BRESSE-TIMOSHENKO SYSTEM : WELL-POSSEDNESS AND STABILITY RESULTS

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Abstract. In this paper, we consider a Bresse-Timoshenko type system with distributed delay term. Under suitable assumptions, we establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates. By using the energy method, we show the exponential stability results for the system with delay in vertical displacement.

1. introduction

In this paper, we deal with a nolinear Bresse-Timoshenko type system with distributed delay, under appropriate assumptions and we study the exponential decay.
It is related to the problem of stability for dissipative models of the Timoshenko type related to the problem of the damage consequences of the so called second spectrum of frequencies, or simply second spectrum. We mention the recent works in [2]-[4] and [10], the distributed delay is not explicitly presented, and therefore it makes sense to consider the problems in this paper. In [18], a Timoshenko system is considered

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu\psi_t = 0 \end{cases} \tag{1}$$

and after an explanation about the meaning of second spectrum based on existing literature, the authors showed that the viscous damping acting on angle rotation of (1).

To the best of our knowledge, the first contribution in that direction was obtained by Manevich and Kolakowski [16]. They analyzed the dynamic of a Timoshenko model where the damping mechanism is viscoelastic. More precisely, they considered the dissipative system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x - \mu_1(\varphi_x + \psi)_{tx} = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) - \mu_2\psi_{ttx} + \mu_1(\varphi_x + \psi)_t = 0 \end{cases} \tag{2}$$

Secondly, based on Elishakoff's papers and collaborators and their studies on truncated versions for classical Timoshenko equations [1] (see also recent contributions of Elishakoff et *al.* [7]-[8]), Almeida Junior and Ramos [2] showed that the total energy for viscous damping acting on angle rotation of the simplified Timoshenko

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system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t = 0 \end{cases} \tag{3}$$

The model is very different from classical Timoshenko system, since it contains three derivatives: two derivatives with respect to time and one derivative with respect to space. The reason behind this is the absence of the second spectrum or non-physical spectrum [1, 8] and its damage consequences for wave propagation speeds [2]. We can find the historical and mathematical observations in [1, 8]. The same results are achieved for a dissipative truncated version where the viscous damping acts on vertical displacement

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{4}$$

Then, in order to get more consistent exponential decay results in light of the absence of second spectrum, Almeida Junior et *al.* [4] considered two cases of dissipative systems for Bresse-Timoshenko type systems with constant delay cases. For the first one, the authors proved the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau) = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{5}$$

For the second one, the authors also proved the exponential decay result for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(x, t-\tau) = 0 \end{cases} \tag{6}$$

Feng et *al.* [10] considered two cases of dissipative systems for Bresse-Timoshenko type systems with time-varying delay cases. For the first one, the authors proved the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau(t)) = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{7}$$

For the second one, the authors also proved the exponential decay result for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(x, t-\tau(t)) = 0 \end{cases} \tag{8}$$

A complement to these works, we are working to establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates. And prove the exponential decay of two cases of dissipative systems for Bresse-Timoshenko type systems with distributed delay, under appropriate assumptions and we prove these results using the energy method and with the help of convex functions. In the following, let c a positive constant.

2. Distributed delay and viscous damping in vertical displacement

Here, we are concerned with the following system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(p)|\varphi_t\,(x, t-p)\,dp = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{9}$$

where

$$(x,p,t)\in(0,1)\times(\tau_1,\tau_2)\times(0,\infty)$$

Additionally, we consider initial conditions given by

$$\begin{cases}\varphi(x,0)=\varphi_0(x),\varphi_t(x,0)=\varphi_1(x),\varphi_{tt}(x,0)=\varphi_2(x)\\ \varphi_{ttt}(x,0)=\varphi_3(x),\ \psi(x,0)=\psi_0(x),x\in(0,1)\end{cases}\tag{10}$$

where $\varphi_0,\varphi_1,\varphi_2,\psi_0,$ are given functions, and boundary conditions of Dirichlet-Dirichlet given by

$$\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0,\ t>0\tag{11}$$

Wherever, φ is the transverse displacement of the beam, ψ is the angle of rotation, and $\rho_1,\rho_2,b,\beta>0$ and the integral represents the distributed delay term with $\tau_1,\tau_2>0$ are a time delay, μ_1 is positive constant, μ_2 is an L^∞function.
(**A1**) $\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2}|\mu_2(p)|dp\leq\mu_1\tag{12}$$

