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Quadruple Inequalities : Between CauchySchwarz and Triangle

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Summary

This study aims to clarify the definition of the quadruple inequality, first in metric spaces, and then in vector spaces and inner product spaces. We also explore some special cases and present related corollaries. In particular, we examine an inequality that connects four points and results in six pairwise distances. Additionally, we investigate the relationships among the four inequalities, including those involving the Cauchy-Schwarz and triangle inequalities.
Keywords : Quadruple inequality- Cauchy-Schwarz inequality-triangle inequality- Metric space.

Résumé

Cette étude vise à clarifier la définition de l'inégalité quadruple, d'abord dans les espaces métriques, puis dans les espaces vectoriels et les espaces à produit scalaire. Nous explorons également certains cas particuliers et présentons des corollaires associés. En particulier, nous examinons une inégalité reliant quatre points et donnant lieu à six distances paires. De plus, nous étudions les relations entre les quatre inégalités, y compris celles impliquant les inégalités de Cauchy-Schwarz et du triangle.
Mot-clés : Inégalité quadruple - Inégalité de Cauchy-Schwarz - Inégalité triangulaire - Espace métrique.

مـلـخص

"تهدف هذه الدراسة الى توضيح تعريف متباينة رباعية, بداية في الفضاءات المترية، ثم في الفضاءات النظيمية ثم فضاءات الجداء السلمي. كما نستعرض بعض الحالات الخاصة و نقدم بعض النتائج التابعة لها. على وجه الخصوص، نوضح العلاقة بين أربعة متباينات، بما في ذلك تلك التي تتضمن متباينتي كوشي-شفارتز و المتباينة المثلثية."
الكلمات المفتاحية :
المتباينة الرباعية - متباينة كوشي- شفارتز - المتباينة المثلثية - الفضاء المتري.

Contents

General Introduction

The aim of our study is clarifying the quadruple inequalities: between Cauchy -Schwarz and triangle.

The inequality is a comparison between two expressions. It is very useful in mathematics, physics, chemistry, ...

In mathematics, inequalities have a storied history; where we know that Egyptian and Mesopotamia used them in the measure land and in the build. Around 30 BC, the philosopher Euclid used geoMetric inequalities to prove his geoMetric theories, which was written in 250 AD. After that, the italian Niccol Fontana Tartaglia (1499-1557) and the french François Viéte (1540-1603) generalise the use of algebraic methods to solve them during the renaissance.

Inequalities have a many mathematical studies like Pierre de Fermat, René Descartes, and Johann bernoulli in the 17th, 18th centuries. These mathematicians help us in current studies. Then in 1800s, inequalities was generalize by a many of mathematicians; most notably: Joseph-Louis Lagrange, Hermann Amandus Schwarz and Augustin- louis Cauchy . Their studies are still use today. And one of them is the quadruple inequalities: between Cauchy -Schwarz and triangle, it studied by Christof Schötz. Which allows the application of triangular inequality when we have four points in CAT(0) spaces. see [4]

In this work, we define the important notation and concepts in our study. Then, we represent the relationship between the Cauchy -Schwarz inequality and triangle. Finally, we give the important corollaries and applied that in Euclidean Metric space: $\mathbb{R}^2$.

General Notations

Let $\mathbb{E}, \mathbb{V}$ be two set, and $\mathbb{A}, \mathbb{B}$ are a sub-set of $\mathbb{E}$.

Notation	:	**Definition**
$\mathbb{A} \subset \mathbb{X}$	:	$\forall x \in A, x \in X$.
$\mathbb{A}^c$	:	the complement of the set $\mathbb{A}$.
$\emptyset$	:	the empty set.
$\mathbb{C}$	:	the set of complex numbers.
$\mathbb{R}$	:	the set of real numbers.
$\mathbb{N}$	:	the set of natural numbers.
$i.e$	:	it means.
$\mathbb{A} \cap \mathbb{B}$	$\doteqdot$	$\{x \in \mathbb{A}$ such that $x \in \mathbb{B}\}$.
$\mathbb{A} \cup \mathbb{B}$	$\doteqdot$	$\{x \in \mathbb{A}$ or $x \in \mathbb{B}\}$.
$\mathbb{X} \times \mathbb{Y}$	$\doteqdot$	$\{(x, y); x \in \mathbb{X}, y \in \mathbb{Y}\}$.
$\inf(\mathbb{A})$	$\doteqdot$	$\{\forall \varepsilon > 0, \exists x \in \mathbb{A}, x < \inf(\mathbb{A}) + \varepsilon\}$.

