



People's Democratic Republic of Algeria



ministry of higher education and scientific research

UNIVERSITY HAMMA LAKHDAR EL OUED

FACULTY OF EXACT SCIENCES

End of Study Thesis

Academic Master

Domain: Mathematics and Informatics

Field: Mathematics

Speciality: Fundamental and Applied Mathematics

Theme:

Partial Hadamard Fractional Integral Equations in Banach Spaces

President by:

Limane Rawan

Guedda Ilham

Before the jury composed of:

Said Beloul

President

Unv .El Oued

Abdelatif Ghendir Aoun

Examiner

Univ. EL Oued

Ahmed Hamrouni

Supervisor

Univ. EL Oued

university year

2021-2022

Acknowledgement

*First and foremost, we would like to praise and thank **God** who helped us so that we had been finally able to accomplish this thesis.*

*We would like to express our highest expression of appreciation and respect to **Dr. Ahmed Hamrouni**, Who accepted the supervision with patience and dedication, especially his precious help, which was the reason for the success of our work.*

We would like to thank all the professors of the Faculty of Exact Sciences who taught us during our university journey.

Thanks also to everyone who contributed to the achievement of this work from near or far.

I dedicate this work to:

My precious mother and dear father.

My grandfather God bless him.

And the beautiful baby Salim.

And everyone who supported me and stood by my side.

Guedda Itham

I dedicate this work to:

My great power in this life my parents.

My brothers and sisters.

Limane Rawan

Contents

Notations	VI
Introduction	VII
1 Preliminaries	1
1.1 Banach Space	1
1.2 Functional reminder	4
1.2.1 Generals and definitions	4
1.2.2 Gamma Function:	5
1.2.3 Arzelà–Ascoli Theorem	6
1.3 Measure of noncompactness	6
1.3.1 The Kuratowski Measure of noncompactness	6
1.4 Contractive conditions (contraction)	8
1.4.1 Applications μ -contractions	8
1.5 Fractional calculus	9
1.5.1 Fractional Integral of Order α	9
1.5.2 Stieltjes Integral	10
1.5.3 Hadamard Fractional Integral	12
1.5.4 Hadamard-Stieltjes Fractional Integral	13
1.6 The fixed point Theorem	13
1.6.1 The fixed point Theorem of Darbo	13
1.6.2 The fixed point Theorem of Monch	14
2 Partial Hadamard Fractional Integral Equations	15

2.1	Existence results for partial Hadamard Fractional Integral Equations with Darbo fixed point	15
2.2	Existence results for partial Hadamard-Stieltjes Fractional Integral Equations with Darbo fixed point	19
2.3	Existence results for partial Hadamard Fractional Integral Equations with Mönch fixed point	24
2.4	Existence results for partial Hadamard-Stieltjes Fractional Integral Equations with Mönch fixed point	25
3	Examples	27
3.1	Example 1	27
3.2	Exemple 2	28
3.3	Exemple 3	29
3.4	Exmeple 4	32
	Conclusion	34

Notations

Ω	: A bounded open of \mathbb{R}^n .
\mathbb{R}	: Space of real number.
$conv\Omega$: The envelope sets convex Ω .
$ \cdot $: Absolute valuer.
$\ \cdot\ _c$: The usual supermum norm.
$\Gamma(\alpha)$: Gamma function.
I^α	: Fractional integral.
${}^H I^\alpha$: Hadamard fractional integral.
${}^{HS} I^\alpha$: Hadamard-Stieltjes fractional integral.
α	: The kuratowski measure of noncompactness.
V_a^b	: The variation of u over $[a,b]$.

Introduction

In the past sixty years, fractional calculus had played a very important role in various fields such as physics, chemistry, mechanics, electricity, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. In the last decade, fractional calculus has been recognized as one of the best tools to describe long-memory processes. Such models are interesting for engineers and physicists but also for pure mathematicians. The most important among such models are those described by differential equations containing fractional-order derivatives.

The theory of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. However, some initial and boundary value domains are fractal curves, which are everywhere continuous but nowhere differentiable. As a result, we cannot employ the classical calculus, which requires that the defined functions should be differentiable, to process ordinary local fractional differential equation and fractional partial differential equation with fractal conditions. The theory of fractional integrals and derivatives (fractal calculus)([22, 37]), as one of useful tools to handle the fractal and continuously non-differentiable functions, was successfully applied in fractional Fokker–Planck equation([22, 23]), anomalous diffusion and relaxation equation in fractal space([32, 33]), the fractal heat conduction equation ([18, 19]), fractal-time dynamical systems([5, 6]), fractal elasticity ([1, 7]), local fractional diffusion equation ([36]), local fractional Laplace equation ([40]), local fractional ordinary differential

equations([37, 38]), local fractional partial differential equation([37, 38, 40, 35]), fractional integral equations([34, 42]), fractal signals ([37, 38, 39, 41]), fractional Brownian motion in local fractional derivatives sense([37]), fractal wave equation ([40, 35]).

This memory deals with the existence of solutions to the following Hadamard partial fractional integral equation of the form:

$$u(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st} dt ds \quad (x, y) \in J, \quad (1)$$

where $J := [1, a] \times [1, b]$, $a, b > 1$, $r_1, r_2 > 0$, $\mu : J \rightarrow E$, $f : J \times E \rightarrow E$ are given continuous function, $(E, \|\cdot\|_E)$ is a Banach space, and $\Gamma(\cdot)$ is the Euler gamma function.

Following that we investigate the existence of solutions to the following Hadamard-Stieltjes partial fractional integral equation of the form.

$$u(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st} d_t g_2(y, t) d_s g_1(x, s) \quad (2)$$

Which has been studied by many authors such as . Abbas et al. ([18, 19]), Benchohra et al. ([43]). We will try to enrich this topic in more detail and give some examples

The main tool used in considerations is the combination of the technique of measure of noncompactness with fixed point theorems of Darbo type and Mönch type.

This memory contains an introduction, three chapters and a list of references.

In **the first chapter**, we present some basic definitions and notions of functional analysis, as metric spaces, Banach spaces, the compactness, convexity. We give also some definitions and properties concerning fractional calculus, like fractional integral and in the sense Hadamard. The Darbo or the Mönch fixed point theorems, the Ascoli Arzella theory, measure of noncompactness and its properties, which have been used in sequel of this work.

The second chapter, includes Existence results for partial Hadamard Fractional Integral Equations with Darbo fixed point. And existence results for partial Hadamard Fractional Integral Equations with Mönch fixed point.

The third chapter, contains practical examples.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, and preliminary facts which are used throughout this memory.

1.1 Banach Space

Since their creation by Banach in 1922, normed spaces have played a central role in functional analysis. Banach spaces are complete normed spaces. Completeness allows to prove the convergence of a sequence or a series without knowing its limit.

To obtain a detailed definition of the Banach space, there are several terms that should be defined first.

Definition 1.1 (Metric spaces). [44] We call distance on a set X any application d defined as:

$$d : X \times X \rightarrow \mathbb{R}^+,$$

having the following properties:

1. $\forall x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
2. $\forall x, y \in X$ $d(x, y) = d(y, x)$ (symmetry)
3. $\forall x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Remark 1.1. The couple (X, d) is called metric space, where X is a set and d a metric.

Definition 1.2 (Open set). A subset M of a metric space X is said to be open if it

$$\forall x \in M, \exists \epsilon > 0, \text{ such that } B(x, \epsilon) \subset M.$$

Definition 1.3 (Closed set). A subset M of X is said to be closed if its complement in X is open, that is, $M^c = X - M$ is open.

Definition 1.4. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of a metric space (E, d) . We say that it converges to $x \in E$ if and only if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow d(x_n, x) < \epsilon.$$

Definition 1.5. (Cauchy Sequence) In a metric space (E, d) , a sequence $(x_n)_{n \in \mathbb{N}}$ is said to be Cauchy if and only if it satisfies:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall p, q > N \Rightarrow d(x_p, x_q) < \epsilon.$$

Definition 1.6. "Continuity and sequences" The function f is continuous at X if and only if, whatever the sequence (X_n) converge to X , then the sequence $f(X_n)$ converge to $f(X)$.

Definition 1.7 (Complete space). we say that (X, D) is complete if all the Cauchy sequences of X are convergent.

Definition 1.8. (Continuity) Let (X, d_X) and (Y, d_Y) be two metric spaces $f : X \rightarrow Y$ an application and $a \in X$ we say that f is continuous at the point a

$$\forall \epsilon > 0, \exists \delta > 0 \forall x \in X, d_X(x, a) \leq \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon.$$

Remark 1.2. We say that f is continuous on X if and only if f is continuous at every point a .

