *of $\mathbf{T}$ is the sum of the diagonal coefficients of A.*

✍**Definition 3.1.2.** *[9] Let $\mathscr{L} : \mathcal{G} \to GL(V)$ be a representation of a finite group $\mathcal{G}$, where V is a finite-dimensional vector space.* ***The character*** *of the representation is the function*

$$\chi : \mathcal{G} \to \mathbb{C}$$

$$\chi(x) = \mathrm{Tr}(\mathscr{L}(x)), \quad \textit{for every } x \in \mathcal{G}$$

.

✎**Example 3.1.1.**

- ✏ *Let $\mathscr{L} : \mathcal{G} \to GL_1(\mathbb{C})$ be the trivial representation $\mathscr{L}(x) = 1$. Then:*

$$\chi(x) = 1, \quad \textit{for all } x \in \mathcal{G}.$$

- ✏ *Consider the trivial representation $\mathscr{L}(x) = 1$ acting on $\mathbb{C}$. Then $\chi(x) = 1$, for every $x \in \mathcal{G}$.*

- ✏ *If χ is the character of $(V, \mathscr{L})$ of $n-$dimensional vector space V, then*

$$\chi(e) = n \quad (\textit{since } \mathscr{L}(e) = \mathbb{I}_n)$$

and the trace of the identity matrix is n.

3.2 Class Functions

✍**Definition 3.2.1.** *[15] Let $\mathcal{G}$ be a group and $x, y \in \mathcal{G}$. We say that x is **conjugate** to y if there exists $t \in \mathcal{G}$ such that:*

$$y = txt^{-1}.$$

The set

$$\mathrm{Cl}(x) = \{txt^{-1} \mid t \in \mathcal{G}\}$$

*is called **the conjugacy class** of x.*

✍**Definition 3.2.2.** *[12] Let $\mathcal{G}$ be a finite group. A function $\mathbf{f} : \mathcal{G} \to \mathbb{C}$ is called a **class function** if it constant on conjugacy classes:*

$$\mathbf{f}(x) = \mathbf{f}(yxy^{-1}), \quad \text{for all } x, y \in \mathcal{G}.$$

➤**Corollary 3.2.1.** *[15] If x and y are conjugate ($x \sim y$), then*

$$\mathbf{f}(x) = \mathbf{f}(y).$$

✭**Remark 3.2.1.** *[19] The set of all class functions on $\mathcal{G}$ forms a vector space over $\mathbb{C}$.*

☛**Proposition 3.2.1.** *[12] If $\mathcal{G}$ is a finite group and has k conjugacy classes, then the space of class functions on $\mathcal{G}$ has dimension k.*

3.3 Characters and Class Functions

▼**Theorem 3.3.1.** *[9] The Character χ of a representation is a class function. In other words;*

$$\chi(txt^{-1}) = \chi(x), \quad x, t \in \mathcal{G}.$$

Proof. Let $\mathscr{L} : \mathcal{G} \to GL(V)$ be a representation with character χ. Suppose that $x \sim y$, then there exists $t \in \mathcal{G}$ such that:

$$y = txt^{-1}.$$

We have

$$\begin{aligned}\chi(y) = \mathrm{Tr}(\mathscr{L}(y)) &= \mathrm{Tr}(\mathscr{L}(txt^{-1})) \\ &= \mathrm{Tr}(\mathscr{L}(t)\mathscr{L}(x)\mathscr{L}(t)^{-1}).\end{aligned}$$

Since

$$\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$$

we obtain:

$$\mathrm{Tr}(\mathscr{L}(t)\mathscr{L}(x)\mathscr{L}(t)^{-1}) = \mathrm{Tr}(\mathscr{L}(x)) = \chi(x).$$

Thus $\chi(y) = \chi(x)$. □

☛**Proposition 3.3.1.** *[10] Every character is a class function; they belong to the vector space of class functions.*

3.4 Character of Direct Sum

✍**Definition 3.4.1.** *[10] Let*

$$\mathscr{L}_1 : \mathcal{G} \to GL(V_1) \quad \mathscr{L}_2 : \mathcal{G} \to GL(V_2)$$

*be two representations of $\mathcal{G}$. **The direct sum representation** is defined as follows:*

$$\mathscr{L}_1 \oplus \mathscr{L}_2 : \mathcal{G} \to GL(V_1 \oplus V_2)$$

$$(\mathscr{L}_1 \oplus \mathscr{L}_2)(x)(v_1, v_2) = (\mathscr{L}_1(x)v_1, \mathscr{L}_2(x)v_2).$$

Which has a matrix of the form

$$(\mathscr{L}_1 \oplus \mathscr{L}_2)(x) = \begin{pmatrix} \mathscr{L}_1(x) & 0 \\ 0 & \mathscr{L}_2(x) \end{pmatrix}.$$

▼**Theorem 3.4.1.** *[12] Let $\mathscr{L}_1 : \mathcal{G} \to GL(V_1)$ and $\mathscr{L}_2 : \mathcal{G} \to GL(V_2)$ be representations of $\mathcal{G}$, χ_1 and χ_2 be their characters respectively. Then the character of the direct sum satisfies:*

$$\chi(\mathscr{L}_1(x) \oplus \mathscr{L}_2(x)) = \chi(\mathscr{L}_1(x)) + \chi(\mathscr{L}_2(x)), \quad \textit{for all } x \in \mathcal{G}.$$

Proof. Let $\dim V_1 = n$ and $\dim V_2 = m$. We choose bases for V_1 and V_2, then for any $x \in \mathcal{G}$, the matrix of $\mathscr{L}_1 \oplus \mathscr{L}_2(x)$ is block diagonal:

$$(\mathscr{L}_1 \oplus \mathscr{L}_2)(x) = \begin{pmatrix} \mathscr{L}_1(x) & 0 \\ 0 & \mathscr{L}_2(x) \end{pmatrix}.$$

The trace of block diagonal matrix is equal to the sum of the traces of its diagonal blocks:

$$\begin{aligned} \chi(\mathscr{L}_1(x) \oplus \mathscr{L}_2(x)) &= \mathrm{Tr}(\mathscr{L}_1(x) \oplus \mathscr{L}_2(x)) \\ &= \mathrm{Tr}(\mathscr{L}_1(x)) + \mathrm{Tr}(\mathscr{L}_2(x)) \\ &= \chi(\mathscr{L}_1(x)) + \chi(\mathscr{L}_2(x)). \end{aligned}$$

□

✎**Example 3.4.1.** ✏ *Let $\mathcal{G} = \mathfrak{S}_3$ and:*

$$\mathscr{L}_1 : \mathfrak{S}_3 \to GL(\mathbb{C}), \quad \mathscr{L}_1(x) = 1$$

$$\mathscr{L}_2 : \mathfrak{S}_3 \to GL(\mathbb{C}), \quad \mathscr{L}_2(x) = \begin{cases} +1, & \textit{if } x \textit{ is even} \\ -1, & \textit{if } x \textit{ is odd.} \end{cases}$$

We have:

$$(\mathscr{L}_1 \oplus \mathscr{L}_2)(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\mathscr{L}_1 \oplus \mathscr{L}_2)((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\mathscr{L}_1 \oplus \mathscr{L}_2)((123)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, for every $x \in \mathfrak{S}_3$:

$$(\mathscr{L}_1 \oplus \mathscr{L}_2)(x) = \begin{pmatrix} \mathscr{L}_1(x) & 0 \\ 0 & \mathscr{L}_2(x) \end{pmatrix}$$

