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Study of Problem Described by Fractional Differential Hemivariational Inequalities

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Dedications

Praise be to Allah who enabled me to complete this work, and brought me to this long-awaited moment.

I dedicate my graduation to:
*To the soul of my dear father **Beyat Ali**, may Allah have mercy on him.*

*To my beloved mother **Fethiza Amara Fatma**, who played both roles, whose prayers accompanied me every step of the way, who shared my worries before my joys, may Allah protect him.*

*To my sisters **Nadjah, Salwa** and **Izdihar**, who supported me with all their love and were a shield to ease my troubles.*

*To my **uncles** and my **aunts**, (especially my aunt **Harkatia**), who instilled hope in me and were my greatest motivators and constant support.*

To everyone who supported me and contributed to my reaching this achievement.
with immense pride and gratitude.

Beyat Malia

Dedications

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kholoud Reguigue

Table of contents

Introduction

Contact problems encompass the investigation of the mechanical behavior exhibited by two or more objects when they are in contact, including the distribution of contact forces, deformation, frictional effects, and other related phenomena. These issues typically arise in the fields of engineering mechanics [27], materials science [11], and physics [3], constituting an essential branch of solid mechanics. Contact problems typically involve complex boundary conditions, which play a crucial role in determining the mechanical behavior of the contact. In recent years, variational inequalities have been extensively employed in the field of contact problems [12, 13, 26]. More recently, hemivariational inequalities are utilized in situations where the interaction between physical quantities in contact phenomena demonstrates a response which is non-monotonic [18, 4, 24, 8]. During the contact process, numerous physical phenomena frequently take place, such as alterations in thermal, electrical and surface properties. The corresponding variational problems result in various hemivariational inequalities systems. In light of a series of results pertaining to the hemivariational inequality problem, researchers have conducted extensive studies on systems of hemivariational inequalities. Papers [16, 23, 30] study systems of hemivariational inequalities that arise from contact problems with adhesion or damage while papers [15, 21] consider hemivariational inequalities systems arising from contact problems with wear terms. Moreover, the references [14, 25] focus on hemivariational inequalities systems resulting from contact problem with thermal effects. Nowadays, due to the powerful adaptability and historical dependence of fractional calculus, the study of hemivariational inequalities in the sense of fractional calculus has attracted considerable attention from many researchers. [17, 32] have studied a fractional hemivariational inequality and employed Rothe method to prove the existence of the weak solution. [19] considered a history-dependent fractional hemivariational inequality and [20] studied a fractional differential hemivariational inequality. Inspired by the aforementioned study, in this research, we *focus on a frictional contact problem with thermal effects as well as long memory term, both the Kelvin-Voigt constitutive law and the heat conduction equation are incorporated into the time-fractional characteristics. By introducing fractional-order derivatives, the model can more accurately describe the mechanical response of materials in the contact region, capturing the complex behavior of materials at the microstructural level. Meanwhile, the corresponding variational formulation leads to a system of a fractional hemivariational inequality and a fractional differential equation. In contrast to previous literature, we prove the existence of a unique weak solution for this system by using Banach fixed point theorem and some results for hemivariational inequality. Moreover, we will conduct further research, considering the corresponding numerical approximation and numerical simulation. This may enrich the theory of numerical methods for hemivariational inequalities. It can also help better understand the physical phenomena arising from contact problems with temperature terms and time-fractional characteristics. In the proof, it is necessary to introduce the generalized versions of the Gronwall lemma.*

The thesis comprises three chapters and are organized as follows: In the first chapter, we begin by defining the physical framework, the behavior laws of different materials, the boundary conditions as well as the constitutive laws and contact condition of the problem to be studied. In the second chapter, we begin by reviewing some results concerning functional spaces, some elements of nonlinear analysis, as well as some definitions about fractional calculus. Hemivariational inquality, Gronwall lemma and Banach fixed point theorem will be of great use for the demonstrations. In the last chapter, we study a contact problem with friction and normal compliance in thermo-viscoelasticity with time. We present a variational formulation of the problem and prove the existence and uniqueness of a solution using fixed-point techniques and Gronwall lemma and Caputo derivative.

General Notations

$\mathbb{N}$	*Set of natural numbers.*
$\mathbb{R}^d$	*The set of real numbers of order d.*
$\mathbb{S}^d$	*The space of second-order symmetric tensors on* $\mathbb{R}^d(d=2,3)$.
Ω	*Open, bounded, and connected set in* $\mathbb{R}^d$ *with Lipschitz boundary.*
$\bar{\Omega}$	*The adherence of* Ω.
Γ	*The boundary of* $\Omega : \Gamma = \partial\Omega$.
Γ_{S_i}	*The boundary parts of* $\Gamma, (i=1,2,3)$.
$mes\Gamma_{S_i}$	*Dimensional* $(d-1)$ *Lebesgue measure of* Γ_{S_i}.
$d\Gamma_{S_i}$	*Surface measure on* Γ_{S_i}.
$\boldsymbol{u}$	*The displacement of the field.*
$\dot{\boldsymbol{u}}$	*The first derivative of* u *with respect to time.*
$\ddot{\boldsymbol{u}}$	*The second derivative of* u *with respect to time.*
$\partial_i u$	*The partial derivative of* u *with respect to time.*
∇u	*The gradient of* u $\nabla u = (\partial_1 u, \dots, \partial_d u)$.
Div	*the divergence operator for tensor fields.*
div	*The divergence operator for vector fields.*
$\boldsymbol{f}_0$	*The volume force density.*
$\boldsymbol{f}_2$	*The surface traction density.*
$\boldsymbol{\nu}$	*The outward unit normal to* Γ.
v_ν, v_τ	*The normal and tangential components of the vector field* v *defined on* $\bar{\Omega}$.
$\boldsymbol{\sigma}$	*Stress tensor corresponding to the displacement* $\boldsymbol{u}$.
$\boldsymbol{\sigma}_\nu, \boldsymbol{\sigma}_\tau$	*The normal and tangential components of the vector field* $\boldsymbol{\sigma}$ *defined on* $\bar{\Omega}$.
$\boldsymbol{\varepsilon}(\boldsymbol{u})$	*Linearized strain tensor:* $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$.
ϑ	*The temperature.*
${}_0^C D_t^\alpha f(t)$	*The Caputo fractional derivative of* f .
${}_0 I_t^\alpha f(t)$	*The Riemann-Liouville fractional integral of order* $\alpha > 0$ *for a function* f .

$\mathcal{A}$	*The viscosity operator.*
$\mathcal{B}$	*The elasticity operator.*
$\mathcal{G}$	*The long memory operator.*
j_ν, j_τ	*The potential functions.*
X	*A real Hilbert space .*
$(\cdot,\cdot)_X$	*The scalar product of* X.
$\|\cdot\|_X$	*The norm of* X.
X^*	*The dual space of* X.
$(\cdot,\cdot)_{X^*\times X}$	*The duality product between* H^* *and* H.
$L^2(\Omega)$	*Space of functions* u *measurable on* Ω *such that* $\int_\Omega \|u\|^2 dx < +\infty$.
$\|\cdot\|_{L^2(\Omega)}$	*The norm of* $L^2(\Omega)$ *defined by* $\| u \|_{L^2(\Omega)} = (\int_\Omega \|u\|^2 dx)^{\frac{1}{2}}$.
$L^\infty(\Omega)$	*Space of functions* u *measurable on* Ω *such that,*
	$\exists c > 0 : \| u \| < c$, *a.e , on* Ω.
$\mathcal{L}(X,Y)$	*The space of linear continuous operators from a normed space* X *to a normed space*
H	*The space* $L^2(\Omega)^d$.
H_1	*The space* $H^1(\Omega)^d$.
$\mathcal{H}$	*The space* $L^2(\Omega)_s^{d\times d}$.
$\mathcal{H}_1$	*The space* $\{\boldsymbol{\sigma} \in \mathcal{H} \quad \| \quad div\boldsymbol{\sigma} \in H\}$.
$H^{\frac{1}{2}}(\Gamma)$	*The Sobolev space of order* $\frac{1}{2}$ *on* Γ.
$H^{-\frac{1}{2}}(\Gamma)$	*The dual space of* $H^{\frac{1}{2}}(\Gamma)$.
$H^1(\Omega)$	*The Sobolev space of order 1 on* Ω.
H_Γ	*Space* $(H^{\frac{1}{2}}(\Gamma))^d$.
H_Γ^*	*The dual space of* H_Γ.
$\gamma : H_1 \to H_\Gamma$	*The trace mapping for vector functions.*
V	*The space* $\{\upsilon = (\upsilon_i) \in H^1(\Omega;\mathbb{R}^d) \| \upsilon = 0$ *on* $\Gamma_{S_1}\}$.
$\mathcal{V}$	*Space* $L^2(0,T;V)$.
E	*The space* $\{\eta \in H^1(\Omega) \| \eta = 0$ *on* $\Gamma_{S_1} \cup \Gamma_{S_2}\}$.
$\mathcal{E}$	*Space* $L^2(0,T;E)$.
W	*The space* $L^2(\Gamma_{S_3};\mathbb{R})$.
$\mathcal{W}$	*Space* $L^2(0,T;W)$.

If in addition $[0,T]$ *a time interval,* $k \in \mathbb{N}$ *and* $1 \le p \le +\infty$, *we note by*

$C(0,T;H)$	*The space of continuous functions from* $[0,T]$ *to* H.
$C^1(0,T;H)$	*The space of continuously differentiable functions on* $[0,T]$ *in* H.
$AC(0,T;X)$	*The space of all absolutely continuous functions from* $(0,T)$ *into* X.
$L^p(0,T;H)$	*The space of measurable functions on* $[0,T]$ *in* H.
$\\| \cdot \\|_{L^p(0,T;H)}$	*The norm of* $L^p(0,T;H)$.
$W^{k,p}(0,T;H)$	*The Sobolev space of parameters* k *and* p.
$\\| \cdot \\|_{W^{k,p}(0,T;H)}$	*The norm of* $W^{k,p}(0,T;H)$.
c	*Strictly positive real constant.*
i.e.	*That is to say.*
a.e.	*Almost everywhere.*

Chapter 1

Modeling and Preliminaries

In this chapter, we present the physical framework for the study of the boundary value problem, some notions of contact mechanics, the equation of motion, equilibrium and the evolution heat, the constitutive laws for the material considered, as well as the boundary conditions of contact.

1.1 Physical framework - Mathematical model

1.1.1 Physical framework

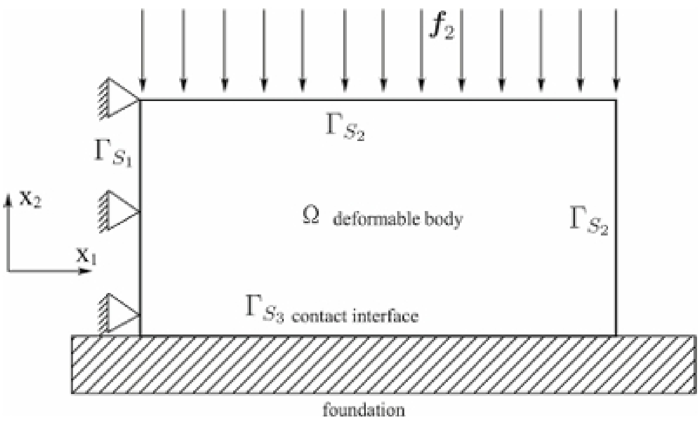

Figure1: The physical configuration of the contact model.

This section presents a comprehensive explanation of the physical model being investigated. Let us analyze a viscoelastic object that interacts with a rigid foundation, occupying an open bounded subset of $\mathbb{R}$ where $d = 2, 3$. That contact process takes into account the time-fractional Kelvin-Voigt constitutive law with long memory effects *and thermal influences. The boundary* $\Gamma = \partial\Omega$ *of* Ω *is assumed to exhibit Lipschitz continuity and can be divided into three distinct and non-overlapping sections:*$\Gamma_{S_1}, \Gamma_{S_2}$ *and* Γ_{S_3}*. The measure of* Γ_{S_1} *denoted as* $m(\Gamma_{S_1})$*, is further postulated to be positive. By fixing the viscoelastic body to the boundary displacement field on* Γ_{S_1} *vanishes there. Moreover, time-varying surface tractions characterized by density* $\boldsymbol{f}_2$ *are applied on* Γ_{S_2}*, while volume forces with density* $\boldsymbol{f}_0$ *operate within* Ω*.*

Our primary focus is to elucidate the evolution of the mechanical state of a body throughout the time period $(0, T)$ *with* $T > 0$*.*

1.1.2 Mathematical model

We commence by introducing the subsequent notions. We will utilise $\boldsymbol{u} = (u_i)$ *the displacement vector,* $\boldsymbol{\sigma} = (\sigma_{ij})$ *stress tensor and* $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ *linearized strain tensor respectively.*

Sometimes, for the sake of simplicity, we neglect the spatial dependence of the variables on $\boldsymbol{x}$*.*

The elements of the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ *can be represented as* $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$*, where* $u_{i,j}$ *denotes the partial derivative of* u_i *with respect to* x_j*.* ***Here,*** *indices* i, j *vary from* 1 *to* d*, and unless otherwise specified, summation convention is applied for respect indexes. The presence of a comma followed by an index signifies a partial derivative with respect to the corresponding element of the variable* $\boldsymbol{x}$*.*

The outward unit normal on $\partial\Omega$ *is expressed by the notation* $\boldsymbol{\nu}$*.*

The notation $\mathbb{S}^d$ *denotes the space comprising second order symmetric tensors in* $\mathbb{R}^d$*.*

$$
\begin{aligned}
u \cdot v &= u_i v_i; \qquad && \|v\| = (v, v)^{\frac{1}{2}}, \quad \forall u, v \in \mathbb{R}^d \\
\sigma \cdot \tau &= \sigma_{ij}\tau_{ij}; \qquad && \|\tau\| = (\tau, \tau)^{\frac{1}{2}}, \quad \forall \sigma, \tau \in \mathbb{S}^d
\end{aligned}
$$

We denote by v_ν *and* $\boldsymbol{v}_\tau$ *the normal and tangential components of* $\boldsymbol{v}$ *on the boundary given by*

$$
v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu} \tag{1.1}
$$

To simplify the notations, we do not explicitly indicate the dependence of the functions on $x \in \bar{\Omega}$ *and* $t \in [0, T]$*.*

For a stress field $\boldsymbol{\sigma}$ *we denote by* σ_ν *and* $\boldsymbol{\sigma}_\tau$ *the normal and tangential components at the boundary given by*

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \sigma\nu - \sigma_\nu \cdot \boldsymbol{\nu} \tag{1.2}$$

using (3.65) *and* (1.2), *we obtain the relation*

$$(\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{v} = \sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau \tag{1.3}$$

which will be used throughout this thesis in establishing variational formulation of mechanical contact problem.

Before we get to the mathematical model that fit the existing physical framework, there are some notes and conventions that we use throughout this period.