In order to deal with the disributed delay feedback term, motivated by [17], let use introduce a new dependent variable

$$y(x,\tau,p,t)=\varphi_t(x,t-p\tau),\tag{13}$$

Using (13), we have

$$\begin{cases}py_t(x,\tau,p,t)=-y_\tau(x,\tau,p,t)\\ y(x,0,p,t)=\varphi_t(x,t).\end{cases}\tag{14}$$

Thus, the problem is equivalent to

$$\begin{cases}\rho_1\varphi_{tt}-\beta(\varphi_x+\psi)_x+\mu_1\varphi_t+\displaystyle\int_{\tau_1}^{\tau_2}|\mu_2(p)|y(x,1,p,t)\,dp=0\\ -\rho_2\varphi_{ttx}-b\psi_{xx}+\beta(\varphi_x+\psi)=0\\ py_t(x,\tau,p,t)+y_\tau(x,\tau,p,t)=0\end{cases}\tag{15}$$

where

$$(x,\tau,p,t)\in(0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Additionally, we consider initial conditions given by

$$\begin{cases}\varphi(x,0)=\varphi_0(x),\varphi_t(x,0)=\varphi_1(x),\varphi_{tt}(x,0)=\varphi_2(x)\\ \varphi_{ttt}(x,0)=\varphi_3(x),\ \psi(x,0)=\psi_0(x),x\in(0,1)\\ y(x,\tau,p,0)=f_0(x,-p\tau),y_t(x,\tau,p,0)=f_1(x,-p\tau),\ in\ \ (0,1)\times(0,1)\times(0,\tau_2),\\ y_{tt}(x,\tau,p,0)=f_2(x,-p\tau),\ in\ \ (0,1)\times(0,1)\times(0,\tau_2)\end{cases}\tag{16}$$

where $\varphi_0,\varphi_1,\varphi_2,\psi_0,f_0,f_1,$ are given functions, and boundary conditions of Dirichlet-Dirichlet given by

$$\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0,\ t>0\tag{17}$$

Next we say that the global well-posedness of problem (15)-(17) given in the following theorem.

Theorem 2.1. *Assume the assumption (12) holds. If the initial data* $(\varphi_0,\varphi_1,\varphi_2,\varphi_3,\psi_0)$ *is in* $(H_0^1(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1)\times H_0^1(0,1))$, $f_0,f_1,f_2\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))$, *then problem (15)-17) has a weak solution such that*

$$\varphi\in C(\mathbb{R}_+,H_0^1(0,1))\cap C^1(\mathbb{R}_+,L^2(0,1)),\ \psi\in C(\mathbb{R}_+,H_0^1(0,1))$$

$$\varphi_t, \varphi_{tt} \in C(\mathbb{R}_+, L^2(0,1)).$$

In addition, we have that the solution $(\varphi, \varphi_t, \varphi_{tt}, \psi)$ depends continuously on the initial data in $H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times H_0^1(0,1)$. In particular, problem (15)-17) has a unique weak solution.

2.1. **The Global Well-Posedness.** In this subsection, we will prove the global existence and the uniqueness of the solution of problem (9)-(17) by using the classical Faedo-Galerkin approximations along with some priori estimates. We only prove the existence of solution in (i). For the existence of stronger solution in (ii), we can use the same method as in (i) and one can refer to Andrade e al. [6] and Jorge Silva and Ma [13] and Feng [12].