Since $\chi(\mathscr{L}_1 \oplus \mathscr{L}_2)(x) = \chi(\mathscr{L}_1(x)) + \chi(\mathscr{L}_2(x))$, then:

$$\chi(e) = 1 + 1 = 2.$$

$$\chi((12)) = 1 + (-1) = 0.$$

$$\chi((123)) = 1 + 1 = 2.$$

☛**Proposition 3.4.1.** *[12] If $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, then the character satisfies:*

$$\chi = \chi_1 + \chi_2 + \cdots + \chi_k.$$

Proof. For any $x \in \mathcal{G}$, since $V = V_1 \oplus V_2 \oplus \cdots V_k$ and each V_i is $\mathcal{G} - invariant$, the linear map $\mathscr{L}(x)$ is block diagonal with respect to a basis:

$$\mathscr{L}(x) = \begin{pmatrix} \mathscr{L}_1(x) & 0 & \cdots & 0 \\ 0 & \mathscr{L}_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathscr{L}_k(x) \end{pmatrix}.$$

The trace of a block matrix is the sum of the traces of the blocks:

$$\mathrm{Tr}(\mathscr{L}(x)) = \mathrm{Tr}(\mathscr{L}_1(x)) + \mathrm{Tr}(\mathscr{L}_2(x)) + \cdots + \mathrm{Tr}(\mathscr{L}_k(x))$$

Using the definition of characters, we obtain:

$$\chi(x) = \chi_1(x) + \chi_2(x) + \cdots \chi_k(x).$$

□

➤**Corollary 3.4.1.** *[10] If a representation decomposes into irreducible representations*

$$V = \mu_1 V_1 \oplus \mu_2 V_2 \oplus \cdots \mu_k V_k$$

then:

$$\chi = \mu_i \chi_1 + \mu_2 \chi_2 + \cdots + \mu_k \chi_k.$$

3.5 Character of Tensor Product

✍**Definition 3.5.1.** *[13] Let*

$$\mathscr{L}_1 : \mathcal{G} \to GL(V_1) \quad \mathscr{L}_2 : \mathcal{G} \to GL(V_2)$$

*be two representations of $\mathcal{G}$. We define **the tensor product representation** by:*

$$\mathscr{L}_1 \otimes \mathscr{L}_2 : \mathcal{G} \to GL(V_1 \otimes V_2)$$

is defined by

$$(\mathscr{L}_1 \otimes \mathscr{L}_2)(x)(v_1, v_2) = \mathscr{L}_1(x)v_1 \otimes \mathscr{L}_2(x)v_2.$$

▼**Theorem 3.5.1.** *[9] Let $\mathscr{L}_1 : \mathcal{G} \to GL(V_1)$ and $\mathscr{L}_2 : \mathcal{G} \to GL(V_2)$ be representations of $\mathcal{G}$, χ_1 and χ_2 be their characters respectively. Then the character of tensor product satisfies:*

$$\chi(\mathscr{L}_1(x) \otimes \mathscr{L}_2(x)) = \chi(\mathscr{L}_1(x)) \cdot \chi(\mathscr{L}_2(x)).$$

Proof. Let $\dim V_1 = n$ and $\dim V_2 = m$. Choose bases $\{\alpha_1, ..., \alpha_n\}$ for V_1 and $\{\beta_1, ..., \beta_m\}$ for V_2. Then $\alpha_i \otimes \beta_j$ forms a basis for $V_1 \otimes V_2$, for any $x \in \mathcal{G}$ we have:

$$(\mathscr{L}_1 \otimes \mathscr{L}_2)(x)(\alpha_i \otimes \beta_i\} = \mathscr{L}_1(x)(\alpha_i) \otimes \mathscr{L}_2(x)(\beta_i).$$

Let $A = \mathscr{L}_1(x)$ and $B = \mathscr{L}_2(x)$, A is an $n \times n$ matrix with coefficients a_{ik} and B is $m \times m$ matrix with coefficients b_{jl}. The matrix $A \otimes B$ with respect to the basis $\{\alpha_i \otimes \beta_i\}$ is the Kronecker product, with coefficients:

$$(A \otimes B) = a_{ik} b_{jl}.$$

The trace of $A \otimes B$ is the sum of its diagonal coefficients. The diagonal coefficients correspond to the indices satisfying $i = k$ and $j = k$:

$$\begin{aligned}
\mathrm{Tr}(A \otimes B) &= \sum_{i=1}^{n} \sum_{i=1}^{m} (A \otimes B) \\
&= \sum_{i=1}^{n} \sum_{i=1}^{m} a_{ii} b_{jj} \\
&= \left(\sum_{i=1}^{n} a_{ii} \right) \left(\sum_{i=1}^{m} b_{jj} \right) \\
&= \mathrm{Tr}(A) \cdot \mathrm{Tr}(B).
\end{aligned}$$

Thus

$$\mathrm{Tr}((\mathscr{L}_1 \otimes \mathscr{L}_2)(x)) = \mathrm{Tr}(\mathscr{L}_1(x)) \cdot \mathrm{Tr}(\mathscr{L}_2(x))$$

by definition of characters:

$$\chi(\mathscr{L}_1 \otimes \mathscr{L}_2)(x) = \chi(\mathscr{L}_1(x)) \cdot \chi(\mathscr{L}_2(x)).$$

□

✯**Remark 3.5.1.** *[10] If* $\dim V_1 = n$ *and* $\dim V_2 = m$*, then* $\dim(V_1 \otimes V_2) = nm$*.*

3.6 First Orthogonality Relation

✍**Definition 3.6.1.** *[10] Let* $\mathcal{G}$ *be a finite group and let*

$$f, g : \mathcal{G} \to \mathbb{C}.$$

The inner product *of* f *and* g *is defined by:*

$$\langle f, g \rangle = \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} f(x) \overline{g(x)}.$$

where $\overline{g(x)}$ *denotes the complex conjugate.*

▼**Theorem 3.6.1.** *[12] Let* χ_1 *and* χ_2 *be characters of irreducible representations of* $\mathcal{G}$*. Then:*

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1, & \text{if } \chi_1 = \chi_2 \\ 0, & \text{if } \chi_1 \neq \chi_2. \end{cases}$$

Thus, disjoint irreducible characters are orthogonal.

Proof. Let V_1 and V_2 be irreducible representations of $\mathcal{G}$ with characters χ_1 and χ_2 respectively. We have:

$$\langle \chi_1, \chi_2 \rangle = \dim \ \mathrm{Hom}_{\mathcal{G}}(V_1, V_2)$$

where $\mathrm{Hom}_{\mathcal{G}}(V_1, V_2)$ is the space of $\mathcal{G}-$homomorphism linear maps.
Using Schur's Lemma(2.8.1):

- If $V_1 \ncong V_2$, then

$$\mathrm{Hom}_{\mathcal{G}}(V_1, V_2) = \{0\} \Longrightarrow \dim \mathrm{Hom}_{\mathcal{G}}(V_1, V_2) = 0.$$

- If $V_1 \cong V_2$, then:

$$\mathrm{Hom}_{\mathcal{G}}(V_1, V_2) \cong \mathbb{C} \Longrightarrow \dim \mathrm{Hom}_{\mathcal{G}}(V_1, V_2) = 1.$$

Thus,

$$\langle \chi_1, \chi_2 \rangle = \dim \mathrm{Hom}_{\mathcal{G}}(V_1, V_2) = \begin{cases} 1, & \text{if } V_1 \cong V_2 \\ 0, & \text{otherwise.} \end{cases}$$

Since irreducible representations are determined by their characters, we get:

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1, & \text{if } \chi_1 = \chi_2 \\ 0, & \text{if } \chi_1 \neq \chi_2. \end{cases} \quad .$$

□

☛**Proposition 3.6.1.** *[9] Since the characters are class functions, the formula can be written as:*

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|\mathcal{G}|} \sum_C |C| \cdot \chi_1(x_C) \cdot \overline{\chi_2(x_C)}.$$

3.7 Second Orthogonality Relation

Let $\mathcal{G}$ be a finite group and:

- $C_1, C_2, ..., C_k$ be conjugacy classes of $\mathcal{G}$.
- x_i be a representative of C_i.
- $\chi_1, ..., \chi_k$ be irreducible characters of $\mathcal{G}$.

