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Theme

Linearization approach to partial differential equations

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Notations

Ω	Open subset of $\mathbb{R}^n$.
$\partial\Omega$	Boundary of Ω.
$\|\Omega\|$	Measure of a set Ω.
$\mathcal{C}([a,b];\mathbb{R})$	Space of continuous functions on $[a,b]$.
$L^p(\Omega)$	Space of p-th power integrable functions on Ω.
$H^1(\Omega)$	Sobolev space of order 1.
$H_0^1(\Omega)$	Subspace of $H^1(\Omega)$ functions with zero trace on the boundary.
$\\| \cdot \\|_\infty$	Norm in the space $\mathcal{C}([a,b];\mathbb{R})$.
$\\| \cdot \\|$	Norm in normed vector space.
$\\| \cdot \\|_{L^{\frac{2}{1-p}}(\Omega)}$	Norm in the space $L^{\frac{2}{1-p}}(\Omega)$.
$\\| \cdot \\|_{H_0^1(\Omega)}$	Norm in the space $H_0^1(\Omega)$.
∇u	$\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right) = \text{grad } u$.
Δu	$\sum_{i=1}^{n} \frac{\partial u^2}{\partial x_i^2} =$ laplacian of u.
$C_c^\infty(\Omega)$	The space of smooth, compactly supported functions on Ω.
$\xrightarrow{w}$	Weak convergence.
$\hookrightarrow$	Compact injection.
$\langle \cdot, \cdot \rangle$	Inner product in a Hilbert space.
C_Ω	The positive Poincaré constant for $H_0^1(\Omega)$.
a.e.	almost everywhere.

Contents

Abstract

In this memory, we explore the application of the linearization technique in some nonlinear elliptic problems. We use this approach together with the fixed point theorem to prove the existence of nontrivial weak solutions to certain nonlinear elliptic problems.

Keywords : linearization, nonlinear elliptic problems, fixed point theorem, existence.

Résumé

Dans cette mémoire, nous explorons l'application de la technique de linéarisation dans certains problèmes elliptiques non linéaires. Nous utilisons cette approche conjointement avec le théorème du point fixe pour prouver l'existence de solutions faibles non triviales à certains problèmes elliptiques non linéaires.

Mots-clés : linéarisation, problèmes elliptiques non linéaires, théorème du point fixe, existence.

ملخص

نستكشف في هذه المذكرة تطبيق تقنية الخطية في بعض المسائل الحدية البيضاوية غير الخطية. نستخدم هذه الطريقة جنباً إلى جنب مع نظرية النقطة الثابتة لإثبات وجود حلول ضعيفة غير تافهة لبعض المسائل الحدية البيضاوية غير الخطية.

الكلمات المفتاحية : تقنية الخطية، مسائل حدية بيضاوية غير خطية، نظرية النقطة الثابتة، الوجود.

Introduction

The linearization is a process creating a linear representation of a nonlinear problem. In the realm of mathematics, we often encounter nonlinear differential equations that impede the finding of analytical solutions. In this case, the method of linearization can contribute to facilitate the study of some nonlinear differential equations. This technique relies on replacing the nonlinear system with a linear one that can be solved, for example, using the fixed point theory.

The memory is structured as follows. The first Chapter covers elements of functional analysis to support subsequent arguments such as Lebesgue spaces, Sobolev spaces, Poincaré inequality, the Lax-Milgram theorem, Schauder and Tychonoff fixed point theorems. In the second Chapter, we use the linearization technique together with the Schauder fixed point theorem to prove the existence of a nontrivial weak solution to the following elliptic problem:

$$\begin{cases} M\left(\|u\|^q_{L^q(\Omega)}\right) div(a(x)\nabla u) = g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $q \in [1,2]$, $M : [0,\infty[\to \mathbb{R}$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are given functions and $a(x)$ is an $n \times n$ matrix. The function g satisfies a generalized Lipschitz condition. By the linearization technique combining with the Tychonoff fixed point theorem, we prove, in the last Chapter, the existence of a nontrivial weak solution of (1) in which the function g satisfies a boundedness condition. The results of this memory can be found in [1].

Chapter 1

Preliminaries

This Chapter covers elements of functional analysis used throughout this memory. The content of this Chapter can be found in [2], [3], [4], [5], [6], [7], [8] and [9].

1.1 Topology of normed vector spaces and operators

Definition 1.1.1. *(Normed vector space)*

Let E be a vector space over the real number. A norm on E is a function $\|\cdot\| : E \to \mathbb{R}$ satisfying :

1. $\|x\| \geq 0$ *for all* $x \in E$, *and* $\|x\| = 0$ *if and only if* $x = 0$.
2. $\|\alpha x\| = |\alpha|\|x\|$ *for all* $\alpha \in \mathbb{R}$ *and* $x \in E$.
3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$ *(triangle inequality).*

The pair $(E, \|\cdot\|)$ is called a normed vector space.

Example 1.1.1.

In a normed vector space $\mathbb{R}^n$, common types of norms for a vector $x = (x_1, ..., x_n)$ are :

1. $\|x\|_1 = \displaystyle\sum_{i=1}^{n} |x_i|$ *(Manhattan norm).*
2. $\|x\|_2 = \sqrt{\displaystyle\sum_{i=1}^{n} x_i^2}$ *(Euclidean norm).*
3. $\|x\|_\infty = \displaystyle\sup_{1 \leq i \leq n} |x_i|$ *(Chebyshev norm).*

Example 1.1.2.

Let Ω be an open subset of $\mathbb{R}^n$. For $1 \leq p < \infty$, the space $L^p(\Omega)$ consists of all Lebesgue *measurable functions* $f : \Omega \to \mathbb{R}$ *such that*

$$\int_{\Omega} |f(x)|^p \, \mathrm{d}x < \infty.$$

The $L^p(\Omega)$ *norm of such function is defined as :*

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

For $p = \infty$*, the* $L^\infty(\Omega)$ *consists of all Lebesgue measurable functions* $f : \Omega \to \mathbb{R}$ *such that*

$$\exists C > 0, \quad |f(x)| \leq C \text{ a.e. } x \in \Omega.$$

The $L^\infty(\Omega)$ *norm, known as the essential supremum norm,*

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|,$$

of a function f *defined on* Ω*, is the smallest number* C *such that* $|f(x)| \leq C$ *for almost every* x *in* Ω*.*

Definition 1.1.2. *(Norm strong convergence)*
Let $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in a normed vector space* E*. The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is said to converge strongly in norm to an element* $x \in E$ *if for every real number* $\epsilon > 0$*, there exists a natural number* N *(depending on* ϵ*) such that for all* $n \geq N$*, the following holds :*

$$\|x_n - x\| < \epsilon.$$

In other words, the sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly in norm to* x *if and only if the norm of the difference between* x_n *and* x *approaches zero as* n *approaches infinity :*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Example 1.1.3.
Let $\Omega =]0, 1[$*. Define a sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$ *on* $]0, 1[$ *as follows:*

$$f_n(x) = \begin{cases} n & \text{if } x \in \left]0, 1/n^2\right], \\ 0 & \text{if } x \in \left]1/n^2, 1\right[. \end{cases}$$

Strong convergence in $L^1(\Omega)$ *:*
We calculate the $L^1(\Omega)$ *norm of the function* f_n *:*

$$\|f_n - 0\|_1 = \|f_n\|_1 = \int_0^1 |f_n(x)| \, \mathrm{d}x = \int_0^{1/n^2} n \, \mathrm{d}x + \int_{1/n^2}^1 0 \, \mathrm{d}x,$$

$$\|f_n\|_1 = n[x]_0^{1/n^2} = n\left(\frac{1}{n^2} - 0\right) = \frac{1}{n}.$$

Since

$$\lim_{n\to\infty} \|f_n - 0\|_1 = \lim_{n\to\infty} \frac{1}{n} = 0,$$

the sequence $\{f_n\}_{n\in\mathbb{N}}$ *converges strongly to the zero function in* $L^1(]0,1[)$.

Definition 1.1.3. *(Cauchy Sequence)*

Let $(E, \|.\|)$ *be a normed vector space. A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* E *is called a Cauchy sequence if for every* $\epsilon > 0$, *there exists an integer* N *such that for all* $m, n > N$, *we have :*

$$\|x_m - x_n\| < \epsilon.$$

Remark 1.1.1. *(Banach space)*

A Banach space is a complete normed vector space. This means it is a vector space with a norm that allows the measurement of vector lengths, and in which any Cauchy sequence of vectors converges to a limit within the space.

Definition 1.1.4. *(Closed ball)*

In E *normed vector space, a closed ball centred at* a *with radius* $r \geq 0$ *is defined as follows :*

$$\overline{B}(a,r) = \{x \in E : \|x - a\|_E \leq r\}.$$

Proposition 1.1.1. *(Closed)*

A subset K *of a normed vector space* E *is said to be closed if every sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* K *that converges to a point in* E *must have its limit in* K. *That is, if* $x_n \in K$ *for all* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} x_n = x,$$

for some $x \in E$, *then* $x \in K$.

Definition 1.1.5. *(Compactness)*

A subset K *of a normed vector space* E *is said to be compact if every open cover of* K *has a finite subcover. That is, for every collection of open sets* $\{U_\alpha\}_{\alpha\in I}$ *in* E *such that*

$$K \subseteq \bigcup_{\alpha\in I} U_\alpha,$$

there exists a finite subset $J \subseteq I$ *such that*

$$K \subseteq \bigcup_{\alpha\in J} U_\alpha.$$

Proposition 1.1.2. *(Relative compactness)*

A subset K of a normed vector space E is said to be relatively compact if every sequence $\{x_n\}_{n\in\mathbb{N}}$ in K has a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ that converges to a point $x \in E$.

Definition 1.1.6. *(Convex)*

A subset K of a real normed vector space E is said to be convex if

$$\forall x, y \in K, \forall t \in [0,1]: \quad (1-t)x + ty \in K.$$

Definition 1.1.7. *(Continuity in normed vector spaces)*

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. An operator $f : E \to F$ is said to be continuous at a point $a \in E$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in E$, if $\|x - a\|_E < \delta$, then $\|f(x) - f(a)\|_F < \epsilon$.

Proposition 1.1.3.

Let E and F be normed vector spaces, and let $T : E \to F$ be a mapping. T is continuous if only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in E that converges to x in E, the sequence $\{T(x_n)\}_{n\in\mathbb{N}}$ converges to $T(x)$ in F.

Example 1.1.4.