2.1.1. *Approximate Problem.* which satisfy the following approximate problem:

$$\begin{aligned}
&\rho_1(\varphi_{mtt}, u_j) + \beta((\varphi_{mx} + \psi_j), u_{mx}) + \mu_1(\varphi_{mt}, u_j) \\
&\qquad\qquad + (\int_{\tau_1}^{\tau_2} |\mu_2(p) y_m(x,1,p,t)\, dp, u_j) = 0, \\
&b(\psi_{mx}, \theta_{jx}) + \rho_2(\varphi_{mtt}, \theta_{jx}) + \beta((\varphi_{mx} + \psi_j), \theta_{mj}) = 0 \\
&(p y_{mt}(x,\tau,p,t), \phi_j) + (y_{m\tau}(x,\tau,p,t), \phi_j) = 0 \\
&(p y_{mtt}(x,\tau,p,t), \phi_j) + (y_{m\tau t}(x,\tau,p,t), \phi_j) = 0
\end{aligned} \tag{18}$$

with initial conditions

$$\begin{aligned}
&\varphi_m(0) = \varphi_0^m, \varphi_{mt}(0) = \varphi_1^m, \varphi_{mtt}(0) = \varphi_2^m \\
&\varphi_{mttt}(0) = \varphi_3^m, \psi_m(0) = \psi_0^m, \psi_{mt}(0) = \psi_1^m, \\
&y_m(0) = y_0^m, y_{mt}(0) = y_1^m, y_{mtt}(0) = y_2^m
\end{aligned} \tag{19}$$

which satisfies

$$\begin{aligned}
&\varphi_0^m \to \varphi_0, \ \ strongly \ \ in \ \ H_0^1(0,1) \\
&\varphi_1^m \to \varphi_1, \ \ strongly \ \ in \ \ L^2(0,1) \\
&\varphi_2^m \to \varphi_2, \ \ strongly \ \ in \ \ L^2(0,1) \\
&\varphi_3^m \to \varphi_3, \ \ strongly \ \ in \ \ L^2(0,1) \\
&\psi_0^m \to \psi_0, \ \ strongly \ \ in \ \ H_0^1(0,1) \\
&\psi_1^m \to \psi_1, \ \ strongly \ \ in \ \ H_0^1(0,1) \\
&y_0^m \to y_0, \ \ strongly \ \ in \ \ L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \\
&y_1^m \to y_1, \ \ strongly \ \ in \ \ L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \qquad (20) \\
&y_2^m \to y_2, \ \ strongly \ \ in \ \ L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \qquad (21)
\end{aligned}$$

By using standard ordinary differential equations theory, the problem (18)-(19) has a solution $(g_{jm}, h_{jm}, f_{jm})_{j=1,m}$ defined on $[0, t_m)$. The following estimate will give the local solution being extended to $[0, T]$, for any given $T > 0$.

2.1.2. *A Priori Estimate I.* It follows from (12), and (**??**) that

$$\begin{aligned}
&\int_0^1 \varphi_{mt}^2 dx + \int_0^1 \varphi_{mtt}^2 dx + \rho_2 \int_0^1 \varphi_{mtx}^2 dx + \int_0^1 (\varphi_{mx} + \psi_m)^2 dx \\
&\qquad + \int_0^1 \psi_{mx}^2 dx + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_m^2(x,\tau,p,t) dp d\tau dx
\end{aligned}$$

$$+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mt}^2(x,\tau,p,t)dpd\tau dx \leq C \tag{22}$$

Thus we can obtain $t_m = T$, for all $T > 0$.

2.1.3. *A Priori Estimate II.* where

$$\begin{aligned}\mathcal{G}_m(t) &= \frac{1}{2}\Big[\rho_1\int_0^1 \varphi_{mtt}^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^1 \varphi_{mttt}^2 dx + \rho_2\int_0^1 \varphi_{mttx}^2 dx \\ &\quad +\beta\int_0^1 (\varphi_{mxt}+\psi_{mt})^2 dx + b\int_0^1 \psi_{mxt}^2 dx\Big] \\ &\quad +\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mt}^2(x,\tau,p,t)dpd\tau dx \\ &\quad +\frac{\rho_2}{2\beta}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mtt}^2(x,\tau,p,t)dpd\tau dx\end{aligned}$$