Chapter 1

Definition and Notations

In this chapter, we recall a many mathematical consepts. we begin by the Metric space, the normed space and inner product space. After that, we present trianglar inequality and quadruple inequalities witch are the important tools in our study. Finally, we represent the relation between them. The main works used in this part of our work are [2], [18] and [13].

1.1 Metric Space

In all this work, we define E a set which

1.1.1 Definitions

Definition 1.1 (see [18])**.** *Let d be a map from $E \times E$ in $\mathbb{R}_+$ verifies:*

1. *Symmetric: $d(x, y) = d(y, x)$;*

2. *Positivity: $d(x, y) \geq 0$ if $x \neq y$, and $d(x, x) = 0$;*

3. *triangular inequality: $d(x, z) \leq d(x, y) + d(y, z)$.*

(each side of a triangle is at most equal to the sum of the other two).

Remark 1.1. *The inequality is immediately deduced:*

$$d(x, z) \geq |d(x, y) - d(y, z)|\,.$$

(each side of a triangle is at least equal to the difference of the other two).

This also implies that if $x_1.x_2, ..., x_n$ are n arbitrary points of E we have:

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{n-1}, x_n).$$

Definition 1.2 (see [18])**.** *Let (E, d) be a Metric space, $a \in E$, and $r > 0$.*

1. *We call* ***open ball*** *of center a and radius r the whole:*

$$B(a, r) = x \in E; d(a, x) < r.$$

2. *We call* ***closed ball*** *of center a and radius r the whole:*

$$B'(a, r) = x \in E; d(a, x) \leq r.$$

3. *We call* ***sphere*** *of center a and radius r the set:*

$$S(a, r) = x \in E; d(a, x) = r.$$

1.1.2 Topology of Metric Spaces

Proposition 1.1 (see [13])**.** *Let (E, d) be a Metric space.*

1. *The set E is the union of open balls.*

2. *Let $a \in E$ and $r > 0$. Let $x \in B(a, r)$ and $\rho = r - d(a, x)$, then $\rho >$ 0 and we have*

$$B(x, \rho) \subset B(a, r).$$

3. *Let $a, b \in E$ and $r_1, r_2 >$ 0. For any $x \in B(a, r_1) \cap B(b, r_2)$, there exists $\rho >$ 0 such as*

$$B(x, \rho) \subset B(a, r_1) \cap B(b, r_2).$$

So the intersection of two open balls is union of open balls.

Proof. 1. Let $a \in E$ and $r > 0$, then we have $a \in B(a, r)$, so E is the union of open balls.

2. Let $y \in B(x, \rho)$, then we have

$$d(x, y) < \rho = r - d(a, x),$$

Hence

$$d(a,x)+d(x,y)<r.$$

According to the triangular inequality, we have

$$d(a,y)\leq d(a,x)+d(x,y).$$

Therefore, we have $d(a,y)<r$. That is, we have $y\in B(x,r)$, so $B(x,\rho)\subset B(x,r)$.

3. Let $\rho=\inf\{r_1-d(a,x),r_2-d(b,x)\}$, then $\rho>0$ and it follows from 2 that we have

$$B(x,\rho)\subset B(a,r_1)\cap B(b,r_2)$$

.

□

Let U be a subset of E, then U is an open for topology $\mathcal{T}_d$ if, and only, if for all $x\in U$, there exists $r>0$ such that $B(x,r)\subset U$.

Proposition 1.2 (see [13])**.** *Let (E,d) be a Metric space. Then it is separated.*

Proof. Let be $x,y\in E$ such that $x\neq y$, then we have $d(x,y)>0$. Let $r=\dfrac{d(x,y)}{2}$, then we have $B(x,r)\cap B(y,r)=\emptyset$.
Therefore, (E,d) is separated. □

Converging sequence

Proposition 1.3 (see [13])**.** *Let $(x_n)_{n\geq 0}$ be a sequence in topological space (E,d).*
We call that the sequence $(x_n)_{n\geq 0}$ is ***convergent*** *if it admits a limit.*
In other word, $(x_n)_{n\geq 0}$ is converging sequence if

$$\lim_{n\longrightarrow+\infty}x_n=l\Leftrightarrow\forall\varepsilon>0,\exists N\in\mathbb{N},\ \textit{tel que}\ \forall n\geq N,d(x_n,l)\leq\varepsilon.$$

1.2 Normed Space

1.2.1 Vector Space

Definition 1.3. *Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every u,v, and w in V and every scalar (real number) c and d, then V is called a vector space.*

- *Addition:*

  (1) $u + v$ *is in* V.