Let $E = \mathcal{C}([0,1];\mathbb{R})$ be the space of continuous real-valued functions on the interval $[0,1]$, equipped with the supremum norm defined by $\|f\|_\infty = \sup\limits_{x\in[0,1]} |f(x)|$.
Consider the mapping $T : (E, \|\cdot\|_\infty) \to (E, \|\cdot\|_\infty)$ defined by $T(f) = f^2$. We want to prove that T is continuous using the sequential definition of continuity.
Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in E that converges to a function $f \in E$ in the space $(E, \|\cdot\|_\infty)$. This means that :

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0,$$

where $\|f_n - f\|_\infty = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)|$.
We need to show that the sequence $\{T(f_n)\}_{n\in\mathbb{N}} = \{f_n^2\}_{n\in\mathbb{N}}$ converges to $T(f) = f^2$ in the space $(E, \|\cdot\|_\infty)$. That is, we need to prove that:

$$\lim_{n\to\infty} \|T(f_n) - T(f)\|_\infty = \lim_{n\to\infty} \|f_n^2 - f^2\|_\infty = 0.$$

Analyze $\|f_n^2 - f^2\|_\infty$:

$$\|f_n^2 - f^2\|_\infty = \|(f_n - f)(f_n + f)\|_\infty.$$

Using the property that $\|hk\|_\infty \leq \|h\|_\infty \|k\|_\infty$ *:*

$$\|(f_n - f)(f_n + f)\|_\infty \leq \|f_n - f\|_\infty \|f_n + f\|_\infty.$$

Using the triangle inequality for norms :

$$\|f_n + f\|_\infty = \|f_n - f + 2f\|_\infty \leq \|f_n - f\|_\infty + \|2f\|_\infty = \|f_n - f\|_\infty + 2\|f\|_\infty.$$

Thus, we have :

$$\|f_n^2 - f^2\|_\infty \leq \|f_n - f\|_\infty(\|f_n - f\|_\infty + 2\|f\|_\infty).$$

Since $\lim\limits_{n\to\infty} \|f_n - f\|_\infty = 0$*, the first factor on the right-hand side goes to zero. The term* $\|f\|_\infty$ *is a fixed non-negative real number because* $f \in E$*.*
Therefore, as $n \to \infty$ *:*

$$\lim_{n\to\infty} \|f_n^2 - f^2\|_\infty \leq \lim_{n\to\infty} \|f_n - f\|_\infty \left(\lim_{n\to\infty} \|f_n - f\|_\infty + 2\|f\|_\infty\right) = 0 \cdot (0 + 2\|f\|_\infty) = 0.$$

Since $\|f_n^2 - f^2\|_\infty \geq 0$ *and* $\lim\limits_{n\to\infty} \|f_n^2 - f^2\|_\infty \leq 0$*, we conclude that :*

$$\lim_{n\to\infty} \|f_n^2 - f^2\|_\infty = 0.$$

This means that the sequence $\{T(f_n)\}_{n\in\mathbb{N}} = \{f_n^2\}_{n\in\mathbb{N}}$ *converges to* $T(f) = f^2$ *in the space* $(E, \|\cdot\|_\infty)$*.*
By the sequential definition of continuity, the mapping T *is continuous.*

Definition 1.1.8. *(Weak convergence)*
Let $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in a normed vector space* E*. We say that* $\{x_n\}_{n\in\mathbb{N}}$ *converges weakly to* $x \in E$ *if :*

$$\lim_{n\to\infty} f(x_n) = f(x),$$

for every continuous linear functional $f : E \to \mathbb{R}$*.*

Proposition 1.1.4. *(Weakly continuous)*
A function $f : E \to F$ *between two normed vector spaces* E *and* F *is weakly continuous at a point* $x \in E$ *if for every sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* E *that converges weakly to* x *(*$x_n \xrightarrow{w} x$*), the sequence* $\{f(x_n)\}_{n\in\mathbb{N}}$ *in* F *converges weakly to* $f(x)$ *(*$f(x_n) \xrightarrow{w} f(x)$*).*
The function $f : E \to F$ *is weakly continuous if it is weakly continuous at every point in* E*.*

Theorem 1.1.1.
Let E be a Banach space and let A be a convex set of E. Then, A is closed if and only if it is weakly closed.

Definition 1.1.9. *(Linear map)*
Let E and F be vector spaces over the real number. A map $T : E \to F$ is called a linear map if for all vectors $u, v \in E$ and scalar $\alpha \in \mathbb{R}$, the following condition hold :

$$T(\alpha u + v) = \alpha T(u) + T(v).$$

Definition 1.1.10. *(Bilinear map)*
Let E, F and G be vector spaces over the real number. A map $B : E \times F \to G$ is called a bilinear map if for all vectors $u, v \in E$, $x, y \in F$, and scalar $\alpha \in \mathbb{R}$, the following conditions hold :

1. $B(\alpha u + v, x) = \alpha B(u, x) + B(v, x)$.
2. $B(u, \alpha x + y) = \alpha B(u, x) + B(u, y)$.

Proposition 1.1.5. *(Bounded operator)*
A linear operator $T : E \to F$ is continuous or bounded if only if there exists a real number $M > 0$ such that for all $x \in E$,

$$\|T(x)\|_F \leq M\|x\|_E.$$

Proposition 1.1.6. *(Continuity of a bilinear map)*
Let E be a normed vector space and let $a(\cdot, \cdot) : E \times E \to \mathbb{R}$ be a bilinear form. We say that a is continuous if it satisfies :
There exists a constant $C > 0$ such that

$$|a(u, v)| \leq C\|u\|\|v\|, \quad \forall u, v \in E.$$

1.2 Hilbert space

Definition 1.2.1. *(Hilbert space)*
Let H be a vector space over the real number. An inner product on H is a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ satisfying :

1. *$\langle x, x\rangle \geq 0$ for all $x \in H$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.*
2. *$\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in H$ (symmetry).*
3. *$\langle \alpha x + y, z\rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$ for all $\alpha \in \mathbb{R}$ and $x, y, z \in H$ (linearity in the first argument).*

A Hilbert space is a complete inner product space. The norm in a Hilbert space is induced by the inner product:

$$\|x\| = \sqrt{\langle x, x\rangle},$$

where $\langle \cdot, \cdot\rangle$ is the inner product.

Definition 1.2.2. *(Weak convergence)*
In Hilbert space H, weak convergence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ to x is defined similarly, but using the inner product

$$\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle, \quad \forall y \in H.$$

Remark 1.2.1.
Strong convergence implies weak convergence, but the converse is not always true in infinite-dimensional spaces. For example, an infinite orthonormal sequence in a Hilbert space converges weakly to zero but does not converge strongly.

Theorem 1.2.1. *(Cauchy-Schwarz inequality)*
Let H be an inner product space. Then for any vectors $u, v \in H$, the following inequality holds:

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Example 1.2.1.
In the real Euclidean space $\mathbb{R}^n$, the Cauchy-Schwarz inequality states that for any two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the inequality is written as follows :

$$|\langle x, y\rangle| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}.$$

Example 1.2.2.
Let f and g be square-integrable functions on the space $L^2(\Omega)$. Then:

$$\left|\int_\Omega f(x)g(x)\,\mathrm{d}x\right| \leq \left(\int_\Omega |f(x)|^2\,\mathrm{d}x\right)^{1/2} \left(\int_\Omega |g(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Theorem 1.2.2. *(Hölder's inequality)*
Let p and q be real numbers such that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let f and g be functions on the space $L^p(\Omega)$. Then :

$$\int_\Omega |f(x)g(x)| \, \mathrm{d}x \leq \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p} \left(\int_\Omega |g(x)|^q \, \mathrm{d}x \right)^{1/q}.$$

Definition 1.2.3. *(Coercivity)*
Let H be a Hilbert space and $a(\cdot, \cdot) : H \times H \to \mathbb{R}$ be a bilinear form. The bilinear form $a(\cdot, \cdot)$ is said to be coercive if there exists a constant $\alpha > 0$ such that:

$$a(u, u) \geq \alpha \|u\|_H^2, \quad \forall u \in H,$$

where $\|u\|_H$ denotes the norm of u in the Hilbert space H.

1.3 Sobolev spaces

Definition 1.3.1. *($H^1(\Omega)$ space)*
Let Ω be an open subset of $\mathbb{R}^n$. The Sobolev space $H^1(\Omega)$ is defined as the space of functions in $L^2(\Omega)$ whose first weak derivatives are also in $L^2(\Omega)$. More precisely, a function $u \in L^2(\Omega)$ belongs to $H^1(\Omega)$ if there exist functions $v_1, ..., v_n \in L^2(\Omega)$ such that for each $i = 1, ..., n$ and for every smooth, compactly supported test function $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega u(x) \frac{\partial \phi}{\partial x_i}(x) \, \mathrm{d}x = - \int_\Omega v_i(x) \phi(x) \, \mathrm{d}x.$$

The functions v_i are the weak partial derivatives of u, denoted by $\frac{\partial u}{\partial x_i}$.

Example 1.3.1.
Prove that $f(x) = e^{-x^2}$ belongs to $H^1(\mathbb{R})$.
The space $H^1(\mathbb{R})$ consists of all functions $f \in L^2(\mathbb{R})$ whose weak derivative f' is also in $L^2(\mathbb{R})$. The $H^1(\mathbb{R})$ norm is defined as :

$$||f||^2_{H^1(\mathbb{R})} = ||f||^2_{L^2(\mathbb{R})} + ||f'||^2_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} |f(x)|^2 \, \mathrm{d}x + \int_{-\infty}^{+\infty} |f'(x)|^2 \, \mathrm{d}x.$$

To prove that $f(x) = e^{-x^2} \in H^1(\mathbb{R})$, we need to show that $f \in L^2(\mathbb{R})$ and its weak derivative $f'(x)$ is also in $L^2(\mathbb{R})$.

Step 1 : *Show that* $f(x) = e^{-x^2} \in L^2(\mathbb{R})$.

We compute the integral of $|f(x)|^2$ *:*

$$\int_{-\infty}^{+\infty} \left|e^{-x^2}\right|^2 \, \mathrm{d}x = \int_{-\infty}^{+\infty} e^{-2x^2} \, \mathrm{d}x.$$

Using the Gaussian integral formula $\displaystyle\int_{-\infty}^{+\infty} e^{-ax^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}}$ *for* $a > 0$, *with* $a = 2$ *:*

$$\int_{-\infty}^{+\infty} e^{-2x^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{2}} < \infty.$$

Since the integral is finite, $f(x) = e^{-x^2} \in L^2(\mathbb{R})$.

Step 2 : *Compute the weak derivative* $f'(x)$.

The function $f(x) = e^{-x^2}$ *is classically differentiable on* $\mathbb{R}$, *and its derivative is :*

$$f'(x) = -2xe^{-x^2}.$$

In this case, the classical derivative is also the weak derivative.