▼**Theorem 3.7.1.** *[12] For all $x, y \in \mathcal{G}$:*

$$\sum_{i=1}^{k} \chi_i(x) \cdot \overline{\chi_i(y)} = \begin{cases} |\mathrm{C}_{\mathcal{G}}(x)|, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

3.8 Consequences of Orthogonality

Results 3.8.1. *[10] Let $(V, \mathscr{L})$ be a representation of $\mathcal{G}$ with character χ and let χ_i be an irreducible character. Then the multiplicity of irreducible representation V_i in V is:*

$$\mu_i = \langle \chi, \chi_i \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner products on class functions.

Results 3.8.2. *[13] The irreducible characters of $\mathcal{G}$ form an orthonormal basis for the vector space of class function on $\mathcal{G}$.*

Results 3.8.3. *[9] The number of irreducible characters of $\mathcal{G}$ is equal to the number of conjugacy classes of $\mathcal{G}$.*

Results 3.8.4. *[15] Let $\chi_1, \chi_2, ..., \chi_k$ be irreducible characters of $\mathcal{G}$ and let $n_i = \chi_i(e)$ be their degrees. Then:*

$$\sum_{i=1}^{k} n_i^2 = |\mathcal{G}|.$$

Results 3.8.5. *[12] A representation with character χ is irreducible if and only if*

$$\langle \chi, \chi \rangle = 1.$$

3.9 Linear Characters

✍**Definition 3.9.1.** *[10] A character χ of a group $\mathcal{G}$ is called **linear** if*

$$\chi(x) \neq 0, \quad \text{for all } x \in \mathcal{G}.$$

✭**Remark 3.9.1.** *[12] Since $GL_1(\mathbb{C}) = \mathbb{C}^\times$, a one-dimensional representation is a group homomorphism $\mathscr{L} : \mathcal{G} \to \mathbb{C}^\times$ with associated the character $\chi(x) = \mathscr{L}(x)$.*

☛**Proposition 3.9.1.** *[9] The set of linear characters of $\mathcal{G}$ forms a group under multiplication.*

▼**Theorem 3.9.1.** *[10] If $\mathcal{G}$ is abelian, then:*

- *All irreducible representations are one-dimensional.*
- *The number of irreducible characters is $|\mathcal{G}|$.*

✎**Example 3.9.1.** ✏ *Let $\mathcal{G} = \mathfrak{C}_3 = \{1, x, x^2\}$. The 1-dimensional representation $\chi : \mathfrak{C}_3 \to \mathbb{C}^\times$. Let:*

$$\chi(x) = t \Longrightarrow t^3 = 1$$

we get:

$$t = 1, \quad \omega = e^{\frac{2\pi i}{3}}, \quad \omega^2.$$

Thus

$$\begin{aligned} \chi_0(1) = 1, &\quad \chi_0(x) = 1, \quad \chi_0(x^2) = 1, \\ \chi_1(1) = 1, &\quad \chi_1(x) = \omega, \quad \chi_1(x^2) = \omega^2, \\ \chi_2(1) = 1, &\quad \chi_2(x) = \omega^2 \quad \chi_2(x^2) = \omega. \end{aligned}$$

Since all the values are nonzero, all the characters are linear. $|\mathfrak{C}_3| = 3$.

3.10 Degree of Irreducible Characters

✍**Definition 3.10.1.** *[10] Let $\mathscr{L} : \mathcal{G} \to GL(V)$ be a representation of $\mathcal{G}$ with character χ.* ***The degree*** *of the character χ is defined by:*

$$\chi(1) = \dim V.$$

☛**Proposition 3.10.1.** *[9] If χ is an irreducible character of $\mathcal{G}$. Then $\chi(1) \geq 1$. Moreover, $\chi(1) = 1 \iff$ the representation is one-dimensional.*

▼**Theorem 3.10.1.** *[15] If χ is an irreducible character of $\mathcal{G}$, then:*

$$\chi(1) \mid |\mathcal{G}|.$$

▼**Theorem 3.10.2.** *[13] If χ is an irreducible character of $\mathcal{G}$, then:*

$$\chi(1)^2 \leq |\mathcal{G}|.$$

▼**Theorem 3.10.3.** *[10] If $\mathcal{G}$ is abelian, then every irreducible character has degree 1. Thus all irreducible representation are one-dimensional.*

3.11 Regular Representation

✍**Definition 3.11.1.** *[9] Let $\mathcal{G}$ be a finite group. $\mathbb{C}[\mathcal{G}]$ is the $n-$dimensional complex vector space consisting of all formal linear combinations of group elements:*

$$\mathbb{C}[\mathcal{G}] = \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \in \mathbb{C} \right\}.$$

We equip this vector space with a multiplication operation from $\mathcal{G}$. For two elements $\sum_{x\in\mathcal{G}} \alpha_x x$ and $\sum_{y\in\mathcal{G}} \beta_y y$ in $\mathbb{C}[\mathcal{G}]$:

$$\left(\sum_{x\in\mathcal{G}} \alpha_x x\right) \cdot \left(\sum_{y\in\mathcal{G}} \beta_y y\right) = \sum_{x,y\in\mathcal{G}} (\alpha_x\beta_y)(xy).$$

✍**Definition 3.11.2.** *[10] Let $\mathcal{G}$ be a finite group and consider the vector space $V = \mathbb{C}[\mathcal{G}]$ whose basis is the set $\{e_x \mid x \in \mathcal{G}\}$. Define:*

$$\mathscr{L} : \mathcal{G} \to GL(\mathbb{C}[\mathcal{G}])$$

$$\mathscr{L}(\rho)(e_x) = e_{\rho x}.$$

This representation is called ***the regular representation****.*

☛**Proposition 3.11.1.** *[12] The degree of the regular representation is $|\mathcal{G}|$.*

▼**Theorem 3.11.1.** *[13] Let χ_{reg} be the character of the regular representation. Then:*

$$\chi_{reg}(x) = \begin{cases} |\mathcal{G}|, & x = e \\ 0, & x \neq e. \end{cases}$$

✎**Example 3.11.1.** ✏ *Let $\mathcal{G} = \mathfrak{C}_2 = \{1, x\}$ with $x^2 = 1$ Since $V = \mathbb{C}[\mathfrak{C}_2]$, if follows that it is a $2-$dimensional vector space with basis:*

$$\{e_1, e_x\}$$

We have:

$$\mathscr{L}(1)(e_1) = e_1$$

$$\mathscr{L}(1)(e_x) = e_{1\cdot x} = e_x$$

Thus, $\mathscr{L}(1)$ is the identity linear map:

$$\mathscr{L}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also:

$$\mathscr{L}(x)(e_1) = e_{x\cdot 1} = e_x$$

$$\mathscr{L}(x)(e_x) = e_{x\cdot x} = e_{x^2} = e_1$$

So

$$\mathscr{L}(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By definition of characters:

$$\chi(1) = \mathrm{Tr}(\mathscr{L}(1)) = 2.$$

$$\chi(x) = \mathrm{Tr}(\mathscr{L}(x)) = 0.$$

Consequently,

$$\chi_{reg}(x) = \begin{cases} 2, & x = 1 \\ 0, & x \neq 1. \end{cases}$$

▼**Theorem 3.11.2.** *[15] Let χ_i be an irreducible character with degree n_i then:*

$$\langle \chi_{reg}, \chi_i \rangle = n_i.$$

Proof. We have:

$$\langle \chi_{reg}, \chi_i \rangle = \frac{1}{|\mathcal{G}|} \sum_{x \in \mathcal{G}} \chi_{reg}(x) \overline{\chi_i(x)}.$$

Using Theorem(3.11.1), we get:

$$\begin{aligned} \langle \chi_{reg}, \chi_i \rangle &= \frac{1}{|\mathcal{G}|} \left(\chi_{reg}(e) \cdot \overline{\chi_i(e)} \right) \\ &= \frac{1}{|\mathcal{G}|} (|\mathcal{G}| \cdot \overline{n_i}) \\ &= n_i. \end{aligned}$$

□

3.12 Decomposition of Regular Representation

Let $\mathscr{L} : \mathcal{G} \to GL(\mathbb{C}[\mathcal{G}])$ be a regular representation of a finite group $\mathcal{G}$. Let $\{V_1, V_2, ..., V_k\}$ be the irreducible representation of $\mathcal{G}$ over $\mathbb{C}$ with dimensions $n_i = \dim V_i$.