Note that the dot above a function represents the derivation with respect to time, for example

$$\dot{\boldsymbol{u}} = \frac{du}{dt} \qquad \ddot{\mathbf{u}} = \frac{d^2u}{dt^2}$$

where $\dot{\mathbf{u}}$ *denotes the velocity field and* $\ddot{\mathbf{u}}$ *denotes the acceleration field.*

The unknown functions of the problem are the displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and the stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$. *Let us denote the mass density by* $\rho : \Omega \to \mathbb{R}_+$ *and the volume force density by* $\boldsymbol{f}_0 : \Omega \times [0, T] \to \mathbb{R}^d$. *The body evolution is described by the Cauchy equation of motion:*

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = \rho\ddot{\mathbf{u}} \qquad in \quad \Omega \times [0, T]. \tag{1.4}$$

The evolution processes modeled by the previous equation are called dynamic processes. In some situations, this equation can be further simplified: for example, in the case where $\dot{\mathbf{u}} = 0$, *it is an equilibrium problem (static processes), or in the case where the velocity field* $\dot{\mathbf{u}}$ *varies very slowly with respect to time, i.e.*
In these two cases, the equation of motion becomes:

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 \qquad in \quad \Omega \times [0, T]. \tag{1.5}$$

The equation is equivalent to scalar relations, and mathematically this equation *is not sufficient to model the body equilibrium problem because, for example, the* $\boldsymbol{u}_i$ *components of the displacement field do not appear in this equation.*

1.2 Thermal phenomena

1.2.1 Thermal condition

Thermal conduction is a mode of temperature or heat transfer between two media in contact. The temperature ϑ *is defined by a parabolic equation with limits condition. We present the evolution of the heat equation and the relevant boundary conditions have been discussed in [11]:*

$$\begin{aligned}
{}_0^C D_t^{\alpha}\vartheta(t) - div(K_c\nabla\vartheta(t)) &= -c_{ij}\frac{\partial\left({}_0^C D_t^{\alpha} u_i(t)\right)}{\partial x_j} + q(t) && in\ \Omega,\\
-k_{ij}\frac{\partial\vartheta(t)}{\partial x_j}\nu_i &= k_e(\vartheta(t)-\vartheta_R) && on\ \Gamma_{S_3},\\
\vartheta(t) &= 0 && on\ \Gamma_{S_1}\cup\Gamma_{S_2}.
\end{aligned}$$

This time fractional heat equation can more precisely demonstrate the material's responsiveness to historical temperature uctuations. The thermal conductivity tensor is represented by $K_c := (k_{ij})$*, while* $q(t)$ *represents the heat source density within a given volume.* ϑ_R *denotes the temperature of the foundation, while* k_e *refers to the coefcient of heat exchange between the body and the obstacle. Additionally, the thermal expansion tensor is denoted as* c_{ij}*.*

1.3 Constitutive laws

1.3.1 Viscoelastic behavior law with long memory

Here we consider a category of material where the stress tensor $\boldsymbol{\sigma}$ *is related by the constitutive law:*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{u}}(t)\right) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(t-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds, \tag{1.6}$$

where $\boldsymbol{u}$*,* $\boldsymbol{\sigma}$ *represent the displacement field and the stress field respectively, where* $\dot{\boldsymbol{u}}$ *denotes the derivative with respect to the time variable. Furthermore,* $\boldsymbol{\varepsilon}(\boldsymbol{u})$ *denotes the* *linearized strain tensor,* $\mathcal{A}$ *and* $\mathcal{B}$ *are linear operators describing the purely viscous and elastic properties of the material from* $\Omega \times \mathbb{S}^d$ *to* $\mathbb{S}^d$*, respectively,* $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ *is a long memory operator.*

1.3.2 Viscoelastic behavior law with short memory

The behavior law of a viscoelastic material with short memory is given by:

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{1.7}$$

1.3.3 Viscoelastic behavior law with time-fractional

The behavior law of a viscoelastic material with time-fractional Kelvin-Voigt is given by:

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}({}_0^C D_t^\alpha \boldsymbol{u}(t))) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{1.8}$$

where ${}_0^C D_t^\alpha \boldsymbol{u}$ *is the* α*-order time fractional derivative of the displacement* $\boldsymbol{u}$ *in the sense of Caputo, where* $\alpha \in (0,1)$*, will be defined later.*

1.3.4 Termo-viscoelastic behavior law with time-fractional

The behavior law of a thermo-viscoelastic material with time-fractional Kelvin-Voigt is given by:

$$\sigma = \mathcal{A}\boldsymbol{\varepsilon}({}_0^C D_t^\alpha \boldsymbol{u}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \vartheta(s))ds, \tag{1.9}$$

where ϑ *denotes the temperature.*
The study conducted in [31] focuses on the similar constitutive equation that neglects the consideration of fractional derivative, while [32] investigates the time-fractional Kelvin-Voigt constitutive law without accounting for thermal influence.

We now move on to the boundary conditions used in chapter 3.

1.4 Boundary conditions

Let us now define the boundary conditions on each of the three parts of Γ.

1.4.1 Displacement boundary condition

The body is fixed in a position on the part Γ_{S_1}*, the displacement field* $\boldsymbol{u}$ *is therefore zero:*

$$\boldsymbol{u} = 0 \qquad on\ \Gamma_{S_1}. \tag{1.10}$$

1.4.2 Traction boundary condition

A surface traction of density $\boldsymbol{f}_2$ *acts on* Γ_{S_2} *and consequently the Cauchy stress vector* $\boldsymbol{\sigma\nu}$ *satisfies:*

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad on\ \Gamma_{S_2}. \tag{1.11}$$

1.5 Contact laws

1.5.1 Contact with normal compliance

We start with the presentation of the conditions in the normal direction, called also contact conditions or contact laws. We consider contact conditions of the form

$$-\sigma_\nu(t) \in \partial j_\nu\,(t, u_\nu(t) - g_0) \quad on\ \Gamma_{S_3} \times (0, T), \tag{1.12}$$

in which j_ν *is a given function, the symbol* ∂j_ν *denotes the Clarke subdifferential of* j_ν *with respect to the last variable and, recall,* g_0 *represents the gap function. Here and below, as usual, we do not indicate the explicit dependence of various functions on the spatial variable* $\boldsymbol{x}$ *and, sometimes, on both* $\boldsymbol{x}$ *and* t*. Note that the explicit dependence of the function* j_ν *with respect to the time variable makes the model more general and allows to describe situations when the contact condition depends on the temperature, which plays the role of a parameter. Note also that in the study of static problems for elastic materials we remove the dependence of the variables with respect the time and, therefore, we replace the contact condition (1.12) with the condition*

$$-\sigma_\nu \in \partial j_\nu\,(u_\nu - g_0) \quad on\ \Gamma_{S_3}. \tag{1.13}$$

The so-called normal compliance contact condition represents an example of contact *condition which can be cast in the subdifferential form (1.12). It describes a deformable foundation and assigns a reactive normal pressure which depends on the interpenetration of the asperities on the body surface and those on the foundation. A general form of this condition is*

$$-\sigma_\nu = k_\nu p_\nu\left(u_\nu - g_0\right) \quad \textit{on } \Gamma_{S_3} \times (0,T), \tag{1.14}$$

where p_ν is a prescribed nonnegative function which vanishes when its argument is negative and k_ν is a nonnegative function, the stiffness coefficient. Equality (1.14) shows that when there is no contact (i.e., when $u_\nu < g_0$) then $\sigma_\nu = 0$ and, therefore, the normal pressure vanishes. When there is contact (i.e., when $u_\nu \geqslant g_0$) then $\sigma_\nu \leqslant 0$ and, therefore, the reaction of the foundation is towards the body; in this case $u_\nu - g_0$ represents a measure of the interpenetration of the surface asperities.

An example of the normal compliance function p_ν is

$$p_\nu(r) = r_+,$$

or, more general,

$$p_\nu(r) = (r_+)^m.$$

Here $m > 0$ is the normal exponent, and $r_+ = \max\{0, r\}$ is the positive part of r. And, finally, in literature one can find condition (1.14) *with the choice*

$$p_\nu(r) = \begin{cases} r_+ & \textit{if} \quad r \le \delta, \\ \delta & \textit{if} \quad r > \delta, \end{cases} \tag{1.15}$$

where δ is a positive coefficient related to the wear and hardness of the surface.

We also consider contact condition of the form

$$-\sigma_\nu(t) \in \partial j_\nu\left(t, \dot{u}_\nu(t)\right) \quad \textit{on } \Gamma_{S_3} \times (0,T), \tag{1.16}$$

in which, again, j_ν is a given function and the symbol ∂j_ν denotes the Clarke subdifferential of j_ν with respect the last variable.

In this thesis, the contact condition is given by

$$-\sigma_\nu(t) \in \partial j_\nu \left({}_0^C D_t^\alpha u_\nu(t)\right) \qquad \text{on } \Gamma_{S_3} \times [0,T], \tag{1.17}$$

is called the multivalued normal compliance condition governed by the subdifferential of a nonconvex potential j_ν. This means the stress σ_ν is not solely contingent upon the immediate deformation, but rather encompasses the pre-existing deformations as well.

1.5.2 Condition in the tangent plane

We turn now to the condition in the tangential direction, called also friction condition or friction law. We consider friction law of the form

$$-\boldsymbol{\sigma}_\tau(t) \in h_\tau\left(u_\nu(t) - g_0\right) \partial j_\tau\left(t, \dot{\boldsymbol{u}}_\tau(t)\right) \qquad \text{on } \Gamma_{S_3} \times (0,T), \tag{1.18}$$

in which h_τ and j_τ are given functions and the symbol ∂j_τ denotes the Clarke subdifferential of j_τ with respect to the last variable. Note that the explicit dependence of the function j_τ with respect to the time variable makes the model more general and allows to describes situations when the friction condition depends on the temperature, which plays the role of a parameter. In the particular case when h_τ is a constant, say $h_\tau \equiv 1$, the friction law (1.18) becomes

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau\left(t, \dot{\boldsymbol{u}}_\tau(t)\right) \qquad \text{on } \Gamma_{S_3} \times (0,T). \tag{1.19}$$

Examples of friction laws which can be cast in the subdifferential form (1.18) or (1.19) abound in the literature. The simplest one is the so-called frictionless condition in which the friction force vanishes during the process, i.e.,

$$\boldsymbol{\sigma}_\tau = 0 \qquad \text{on } \Gamma_{S_3} \times (0,T). \tag{1.20}$$

This is an idealization of the process, since even completely lubricated surfaces generate shear resistance to tangential motion. However, (1.20) is a sufficiently good approximation of the reality in some situations. Clearly, the frictionless condition (1.20) is of the form (1.19) with $j_\tau \equiv 0$.

Based on the examples and comments above, next we consider contact problem *involving subdifferential frictional condition of the forms (1.18) and (1.19). Note that all these la model dynamic or quasistatic frictional processes. However, sometimes we shall consider static versions of these laws which are obtained by replacing the tangential velocity with the tangential displacement. Thus, the static version of the friction law (1.18) is given by*

$$-\boldsymbol{\sigma}_\tau(t) \in h_\tau\left(u_\nu(t) - g_0\right) \partial j_\tau\left(t, \boldsymbol{u}_\tau(t)\right) \qquad \text{on } \Gamma_{S_3} \times (0, T), \tag{1.21}$$

and the static version of the friction law (1.19) is given by

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau\left(t, \boldsymbol{u}_\tau(t)\right) \qquad \text{on } \Gamma_{S_3} \times (0, T). \tag{1.22}$$

Also, when the displacement field and the stress field do not depend on time, the friction laws (1.21) and (1.22) read

$$-\boldsymbol{\sigma}_\tau \in h_\tau\left(u_\nu - g_0\right) \partial j_\tau\left(\boldsymbol{u}_\tau\right) \qquad \text{on } \Gamma_{S_3}, \tag{1.23}$$

and

$$-\boldsymbol{\sigma}_\tau \in \partial j_\tau\left(\boldsymbol{u}_\tau\right) \qquad \text{on } \Gamma_{S_3}, \tag{1.24}$$

respectively.

In this thesis, we consider friction law of the form

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau\left({}_0^C D_t^\alpha \boldsymbol{u}_\tau(t)\right) \qquad \text{on } \Gamma_{S_3} \times [0, T], \tag{1.25}$$

where j_τ *represents a given function* ∂j_τ *signifies the Clarke subdifferential of* j_τ*, this suggests the existence of a non-monotonic correlation where the function* j_τ *is not convex.*

This can more accurately capture the historical effects and complex time-dependent behaviors that exist in the actual friction process.

For more details on contact conditions and friction laws see [4].

Chapter 2

Mathematical Tools

In this chapter, we give some reminders about Lebesgue, Sobolev and Hilbert spaces, such as we introduce some functional spaces for thermo mechanics that are needed in the study of the contact problem, we address in the rest of the manuscript. We also provide a brief overview of strongly monotone operators. Banach fixed point theorem, Gronwall lemma. As well as some analytical background on fractional calculus and hemivariational inequality involving fractional operators.

Firsty, we present preliminary material functional analysis which will be used in subsequent chapter. For more details refer to[1, 7, 6].

2.1 Lebesgue spaces L^p

For $p \in [1,\infty[, L^p(\Omega)$ *denotes the space of measurable functions in the Lebesgue sense defined on* Ω *with values in* $\mathbb{R}$

$L^p(\Omega) = \{u : \Omega \to \mathbb{R}$: *u Lebesgue measurable and* $|u|^p$ *Lebesgue integrable on* $\Omega\}$.

It is a Banach space, the norm of which is given by

$$\| u \|_{L^p} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$ *and* $u : \Omega \to \mathbb{R}$ *is measurable, then*

$$\| u \|_{L^\infty} = \sup ess(u) = \inf\{c : |u(x)| \leq c\}.$$

Then $L^\infty(\Omega)$ *space is also Banach space.*

The space $L^2(\Omega)$ endowed with the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx \quad \forall u, v \in L^2(\Omega),$$

is a Hilbert space. In addition, the Cauchy-Schwarz inequality corresponding to the Holder inequality is satisfied, i.e.

$$\int_\Omega |u(x)v(x)|dx \leq \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} .$$

Young's convolution inequality

Suppose $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ and $(\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1)$ with $1 \leqslant p, q, r < +\infty$. Then,

$$\| f * g \|_r \leqslant \| f \|_p \| g \|_q$$

where

$$(f * g)(x) = \int_{\mathbb{R}}^d f(x - y)g(y)\, dy.$$

2.2 Sobolev spaces

Sobolev spaces were introduced at the beginning of the century and made it possible to solve many problems concerning partial differential equations that had previously been unanswered.

We begin with a brief review of some results on the Sobolev space $H^1(\Omega)$ defined by:

$$H^1(\Omega) = \Big\{ u \in L^2(\Omega) \quad | \quad \partial_i u \in L^2(\Omega)\, i = 1, \cdots, d \Big\}.$$

We denote by ∇u the component vector $\partial_i u$.

We have $\nabla u \in L^2(\Omega)^d$ for all $u \in H^1(\Omega)$.

We know that $H^1(\Omega)$ is a Hilbert space for the scalar product:

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\partial_i u, \partial_i v)_{L^2(\Omega)},$$

and the associated norm:

$$\| u \|_{H^1(\Omega)} = (u, u)^{\frac{1}{2}}_{H^1(\Omega)}, \text{ and we write } \| u \|^2_{H^1(\Omega)} = \| u \|^2_{L^2(\Omega)} + \| \nabla u \|^2_{L^2(\Omega)^d} .$$

We have the following results:

$$C^1(\bar{\Omega}) \textit{ is dense in } H^1(\Omega).$$

Theorem 2.2.1 ***(Rellich)***

$$H^1(\Omega) \subset L^2(\Omega) \textit{ with compact injection.}$$

Theorem 2.2.2 ***(Sobolev trace)***

There exists a linear and continuous map $\gamma : H^1(\Omega) \to L^2(\Gamma)$ *such that* $\gamma u = u|_\Gamma$ *for all* $u \in C^1(\bar{\Omega})$.

Remark 2.2.1 *The space* $L^2(\Gamma)$ *above represents the space of real functions on* Γ *which are* L^2 *for the superficial measure* $d\Gamma$. *The map* γ *is called a trace map, it is defined as the density extension of the map* $u \to u|_\Gamma$ *defined for* $u \in C^1(\bar{\Omega})$.

Remark 2.2.2 *Note that the trace map* $\gamma : H^1(\Omega) \to L^2(\Gamma)$ *is a compact operator.*

Definition 2.2.1 *For all* $k \in \mathbb{N}$ *and for all* $p \in [1, +\infty]$, *we define the Sobolev space* $W^{k,p}(\Omega)$ *by*

$$W^{k,p}(\Omega) = \Big\{ u \in L^p(\Omega) \;\; \forall \alpha, \, |\alpha| \leq k; \;\; \exists v_\alpha \in L^p(\Omega), \;\; \textit{such that}\, v_\alpha = D^\alpha u \Big\}.$$

Remark 2.2.3 *We will very often abuse the notation by identifying* $D^\alpha u$ *and* v_α.