Similarly to **A Priori Estimate I**, we can get there exists a positive constant C independent on m such that

$$\mathcal{G}_m(t) \leq C, \quad t \geq 0. \tag{23}$$

2.1.4. *Passage to Limit.* From (22) and (23), we conclude that for any $m \in \mathbb{N}$,

$$\begin{array}{lll} \varphi_m & \textit{weakly star in} & L^2(\mathbb{R}_+, H_0^1) \\ \varphi_{mt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2) \\ \varphi_{mtt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2) \\ \psi_m & \textit{weakly star in} & L^2(\mathbb{R}_+, H_0^1) \\ \psi_{mt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2) \\ y_m & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ y_{mt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \end{array} \tag{24}$$

By (24), we can also deduce that φ_m, ψ_m is bounded in $L^2(\mathbb{R}_+, H_0^1)$ and $\varphi_{mt}, \varphi_{mtt}$ is bounded in $L^2(\mathbb{R}_+, L^2)$. Then from Aubin-Lions theorem [15], we infer that for and, $T > 0$,

$$\begin{array}{lll} \varphi_m & \textit{strongly in} & L^\infty(0,T,H_0^1(0,1)) \\ \psi_m & \textit{strongly in} & L^\infty(0,T,H_0^1(0,1)) \end{array} \tag{25}$$

We also obtain by Lemma 1.4 in Kim [14] that

$$\begin{array}{lll} \varphi_m & \textit{strongly in} & C(0,T,H_0^1(0,1)) \\ \psi_m & \textit{strongly in} & C(0,T,H_0^1(0,1)) \end{array} \tag{26}$$

Then we can pass to limit the approximate problem (18)-(19) in order to get a weak solution of problem (15)-(17).

2.1.5. *Continuous Dependence and Uniqueness.* Firstly we prove the continuous dependence and uniqueness for stronger solutions of problem (15)-(17).
Let $(\varphi, \varphi_t, \varphi_{tt}, \varphi, \Upsilon, \Upsilon_t)$, and $(\Gamma, \Gamma_t, \Gamma_{tt}, \Xi, \Pi, \Pi_t)$ be two global solutions of problem (15)-(17) with respect to initial data $(\varphi_0, \varphi_1, \varphi_2, \varphi_0, \Theta_0, \Theta_1)$, and $(\Gamma_0, \Gamma_1, \Gamma_0, \Xi_0, \Phi_0, \Phi_1)$ respectively. Let

$$\Lambda(t) = \varphi - \Gamma$$

$$\Sigma(t) = \varphi - \Xi$$
$$\chi(t) = \Pi - \Phi \tag{27}$$

Then (Λ, Σ, χ) verifies (15)-(17), and we have

$$\begin{aligned}
&\rho_1 \Lambda_{tt} - \beta(\Lambda_x + \Sigma)_x + \mu_1 \Lambda_t + \int_{\tau_1}^{\tau_2} |\mu_2(p)| \Lambda_t\,(x, t-p)\,dp \\
&-\rho_2 \Lambda_{ttx} - b\Sigma_{xx} + \beta(\Lambda_x + \Sigma) = 0 \\
&p\chi_t(x, \tau, p, t) + \chi_\tau(x, \tau, p, t) = 0
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2}\Big[\rho_1 \int_0^1 \Lambda_t^2 dx + \frac{\rho_1 \rho_2}{\beta} \int_0^1 \Lambda_{tt}^2 dx + \rho_2 \int_0^1 \Lambda_{tx}^2 dx + \beta \int_0^1 (\Lambda_x + \Sigma)^2 dx \\
&\quad + b \int_0^1 \Sigma_x^2 dx\Big] + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| \chi^2\,(x, 1, p, t)\,dp d\tau dx
\end{aligned} \tag{29}$$

Applying Gronwall's inequality to (**??**), we get

$$\begin{aligned}
&(\|\Lambda_t\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{tx}\|^2 + \|\Sigma_x\|^2 + \|(\Lambda_x + \Sigma)\|^2 \\
&+ \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| \|\chi\,(x, 1, p, t)\,\|^2 dp d\tau) \le e^{C_2 t} \mathcal{E}(0)
\end{aligned} \tag{30}$$

This shows that solution of problem (15)-(17) depends continuously on the initial data.