  (2) $u + v = v + u$.

  (3) $u + (v + w) = (u + v) + w$.

  (4) V *has a zero vector* 0 *such that for every* u *in* V, $u + 0 = u$.

  (5) *For every* u *in* V, *there is a vector in* V *denoted by* $-u$. *such that* $u + (-u) = 0$.

- *Scalar multiplication:*

(6) cu *is in* V.

(7) $c(u + v) = cu + cv$.

(8) $(c + d)u = cu + du$.

(9) $c(du) = (cd)u$.

(10) $1(u) = u$.

Example 1.1. 1. *n-tuple space:* $\mathbb{R}^n$ *with: vector addition:* $(u_1, u_2, ..., u_n) + (v_1, v_2, ..., v_n) = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n)$.
scalar multiplication: $k(u_1, u_2, ..., u_n) = (ku_1, ku_2, ..., ku_n)$.

2. *Matrix space:* $V = M_{m\times n}$ *(the set of all* $m \times n$ *matrices with real values).*
Ex: $(m = n = 2)$ *vector addition:*

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22}. \end{bmatrix}.$$

scalar multiplication:

$$k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22}. \end{bmatrix}.$$

3. *n-th degree polynomial space:* $V = P_n(x)$ *(the set of all real polynomials of degree* n *or less).*
Vector addition:

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + ... + (a_n + b_n)x^n.$$

Scalar multiplication:

$$kp(x) = ka_0 + ka_1x + ... + ka_nx^n.$$

4. *Function space:* $c(-\infty, +\infty)$ *(the set of all real-valued continuous functions defined on the entire real line).*

$$(f + g)(x) = f(x) + g(x).$$

$$(kf)(x) = kf(x).$$

1.2.2 Normed Space

Definition 1.4 (see [18])**.** *A **norm** on a* $\mathbb{K}$*- vector space* E *is a map:*

$$\begin{array}{rcl} \|\| \;:\; E & \rightarrow & \mathbb{R}_+ \\ x & \rightarrow & \| x \| \end{array}$$

verifies the following properties:

(N1) For all $x \in E$ *non null, we have* $\| x \| \neq 0$*;*

(N2) For all $x \in E$ *and all* $\lambda \in \mathbb{K}$*, we have* $\| \lambda x \| = |\lambda| \, \| x \|$*;*

(N3) For all $x, y \in E$*, we have* $\| x + y \| \leq \| x \| + \| y \|$*.* *(**convexity inequality**).*

The space E*, provided with a norm* $\|\|$*, is called **normed space** or **normed vector space** (or* $\mathbb{K}$*-**normed vector space**, if we want to specify the (corps)* $\mathbb{K}$*). We often denote such a space* $(E, \|\|)$*. We deduce from the property (N2) that we have* $\| 0 \| = 0$*.*

Proposition 1.4 (see [18])**.** *Let* $(E, \|\|)$ *be a normed space.*

1. *The above property (N2) is equivalent to the following weakened form:*
   (N2') For all $x \in E$ *and all* $\lambda \in \mathbb{K}$*, we have* $\| \lambda x \| \leq |\lambda| \, \| x \|$*.*

2. *We can replace the property (N3) above by the following property :*
   (N3') For all $x, y \in E$ *and all* $t \in [0, 1]$*, we have:*

$$\| tx + (1 - t)y \| \leq t \| x \| + (1 - t) \| y \| .$$

3. *For all* $x, y \in E$*, we have the inequality* $|\| x \| - \| y \|| \leq \| x - y \|$*.*

4. *The inequality (N3) generalizes to n points. For all $x_1, x_2, ..., x_n \in E$, we have:*

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} \| x_i \| .$$

Proof. 1. It's clear that (N2)$\Longrightarrow$ (N2').Reciprocally, we suppose (N2') verified.

Let $x \in E$ and $\lambda \in \mathbb{K}$.

If $\lambda = 0$, so

$$\begin{aligned} 0 &\leq \| \lambda x \| \\ &\leq |\lambda| \| x \| = 0. \end{aligned}$$

If $\lambda \neq 0$, so we have

$$\begin{aligned} |\lambda| \| x \| &= |\lambda| \| \lambda^{-1}(\lambda x) \| \\ &= \| \lambda x \| \\ &\leq |\lambda| \| x \| . \end{aligned}$$

We deduce (N2).