The norm on the space $W^{k,p}(\Omega)$ *is given by*

$$\| u \|_{W^{k,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \| D^\alpha u \|_{L^p(\Omega)} \right)^{\frac{1}{p}} & \textit{if} \quad 1 \leq p < \infty, \\ \displaystyle\max_{|\alpha| \leq k} \| D^\alpha u \|_{L^\infty(\Omega)} & \textit{if} \quad p = \infty. \end{cases}$$

For $p = 2$, *we denote by* $H^k(\Omega)$ *the space* $W^{k,2}(\Omega)$ *and the norm is induced by the inner product:*

$$(u, v) = \int_\Omega \sum_{|\alpha| \leq k} D^\alpha u(x) D^\alpha v(x) dx, \quad u, v \in H^k(\Omega).$$

Theorem 2.2.3 *The Sobolev spaces $W^{k,p}(\Omega)$, for $k \in \mathbb{N}$ and $p \in [1, +\infty]$, equipped with the norm $\|\cdot\|$, are Banach spaces. Moreover, the spaces $H^k(\Omega)$, for any integer k, are Hilbert spaces.*

2.3 Function spaces in contact mechanics

Let us introduce the following Hilbert spaces, associated with the mechanical unknowns $\boldsymbol{u}$ and $\boldsymbol{\sigma}$:

$$\begin{cases} H = \Big\{\boldsymbol{u} = (u_i) \quad | \quad u_i \in L^2(\Omega)\Big\} = L^2(\Omega)^d, \\ \mathcal{H} = \Big\{\boldsymbol{\sigma} = (\sigma_{ij}) \quad | \quad \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\Big\} = L^2(\Omega)_s^{d\times d}, \\ H_1 = \Big\{\boldsymbol{u} = (u_i) \quad | \quad u_i \in H^1(\Omega)\Big\} = (H^1(\Omega))^d, \\ \mathcal{H}_1 = \Big\{\boldsymbol{\sigma} \in \mathcal{H} \quad | \quad Div\boldsymbol{\sigma} \in H\Big\}. \end{cases} \tag{2.1}$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces equipped with the following scalar products:

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_H = \displaystyle\int_\Omega u_i v_i dx, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \displaystyle\int_\Omega \sigma_{ij}\tau_{ij} dx, \\ (\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H. \end{cases} \tag{2.2}$$

The norms on the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H, \|\cdot\|_{\mathcal{H}}, \|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

For $v \in H^1(\Omega; \mathbb{R}^d)$, the same notation v is used to represent the trace of v on $\partial\Omega$.

Additionally, we introduce the spaces V by:

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \quad | \quad \boldsymbol{v} = 0 \ \text{on}\ \Gamma_{S_1}\}. \tag{2.3}$$

Here, the condition $\boldsymbol{v} = 0$ a.e. on Γ_{S_1} is in the sense of the trace , that is to say $\gamma\boldsymbol{v} = 0$ a.e. on Γ_{S_1} . Since the trace operator is linear and continuous of $H^1(\Omega; \mathbb{R}^d)$.

Korn's inequality: *Is satisfied on V that is to say, let $mes(\Gamma_{S_1}) > 0$, there exists a constant $c_k > 0$ depending only on Ω and Γ_{S_1} such that*

$$\| \varepsilon(\boldsymbol{v}) \|_{\mathcal{H}} \geq c_k \| \boldsymbol{v} \|_{H^1(\Omega;\mathbb{R}^d)} \quad \forall \boldsymbol{v} \in V. \tag{2.4}$$

A proof of this inequality can be found in ([5] p.79).

The space V is furnished with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}.$$

And

$$\| \boldsymbol{v} \|_V = \| \boldsymbol{\varepsilon}(\boldsymbol{v}) \|_{\mathcal{H}} . \tag{2.5}$$

The norms $\| \boldsymbol{v} \|_V$ and $\| \cdot \|_{H^1(\Omega;\mathbb{R}^d)}$ are equivalent on V, therefore $(V, \| \cdot \|_V)$ is a Hilbert space.

Moreover, under the Sobolev trace therem,

$$\| \boldsymbol{v} \|_{L^2(\Gamma_{S_3};\mathbb{R}^d)} \leq \| \gamma \| \| \boldsymbol{v} \|_V \quad \forall \boldsymbol{v} \in V, \tag{2.6}$$

where $\| \gamma \|$ signifies the norm of the trace operator $\gamma : V \to L(\Gamma_{S_3}; \mathbb{R}^d)$.

Next, we let E be a closed subspace of $H^1(\Omega)$ given by:

$$E = \{\eta \in H^1(\Omega) \quad | \quad \eta = 0 \ \text{on} \ \Gamma_{S_1} \cup \Gamma_{S_2}\}. \tag{2.7}$$

*Moreover, when $\delta \in \mathcal{H}_1$ is a regular function, the following **Green's formula** holds:*

$$(\varphi, \nabla\psi)_{L^2(\Omega)^d} + (div\varphi, \psi)_{L^2(\Omega)} = \int_\Gamma \varphi \cdot \boldsymbol{\nu}\psi da \quad \forall \psi \in H^1(\Omega). \tag{2.8}$$

Poincaré inequality: *There exists a constant C such that*

$$\| \nabla\eta \|_H \geq C \| \eta \|_{H^1(\Omega)} \quad \forall \eta \in E, \tag{2.9}$$

where C is a positive constant depending on the problem data but independent of the solution; its value may change from none line to another.

We define the inner product on E *by:*

$$(\eta, \delta)_E = (\nabla\eta, \nabla\delta)_H \quad \forall \eta, \delta \in E.$$

The norms $\| \cdot \|_E$ *and* $\| \cdot \|_{H^1(\Omega)}$ *are equivalent on* E*, therefore* $(E, \| \cdot \|_E)$ *is a Hilbert space.*

Furthermore, by the Sobolev trace theorem, there exists a constant $\tilde{C}_\Gamma > 0$ *which depends only on* $\Omega, \Gamma_{S_1}, \Gamma_{S_2}$ *and* Γ_{S_3} *such that*

$$\| \eta \|_{L^2(\Gamma_{S_3})} \leq \tilde{C} \| \eta \|_E \quad \forall \eta \in E. \tag{2.10}$$

E^* *is the dual space of* E*. Identifing* $L^2(\Omega)$ *with its own dual, we can write*

$$E \subset L^2(\Omega) \subset E^*.$$

We denote by $\langle \cdot, \cdot \rangle$ *the duality pairing between* E^* *and* E*, and* $\| \cdot \|_{E^*}$ *denotes the norm on* E^**. Also*

$$\langle \theta, \eta \rangle_{E^* \times E} = (\theta, \eta)_{L^2(\Omega)}$$

for $\theta \in L^2(\Omega)$ *and* $\eta \in E$*.*

Let us introduce space $\mathcal{V} = L^2(0, T; V)$ *along with* $\mathcal{V}^* = L^2(0, T; V^*)$ *as its dual, and introduced* $\mathcal{E} = L^2(0, T; E)$ *and its dual space is* $\mathcal{E}^* = L^2(0, T; E^*)$ *respectively, where* $\boldsymbol{\varepsilon} : H_1 \to \mathcal{H}$ *and* $Div : \mathcal{H}_1 \to H$ *are respectively the deformation and divergence operators, defined by*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Since the boundary Γ *is Lipschitzian, the exterior normal vector* $\boldsymbol{\nu}$ *to the boundary is defined a.e. For any vector field* $\boldsymbol{v} \in H_1$ *we use the notation* $\boldsymbol{v}$ *to denote the trace* $\gamma\boldsymbol{v}$ *of* $\boldsymbol{v}$ *on* Γ*. Recall that the trace map* $\gamma : H_1 \to L^2(\Gamma)^d$ *is linear and continuous, but is not surjective. Let* H_Γ^* *be the dual of* H_Γ*, and* $(\cdot, \cdot)$ *be the duality product between* H_Γ^* *and* H_Γ*. For any* $\boldsymbol{\sigma} \in \mathcal{H}_1$*, there exists an element* $\boldsymbol{\sigma\nu} \in H_\Gamma^*$ *such that:*

$$(\boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}) = (\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \boldsymbol{v})_H \;\; \forall \boldsymbol{v} \in H_1. \tag{2.11}$$

Furthermore, if $\boldsymbol{\sigma}$ *is regular, we have the formula*

$$(\boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}) = \int_{\Gamma} \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} da \ \ \forall \boldsymbol{v} \in H_1. \tag{2.12}$$

So, for $\boldsymbol{\sigma}$ *fairly regular we have the following* ***Green's formula****:*

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_{\Gamma} \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} da \ \ \forall \boldsymbol{v} \in H_1, \tag{2.13}$$

where da is an element of surface measurement.

2.4 Reminders on Hilbert spaces

Let H *be a real vector space and* $(.,.)_H$ *be a scalar product on* H*, that is,* $(.,.)_H : H \times H \to \mathbb{R}$ *is a symmetric and positive definite bilinear map. We relied on the thesises[35, 33].*

We denote by $\| \cdot \|_H$ *the application of* $H \to \mathbb{R}_+$ *defined by:*

$$\| u \|_H = (u, u)_H^{\frac{1}{2}}, \tag{2.14}$$

and we recall that $\| \cdot \|_H$ *is a norm on* H *that satisfies the Cauchy-Schwartz inequality:*

$$|(u, v)_H| \leq \| u \|_H \| v \|_H, \ \ \forall u, v \in H. \tag{2.15}$$

We say that H *is a Hilbert space if* H *is complete for the norm defined by (2.14). Let* H^* *be the dual space of* H*, that is, the space of linear and continuous functionals on* H *equipped with the norm:*

$$\| \eta \|_{H^*} = \sup_{v \in H - \{0\}} \frac{< \eta, v >_{H^* \times H}}{\| v \|_H},$$

where $\langle ., . \rangle_{H^* \times H}$ *represents the duality between* H^* *and* H*.*

Theorem 2.4.1 ***(Riesz-Fréchet representation theorem)****:*

Let H *be a Hilbert space and let* H^* *be its dual space. Then, for all* $\phi \in H^*$ *there exists unique* $f \in H$ *such that*

$$\langle \phi, v \rangle_{H^* \times H} = (f, v)_H \quad \forall v \in H.$$

By additionally

$$\| \phi \|_{H^*} = \| f \|_H .$$

The importance of this theorem is that any continuous linear form on H can be represented using the scalar product. The map $\phi \mapsto f$ is an isometric isomorphism that allows us to identify H and H^*.

Now, we will recall some notions of nonlinear analysis in Hilbert spaces and some results concerning variational evolution equations and inequalities that are used in the study of mechanical problems.

Linear operators

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be two normed spaces and let $L : X \to Y$ be an operator.

We say that L is linear operator if

$$L(\alpha u_1 + \beta u_2) = \alpha u_1 + \beta u_2 \quad \forall u_1, u_2 \in X, \quad \alpha, \beta \in \mathbb{R}.$$

A linear operator is continuous if and only if it is bounded, i.e, there exists a constant $M > 0$ such that

$$\| L(u) \|_Y \leq M \| u \|_X \quad \forall u \in X.$$

We use the notation $\mathcal{L}(X, Y)$ for the set of all continuous linear operators from X to Y.

Nonlinear operators

In the study of the mechanical problems approached in the thesis, the non-linearity of the operators used implies the use of properties that we described below.

We begin here with a brief reminder of the strongly monotonic and Lipschitz operators. For this, we consider a Hilbert space X endowed with the scalar product $(\cdot, \cdot)_X$ and the associated norm $\| \cdot \|_X$.

Let $A : X \to X$ a non-linear operator

Definition 2.4.1 *Let $A : X \to X$ be a nonlinear operator, the operator A is said:*

1. *Monotone: if*

$$(Au - Av, u - v)_X \geq 0, \quad \forall u, v \in X.$$

2. *Strongly monotonic: if there exists $m > 0$ such that*

$$(Au - Av, u - v)_X \geq m \parallel u - v \parallel_X^2, \quad \forall u, v \in X.$$

3. *Lipschitz: if there exists $L > 0$ such that*

$$\parallel Au - Av \parallel_X \leq L \parallel u - v \parallel_X, \quad \forall u, v \in X.$$

In the study of evolutionary nonlinear equations we will consider operators defined on a normed space with values in its dual. X^* the dual space of X, denoting by $(\cdot, \cdot)_{X^* \times X}$ for the duality product between X^* and X.

Definition 2.4.2 *Let $A : X \to X^*$ an operator defined on X. Operator A is said*

1. *Monotonous: if*

$$(Au - Av, u - v)_{X^* \times X} \geq 0 \quad \forall u, v \in X.$$

2. *Hemicontinuous: if*

$$\forall u, v \in X \text{ the application } t \to A(u + tv) : \mathbb{R} \to X^* \text{ is continuous.}$$

The Banach fixed-point theorem will be used several times in this work to prove the existence of solutions to variational problems in contact mechanics.

Theorem 2.4.2 (Banach Fixed Point Theorem)

Let K be a closed and nonempty subset of the Banach space $(X, \parallel \cdot \parallel_X)$. Suppose that $\mathbf{\Pi} : K \to K$ is a contraction, i.e. there exists $c \in]0, 1[$ such that

$$\parallel \mathbf{\Pi}(u) - \mathbf{\Pi}(v) \parallel_X \leq c \parallel u - v \parallel_X \quad \forall u, v \in K. \tag{2.16}$$

Then, there exists a unique element $u \in K$ such that $\mathbf{\Pi}(u) = u$, i.e., has a unique fixed point in K.

Proof.

Let $u_0 \in K$ *and let* $\{u_n\}$ *be the sequence defined by*

$$u_{n+1} = \mathbf{\Pi} u_n \quad \forall n = 0, 1, 2, \ldots$$

Since $\mathbf{\Pi} : K \to K$ *, the sequence* $\{u_n\}$ *is well-defined. Let us first show that* $\{u_n\}$ *is a Cauchy sequence. Using the fact that* $\mathbf{\Pi}$ *is a contraction, we have*

$$|u_{n+1} - u_n|_X \leq c|u_n - u_{n-1}|_X \leq \cdots \leq c^n |u_1 - u_0|_X.$$

Then for all $m > n \geq 1$,

$$\begin{aligned} |u_m - u_n|_X &\leq \sum_{j=0}^{m-n-1} |u_{n+j+1} - u_{n+j}|_X \\ &\leq \sum_{j=0}^{m-n-1} c^{n+j} |u_1 - u_0|_X \\ &\leq \frac{c^n}{1-c} |u_1 - u_0|_X. \end{aligned}$$

Since $c \in [0, 1)$, *it follows that*

$$|u_m - u_n|_X \to 0 \quad \textit{as} \quad m, n \to \infty.$$

Thus, $\{u_n\}$ *is a Cauchy sequence and has a limit* $u \in K$, *since* K *is a closed subset of the Banach space* X. *Furthermore, since* $u_n \to u$ *in* X, *it follows from (2.4.2) that* $\mathbf{\Pi} u_n \to \mathbf{\Pi} u$ *in* X.

Consequently, by taking the limit in $u_{n+1} = \mathbf{\Pi} u_n$, *we have* $\mathbf{\Pi} u = u$, *which concludes the existence part of the theorem.*

Suppose that $u_1, u_2 \in K$ *are fixed points of* $\mathbf{\Pi}$. *Then from* $u_1 = \mathbf{\Pi} u_1$ *and* $u_2 = \Pi u_2$, *we have*

$$u_1 - u_2 = \mathbf{\Pi} u_1 - \mathbf{\Pi} u_2,$$

and since $\mathbf{\Pi}$ *is a contraction*

$$|u_1 - u_2|_X = |\mathbf{\Pi} u_1 - \mathbf{\Pi} u_2|_X \leq c|u_1 - u_2|_X,$$

which implies that $|u_1 - u_2|_X = 0$, *since* $c \in [0, 1)$.*Therefore, if* $\mathbf{\Pi}$ *admits a fixed point, which concludes the proof.* ■

Corollary 2.4.1 *Take* $\mathbf{\Pi} : L^2(0,T;X) \to L^2(0,T;X)$ *as the operator satisfying*

$$\|(\mathbf{\Pi}\mu_1)(t) - (\mathbf{\Pi}\mu_2)(t)\|_X^2 \leqslant c \int_0^t \|\mu_1(s) - \mu_2(s)\|_X^2 ds \tag{2.17}$$

for every $\mu_1, \mu_2 \in L^2(0,T;X)$,*a.e.* $t \in (0,T)$. *Then,* $\mathbf{\Pi}$ *admits a unique fixed point in* $L^2(0,T;X)$, *that is, there is a unique* $\mu^* \in L^2(0,T;X)$ *which satsifies* $\mathbf{\Pi}\mu^* = \mu^*$.

For the operator $\mathbf{\Pi}^m : K \to K$ *defined by the relation*

$$\mathbf{\Pi}^m = \mathbf{\Pi}(\mathbf{\Pi}^{m-1}) \qquad m \geq 2,$$

we have the following version of the fixed point theorem.