2.2. **Exponential stability.** In this subsection, we will prove an exponential stability estimate for problem (15) − (17), under the assumption (12), and by using a multiplier technique.

We define the energy of solution

$$\begin{aligned}
\mathcal{E}\,(t) &= \frac{1}{2} \int_0^1 \left[\rho_1 \varphi_t^2 + b\psi_x^2 + \beta(\varphi_x + \psi)^2 + \frac{\rho_1 \rho_2}{\beta} \varphi_{tt}^2 + \rho_2 \varphi_{tx}^2\right] dx \\
&\quad + \frac{1}{2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2\,(x, \tau, p, t)\,dp d\tau dx \\
&\quad + \frac{1}{2} \frac{\rho_2}{\beta} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_t^2\,(x, \tau, p, t)\,dp d\tau dx
\end{aligned} \tag{31}$$

Then we have the following lemma.

Lemma 2.2. *The energy $\mathcal{E}(t)$ satisfies*

$$\begin{aligned}
\mathcal{E}'\,(t) &\le -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)| dp\right) \int_0^1 \varphi_t^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)| dp\right) \int_0^1 \varphi_{tt}^2 dx \\
&\le -\eta_0 \int_0^1 \varphi_t^2 dx - \eta_0 \frac{\rho_2}{\beta} \int_0^1 \varphi_{tt}^2 dx \le 0
\end{aligned} \tag{32}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)| dp \ge 0$.

Lemma 2.3. *The functional*

$$F_1\,(t) := -\frac{\mu_1}{2} \int_0^1 \varphi_t^2 dx - \beta \int_0^1 \varphi_{tx} \varphi_x dx \tag{33}$$

satisfies

$$\begin{aligned} F_1'(t) &\leq -\beta \int_0^1 \varphi_{tx}^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx + c(1+\frac{1}{\varepsilon_1}) \int_0^1 \varphi_{tt}^2 dx \\ &\quad + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t) dp dx \end{aligned} \tag{34}$$

Lemma 2.4. *The functional*

$$F_2(t) := \rho_1 \int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx + \frac{\mu_1\rho_2}{2\beta}\int_0^1 \varphi_t^2 dx + \rho_2 \int_0^1 \varphi_{tx}\varphi_x dx$$

satisfies,

$$\begin{aligned} F_2(t) &\leq -\frac{\rho_1\rho_2}{2\beta}\int_0^1 \varphi_{tt}^2 dx - \frac{\beta}{2}\int_0^1 (\varphi_x+\psi)^2 dx - b\int_0^1 \psi_x^2 dx \\ &\quad +\rho_2 \int_0^1 \varphi_{tx}^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ &\quad + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t) dp dx. \end{aligned} \tag{35}$$

Lemma 2.5. *The functional*

$$\begin{aligned} F_3(t) &:= \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y^2(x,\tau,p,t)\, dp d\tau dx \\ &\quad + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y_t^2(x,\tau,p,t)\, dp d\tau dx. \end{aligned}$$

satisfies,

$$\begin{aligned} F_3'(t) &\leq -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2(x,\tau,p,t)\, dp d\tau dx + \mu_1 \int_0^1 \varphi_t^2 dx \\ &\quad -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t)\, dp dx \\ &\quad -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_t^2(x,\tau,p,t)\, dp d\tau dx + \mu_1 \int_0^1 \varphi_{tt}^2 dx \\ &\quad -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_t^2(x,1,p,t)\, dp dx \end{aligned} \tag{36}$$

where $\eta_1 > 0$.