2. We suppose (N3) verified. So for all $x, y \in E$ and all $t \in [0, 1]$, we have:

$$\begin{aligned} \| tx + (1-t)y \| &\leq \| tx \| + \| (1-t)y \| \\ &= t \| x \| + (1-t) \| y \| . \end{aligned}$$

Reciprocally, we suppose (N3') verified. Let $x, y \in E$, taking $t = \frac{1}{2}$, we get:

$$\begin{aligned} \| x + y \| &= \| \tfrac{1}{2}(2x) + \tfrac{1}{2}(2y) \| \leq \tfrac{1}{2} \| 2x \| + \tfrac{1}{2} \| 2y \| \\ &= \tfrac{1}{2} 2 \| x \| + \tfrac{1}{2} 2 \| y \| \\ &= \| x \| + \| y \| . \end{aligned}$$

3. We have

$$\begin{aligned} \| x \| &= \| y + (x-y) \| \\ &\leq \| y \| + \| x - y \|, \end{aligned}$$

so

$$\| x \| - \| y \| \leq \| x - y \| .$$

Also, we have

$$\begin{aligned} \| y \| - \| x \| &\leq \| y - x \| \\ &= \| x - y \| . \end{aligned}$$

Consequently, we have

$$| \| x \| - \| y \| | \leq \| x - y \|$$

. the last propriety is trivial. We verify that by induction on n.

$\square$

Topology of Normed Space

We have already introduced open balls in normed spaces.

Definition 1.5. *Let $(E, \|\|)$ be a normed space. A subset $G \subset E$ is called open (in $(E, \|\|)$) if for all $x \in G$ there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. A subset $F \subset E$ is called closed if $F^c := X\ F$ is open. The interior of $A \subset E$ is the union of all open subsets contained in A, i.e.,*

$$intA := A^\circ := \cup_{G \subset A, G\ open} G.$$

The closure of $A \subset E$ is the intersection of all closed supersets of A, i.e.,

$$clA := \overline{A} := \cap_{F \subset A, F\ closed} F.$$

The following properties now follow straight from the definitions and de Morgan's laws.

Proposition 1.5. *Let $(E, \|\|)$ be a normed space.*

1. *The open balls in E are open.*
2. *$\emptyset$ and E are open.*
3. *If $U_1, ..., U_n$ are open, then $\cap_{k=1}^n U_k$ is open.*
4. *If $(U_j)_{j \in J}$ is a family of open sets, then $\cup_{j \in J} U_j$ is open.*
5. *$\emptyset$ and E are closed.*
6. *If $A_1, ..., A_n$ are closed, then $\cup_{k=1}^n A_k$ is closed.*
7. *If $(A_j)_{j \in J}$ is a family of closed sets, then $\cap_{j \in J} A_j$ is closed.*
8. *Let $A \subset E$. Then the interior of A is open, the closure of A is closed, and $intA \subset A \subset clA$. Moreover, A is open if, and only, if $A = intA$, and A is closed if, and only, if $A = clA$.*

The definition of closed sets and the closure is inconvenient for practical purposes. The following result connects these notions with the convergence of sequences.

Proposition 1.6. *Let $(E, \|\|)$ be a normed space and $A \subset E$. Then $x \in clA$ if, and only, if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that $x_n \longrightarrow x$ in E.*

Proof. Let $x \in clA$. We claim that then for all $n \in \mathbb{N}$ one has $B\left(x, \dfrac{1}{n}\right) \cap A = \emptyset$. In fact, assume for contradiction that there exists an $n_0 \in \mathbb{N}$ such that $B\left(x, \dfrac{1}{n_0}\right) \cap A = \emptyset$. Then$B\left(x, \dfrac{1}{n_0}\right)^c$ is closed and contains A. Hence $clA \subset B\left(x, \dfrac{1}{n_0}\right)^c$, and hence $x \notin A$, which is a contradiction. So ther exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_n \in B\left(x, \dfrac{1}{n}\right)$ for all $n \in \mathbb{N}$. Then $x_n \longrightarrow x$ in E.

Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in A that converges to $x \in E$. We need to show that $x \in clA$. Let F be closed such that $A \subset F$. It suffices to show $x \in F$ as then by generalisation x is in the intersection of all closed supersets of A.
Assume for contradiction that $x \in F^c$. As F^c is open, there exists an $\varepsilon > 0$ such that $B(x, \vartheta) \subset F^c$. But there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has $x_n \in B(x, \varepsilon) \subset F^c$. This is a contradiction since $B(x, \varepsilon) \cap F = \emptyset$, but $x_{n_0} \in B(x, \varepsilon) \cap F$. We have proved that $x \in clA$. $\square$

We obtain the following consequence.

Corollary 1.1. *Let $(E, \|\|)$ be a normed space and $A \subset E$. Then A is closed if, and only, if it contains the limit of every convergent sequence of elements in A,
i.e., if $(x_n)_{n\in\mathbb{N}}$ is a sequence in A such that $x_n \longrightarrow x$ in E, then $x \in A$.*

1.3 Distance associated by a norm

Let $(E, \|\|)$ be a normed space. For all $x, y \in E$, we put $d(x, y) = \| x - y \|$. It is easy to verify that d is a distance on E, called **the distance associated by a norm** . The corresponding topology will be called **normal topology** . Except mention of contrary, we provide all normed spaces to the normed topology. The distance d norm associated by the norm has the following properties:

(a) $d(x + z, y + z) = d(x, y)$ for each $x, y, z \in E$.

(b) $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x \in E$, and all $\lambda \in \mathbb{K}$.
Conversely, if E be a $\mathbb{K}$-vector space provided with a distance d having the properties (a) and (b) above, so d is defined from a norm; It suffices to put $\| x \| = d(x, 0)$, for all $x \in E$.

Proposition 1.7 (see [18]). *Let $(E, \|\|)$ be a normed space. Then we have*

1. *The function:*

$$\begin{array}{ccc} E \times E & \longrightarrow & E \\ (x, y) & \longmapsto & x + y \end{array}$$

is lipschitzien, so uniformly continuous.

2. *The following function is continuous.*

$$\begin{array}{ccc} \mathbb{K} \times E & \longrightarrow & E \\ (\lambda, x) & \longmapsto & \lambda x \end{array}$$

3. *The following function is lipschitzien, so uniformly continuous.*

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \| x \| \end{array}$$

4. *The translation, and the homotheties of non-null ratio are homeomorphisms of E. In the other words, for all $a \in E$ and all non-null $\lambda \in \mathbb{K}$, the following function are homeomorphisms of E.*

$$\begin{array}{ccc} E & \longrightarrow & E \\ x & \longmapsto & x + a \end{array}$$

and

$$\begin{array}{ccc} E & \longrightarrow & E \\ x & \longmapsto & \lambda x \end{array}$$

Corollary 1.2 (see [13])**.** *Let $(E, \|\|)$ a $\mathbb{K}$-normed vector space, $(x_n)_{n\geq 0}$,$(y_n)_{n\geq 0}$ two converging sequences in E and $(\lambda)_{n\geq 0}$ be converging sequence in $\mathbb{K}$. Then $(x_n + y_n)_{n\geq 0}$, $(\lambda_n x_n)_{n\geq 0}$ and $(\| x_n \|)_{n\geq 0}$ are convergent in their respective spaces, and we have $\lim_{n\longrightarrow+\infty} x_n + y_n = \lim_{n\longrightarrow+\infty} x_n + \lim_{n\longrightarrow+\infty} y_n$, $\lim_{n\longrightarrow+\infty} \lambda_n x_n = (\lim_{n\longrightarrow+\infty} \lambda_n)(\lim_{n\longrightarrow+\infty} x_n)$ et $\lim_{n\longrightarrow+\infty} \| x_n \| = \| \lim_{n\longrightarrow+\infty} x_n \|$.*

Definition 1.6 (see [13])**.** *Tow norms $\|\|$ and $\|\|'$ on a vector space E are called* ***equivalents*** *if there exists $\alpha, \beta \in]0, +\infty[$ such that for all $x \in E$, we have $\alpha \| x \| \leq \| x \|' \leq \beta \| x \|$.*

We will see that tow norms on E are equivalents if, and only, if they defined the same topology on E.

Remark 1.2. *Let E be vector space and $\|\|$,$\|\|'$ two norms on E.If there exists a sequence $(x_n)_{n\geq 0}$ in E such that the sequence $(\| x_n \|)_{n\geq 0}$ is bounded and the sequence $(\| x_n \|')_{n\geq 0}$ isn't bounded so two norms $\|\|$ and $\|\|'$ aren't equivalents.*

Cauchy sequence

Definition 1.7 (see [13]). *Let (E,d) be a Metric space.*
The sequence $(x_n)_{n\geq 0}$ in (E,d) is Cauchy sequence if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, m \geq n_0$, we have $d(x_m, x_n) < \varepsilon$.