Theorem 2.4.3 *Let* K *be a closed and nonempty subset of the Banach space* $(X, \|\cdot\|_X)$. *Suppose that* $\mathbf{\Pi}^m : K \to K$ *is a contraction for* m *a positive integer. Then* $\mathbf{\Pi}$ *has a unique fixed point in* K.

Proof.

According to theorem(2.4.2), $\mathbf{\Pi}$ *has a unique fixed point* $u \in K$. *From* $\mathbf{\Pi}^m(u) = u$, *we obtain:*

$$\mathbf{\Pi}^m(\mathbf{\Pi}(u)) = \mathbf{\Pi}(\mathbf{\Pi}^m(u)) = \mathbf{\Pi}u.$$

Thus, $\mathbf{\Pi}u \in K$ *is also a fixed point of* $\mathbf{\Pi}^m$. *Since* $\mathbf{\Pi}^m$ *admits a unique fixed point, we have:*

$$\mathbf{\Pi}u = u$$

that is, u *a fixed point of* $\mathbf{\Pi}^m$. ■

Definition 2.4.3 *A bilinear form* $a : X \times X \longrightarrow \mathbb{R}$ *is said to be*

- *Continuous : if there exists a real* $M > 0$ *such that:*

$$|a(u,v)| \leq M \parallel u \parallel_X \parallel v \parallel_X, \quad \forall u, v \in X.$$

- *Coercive : if there exists a constant* $m > 0$ *such that:*

$$a(u,u) \geq m \parallel u \parallel_X^2, \quad \forall u \in X.$$

Under differentiability

We consider throughout this paragraph that X is a Hilbert space and K a subset of the space X.

we consider a function φ defined on a real vector space X and with value in $]-\infty, +\infty]$. Such a function is said to be proper if it is not identically equal to $+\infty$, i.e. if there exists $v_0 \in X$ such that $\varphi(v_0) < +\infty$. The function φ is said to be convex if

$$\varphi(tv + (1-t)w) \leq t\varphi(v) + (1-t)\varphi(w) \quad \forall v, w \in X, \quad t \in [0,1].$$

Definition 2.4.4 *The indicator function of K is the function $\Psi_K : X \to]-\infty, +\infty]$ defined by*

$$\Psi_K = \begin{cases} 0 & if \quad u \in K, \\ +\infty & if \quad u \notin K. \end{cases}$$

Definition 2.4.5 *Let j be a function $j : X \longrightarrow \mathbb{R}$ and u be an element of the space X such that $j(u) \neq \pm\infty$.*

A function j is said to be Gateaux-differentiable at the point u if there is an element $\nabla j(u) \in X$ such that:

$$\lim_{t \to 0} \frac{j(u+tv) - j(u)}{t} = (\nabla j(u), v)_X, \quad \forall v \in X. \tag{2.18}$$

The element $\nabla j(u)$ is called the differential in the sense of Cakes of j in u. The function j is said to be Gateaux-differentiable if it is Gateaux-differentiable at any point of X, in this case the operator $\nabla j : X \to X$ which associates each element $u \in X$ with the element $\nabla j(u)$ is called the gradient operator of j.

The subdifferential of the function j at u, denoted $\partial j(u)$, is the set defined by

$$\partial j(u) = \{\dot{u} \in X^* \mid \; j(v) \geq j(u) + (\dot{u}, v - u), \; \forall v \in K\}. \tag{2.19}$$

The bracket $(\cdot, \cdot)$ denotes the duality between X^* and X. Any element $\dot{u}$ of the set $\partial j(u)$ is called a subgradient of the function j at u. The function j is said to be subdifferentiable at u if $\partial j(u) \neq \emptyset$. It is said to be subdifferentiable if it is subdifferentiable at every point u of the space X. We can characterize the subdifferential $\partial \Psi_K$ of an *indicator function* Ψ_K *of a non-empty convex set.*

$$\partial\Psi_K = \{\dot{u} \in X^* \mid \ (\dot{u}, v-u) \leq 0, \ \forall v \in K\}. \tag{2.20}$$

2.5 Preliminaries on fractional calculus

In this section, we recall some known definitions and properties on nonlinear analysis and fractional calculus, which can be found in [2, 4, 32]. Then, we investigate a general elliptic time-dependent hemivariational inequality in the framework of evolution triple of spaces. For this inequality, we will provide a result on existence of solutions. The method of proof relies on the Rothe method. See[4, 32].

Definition 2.5.1 *(The RiemannLiouville fractional integral).*

Let X *be a Banach space and* $(0,T)$ *be a finite time interval. The Riemann-Liouville fractional integral of order* $\alpha > 0$ *for a given function* $f \in L^1(0,T;X)$ *is defined by*

$${}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \quad \text{for all } t \in (0,T),$$

where $\Gamma(\cdot)$ *stands for the Gamma function defined by*

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt.$$

To complement the definition, we set ${}_0I_t^0 = I$*, where* I *is the identity operator, which means that* ${}_0I_t^0 f(t) = f(t)$ *for a.e.* $t \in (0,T)$.

Definition 2.5.2 *(The Caputo derivative of order,* $0 < \alpha \leq 1$*).*

Let X *be a Banach space,* $0 < \alpha \leq 1$*, and* $(0,T)$ *be a finite time interval. For a given function* $f \in AC(0,T;X)$*, the Caputo fractional derivative of* f *is defined by:*

$${}_0^C D_t^{\alpha} f(t) = {}_0I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds \quad \text{for all } t \in (0,T).$$

The notation $AC(0,T;X)$ *refers to the space of all absolutely continuous functions from* $(0,T)$ *into* X.

It is obvious that if $\alpha = 1$*, then the Caputo derivative reduces to the classical* *first-order derivative, that is, we have:*

$$ {}_0^C D_t^1 f(t) = I f'(t) = f'(t) \quad \textit{for a.e. } t \in (0,T). $$

Proposition 2.5.1 *Let X be a Banach space and $\alpha, \beta > 0$. Then the following statements hold:*

(i) For $y \in L^1(0,T;X)$, we have

$$ {}_0I_t^\alpha {}_0I_t^\beta y(t) = {}_0I_t^{\alpha+\beta} y(t) \quad \textit{for } t \in (0,T). $$

(ii) For $y \in AC(0,T;X)$ and $\alpha \in (0,\alpha]$, we have

$$ {}_0I_t^\alpha {}_0^C D_t^\alpha y(t) = y(t) - y(0) \quad \textit{for a.e. } t \in (0,T). $$

(iii) For $y \in L^1(0,T;X)$, we have

$$ {}_0^C D_t^\alpha {}_0I_t^\alpha y(t) = y(t) \quad \textit{for a.e. } t \in (0,T). $$

Definition 2.5.3 *(The Clarke generalized directional derivative and the generalized gradient).*

Let $\Psi : X \to \mathbb{R}$ be a locally Lipschitz function. We denote by $\Psi^0(u,v)$ the Clarke generalized directional derivative of Ψ at the point $x \in X$ in the direction $v \in X$ defined by

$$ \Psi^0(x,v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\Psi(y+\lambda v) - \Psi(y)}{\lambda}. $$

Additionally, the Clarke subdifferential of Ψ at x, denoted as $\partial\Psi(x)$, is defined by a subset of the space X^*, and

$$ \partial\Psi(x) = \left\{ \zeta \in X^* \mid \Psi^0(x,v) \geq \langle \zeta, v \rangle_{X^* \times X} \quad \forall v \in X \right\}. $$

A locally Lipschitz function Ψ is called regular in the sense of Clarke at $x \in X$ if for all $v \in X$ the one-sided directional derivative $\Psi'(x;v)$ exists and satisfies $\Psi^0(x;v) = \Psi'(x;v)$ for all $v \in X$.

For the hemivariational inequality we make the following hypotheses

For the function $j : \Gamma_{S_3} \times \mathbb{R}^d \to \mathbb{R}$ *is such that*

$$\begin{cases} (a)\ j(\cdot,\eta,\xi) \text{ is measurable on } \Gamma_{S_3} \text{ for all } \eta,\xi \in \mathbb{R}^d \\ \quad \text{and there exists } e \in L^2(\Gamma_{S_3};\mathbb{R}^d) \text{ such that for all } w \in L^2(\Gamma_{S_3};\mathbb{R}^d) \\ \quad \text{we have } j(\cdot,w(\cdot),e(\cdot)) \in L^1(\Gamma_{S_3}). \\ (b)\ j(x,\cdot,\xi) \text{ is continuous on } \mathbb{R}^d \text{ for all } \xi \in \mathbb{R}^d,\ \text{a.e. } x \in \Gamma_{S_3} \\ \quad \text{and } j(x,\eta,\cdot) \text{ is locally Lipschitz continuous on } \mathbb{R}^d \text{ for all } \eta \in \Gamma_{S_3}, \\ \quad \text{for a.e. } x \in \Gamma_{S_3}. \\ (c)\ \| \partial j(x,\eta,\xi) \|_{\mathbb{R}^d} \leq \bar{c}_0 + \bar{c}_1 \| \xi \|_{\mathbb{R}^d} + \bar{c}_2 \| \eta \|_{\mathbb{R}^d},\ \text{for all } \eta,\xi \in \mathbb{R}^d, \\ \quad \text{for a.e. } x \in \Gamma_{S_3} \text{ with } \bar{c}_0,\bar{c}_1,\bar{c}_2 \geq 0. \\ (d)\ \text{Either } j(x,\eta,\cdot) \text{ or } -j(x,\eta,\cdot) \text{ is regular , for all} \\ \quad \eta \in \mathbb{R}^d,\ \text{a.e.} x \in \Gamma_{S_3}. \end{cases} \tag{2.21}$$

Lemma 2.5.1 *Let* X_1 *and* X_2 *be Banach spaces,* $(x_1,x_2) \in X_1 \times X_2$, $g : X_1 \to \mathbb{R}$ *be locally Lipschitz near* x_1 *and* $h : X_2 \to \mathbb{R}$ *be locally Lipschitz near* x_2.

Let $\theta : X_1 \times X_2 \to \mathbb{R}$ *be given by*

$$\theta(y_1,y_2) = g(y_1) + h(y_2) \quad \text{for all } y_1,y_2 \in X_1 \times X_2. \tag{2.22}$$

Then θ *is locally Lipschitz near* (x_1,x_2). *Moreover, if* g *(respectively,* $-g$*) is regular at* x_1 *and* h *(respectively,* $-h$*) is regular at* x_2 *then* θ *(respectively,* $-\theta$*) is regular at* (x_1,x_2) *and*

(i) $\theta^0(x_1,x_2;v_1,v_2) = g^0(x_1;v_1) + h^0(x_2;v_2)$ *for all* $(v_1,v_2) \in X_1 \times X_2$.

(ii) $\partial\theta(x_1,x_2) = \partial g(x_1) \times \partial h(x_2)$.

We introduce the functional $J : L^2(\Gamma_{S_3};\mathbb{R}^d) \times L^2(\Gamma_{S_3};\mathbb{R}^d) \to \mathbb{R}$ *defined by*

$$J(w,u) = \int_{\Gamma_{S_3}} j(x,w(x),u(x))d\Gamma \quad \text{for } w,u \in L^2(\Gamma_{S_3};\mathbb{R}^d). \tag{2.23}$$

Corollary 2.5.1 *Assume that* (2.21)*(a)-(d) hold. Then the functional* J *defined by* (2.23) *satisfies*

(i) J *is well defined and proper on* $L^2(\Gamma_{S_3};\mathbb{R}^d) \times L^2(\Gamma_{S_3};\mathbb{R}^d)$.

(ii) $J(w,\cdot)$ *is Lipschitz continuous on bounded subsets of* $L^2(\Gamma_{S_3};\mathbb{R}^d)$.

(iii) *For all* $w,u,v \in L^2(\Gamma_{S_3};\mathbb{R}^d)$, *we have*

$$J^0(w,u;v) \le \int_{\Gamma_{S_3}} j^0(x,w(x),u(x);v(x))d\Gamma.$$

(iv) *For all* $w,u \in L^2(\Gamma_{S_3};\mathbb{R}^d)$, *we have*

$$\partial J(w,u) \subset \int_{\Gamma_{S_3}} \partial j(x,w(x),u(x);v(x))d\Gamma.$$

(v) *For all* $w,u \in L^2(\Gamma_{S_3};\mathbb{R}^d)$, *we have*

$$\| \partial J(w,u)_{L^2(\Gamma_{S_3};\mathbb{R}^d)} \| \le c_0 + c_1 \| u \|_{L^2(\Gamma_{S_3},\mathbb{R}^d)} + c_2 \| w \|_{L^2(\Gamma_{S_3},\mathbb{R}^d)}$$

with $c_0 = \sqrt{3mes(\Gamma_{S_3})}\bar{c}_0, \quad c_1 = \sqrt{3}\bar{c}_1, \quad c_2 = \sqrt{3}\bar{c}_2.$

(vi) *If, in addition,* (2.21)*(d) is satisfied, then (iii) and (iv) hold with equalities.*

(vii) *If, in addition,* (2.21)*(d) is satisfied, then* $J(w,\cdot)$ *or* $-J(w,\cdot)$ *is regular on* $L^2(\Gamma_{S_3};\mathbb{R}^d)$ *for all* $w \in L^2(\Gamma_{S_3};\mathbb{R}^d)$, *respectively.*

Let $V \subset H \subset V^*$ *be an evolution triple of spaces, where* V *is a reflexive and separable Banach space,* H *is a separable Hilbert space, and the embedding* $V \to H$ *is dense and continuous. Let* W *be another separable and reflexive Banach space,* $\mathcal{W} = L^2(0,T;W)$ *and* $\mathcal{W}^* = L^2(0,T;W^*)$.

Given bounded linear operators $A,B : V \to V^*, M : V \to W$, *a locally Lipschitz functional* $J : W \to \mathbb{R}, f \in \mathcal{V}^*$ *and* $u_0 \in V$.

We consider the generalized hemivariational inequality of the following form:

Problem 2.5.1 *Find* $u \in \mathcal{V}$ *such that*

$$\langle Au(t) + B(u_0 +_0 I_t^\alpha u(t)), \boldsymbol{v}\rangle + J^0(M(u_0 +_0 I_t^\alpha u(t)); M\boldsymbol{v}) \ge \langle f(t), \boldsymbol{v}\rangle$$

for all $\boldsymbol{v} \in V$, *a.e.* $t \in (0,T)$.
Here, ${}_0I_t^\alpha u(t)$ *denotes the fractional integral of order* $\alpha > 0$ *of* u. *It is useful to observe that Problem 2.5.1 has another equivalent operator inclusion formulation.*

Problem 2.5.2 *Find $u \in \mathcal{V}$ such that*

$$Au(t) + B(u_0 +_0 I_t^\alpha u(t)) + M^* \partial J(M(u_0 +_0 I_t^\alpha u(t))) \ni f(t)$$

for a.e. $t \in (0,T)$.

Definition 2.5.4 *We say that an element $u \in \mathcal{V}$ is called a solution to Problem 2.5.1 (or Problem 2.5.2) if and only if there exists $\xi \in \mathcal{E}^*$ such that*

$$Au(t) + B(u_0 +_0 I_t^\alpha u(t)) + M^* \xi(t) = f(t) \quad \text{for a.e. } t \in (0,T)$$

with $\xi(t) \in \partial J(M(u_0 +_0 I_t^\alpha u(t)))$ for a.e. $t \in (0,T)$.

Remark 2.5.1 *We point out that if $\alpha = 1$, then the generalized hemivariational inequality in Problem 2.5.1 reduces to the following classical hemivariational inequality: find an element $u \in \mathcal{V}$ such that*

$$\langle Au(t) + B(u_0 + \int_0^t u(s)ds), \boldsymbol{v}\rangle + J^0\left(M(u_0 + \int_0^t u(s)ds); M\boldsymbol{v}\right) \geq \langle f(t), \boldsymbol{v}\rangle$$

for all $\boldsymbol{v} \in V$, a.e. $t \in (0,T)$.

We impose the following assumptions on the data of Problem 2.5.1.

$H(A)$: $A \in \mathcal{L}(V, V^*)$ is coercive , i.e. , there exists a constant $m_A > 0$ such that

$$\langle Av, v\rangle \geq m_A \parallel v \parallel^2 \quad \text{for all } v \in V.$$

$H(B)$: $B \in \mathcal{L}(V, V^*)$.