Definition 1.9 (Norm vector space). [44] Let E be a vector space on $K(\mathbb{C}$ or $\mathbb{R})$, a norm on E is an application:

$$\begin{aligned} \|\cdot\| : E &\rightarrow [0, +\infty[\\ x &\rightarrow \|x\|, \end{aligned}$$

having the following properties:

(i) $x \in E$ et $\|x\| = 0 \Rightarrow x = 0$

(ii) $\forall x \in E, \forall \lambda \in \mathbb{K}$, we have $\|\lambda x\| = |\lambda| \|x\|$, (homogeneity).

(iii) $\forall x \in E, \forall y \in E : \|x + y\| \leq \|x\| + \|y\|$, (triangle inequality).

The couple $(E, \|\cdot\|)$ is called normed vector space.

Definition 1.10 (Banach space). [44] We call a Banach space a normed vector space which is complete for the distance resulting from its norm.

Definition 1.11. (Compact parts)[3] We say that $C \subset \mathbb{R}$ is compact if for any cover of C by open we can extract a finite undercoverage. This translates as follows: if $(U_i)_{i \in I}$ is an open family such that $C \subset (U_i)_{i \in I}$ then there exists a finite subset $J \subset I, C \subset \bigcup_{i \in J} U_i$.

Definition 1.12 (Relatively compact). [27] Let U be a set of a normed space X . We say that U is relatively compact if its closure \bar{U} is compact.

Definition 1.13 (Equicontinuity). Let $\mathcal{F} \subset C(J, E)$ ($J = [1, a] \times [1, b]$) (E is a banach space). \mathcal{F} is an equicontinuity family given any $\epsilon > 0$ we can find a $\delta > 0$ such that if

$$\forall x, y \in J, |x - y| < \delta \text{ then } \|f(x) - f(y)\|_E < \epsilon \forall f \in \mathcal{F}.$$

Definition 1.14. [31] (Convexity) Let $A \in E$, We say that A is convex if

$$\forall x, y \in A \forall \lambda \in [0, 1]; \lambda x + (1 - \lambda)y \in A.$$

1.2 Functional reminder

1.2.1 Generals and definitions

Theorem 1.1. (*Dominated convergence theorem, Lebesgue, [14]*) Let Ω a nonempty set and (f_n) be a sequence of functions in L^1 that satisfy

i) $f_n(x) \rightarrow f(x)$ a.e.on Ω ,

ii) there is a function $g \in L^1$ such that $\forall n, |f_n(x)| \leq g(x)$ a.e.on Ω .

Then $f \in L^1$ and $\|f_n - f\|_1 \rightarrow 0$. Where $\|f\|_1 = \int_{\Omega} |f(x)| dx$.

Definition 1.15 (Bounded subset). We say that A is a bounded subset of X if:

$$\exists M > 0, \text{ such as } \forall x \in A, \|x\| \leq M.$$

Definition 1.16 (Bounded operator). A linear operator N defined on X in Y is said to be bounded if there is a positive constant $C > 0$. such that

$$\|T(x)\|_Y \leq C \|x\|_X \quad \forall x \in X.$$

Definition 1.17. Let $C := C(J, E)$ be the space of E valued continuous functions on J with the usual supremum norm

$$\|u\|_C = \sup_{(x,y) \in J} \|u(x, y)\|_E.$$

Definition 1.18. Let $L^1(J, E)$ ($J = [1, a] \times [1, b]$) the Banach space of functions $u : J \rightarrow E$ that are Bochner integrable with the norm

$$\|u\|_{L^1} = \int_1^a \int_1^b |u(x, y)| dy dx.$$

For a given set V of function $v : J \rightarrow E$ Let us denote by

$$V(x, y) = v(x, y) : v \in V, (x, y) \in J,$$

and

$$V(J) = v(x, y) : v \in V, (x, y) \in J.$$

1.2.2 Gamma Function:

Definition 1.19. [16] The gamma function $\Gamma(z)$ is defined by:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, (\Re(z) > 0),$$

where $t^{z-1} = e^{(z-1)\log(t)}$, which converges in the right half of the complex plane $\Re(z) > 0$.

Indeed, we have:

$$\begin{aligned} \Gamma(x + iy) &= \int_0^{\infty} e^{-t} t^{x-1+iy} dt \\ &= \int_0^{\infty} e^{-t} t^{x-1} + e^{iy \log(t)} dt \\ &= \int_0^{\infty} e^{-t} t^{x-1} [\cos(y \log(t)) + i \sin(y \log(t))] dt. \end{aligned} \tag{1.1}$$

The expression in the square brackets in (1.1) is bounded for all t , convergence at infinity is provided by e^{-t} , and for the convergence at $t = 0$ we must have $x = \Re(z) > 0$.

Proposition 1. [16]

1. $\Gamma(z + 1) = z\Gamma(z)$ ($\Re(z) > 0$).
2. $\Gamma(n + 1) = n!$, $\forall n \in \mathbb{N}$.
3. $\Gamma(1) = 1$.
4. $\Gamma(-m) = \mp\infty$, $\forall m \in \mathbb{N}$.
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
6. $\Gamma(z + 1) = \lim_{n \rightarrow +\infty} \frac{n! n^z}{z(z+1)\dots(z+n)}$, ($\Re(z) > 0$).

1.2.3 Arzelà–Ascoli Theorem

Theorem 1.2. [3] *Let X be a compact metric space. Let $C(X, \mathbb{K})$ be given the sup norm metric. (\mathbb{K} is either \mathbb{R} or \mathbb{C}). Then a set $\mathcal{B} \subset C(X)$ is compact iff \mathcal{B} is bounded, closed and equicontinuous.*

1.3 Measure of noncompactness

The measures of noncompactness play an important role in showing that there are solutions to differential equations, especially for implicit differential equations, integral equations, and integro-differential equations.[8, 26]

1.3.1 The Kuratowski Measure of noncompactness

Now, we define the Kuratowski measure of noncompactness and give its basic properties.

Definition 1.20. [8] *Let E be a Banach space and Ω_E the bounded subsets of E . The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \rightarrow [0, \infty]$ defined by*

$$\alpha(B) = \inf\{\varepsilon > 0 : B \subseteq \cup_{i=1}^n B_i, \text{diam}(B_i) \leq \varepsilon\}, \quad B \in \Omega_E,$$

where

$$\text{diam}(B_i) = \sup\{\|u - v\|_E : u, v \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties.

Lemma 1.1. ([8, 21]) *Let A and B bounded sets.*

- (a) $\alpha(B) = 0 \iff \overline{B}$ is compact (B is relatively compact), where \overline{B} denotes the closure of B .
- (b) *Nonsingularity:* α is equal to zero on every one element-set.
- (c) If B is a finite set, then $\alpha(B) = 0$.
- (d) $\alpha(B) = \alpha(\overline{B}) = \alpha(\text{conv}B)$, where $\text{conv}B$ is the convex hull of B .

(e) *Monotonicity*: $A \subset B \implies \alpha(A) \leq \alpha(B)$.

(f) *Algebraic semi-additivity*: $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where

$$A + B = \{x + y : x \in A, y \in B\}.$$

(g) *Semi-homogeneity*: $\alpha(\lambda B) = |\lambda| \alpha(B)$, $\lambda \in \mathbb{R}$, where

$$\lambda(B) = \{\lambda x : x \in B\}.$$

(h) *Semi-additivity*: $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(i) $\alpha(A \cap B) = \min\{\alpha(A), \alpha(B)\}$.

(j) *Invariance under translations*: $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in E$.

Lemma 1.2. ([13]) *Let $V \subset C(J, E)$ be a bounded and equicontinuous set, then*

(i) *the function $(x, y) \rightarrow \alpha(V(x, y))$ is continuous on J , and*

$$\alpha_C(V) = \sup_{(x,y) \in J} \alpha(V(x, y)).$$

(ii) $\alpha \left(\int_1^a \int_1^b u(s, t) dt ds : u \in V \right) \leq \int_1^a \int_1^b \alpha(V(s, t)) dt ds$, where

$$V(s, t) = \{u(s, t) : u \in V\}, (s, t) \in J.$$

From the above Lemma, we can conclude the following Lemma.

Lemma 1.3. *Let $V \subset C(J, E)$ be a bounded and equicontinuous set. then*

$$\alpha \{({}^H I_\sigma^r u)(x, y) : u \in V\} \leq ({}^H I_\sigma^r \alpha(u))(x, y) : u \in V.$$

1.4 Contractive conditions (contraction)

Definition 1.21 ([29]). Let (M, d) be a metric space. The map $T : M \rightarrow M$ is said to be Lipschitzian if there exists a constant $k > 0$ (called a Lipschitz constant) such that

$$d(T(x), T(y)) \leq kd(x, y) \quad \text{for all } x, y \in M.$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called a contraction.