Step 3 : *Show that* $f'(x) = -2xe^{-x^2} \in L^2(\mathbb{R})$.

We compute the integral of $|f'(x)|^2$ *:*

$$\int_{-\infty}^{+\infty} \left|-2xe^{-x^2}\right|^2 \, \mathrm{d}x = \int_{-\infty}^{+\infty} 4x^2e^{-2x^2} \, \mathrm{d}x.$$

Using integration by parts with $u = x$ *and* $\mathrm{d}v = 4xe^{-2x^2} \, \mathrm{d}x$, *so* $\mathrm{d}u = \mathrm{d}x$ *and* $v = -e^{-2x^2}$*:*

$$\int_{-\infty}^{+\infty} 4x^2e^{-2x^2} \, \mathrm{d}x = \left[-xe^{-2x^2}\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -e^{-2x^2} \, \mathrm{d}x.$$

We evaluate the first term with L'Hôpital's rule :

$$\lim_{x\to\pm\infty} -xe^{-2x^2} = \lim_{x\to\pm\infty} \frac{-x}{e^{2x^2}} = \lim_{x\to\pm\infty} \frac{-1}{4xe^{2x^2}} = 0.$$

So the first term is 0. *We are left with :*

$$\int_{-\infty}^{+\infty} e^{-2x^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{2}}.$$

Therefore :

$$\int_{-\infty}^{+\infty} |f'(x)|^2 \, \mathrm{d}x = \sqrt{\frac{\pi}{2}} < \infty.$$

Since the integral of $|f'(x)|^2$ *is finite,* $f'(x) = -2xe^{-x^2} \in L^2(\mathbb{R})$.

Finally, since $f(x) = e^{-x^2} \in L^2(\mathbb{R})$ *and its weak derivative* $f'(x) = -2xe^{-x^2} \in L^2(\mathbb{R})$, *we conclude that* $f(x) = e^{-x^2}$ *belongs to the Sobolev space* $H^1(\mathbb{R})$.

Example 1.3.2.

Showing that $g(x) = e^{-|x|}$ belongs to $H^1(\mathbb{R})$.

To prove that the function $g(x) = e^{-|x|}$ belongs to the Sobolev space $H^1(\mathbb{R})$, we need to demonstrate that both the function and its weak derivative are square-integrable over the real line, i.e., they belong to $L^2(\mathbb{R})$.

1. *Showing that $g \in L^2(\mathbb{R})$:*

   The function g is defined piecewise as :

$$g(x) = \begin{cases} e^{-x} & \textit{if } x \geq 0, \\ e^{x} & \textit{if } x < 0. \end{cases}$$

   We compute the $L^2(\mathbb{R})$ norm squared of g :

$$\begin{aligned} \|g\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{+\infty} |g(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{0} (e^x)^2 \, \mathrm{d}x + \int_{0}^{+\infty} (e^{-x})^2 \, \mathrm{d}x \\ &= \int_{-\infty}^{0} e^{2x} \, \mathrm{d}x + \int_{0}^{\infty} e^{-2x} \, \mathrm{d}x. \end{aligned}$$

   Evaluating the first integral :

$$\int_{-\infty}^{0} e^{2x} \, \mathrm{d}x = \lim_{a \to -\infty} \int_{a}^{0} e^{2x} \, \mathrm{d}x = \lim_{a \to -\infty} \left[\frac{1}{2} e^{2x} \right]_a^0 = \frac{1}{2} e^0 - \lim_{a \to -\infty} \frac{1}{2} e^{2a} = \frac{1}{2} - 0 = \frac{1}{2}.$$

   Evaluating the second integral :

$$\begin{aligned} \int_{0}^{+\infty} e^{-2x} \, \mathrm{d}x &= \lim_{b \to +\infty} \int_{0}^{b} e^{-2x} \, \mathrm{d}x = \lim_{b \to +\infty} \left[-\frac{1}{2} e^{-2x} \right]_0^b \\ &= -\lim_{b \to +\infty} \frac{1}{2} e^{-2b} - \left(-\frac{1}{2} e^0 \right) = 0 + \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

   Therefore,

$$\|g\|^2_{L^2(\mathbb{R})} = \frac{1}{2} + \frac{1}{2} = 1 < \infty.$$

   This shows that $g \in L^2(\mathbb{R})$.

2. *Finding the weak derivative g' and showing it is in $L^2(\mathbb{R})$:*

   The classical derivative of g for $x \neq 0$ is :

$$g'(x) = \begin{cases} -e^{-x} & \textit{if } x > 0, \\ e^{x} & \textit{if } x < 0. \end{cases}$$

We can write this as $g'(x) = -\text{sgn}(x)e^{-|x|}$, *where*

$$\text{sgn}(x) = \begin{cases} 1 & \textit{if } x > 0, \\ -1 & \textit{if } x < 0. \end{cases}$$

To show that g' *is the weak derivative, we need to verify that for all test functions* $\phi \in C_c^\infty(\mathbb{R})$ *:*

$$\int_{-\infty}^{+\infty} g(x)\phi'(x)\ \mathrm{d}x = -\int_{-\infty}^{+\infty} g'(x)\phi(x)\ \mathrm{d}x.$$

Integrating by parts on the left-hand side :

$$\begin{aligned}
\int_{-\infty}^{+\infty} g(x)\phi'(x)\ \mathrm{d}x &= \int_{-\infty}^{0} e^x\phi'(x)\ \mathrm{d}x + \int_0^{+\infty} e^{-x}\phi'(x)\ \mathrm{d}x \\
&= [e^x\phi(x)]_{-\infty}^0 - \int_{-\infty}^0 e^x\phi(x)\ \mathrm{d}x + [e^{-x}\phi(x)]_0^{+\infty} - \int_0^{+\infty} -e^{-x}\phi(x)\ \mathrm{d}x \\
&= (\phi(0) - 0) - \int_{-\infty}^0 e^x\phi(x)\ \mathrm{d}x + (0 - \phi(0)) + \int_0^{+\infty} e^{-x}\phi(x)\ \mathrm{d}x \\
&= -\int_{-\infty}^0 e^x\phi(x)\ \mathrm{d}x + \int_0^{+\infty} e^{-x}\phi(x)\ \mathrm{d}x \\
&= -\int_{-\infty}^{+\infty} g'(x)\phi(x)\ \mathrm{d}x.
\end{aligned}$$

Thus, the weak derivative is indeed $g'(x) = -\text{sgn}(x)e^{-|x|}$.

Now, we compute the $L^2(\mathbb{R})$ *norm squared of* g' *:*

$$\begin{aligned}
\|g'\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{+\infty} |g'(x)|^2\ \mathrm{d}x = \int_{-\infty}^{+\infty} |-\text{sgn}(x)e^{-|x|}|^2\ \mathrm{d}x = \int_{-\infty}^{+\infty} e^{-2|x|}\ \mathrm{d}x \\
&= \int_{-\infty}^0 e^{2x}\ \mathrm{d}x + \int_0^{+\infty} e^{-2x}\ \mathrm{d}x = \frac{1}{2} + \frac{1}{2} = 1 < \infty.
\end{aligned}$$

This shows that $g' \in L^2(\mathbb{R})$.

3. *Conclusion :*

   Since $g \in L^2(\mathbb{R})$ *and its weak derivative* $g' \in L^2(\mathbb{R})$, *the function* $g(x) = e^{-|x|}$ *belongs to the Sobolev space* $H^1(\mathbb{R})$.

Proposition 1.3.1.

The Sobolev space $H^1(\Omega)$ *is a Hilbert space when equipped with the inner product*

$$\forall u, v \in H^1(\Omega) : \quad \langle u, v\rangle_{H^1(\Omega)} = \int_\Omega (u(x)v(x) + \nabla u(x)\cdot\nabla v(x))\ \mathrm{d}x,$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$ *is the gradient of* u. *The norm induced by this inner product is the* $H^1(\Omega)$ *norm, given by*

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega (|u(x)|^2 + |\nabla u(x)|^2)\ \mathrm{d}x\right)^{1/2} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{1/2},$$

where

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega |\nabla u(x)|^2\ \mathrm{d}x = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^2\ \mathrm{d}x.$$

Theorem 1.3.1. *(The trace theorem)*

For a sufficiently regular boundary $\partial\Omega$, *the trace theorem defines a continuous linear operator* γ *(the trace operator) that maps functions in* $H^1(\Omega)$ *to a function space on the boundary, typically* $L^2(\partial\Omega)$ *or a related Sobolev space on the boundary. This trace operator can be thought of as the "boundary value" of the function in* $H^1(\Omega)$.

Definition 1.3.2. *($H^1_0(\Omega)$ Space)*

Let Ω *be a bounded domain in* $\mathbb{R}^n$. *The* $H^1_0(\Omega)$ *space is defined as the completion of the space of smooth functions with compact support in* Ω, *denoted by* $C^\infty_c(\Omega)$, *with respect to the* $H^1(\Omega)$ *norm. The norm on* $H^1_0(\Omega)$ *is the* $H^1(\Omega)$ *norm,*

$$\|u\|_{H^1_0(\Omega)} = \left(\int_\Omega (|u(x)|^2 + |\nabla u(x)|^2)\ \mathrm{d}x\right)^{1/2}.$$

Corollary 1.3.1.

The trace theorem states that for $u \in H^1_0(\Omega)$, *its trace* $\gamma(u)$ *on the boundary* $\partial\Omega$ *is zero in the appropriate function space on the boundary.*

Theorem 1.3.2. *(Poincaré inequality)*

Let Ω *be a bounded domain in* $\mathbb{R}^n$. *A fundamental inequality that holds for functions in* $H^1_0(\Omega)$. *It states that there exists a constant* $C_\Omega > 0$ *such that :*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|u\|_{H^1_0(\Omega)}, \quad \forall u \in H^1_0(\Omega).$$

Remark 1.3.1.