$H(J)$: $J : W \to \mathbb{R}$ is locally Lipschitz and there exists $m_j > 0$ such that

$$\parallel \partial J(v) \parallel_{W^*} \leq m_j(1 + \parallel v \parallel_W) \quad \text{for all } v \in W.$$

$H(M)$: $M \in \mathcal{L}(V, W)$ is compact.

$H(f)$: $f \in C(0,T; V^*)$.

Theorem 2.5.1 .

Assume that $H(A), H(B), H(f), H(J)$ *and* $H(M)$ *hold. Let* $\{\tau_n\}$ *be a sequence such that* $\tau_n \longrightarrow 0$, *as* $n \longrightarrow \infty$. *Then, for a subsequence still denoted by* τ, *we have*
$\bar{u}_\tau \longrightarrow u$ *weakly in* $\mathcal{V}$ *and* $\xi_\tau \longrightarrow \xi$ *weakly in* $\mathcal{W}^*$ *as* $\tau \longrightarrow 0$,
where $(u, \xi) \in \mathcal{V} \times \mathcal{E}^*$ *is a solution to Problem2.5.1 in the sense of Definition2.5.4.*

for all $\boldsymbol{v} \in V$, *a.e.* $t \in (0, T)$,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega.$$

2.6 Gronwall Lemma

We now recall a generalized version of the Gronwall inequality that has been established in [4].

Lemma 2.6.1 *Let a,g be non-decreasing and* $g(t) \leqslant C$ *for all* $t \in [0, T]$
and let $0 < \alpha \leqslant 1$.
if $x \in L^\infty_+[0, T]$ *satisfies the inequality*

$$x(t) \leqslant a(t) + g(t) \int_0^t (t-s)^{\alpha-1} x(s) ds, \qquad \in [0, T],$$

then

$$x(t) \leqslant a(t) E_\alpha[g(t)\Gamma(\alpha) t^\alpha],$$

where E_α *is the Mittag-Leffler function defined by* $E_\alpha(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$.

Chapter 3

Study of thermo-viscoelastic problem with time fractional

In this chapter, we consider a mathematical model in a quasistatic process of a contact problem thermo-viscoelastic with normal compliance and friction, For this problem, the contact is modeled by friction. Using Green's formula, we propose a variational formulation of the thermo-mechanical problem and then present an existence and uniqueness result of the solution. The proofs are based on the hemivariational inequality and Banach fixed point.

3.1 Position of problem

Problem $\mathcal{P}_1$

Find a displacement field $\boldsymbol{u} : \Omega\times]0,T[\longrightarrow \mathbb{R}^d$, *a stress* $\boldsymbol{\sigma} = \Omega\times]0,T[\longrightarrow \mathbb{S}^d$ *and a temperature* $\vartheta : \Omega\times]0,T[\longrightarrow \mathbb{R}_+$ *such that for all* $t \in (0,T)$,

$$\begin{aligned}
\boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}\left({}_0^C D_t^\alpha \boldsymbol{u}(t)\right) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \\
&\quad + \int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \vartheta(s))\, ds && \text{in } \Omega, && (3.1)\\
Div\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) &= \boldsymbol{0} && \text{in } \Omega, && (3.2)\\
\boldsymbol{u}(t) &= \boldsymbol{0} && \text{on } \Gamma_{S_1}, && (3.3)\\
\boldsymbol{\sigma}(t)\boldsymbol{\nu} &= \boldsymbol{f}_2(t) && \text{on } \Gamma_{S_2}, && (3.4)\\
-\sigma_\nu(t) &\in \partial j_\nu\left({}_0^C D_t^\alpha u_\nu(t)\right) && \text{on } \Gamma_{S_3}, && (3.5)\\
-\boldsymbol{\sigma}_\tau(t) &\in \partial j_\tau\left({}_0^C D_t^\alpha \boldsymbol{u}_\tau(t)\right) && \text{on } \Gamma_{S_3}, && (3.6)\\
{}_0^C D_t^\alpha \vartheta(t) &- div(K_c \nabla\vartheta(t)) \\
&= -c_{ij}\frac{\partial\left({}_0^C D_t^\alpha u_i(t)\right)}{\partial x_j} + q(t) && \text{in } \Omega, && (3.7)\\
-k_{ij}\frac{\partial\vartheta(t)}{\partial x_j}\nu_i &= k_e(\vartheta(t) - \vartheta_R) && \text{on } \Gamma_{S_3}, && (3.8)\\
\vartheta(t) &= 0 && \text{on } \Gamma_{S_1}\cup\Gamma_{S_2}, && (3.9)\\
\boldsymbol{u}(0) &= \boldsymbol{u}_0 && \text{in } \Omega, && (3.10)\\
\vartheta(0) &= \vartheta_0 && \text{in } \Omega. && (3.11)
\end{aligned}$$

Here (3.1) represents the time-fractional Kelvin-Voigt thermo-viscoelastic constitutive law of Caputo type. Note that since the contact process is assumed to be quasistatic, the acceleration term is negligible with equilibrium equation (3.2), where $\boldsymbol{f}_0$ *denotes the time dependent density of volume forces. where the displacement and traction boundary conditions are (3.3) and (3.4), respectively. Equation (3.5) represents the normal compliance contact condition. The friction law of contact is considered in (3.6). (3.7) represents the evolution of the heat equation . (3.8)and (3.9) represent limits boundary conditions for the temperature on* Γ. *Finally, (3.10) and (3.11) represent the initial condition.*

To obtain a variational formulation of the mechanical problem $\mathcal{P}_1$, *we need to in-*

troduce some assumptions about the data.

For the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$

$$\begin{cases} \textit{(i) } \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = a(\boldsymbol{x})\boldsymbol{\varepsilon} \textit{ for a.e. } \boldsymbol{x} \in \Omega \textit{ and all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \textit{(ii) } a(\boldsymbol{x}) = (a_{ijkl}(\boldsymbol{x})) \textit{ with } a_{ijkl} \in L^\infty(\Omega), \\ \textit{(iii) } a_{ijkl}(\boldsymbol{x})\varepsilon_{ij}\varepsilon_{kl} \geqslant m_a \parallel \boldsymbol{\varepsilon} \parallel^2_{\mathbb{S}^d} \textit{ for a.e. } \boldsymbol{x} \in \Omega, \\ \quad \textit{and all } \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d \textit{ with } m_a > 0. \end{cases} \tag{3.12}$$

For the elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$

$$\begin{cases} \textit{(i) } \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = b(\boldsymbol{x})\boldsymbol{\varepsilon} \textit{ for a.e. } \boldsymbol{x} \in \Omega \textit{ and all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \textit{(ii) } b(\boldsymbol{x}) = (b_{ijkl}(\boldsymbol{x})) \textit{ with } b_{ijkl} \in L^\infty(\Omega). \\ \textit{(iii) } b_{ijkl}(\boldsymbol{x})\varepsilon_{ij}\varepsilon_{kl} \leqslant c_b \parallel \boldsymbol{\varepsilon} \parallel_{\mathbb{S}^d} \textit{ for a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{3.13}$$

For the long memory operator $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$

$$\begin{cases} \textit{(i) there exists a positive constant } L_{\mathcal{G}} \textit{ which satisfies} \\ \quad \parallel \mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \vartheta_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \vartheta_2) \parallel_{\mathbb{S}^d} \leq L_{\mathcal{G}}(\parallel \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \parallel_{\mathbb{S}^d} + |\vartheta_1 - \vartheta_2|), \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \vartheta_1, \vartheta_2 \in \mathbb{R} \textit{ and a.e. } \boldsymbol{x} \in \Omega, \\ \textit{(ii) } \mathcal{G}(\cdot, \boldsymbol{\varepsilon}, \vartheta) \textit{ is measurable on } \Omega, \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d \textit{ and } \vartheta \in \mathbb{R}, \\ \textit{(iii) } \mathcal{G}(\boldsymbol{x}, \mathbf{0}, 0) = \mathbf{0} \textit{ a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{3.14}$$

For the potential function $j_\tau : \Gamma_{S_3} \times \mathbb{R}^d \to \mathbb{R}$

$$\begin{cases} \textit{(i) } j_\tau(\cdot, \boldsymbol{\xi}) \textit{ is measurable on } \Gamma_{S_3} \textit{ for all } \boldsymbol{\xi} \in \mathbb{R}^d \\ \quad \textit{and there exists } \boldsymbol{e} \in L^2(\Gamma_{S_3}; \mathbb{R}^d) \textit{ which satisfies } j_\tau(\cdot, \boldsymbol{e}(\cdot)) \in L^1(\Gamma_{S_3}), \\ \textit{(ii) } j_\tau(\boldsymbol{x}, \cdot) \textit{ is Lipschitz continuous on } \mathbb{R}^d \textit{ for a.e. } \boldsymbol{x} \in \Gamma_{S_3}, \\ \textit{(iii) } |\partial j_\tau(\boldsymbol{x}, \boldsymbol{\xi})| \leq \bar{c}_0 \textit{ for a.e. } \boldsymbol{x} \in \Gamma_{S_3}, \textit{ all } \boldsymbol{\xi} \in \mathbb{R}^d \textit{ with } \bar{c}_0 \geq 0, \\ \textit{(iv) } j^0_\tau(\boldsymbol{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j^0_\tau(\boldsymbol{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leq \bar{\beta}_1 \parallel \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \parallel^2_{\mathbb{R}^d} \\ \quad \textit{for a.e. } \boldsymbol{x} \in \Gamma_{S_3}, \textit{ all } \boldsymbol{\xi}_i \in \mathbb{R}^d, i = 1, 2 \textit{ with } \bar{\beta}_1 \geq 0, \\ \textit{(v) } j_\tau(\boldsymbol{x}, \cdot) \textit{ or } -j_\tau(\boldsymbol{x}, \cdot) \textit{ is regular , for a.e. } \boldsymbol{x} \in \Gamma_{S_3}. \end{cases} \tag{3.15}$$

For the potential function $j_\nu : \Gamma_{S_3} \times \mathbb{R}^d \to \mathbb{R}$

$$\begin{cases} \textit{(i) } j_\nu(\cdot, r) \textit{ is measurable on } \Gamma_{S_3} \textit{ for all } r \in \mathbb{R}^d \\ \quad \textit{and there exists } \boldsymbol{e} \in L^2(\Gamma_{S_3}; \mathbb{R}^d) \textit{ which satisfies } j_\nu(\cdot, \boldsymbol{e}(\cdot)) \in L^1(\Gamma_{S_3}), \\ \textit{(ii) } j_\nu(\boldsymbol{x}, \cdot) \textit{ is Lipschitz continuous on } \mathbb{R} \textit{ for a.e. } \boldsymbol{x} \in \Gamma_{S_3}, \\ \textit{(iii) } |\partial j_\nu(\boldsymbol{x}, r)| \leq \bar{c}_1 \textit{for a.e. } \boldsymbol{x} \in \Gamma_{S_3}, \ \textit{all } r \in \mathbb{R} \textit{ with } \bar{c}_1 \geq 0, \\ \textit{(iv) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \leq \bar{\beta}_2 \parallel r_1 - r_2 \parallel_{\mathbb{R}}^2 \\ \quad \textit{for a.e. } \boldsymbol{x} \in \Gamma_{S_3}, \ \textit{all } r_i \in \mathbb{R}, \quad i = 1, 2 \textit{ with } \bar{\beta}_2 \geq 0, \\ \textit{(v) } j_\nu(\boldsymbol{x}, \cdot) \textit{ or } -j_\nu(\boldsymbol{x}, \cdot) \textit{ is regular , for a.e. } \boldsymbol{x} \in \Gamma_{S_3}. \end{cases} \tag{3.16}$$

The densities of body forces and surface tractions are assumed to satisfy conditions

$$\boldsymbol{f}_0 \in C(0, T; L^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{f}_2 \in C(0, T; L^2(\Gamma_{S_2}; \mathbb{R}^d)). \tag{3.17}$$

Regarding the thermal tensors and the heat source density, we assume that their properties adhere to the necessary regularity conditions

$$c_{ij} = c_{ji} \in L^\infty(\Omega), \quad q \in H^1(0, T; L^2(\Omega)). \tag{3.18}$$

For a constant $m_K > 0$ *and for* $\forall(\xi_i) \in \mathbb{R}^d$*,*

$$k_{ij} = k_{ji} \in L^\infty(\Omega), \quad k_{ij}\xi_i\xi_j \geq m_K \xi_i \xi_j. \tag{3.19}$$

The boundary thermic data satisfy

$$k_e \in L^\infty(\Gamma_{S_3}; \mathbb{R}_+), \quad \vartheta_R \in H^1(0, T; L^2(\Gamma_{S_3})). \tag{3.20}$$

The initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \vartheta_0 \in E. \tag{3.21}$$

Define $\boldsymbol{f} : (0, T) \to V^*$ *as*

$$\begin{gathered} \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V^* \times V} = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_{S_2}; \mathbb{R}^d)}, \\ \forall \boldsymbol{v} \in V, \ \textit{a.e. } t \in (0, T). \end{gathered} \tag{3.22}$$

The functions $K : E \to E^*, R : V \to E^*$ *and* $\mathrm{Q} : (0,T) \to E^*$ *are defined by* $\forall \boldsymbol{w} \in V, \tau \in E, \eta \in E$

$$\langle K\tau, \eta\rangle_{E^*\times E} = \sum_{i,j=1}^{d} \int_{\Omega} k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i} dx + \int_{\Gamma_{S_3}} K_e \tau \cdot \eta d\Gamma. \tag{3.23}$$

$$\langle Rw, \eta\rangle_{E^*\times E} = -\int_{\Omega} c_{ij} \frac{\partial w_i}{\partial x_j} \eta dx. \tag{3.24}$$

$$\langle \mathrm{Q}(t), \eta\rangle_{E^*\times E} = -\int_{\Gamma_{S_3}} k_e \vartheta_R \eta d\Gamma + \int_{\Omega} q(t)\eta dx. \tag{3.25}$$

The operator $K : E \to E^*$ *satisfies that*

$$\| K\eta_1 - K\eta_2 \|_{E^*} \leq L_K \| \eta_1 - \eta_2 \|_E, \quad \forall \eta_1, \eta_2 \in E \text{ with } L_K > 0, \tag{3.26}$$

and the operator $R : V \to E^*$ *satisfies that*

$$\| R\boldsymbol{v}_1 - R\boldsymbol{v}_2 \|_{E^*} \leq L_R \| \boldsymbol{v}_1 - \boldsymbol{v}_2 \|_V, \quad \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V \text{ with } L_R > 0. \tag{3.27}$$

3.2 Variational Formulation

Let $(\boldsymbol{u}, \boldsymbol{\sigma}, \theta)$ *be a triple of sufficiently regular functions that satisfy* (3.1)-(3.11). *We use Green's formula 2.13 we have*

$$\int_{\Omega} \boldsymbol{\sigma}\varepsilon(\boldsymbol{v})dx + \int_{\Omega} Div\boldsymbol{\sigma}\boldsymbol{v}dx = \int_{\Gamma} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}da.$$

For a.e. $t \in (0,T)$, *recalling that*

$$\int_{\Omega} \boldsymbol{\sigma}\varepsilon(\boldsymbol{v})dx + \int_{\Omega} Div\boldsymbol{\sigma}\boldsymbol{v}dx = \int_{\Gamma_{S_1}} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}da + \int_{\Gamma_{S_2}} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}da + \int_{\Gamma_{S_3}} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}da, \quad \forall \boldsymbol{v} \in V.$$

According to (3.2), (3.4) *and for* $\boldsymbol{v} \in V$ *we have*

$$\int_{\Omega} \boldsymbol{\sigma}\varepsilon(\boldsymbol{v})dx - \int_{\Omega} f_0.\boldsymbol{v}dx = \int_{\Gamma_{S_2}} f_2\boldsymbol{v}da + \int_{\Gamma_{S_3}} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}da, \quad \forall \boldsymbol{v} \in V.$$

By use (3.22), *we find*

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} = (f(t), \boldsymbol{v})_{\boldsymbol{V}^*\times\boldsymbol{V}} + \int_{\Gamma_{S_3}} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}da, \quad \forall \boldsymbol{v} \in V. \tag{3.28}$$