1.4.1 Applications μ -contractions

Let E and F be two Banach spaces.

Definition 1.22. [9, 10] Let $T : E \rightarrow F$ be a continuous and bounded map on each bounded subset of E .

We say that T is:

1. μ -Lipschitz with the constant $k \geq 0$, if:

$$\mu(T(\Omega)) \leq k\mu(\Omega), \quad \forall \Omega \in M_E \text{ when } M_E \text{ subset.}$$

2. completely continuous if it is μ -Lipschitz with $k = 0$.

3. μ -contraction if it is μ -Lipschitz with $k < 1$.

4. condensing if it is μ -Lipschitz with $k = 1$ and

$$\mu(T(\Omega)) \leq \mu(\Omega)$$

for all Ω bounded and non relatively compact ($\mu(\Omega) > 0$).

When the μ is an arbitrary measure of noncompactness.

1.5 Fractional calculus

1.5.1 Fractional Integral of Order α

Definition 1.23. [2] *The fractional order integral of the function $f \in C([a, b])$ of order $\alpha \in \mathbb{R}_+$ is defined by:*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad a \leq s \leq t.$$

Definition 1.24. *For every $\alpha > 0$ and a local integrable function $f(t)$, the right of order α is defined:*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \lim_{t \rightarrow \infty} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad -\infty \leq a < t < \infty. \quad (1.2)$$

Alternatively, it can be defined also the left as:

$$I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \lim_{t \rightarrow -\infty} \int_t^b (t-s)^{\alpha-1} f(s) ds, \quad -\infty < t < b \leq \infty. \quad (1.3)$$

Proposition 2. *we have the following properties:*

(i) $I^0 f(t) = f(t)$.

(ii) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$.

(iii) *integral Operator I^α is linear:*

$$I^\alpha(\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t), \quad \lambda \in (\mathbb{R}_+ \text{ or } \mathbb{C}).$$

(iiii) $\frac{d}{dt}(I^\alpha f)(t) = I^{\alpha-1} f(t)$

1.5.2 Stieltjes Integral

We start with the definition of variation and the bounded variation of a function.

Definition 1.25. *If u is a real function defined on the interval $[a, b]$. Given any finite partition*

$$\tau = \{a = x_0 < \cdots < x_n = b\},$$

of $[a, b]$. *The variation of u over $[a, b]$ is:*

$$\bigvee_a^b u = \sup \sum_{i=1}^n |f(x_i - x_{i-1})|.$$

- *If $w : [a, b] \times [c, d] \rightarrow \mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t, s)$ indicates the variation of the function $t \rightarrow w(t, s)$ on the interval $[p, q] \subset [a, b]$, where s is arbitrarily fixed in $[c, d]$. In the same way we define $\bigvee_{s=p}^q w(t, s)$.*

Remark 1.3. *[17, 15] We say that u is of bounded variation on the interval $[a, b]$ whenever $\bigvee_a^b u$ is finite.*

Theorem 1.3. *If $u : [a, b] \rightarrow \mathbb{R}$ is monotone on $[a, b]$ then u is of bounded variation on $[a, b]$ and $\bigvee_a^b(u) = |u(b) - u(a)|$.*

Now, we have if u and φ are two real function defined on the interval $[a, b]$, then under some conditions we can define the **Stieltjes integral**

$$\int_a^b u(t) d\varphi(t)$$

of the function u with respect to φ . In this case we say that u is Stieltjes integrable on $[a, b]$ with respect to φ . The Riemann integral is a special case of the Stieltjes integral, and that's when we put $\varphi(t) = t$ and several conditions.[25]

Theorem 1.4. *The integral*

$$\int_a^b u(t)d\varphi(t),$$

exists if the function u is continuous on $[a,b]$ and φ is of finite variation on $[a,b]$, and we have

$$\left| \int_a^b u(t)d\varphi(t) \right| \leq \sup_{t \in [a,b]} |u(t)| \bigvee_a^b(\varphi).$$

Properties 1. [28]

- *If u is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then*

$$\left| \int_a^b u(t)d\varphi(t) \right| \leq \int_a^b |u(t)|d \left(\bigvee_a^t \varphi \right).$$

- *If u and v are Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function φ such that $u(t) \leq v(t)$ for $t \in [a,b]$. Then*

$$\int_a^b u(t)d\varphi(t) \leq \int_a^b v(t)d\varphi(t).$$

Remark 1.4. *In the sequel we consider Stieltjes integrals of the form*

$$\int_a^b u(t)d_s g(t, s).$$

1.5.3 Hadamard Fractional Integral

A construction of fractional itegro-differentiation which is a fractional power of the type $(x \frac{d}{dx})^\alpha$ was suggested by Hadamard [20]. This construction is well suited to the case of the half-axis and is invariant relative to dilation.[30]

Definition 1.26 (Hadamard fractional integral). *Let (a, b) ($0 \leq a < b \leq \infty$) be a finite or infinite interval of the half-axis \mathbb{R}_+ and let $r > 0$, the left-sided and right-sided Hadamard fractional integrals of order r defined by*

$$({}^H I_{a+}^r g)(x) = \frac{1}{\Gamma(r)} \int_a^x \left(\log \frac{x}{s}\right)^{r-1} \frac{g(s)}{s} ds, \quad 0 < a < x \leq b, \quad (1.4)$$

and

$$({}^H I_{b-}^r g)(x) = \frac{1}{\Gamma(r)} \int_x^b \left(\log \frac{s}{x}\right)^{r-1} \frac{g(s)}{s} ds, \quad 0 < a \leq x < b, \quad (1.5)$$

respectively.

So for order $r > 0$ and a function $g \in L^1([1, a], E)$, is defined as :

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(r)} \int_1^x \left(\log \frac{x}{s}\right)^{r-1} \frac{g(s)}{s} ds.$$

Definition 1.27. *Let $r_1, r_2 \geq 0$, $\sigma = (1, 1)$ and $r = (r_1, r_2)$. For $w \in L^1(J, E)$, defined the Hadamard partial fractional integral of order r by the expression*

$$({}^H I_\sigma^r w)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{w(s, t)}{st} dt ds.$$

1.5.4 Hadamard-Stieltjes Fractional Integral

Definition 1.28. *Hadamard-Stieltjes integral of fractional order of the form*

$$({}^{HS}I_1^r g)(x) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} u(s) d_s g(t, s),$$

where $g : [1, \infty) \times [1, \infty) \rightarrow \mathbb{R}$, $q \in (0, \infty)$ and the symbol d_s indicates the integration with respect to s .

Definition 1.29. *Let $r_1, r_2 \geq 0$, $\sigma = (1, 1)$ and $r = (r_1, r_2)$. For $w \in L^1(J, E)$, defined the Hadamard-Stieltjes partial fractional integral of order r by the expression*

$$({}^{HS}I_\sigma^r w)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s} \right)^{r_1-1} \left(\log \frac{y}{t} \right)^{r_2-1} \frac{w(s, t)}{st} d_t g_2(y, t) d_s g_1(x, s)$$

where $g_1, g_2 : [1, \infty) \times [1, \infty) \rightarrow \mathbb{R}$.

1.6 The fixed point Theorem

Fixed point theorems have always a major role in various fields, specially, in fields of differential, integral and functional equations. Fixed point theorems constitute a topological tool for the qualitative investigations of solution of linear and nonlinear equations. The theory of fixed points is concerned with the conditions which guarantee that a map $N : E \rightarrow E$ of a topological space C into it self admits one or more fixed points, that is, points x of C for which $x = Nx$. In this section, we present some fixed point theorems involving an arbitrary measure of noncompactness.

1.6.1 The fixed point Theorem of Darbo

Theorem 1.5 (Darbo's Fixed Point Theorem [11, 12]). *Let X be a Banach space and C be a nonempty, bounded, closed and convex subset of X . Suppose a continuous mapping $N : C \rightarrow C$ is such that for all nonempty subsets D of C*

$$\alpha(N(D)) \leq k\alpha(D), \tag{1.6}$$

where $0 \leq k \leq 1$, and α is the Kuratowski measure of noncompactness. The N has a fixed point in C .

Remark 1.5. Mappings satisfying the Darbo condition (1.6) have subsequently been called K -set contractions.

1.6.2 The fixed point Theorem of Monch

Theorem 1.6 (Mönch's Fixed Point Theorem [24]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{\text{conv}}N(V) \text{ or } V = N(V) \cup \{0\} \implies \alpha(V) = 0,$$

holds for every subset V of D , then N has a fixed point.

