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Green's Function Approach to Entanglement Entropy on Lattices : 2-Coupled Harmonic Oscillators Application

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Abstract

We develop a Green's function approach to compute Rényi entanglement entropy on lattices. The Rényi entropy resulting from tracing out an arbitrary collection of subsets of coupled harmonic oscillators is written as zero temperature partition function generated by an Euclidean action with n-fold step potential. The associated Green's function is explicitly constructed and an alternative new formula for the Rényi entropy is obtained. To show the applicability of the method we take the example of 2-coupled harmonic oscillator as an application.

Keywords: entanglement entropy, Rényi entropy, harmonic oscillator, Green's function, step potential.

1. Introduction

Quantum entanglement plays a major role in number of physics fields. Evaluating entanglement therefore is highly important, many entanglement measures have been developed [1, 2], but among them all entanglement entropy is the most complete and coherent one.

There are many approaches to evaluate the entanglement entropy[2, 3], one of them is to use Green's function. For lattices this method has not been used before, thence evaluating it will give another perspective to this issue.

2. Rényi entropy for N-coupled harmonic oscillator

The lattice is taken to be formed of N-coupled harmonic oscillator, its standard Lagrangian is [4]:

$$L = \frac{1}{2}(\dot{Q}^T.\dot{Q} + Q^T.V.Q)$$

Where Q is a N-component column vector ((Q)i = qi), and V is N∗N positive symmetric matrix.

If we take the global state to be in the pure ground state, then the density matrix of the subsystem formed by the first (p) harmonic oscillator is -using path integral formulation- :

$$\rho_A(q_1..q_p;q_1'..q_p') = \int D_Q \exp(-\int_{-\infty}^{0^-\left(Q=(q_1..q_p,q_{p+1}..q_N)\right)} L\,dt - \int_{0^+\left(Q=(q_1'..q_p',q_{p+1}..q_N)\right)}^{\infty} L\,dt)$$

Rényi entropy for the system is :

$$\ln Tr(\rho_A^n) = \frac{-1}{2}.\frac{\det(-\frac{d^2}{dt^2}+\boldsymbol{V}(t))}{\det(-\frac{d^2}{dt^2}+V(t))^n}$$

Where :

$$\boldsymbol{V}(t) = \boldsymbol{V}.\theta(-t) + \boldsymbol{V_p}.\theta(t)$$

$$\boldsymbol{V} = V \otimes I_n;\ \boldsymbol{V_p} = \boldsymbol{P_\pi^T}.\boldsymbol{V}.\boldsymbol{P_\pi}$$

$\boldsymbol{V}$ and $\boldsymbol{V}_p$ are nN ∗ nN matrices, $\theta(t)$ is the step function, I_n is n ∗ n unit matrix, and $\boldsymbol{P_\pi}$ is the permutation matrix. we write $\boldsymbol{V}$ as :

$$V = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

Where A and C are respectively p ∗ p and ((N − p) ∗ (N − p)) matrices. Using this decomposition we can write $\boldsymbol{V}_p$ as :

$$\boldsymbol{V}_p = \begin{bmatrix} A & 0 & 0 & 0 & 0 & \dots & B \\ 0 & C & B^T & 0 & 0 & \dots & 0 \\ 0 & B & A & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & C & B^T & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ B^T & 0 & 0 & 0 & 0 & 0 & C \end{bmatrix}$$

To this point evaluating the entropy is fulfilled by evaluating the determinant of the operator $(-\frac{d^2}{dt^2} + \boldsymbol{V}(t))$, which can be done if we determine Green's function $\boldsymbol{G}(t, t_0)$:

$$\left(-\frac{d^2}{dt^2} + \boldsymbol{V}(t) + E\right)\boldsymbol{G}(t, t_0) = \delta(t, t_0)$$

The explicit formula of Green's function that fulfils the previous equation is :

$$\boldsymbol{G}(t, t_0)_{\mp\mp} = \frac{1}{2\boldsymbol{W}_\mp}\exp(-\boldsymbol{W}_\mp|t - t_0|) + \frac{1}{2\boldsymbol{W}_\mp}\exp(\pm\boldsymbol{W}_\mp t)\frac{\boldsymbol{W}_\mp - \boldsymbol{W}_\pm}{\boldsymbol{W}_+ + \boldsymbol{W}_-}\exp(\pm\boldsymbol{W}_\mp t_0)$$

$$\boldsymbol{G}(t, t_0)_{\pm\mp} = \exp(\mp\boldsymbol{W}_\pm t)\frac{1}{\boldsymbol{W}_+ + \boldsymbol{W}_-}\exp(\pm\boldsymbol{W}_\mp t_0)$$