Using the decomposition of the displacement and the stress fields on the contact boundary Γ_{S_3}, we find

$$\int_{\Gamma_{S_3}} \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}da = \int_{\Gamma_{S_3}} (\sigma_\nu\boldsymbol{\nu} + \boldsymbol{\sigma}_\tau\boldsymbol{\nu})\cdot(v_\nu\boldsymbol{\nu} + \boldsymbol{v}_\tau)da. \tag{3.29}$$

$$\int_{\Gamma_{S_3}} \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}da = \int_{\Gamma_{S_3}} \sigma_\nu v_\nu da + \int_{\Gamma_{S_3}} \boldsymbol{\sigma}_\tau\boldsymbol{\nu}\cdot\boldsymbol{v}_\tau da. \tag{3.30}$$

From (3.30)*and* (3.28)

$$\langle\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{v})\rangle_{\mathcal{H}} = \langle\boldsymbol{f}(t),\boldsymbol{v}\rangle + \int_{\Gamma_{S_3}} (\sigma_\nu(t)v_\nu + \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau)da. \tag{3.31}$$

On the other hand, from (3.5) *and* (3.6)*, by the definition of the subdifferential, we have*

$$j_\nu^0(u_\nu(t);v_\nu) \geq -\sigma_\nu(t)v_\nu. \tag{3.32}$$

$$j_\tau^0(\boldsymbol{u}_\tau(t);\boldsymbol{v}_\tau) \geq -\boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau. \tag{3.33}$$

Inserting(3.32) *and* (3.33) *in* (3.31)*, we obtain the following inequality*

$$\begin{aligned}(\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{v}))_{\mathcal{H}} + \int_{\Gamma_{S_3}} j_\nu^0({}_0^C D_t^\alpha u_\nu(t);v_\nu)d\Gamma + \int_{\Gamma_{S_3}} j_\tau^0({}_0^C D_t^\alpha \boldsymbol{u}_\tau;\boldsymbol{v}_\tau)da \\ \geq \langle\boldsymbol{f}(t),\boldsymbol{v}\rangle_{V^*\times V} \quad \forall\boldsymbol{v}\in V.\end{aligned} \tag{3.34}$$

Now, for all $t\in[0,T]$ and (3.7)*, we obtain*

$$\begin{aligned}&\left({}_0^C D_t^\alpha\vartheta(t),\eta\right)_{L^2(\Omega)} - (div(K_c\nabla\vartheta(t)),\eta)_{L^2(\Omega)} \\ &\qquad = (-c_{ij}\frac{\partial({}_0^C D_t^\alpha u_i(t))}{\partial x_j},\eta)_{L^2(\Omega)} + (q(t),\eta)_{L^2(\Omega)} \quad \forall\eta\in E.\end{aligned} \tag{3.35}$$

Using Green's formula 2.13 *to obtain*

$$-(div(K_c\nabla\vartheta(t),\eta)_{L^2(\Omega)} = \int_\Omega K_c\nabla\vartheta(t)\nabla\eta\,dx - \int_\Gamma K_c\frac{\partial\vartheta(t)}{\partial x}\nu\eta\,da, \quad \forall\eta\in E.$$

By $K_c = K_{ij}$*, and for* $\eta \in E$*, we have*

$$
\begin{aligned}
-\left(div(K_c \nabla \vartheta(t), \eta\right)_{L^2(\Omega)} &= \int_\Omega K_{ij} \frac{\partial \vartheta(t)}{\partial x_i} \frac{\partial \eta}{\partial x_j}\, dx \\
&\quad - \int_{\Gamma_{S_3}} K_{ij} \frac{\partial \vartheta(t)}{\partial x_i} \nu_j \eta\, da, \quad \forall \eta \in E. \qquad (3.36)
\end{aligned}
$$

According to(3.8, we obtain

$$
\begin{aligned}
-\left(div(K_c \nabla \vartheta(t)), \eta\right)_{L^2(\Omega)} &= \int_\Omega K_{ij} \frac{\partial \vartheta(t)}{\partial x_i} \frac{\partial \eta}{\partial x_j}\, dx \\
&\quad + \int_{\Gamma_{S_3}} K_e(\vartheta(t) - \vartheta_R)\eta\, da, \quad \forall \eta \in E. \qquad (3.37)
\end{aligned}
$$

Inserting (3.37) in (3.35), we find

$$
\begin{aligned}
&\int_\Omega {}_0^C D_t^\alpha \vartheta(t) \eta\, dx + \int_\Omega K_{ij} \frac{\partial \vartheta(t)}{\partial x_i} \frac{\partial \eta}{\partial x_j}\, dx + \int_{\Gamma_{S_3}} K_e(\vartheta(t) - \vartheta_R)\eta\, da \\
&\qquad = \int_\Omega -c_{ij} \frac{\partial \left({}_0^C D_t^\alpha u_i(t)\right)}{\partial x_j} \eta\, dx + \int_\Omega q(t)\eta\, dx \quad \forall \eta \in E. \qquad (3.38)
\end{aligned}
$$

So

$$
\begin{aligned}
&\int_\Omega {}_0^C D_t^\alpha \vartheta(t) \eta\, dx + \int_\Omega K_{ij} \frac{\partial \vartheta(t)}{\partial x_i} \frac{\partial \eta}{\partial x_j}\, dx + \int_{\Gamma_{S_3}} K_e \vartheta(t)\eta\, da = \int_\Omega -c_{ij} \frac{\partial \left({}_0^C D_t^\alpha u_i(t)\right)}{\partial x_j} \eta\, dx \\
&\qquad + \int_\Omega q(t)\eta\, dx + \int_{\Gamma_{S_3}} K_e \vartheta_R \eta\, da \quad \forall \eta \in E. \qquad (3.39)
\end{aligned}
$$

From (3.23), (3.24) and (3.25), we obtain

$$
\begin{aligned}
&\langle {}_0^C D_t^\alpha \vartheta(t), \eta\rangle_{E^* \times E} + \langle K\vartheta(t), \eta\rangle_{E^* \times E} \\
&\quad = \langle R({}_0^C D_t^\alpha \boldsymbol{u}(t), \eta\rangle_{E^* \times E} + \langle Q(t), \eta\rangle_{E^* \times E}, \quad \forall \eta \in E. \qquad (3.40)
\end{aligned}
$$

Problem$\mathcal{PV}$.

Find $u \in H^1(0,T;V), \vartheta \in H^1(0,T;E)$ such that for a.e. $t \in (0,T)$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}\left({}_0^C D_t^\alpha \boldsymbol{u}(t)\right) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \vartheta(s))\, ds, \tag{3.41}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} &+ \int_{\Gamma_{S_3}} j_\nu^{,0}({}_0^C D_t^\alpha u_\nu(t); v_\nu) da + \int_{\Gamma_{S_3}} j_\tau^{,0}({}_0^C D_t^\alpha \boldsymbol{u}_\tau; \boldsymbol{v}_\tau) da \\ &\geq \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V^* \times V} \quad \forall \boldsymbol{v} \in V,\end{aligned} \tag{3.42}$$

$$\begin{aligned}\langle {}_0^C D_t^\alpha \vartheta(t), \eta \rangle_{E^* \times E} &+ \langle K\vartheta(t), \eta \rangle_{E^* \times E} \\ &= \langle R({}_0^C D_t^\alpha \boldsymbol{u}(t), \eta \rangle_{E^* \times E} + \langle Q(t), \eta \rangle_{E^* \times E}, \quad \forall \eta \in E,\end{aligned} \tag{3.43}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad {}_0^C D_t^\alpha \boldsymbol{u}(0) = \boldsymbol{w}_0, \quad \vartheta(0) = \vartheta_0, \quad {}_0^C D_t^\alpha \vartheta(0) = \rho_0.$$

Here $\boldsymbol{w}_0$ is the initial value of ${}_0^C D_t^\alpha \boldsymbol{u}(t)$, ρ_0 is the initial value of ${}_0^C D_t^\alpha \vartheta(t)$.

The unique solvability of Problem $\mathcal{PV}$ is stated as follows and the proof of it will be given in the next section.

Theorem 3.2.1 *Assume (3.12)-(3.21). If*

$$m_{\mathcal{A}} > (\bar{\beta}_1 + \bar{\beta}_2) \parallel \gamma \parallel^2,$$

then problem$\mathcal{PV}$ guarantees a unique solution $(\boldsymbol{u}, \vartheta)$ that satisfies the regularity $u \in H^1(0,T;V), \vartheta \in H^1(0,T;E)$.

3.3 Proof of main results

The subsequent step, to streamline the existence and uniqueness, the fractional history-dependent hemivariational inequality is reformulated concerning the variable $\boldsymbol{w}(t) = {}_0^C D_t^\alpha \boldsymbol{u}(t)$. By considering the initial value condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$, and applying Proposition 2.5.1, we can recover $\boldsymbol{u}$ from $\boldsymbol{w}$

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + {}_0I_t^\alpha \boldsymbol{w}(t), \tag{3.44}$$

where

$${}_0I_t^\alpha \boldsymbol{w}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \boldsymbol{w}(s) ds. \tag{3.45}$$

Similarly, define $\rho(t) = {}_0^C D_t^\alpha \vartheta(t)$, *and then*

$$\vartheta(t) = \vartheta_0 + {}_0I_t^\alpha \rho(t), \quad {}_0I_t^\alpha \rho(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) ds. \tag{3.46}$$

By applying equations (3.44) and (3.46) to Problem$\mathcal{PV}$, *the problem can be reformulated as follows:*

Problem PV1.

Find $\boldsymbol{w} \in \mathcal{V}$, $\rho \in \mathcal{E}^*$ *such that for a.e.* $t \in (0,T)$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 + {}_0I_t^\alpha \boldsymbol{w}(t)) + \int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \vartheta(s)) ds, \tag{3.47}$$

$$\begin{aligned} &\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathcal{H}} + \int_{\Gamma_{S_3}} j_\nu^0(w_\nu(t); v_\nu) da + \int_{\Gamma_{S_3}} j_\tau^0(\boldsymbol{w}_\tau(t); \boldsymbol{v}_\tau) da \\ &\quad \geq \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V^* \times V} \quad \forall \boldsymbol{v} \in V, \end{aligned} \tag{3.48}$$

$$\begin{aligned} &\langle \rho(t), \eta \rangle_{E^* \times E} + \langle K(\vartheta_0 + {}_0I_t^\alpha \rho(t)), \eta \rangle_{E^* \times E} \\ &\quad = \langle R\boldsymbol{w}(t), \eta \rangle_{E^* \times E} + \langle Q(t), \eta \rangle_{E^* \times E}, \quad \forall \eta \in E, \end{aligned} \tag{3.49}$$

and

$$\boldsymbol{w}(0) = \boldsymbol{w}_0, \quad \rho(0) = \rho_0. \tag{3.50}$$

Therefore, we deduce that $(\boldsymbol{u}, \vartheta) \in H^1(0,T;V) \times H^1(0,T;E)$ *is a solution to Problem*$\mathcal{PV}$ *if and only if* $(\boldsymbol{w}, \rho) \in \mathcal{V} \times \mathcal{E}^*$ *solves Problem PV1.*

The proof of Theorem 3.2.1 will be conducted through four sequential procedures.

Step 1.

Given $\omega \in \mathcal{V}^*$, *the subsequent hemivariational inequality can be obtained.*

Problem PV2.

Find $\boldsymbol{w}_\omega \in \mathcal{V}$ such that

$$\langle \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_\omega(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 + {}_0I_t^\alpha \boldsymbol{w}_\omega(t)), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{H}} + \int_{\Gamma_{S_3}} j_\nu^0(w_{\omega\nu}(t); v_\nu)da$$
$$+ \int_{\Gamma_{S_3}} j_\tau^0(\boldsymbol{w}_{\omega\tau}(t); \boldsymbol{v}_\tau)da \geq \langle \boldsymbol{f}(t) - \omega(t), \boldsymbol{v}\rangle_{V^*\times V} \quad \forall \boldsymbol{v} \in V, \ a.e.\ t \in (0,T). \tag{3.51}$$

Lemma 3.3.1 *For $\omega \in \mathcal{V}^*$, there exists a unique solution $\boldsymbol{w}_\omega \in \mathcal{V}$ to Problem PV2. Furthermore, given $\omega_i \in \mathcal{V}^*$ and their corresponding unique solutions $\boldsymbol{w}_{\omega_i}$, $i = 1, 2$, the following inequality can be established*

$$\int_0^t \|\boldsymbol{w}_{\omega 1}(s) - \boldsymbol{w}_{\omega 2}(s)\|_V^2 ds \leq c \int_0^t \|\omega_1(s) - \omega_2(s)\|_{V^*}^2 ds, \tag{3.52}$$

for all $t \in [0,T]$ with $c > 0$.

Proof.

The proof is based on Theorem 2.5.1.
We consider the space $W = L^2(\Gamma_{S_3}; \mathbb{R}^d)$, and defined operators $A, B : V \to V^*$ by

$$\langle A\boldsymbol{w}, \boldsymbol{v}\rangle_{V\times V^*} = \langle \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{w})), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{H}} \quad \text{for } \boldsymbol{w}, \boldsymbol{v} \in V. \tag{3.53}$$
$$\langle B\boldsymbol{w}, \boldsymbol{v}\rangle_{V\times V^*} = \langle \mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{w})), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{H}} \quad \text{for } \boldsymbol{w}, \boldsymbol{v} \in V. \tag{3.54}$$

We consider the functional $J : W \to \mathbb{R}$ defined by

$$J(\boldsymbol{w}) = \int_{\Gamma_{S_3}} (j_\nu(\boldsymbol{x}, w_\nu) + j_\tau(\boldsymbol{x}, \boldsymbol{w}_\tau))da, \quad \forall \boldsymbol{w} \in W. \tag{3.55}$$

It follows from hypotheses (3.15)(v) and (3.16)(v) and Corollary2.5.1(vii), we obtain that $J(\cdot)$ or $-J(\cdot)$ is regular on W. And then, applying Corollary2.5.1(vi) and Lemma2.5.1, we obtain

$$J^0(\boldsymbol{w}) = \int_{\Gamma_{S_3}} (j_\nu^0(\boldsymbol{x}, w_\nu) + j_\tau^0(\boldsymbol{x}, \boldsymbol{w}_\tau))da, \quad \forall \boldsymbol{w} \in W. \tag{3.56}$$

$$\partial J(\boldsymbol{w}) \subseteq \int_{\Gamma_{S_3}} (\partial j_\nu(\boldsymbol{x}, w_\nu) + \partial j_\tau(\boldsymbol{x}, \boldsymbol{w}_\tau))da, \quad \forall \boldsymbol{w} \in W. \tag{3.57}$$

Moreover, let $M = \gamma$, where $\gamma : V \to W$ represents the trace operator.

Then (3.51) turns to be

$$\begin{aligned}&\langle A(\boldsymbol{w}_\omega(t)) + B(\boldsymbol{u}_0 + {}_0I_t^\alpha \boldsymbol{w}_\omega(t)), \boldsymbol{v}\rangle_{\mathcal{H}}\\&\quad + J^0(\gamma \boldsymbol{w}_\omega(t); \gamma \boldsymbol{v}) \geq \langle \boldsymbol{f}_\omega(t), \boldsymbol{v}\rangle_{V^*\times V} \quad \forall \boldsymbol{v} \in V, \ a.e.\ t \in (0,T),\end{aligned} \tag{3.58}$$

where $\boldsymbol{f}_\omega(t) = \boldsymbol{f}(t) - \omega(t)$.