Chapter 2

Partial Hadamard Fractional Integral Equations

2.1 Existence results for partial Hadamard Fractional Integral Equations with Darbo fixed point

In this section, we discuss the existence of solutions for our Hadamard integral equation (1).

The following hypothesis will be used in the sequel.

(\mathcal{H}) There exist functions $p_1, p_2 \in C(J, [0, \infty))$ such that for any $u \in E$ and $(x, y) \in J$,

$$\|f(x, y, u)\|_E \leq p_1(x, y) + \frac{p_2(x, y)}{1 + \|u(x, y)\|_E} \|u(x, y)\|_E.$$

We are now in a position to state and prove our first existence result for the integral equation (1) based on the concept of measure of noncompactness and Darbo's fixed point theorem.

Set

$$p_i^* = \sup_{(x, y) \in J} p_i(x, y), i = 1, 2.$$

Remark 2.1. ([4]) *The hypothesis (H) implies the inequality*

$$\alpha(f(x, y, B)) \leq p_2^* \alpha(B),$$

for any bounded set $B \subseteq E$ and for each $(x, y) \in J$.

Theorem 2.1. *Assume that the hypothesis (H) holds. If*

$$\frac{(\log a)^{r_1} (\log b)^{r_2} p_2^*}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \quad (2.1)$$

then the integral equation (1) has a solution defined on J.

Proof Let $\rho > 0$ be a constant such that

$$\rho > \frac{\|\mu\|_C + \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} p_1^*}{1 - \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} p_2^*}. \quad (2.2)$$

We shall use Darbo's theorem (Theorem 1.5), to prove that the operator $N : C \rightarrow C$ defined by

$$(Nu)(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st} dt ds \quad (2.3)$$

has a fixed point. The proof will be given in several steps.

Step 1. N is continuous

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \rightarrow u$ in C . Then

$$\begin{aligned} \|(Nu_n)(x, y) - (Nu)(x, y)\|_E &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \frac{\|f(s, t, u_n(s, t)) - f(s, t, u(s, t))\|_E}{st} dt ds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \frac{\sup_{(s,t) \in J} \|f(s, t, u_n(s, t)) - f(s, t, u(s, t))\|_E}{st} dt ds \\ &\leq \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|f(\cdot, \cdot, u_n(\cdot, \cdot)) - f(\cdot, \cdot, u(\cdot, \cdot))\|_C. \end{aligned}$$

From Lebesgue's dominated convergence theorem and the continuity of the function f , we get

$$\|(Nu_n)(x, y) - (Nu)(x, y)\|_E \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Step 2. N transforms the ball $B_\rho := \{u \in \mathcal{C} : \|u\|_C \leq \rho\}$ into itself.

For any $u \in B_\rho$ and each $(x, y) \in J$, we have

$$\begin{aligned} \|(Nu)(x, y)\|_E &\leq \|\mu(x, y)\|_E \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{p_1(s, t) + p_2(s, t)}{st} \leq \|\mu\|_C dt ds \\ &\leq \|\mu\|_C + \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} (p_1^* + p_2^* \rho). \end{aligned}$$

Thus, by (2.1) and the definition of ρ we get $\|(Nu)\|_C \leq \rho$. This implies that N transforms the ball B_ρ into itself.

Step 3. $N(B_\rho)$ is bounded.

This is clear since $N(B_\rho) \subset B_\rho$ and B_ρ is bounded.

Step 4. $N(B_\rho)$ is equicontinuous.

Let $(x_1, y_1), (x_2, y_2) \in J$, $x_1 < x_2$, $y_1 < y_2$. Then

$$\begin{aligned} \|(Nu)(x_2, y_2) - (Nu)(x_1, y_1)\|_E &\leq \|\mu(x_1, y_1) - \mu(x_2, y_2)\|_E \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_1^{x_1} \int_1^{y_1} \left[\left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} - \left| \log \frac{x_1}{s} \right|^{r_1-1} \left| \log \frac{y_1}{t} \right|^{r_2-1} \right] \frac{\|f(s, t, u(s, t))\|_E}{st} dt ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{\|f(s, t, u(s, t))\|_E}{st} dt ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_1^{x_1} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{\|f(s, t, u(s, t))\|_E}{st} dt ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_{x_1}^{x_2} \int_1^{y_1} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{\|f(s, t, u(s, t))\|_E}{st} dt ds. \end{aligned}$$

Thus

$$\begin{aligned}
& \| (Nu)(x_2, y_2) - (Nu)(x_1, y_1) \|_E \leq \| \mu(x_1, y_1) - \mu(x_2, y_2) \|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
& \times \int_1^{x_1} \int_1^{y_1} \left[\left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} - \left| \log \frac{x_1}{s} \right|^{r_1-1} \left| \log \frac{y_1}{t} \right|^{r_2-1} \right] \frac{p_1^* + p_2^* \rho}{st} dt ds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \times \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{p_1^* + p_2^* \rho}{st} dt ds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \times \int_1^{x_1} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{p_1^* + p_2^* \rho}{st} dt ds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \times \int_{x_1}^{x_2} \int_1^{y_1} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{p_1^* + p_2^* \rho}{st} dt ds \\
& \leq \frac{p_1^* + p_2^* \rho}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
& \times [2(\log y_2)^{r_2}(\log x_2 - \log x_1)^{r_1} + 2(\log x_2)^{r_1}(\log y_2 - \log y_1)^{r_2} \\
& + (\log x_1)^{r_1}(\log y_1)^{r_2} - (\log x_2)^{r_1}(\log y_2)^{r_2} - 2(\log x_2 - \log x_1)^{r_1}(\log y_2 - \log y_1)^{r_2}].
\end{aligned}$$

As $x_1 \rightarrow x_2$ and $y_1 \rightarrow y_2$, the right-hand side of the above inequality tends to zero.

Step 5. The operator $N : B_\rho \rightarrow B_\rho$ is a K-set contraction.

Let $V \subset B_\rho$ and $(x, y) \in J$. Then we have,

$$\begin{aligned}
& \alpha((NV)(x, y)) = \alpha((Nu)(x, y) : u \in V) \\
& \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s} \right)^{r_1-1} \left(\log \frac{y}{t} \right)^{r_2-1} \frac{\alpha(f(s, t, u(s, t)))}{st} dt ds : u \in V \right\}.
\end{aligned}$$

Then Remark 2.1 implies that, for each $(s, t) \in J$,

$$\alpha(\{f(s, t, u(s, t)) : u \in V\}) \leq p_2^* \alpha(\{u(s, t) : u \in V\}).$$

Then

$$\begin{aligned}
& \alpha((NV)(x, y)) \\
& \leq \frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)} \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s} \right)^{r_1-1} \left(\log \frac{y}{t} \right)^{r_2-1} \frac{\alpha(u(s, t))}{st} dt ds : u \in V \right\} \\
& \leq \frac{(\log a)^{r_1} (\log b)^{r_2} p_2^*}{\Gamma(1+r_1)\Gamma(1+r_2)} \alpha_c(V).
\end{aligned}$$

Therefore

$$\alpha_c(NV) \leq \frac{(\log a)^{r_1} (\log b)^{r_2} p_2^*}{\Gamma(1+r_1)\Gamma(1+r_2)} \alpha_c(V).$$

So, by 2.1, the operator N is a k-set contraction, where

$$k = \frac{(\log a)^{r_1} (\log b)^{r_2} p_2^*}{\Gamma(1+r_1)\Gamma(1+r_2)}.$$

As a consequence of steps 1-5 together with Theorem 1.5, we can conclude that N has a fixed point which is solution of the integral equation 1.

2.2 Existence results for partial Hadamard-Stieltjes Fractional Integral Equations with Darbo fixed point

Now, we discuss the existence of solutions for the Hadamard integral equation 2. The following hypotheses will be used in the sequel.

(H₁) There exist functions $p_1, p_2 \in C(J, \mathbb{R}_+)$ such that for any $u \in \mathbb{R}$ and $(x, y) \in J$,

$$\|f(x, y, u)\|_E \leq p_1(x, y) + \frac{p_2(x, y)}{1 + \|u(x, y)\|_E} \|u(x, y)\|_E,$$

with

$$p_i^* = \sup_{(x,y) \in J} \sup_{(s,t) \in [1,x] \times [1,y]} \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{p_i(s, t)}{st\Gamma(r_1)\Gamma(r_2)}, i = 1, 2.$$

(H₂) For all $x_1, x_2 \in [1, a]$ such that $x_1 < x_2$, the function $s \mapsto g(x_2, s) - g(x_1, s)$ is nondecreasing on $[1, a]$. Also, for all $y_1, y_2 \in [1, b]$ such that $y_1 < y_2$, the function $t \mapsto g(y_2, t) - g(y_1, t)$ is nondecreasing on $[1, b]$.