By the Poincaré inequality, the norm of $H^1_0(\Omega)$ *is equivalent to the norm*

$$\|u\|_{H^1_0(\Omega)} = \left(\int_\Omega |\nabla u(x)|^2)\ \mathrm{d}x\right)^{1/2} = \|\nabla u\|_{L^2(\Omega)},$$

which is derived from the inner product

$$\forall u, v \in H^1_0(\Omega) : \quad \langle u, v\rangle_{H^1_0(\Omega)} = \int_\Omega \nabla u(x) \cdot \nabla v(x)\ \mathrm{d}x.$$

Theorem 1.3.3. *(Rellich-Kondrachov Theorem)*
The Rellich-Kondrachov Theorem, when applied to the Sobolev space $H_0^1(\Omega)$, provides a crucial result regarding the compactness of the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ spaces.
A compact embedding (or compact injection) of $H_0^1(\Omega)$ into $L^p(\Omega)$ is a continuous linear embedding $i : H_0^1(\Omega) \to L^p(\Omega),\ u \to i(u) = u$ such that for any bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ to a limit $u \in H_0^1(\Omega)$ and the subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ converges strongly to the same limit u in $L^p(\Omega)$.
Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary. Then :

1. *If $n > 2$, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $1 \leq p < \frac{2n}{n-2}$.*
2. *If $n = 2$, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $1 \leq p < \infty$.*
3. *If $n = 1$, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $1 \leq p \leq \infty$.*

The Lax-Milgram theorem is a fundamental result in functional analysis that provides conditions for the existence and uniqueness of solutions to certain variational problems. It is widely used in the study of partial differential equations, particularly in the context of weak solutions.

Theorem 1.3.4.
Let H be a Hilbert space. For any bounded linear functional $F : H \to \mathbb{R}$ and $a(\cdot,\cdot) : H\times H \to \mathbb{R}$ be a bilinear, continuous and coercive, there exists a unique $u \in H$ such that

$$a(u, v) = F(v), \quad \forall v \in H.$$

Moreover, the solution u satisfies the estimate

$$\|u\|_H \leq \frac{1}{\alpha}\|F\|_{H'},$$

where H' denotes the dual space of H.

1.4 Fixed point theorems

Definition 1.4.1.
If we have an operator T from a set E normed vector space to itself $(T : E \to E)$, then a point $x \in E$ is called a fixed point of the operator T if the condition $T(x) = x$ is satisfied.

Example 1.4.1.

Show that $T(x) = \frac{1}{4}x^2 + \frac{1}{4}$ *has a fixed point :*

To prove that the function $T(x) = \frac{1}{4}x^2 + \frac{1}{4}$ *has a fixed point, we need to find a value* x *such that* $T(x) = x$.

If x *is a fixed point of the function* T*, then it satisfies the equation :*

$$x = T(x) \Longleftrightarrow x = \frac{1}{4}x^2 + \frac{1}{4}.$$

We have two real solutions for the equation $T(x) = x$ *:*

$$x_1 = 2 + \sqrt{3},$$

$$x_2 = 2 - \sqrt{3}.$$

Since we have found values of x *that satisfy* $T(x) = x$*, this proves that the function* $T(x) = \frac{1}{4}x^2 + \frac{1}{4}$ *has two fixed points.*

Finally, the function $T(x) = \frac{1}{4}x^2 + \frac{1}{4}$ *has two fixed points at* $x = 2 + \sqrt{3}$ *and* $x = 2 - \sqrt{3}$.

Example 1.4.2.

Example of a function with a unique fixed point. Consider the function :

$$T(x) = \frac{1}{2}x + 1.$$

To find the fixed point(s), we solve the equation $T(x) = x$*, which is*

$$x = \frac{1}{2}x + 1.$$

We find one real solutions $x = 2$ *for the equation* $T(x) = x$*. Thus, the unique fixed point of the function* $T(x) = \frac{1}{2}x + 1$ *is* $x = 2$.

Example 1.4.3.

Consider the function :

$$g(x) = x^2 + 1.$$

To find the fixed point(s), we solve the equation $g(x) = x$ *:*

$$x = x^2 + 1.$$

Since the discriminant is negative $(\Delta < 0)$*, the quadratic equation* $x^2 - x + 1 = 0$ *has no real solutions.*

Finally, the function $g(x) = x^2 + 1$ *has no fixed points in the set of real numbers.*

Theorem 1.4.1. *(Schauder fixed point theorem)*
Let E be a Banach space, and let K be a nonempty, convex, closed and $T(K)$ relatively compact subset of E. If $T : K \to K$ is a continuous mapping, then T has at least one fixed point, i.e., there exists $x \in K$ such that $T(x) = x$.

Example 1.4.4.
Let $E = \mathbb{R}$ be the normed vector space of real numbers with the absolute value as the norm. Let $K = [0, 1]$ be the closed interval from 0 to 1. This set is non-empty, convex, and relatively compact in $\mathbb{R}$. Define the function $T : K \to K$ by

$$T(x) = \frac{1}{2}\sin(x) + \frac{1}{2}$$

$T(K) = K$:
For any $x \in [0, 1]$, we have $-1 \leq \sin(x) \leq 1$, which implies $-\frac{1}{2} \leq \frac{1}{2}\sin(x) \leq \frac{1}{2}$. Adding $\frac{1}{2}$ gives $0 \leq \frac{1}{2}\sin(x) + \frac{1}{2} \leq 1$. Thus, $0 \leq T(x) \leq 1$, so T maps K into itself.
By the Schauder fixed point theorem, since K is a non-empty, convex, closed and $T(K)$ relatively compact subset of the normed vector space $\mathbb{R}$, and $T : K \to K$ is continuous, there exists at least one fixed point $x^* \in K$ such that $T(x^*) = x^*$, i.e.,

$$x^* = \frac{1}{2}\sin(x^*) + \frac{1}{2}.$$

Theorem 1.4.2. *(Tychonoff fixed point theorem)*
Let E be a Banach space. If K is a nonempty, convex and closed subset of E, and $S : K \to K$ is a weakly continuous function, then S has a fixed point in K.

Chapter 2

Existence of a solution provided that the function g is Lipschitz continuous

In this Chapter, using the linearization approach together with the Schauder fixed point theorem to prove the existence of a solution of $(\mathcal{P}_1)$.

2.1 Statement of the problem

We are presented with the following problem :

$$(\mathcal{P}_1) : \begin{cases} M\left(\|u\|^q_{L^q(\Omega)}\right) div\,(a(x)\nabla u) = g(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $1 \leq q \leq 2$ and $a(x)$ is an $n \times n$ matrix satisfying for some constants λ and Λ the conditions :

$$\forall \xi \in \mathbb{R}^n : \quad \lambda|\xi|^2 \leq a(x)\xi \cdot \xi \qquad \text{a.e. } x \in \Omega, \tag{2.1}$$

$$\forall \xi \in \mathbb{R}^n : \quad |a(x)\xi| \leq \Lambda|\xi| \qquad \text{a.e. } x \in \Omega. \tag{2.2}$$

Given that M is a continuous function from $\mathbb{R}$ to $\mathbb{R}$, for a strictly positive constant m_0, we have :

$$M(t) \geq m_0 \quad \forall t \in \mathbb{R}, \tag{2.3}$$

with the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous such that the mapping $x \mapsto g(x,t)$ is measurable for every real number t, and $g(\cdot,0) \in L^2(\Omega)$ with $g(\cdot,0) \neq 0$ a.e. in Ω. Furthermore,

$$\forall t,s \in \mathbb{R}, x \in \Omega : \quad |g(x,t) - g(x,s)| \ \leq \ \begin{cases} b(x)|t-s|^p, & 0 < p < 1; \\ C|t-s|, & p = 1, \end{cases} \tag{2.4}$$

where $b \in L^{\frac{2}{1-p}}(\Omega)$ and C is a positive constant such that $C < \dfrac{\lambda m_0}{C_\Omega^2}$.

2.2 Existence of a solution

Theorem 2.2.1. *Suppose that* (2.1)-(2.4) *are true. Then, the problem identified as* ($\mathcal{P}_1$) *has at least one solution that is not simply the zero solution.*

Proof.

Firstly, by integrating both sides of problem ($\mathcal{P}_1$), we find

$$M\left(\|u\|^q_{L^q(\Omega)}\right) \int_\Omega a(x)\nabla u \cdot \nabla \xi \ \mathrm{d}x = \int_\Omega g(x,u)\xi \ \mathrm{d}x, \quad \forall \xi \in H_0^1(\Omega). \tag{2.5}$$

Then, let $v \in L^2(\Omega)$, we are now going to solve the following equation :

$$M\left(\|v\|^q_{L^q(\Omega)}\right) \int_\Omega a(x)\nabla u \cdot \nabla \xi \ \mathrm{d}x = \int_\Omega g(x,v)\xi \ \mathrm{d}x.$$

Consequently, dividing by $M\left(\|v\|^q_{L^q(\Omega)}\right)$ results in the equation :

$$\int_\Omega a(x)\nabla u \cdot \nabla \xi \ \mathrm{d}x = \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v)\xi \ \mathrm{d}x. \tag{2.6}$$

We can write it concisely as follows :

$$\exists! u \in H_0^1(\Omega) : \qquad A(u,\xi) = L_v(\xi), \quad \forall \xi \in H_0^1(\Omega),$$

where

$$\begin{aligned} A(u,\xi) &= \int_\Omega a(x)\nabla u \cdot \nabla \xi \ \mathrm{d}x, & \forall \xi \in H_0^1(\Omega), \\ L_v(\xi) &= \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v)\xi \ \mathrm{d}x, & \forall \xi \in H_0^1(\Omega). \end{aligned}$$

We will now begin to check the Lax-Milgram theorem's conditions :

1. Proving A is bilinearity : Let α be real constant. For all $u, w, \xi, \eta \in H_0^1(\Omega)$,

$$\begin{aligned}
A(\alpha u + w, \xi) &= \int_\Omega a(x)\nabla(\alpha u + w) \cdot \nabla\xi \, \mathrm{d}x \\
&= \int_\Omega a(x)(\alpha\nabla u + \nabla w) \cdot \nabla\xi \, \mathrm{d}x \\
&= \alpha \int_\Omega a(x)\nabla u \cdot \nabla\xi \, \mathrm{d}x + \int_\Omega a(x)\nabla w \cdot \nabla\xi \, \mathrm{d}x \\
&= \alpha A(u, \xi) + A(w, \xi).
\end{aligned}$$

$$\begin{aligned}
A(u, \alpha\xi + \eta) &= \int_\Omega a(x)\nabla u \cdot \nabla(\alpha\xi + \eta) \, \mathrm{d}x \\
&= \int_\Omega a(x)\nabla u \cdot (\alpha\nabla\xi + \nabla\eta) \, \mathrm{d}x \\
&= \alpha \int_\Omega a(x)\nabla u \cdot \nabla\xi \, \mathrm{d}x + \int_\Omega a(x)\nabla u \cdot \nabla\eta \, \mathrm{d}x \\
&= \alpha A(u, \xi) + A(u, \eta).
\end{aligned}$$

2. We prove that A is continuous. We have :

$$\begin{aligned}
\forall u, \xi \in H_0^1(\Omega) : \quad |A(u, \xi)| &= \left| \int_\Omega a(x)\nabla u \cdot \nabla\xi \, \mathrm{d}x \right| \\
&\leq \int_\Omega |a(x)\nabla u \cdot \nabla\xi| \, \mathrm{d}x.
\end{aligned}$$