▼**Theorem 3.12.1.** *[10] The regular representation decomposes as:*

$$\mathbb{C}[\mathcal{G}] \cong \bigoplus_{i=1}^{k} n_i V_i.$$

Proof. It suffices to prove:

$$\chi_{reg} = \sum_{i=1}^{k} n_i \chi_i.$$

Using Theorems(3.11.1) and (3.11.2) we have:

$$\langle \chi_{reg}, \chi_i \rangle = n_i.$$

Any character χ can be written uniquely as:

$$\chi = \sum_{i=1}^{k} \mu_i \chi_i$$

where $\mu_i = \langle \chi, \chi_i \rangle$. We apply this to χ_{reg} we get:

$$\mu_i = \langle \chi_{reg}, \chi_i \rangle = n_i.$$

Therefore:

$$\chi_{reg} = \sum_{i=1}^{k} n_i \chi_i.$$

□

➤**Corollary 3.12.1.** *[13] From the decomposition*

$$\mathbb{C}[\mathcal{G}] \cong \bigoplus_{i=1}^{k} n_i V_i.$$

we get:

$$\dim(\mathbb{C}[\mathcal{G}]) \cong \dim\left(\bigoplus_{i=1}^{k} n_i V_i\right)$$

$$\sum_{i=1}^{k} n_i^2 = |\mathcal{G}|.$$

Chapter 4

Character Table

In this chapter, we study character table and their role in describing the representation theory of finite groups. We also use the orthogonality relations of characters to obtain important structural results and to explain how character tables can be constructed in practice.

4.1 Character Table

✍**Definition 4.1.1.** *[17] Let $\mathcal{G}$ be a finite group, χ_i be an irreducible character and C_i the conjugacy classes.* ***The character table*** *of $\mathcal{G}$ is a square matrix in which the rows correspond to irreducible characters and the columns correspond to conjugacy classes.*

✯**Remark 4.1.1.** *[19] Because characters are constant on class functions, we have:*

$$\chi(x) = \chi(txt^{-1}).$$

✍**Definition 4.1.2.** *[18] The character table has the form:*

	C_1	C_2	C_3	$\cdots$	C_k
χ_1	$\chi_1(\mathrm{C}_1)$	$\chi_1(\mathrm{C}_2)$	$\chi_1(\mathrm{C}_3)$	$\cdots$	$\chi_1(\mathrm{C}_k)$
χ_2	$\chi_2(\mathrm{C}_1)$	$\chi_2(\mathrm{C}_2)$	$\chi_2(\mathrm{C}_3)$	$\cdots$	$\chi_2(\mathrm{C}_k)$
χ_3	$\chi_3(\mathrm{C}_1)$	$\chi_3(\mathrm{C}_2)$	$\chi_3(\mathrm{C}_3)$	$\cdots$	$\chi_3(\mathrm{C}_k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_k	$\chi_k(\mathrm{C}_1)$	$\chi_k(\mathrm{C}_2)$	$\chi_k(\mathrm{C}_3)$	$\cdots$	$\chi_k(\mathrm{C}_k)$

4.2 Properties of The Character Table

❖**Properties 4.2.1.** *[16]*

(i) *Each character satisfies:*

$$\chi(x) = \chi(txt^{-1}), \quad \textit{for all } x, t \in \mathcal{G}.$$

The coefficients depend only on conjugacy classes.

(ii) *The number of irreducible characters is equal to the number of conjugacy classes. Hence, the character table is always square.*

	C_1	$\cdots$	C_k
χ_1	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_k	$\cdots$	$\cdots$	$\cdots$

(iii) *For the first row:*

$$\chi_1(x) = 1, \quad \text{for all } x \in \mathcal{G}.$$

For the first column:

$$\chi_i(e) = \deg\ (\chi_i).$$

(iv)

$$\sum_{i=1}^{k} (\deg\ \chi_i)^2 = |\mathcal{G}|.$$

(v) *For each character:*

$$\chi(x^{-1}) = \overline{\chi(x)}$$

so:
. Real-valued characters $\Longrightarrow$ symmetric table.
. Complex-valued characters occur in conjugate pairs.

(iv) *For any irreducible character:*

$$|\chi(x)| \leq \chi(1).$$

(vii) *Each degree divides the group order:*

$$\deg\ (\chi_i) \mid |\mathcal{G}|.$$

(viii) *The character table reflects the class structure:*

$$|\mathcal{G}| = \sum_j |C_j|$$

each column correspond to a conjugacy class of the group.

(ix) *All character values are algebraic integers.*

4.3 Orthogonality of Rows

✯**Remark 4.3.1.** *[17] All characters are class functions, then the inner product may be written in terms of conjugacy classes as follows:*

$$\langle f, g \rangle = \frac{1}{|\mathcal{G}|} \sum_{i=1}^{k} |C_j| \cdot f(C_j) \cdot \overline{g(C_j)}.$$

▼**Theorem 4.3.1.** *[9] Let $\chi_1, \chi_2, ..., \chi_k$ be irreducible characters of $\mathcal{G}$. Then:*

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

☛**Proposition 4.3.1.** *[18] The orthogonality relation in* The Theorem(4.3.1) *can be expressed in the following form:*

$$\frac{1}{|\mathcal{G}|}\sum_{j=1}^{k}|\mathrm{C}_j|\cdot\chi_i(\mathrm{C}_j)\cdot\overline{\chi_j(\mathrm{C_j})}=\delta_{ij}$$

where δ_{ij} is the Kronecker delta.

Results 4.3.1. *[12]*

1. *Irreducible characters are linearly independent:*

$$\alpha_1\chi_1+\alpha_2\chi_2+\cdots+\alpha_k\chi_k=0\Longrightarrow\alpha_1=\alpha_2=\cdots=\alpha_k=0.$$

2. *Irreducible characters form an orthonormal basis of class functions, since every class function $\mathbf{f}:\mathcal{G}\to\mathbb{C}$ can be written as:*

$$\mathbf{f}=\sum_{i=1}^{k}\alpha_i\chi_i.$$

3. *If V is a representation with character χ, then the multiplicity of the irreducible representation V_i in V is given by:*

$$\mu_i=\langle\chi,\chi_i\rangle.$$

4. *For any character χ:*

$$\chi=\sum_i\langle\chi,\chi_i\rangle\chi_i.$$

5. *For any character χ:*

$$\langle\chi,\chi\rangle=\sum_i\mu_i^2.$$

6. *The decomposition of a character χ, into irreducible components is unique:*

$$\chi=\sum_i\mu_i\chi_i=\sum_i n_i\chi_i.$$

4.4 Orthogonality of Columns

▼**Theorem 4.4.1.** *[10] Let $\mathrm{C}_i,\mathrm{C}_j$ be conjugacy classes of $\mathcal{G}$. Then:*

$$\sum_{i=1}^{k}\chi_n(\mathrm{C}_i)\cdot\overline{\chi_m(\mathrm{C}_j)}=\begin{cases}\frac{|\mathcal{G}|}{|\mathrm{C}_i|}, & \text{if } i=j\\ 0, & \text{if } i\neq j.\end{cases}$$

☛**Proposition 4.4.1.** *[9] Define:*

$$\tilde{\chi}_n(\mathrm{C}_j)=\sqrt{|\mathrm{C}_j|}\chi_n(\mathrm{C}_j).$$

Then:

$$\sum_{i=1}^{k}\tilde{\chi}_n(\mathrm{C}_i)\overline{\tilde{\chi}_n(\mathrm{C}_j)}=|\mathcal{G}|\delta_{ij}.$$

Results 4.4.1. *[18]*

1. *The number of irreducible characters is equal to the number of conjugacy classes.*
2. *Columns of the character table are linearly independent.*
3. *Two elements $x, y \in \mathcal{G}$ are conjugate if and only if:*

$$\chi_i(x) = \chi_i(y).$$

4. $$\sum_i |\chi_i(\mathrm{C}_j)|^2 = \frac{|\mathcal{G}|}{|\mathrm{C}_j|}.$$

5. $$\sum_{i=1}^{k} |\chi_i(x)|^2 = |\mathrm{C}_{\mathcal{G}}(x)|$$

where $\mathrm{C}_{\mathcal{G}}(x)$ is the centralizer of x.