(H₃) The function $s \mapsto g_1(0, s)$ and $t \mapsto g_2(0, t)$ are nondecreasing on $[1, a]$ or $[1, b]$ respectively.

(H₄) The function $s \mapsto g_1(x, s)$ and $x \mapsto g_1(x, s)$ are continuous $[1, a]$ for each fixed

$x \in [1, a]$ or $s \in [1, a]$, respectively. Also, the function $t \mapsto g_2(y, t)$ and $y \mapsto g_2(y, t)$ are continuous $[1, b]$ for each fixed $y \in [1, b]$ or $t \in [1, b]$, respectively.

Set

$$g^* = \sup_{(x,y) \in J} \bigvee_{k_2=1}^y g_2(y, k_2) \bigvee_{k_1=1}^x g_1(x, k_1).$$

Theorem 2.2. *Assume that the hypotheses $(H_1) - (H_4)$ hold. If*

$$g^* p_2^* < 1, \tag{2.4}$$

then the integral equation (2) has a solution defined on J .

Proof Consider the operator $N' : C \rightarrow C$ defined by

$$\begin{aligned} (N'u)(x, y) &= \mu(x, y) \\ &+ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t) d_s g_1(x, s). \end{aligned} \tag{2.5}$$

The fixed points of the operator N' are solution of the integral equation (2).

Let $\rho > 0$ be a constant such that

$$\rho > \|\mu\|_C + g^*(p_1^* + p_2^*).$$

We shall use Darbo's fixed point theorem to prove that the operator N' has a fixed point.

The proof will be given in several steps.

Step 1. $N' : C \rightarrow C$ is continuous.

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \rightarrow u$ in C . Then

$$\begin{aligned}
& |(N'u_n(x, y) - (N'u(x, y))| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
& \times \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{|f(s, t, u_n(s, t)) - f(s, t, u(s, t))|}{st} d_t g_2(y, t) d_s g_1(x, s) \\
& \leq \frac{\sup_{(s,t) \in J} |f(s, t, u_n(s, t)) - f(s, t, u(s, t))|}{\Gamma(r_1)\Gamma(r_2)} \\
& \times \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} d_t \bigvee_{k_2=1}^t g_2(y, k_2) d_s \bigvee_{k_1=1}^s g_1(x, k_1) \\
& \leq g^* \|f(\cdot, \cdot, u_n(\cdot, \cdot)) - f(\cdot, \cdot, u(\cdot, \cdot))\|_C.
\end{aligned}$$

From Lebesgue's dominated convergence theorem and the continuity of the function f , we get

$$|(N'u_n(x, y) - (N'u(x, y))| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Step 2. N' transforms the ball $B_\rho := \{u \in \mathcal{C} : \|u\|_C \leq \rho\}$ into itself.

For any $u \in B_\rho$ and each $(x, y) \in J$. we have

$$\begin{aligned}
& |(N'u)(x, y)| \leq |\mu(x, y)| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \times \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{p_1(s, t)}{st} |d_t g_2(y, t) d_s g_1(x, s)| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \times \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{p_2(s, t)|u(s, t)|}{st(1 + |u(s, t)|)} |d_t g_2(y, t) d_s g_1(x, s)| \\
& \leq \|\mu\|_C + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
& \times \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{p_1(s, t) + p_2(s, t)\rho}{st} d_t \bigvee_{k_2=1}^t g_2(y, k_2) d_s \bigvee_{k_1=1}^s g_1(x, k_1) \\
& \leq \|\mu\|_C + (p_1^* + p_2^*) \int_1^x \int_1^y d_t \bigvee_{k_2=1}^t g_2(y, k_2) d_s \bigvee_{k_1=1}^s g_1(x, k_1) \\
& \leq \|\mu\|_C + g^*(p_1^* + p_2^*) \\
& \leq \rho.
\end{aligned}$$

Thus, $\|(N'u)\|_C \leq \rho$. This implies that N' transforms the ball B_ρ into itself.

Step 3. $N'(B_\rho)$ is bounded.

This is clear since $N(B_\rho) \subset B_\rho$ and B_ρ is bounded.

Step 4. $N'(B_\rho)$ is equicontinuous.

Let $(x_1, y_1), (x_2, y_2) \in J$, $x_1 < x_2$, $y_1 < y_2$. Then

$$\begin{aligned}
& |(N'u)(x_2, y_2) - (N'u)(x_1, y_1)| \\
& \leq |\mu(x_1, y_1) - \mu(x_2, y_2)| \\
& + \left| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \times \frac{f(s, t, u(s, t))}{st} d_t g_2(y_2, t) d_s g_1(x_2, s) \right. \\
& - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left| \log \frac{x_1}{s} \right|^{r_1-1} \left| \log \frac{y_1}{t} \right|^{r_2-1} \times \frac{f(s, t, u(s, t))}{st} d_t g_2(y_1, t) d_s g_1(x_1, s) \left. \right| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{|f(s, t, u(s, t))|}{st} |d_t g_2(y_2, t) d_s g_1(x_2, s)| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{|f(s, t, u(s, t))|}{st} |d_t g_2(y_2, t) d_s g_1(x_2, s)| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_1^{y_1} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{|f(s, t, u(s, t))|}{st} |d_t g_2(y_2, t) d_s g_1(x_2, s)|.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
& |(N'u)(x_2, y_2) - (N'u)(x_1, y_1)| \\
& \leq |\mu(x_1, y_1) - \mu(x_2, y_2)| \\
& + (p_1^* + p_2^*) \int_1^{x_1} \int_1^{y_1} \left| d_t \bigvee_{k_2=1}^t g_2(y_2, k_2) d_s \bigvee_{k_1=1}^s g_1(x_2, k_1) - d_t \bigvee_{k_2=1}^t g_2(y_1, k_2) d_s \bigvee_{k_1=1}^s g_1(x_1, k_1) \right| \\
& + (p_1^* + p_2^*) \int_{x_1}^{x_2} \int_{y_1}^{y_2} d_t \bigvee_{k_2=1}^t g_2(y_2, k_2) d_s \bigvee_{k_1=1}^s g_1(x_2, k_1) \\
& + (p_1^* + p_2^*) \int_1^{x_1} \int_{y_1}^{y_2} d_t \bigvee_{k_2=1}^t g_2(y_2, k_2) d_s \bigvee_{k_1=1}^s g_1(x_2, k_1) \\
& + (p_1^* + p_2^*) \int_{x_1}^{x_2} \int_1^{y_1} d_t \bigvee_{k_2=1}^t g_2(y_2, k_2) d_s \bigvee_{k_1=1}^s g_1(x_2, k_1).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
& |(N'u)(x_2, y_2) - (N'u)(x_1, y_1)| \\
& \leq |\mu(x_1, y_1) - \mu(x_2, y_2)| \\
& + (p_1^* + p_2^*) \left| \prod_{k_2=1}^{y_1} g_2(y_2, k_2) \prod_{k_1=1}^{x_1} g_1(x_2, k_1) - \prod_{k_2=1}^{y_1} g_2(y_1, k_2) \prod_{k_1=1}^{x_1} g_1(x_1, k_1) \right| \\
& + (p_1^* + p_2^*) \prod_{k_2=y_1}^{y_2} g_2(y_2, k_2) \prod_{k_1=x_1}^{x_2} g_1(x_2, k_1) \\
& + (p_1^* + p_2^*) \prod_{k_2=y_1}^{y_2} g_2(y_2, k_2) \prod_{k_1=1}^{x_2} g_1(x_2, k_1) \\
& + (p_1^* + p_2^*) \prod_{k_2=1}^{y_2} g_2(y_2, k_2) \prod_{k_1=x_1}^{x_2} g_1(x_2, k_1).
\end{aligned}$$

As $x_1 \rightarrow x_2$ and $y_1 \rightarrow y_2$, the right-hand side of the above inequality tends to zero.

Step 5. The operator $N' : B_\rho \rightarrow B_\rho$ is a K-set contraction.