By applying the Cauchy-Schwarz inequality, we find :

$$|A(u, \xi)| \leq \int_\Omega |a(x)\nabla u||\nabla\xi| \, \mathrm{d}x.$$

Subsequently, by condition (2.2), we deduce :

$$|A(u, \xi)| \leq \Lambda \int_\Omega |\nabla u||\nabla\xi| \, \mathrm{d}x.$$

Thereafter, applying the Cauchy-Schwarz inequality once more :

$$|A(u, \xi)| \leq \Lambda \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla\xi|^2 \, \mathrm{d}x \right)^{1/2},$$

in other words

$$\begin{aligned}
|A(u, \xi)| \leq \quad & \Lambda \left(\int_\Omega \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n} \right)^2 \right) \mathrm{d}x \right)^{1/2} \\
& \times \left(\int_\Omega \left(\left(\frac{\partial \xi}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial \xi}{\partial x_n} \right)^2 \right) \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

Therefore

$$|A(u,\xi)| \leq \quad \Lambda\left(\left\|\frac{\partial u}{\partial x_1}\right\|^2_{L^2(\Omega)}+\cdots+\left\|\frac{\partial u}{\partial x_n}\right\|^2_{L^2(\Omega)}\right)^{1/2} \times\left(\left\|\frac{\partial \xi}{\partial x_1}\right\|^2_{L^2(\Omega)}+\cdots+\left\|\frac{\partial \xi}{\partial x_n}\right\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Hence

$$|A(u,\xi)| \leq \Lambda\|u\|_{H_0^1(\Omega)}\|\xi\|_{H_0^1(\Omega)}.$$

3. Demonstrate that A is coercive. We possess :

$$\forall u \in H_0^1(\Omega): \quad A(u,u) \quad = \quad \int_\Omega a(x)\nabla u\cdot\nabla u \,\mathrm{d}x.$$

Based on condition (2.1), we arrive at :

$$\begin{aligned}\forall u \in H_0^1(\Omega): \quad A(u,u) \quad &\geq \quad \lambda\int_\Omega |\nabla u|^2 \,\mathrm{d}x\\ &\geq \quad \lambda\|u\|^2_{H_0^1(\Omega)}.\end{aligned}$$

4. The objective is to show that L_v is linear : Let α be real constant. For all $v,\xi,\eta \in H_0^1(\Omega)$,

$$\begin{aligned}L_v(\alpha\xi+\eta) \quad &= \quad \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\int_\Omega g(x,v)(\alpha\xi+\eta)\,\mathrm{d}x\\ &= \quad \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\left(\alpha\int_\Omega g(x,v)\xi \,\mathrm{d}x+\int_\Omega g(x,v)\eta \,\mathrm{d}x\right)\\ &= \quad \alpha L_v(\xi)+L_v(\eta).\end{aligned}$$

5. We aim to show that L_v is continuous :

$$\begin{aligned}\forall \xi \in H_0^1(\Omega): \quad |L_v(\xi)| \quad &= \quad \left|\frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\int_\Omega g(x,v)\xi \,\mathrm{d}x\right|\\ &\leq \quad \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\int_\Omega |g(x,v)||\xi| \,\mathrm{d}x\\ &\leq \quad \frac{1}{m_0}\int_\Omega |g(x,v)||\xi| \,\mathrm{d}x. \qquad (2.7)\end{aligned}$$

Taking $s=0$ in (2.4) to find :

$$|g(x,t)-g(x,0)| \leq b(x)|t|^p.$$

Applying the following property of the absolute value :

$$\forall a, b \in \mathbb{R}: \quad ||a| - |b|| \leq |a-b|,$$

we acquire :

$$||g(x,t)| - |g(x,0)|| \leq b(x)|t|^p,$$

then :

$$-b(x)|t|^p \leq |g(x,t)| - |g(x,0)| \leq b(x)|t|^p,$$

after that, yields,

$$|g(x,t)| \leq b(x)|t|^p + |g(x,0)|. \tag{2.8}$$

For $t = v$ in (2.8), thereafter substituting $|g(x,v)|$ into (2.7), we obtain :

$$\forall \xi \in H_0^1(\Omega): \quad |L_v(\xi)| \ \leq \ \frac{1}{m_0}\int_\Omega (b(x)|v|^p + |g(x,0)|)|\xi| \ \mathrm{d}x.$$

Applying Hölder's inequality,

$$|L_v(\xi)| \ \leq \ \frac{1}{m_0}\left(\left(\int_\Omega |b(x)|^2|v|^{2p} \ \mathrm{d}x\right)^{1/2} + \|g(\cdot,0)\|_{L^2(\Omega)}\right)\|\xi\|_{L^2(\Omega)}. \tag{2.9}$$

Thereafter, by using Hölder's inequality again on $\left(\int_\Omega |b(x)|^2|v|^{2p} \ \mathrm{d}x\right)^{1/2}$, we observe

$$\left(\int_\Omega |b(x)|^2|v|^{2p} \ \mathrm{d}x\right)^{1/2} \leq \left(\left(\int_\Omega |b(x)|^{2r'} \ \mathrm{d}x\right)^{1/r'}\left(\int_\Omega |v|^2 \ \mathrm{d}x\right)^{1/r}\right)^{1/2},$$

by considering $r = \dfrac{1}{p}$ and $r' = \dfrac{1}{1-p}$, we observe :

$$\begin{aligned}\left(\int_\Omega |b(x)|^2|v|^{2p} \ \mathrm{d}x\right)^{1/2} \ &\leq \ \left(\left(\int_\Omega |b(x)|^{\frac{2}{1-p}} \ \mathrm{d}x\right)^{1-p}\left(\int_\Omega |v|^2 \ \mathrm{d}x\right)^{p}\right)^{1/2} \\ &\leq \ \left(\int_\Omega |b(x)|^{\frac{2}{1-p}} \ \mathrm{d}x\right)^{\frac{1-p}{2}}\left(\int_\Omega |v|^2 \ \mathrm{d}x\right)^{\frac{p}{2}} \\ &\leq \ \|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)}.\end{aligned}$$

With substitution into (2.9),

$$|L_v(\xi)| \ \leq \ \frac{1}{m_0}\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)} + \|g(\cdot,0)\|_{L^2(\Omega)}\right)\|\xi\|_{L^2(\Omega)}. \tag{2.10}$$

Thereafter, utilizing Poincaré's inequality, we arrive for all $0 < p < 1$:

$$\forall \xi \in H_0^1(\Omega): \quad |L_v(\xi)| \;\leq\; \frac{C_\Omega}{m_0}\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)} + \||g(\cdot,0)\|_{L^2(\Omega)}\right)\|\xi\|_{H_0^1(\Omega)}.$$

If $p = 1$, we find :

$$\forall \xi \in H_0^1(\Omega): \quad |L_v(\xi)| \;\leq\; \frac{C_\Omega}{m_0}\left(C\|v\|_{L^2(\Omega)} + \|g(\cdot,0)\|_{L^2(\Omega)}\right)\|\xi\|_{H_0^1(\Omega)}.$$

Finally, by the Lax-Milgram theorem, for every $v \in L^2(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega)$ satisfying the variational equality.

Next, let the operator T be defined as follows :

$$T : L^2(\Omega) \to H_0^1(\Omega), \quad v \mapsto u = T(v).$$

T is well-defined where each pre-image $v \in L^2(\Omega)$ has a unique image $u \in H_0^1(\Omega)$.

Let $\overline{B}(0,R)$ denote the closed ball in the space $L^2(\Omega)$ with center 0 and radius R. Then, we have there is an $R > 0$ such that :

$$\left|\begin{array}{ll} (i) & \exists R > 0 : T(\overline{B}(0,R)) \subset \overline{B}(0,R), \\ (ii) & T : \overline{B}(0,R) \to \overline{B}(0,R) \text{ is continuous}, \\ (iii) & T(\overline{B}(0,R)) \text{ is relatively compact in } L^2(\Omega). \end{array}\right.$$

We now embark on proving :

(i) Let $v \in \overline{B}(0,R)$, which implies $\|v\|_{L^2(\Omega)} \leq R$. We want to show that $u = T(v) \in \overline{B}(0,R)$, i.e., $\|u\|_{L^2(\Omega)} \leq R$.

Choose $\xi = u$ as a test function in (2.6),

$$\int_\Omega a(x)\nabla u \cdot \nabla u \,\mathrm{d}x = \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\int_\Omega g(x,v)u \,\mathrm{d}x. \tag{2.11}$$

From (2.1) and utilizing Poincaré's inequality, we deduce :

$$\int_\Omega a(x)\nabla u \cdot \nabla u \,\mathrm{d}x \geq \frac{\lambda}{C_\Omega^2}\|u\|^2_{L^2(\Omega)}. \tag{2.12}$$

For the other side, we deduce from (2.10),

$$\begin{aligned} &\frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\int_\Omega g(x,v)u \,\mathrm{d}x \\ &\qquad \leq \frac{1}{m_0}\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)} + \|g(\cdot,0)\|_{L^2(\Omega)}\right)\|u\|_{L^2(\Omega)}, \quad 0 < p < 1. \end{aligned} \tag{2.13}$$

From (2.11), (2.12) and (2.13), we get :

$$\frac{\lambda}{C_\Omega^2}\|u\|^2_{L^2(\Omega)} \leq \frac{1}{m_0}\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)} + \|g(\cdot,0)\|_{L^2(\Omega)}\right)\|u\|_{L^2(\Omega)}.$$

Then, we obtain the following inequality :

$$\|u\|_{L^2(\Omega)} \leq \frac{C_\Omega^2}{\lambda m_0}\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)} + \|g(\cdot,0)\|_{L^2(\Omega)}\right), \quad 0<p<1.$$

On the flip side, if $p=1$, we achieve :

$$\|u\|_{L^2(\Omega)} \leq \frac{C_\Omega^2}{\lambda m_0}\left(C\|v\|_{L^2(\Omega)} + \|g(\cdot,0)\|_{L^2(\Omega)}\right).$$

Then, for $\|v\|_{L^2(\Omega)} \leq R$, we write

$$\begin{aligned}
\frac{C_\Omega^2}{\lambda m_0}\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)} + \|g(\cdot,0)\|_{L^2(\Omega)}\right) &\leq \frac{C_\Omega^2}{\lambda m_0}\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}R^p + \|g(\cdot,0)\|_{L^2(\Omega)}\right), \\
&\leq R.
\end{aligned}$$