Where $\boldsymbol{G}(t, t_0)_{+-}$ is for (t > 0 and t_0 < 0)..etc, also:

$$\boldsymbol{W}_\pm = \sqrt{\boldsymbol{V}_\pm} \;\; ; \;\; \boldsymbol{V}_- = \boldsymbol{V} + E \;\; ; \;\; \boldsymbol{V}_+ = \boldsymbol{V}_{\boldsymbol{p}} + E$$

Green's function formula allows one to evaluate Rényi's entropy :

$$\frac{\partial \ln(Tr(\rho^n))}{\partial E} = -\frac{1}{2}\int dt\, Tr(\boldsymbol{G}(t, t) - nG(t, t))$$

Where G is the same as **G** but for n = 1.

Finally :

$$\ln\big(Tr(\rho^n)\big) = \frac{1}{8}\int_0^\infty dE\, (\boldsymbol{V}_-^{-1} - \boldsymbol{V}_+^{-1})(\boldsymbol{W}_- - \boldsymbol{W}_+)(\boldsymbol{W}_- + \boldsymbol{W}_+)^{-1}$$

2. 2-coupled harmonic oscillators

consider the following Hamiltonian [4]

$$H = \frac{1}{2}(p_1^2 + p_2^2 + Q^T.\boldsymbol{V}.Q)$$

where the potential is of the form

$$\boldsymbol{V} = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$$

It is easy to show that

$$(\boldsymbol{V}_-^{-1} - \boldsymbol{V}_+^{-1})(\boldsymbol{W}_- - \boldsymbol{W}_+) = \frac{\sqrt{x-1} - \sqrt{x+1}}{2(x^2-1)}\boldsymbol{C}$$

where $x = a + E$, $\boldsymbol{C}$ is a 2n × 2n circulant matrix with elements $\boldsymbol{C}_{\boldsymbol{j}}$ $(j = 0..2n-1)$, $\boldsymbol{C}_{\boldsymbol{0}} = 2, \boldsymbol{C}_{2n-2} = C_2 = -1$... and zeros otherwise. On the other hand $(\boldsymbol{W}_- + \boldsymbol{W}_+)$ is in this particular case another circulant matrix C' with elements $\boldsymbol{C'}_0 = 2\alpha_+, \boldsymbol{C'}_{2n-1} = \boldsymbol{C'}_{\boldsymbol{1}} = \alpha_-, \alpha_\mp(x) = \sqrt{x+1} \mp \sqrt{x-1}$ and zeros otherwise. Because circulant matrices form a commutative subalgebra $\boldsymbol{C}$ and $\boldsymbol{C'^{-1}}$ can simultaneously be diagonalized. The eigenvalues of **C** and **C'** are given by

$$\lambda_j = 2\left(1 - \cos\left(\frac{2\pi j}{n}\right)\right) \;\; ; \;\; \lambda'_j = \alpha_+ + \alpha_- \cos(\frac{\pi j}{n})$$

then it follows that :

$$\ln\left(Tr(\rho^n)\right) = \frac{-1}{8}\sum_{j=0}^{2n-1}(1-\cos\left(\frac{2\pi j}{n}\right))\int_a^\infty dx \frac{\alpha_-(x)}{(x^2-1)(\alpha_+ + \alpha_- \cos\left(\frac{\pi j}{n}\right))}$$

The integrals inside the sum can be exactly evaluated and after some arrangements and simplifications we end up with

$$\ln\left(Tr(\rho^n)\right) = \sum_{j=1}^{2n} \ln(2\frac{\beta+\cos\left(\frac{\pi j}{n}\right)}{\beta^2-1}) - \frac{\cos\left(\frac{\pi j}{n}\right)}{4}\ln(\frac{a+1}{a-1})$$

Where $\beta = a + \sqrt{a^2-1}$.

The sum can be evaluated exactly using standard products and sum formulas leading to

$$\ln\left(Tr(\rho_A^n)\right) = \frac{(\alpha^n - \alpha^{-n})^2}{(\beta^2-1)^{2n}},\ \ \beta = \frac{\alpha^2-1}{2\alpha}$$

Here our α turns out to be related to ξ defined in [5] by $\alpha = 1/\sqrt{\xi}$.

Using the replica trick and the definition of α it is straightforward to show that

$$-\frac{d}{dn}\ln\left(Tr(\rho_A^n)\right)\Big|_{n=1} = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln(\xi)$$

References

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