Let $V \subset B_\rho$ and $(x, y) \in J$. Then we have,

$$\begin{aligned}
& \alpha((N'V)(x, y)) \\
& = \alpha((N'u)(x, y) : u \in V) \\
& \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
& \times \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{\alpha(f(s, t, u(s, t)))}{st} d_t g(y_2, t) d_s g(x_2, s) : u \in V \right\}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
& \alpha((N'V)(x, y)) \\
& \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
& \times \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{p_2(s, t)\alpha(u(s, t))}{st} d_t g_2(y, t) d_s g_1(x, s) : u \in V \right\} \\
& \leq p_2^* \left\{ \int_1^x \int_1^y d_t \bigvee_{k_2=1}^t g_2(y, k_2) d_s \bigvee_{k_1=1}^s g_1(x, k_1) \alpha(u(s, t)) u \in V \right\} \\
& \leq g^* p_2^* \alpha_c(V).
\end{aligned}$$

Therefore

$$\alpha_c(N'V) \leq g^* p_2^* \alpha_c(V).$$

So, by (2.2), the operator N' is a k -set contraction, where

$$k = g^* p_2^*.$$

As a consequence of steps 1-5 together with Theorem (1.5), we can conclude that N' has a fixed point which is solution of the integral equation (2).

2.3 Existence results for partial Hadamard Fractional Integral Equations with Mönch fixed point

Our next existence result for the problem (1) is based on the concept of measures of noncompactness and Mönch's fixed point theorem.

Theorem 2.3. *Assume that the hypothesis (H) and the condition (2.1) hold. Then the integral equation (1) has at least one solution defined on J .*

Proof Consider the operator N defined in (2.3). We shall show that N satisfies the assumption of Mönch's fixed point theorem (Theorem 1.6).

We know that $N : B_\rho \rightarrow B_\rho$ is bounded and continuous, we need to prove that the

implication

$$V = \overline{\text{conv}}N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \implies \alpha(V) = 0$$

holds for every subset V of B_ρ .

Now let V be a subset of B_ρ such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$. Since V is bounded and equicontinuous, the function $(x, y) \longrightarrow v(x, y) = \alpha(V(x, y))$ is continuous on J . By Remark (2.1), Lemma (1.2) and the properties of the measure α we have for each $(x, y) \in J$,

$$\begin{aligned} V(x, y) &\leq \alpha((NV)(x, y) \cup \{0\}) \\ &\leq \alpha((NV)(x, y)) \\ &\leq \alpha\{(Nu)(x, y) : u \in V\} \\ &\leq \frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)} \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{\alpha(u(s, t))}{st} dt ds : u \in V \right\} \\ &\leq \frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{v(s, t)}{st} dt ds \end{aligned}$$

This implies that $v(x, y) = 0$ for each $(x, y) \in J$, and then $V(x, y)$ is relatively compact in E . In view of the Ascoli-Arzelà theorem, V is relatively compact in B_ρ .

Consequently, from Theorem 1.6 we conclude that N has a fixed point $u \in B_\rho$. Hence N has a fixed point which is a solution to our integral equation (1).

2.4 Existence results for partial Hadamard-Stieltjes Fractional Integral Equations with Mönch fixed point

Our last existence result for the problem (2) is based on Mönch's fixed point theorem.

Theorem 2.4. *Assume that the hypotheses $(H_1) - (H_4)$ and the condition (2.2) hold. Then the integral equation (2) has at least one solution defined on J .*

Proof Consider the operator N' defined in (2.5). We shall show that N' satisfies the assumption of Mönch's fixed point theorem. Knowing that $N' : B_\rho \rightarrow B_\rho$ is bounded and

continuous, we need to prove that the implication

$$V = \overline{\text{conv}}N'(V) \quad \text{or} \quad V = N'(V) \cup \{0\} \implies \alpha(V) = 0,$$

holds for every subset V of B_ρ .

Let V be a subset of B_ρ such that $V \subset \overline{\text{conv}}(N'(V) \cup \{0\})$. V is bounded and equicontinuous therefore the function $(x, y) \rightarrow v(x, y) = \alpha(V(x, y))$ is continuous on J . By Remark (2.1), Lemma (1.2) and the properties of the measure α we have for each $(x, y) \in J$,

$$\begin{aligned} V(x, y) &\leq \alpha((N'V)(x, y) \cup \{0\}) \\ &\leq \alpha((N'V)(x, y)) \\ &\leq \alpha\{(N'u)(x, y) : u \in V\} \\ &\leq \frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)} \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{\alpha(u(s, t))}{st} d_t g_2(y, t) d_s g_1(x, s) : u \in V \right\} \\ &\leq \frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{v(s, t)}{st} d_t g_2(y, t) d_s g_1(x, s) \end{aligned}$$

This implies that $v(x, y) = 0$ for each $(x, y) \in J$, and then $V(x, y)$ is relatively compact in E . In view of the Ascoli-Arzelà theorem, V is relatively compact in B_ρ .

Consequently, from Theorem 1.6 we conclude that N' has a fixed point $u \in B_\rho$. Hence N' has a fixed point which is a solution to our integral equation (2).

Chapter 3

Examples

3.1 Example 1

Exemple 3.1. *As an application of our results we consider the following partial Hadamard integral equation of the form*

$$u_n(x, y) = \mu(x, y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f_n(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dt ds \quad (x, y) \in [1, e] \times [1, e], \quad (3.1)$$

where

$$r_1, r_2 > 0, \mu(x, y) = x + y^2, \quad (x, y) \in [1, e] \times [1, e],$$

and

$$f_n(x, y, u(x, y)) = \frac{cxy^2}{1 + \|u\|_E} \left(e^{-7} + \frac{1}{e^{x+y+5}} \right) u_n(x, y), \quad (x, y) \in [1, e] \times [1, e],$$

with

$$u = (u_1, u_2, \dots, u_n, \dots) \text{ and } c := \frac{e^4}{2} \Gamma(1 + r_1) \Gamma(1 + r_2).$$

Set

$$f = (f_1, f_2, \dots, f_n, \dots).$$

Clearly, the function f is continuous. Moreover, for each $u \in E$ and $(x, y) \in [1, e] \times [1, e]$, we have

$$\|f(x, y, u(x, y))\|_E \leq ce^{-4}(1 + \|u\|_E).$$

Hence, the hypothesis (**H**) is satisfied with $p_1^* = p_2^* = ce^{-4}$.

We shall show that condition (2.1) hold with $a=b=e$. Ended,

$$\frac{(\log a)^{r_1}(\log b)^{r_2}p_2^*}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{c}{e^4\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{2} < 1.$$

From Theorem 2.1, it follows that the integral equation (3.1) has at least one solution on $[1, e] \times [1, e]$.

3.2 Exemple 2

Exemple 3.2. Consider now the following partial Hadamard-Stieltjes integral equation of the form

$$u_n(x, y) = \mu(x, y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f_n(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t) d_s g_1(x, s) \quad (x, y) \in [1, e] \times [1, e], \quad (3.2)$$

where

$$r_1, r_2 > 0, \mu(x, y) = x + y^2, \quad : (x, y) \in [1, e] \times [1, e], g_1(x, s) = s, \quad g_2(y, t) = t, \quad s, t \in [1, e]$$

and

$$f_n(x, y, u(x, y)) = xy^2 \left(e^{-7} + \frac{u_n(x, y)}{e^{x+y+5}} \right), \quad (x, y) \in [1, e] \times [1, e].$$

The condition (H_1) is satisfied with $p_1(x, y) = xy^2e^{-7}$ and $p_2(x, y) = \frac{xy^2}{e^{x+y+5}}$. Also, the function g_1 and g_1 satisfy (H_2) – (H_4).

We shall show that condition (2.4) holds with $a=b=e$. Indeed,

$$g^*p_2^* = (e-1)^2e^{-4} < e^{-2} < 1.$$

Hence, Theorem 2.2 shows that the integral equation (3.2) has at least one solution on $[1, e] \times [1, e]$

3.3 Exemple 3

Exemple 3.3. We consider the following Hadamard-Volterra-Stieltjes integral equation

$$\begin{aligned}
 u_n(x, y) &= \frac{\ln(x + y + 1)}{1 + e^{x+y}} \\
 &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{se^{-s-t}}{t|u(s, t)| + st} \frac{d_t \ln([y] + t) d_s e^{-[x]-s}}{st}
 \end{aligned} \tag{8}$$

where $(x, y) \in J = [1, +\infty) \times [1, \pi]$, and $r_1, r_2 > 0$.