Theorem 2.6. *Assume (A1), there exist positive constants* λ_1 *and* λ_2 *such that the energy functional (31) satisfies*

$$\mathcal{E}(t) \leq \lambda_2 e^{-\lambda_1 t}, \forall t \geq 0 \tag{37}$$

References

[1] H. Abramovich, I. Elishakoff, Application of the Krein's method for determination of natural frequencies of periodically supported beam based on simplified Brese-Timoshenko equations, Acta Mech. 66, 39-59, (1987).

[2] D. S. Almeida Junior, A. J. A. Ramos, On the nature of dissipative Timoshenko systems at light of the second spectrum, Z. Angew. Math. Phys. 68, art. no. 145, (2017).

[3] D. S. Almeida Junior, A. J. A. Ramos, M. L. Santos, R. M. L. Gutemberg, Asymptotic behavior of weakly dissipative Bresse-Timoshenko system on influence of the second spectrum of frequency, Z. Angew. Math. Mech., 98, 1320-1333, (2018).

[4] D. S. Almeida Junior, I. Elishakoff, A. J. A. Ramos, R. M. L. Gutemberg, The hypothesis of equal wave speeds for stabilization of Bresse- Timoshenko system is not necessary anymore: the time delay cases, IMA J. Appl. Math. 84, 763-796, (2019).

[5] D. S. Almeida Junior, I. Elishakoff, A. J. A. Ramos, Asymptotic behavior of weakly dissipative Bresse-Timoshenko system on influence of the second spectrum of frequency, ZAAM, (2018).DOI: 10.1002/zamm.201700211.

[6] D. Andrade, M. A. Jorge Silva, and T. F. Ma, "Exponential stability for a plate equation with p-Laplacian and memory terms," Mathematical Methods in the Applied Sciences, vol. 35, no. 4, pp. 417-426, 2012.

[7] I. Elishakoff, E. Lubliner, Random vibration of a structure via classical and non-classical theories, Proceedings of the IUTAM Symposium on Probabilistic Mechanics of Structures, Springer, New York 1985.

[8] I. Elishakoff, An equation both more consistent and simpler than the Bresse-Timoshenko equation, Advances in Mathematical Modeling and Experimental Methods for Materials and Structures, Solid Mechanics and Its Applications, Springer, Berlim 2010, pp. 249-254.

[9] B. Feng, X-G. Yang, Long-time dynamics for a nonlinear Timoshenko system with delay, Appl. Anal. 96, 606-625, (2017).

[10] B. Feng, D. S. Almeida Junior, M. J. dos Santos and L. G. Rosario Miranda, A new scenario for stability of nonlinear Bresse-Timoshenko type systems with time dependent delay, Z Angew Math. Mech. 2019. DOI: 10.1002/zamm.201900160

[11] B. Feng, M. Pelicer, Global existence and exponential stability for a nonlinear Timoshenko system with delay, Boundary Value Problems, 206, 1-13, (2015).

[12] B. Feng, Global well-posedness and stability for a viscoelastic plate equation with a time delay, Mathematical Problems in Engineering, Volume (2015), Article ID 585021, http://dx.doi.org/10.1155/2015/585021.

[13] M. A. Jorge Silva and T. F. Ma, On a viscoelastic plate equation with history setting and perturbation of p-Laplacian type," IMA Journal of Applied Mathematics, vol. 78, no. 6, pp. 1130–1146, 2013.

[14] J. U. Kim, "A boundary thin obstacle problem for a wave equation," Communications in Partial Differential Equations, vol. 14, no. 8-9, pp. 1011–1026, 1989.

[15] J. L. Lions, Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires, Dunod Gauthier-Villars, Paris, France, 1969.

[16] A. Manevich, Z. Kolakowski, Free and forced oscillations of Timoshenko beam made of viscoelastic material, J. Theor. Appl. Mech. 49, 3-16, (2011).

[17] S. Nicaise, C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Diff. Int. Equs., 21 (9-10), 935-958 (2008).

[18] S. P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, Philos. Mag. [Ser. 6] 1921, 41, 744.

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