Now, we will apply Theorem 2.5.1 to prove that problem(3.58) has a solution. To do so, we will verify the hypotheses $H(A), H(B), H(f), H(J)$ *and* $H(M)$ *of Theorem 2.5.1. It is easy to see that under hypothesis(3.12(ii)), the operator* A *dened by (3.53) is coercive with* $m_A = m_a$ *and* $A \in \mathcal{L}(V, V^*)$*, i.e.,* $H(A)$ *holds. Since* $\mathcal{B}$ *is bounded and linear, the operator* B *dened by (3.54) satisfies the condition* $H(B)$*. Starting from (3.55) and hypothesis (3.15),(3.16), and from (3.57) implies that the condition* $H(J)$ *is satised with* $m_J = mes(\Gamma_{S_3})(\bar{c}_0 + \bar{c}_1)$*. By the regularity hypothesis (3.17), we obtain that* f *satises* $H(f)$*. Next, by Theorem2.2.2 and Remark2.2.2, it is clear that the trace operator* γ *satises* $H(M)$*.*

Having veried the hypotheses of Theorem 2.5.1, we deduce that Problem $PV1$ *has a unique solution* $\boldsymbol{w}_\omega \in \mathcal{V}$*.*

Moving on to the second part of this problem, suppose $\boldsymbol{w}_{\omega 1}, \boldsymbol{w}_{\omega 2} \in \mathcal{V}$ *to be two solutions to Problem* $PV2$*.*

We start from the variational inequalities satisfied by $\boldsymbol{w}_{\omega_1}$ *and* $\boldsymbol{w}_{\omega_2}$*. For a.e.* $t \in (0,T)$ *and all* $\boldsymbol{v} \in V$*,*

$$\begin{aligned}&\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_1}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 +_0 I_t^\alpha \boldsymbol{w}_{\omega_1}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} + J^0(\gamma \boldsymbol{w}_{\omega_1}(t); \gamma \boldsymbol{v})\\&\quad \geq \langle \boldsymbol{f}(t) - \omega_1(t), \boldsymbol{v}\rangle_{V^*\times V},\end{aligned} \tag{3.59}$$

$$\begin{aligned}&\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_2}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 +_0 I_t^\alpha \boldsymbol{w}_{\omega_2}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} + J^0(\gamma \boldsymbol{w}_{\omega_2}(t); \gamma \boldsymbol{v})\\&\quad \geq \langle \boldsymbol{f}(t) - \omega_2(t), \boldsymbol{v}\rangle_{V^*\times V},\end{aligned} \tag{3.60}$$

where $J(\boldsymbol{w}) = \int_{\Gamma_{S_3}} (j_\nu(\boldsymbol{w}_\nu) + j_\tau(\boldsymbol{w}_\tau))\, da$ *and* J^0 *denotes its Clarke directional derivative.*

Choose $\boldsymbol{v} = \boldsymbol{w}_{\omega_2}(t) - \boldsymbol{w}_{\omega_1}(t)$ *in (3.59) and* $\boldsymbol{v} = \boldsymbol{w}_{\omega_1}(t) - \boldsymbol{w}_{\omega_2}(t)$ *in (3.60). Adding* *the two resulting inequalities gives*

$$\begin{aligned}
&\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}),\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2})\big)_{\mathcal{H}}\\
&\leq \big(\mathcal{B}\boldsymbol{\varepsilon}({}_0I_t^{\alpha}\boldsymbol{w}_{\omega_1}-{}_0I_t^{\alpha}\boldsymbol{w}_{\omega_2}),\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_2}-\boldsymbol{w}_{\omega_1})\big)_{\mathcal{H}}\\
&\quad + J^0(\gamma\boldsymbol{w}_{\omega_1};\gamma\boldsymbol{w}_{\omega_2}-\gamma\boldsymbol{w}_{\omega_1}) + J^0(\gamma\boldsymbol{w}_{\omega_2};\gamma\boldsymbol{w}_{\omega_1}-\gamma\boldsymbol{w}_{\omega_2})\\
&\quad + \langle\omega_1-\omega_2,\boldsymbol{w}_{\omega_2}-\boldsymbol{w}_{\omega_1}\rangle_{V^*\times V}.
\end{aligned} \tag{3.61}$$

Now, we estimate each term using the given assumptions.

Term with $\mathcal{A}$: *By assumption (3.12)(iii) (strong monotonicity) we have*

$$\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}),\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2})\big)_{\mathcal{H}} \geq m_a\|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V^2. \tag{A}$$

Term with $\mathcal{B}$: *Using the Lipschitz property (3.13)(iii) and the Cauchy-Schwarz inequality,*

$$\begin{aligned}
&\big(\mathcal{B}\boldsymbol{\varepsilon}({}_0I_t^{\alpha}\boldsymbol{w}_{\omega_1}-{}_0I_t^{\alpha}\boldsymbol{w}_{\omega_2}),\boldsymbol{\varepsilon}(\boldsymbol{w}_{\omega_2}-\boldsymbol{w}_{\omega_1})\big)_{\mathcal{H}}\\
&\leq c_b\|{}_0I_t^{\alpha}(\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2})\|_V\ \|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V.
\end{aligned} \tag{B}$$

Terms with J^0: *The properties (3.15)(iv) and (3.16)(iv) together with the definition of J imply (see Corollary 2.5.1)*

$$J^0(\gamma\boldsymbol{w}_1;\gamma\boldsymbol{w}_2-\gamma\boldsymbol{w}_1)+J^0(\gamma\boldsymbol{w}_2;\gamma\boldsymbol{w}_1-\gamma\boldsymbol{w}_2) \leq (\overline{\beta}_1+\overline{\beta}_2)\|\gamma\|^2\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V^2. \tag{C}$$

Term with $\omega_1-\omega_2$: *By duality,*

$$\langle\omega_1-\omega_2,\boldsymbol{w}_{\omega_2}-\boldsymbol{w}_{\omega_1}\rangle_{V^*\times V} \leq \|\omega_1-\omega_2\|_{V^*}\ \|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V. \tag{D}$$

Inserting (A)(D) into (3.61) yields

$$\begin{aligned}
m_a\|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V^2 &\leq c_b\|{}_0I_t^{\alpha}(\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2})\|_V\|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V\\
&\quad + (\overline{\beta}_1+\overline{\beta}_2)\|\gamma\|^2\|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V^2 + \|\omega_1-\omega_2\|_{V^*}\|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V.
\end{aligned}$$

Because the theorem assumes $m_a > (\overline{\beta}_1+\overline{\beta}_2)\|\gamma\|^2$, we can move the middle term to the left:

$$\big(m_a-(\overline{\beta}_1+\overline{\beta}_2)\|\gamma\|^2\big)\|\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2}\|_V \leq c_b\|{}_0I_t^{\alpha}(\boldsymbol{w}_{\omega_1}-\boldsymbol{w}_{\omega_2})\|_V + \|\omega_1-\omega_2\|_{V^*}. \tag{3.62}$$

Define the positive constants

$$M := m_a - (\overline{\beta}_1 + \overline{\beta}_2)\|\gamma\|^2 > 0, \qquad C := \frac{c_b}{M}.$$

Let $e(t) := \|\boldsymbol{w}_{\omega_1}(t) - \boldsymbol{w}_{\omega_2}(t)\|_V$ *and* $a(t) := \frac{1}{M}\|\omega_1(t) - \omega_2(t)\|_{V^*}$. *Then* (3.62) *becomes*

$$e(t) \leq a(t) + C\,\|{}_0I_t^\alpha(\boldsymbol{w}_{\omega_1} - \boldsymbol{w}_{\omega_2})(t)\|_V.$$

Using the definition of the fractional integral and the fact that $e(s) \geq 0$,

$$\|{}_0I_t^\alpha(\boldsymbol{w}_{\omega_1} - \boldsymbol{w}_{\omega_2})(t)\|_V \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e(s)\,ds.$$

Hence

$$e(t) \leq a(t) + \frac{C}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e(s)\,ds.$$

Now apply the fractional Gronwall inequality (Lemma 2.6.1). Since $a(t)$ *and the constant* $g(t) = C/\Gamma(\alpha)$ *are nondecreasing, Lemma 2.6.1 gives*

$$e(t) \leq a(t)\,E_\alpha\big(g(t)\Gamma(\alpha)\,t^\alpha\big) = a(t)\,E_\alpha(C\,t^\alpha).$$

Because $C = c_b/M$, *we have*

$$e(t) \leq \frac{E_\alpha(c_b t^\alpha)}{M}\,\|\omega_1(t) - \omega_2(t)\|_{V^*}.$$

Finally, square both sides and integrate from 0 *to* t*:*

$$\int_0^t e(s)^2\,ds \leq \left(\frac{E_\alpha(c_b T^\alpha)}{M}\right)^2 \int_0^t \|\omega_1(s) - \omega_2(s)\|_{V^*}^2\,ds.$$

Thus (3.52) *holds with the constant*

$$c = \left(\frac{E_\alpha(c_b T^\alpha)}{m_a - (\overline{\beta}_1 + \overline{\beta}_2)\|\gamma\|^2}\right)^2.$$

This completes the proof of Lemma 3.3.1. ■

Step 2.

Utilizing the velocity field $\boldsymbol{w}_\omega$ obtained in Lemma 3.3.1, our attention is directed towards the subsequent hemivariational inequality.

Problem PV3.

Find $\rho_\omega \in \mathcal{E}^*$ such that

$$\langle \rho_\omega(t), \eta\rangle_{E^*\times E} + \langle K(\vartheta_0 + {}_0I_t^\alpha \rho_\omega(t)), \eta\rangle_{E^*\times E} = \langle R\boldsymbol{w}_\omega(t) + Q(t), \eta\rangle_{E^*\times E} \quad \forall \eta \in E. \tag{3.63}$$

Lemma 3.3.2 *For $\omega \in \mathcal{V}^*$, there exists a unique solution $\rho_\omega \in \mathcal{E}^*$ for Problem PV3. Moreover, if $\rho_{\omega i}$, $i = 1, 2$ represent two solutions to Problem PV3 with respect to $\omega = \omega_i$, then there exists a constant k which satisfies*

$$\|\rho_{\omega 1}(t) - \rho_{\omega 2}(t)\|_E^2 \leq k\|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V^2. \tag{3.64}$$

Proof. *With the identification, equation (3.49) becomes:*

$$\rho_\omega(t) + K({}_0I_t^\alpha \rho_\omega(t)) = F_\omega(t), \tag{3.65}$$

where

$$F_\omega(t) = R\boldsymbol{w}_\omega(t) + Q(t) - K\vartheta_0.$$

*Equation(3.65) is a **linear** fractional integral equation.*

Define the maping $\mathcal{L} : \mathcal{E}^* \longrightarrow \mathcal{E}^*$ by

$$(\mathcal{L}_\omega \rho)(t) = F_\omega(t) - K({}_0I_t^\alpha \rho(t)), \tag{3.66}$$

Then (3.65) is equivalent to the fixed-point equation $\rho = \mathcal{L}_\omega \rho$.

We work in $L^2(0, T; E^*)$ with the following weighted norm

$$\|\rho\|_\lambda^2 = \int_0^T e^{-2\lambda t}\|\rho(t)\|_{E^*}^2\, dt, \qquad \lambda > 0.$$

For any $\rho_1, \rho_2 \in L^2(0,T;E^*)$, using the Lipschitz property of K,

$$\begin{aligned}\| (\mathcal{L}_\omega \rho_1)(t) - (\mathcal{L}_\omega \rho_2)(t) \|_{E^*} &\leq L_K \|_0 I_t^\alpha (\rho_1 - \rho_2)(t) \|_{E^*} . \\ &\leq L_K \, {}_0I_t^\alpha \| (\rho_1 - \rho_2)(t) \|_{E^*}\end{aligned} \tag{3.67}$$

and we have

$${}_0I_t^\alpha \| (\rho_1 - \rho_2)(t) \|_{E^*} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} \| \rho_1(s) - \rho_2(s) \|_{E^*} \, ds. \tag{3.68}$$

Let $f(t) = \begin{cases} \| (\rho_1(t) - \rho_2(t) \|_{E^*} & t \geq 0 \\ 0 & t < 0, \end{cases}$ *and* $g(t) = \begin{cases} \dfrac{(t-s)^{(\alpha-1)}}{\Gamma(\alpha)} & t \geq 0 \\ 0 & t < 0. \end{cases}$

Then, we can write

$$f * g(s) = \int_{\mathbb{R}} g(t-s) f(s) ds.$$

Introduce the exponential weight $e^{-\lambda t}$ with $\lambda > 0$, we obtain

$$\begin{aligned} e^{-\lambda t}(f*g)(t) &= \int_{\mathbb{R}} e^{-\lambda(t-s)} g(t-s) e^{-\lambda s} f(s) ds \\ &= f_\lambda * g_\lambda \end{aligned}$$

where $f_\lambda(t) = e^{-\lambda t} f(t)$ and $g_\lambda(t) = e^{-\lambda t} g(t)$.

Now, $g_\lambda(t) \in L^1(\mathbb{R}_+)$ and $f_\lambda(t) \in L^2(\mathbb{R}_+)$.

Therefore, Young's inequality for convolutions (with $p = 1, q = 2$ and $r = 2$) gives

$$\| f_\lambda * g_\lambda \|_{L^2(\mathbb{R}_+)} \leq \| f_\lambda \|_{L^2(\mathbb{R}_+)} \cdot \| g_\lambda \|_{L^1(\mathbb{R}_+)} .$$

The L^1 norm of g_λ is compated as

$$\| g_\lambda \|_{L^1(\mathbb{R}_+)} = \int_0^\infty e^{-\lambda t} \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} \, dt = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{\lambda^\alpha} = \lambda^{-\alpha}.$$

From(3.68) we have

$$\left(\int_0^\infty e^{-2\lambda t} \, {}_0I_t^\alpha \mid f(t) \mid^2 dt \right)^{\frac{1}{2}} \leq \lambda^{-\alpha} \left(\int_0^\infty \mid f(t) \mid^2 dt \right)^{\frac{1}{2}} . \tag{3.69}$$

Squaring and resting the integrale to $[0,T]$, yeilds

$$\int_0^T e^{-2\lambda t} \mid_0 I_t^\alpha f(t) \mid^2 \, dt \leq \lambda^{-2\alpha} \int_0^T \mid f(t) \mid^2 \, dt. \tag{3.70}$$

Appling (3.70) to $f(t) = \| \rho_1(t) - \rho_2(t) \|_{E^*}$ gives

$$\int_0^T e^{-2\lambda t} \|_0 \, I_t^\alpha (\rho_1 - \rho_2)(t) \|_{E^*}^2 \, dt \leq \lambda^{-2\alpha} \int_0^T e^{-2\lambda t} \| \rho_1(t) - \rho_2(t) \|_{E^*}^2 \, dt. \tag{3.71}$$

From (3.67) and (3.71) we have

$$\| \mathcal{L}_\omega \rho_1(t) - \mathcal{L}_\omega \rho_2(t) \|_\lambda^2 \leq \frac{L_K^2}{\lambda^{2\alpha}} \| \rho_1(t) - \rho_2(t) \|_\lambda^2$$

Choosing λ sufficiently large so that $\frac{L_K^2}{\lambda^{2\alpha}} < 1$, the operator $\mathcal{L}_\omega$ becomes a contraction. By the Banach fixed-point theorem, there exists a unique $\rho_\omega \in L^2(0,T;E^*)$.

Since that Problem $PV3$ is an ordinary fractional integral equation, it has a unique solution.

For the second part, for $t \in [0,T]$ and any $\eta \in E$, let $\rho_{\omega i} \in \mathcal{E}^*$, ProblemPV3 gives

$$\langle \rho_{\omega_i}(t), \eta \rangle_{E^* \times E} + \langle K(\vartheta_0 +_0 I_t^\alpha \rho_{\omega_i}(t)), \eta \rangle_{E^* \times E} = \langle R \boldsymbol{w}_{\omega_i}(t) + Q(t), \eta \rangle_{E^* \times E}, \qquad i = 1, 2.$$

Subtracting the two equations yields

$$\langle \rho_1(t) - \rho_2(t), \eta \rangle + \langle K(\vartheta_0 +_0 I_t^\alpha \rho_1(t)) - K(\vartheta_0 +_0 I_t^\alpha \rho_2(t)), \eta \rangle = \langle R(\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)), \eta \rangle,$$

where we write $\rho_i = \rho_{\omega_i}$ and $\boldsymbol{w}_i = \boldsymbol{w}_{\omega_i}$ for brevity.