Set

$$\mu(x, y) = \frac{\ln(x + y + 1)}{1 + e^{x+y}}; \quad g_1(x, s) = e^{-[x]-s}; \quad g_2(y, t) = \ln([y] + t),$$

and

$$f(s, t, u(s, t)) = \frac{se^{-s-t}}{t|u(s, t)| + st}; \quad (s, t) \in J.$$

The symbol $[y]$ indicates the integer value of y . It is clear that equation (8) can be written as equation (2). Let us show that conditions $(H_1) - (H_3)$ hold. The function $t \rightarrow \ln([y] + t)$ is continuous and nondecreasing for each fixed $y_0 \in [1, \pi]$, so it is of bounded variation on $[1, \pi] \times [1, \pi]$, and

$$\bigvee_{t=1}^{t=\pi} g_1(y, t) \leq \ln(3 + \pi) - \ln 2.$$

The function $s \rightarrow e^{-[x]-s}$ is continuous and decreasing for each fixed $x_0 \in [1, +\infty)$, and

$$\lim_{s \rightarrow +\infty} e^{-[x]-s} = 0.$$

So it is of bounded variation on $[1, +\infty) \times [1, +\infty)$, and for each fixed $x_0 \in [1, +\infty)$

$$d_s g_2(x_0, s) = -e^{-[x_0]-s} ds.$$

$$\begin{aligned} |f(s, t, u_1) - f(s, t, u_2)| &= \left| \frac{se^{-s-t}}{t|u_1| + st} - \frac{se^{-s-t}}{t|u_2| + st} \right| \\ &= \left| \frac{tse^{-s-t}(|u_1| - |u_2|)}{(t|u_1| + st)(t|u_2| + st)} \right| \\ &\leq \frac{tse^{-s-t}|u_1 - u_2|}{u_1 + |u_2| + 1}. \end{aligned}$$

So $g(s, t) = tse^{-s-t}$; $f(s, t, 0) = \frac{e^{-s-t}}{t}$ and for a fixed $x_0 \in [1, +\infty)$ we have :

$$\begin{aligned} &\left| \int_1^{x_0} \int_1^y \left(\ln \frac{x_0}{s} \right)^{r_1-1} |f(s, t, 0)| d_t \ln([y] + t) d_s e^{-[x_0]-s} \right| \\ &= \left| \int_1^{x_0} \int_1^y \left(\ln \frac{x_0}{s} \right)^{r_1-1} \frac{e^{-s-t}}{t} d_t \ln([y] + t) d_s e^{-[x_0]-s} \right| \\ &\leq \left| \ln \pi [\ln(3 + \pi) - \ln 2] \int_1^{x_0} - \left(\ln \frac{x_0}{s} \right)^{r_1-1} e^{-s-[x_0]} e^{-[x_0]-s} ds \right| \\ &\leq \left| \ln \pi [\ln(3 + \pi) - \ln 2] \ln x_0 \int_1^{x_0} e^{-2s-2[x_0]} ds \right| \\ &\leq \left| \ln \pi [\ln(3 + \pi) - \ln 2] \ln x_0 \frac{1}{2} [e^{-2s-2[x_0]}]_1^{x_0} \right| \\ &\leq \frac{1}{2} \left| \ln \pi [\ln(3 + \pi) - \ln 2] \ln x_0 [e^{-2x_0-2[x_0]} - e^{-2[x_0]-2}] \right|. \end{aligned}$$

The function

$$\varphi(x) = \ln \pi [\ln(3 + \pi) - \ln 2] \ln x [e^{-2x-2[x]} - e^{-2[x]-2}],$$

is bounded on $[1, +\infty)$. Then there exist $\lambda > 0$ such that

$$\sup_{x \geq 1; 1 \leq y \leq \pi} \left| \int_1^x \int_1^y \left(\ln \frac{x}{s} \right)^{r_1-1} |f(s, t, o)| d_t \ln([y] + t) d_s e^{-[x]-s} \right| \leq \lambda.$$

We have also

$$\begin{aligned} \left| \int_1^{x_0} \left(\ln \frac{x_0}{s} \right)^{r_1-1} d_s e^{-[x_0]-s} \right| &\leq \left| \ln x_0 \int_1^{x_0} e^{-[x_0]-s} ds \right| \\ &\leq \left| \ln x_0 [e^{-[x_0]-x_0} - e^{-[x_0]-1}] \right|. \end{aligned}$$

The function

$$\psi(x) = \ln x [e^{-[x]-x} - e^{-[x]-1}],$$

is bounded on $[1, +\infty)$. Then there exist $\eta > 0$ such that

$$\sup_{x \geq 1} \left| \int_1^x \left(\ln \frac{x}{s} \right)^{r_1-1} d_s e^{-[x]-s} \right| \leq \eta.$$

It follows that

$$k = \frac{\eta \ln b \|g\|}{\Gamma(r_1)\Gamma(r_2)} \bigvee_{\alpha=1}^{\alpha=b} g_1(y, \alpha) = \frac{\eta e^{-2} \ln \pi}{\Gamma(r_1)\Gamma(r_2)} [\ln(3 + \pi) - \ln 2] < 1.$$

Thus, from Theorem (2.4) the Eq (8) has at least solution in BC and solution of Eq (8) are globally attractive.

3.4 Exmpele 4

We consider the following Hadamard-Volterra integral equation

$$u(x, y) = \frac{x+1}{y^2 e^x} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{e^{-t}+s}{t^2 s^3 + 1} \sqrt{u^2(t, s) + e^{-t}} \frac{dt ds}{st}, \quad (3.3)$$

where $(x, y) \in J = [1, +\infty) \times [1, b], b > 1$ and $r_1, r_2 > 0$.

Set

$$\mu(x, y) = \frac{x+1}{y^2 e^x}, \quad f(t, s, u(t, s)) = \frac{e^{-t}+s}{t^2 s^3 + 1} \sqrt{u^2(t, s) + e^{-t}}; (t, s) \in J.$$

It is clear that equation 3.3 can be written as equation 1. Let us show that the conditions hold.

$$\begin{aligned} |f(t, s, u_1(t, s)) - f(t, s, u_2(t, s))| &\leq \frac{e^{-t}+s}{t^2 s^3 + 1} \left| \sqrt{u_2^2(t, s) + e^{-t}} - \sqrt{u_1^2(t, s) + e^{-t}} \right| \\ &\leq \frac{e^{-t}+s}{t^2 s^3 + 1} \left| \sqrt{u_2^2(t, s)} - \sqrt{u_1^2(t, s)} \right| \\ &\leq \frac{e^{-t}+s}{t^2 s^3 + 1} |u_2(t, s) - u_1(t, s)|. \end{aligned}$$

Take

$$b(t, s) = \frac{e^{-t}+s}{t^2 s^3 + 1} \quad f(t, s, 0) = \frac{e^{-t}+s}{t^2 s^3 + 1} e^{-\frac{1}{2}t}.$$

We have

$$\begin{aligned} &\int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{e^{-t}+s}{t^2 s^3 + 1} dt ds \\ &\leq (\ln x)^{r_1} (\ln y)^{r_2} \int_1^x \int_1^y \frac{e^{-t}+s}{t^2 s^3} dt ds \\ &= (\ln x)^{r_1} (\ln y)^{r_2} \int_1^x \int_1^y \left(\frac{e^{-t}}{s^3} + \frac{1}{t^2 s^2} \right) dt ds \\ &= (\ln x)^{r_1} (\ln y)^{r_2} \left[(-e^{-t} + e^{-1}) \left(-\frac{1}{2x^2} + \frac{1}{2} \right) + \left(-\frac{1}{y} + 1 \right) \left(-\frac{1}{x} + 1 \right) \right] \\ &= h(x, y), \end{aligned}$$

And

$$\begin{aligned}
& \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} |f(t, s, 0)| dt ds \\
& \leq (\ln x)^{r_1} (\ln y)^{r_2} \int_1^x \int_1^y \frac{e^{-t} + s}{t^2 s^3 + 1} e^{-\frac{1}{2}t} dt ds \\
& \leq (\ln x)^{r_1} (\ln y)^{r_2} \int_1^x \int_1^y \frac{e^{-t} + s}{t^2 s^3} e^{-\frac{1}{2}t} dt ds \\
& = (\ln x)^{r_1} (\ln y)^{r_2} \int_1^x \int_1^y \left(\frac{e^{-\frac{3}{2}t}}{s^3} + \frac{e^{-\frac{1}{2}t}}{s^2} \right) dt ds \\
& = (\ln x)^{r_1} (\ln y)^{r_2} \left[\left(-\frac{2}{3} e^{-\frac{3}{2}y} + \frac{2}{3} e^{-\frac{3}{2}} \right) \left(-\frac{1}{2x^2} + \frac{1}{2} \right) + \left(-\frac{1}{x} + 1 \right) \left(-2e^{-\frac{1}{2}y} + 2e^{-\frac{1}{2}} \right) \right] \\
& = g(x, y).
\end{aligned}$$

It is clear that $h(x, y), g(x, y)$ are bounded functions on J . And we have also

$$k = \frac{\|h\|}{\Gamma(r_1)\Gamma(r_2)} = \frac{\frac{1}{2e} + 1}{\Gamma(r_1)\Gamma(r_2)} < 1,$$

for an arbitrary r_1, r_2 . Then from Theorem 2.1 the eq 3.3 has at least solution in X .