4.5 Steps to Construct a Character Table

The following procedure is commonly used when constructing a character table:

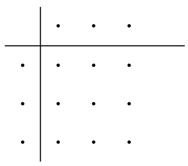
Group Structure ⟶ Classes ⟶ Degrees ⟶ Orthogonality ⟶ Completion

1. ***First***, *Note that the number of classes equals number of irreducible characters. Find all conjugacy classes:*

$$\mathrm{C}_1, \mathrm{C}_2, ..., \mathrm{C}_k$$

and their sizes,these determine the dimensions of the character table.

2. ***Second***, *Construct the initial frameork of the table, by creating columns(conjugacy classes) and rows(irreducible characters) with:*
- First row: for the trivial representation.
- First column: degree of $\chi_i(e)$.



3. ***Third***, *Apply the following formula:*

$$\sum_{i=1}^{k} (\deg\ \chi_i)^2 = |\mathcal{G}|$$

so we can determine possible degrees.

4. ***Fourth***, *Next use familiar representations such as the trivial, sign, or permutation representations to determine some rows explicitly.*

5. ***Fifth***, *Apply orthogonality of rows:*

$$\frac{1}{|\mathcal{G}|}\sum_{j=1}^{k}|\mathrm{C}_j|\cdot\chi_i(\mathrm{C}_j)\cdot\overline{\chi_l(\mathrm{C_j})}=\delta_{il}.$$

6. ***Sixth***, *Apply orthogonality of columns:*

$$\sum_{l=1}^{k}\chi_l(\mathrm{C}_i)\cdot\overline{\chi_l(\mathrm{C}_j)}=\frac{|\mathcal{G}|}{|\mathrm{C}_i|}\delta_{ij}.$$

7. ***Finally***, *Keep in mind the following constraints on character value:*
   - Character bounds:

$$|\chi(x)|\leq\chi(1).$$

   - Character values must be:
   - *Integers.*
   - *Roots of unity combinations.*
   - $\chi(x^{-1})=\overline{\chi(x)}$.

4.6 Computation of Character Tables of Small Groups

4.6.1 Cyclic Groups $\mathfrak{C}_4$

(I) Consider the cyclic group $\mathfrak{C}_4$

$$\mathfrak{C}_4=\langle x\mid x^4=e\rangle$$

its elements:

$$\mathfrak{C}_4=\{e,x,x^2,x^3\}.$$

The group $\mathfrak{C}_4$ is abelian and has order 4.

(II) Since $\mathfrak{C}_4$ is an abelian group, it follows that:

$$yxy^{-1}=x,\quad \text{for all } x,y\in\mathfrak{C}_4.$$

Thus, each element forms its own conjugacy class:

$$yey^{-1}=e,\quad yxy^{-1}=x,\quad yx^2y^{-1}=x^2,\quad yx^3y^{-1}=x^3.$$

Therefore, we have 4 conjugacy classes:

$$\{e\},\{x\},\{x^2\},\{x^3\}.$$

(III) Applying the formula:

$$\sum_{i=1}^{k} (\deg\ \chi_i)^2 = |\mathcal{G}|.$$

But since $\mathfrak{C}_4$ is abelian, then all irreducible representations are 1−dimensional. Then:

$$(\deg\ \chi_i)^2 = 1 \quad \text{for all } i = 0, 1, 2, 3.$$

(IV) In cyclic groups, every irreducible representation is determined by:

$$\mathscr{L}_k(x) = \omega^k, \quad \text{where } \omega = e^{\frac{2\pi k}{4}}.$$

We have $x^4 = 1$, so we get 4 roots of unity:

$$\omega^0 = 1, \quad \omega^1 = i, \quad \omega^2 = -1, \quad \omega^3 = -i.$$

Now, Using this representation, the characters are computed by:

$$\chi_k(x^n) = \omega^{kn} \quad \text{for } k = 0, 1, 2, 3.$$

(V) We set up the following character table:

$\mathfrak{C}_4$	e	x	x^2	x^3
χ_0	1	1	1	1
χ_1	1	i	-1	$-i$
χ_2	1	-1	1	-1
χ_3	1	$-i$	-1	i

4.6.2 Symmetric Groups $\mathfrak{S}_3$

(I) The symmetric group $\mathfrak{S}_3$ consists of all permutations of the set $1, 2, 3$. Its elements:

$$\mathfrak{S}_3 = \{e, (12), (13), (23), (123), (132)\}$$

with $|\mathfrak{S}_3| = 6$.

(II) In $\mathfrak{S}_3$, conjugacy classes are determined by cycle types. Therefore:

$$\mathrm{C}_1 : \{e\}.$$

$$\mathrm{C}_2 : \{(12), (13), (23)\}.$$

$$\mathrm{C}_3 : \{(123), (132)\}.$$

Hence, we have 3 conjugacy classes.

(III) We have:

$$\sum_{i\geq 1} (\deg\ \chi_i)^2 = |\mathfrak{S}_3| = 6.$$

Then:

$$1^2 + 1^2 + 2^2 = 6.$$

(IV) Now, we determine the characters:

(i) Consider the trivial representation $\mathscr{L}(x) = 1$. Then:

$$\chi_1 = (1, 1, 1).$$

(ii) Define the sign representation:

$$\mathscr{L}(x) = \begin{cases} 1, & \text{if } x \text{ is even}, \\ -1, & \text{if } x \text{ is odd}. \end{cases}$$

Then:

$$\chi_2 = (1 - 1, 1).$$

(iii) We use the orthogonality to determine the 2−dimensional representation. Let $\chi_3 = (2, x, y)$. We have:

$$\bullet \frac{1}{6}[1(2) + 3x + 2y] = 0 \Longrightarrow 2 + 3x + 2y = 0.$$

$$\bullet \frac{1}{6}[1(2) + 3(-x) + 2y] = 0 \Longrightarrow 2 - 3x + 2y = 0.$$

This gives the following system of equations:

$$\begin{cases} 2 + 3x + 2y = 0 \\ 2 - 3x + 2y = 0 \end{cases}$$

By subtraction we get

$$x = 0, y = -1.$$

Therefore:

$$\chi_3 = (2, 0, -1).$$

(V) The character table of $\mathfrak{S}_3$ is:

$\mathfrak{S}_3$	e	(12)	(123)
χ_1	1	1	1
χ_2	1	−1	1
χ_3	2	0	−1

4.6.3 Dihedral Groups $\mathfrak{D}_{2n}$

We will compute the character table of $\mathfrak{D}_{2n}$ in two cases:

1. n is odd :

(I) Consider the case $n = 3$. The dihedral group:

$$\mathfrak{D}_6 = \langle r, s \mid r^3 = e, s^2 = e, srs = r^{-1} \rangle$$

its elements:

$$\mathfrak{D}_6 = \{e, r, r^2, s, sr, sr^2\}$$

with $|\mathfrak{D}_6| = 6$.

(II) Since $sr^k s = r^{-k}$, we have:

$$\{e\}, \quad \{r, r^2\}, \quad \{s, sr, sr^2\}.$$

Hence, there are three conjugacy classes and therefore three irreducible characters.