Now, we proceed to find $R>0$ that satisfies $\|u\|_{L^2(\Omega)} \leq R$, provided that $0<p<1$,

$$\begin{aligned}
&\frac{C_\Omega^2}{\lambda m_0}(CR^p + \|g(\cdot,0)\|_{L^2(\Omega)}) \leq R \\
\Longleftrightarrow\ &\frac{C_\Omega^2}{\lambda m_0}CR^p + \frac{C_\Omega^2}{\lambda m_0}\|g(\cdot,0)\|_{L^2(\Omega)} \leq R \\
\Longleftrightarrow\ &\frac{C_\Omega^2}{\lambda m_0}\|g(\cdot,0)\|_{L^2(\Omega)} \leq R - \frac{C_\Omega^2}{\lambda m_0}CR^p \\
\Longleftrightarrow\ &\|g(\cdot,0)\|_{L^2(\Omega)} \leq \frac{\lambda m_0}{C_\Omega^2}R - CR^p \\
\Longleftrightarrow\ &\|g(\cdot,0)\|_{L^2(\Omega)} \leq R\left(\frac{\lambda m_0}{C_\Omega^2} - CR^{p-1}\right) \\
\Longleftrightarrow\ &\|g(\cdot,0)\|_{L^2(\Omega)} \leq R\left(\frac{\lambda m_0}{C_\Omega^2} - \frac{C}{R^{1-p}}\right).
\end{aligned}$$

It is possible to choose a sufficiently large R that satisfies

$$\|u\|_{L^2(\Omega)} \leq R.$$

From another perspective, if $p=1$ and taking into account $C < \dfrac{\lambda m_0}{C_\Omega^2}$, we can choose

$$R = \frac{\|g(\cdot,0)\|_{L^2(\Omega)}}{\dfrac{\lambda m_0}{C_\Omega^2} - C},$$

that satisfies

$$\|u\|_{L^2(\Omega)} \leq R.$$

Therefore, $T(\overline{B}(0,R)) \subset \overline{B}(0,R)$.

(ii) Let $\{v_n\}_{n\in\mathbb{N}} \subset \overline{B}(0,R)$ be a sequence such that

$$\lim_{n\to+\infty} \|v_n - v\|_{L^2(\Omega)} = 0. \tag{2.14}$$

We want to show that

$$\lim_{n\to+\infty} \|u_n - u\|_{L^2(\Omega)} = \lim_{n\to+\infty} \|T(v_n) - T(v)\|_{L^2(\Omega)} = 0.$$

By the definition of T, we have :

$$\int_\Omega a(x)\nabla u_n \cdot \nabla\xi \, \mathrm{d}x = \frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v_n)\xi \, \mathrm{d}x, \quad \forall\xi \in H_0^1(\Omega), \tag{2.15}$$

and

$$\int_\Omega a(x)\nabla u \cdot \nabla\xi \, \mathrm{d}x = \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v)\xi \, \mathrm{d}x, \quad \forall\xi \in H_0^1(\Omega). \tag{2.16}$$

Subtract (2.15) from (2.16) :

$$\int_\Omega a(x)\nabla(u_n - u) \cdot \nabla\xi \, \mathrm{d}x = \frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v_n)\xi \, \mathrm{d}x - \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v)\xi \, \mathrm{d}x.$$

Choose $\xi = u_n - u$ in the previous equation

$$\begin{aligned}\int_\Omega a(x)\nabla(u_n - u) \cdot \nabla(u_n - u) \, \mathrm{d}x &= \frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v_n)(u_n - u) \, \mathrm{d}x \\ &- \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x,v)(u_n - u) \, \mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Omega a(x)\nabla(u_n - u) \cdot \nabla(u_n - u)\mathrm{d}x \\ &\qquad = \int_\Omega \left\{ \frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)} g\left(x, v_n\right) - \frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)} g(x,v) \right\} (u_n - u) \, \mathrm{d}x.\end{aligned}$$

This can be expressed as follows :

$$\begin{aligned}
&\int_\Omega a(x)\nabla(u_n-u)\cdot\nabla(u_n-u)\mathrm{d}x\\
&\qquad=\frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)}\int_\Omega\left(g\left(x,v_n\right)-g(x,v)\right)\left(u_n-u\right)\mathrm{d}x\\
&\qquad+\left\{\frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)}-\frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\right\}\int_\Omega g(x,v)\left(u_n-u\right)\mathrm{d}x.
\end{aligned}$$

From (2.1), Poincaré's inequality from one side and on the other side from (2.10), we get

$$\begin{aligned}
&\frac{\lambda}{C_\Omega^2}\|u_n-u\|^2_{L^2(\Omega)}\\
&\quad\le\left\{\frac{\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v_n-v\|^p_{L^2(\Omega)}}{m_0}\right\}\|u_n-u\|_{L^2(\Omega)}\\
&\quad+\left\{\left|\frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)}-\frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\right|\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)}\right)\right\}\|u_n-u\|_{L^2(\Omega)}\\
&\quad+\left\{\left|\frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)}-\frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\right|\right\}\|g(\cdot,0)\|_{L^2(\Omega)}\|u_n-u\|_{L^2(\Omega)}.
\end{aligned}$$

This leads us to, if $0<p<1$:

$$\begin{aligned}
&\|u_n-u\|_{L^2(\Omega)}\\
&\quad\le\frac{C_\Omega^2}{\lambda}\left\{\frac{\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v_n-v\|^p_{L^2(\Omega)}}{m_0}\right\}\\
&\quad+\frac{C_\Omega^2}{\lambda}\left\{\left|\frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)}-\frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\right|\left(\|b\|_{L^{\frac{2}{1-p}}(\Omega)}\|v\|^p_{L^2(\Omega)}+\|g(\cdot,0)\|_{L^2(\Omega)}\right)\right\}.
\end{aligned}\tag{2.17}$$

Conversely, if $p=1$, we observe :

$$\begin{aligned}
\|u_n-u\|_{L^2(\Omega)}\ &\le\ \frac{C_\Omega^2}{\lambda}\left\{\frac{C}{m_0}\|v_n-v\|_{L^2(\Omega)}\right\}\\
&+\ \frac{C_\Omega^2}{\lambda}\left\{\left|\frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)}-\frac{1}{M\left(\|v\|^q_{L^q(\Omega)}\right)}\right|\left(C\|v\|_{L^2(\Omega)}+\|g(\cdot,0)\|_{L^2(\Omega)}\right)\right\}.
\end{aligned}\tag{2.18}$$

Conversely, with the application of Hölder's inequality and the selection of $r=\dfrac{2}{q}$ and

$r' = \dfrac{2}{2-q}$

$$\begin{aligned}
\int_\Omega |v_n - v|^q \, \mathrm{d}x &\leq \left(\int_\Omega |v_n - v|^2 \, \mathrm{d}x\right)^{1/r} \left(\int_\Omega 1^{r'} \, \mathrm{d}x\right)^{1/r'} \\
&\leq \left(\int_\Omega |v_n - v|^2 \, \mathrm{d}x\right)^{\frac{q}{2}} \left(\int_\Omega 1^{\frac{2}{2-q}} \, \mathrm{d}x\right)^{\frac{2-q}{2}} \\
&\leq \|v_n - v\|^q_{L^2(\Omega)} \, |\Omega|^{\frac{2-q}{2}} .
\end{aligned}$$

From (2.14), we find that

$$\lim_{n\to+\infty} \left(\int_\Omega |v_n - v|^2 \, \mathrm{d}x\right)^{\frac{q}{2}} = 0,$$

thus

$$\lim_{n\to+\infty} \int_\Omega |v_n - v|^q \, \mathrm{d}x = 0,$$

this leads us to, $\{v_n\}_{n\in\mathbb{N}}$ converges to v in $L^q(\Omega)$, where $1 \leq q \leq 2$, that is to say

$$\lim_{n\to+\infty} \|v_n - v\|_{L^q(\Omega)} = 0. \tag{2.19}$$

Subsequently, via the following inequality

$$\left| \|v_n\|_{L^q(\Omega)} - \|v\|_{L^q(\Omega)} \right| \leq \|v_n - v\|_{L^q(\Omega)},$$

we get

$$\lim_{n\to+\infty} \|v_n\|_{L^q(\Omega)} = \|v\|_{L^q(\Omega)}.$$

Since M is a continuous function, then

$$\lim_{n\to+\infty} M\left(\|v_n\|^q_{L^q(\Omega)}\right) = M\left(\|v\|^q_{L^q(\Omega)}\right). \tag{2.20}$$

Hence, by passing to the limit in both equations (2.17) and (2.18), and by utilizing the results of (2.19) and (2.20), we conclude that

$$\lim_{n\to+\infty} \|u_n - u\|_{L^2(\Omega)} = 0.$$

Thus, T is continuous on $\overline{B}(0, R)$.

(iii) Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $T(\overline{B}(0, R))$. Then there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \overline{B}(0, R)$ such that $u_n = T(v_n)$ for all $n \in \mathbb{N}$.

By the definition of T, we have :

$$\int_\Omega a(x)\nabla u_n \cdot \nabla\xi \, \mathrm{d}x = \frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x, v_n)\xi \, \mathrm{d}x, \quad \forall\xi \in H^1_0(\Omega). \tag{2.21}$$

Choosing $\xi = u_n$ in (2.21) :

$$\int_\Omega a(x)\nabla u_n \cdot \nabla u_n \,\mathrm{d}x = \frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x, v_n)u_n \,\mathrm{d}x. \tag{2.22}$$

On the one hand, from (2.1), we deduce :

$$\begin{aligned} \int_\Omega a(x)\nabla u_n \cdot \nabla u_n \,\mathrm{d}x &\geq \lambda \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x \\ &\geq \lambda \|u_n\|^2_{H^1_0(\Omega)}. \end{aligned} \tag{2.23}$$

From the other hand, using (2.10) and Poincaré's inequality

$$\begin{aligned} &\frac{1}{M\left(\|v_n\|^q_{L^q(\Omega)}\right)} \int_\Omega g(x, v_n)u_n \,\mathrm{d}x \\ &\qquad\qquad \leq \frac{C_\Omega}{m_0} \left\{ \|b\|_{L^{\frac{2}{1-p}}(\Omega)} \|v_n\|^p_{L^2(\Omega)} + \|g(\cdot, 0)\|_{L^2(\Omega)} \right\} \|u_n\|_{H^1_0(\Omega)}. \end{aligned} \tag{2.24}$$