Conclusion

In this work, the objective is the study of existence of solutions for Partial Hadamard Fractional Integral Equations in Banach Spaces. And specifically by using in this study the technique of measure of noncompactness and the fixed point theorems of Darbo and Mönch.

This technique is often used in several branches of nonlinear analysis. Especially this technique has proven to be a very useful tool in the existence of solutions of several types of integral equations.

Adopting the non measure of noncompactness technique transforming the problem from integral equation to fixed point problem by adapting the hypotheses of the two problems. And we concluded this work by giving practical examples of this problem to confirm this work.

Bibliography

- [1] A. Carpinteri, B.Cornetti, K. M. Kolwankar, Calculation of the tensile and flexural strength of disordered materials using fractional calculus, *Chaos, Solitons, Fractals*, 21 (2004) 623-632.
- [2] A.A. Kilbas,H.M. Srivastava,J.J. Trujillo ,*Theory And Applications Of Fractional Differentail Equations*,2006.
- [3] A. Kirillov, A. Gvichiani, *Théorèmes Et Problèmes D'analyse Fonctionnelle*, Éditions Mir Moscou , 1982.
- [4] Appell, J.: Implicit functions, nonlinear integral equations, and the measure of non-compactness of the superposition operator. *J. Math. Anal. Appl.* 83, 251–263 (1981).
- [5] A. Parvate, A. D.Gangal, Fractal differential equations and fractal-time dynamical systems, *Pramana J. Phys.*, 64 (3) (2005) 389-409.
- [6] A. Parvate, A. D.Gangal, Calculus on fractal subsets of real line - I: formulation, *Fractals*, 17 (1) (2009) 53-81.
- [7] A.V. Dyskin, Effective characteristics and stress concentration materials with self-similar microstructure, *Int. J. Sol. Struct.*, 42 (2005) 477-502.
- [8] Banaś, J., Goebel, K.: *Measures of Noncompactness in Banach Spaces*. Marcel Dekker, New York (1980).
- [9] B.Bojareski, *Dissertationes Mathematicae*, Państwowe Wydawnictwo Naukowe-Polish Scientific Publishers, Warszawa 1988.

- [10] D. Guo, V. Lakshmikantham and X. Liu, Nonlinear integral equations in abstract spaces, Kluwer Acad. Publishers, Dordrecht, 1996.
- [11] Goebel, K.: Concise Course on Fixed Point Theorems. Yokohama Publishers, Japan (2002).
- [12] Granas, A., Dugundji, J.: Fixed Point Theory. Springer-Verlag, New York (2003).
- [13] Guo, D.J., Lakshmikantham, V., Liu, X.: Nonlinear Integral Equations in Abstract Spaces. Kluwer Academic Publishers, Dordrecht (1996).
- [14] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.
- [15] I. P. Natanson, Theory of Functions of a Real Variable, Ungar, New York, 1960.
- [16] I. Podlubny. Fractional differential equations. Academic Press , 1999.
- [17] J. Appell, J. Banaś, N. Merentes, Bounded Variation and Around, De Gruyter Series in Nonlinear Analysis and Applications 17. Walter de Gruyter, Berlin, 2014.
- [18] J.H. He, A new fractal derivation, Thermal Science, 15(1) (2011) 2011-147.
- [19] J.H. He, S.K. Elagan, Z.B. Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, Phy. Lett. A, 376(4) (2012) 257-259.
- [20] J. Hadamard, Essai sur l'étude des fonctions données par leur développement de Taylor, Journal of pure and applied mathematics, 4(8) (1892) 101?-186.
- [21] Kirk, W.A., Sims, B.: Handbook of Metric Fixed Point Theory. Springer-Science + Business Media, B.V., Dordrecht (2001).
- [22] K.M. Kolwankar, A.D. Gangal, Hölder exponents of irregular signals and local fractional derivatives, Pramana J. Phys., 48 (1997) 49-68.
- [23] K.M. Kolwankar, A.D. Gangal, Local fractional Fokker–Planck equation, Phys. Rev. Lett., 80 (1998) 214-217.

- [24] Mönch, H.: Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal.* 4, 985–999 (1980).
- [25] Natanson, I.P.: *Theory of Functions of a Real Variable*. Ungar, New York (1960).
- [26] Nieto, J.J.; Samet, B. Solvability of an implicit fractional integral equation via a measure of noncompactness argument. *Acta Math. Sci.* 2017, 37, 195–204.
- [27] P.Lissy, utilisation de la notion de compacité, May 29, 2010.
- [28] Rudin, W.: *Real and Complex Analysis*. McGraw-Hill, New York (1970).
- [29] S. Almezal, Q. H. Ansari and M. A. Khamsi, *Topics in Fixed Point Theory*, Springer-Verlag, New York, 2014.
- [30] Samko, S.G., Kilbas, A.A., Marichev, O.I: *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, Yverdon et alibi (1993).
- [31] S. Carl & S.Heikkilä ,« Fixed point theory in ordered sets and applications » , Springer New York Dordrecht Heidelberg London, 2010.
- [32] W. Chen, Time–space fabric underlying anomalous diffusion, *Chaos, Solitons, Fractals*, 28 (2006) 923–929.
- [33] W. Chen, X.D. Zhang, D. Korosak, Investigation on fractional and fractal derivative relaxation- oscillation models. *Int. J. Nonlin. Sci. Num.*, 11 (2010) 3-9.
- [34] W.P. Zhong, F. Gao, X.M. Shen, Applications of Yang-Fourier transform to local Fractional equations with local fractional derivative and local fractional integral, *Advanced Materials Research*, 416 (2012) 306-310.
- [35] W. P. Zhong, F. Gao, Application of the Yang-Laplace transforms to solution to nonlinear fractional wave equation with local fractional derivative. In: *Proc. of the 2011 3rd International Conference on Computer Technology and Development*, ASME, 2011, pp.209-213.

- [36] X.J. Yang, Applications of local fractional calculus to engineering in fractal time-space: Local fractional differential equations with local fractional derivative, ArXiv:1106.3010v1 [math-ph], 2011.
- [37] X.J Yang, Local Fractional Integral Transforms, Progr. in Nonlinear Sci., 4 (2011) 1-225.
- [38] X.J Yang, Local Fractional Functional Analysis and Its Applications, Asian Academic publisher Limited, Hong Kong, 2011.
- [39] X.J. Yang, M.K. Liao, J.W. Chen, A novel approach to processing fractal signals using the Yang-Fourier transforms, Procedia Eng., 29 (2012) 2950-2954.
- [40] X.J. Yang, Local fractional partial differential equations with fractal boundary problems, Advances in Computational Mathematics and its Applications, 1(1) (2012) 60-63.
- [41] X.J. Yang, Generalized Sampling Theorem for Fractal Signals, Advances in Digital Multimedia, 1(2) (2012)88-92.
- [42] X. J. Yang, Local Fractional Kernel Transform in Fractal Space and Its Applications, Advances in Computational Mathematics and its Applications, 1(2) (2012) 86-93.
- [43] X.R. Li, Fractional Calculus, Fractal Geometry, and Stochastic Processes, Ph.D. Thesis, University of Western Ontario, 2003.
- [44] Yves SONNTAG :Topologie et ANALYSE FONCTIONNELLE, cours de licence avec 240 exercices et 30 problèmes corrigés, ellipses, universités mathématiques de Provence.

Summary

In this work, we have tried to present and enrich the results obtained by other authors in the study Having solutions to a row of Hadamard's partial integral equations.

The main tool used in considerations is the combination of the technique of measure of noncompactness with fixed point theorems of Darbo type and Mönch type.

Résumé

Dans ce travail, nous avons essayé de présenter et d'enrichir les résultats obtenus par d'autres auteurs dans l'étude l'existence des solutions à une rangée d'équations intégrales partielles d'Hadamard.

Le principal outil est la technique de la mesure de non compacité combiné avec les théorèmes du point fixe et en particulier le point fixe de Darbo et de Mönch.

ملخص

في هذا العمل ، حاولنا تقديم وإثراء النتائج المتحصل عليها من قبل كتاب آخرون في دراسة وجود حلول لصف من المعادلات التكاملية الجزئية لـ Hadamard. الأداة الرئيسية المستخدمة في هذه الدراسة هي تقنية القياس غير المتراص مع نظريات النقطة الثابتة و خاصة النقطة الثابتة لـ Darbo و النقطة الثابتة لـ Mönch.