(III) Since $\mathfrak{D}_6 \cong \mathfrak{S}_3$ we get:

$$\sum_{i\geq 1} (\deg \chi_i)^2 = |\mathfrak{D}_6| = 6 \Longrightarrow 1^2 + 1^2 + 2^2 = 6.$$

(IV) We compute the characters:

(i) We define the trivial representation:

$$\mathscr{L}_1(r) = 1, \quad \mathscr{L}_1(s) = 1.$$

Then:

$$\chi_1 = (1, 1, 1).$$

(ii) Define the sign representation:

$$\mathscr{L}_2(r) = 1, \quad \mathscr{L}_2(s) = -1.$$

Then:

$$\mathscr{L}_2(sr^k) = \mathscr{L}_2(s)\mathscr{L}_2(r^k) = -1.$$

Therefore:

$$\chi_2 = (1, 1, -1).$$

(iii) Now let $\mathfrak{D}_6$ acts on $V = \mathbb{R}^2$.The rotation r is represented by:

$$\mathscr{L}_3(r) = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} \Longrightarrow \mathscr{L}_3(r) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

The representation of s:

$$\mathscr{L}_3(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since this representation has dimension 2, we obtain:

$$\chi_3(e) = 2, \quad \chi_3(r) = -1, \quad \chi_3(s) = 0.$$

Thus,

$$\chi_3 = (2, -1, 0).$$

(V) We can set up the character table:

$\mathfrak{D}_6$	e	$\{r, r^2\}$	$\{s, sr, sr^2\}$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

1. n is even :

(I) Take $n = 4$. We have:

$$\mathfrak{D}_8 = \langle r, s \mid r^4 = e, s^2 = e, srs = r^{-1} \rangle$$

its elements:

$$\mathfrak{D}_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

with $|\mathfrak{D}_8| = 8$.

(II) We have the following conjugacy classes:

$$\{e\}, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}.$$

(III) Since we have 5 irreducible characters:

$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8.$$

(IV) Computing the characters:

(i) We define the trivial representation:

$$\mathscr{L}_1(r) = 1, \quad \mathscr{L}_1(s) = 1.$$

Hence:

$$\chi_1 = (1, 1, 1, 1, 1).$$

(ii) Since $r^4 = e$, the one-dimensional representation send r to either 1 or -1;

$$\mathscr{L}_2(r) = 1, \quad \mathscr{L}_2(s) = -1.$$

Thus:

$$\chi_2 = (1, 1, 1, -1, -1).$$

And

$$\mathscr{L}_3(r) = -1, \quad \mathscr{L}_3(s) = 1$$
$$\Longrightarrow \chi_3 = (1, 1, -1, 1, -1).$$

And

$$\mathscr{L}_3(r) = -1, \quad \mathscr{L}_3(s) = -1$$
$$\chi_4 = (1, 1, -1, -1, 1).$$

(iii) We have the 2−dimensional representations:

$$\mathscr{L}_5(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathscr{L}_5(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consequently:

$$\chi_5 = (2, -2, 0, 0, 0).$$

(V) Therefore, the character table of $\mathfrak{D}_8$ is given by:

$\mathfrak{D}_8$	e	r^2	$\{r, r^3\}$	$\{s, sr^2\}$	$\{sr, sr^3\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	1	−1	1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

4.6.4 Quaternion Group $\mathfrak{Q}_8$

(I) Consider the quaternion group $\mathfrak{Q}_8$

$$\mathfrak{Q}_8 = \{1, -1, i - i, j, -j, k, -k\}$$

whose elements satisfying:

$$i^2 = j^2 = k^2 = -1$$

with $|\mathfrak{Q}_8| = 8$.

(II) Using the definition of conjugacy classes, we obtain:

$$\{1\}, \quad \{-1\}, \quad \{i, -i\}, \quad \{j, -j\}, \quad \{k, -k\}.$$

(III) Using the formula $\sum_i (\deg \chi)^2 = |\mathfrak{Q}_8| = 8$, since $\mathfrak{Q}_8$ has four 1−dimensional representations and one 2−representation then:

$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8.$$

(IV) Computing the characters:

(i) We have:

$$\mathscr{L}_1(i) = \mathscr{L}_1(j) = \mathscr{L}_1(j) = 1$$

consequently:

$$\chi_1 = (1, 1, 1, 1).$$

(ii) Consider $\mathscr{L}_2(i) = 1, \mathscr{L}_2(j) = -1$, then:

$$\mathscr{L}_2(k) = \mathscr{L}_2(i)\mathscr{L}_2(j) = -1$$

$$\Longrightarrow \chi_2 = (1, 1, 1, -1, -1).$$

And $\mathscr{L}_3(i) = -1, \mathscr{L}_3(j) = 1$, then $\mathscr{L}_3(k) = -1$, so $\chi_3 = (1, 1, -1, 1, -1)$. With the same argument; $\mathscr{L}_4(i) = -1, \mathscr{L}_4(j) = -1$, it follows that $\mathscr{L}_4(k) = 1$ then $\chi_4 = (1, 1, -1, -1, 1)$.
We define the 2−dimensional representation:

$$\mathscr{L}_5 : \mathfrak{Q}_8 \to GL_2(\mathbb{C})$$

$$\mathscr{L}_5(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathscr{L}_5(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathscr{L}_5(k) = \mathscr{L}_5(i)\mathscr{L}_5(j) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\Longrightarrow \chi_5 = (2, -2, 0, 0, 0).$$

(V) Consequently, we obtain the following character table:

$\mathfrak{Q}_8$	1	-1	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

✯**Remark 4.6.1.** *We observe that $\mathfrak{D}_8$ and $\mathfrak{Q}_8$ have the same character degrees and character tables, both are not abelian, so they are not isomorphic ($\mathfrak{D}_8 \ncong \mathfrak{Q}_8$).*

Chapter 5

Applications

In this chapter, we apply character theory to several important problems in finite group theory. In particular, orthogonality relations will be used to prove Burnside's Theorem, study simplicity of groups, and investigate linear characters and the center of a group.

5.1 Burnside's $p^a \cdot q^b-$Theorem

✍**Definition 5.1.1.** *[12] A complex number $z \in \mathbb{C}$ is called* ***an algebraic integer*** *if it is a root of a monic polynomial (leading coefficient equals 1) with integer coefficients.*

✎**Example 5.1.1.**

- ✏ *Since i is a root of x^2+1, we have that it is an algebraic integer.*
- ✏ *$\sqrt{2}$ is an algebraic integer (root of x^2-2).*
- ✏ *Since $2x-1$ is not a monic polynomial, it does not satisfy the definition; therefore $\frac{1}{2}$ is not an algebraic integer.*

✍**Definition 5.1.2.** *[1] A finite group $\mathcal{G}$ is said to be* ***solvable*** *if there exists a sequence of normal subgroups:*

$$\{e\} = \mathcal{G}_0 \trianglelefteq \mathcal{G}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{G}_n = \mathcal{G}$$

such that each quotient $\mathcal{G}_{i+1}/\mathcal{G}_i$ is abelian.

◆**Lemma 5.1.1.** *[10] Let χ be a character of a finite group $\mathcal{G}$. Then for every $x \in \mathcal{G}$*

$$\chi(x) \quad \text{is an algebraic integer.}$$

Proof. Let χ be the character of a representation $(V, \mathscr{L})$ where $\dim V = d$. Let n be the order of $x \in \mathcal{G}$, so that $x^n = e$.
Since $\mathscr{L}$ is a homomorphism $\mathscr{L}(x)^n = \mathscr{L}(x^n) = \mathbb{I}_d$, which satisfies the equation $T^n - \mathbb{I}_d = 0$.
Therefore, every eigenvalue λ of $\mathscr{L}(x)$ is an $n-$th root of unity. Consequently, λ is an algebraic integer. Let $\lambda_1, \lambda_2, ..., \lambda_d$ be the eigenvalues of $\mathscr{L}(x)$. By definition:

$$\chi(x) = \text{Tr }(\mathscr{L}(x)) = \lambda_1 + \lambda_2 + \cdots + \lambda_d.$$

Since each λ_i is an algebraic integer, their sum $\chi(x) = \sum_{i=1}^{d} \lambda_i$ is also an algebraic integer. □

☛**Proposition 5.1.1.** *[12] If $\frac{|\mathrm{C}|}{\chi(1)}\chi(x)$ is both a rational number and an algebraic integer, then it is an integer.*

◆**Lemma 5.1.2.** *[1] Let $\mathcal{G}$ be a finite group and C a conjugacy class, then:*

$$|\mathrm{C}| = [\mathcal{G} : \mathrm{C}_{\mathcal{G}}(x)]$$

where $\mathrm{C}_{\mathcal{G}}(x)$ is the centralizer of x.