Then, from (2.22), (2.23) and (2.24), we obtain

$$\|u_n\|_{H^1_0(\Omega)} \leq \frac{C_\Omega}{\lambda m_0} \left\{ \|b\|_{L^{\frac{2}{1-p}}(\Omega)} \|v_n\|^p_{L^2(\Omega)} + \|g(\cdot, 0)\|_{L^2(\Omega)} \right\}.$$

Thus, let $v_n \in \overline{B}(0, R)$, which implies $\|v_n\|_{L^2(\Omega)} \leq R$

$$\|u_n\|_{H^1_0(\Omega)} \leq \frac{C_\Omega}{\lambda m_0} \left\{ \|b\|_{L^{\frac{2}{1-p}}(\Omega)} R^p + \|g(\cdot, 0)\|_{L^2(\Omega)} \right\} := R_p, \quad 0 < p < 1.$$

Conversely, if $p = 1$:

$$\begin{aligned} \|u_n\|_{H^1_0(\Omega)} &\leq \frac{C_\Omega}{\lambda m_0} \left\{ \|v_n\|_{L^2(\Omega)} + \|g(\cdot, 0)\|_{L^2(\Omega)} \right\} \\ &\leq \frac{C_\Omega}{\lambda m_0} \left\{ R + \|g(\cdot, 0)\|_{L^2(\Omega)} \right\} := R_1. \end{aligned}$$

By Rellich-Kondrachov Theorem, the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Therefore, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} \|u_{n_k} - u\|_{L^2(\Omega)} = 0.$$

Thus, $T(\overline{B}(0, R))$ is relatively compact in $L^2(\Omega)$. □

Conclusively, by utilizing Schauder's fixed-point theorem, we can definitively conclude that the mapping T possesses a fixed point $u \in H^1_0(\Omega)$. This fixed point u is a solution of the problem $(\mathcal{P}_1)$, a direct consequence of $g(x, 0)$ being non-zero for almost all points within Ω, because if we assume that $u = 0$ in (2.5) we find

$$M\left(\|0\|^q_{L^q(\Omega)}\right) \int_\Omega a(x)\nabla 0 \cdot \nabla \xi \,\mathrm{d}x = \int_\Omega g(x, 0)\xi \,\mathrm{d}x, \quad \forall \xi \in H^1_0(\Omega),$$

which gives

$$\int_{\Omega} g(x,0)\xi \, \mathrm{d}x = 0, \quad \forall \xi \in H_0^1(\Omega).$$

Thus, $g(x,0) = 0$ a.e. $x \in \Omega$, which contradicts the fact that $g(x,0) \neq 0$ a.e. $x \in \Omega$.

Chapter 3

Existence of a solution under condition of boundedness

In this Chapter, we use the linearization technique combining with the Tychonoff fixed point theorem to prove the existence of a solution of $(\mathcal{P}_2)$.

3.1 Statement of the problem

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $x \mapsto f(x,t)$ is measurable on Ω for all $t \in \mathbb{R}$, $t \mapsto f(x,t)$ on $\mathbb{R}$ for almost everywhere $x \in \Omega$ and $g(.,0) \neq 0$. Suppose there exists $h \in L^2(\Omega)$ satisfying

$$|g(x,t)| \leq h(x) \tag{3.1}$$

for all $t \in \mathbb{R}$ and almost everywhere $x \in \Omega$. By (3.1) and the Cauchy-Schwarz inequality, the integral

$$\int_\Omega g(x,w)\varphi(x)\ \mathrm{d}x,$$

where $w, \varphi \in L^2(\Omega)$, since

$$\begin{aligned}\left|\int_\Omega g(x,w)\varphi(x)\ \mathrm{d}x\right| &\leq \int_\Omega |g(x,w)\varphi(x)|\ \mathrm{d}x \\ &= \int_\Omega |g(x,w)||\varphi(x)|\ \mathrm{d}x \\ &\leq \int_\Omega h(x)|\varphi(x)|\ \mathrm{d}x \\ &\leq \|h\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}.\end{aligned} \tag{3.2}$$

Assume that (2.1)-(2.3) and (3.1) hold. We study the existence of a solution of the following elliptic problem

$$(\mathcal{P}_2) \quad \begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that :} \\ M\left(\|u\|^q_{L^q(\Omega)}\right) \displaystyle\int_\Omega a(x)\nabla u \cdot \nabla\varphi \, \mathrm{d}x = \int_\Omega g(x,u)\varphi \, \mathrm{d}x, \\ \quad \forall \varphi \in H_0^1(\Omega). \end{cases}$$

3.2 Existence of a solution

Theorem 3.2.1. *If (2.1)-(2.3) and (3.1) hold, the problem $(\mathcal{P}_2)$ has a nontrivial solution.*

Proof. Let $v \in H_0^1(\Omega)$. Consider

$$\forall u, \varphi \in H_0^1(\Omega) : \quad A(u,\varphi) = M\left(\|v\|^q_{L^q(\Omega)}\right) \int_\Omega a(x)\nabla u \cdot \nabla\varphi \, \mathrm{d}x,$$
$$\forall \varphi \in H_0^1(\Omega) : \quad L_v(\varphi) = \int_\Omega g(x,v)\varphi \, \mathrm{d}x.$$

By (3.1) and Poincaré's inequality, we have

$$\begin{aligned} \left|\int_\Omega g(x,v)\varphi(x) \, \mathrm{d}x\right| &\le \int_\Omega |g(x,v)\varphi(x)| \, \mathrm{d}x \\ &= \int_\Omega |g(x,v)||\varphi(x)| \, \mathrm{d}x \\ &\le \int_\Omega h(x)|\varphi(x)| \, \mathrm{d}x \\ &\le \|h\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \\ &\le C_\Omega\|h\|_{L^2(\Omega)}\|\varphi\|_{H_0^1(\Omega)}. \end{aligned} \tag{3.3}$$

Therefore, since L_v is a linear form on $H_0^1(\Omega)$, it follows that L_v is continuous on $H_0^1(\Omega)$. Since A is a continuous, coercive bilinear form on $H_0^1(\Omega)$, we deduce, using the Lax-Milgram Theorem, that there exists a unique solution $u \in H_0^1(\Omega)$ of the variational formulation

$$\forall \varphi \in H_0^1(\Omega) : \quad A(u,\varphi) = L_v(\varphi),$$

which is equivalent to

$$\forall \varphi \in H_0^1(\Omega) : \quad M\left(\|v\|^q_{L^q(\Omega)}\right) \int_\Omega a(x)\nabla u \cdot \nabla\varphi \, \mathrm{d}x = \int_\Omega g(x,v)\varphi \, \mathrm{d}x. \tag{3.4}$$

Let us define the following mapping

$$S : H_0^1(\Omega) \to H_0^1(\Omega),\ v \mapsto S(v) = u,$$

where u is the unique solution of (3.4). We apply the Tychonoff fixed point theorem to the mapping S to obtain a solution to $(\mathcal{P}_2)$.

(i) Let $\overline{B}(0,R)$ be the closed ball in $H_0^1(\Omega)$ with center 0 and radius $R > 0$. We want to find $R \in]0,\infty[$ such that $S(\overline{B}(0,R)) \subset \overline{B}(0,R)$. Taking $\varphi = u$ in (3.4) to get

$$M\left(\|v\|^q_{L^q(\Omega)}\right) \int_\Omega a(x)\nabla u \cdot \nabla u \, \mathrm{d}x = \int_\Omega g(x,v)u \, \mathrm{d}x. \tag{3.5}$$

By (2.1) and (2.3), we have

$$\begin{aligned} M\left(\|v\|^q_{L^q(\Omega)}\right) \int_\Omega a(x)\nabla u \cdot \nabla u \, \mathrm{d}x &\geq \lambda m_0 \int_\Omega |\nabla u|^2 \, \mathrm{d}x \\ &= \lambda m_0 \|u\|^2_{H_0^1(\Omega)}. \end{aligned} \tag{3.6}$$

On the other hand, in view of the estimate (3.3), we obtain

$$\left|\int_\Omega g(x,v)u(x) \, \mathrm{d}x\right| \leq C_\Omega \|h\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)}. \tag{3.7}$$

Using (3.5)-(3.7), we find

$$\lambda m_0 \|u\|^2_{H_0^1(\Omega)} \leq C_\Omega \|h\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)},$$

which implies

$$\|u\|_{H_0^1(\Omega)} \leq \frac{C_\Omega \|h\|_{L^2(\Omega)}}{\lambda m_0}.$$

Hence, it suffices to choose

$$R = \frac{C_\Omega \|h\|_{L^2(\Omega)}}{\lambda m_0}$$

to have $S(\overline{B}(0,R)) \subset \overline{B}(0,R)$.

(ii) $S : \overline{B}(0,R) \to \overline{B}(0,R)$ is weakly continuous. Let $\{v_j\}_{j\in\mathbb{N}} \subset \overline{B}(0,R)$ and $v^* \in \overline{B}(0,R)$ such that

$$\{v_j\}_{j\in\mathbb{N}} \text{ converges weakly to } v^* \text{ in } H_0^1(\Omega). \tag{3.8}$$

From (3.8),

$$\{v_j\}_{j\in\mathbb{N}} \text{ is bounded in } H_0^1(\Omega). \tag{3.9}$$

Using (3.9) and the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we can find $\overline{v} \in H_0^1(\Omega)$ and a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ of $\{v_j\}_{j\in\mathbb{N}}$ such that

$$\{v_{j_k}\}_{k\in\mathbb{N}} \text{ converges strongly to } \overline{v} \text{ in } L^2(\Omega), \tag{3.10}$$

$$\{v_{j_k}\}_{k\in\mathbb{N}} \text{ converges almost everywhere on } \Omega \text{ to } \overline{v}. \tag{3.11}$$

In view of (3.8), (3.10), (3.11) and the uniqueness of the limit, one has

$$\{v_{j_k}\}_{k\in\mathbb{N}} \text{ converges strongly to } v^* \text{ in } L^2(\Omega), \tag{3.12}$$

$$\{v_{j_k}\}_{k\in\mathbb{N}} \text{ converges almost everywhere on } \Omega \text{ to } v^*. \tag{3.13}$$

From (3.12), we obtain

$$\{v_{j_k}\}_{k\in\mathbb{N}} \text{ converges strongly to } v^* \text{ in } L^q(\Omega),\ q \in [1,2],$$

which leads to

$$\lim_{k\to\infty} \|v_{j_k}\|_{L^q(\Omega)} = \|v^*\|_{L^q(\Omega)}. \tag{3.14}$$