Because E is a Hilbert space we identify E^* with E and choose $\eta = \rho_1(t) - \rho_2(t) \in E$. Then

$$\|\rho_1 - \rho_2\|_E^2 + \langle K(\vartheta_0 +_0 I_t^\alpha \rho_1) - K(\vartheta_0 +_0 I_t^\alpha \rho_2), \, \rho_1 - \rho_2 \rangle = \langle R(\boldsymbol{w}_1 - \boldsymbol{w}_2), \, \rho_1 - \rho_2 \rangle.$$

Using the Lipschitz properties of K and R (assumptions (3.26) and (3.27)), we obtain

$$\left| \langle K(\vartheta_0 +_0 I_t^\alpha \rho_1) - K(\vartheta_0 +_0 I_t^\alpha \rho_2), \rho_1 - \rho_2 \rangle \right| \leq L_K \|_0 I_t^\alpha (\rho_1 - \rho_2)\|_E \, \|\rho_1 - \rho_2\|_E,$$

$$\left| \langle R(\boldsymbol{w}_1 - \boldsymbol{w}_2), \rho_1 - \rho_2 \rangle \right| \leq L_R \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_V \, \|\rho_1 - \rho_2\|_E.$$

Hence

$$\|\rho_1 - \rho_2\|_E^2 \leq L_K \|{}_0I_t^\alpha(\rho_1 - \rho_2)\|_E \, \|\rho_1 - \rho_2\|_E + L_R \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_V \, \|\rho_1 - \rho_2\|_E.$$

If $\|\rho_1 - \rho_2\|_E = 0$ *the desired inequality holds trivially.*
Otherwise divide by this norm to get

$$\|\rho_1(t) - \rho_2(t)\|_E \leq L_K \|{}_0I_t^\alpha(\rho_1 - \rho_2)(t)\|_E + L_R \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V. \tag{3.72}$$

The fractional integral satisfies the standard estimate

$$\|{}_0I_t^\alpha(\rho_1 - \rho_2)(t)\|_E \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\rho_1(s) - \rho_2(s)\|_E \, ds.$$

Define $e(t) = \|\rho_1(t) - \rho_2(t)\|_E$ *and* $a(t) = L_R \|w_1(t) - w_2(t)\|_V$. *Then* (3.72) *becomes*

$$e(t) \leq a(t) + \frac{L_K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e(s) \, ds. \tag{3.73}$$

The functions $a(t)$ *and* $g(t) = L_K/\Gamma(\alpha)$ *are nonnegative and* g *is constant (hence nondecreasing). Lemma 2.6.1 (the fractional Gronwall inequality) applied to* (3.73) *yields*

$$e(t) \leq a(t)\, E_\alpha\big(g(t)\,\Gamma(\alpha)\, t^\alpha\big) = L_R \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V \; E_\alpha\big(L_K t^\alpha\big).$$

Since E_α *is increasing in its argument, for all* $t \in [0, T]$ *we have* $E_\alpha(L_K t^\alpha) \leq E_\alpha(L_K T^\alpha)$. *Therefore*

$$e(t) \leq L_R E_\alpha(L_K T^\alpha) \; \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V.$$

Squaring both sides gives precisely

$$\|\rho_{\omega_1}(t) - \rho_{\omega_2}(t)\|_E^2 \leq k \, \|\boldsymbol{w}_{\omega_1}(t) - \boldsymbol{w}_{\omega_2}(t)\|_V^2$$

with $k = \big(L_R E_\alpha(L_K T^\alpha)\big)^2$. ■

Step 3.

Let $G : (0, T) \times V \times E \to V^*$ *be defined by*

$$\langle G(t, \boldsymbol{u}, \vartheta), \boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{G}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}), \vartheta), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V,\ \vartheta \in E. \tag{3.74}$$

We define the operator Φ *as*

$$\Phi\omega = \int_0^t \mathcal{G}(t - s, \boldsymbol{u}_\omega(s), \vartheta_\omega(s))ds. \tag{3.75}$$

Lemma 3.3.3 *The operator* Φ *admits a unique fixed point* $\omega^* \in \mathcal{V}^*$.

Proof.

Let $\omega_1, \omega_2 \in \mathcal{V}^*$. *Combining (3.52), (3.64) and (3.75), we have the next inequality*

$$\begin{aligned}
&\|\Phi\omega_1(t) - \Phi\omega_2(t)\|^2_{V^*}\\
&= \left\|\int_0^t G(t - s, \boldsymbol{u}_{\omega 1}(s), \vartheta_{\omega 1}(s))ds - \int_0^t G(t - s, \boldsymbol{u}_{\omega 2}(s), \vartheta_{\omega 2}(s))ds\right\|^2_{V^*}\\
&= \left\|\int_0^t \mathcal{G}(t - s, \boldsymbol{\varepsilon}(\boldsymbol{u}_{\omega 1}(s)), \vartheta_{\omega 1}(s))ds - \int_0^t \mathcal{G}(t - s, \boldsymbol{\varepsilon}(\boldsymbol{u}_{\omega 2}(s)), \vartheta_{\omega 2}(s))ds\right\|^2_{\mathcal{H}}\\
&\le 2t \cdot L^2_{\mathcal{G}} \left(\int_0^t (\|\boldsymbol{u}_{\omega 1}(s) - \boldsymbol{u}_{\omega 2}(s)\|^2_V + \|\vartheta_{\omega 1}(s) - \vartheta_{\omega 2}(s)\|^2_E)ds\right)\\
&\le 2tL^2_{\mathcal{G}} \left(\int_0^t \|{}_0I^\alpha_s(\boldsymbol{w}_{\omega 1}(s) - \boldsymbol{w}_{\omega 2}(s))\|^2_V ds + \int_0^t \|{}_0I^\alpha_s(\rho_{\omega 1}(s) - \rho_{\omega 2}(s))\|^2_E ds\right)\\
&\le 2tL^2_{\mathcal{G}} \left(\frac{t^{2\alpha}}{2\alpha(\Gamma(\alpha))^2}\left(\int_0^t \|\boldsymbol{w}_{\omega 1}(s) - \boldsymbol{w}_{\omega 2}(s)\|^2_V ds + \int_0^t \|\rho_{\omega 1}(s) - \rho_{\omega 2}(s)\|^2_E ds\right)\right)\\
&\le 2tL^2_{\mathcal{G}} \frac{t^{2\alpha}}{2\alpha(\Gamma(\alpha))^2} \cdot (c + ck) \int_0^t \|\omega_1(s) - \omega_2(s)\|^2_{V^*} ds\\
&= \frac{(c + ck)L^2_{\mathcal{G}} t^{2\alpha+1}}{\alpha(\Gamma(\alpha))^2} \int_0^t \|\omega_1(s) - \omega_2(s)\|^2_{V^*} ds,
\end{aligned}$$

for all $t \in [0, T]$ *with* $\frac{(c+ck)L^2_{\mathcal{G}} t^{2\alpha+1}}{\alpha(\Gamma(\alpha))^2} > 0$.

By utilizing Lemma 2.4.1, it can be inferred that there exists a unique $\omega^* \in \mathcal{V}^*$ *such that* $\Phi\omega^* = \omega^*$, *thereby completing the proof of the lemma.* ■

Step 4.

We now have all the ingredients to prove Theorem 3.2.1.

Proof.

Consider $\omega^* \in \mathcal{V}^*$ *to be the unique fixed point of the operator* Φ*, that is*

$$\omega^*(t) = \int_0^t \mathcal{G}(t-s, \boldsymbol{u}_{\omega^*}(s), \vartheta_{\omega^*}(s))ds,$$

a.e. $t \in (0,T)$*. Consider* $\boldsymbol{w}^* = \boldsymbol{w}_{\omega^*}$ *as the unique solution to Problem* $PV2$*, corresponding to* ω^* *obtained from Lemma 3.3.1. Additionally, let* $\rho^* = \rho_{\omega^*}$ *be the unique solution of Problem* $PV3$ *as proven in Lemma3.3.2. Hence,* $(\boldsymbol{w}^*, \rho^*)$ *can serve as the unique solution for Problem* $PV1$*, with* $\boldsymbol{w}^* \in \mathcal{V}$ *and* $\rho^* \in \mathcal{E}^*$*. The unique result of this theorem is a direct consequence of the uniqueness of the fixed point of* Φ *as well as Lemma3.3.1 and 3.3.2. We conclude that Problem*$\mathcal{PV}$ *possesses a unique solution, thus completing the proof.* ■

Bibliography

[1] H. Brézis.: Analyse fonctionnelle, Théorie et Application*, Masson, 1987.*

[2] Kilbas, A., Srivastava, H., Trujillo, J.: Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006)*

[3] Lurie, A.I., Belyaev, A.: Theory of Elasticity*. Springer, Berlin (2005)*

[4] Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities*. Springer, New York (2013)*

[5] J. Necas , I. Hlavácek.: Mathematical Theory of Elastic and Elastoplastic Bodies*. An Introduction, Elsevier, Amsterdam, 1981.*

[6] R. S.: Adams Sobolev spaces*. Academic press, New York (1975).*

[7] L. Schwartz.: Théorie des Distributions*, Hermann, Paris (1967).*

[8] Sofonea, M., Migórski, S.: VariationalHemivariational Inequalities with Applications *. Chapman and Hall/CRC Press, New York (2018)*

[9] Bartosz, K., Cheng, X., Kalita, P., et al.: Roth method for parabolic variational-hemivariational inequalities*. J. Math. Anal. Appl.* ***423****, 841-862 (2015)*

[10] Chau, O.: Numerical analysis of a thermal contact problem with adhesion*. Comput. Appl. Math.* ***37****, 5424-5455 (2018)*

[11] Chudzikiewicz, A.,My' sli' nski, A.: Thermoelastic wheel-rail contact problem with elastic gradedmaterials*. Wear* ***271****, 417-425 (2011)*

[12] Cocou, M.: A class of dynamic contact problems with Coulomb friction in viscoelasticity*. Nonlinear Anal. Real World Appl.* ***22****, 508-519 (2015)*

[13] Essoufi, E., Zafrar, A.: Optimal control of friction coefficient in Signorini contact problems. *Opt. Control Appl. Methods **42**, 1794-1811 (2021)*

[14] Guo, F., Wang, J., Han, J.: Dynamic viscoelastic unilateral constrained contact problems with thermal *effects. Appl. Math. Comput. **424**, 127034 (2022)*

[15] Han, D., Han, W., Jureczka, M., Ochal, A.: Numerical analysis of a contact problem with wear. *Comput. Math. Appl. **79**, 2942-2951 (2020)*

[16] Han, J., Migórski, S.: A quasistatic viscoelastic frictional contact problem with multivalued normal compliance, unilateral constraint and material damage. *J. Math. Anal. Appl. **443**, 57-80 (2016)*

[17] Han, J., Migórski, S., Zeng, H.: Weak solvability of a fractional viscoelastic frictionless contact problem. *Appl. Math. Comput. **303**, 1-18 (2017)*

[18] Han, W., Migórski, S., Sofonea, M.: A class of variationalhemivariational inequalities with applications to frictional contact problems. *SIAM J. Math. Anal. **46**, 3891-3912 (2014)*

[19] Hao, J., Wang, J., Han, J.: History-dependent fractional hemivariational inequality with time-delay system for a class of new frictionless quasistatic contact problems. *Math. Mech. Solids (2021)*

[20] Hao, J., Wang, J., Lu, L.: Coupled system of fractional hemivariational inequalities with applications. *Optimization **73**, 969-994 (2024)*

[21] Kalita, P., Szafraniec, P., Shillor, M.: A frictional contact problem with wear diffusion. *Z. Angew. Math. Phys. **70**, 1-17 (2019)*

[22] Kulig, A., Migórski, S.: Solvability and continuous dependence results for second order nonlinear evolution inclusions with a Volterra-type operator. Nonlinear Anal. ***75**, 4729-4746 (2012)*

[23] Kulig, A.: Variationalhemivariational approach to quasistatic viscoplastic contact problem with normal compliance, unilateral constraint, memory term, friction and damage. *Nonlinear Anal. Real World Appl. **44**, 401-416 (2018)*

[24] Migorski, S., Cai, D.: A general differential quasi variational-hemivariational inequality: well-posedness and application . *Sci. Numer. Simul. **125**, 107379 (2023)*

[25] *Migórski, S., Szafraniec, P.:* A class of dynamic frictional contact problems governed by a system of hemivariational inequalities in thermoviscoelasticity. *Nonlinear Anal. Real World Appl.* ***15****, 158-171 (2014)*

[26] *Peng, Z.:* Existence of a class of variational inequalities modelling quasi-static viscoelastic contact problems. *Z. Angew. Math. Mech.* ***99****,* ***e201800172*** *(2019)*

[27] *Poluektov, M., Figiel, L.:* A cut finite-element method for fracture and contact problems in largedeformation solid mechanics. *Comput. Methods Appl. Mech. Eng.* ***388****,* ***114234*** *(2022)*

[28] *Webb, J.:* A fractional Gronwall inequality and the asymptptic behaviour of global solutions of caputo fractional problems. *Electron. J. Differ. Equ.* ***2021****, 1-22 (2021)*

[29] *Weng, Y., Chen, T., Li, X., Huang, N.:* Rothe method and numerical analysis for a new class of fractional differential hemivariational inequality with an application. *Comput. Math. Appl.* ***98****, 118-138 (2021)*

[30] *Xuan, H., Cheng, X., Xiao, Q.:* Analysis of a doubly-history dependent variational-hemivariational inequality arising in adhesive contact problem. *Appl. Anal.* ***103****, 1224-1240 (2024)*

[31] *Xuan, H., Cheng, X., Han, W., Xiao, Q.:* Numerical analysis of a dynamic contact problem with historydependent operators. *Numer. Math. Theory Methods Appl.* ***13****, 569-594 (2020)*

[32] *Zeng, S., Migórski, S.:* A class of time-fractional hemivariational inequalities with application to frictional contact problem. *Commun. Nonlinear Sci. Numer. Simul.* ***56****, 34-48 (2018)*

[33] *A. Aissaoui .:* Analyse Mathématique dun système aux dérivée partielles modélisant des problèmes de contact en phénomènes mécaniques*, Thèse de doctorat Université de Echahid Hama Lakhdar, El Oued (2023).*

[34] *K. Rimi.:* Mathematical analysis of some contact problems involving fractional derivatives , *Theme of master University of Echahid Hamma Lakhdar, El-Oued*

[35] *L. Selmani.:* Analyse variationnelle de différents problèmes aux limites en mécanique du contact*, Thèse de doctorat Université Ferhat Abbas, sétif (2018).*

ملخص المذكرة

الملخص: الهدف من هذه المذكرة هو إثبات وجود ووحدانية الحل الضعيف لمسألة تلامس ميكانيكي حراري لجسم مرن، لزج، مع مراعاة تأثير الذاكرة الطويلة مع قاعدة بالإضافة إلى وجود استجابة عادية واحتكاك، وهذه المسألة يتم نمذجتها ميكانيكياً باستخدام المتراجحات التفاضلية الكسرية والشبه تغيرية بمفهوم ''كابوتو'' حيث ركزت دراستنا على الجانب التحليلي معتمدين على مبرهنة بناخ للنقطة الثابتة ومتراجحة غرونوال.

الكلمات المفتاحية: تلامس ميكانيكي حراري، لزوجة مرنة، ذاكرة طويلة، تفاضل كسري بمفهوم كابوتو، متراجحات شبه تغيرية، مبرهنة بناخ، متراجحة غرونوال، الحل الضعيف.

Abstract

Abstract: The objective of this thesis is to prove the existence and uniqueness of the weak solution for a thermomechanical contact problem of a viscoelastic body, taking into account the long-term memory effect along with a normal compliance and friction law. This problem is mechanically modeled using fractional and quasi-variational differential inequalities in the sense of Caputo. Our study focused on the analytical aspect, relying on Banach's fixed-point theorem and Gronwall's inequality.

Keywords: Thermomechanical contact, Viscoelasticity, Long-term memory, Caputo fractional derivative, Quasi-variational inequalities, Banach fixed-point theorem, Gronwall's inequality.

Résumé

Résumé : L'objectif de ce mémoire est de pouver l'existence et l'unicité de la solution faible d'un problème de contact thermomécanique pour un corps viscoélastique, en prenant en compte l'effet de la mémoire longue avec une loi de réponse normale et de frottement. Ce problème est modélisé mécaniquement à l'aide d'inégalités différentielles fractionnaires et quasi-variationnelles au sens de Caputo. Notre étude s'est focalisée sur l'aspect analytique, en nous basant sur le théorème du point fixe de Banach et l'inégalité de Gronwall.

Mots-clés : Contact thermomécanique, Viscoélasticité, Mémoire longue, Dérivée fractionnaire de Caputo, Inégalités quasi-variationnelles, Théorème du point fixe de Banach, Inégalité de Gronwall.