☛**Proposition 5.1.2.** *[12] Let C be a conjugacy class of a finite group $\mathcal{G}$, and let p be a prime. If $|\mathrm{C}|$ is not divisible by p, then the centralizer $\mathrm{C}_{\mathcal{G}}(x)$ has index not divisible by p. Consequently, $\mathrm{C}_{\mathcal{G}}(x)$ contains a Sylow $p-$subgroup of $\mathcal{G}$.*

✍**Definition 5.1.3.** *[1] A finite group $\mathcal{G}$ is called **simple** if its only normal subgroups are*

$$\{e\} \quad \text{and} \quad \mathcal{G}.$$

✎**Example 5.1.2.** ✏ *Let p be a prime number. Then:*

$$\mathcal{G} = \mathbb{Z}/p\mathbb{Z}$$

is a simple group.

✏ *The group*

$$A_n = \{\text{even permutations in } \mathfrak{S}_n\}$$

is simple for all $n \geq 5$.

➤**Corollary 5.1.1.** *[3] Let p, q be prime numbers, and let $\mathcal{G}$ be a finite group of order $p^a q^b$. Then $\mathcal{G}$ is not simple unless it is cyclic of prime order.*

▼**Theorem 5.1.1.** *[9](Burnside's Theorem)*
Let $\mathcal{G}$ be a finite group such that:

$$|\mathcal{G}| = p^a \cdot q^b$$

where p, q are prime numbers. Then $\mathcal{G}$ is solvable.

Proof. Assume there are primes p, q such that there exists a group of order $p^a q^b$ which is not solvable. Let $\mathcal{G}$ be such a group smallest possible order.
Since every abelian group is solvable, the group $\mathcal{G}$ cannot be abelian. By Corollary(5.1.1), the group $\mathcal{G}$ is not simple, so it contains a nontrivial normal subgroup $\mathcal{N}$. By the minimality of $|\mathcal{G}|$, both $\mathcal{N}$ and $\mathcal{G}/\mathcal{N}$ must to be solvable.
Notice that if $|\mathcal{G}|p^a q^b$, then $|\mathcal{N}| = p^{a_1} q^{b_1}$ and $|\mathcal{G}/\mathcal{N}| = p^{a-a_1} q^{b-b_1}$.
Moreover, the class of solvable groups is closed under extensions, so if $\mathcal{G}$ and $\mathcal{G}/\mathcal{N}$ are solvable, it follows that $\mathcal{G}$ is solvable. □

5.2 The Use of Characters to Study Simplicity

✍**Definition 5.2.1.** *[12] Let χ be the character associated with representation $(V, \mathscr{L})$.* ***The kernel*** *of χ is defined by:*

$$\ker(\chi) = \{x \in \mathcal{G} \mid \chi(x) = \chi(1)\}.$$

☛**Proposition 5.2.1.** *[10] Let $(V, \mathscr{L})$ be a representation of a finite group $\mathcal{G}$, and let χ be its character. Then:*

$$\ker(\chi) = \ker(\mathscr{L}).$$

In particular, $\ker(\chi) \trianglelefteq \mathcal{G}$.

Proof. Let $x \in \ker(\mathscr{L})$ then:

$$\mathscr{L}(x) = \mathbb{I}d_V$$

Taking the trace on both sides gives:

$$\chi(x) = \mathrm{Tr}(\mathscr{L}(x)) = \mathrm{Tr}(\mathbb{I}d_V) = \dim(V) = \chi(1).$$

So

$$x \in \ker(\chi) \Longrightarrow \ker(\mathscr{L}) \subseteq \ker(\chi).$$

Let $x \in \ker(\chi)$, then:

$$\chi(x) = \chi(1) = \dim(V).$$

Since $\mathscr{L}(x)$ is a linear operator of finite order, its eigenvalues are roots of unity.
Let $\lambda_1, ..., \lambda_n$ be the eigenvalues;

$$\chi(x) = \lambda_1 + \cdots + \lambda_n$$

since each eigenvalue satisfies $|\lambda_i| = 1$ and their sum equals n, it follows that:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1.$$

Therefore

$$\mathscr{L}(x) = \mathbb{I}d_V.$$

Consequently,

$$x \in \ker(\mathscr{L}).$$

□

☛**Proposition 5.2.2.** *[9] Every linear character defines a group homomorphism from $\mathcal{G}$ into $\mathbb{C}^\times$.*

▼**Theorem 5.2.1.** *[12] If a non-abelian group $\mathcal{G}$ admits a nontrivial linear character, then $\mathcal{G}$ is not simple.*

Proof. Let $\chi : \mathcal{G} \to \mathbb{C}^\times$ be a nontrivial linear character. Since $\ker(\chi) \trianglelefteq \mathcal{G}$, then:
Since χ is nontrivial, there exists an element $x \in \mathcal{G}$ for which $\chi(x) \neq 1$. Hence:

$$\ker(x) \neq \mathcal{G}.$$

Suppose that $\ker(\chi) = \{e\}$. Then χ is injective, which implies that $\mathcal{G}$ is isomorphic to a subgroup of $\mathbb{C}^\times$. Since $\mathbb{C}^\times$ is abelian, $\mathcal{G}$ must also be abelian, this contradicts the assumption that $\mathcal{G}$ is non-abelian, so:

$$\ker(\chi) \neq \{e\}.$$

□

➤**Corollary 5.2.1.** *[11] If a group has more than one irreducible character of degree* 1*, then it is not simple.*

▼**Theorem 5.2.2.** *[10] Let* χ *be irreducible character. If there exists* $x \neq e$ *such that* $\chi(x) = \chi(1)$*, then* $x \in \ker(\chi)$*, hence, it generates a nontrivial normal subgroups of* $\mathcal{G}$*.*

✎**Example 5.2.1.** ✏ *Consider the group* $\mathfrak{S}_3$*, then the character table:*

$\mathfrak{S}_3$	e	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

- χ_2 *is a nontrivial linear character. Hence* $\mathfrak{S}_3$ *is not simple.*

5.3 Sum of Squares Formula

✍**Definition 5.3.1.** *[10] Let* $\mathcal{G}$ *be a finite group.* ***A character*** χ *is called irreducible if it arises from an irreducible representation of* $\mathcal{G}$*.*

✍**Definition 5.3.2.** *[12] The degree of* χ *is defined by*

$$\deg(\chi) = \chi(1)$$

✍**Definition 5.3.3.** *[13] Let* $\mathcal{G}$ *be a finite group. We define* ***the regular representation*** *of the group* $\mathcal{G}$*:*

$$\mathscr{L} : \mathcal{G} \to GL(\mathbb{C}[\mathcal{G}])$$
$$\mathscr{L}(\rho)(x) = \rho x.$$

☛**Proposition 5.3.1.** *The dimension of the regular representation is:*

$$\dim(\mathbb{C}[\mathcal{G}]) = |\mathcal{G}|.$$

☛**Proposition 5.3.2.** *The character of the regular representation satisfies:*

$$\chi_{reg}(x) = \begin{cases} |\mathcal{G}|, & \text{if } x = e \\ 0, & \text{if } x \neq e. \end{cases}$$

➢**Notation 10.** *We denote by* $\mathscr{L}_{reg}$ *the regular representation.*