Using (3.5), one can write

$$M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)\int_\Omega a(x)\nabla u_{j_k}\cdot\nabla\varphi \,\mathrm{d}x = \int_\Omega g(x,v_{j_k})\varphi \,\mathrm{d}x, \tag{3.15}$$

$$M\left(\|v^*\|^q_{L^q(\Omega)}\right)\int_\Omega a(x)\nabla u^*\cdot\nabla\varphi \,\mathrm{d}x = \int_\Omega g(x,v^*)\varphi \,\mathrm{d}x, \tag{3.16}$$

with $u_{j_k} = S(v_{j_k})$ and $u^* = S(v^*)$, for all $\varphi \in H_0^1(\Omega)$. Subtracting (3.15) from and (3.16), we find

$$\begin{aligned}&\int_\Omega a(x)\Big\{M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)\nabla u_{j_k} - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\nabla u^*\Big\}\cdot\nabla\varphi \,\mathrm{d}x\\ &= \int_\Omega (g(x,v_{j_k}) - g(x,v^*))\varphi \,\mathrm{d}x,\end{aligned}$$

which can be written as

$$\begin{aligned}&M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)\int_\Omega a(x)\nabla(u_{j_k}-u^*).\nabla\varphi \,\mathrm{d}x\\ &+\left\{M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right\}\int_\Omega a(x)\nabla u^*.\nabla\varphi \,\mathrm{d}x\\ &= \int_\Omega (g(x,v_{j_k}) - g(x,v^*))\varphi \,\mathrm{d}x.\end{aligned}$$

Taking $\varphi = u_{j_k} - u^*$, we get

$$\begin{aligned}
&M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)\int_\Omega a(x)\nabla(u_{j_k}-u^*).\nabla(u_{j_k}-u^*)\ \mathrm{d}x\\
&+\left\{M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)-M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right\}\int_\Omega a(x)\nabla u^*.\nabla(u_{j_k}-u^*)\ \mathrm{d}x\\
&=\int_\Omega (g(x,v_{j_k})-g(x,v^*))(u_{j_k}-u^*)\ \mathrm{d}x.
\end{aligned}\tag{3.17}$$

Using (3.1) and (3.2), we obtain

$$M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)\int_\Omega a(x)\nabla(u_{j_k}-u^*).\nabla(u_{j_k}-u^*)\ \mathrm{d}x \geq \lambda m_0\|u_{j_k}-u^*\|^2_{H^1_0(\Omega)}.\tag{3.18}$$

Combining (3.17) and (3.18), we get

$$\begin{aligned}
&\|u_{j_k}-u^*\|^2_{H^1_0(\Omega)} \leq \frac{1}{\lambda m_0}\int_\Omega (g(x,v_{j_k})-g(x,v^*))(u_{j_k}-u^*)\ \mathrm{d}x\\
&+\frac{1}{\lambda m_0}\left\{M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)-M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right\}\int_\Omega a(x)\nabla u^*.\nabla(u_{j_k}-u^*)\ \mathrm{d}x\\
&:=\frac{1}{\lambda m_0}(I_{j_k}+J_{j_k})
\end{aligned}\tag{3.19}$$

with

$$\begin{aligned}
I_{j_k} &= \int_\Omega (g(x,v_{j_k})-g(x,v^*))(u_{j_k}-u^*)\ \mathrm{d}x,\\
J_{j_k} &= \left\{M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right)-M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right\}\int_\Omega a(x)\nabla u^*.\nabla(u_{j_k}-u^*)\ \mathrm{d}x.
\end{aligned}$$

Using (3.2), Cauchy-Schwarz and Poincaré's inequalities, and the fact that $u_{j_k} \in \overline{B}(0,R)$, we obtain

$$\begin{aligned}
|I_{j_k}| &= \left|\int_\Omega (g(x,v_{j_k})-g(x,v^*))(u_{j_k}-u^*)\ \mathrm{d}x\right|\\
&\leq \int_\Omega |g(x,v_{j_k})-g(x,v^*)||u_{j_k}-u^*|\ \mathrm{d}x\\
&\leq \|g(.,v_{j_k})-g(.,v^*)\|_{L^2(\Omega)}\|u_{j_k}-u^*\|_{L^2(\Omega)}\\
&\leq (\|u_{j_k}\|_{L^2(\Omega)}+\|u^*\|_{L^2(\Omega)})\|g(.,v_{j_k})-g(.,v^*)\|_{L^2(\Omega)}\\
&\leq (C_\Omega\|u_{j_k}\|_{H^1_0(\Omega)}+\|u^*\|_{L^2(\Omega)})\|g(.,v_{j_k})-g(.,v^*)\|_{L^2(\Omega)}\\
&\leq (C_\Omega R+\|u^*\|_{L^2(\Omega)})\|g(.,v_{j_k})-g(.,v^*)\|_{L^2(\Omega)}
\end{aligned}\tag{3.20}$$

and

$$
\begin{aligned}
|J_{j_k}| &= \left|M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right|\left|\int_\Omega a(x)\nabla u^*.\nabla(u_{j_k}-u^*)\ \mathrm{d}x\right| \\
&\le \left|M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right|\int_\Omega |a(x)\nabla u^*.\nabla(u_{j_k}-u^*)|\ \mathrm{d}x \\
&\le \left|M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right|\int_\Omega |a(x)\nabla u^*||\nabla(u_{j_k}-u^*)|\ \mathrm{d}x \\
&\le \Lambda\left|M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right|\int_\Omega |\nabla u^*||\nabla(u_{j_k}-u^*)|\ \mathrm{d}x \\
&\le \Lambda\|u^*\|_{H^1_0(\Omega)}\|u_{j_k}-u^*\|_{H^1_0(\Omega)}\left|M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right| \\
&\le \Lambda\|u^*\|_{H^1_0(\Omega)}(\|u_{j_k}\|_{H^1_0(\Omega)}+\|u^*\|_{H^1_0(\Omega)})\left|M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right| \\
&\le \Lambda\|u^*\|_{H^1_0(\Omega)}(R+\|u^*\|_{H^1_0(\Omega)})\left|M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) - M\left(\|v^*\|^q_{L^q(\Omega)}\right)\right|.
\end{aligned} \tag{3.21}
$$

From (3.1), we have

$$\forall k \in \mathbb{N}: \quad |g(x, v_{j_k})| \le h(x) \ \text{ a.e. } x \in \Omega \tag{3.22}$$

with $h \in L^2(\Omega)$. On the other hand, by (3.13) and the continuity of $t \mapsto g(., t)$, one can write

$$\lim_{k\to+\infty} g(x, v_{j_k}) = g(x, v^*) \ \text{ a.e. } x \in \Omega. \tag{3.23}$$

Using (3.22), (3.23) and the dominated convergence theorem, we obtain

$$\lim_{k\to+\infty} \|g(., v_{j_k}) - g(., v^*)\|_{L^2(\Omega)} = 0. \tag{3.24}$$

Moreover, by (3.14) and the continuity of M, we have

$$\lim_{k\to+\infty} M\left(\|v_{j_k}\|^q_{L^q(\Omega)}\right) = \left(\|v^*\|^q_{L^2(\Omega)}\right). \tag{3.25}$$

Letting $k \to +\infty$ in (3.20) and (3.21), we obtain by using (3.24) and (3.25),

$$\lim_{k\to+\infty} I_{j_k} = J_{j_k} = 0. \tag{3.26}$$

Passing to the limit in (3.19) as $k \to +\infty$ and taking into account (3.26), we find

$$\lim_{k\to+\infty} \|u_{j_k} - u^*\|_{H^1_0(\Omega)} = 0. \tag{3.27}$$

Thus, u^* is a weak limit point for $\{u_j\}_{j\in\mathbb{N}}$ in $\overline{B}(0, R)$. If $\overline{u} \in \overline{B}(0, R)$ is other weak limit point for $\{u_j\}_{j\in\mathbb{N}}$, there exists a subsequence $\{u_{j_m}\}_{m\in\mathbb{N}} \subset \{u_j\}_{j\in\mathbb{N}}$ such that

$$\{u_{j_m}\}_{m\in\mathbb{N}} \text{ converges weakly to } \overline{u} \text{ in } H^1_0(\Omega). \tag{3.28}$$

In view of the above, $\{u_{j_m}\}_{m\in\mathbb{N}}$ has a further sequence which converges to u^* in $H_0^1(\Omega)$. Therefore, by (3.28), we obtain $\overline{u} = u^*$. Hence, u^* is the unique weak limit point of $\{u_j\}_{j\in\mathbb{N}}$ in $\overline{B}(0,R)$. Since $\overline{B}(0,R)$ is compact with respect to the weak topology in $H_0^1(\Omega)$, we deduce that $\{u_j\}_{j\in\mathbb{N}}$ converges weakly to $\overline{u}$ in $H_0^1(\Omega)$. Thus, $S : \overline{B}(0,R) \to \overline{B}(0,R)$ is weakly continuous.

From (i), (ii) and the Tychonoff fixed point theorem, S has a fixed point. Thus, the problem $(\mathcal{P}_2)$ has at least one solution.

Remark 3.2.1. *If $g(.,0) = 0$, then, $u = 0$ can be the unique solution of the problem $(\mathcal{P}_2)$. Let us consider $a(x) = I_n$ the identity matrix,*

$$\forall t \in \mathbb{R} : \quad g(t) = -\frac{t}{1+t^2}$$

and

$$\forall t \in [0,+\infty[: \quad M(t) = e^t.$$

Notice that g and M are continuous functions, $g(0) = 0$ and

$$\forall t \in \mathbb{R} : \quad |g(t)| = \left| -\frac{t}{1+t^2} \right| = \frac{|t|}{1+t^2} \leq 1 \in L^2(\Omega)$$

and

$$\forall t \in [0,+\infty[: \quad M(t) = \exp(t^{\frac{1}{q}}) \geq 1 = m_0.$$

Let $u \in H_0^1(\Omega)$ be a solution of the equation

$$\exp(\|u\|_{L^q(\Omega)})\Delta u = \frac{u}{1+u^2} \quad \text{in } \Omega. \tag{3.29}$$

Multiplying the equation (3.29) by u and integrating over Ω, we obtain

$$\exp(\|u\|_{L^q(\Omega)}) \int_\Omega |\nabla u|^2\, dx = -\int_\Omega \frac{u^2}{1+u^2}\, dx \quad \text{in } \Omega,$$

which gives

$$\int_\Omega \frac{u^2}{1+u^2}\, dx = 0.$$

Therefore, $u = 0$. Thus, $u = 0$ is the unique solution of the problem

$$\exp(\|u\|_{L^q(\Omega)})\Delta u = \frac{u}{1+u^2} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Conclusion

In this memory, we studied how to apply the linearization method in nonlinear elliptic problems to prove the existence of solutions. We see that this technique turns the problem into a linear one, which makes it easier to solve.

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