▼**Theorem 5.3.1.** *[10] Let* $\{\chi_1, \cdots \chi_k\}$ *be irreducible characters of* $\mathcal{G}$*, with degrees* $n_i = \chi_i(1)$*. Then:*

$$\mathscr{L}_{reg} \cong \bigoplus_{i=1}^{k} n_i \mathscr{L}_i$$

▼**Theorem 5.3.2.** *[12](Sum of Squares Formula)*
Let $\mathcal{G}$ *be a finite group and* $\{\chi_1, \cdots \chi_k\}$ *its irreducible characters. Then:*

$$\sum_{i=1}^{k} (\deg \chi_i)^2 = |\mathcal{G}|.$$

Proof. From Theorem(5.3.1), we have:

$$\mathscr{L}_{reg} \cong \bigoplus_{i=1}^{k} n_i \mathscr{L}_i$$

By comparing dimensions on both sides, we obtain:

$$|\mathcal{G}| = \sum_{i=1}^{k} n_i \cdot \dim(\mathscr{L}_i)$$

because $\dim(\mathscr{L}_i) = n_i$, this simplifies to:

$$|\mathcal{G}| = \sum_{i=1}^{k} n_i^2.$$

□

➤**Corollary 5.3.1.** *[10] For every irreducible character χ,*

$$\deg(\chi) \mid |\mathcal{G}|.$$

✎**Example 5.3.1.** ✏ *Take $\mathcal{G} = \mathfrak{S}_3$. We have:*

$$|\mathfrak{S}_3| = 6$$

we get the following decomposition:

$$1^2 + 1^2 + 2^2 = 6.$$

Thus, the only degrees are:

$$1, 1, 2.$$

5.4 Center of a Group and Linear Characters

✍**Definition 5.4.1.** *[1] Let $\mathcal{G}$ be a group.* ***The center*** *of $\mathcal{G}$ is defined by:*

$$\mathrm{Z}(\mathcal{G}) = \{x \in \mathcal{G} \mid xy = yx \quad \textit{for all } y \in \mathcal{G}\}.$$

✎**Example 5.4.1.** ✏ *Consider the general linear group $GL_2(\mathbb{R})$, then*

$$\mathrm{Z}(GL_2(\mathbb{R})) = \{\lambda \mathbb{I} \mid \lambda \in \mathbb{R}^\times\}.$$

✏ *If $\mathcal{G}$ is abelian, then very element commutes with every other element so:*

$$\mathrm{Z}(\mathcal{G}) = \mathcal{G}.$$

Take $\mathcal{G} = \mathbb{Z}_6$, then $\mathrm{Z}(\mathcal{G}) = \mathbb{Z}_6$.

☛**Proposition 5.4.1.** *[9] The center $\mathrm{Z}(\mathcal{G})$ is a normal subgroup of $\mathcal{G}$.*

☛**Proposition 5.4.2.** *[8] A group $\mathcal{G}$ is abelian if and only if:*

$$Z(\mathcal{G}) = \mathcal{G}.$$

✍**Definition 5.4.2.** *[1]* ***A linear character*** *of a finite group $\mathcal{G}$ is an irreducible character of degree 1:*

$$\chi : \mathcal{G} \to \mathbb{C}^{\times}.$$

✎**Example 5.4.2.** ✏ *Take $\mathcal{G} = \mathfrak{S}_3$, then $\mathfrak{S}_3$ has two linear characters:*

- *Trivial character $\chi(x) = 1$.*
- *Sign character $\chi(\sigma) = sgn(\sigma)$.*

✏ *Let $\mathcal{G} = \mathfrak{C}_3 = \langle x \mid x^3 = e \rangle$. we have: $\omega = e^{\frac{2\pi i}{3}}$, then:*

$$\chi_0 : e \mapsto 1, \quad x \mapsto 1, \quad x^2 \mapsto 1.$$

$$\chi_1 : e \mapsto 1, \quad x \mapsto \omega, \quad x^2 \mapsto \omega^2.$$

$$\chi_2 : e \mapsto 1, \quad x \mapsto \omega^2, \quad x^2 = \omega^4 = \omega.$$

✮**Remark 5.4.1.** *[13] Every linear character is a group homomorphism.*

☛**Proposition 5.4.3.** *[9] The kernel of a linear character is a normal subgroup of $\mathcal{G}$.*

▼**Theorem 5.4.1.** *[12] The set of all linear characters of $\mathcal{G}$ forms a group under multiplication:*

$$(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x).$$

This group is called ***The dual group*** *of $\mathcal{G}$.*

✍**Definition 5.4.3.** *[1]* ***The commutator subgroup*** *is the subgroup generated by all commutators of elements of $\mathcal{G}$:*

$$[\mathcal{G}, \mathcal{G}] = \langle xyx^{-1}y^{-1} \mid x, y \in \mathcal{G} \rangle.$$

☛**Proposition 5.4.4.** *[9] $[\mathcal{G}, \mathcal{G}]$ is a normal subgroup of $\mathcal{G}$:*

$$[\mathcal{G}, \mathcal{G}] \trianglelefteq \mathcal{G}.$$

❖**Properties 5.4.1.** *[16]*

(i) *The quotient group $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$ is abelian; it's the largest abelian quotient of $\mathcal{G}$ called* ***the abelianization*** *of $\mathcal{G}$ denoted $\mathcal{G}^{ab}$.*

(ii) *$\mathcal{G}$ is abelian if and only if $[\mathcal{G}, \mathcal{G}] = \{e\}$.*

✎**Example 5.4.3.** ✏ *If $\mathcal{G}$ is abelian, then $xyx^{-1}y^{-1} = e$. Thus:*

$$[\mathcal{G}, \mathcal{G}] = \{e\}.$$

✏ *Any simple non-abelian group $\mathcal{G}$:*

$$[\mathcal{G} : \mathcal{G}] = \mathcal{G}.$$

✏ *Consider the general linear group* $GL_n(\mathbb{R})$*. Then for* $n \geq 2 : [GL_n(\mathbb{R}), GL_n(\mathbb{R})] = SL_n(\mathbb{R})$

▼**Theorem 5.4.2.** *[10] Every linear character is trivial on* $[\mathcal{G} : \mathcal{G}]$*;*

$$[\mathcal{G} : \mathcal{G}] \subseteq \ker(\chi).$$

☛**Proposition 5.4.5.** *[17] Let* $[\mathcal{G}, \mathcal{G}]$ *be the commutator subgroup of* $\mathcal{G}$*. Then:*

$$|\mathcal{G} : [\mathcal{G}, \mathcal{G}]| = \text{ number of linear characters.}$$

◆**Lemma 5.4.1.** *[13](Schur's Lemma Consequence)*
Let $\mathscr{L}$ *be an irreducible representation of* $\mathcal{G}$*. Then for all* $x \in \mathrm{Z}(\mathcal{G})$*,*

$$\mathscr{L}(x) = \lambda \mathbb{I}, \quad \lambda \in \mathbb{C}.$$

▼**Theorem 5.4.3.** *[19] Let* $\mathcal{G}$ *be a finite group, for every irreducible character* χ*, if* $\chi(1) \neq 1$ *then there exists* $x \in \mathcal{G}$ *such that* $\chi(x) = 0$*.*

Proof. We will use the column orthogonality relation;

$$\sum_{x \in Irr(\mathcal{G})} |\chi(x)|^2 = |\mathrm{C}_{\mathcal{G}}(x)|.$$

Suppose $\chi(x) \neq 0$ for every $x \in \mathcal{G}$. Then:

$$|\chi(x)|^2 = \chi(1)$$

for $x = 1$ we have:

$$\chi(1)^2 = \chi(1) \Longrightarrow \chi(1) = 1$$

which contradict our assumption.
Consequently, $\chi(x) = 0$. □

☛**Proposition 5.4.6.** *[17] For every irreducible character* χ *of* $\mathcal{G}$*:*

$$\chi(1) \mid |\mathcal{G} : \mathrm{Z}(\mathcal{G})|.$$

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