It amounts to the same thing to say that for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all $p \in \mathbb{N}$, we have $d(x_{n+p}, x_n) < \varepsilon$.

Proposition 1.8 (see [13]). *Let (E,d) be a Metric space.*

1. *All converging sequence in E is Cauchy .*
2. *All Cauchy sequence in E is bounded.*
3. *All sub-sequence of Cauchy sequence is Cauchy .*

Proposition 1.9. *Let (E,d) be a Metric space and $(x_n)_{n\geq 0}$ be Cauchy sequence in E.*

The sequence $(x_n)_{n\geq 0}$ is convergent if, and only, if it has a convergent subsequence. In other words, a Cauchy sequence in E with an **adherence value** *is convergent. The adherence value is then unique. This is the limit of the sequel.*

Complete space

Definition 1.8 (see [13]). *Let (E,d) be a Metric space.*

1. *We say that (E,d) is **complete** if every Cauchy sequence in (E,d) is convergent.*
2. *A subset A of (E,d) is said to be complete if A with the induced distance is a complete Metric space.*

The fundamental importance of complete Metric spaces is in the fact that, to demonstrate that a sequence is convergent in an abstract space, it isn't necessary to know in advance the value of the limit.

Example 1.2. *Let (E,d) be a discrete Metric space, so (E,d) is complete because we have $d(x,y) = 0$ if $x \neq y$, so all Cauchy sequence in E is stationary, then convergent.*

Banach space

Definition 1.9 (see [13]). *A **Banach** space be a normed space $(E, \|\|)$, witch is complete for the distance associated to the norm.*

Remark 1.3 (see [13]). *Let E be a vector space and $\|\|, \|\|'$ two equivalent norms on E. So $(E, \|\|)$ is a Banach space if, and only, if $(E, \|\|')$ be a Banach space.*

1.4 Inner Product Space

In the following all vector spaces are assumed to be over the real or complex field.

Definition 1.10. *Let V be a complex or real vector space. The function $\langle .,. \rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$) is an inner product if it satisfies the following properties for all vectors $x, y, z \in V$ and all scalars $\alpha, \beta \in \mathbb{C}$ (or $\mathbb{R}$).*

1. $\langle x, x \rangle \in \mathbb{R} \geq 0$ *and* $\langle x, x \rangle = 0 \Longrightarrow x = 0$,
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, *(for a real vector space $\langle x, y \rangle = \langle y, x \rangle$).*

Definition 1.11. *A vector space with a particular inner product defined on it is called an **inner product space**, sometimes a real or complex inner product space depending on the field over which the vector space is defined.*

Example 1.3. *A first example is the real or complex (standard) Euclidean inner product over $\mathbb{R}^n$ or $\mathbb{C}^n$:*

$$x, y \in \mathbb{R}^n : \langle x, y \rangle = y^T x = \sum_{i=1}^{n} x_i y_i.$$

$$x, y \in \mathbb{C}^n : \langle x, y \rangle = y^T x = \sum_{i=1}^{n} x_i \overline{y}_i.$$

We show the properties for the complex case, $x, y \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$ (Note that this constitutes a proof that the Euclidean inner product is indeed an inner product).

1. $\langle x, x \rangle = x^T x = \sum_{i=1}^{n} x_i \overline{x}_i = \sum_{i=1}^{n} |x_i|^2 \geq 0$ Moreover, $\langle x, x \rangle = |x_i|^2 = 0 \Leftrightarrow x_i = 0$ for $i = 1, ..., n \Leftrightarrow x = 0$
2. $\langle x + y, z \rangle = z^T (x + y) = \sum_{i=1}^{n} (x_i + y_i) \overline{z}_i = \sum_{i=1}^{n} (x_i \overline{z}_i + y_i \overline{z}_i) = \sum_{i=1}^{n} (x_i \overline{z}_i) + \sum_{i=1}^{n} (y_i \overline{z}_i) = \langle x, z \rangle + \langle y, z \rangle$.

3. $\langle \alpha x, y\rangle = y^T(\alpha x) = \sum_{i=1}^{n}(\alpha x_i)\overline{y}_i = \alpha \sum_{i=1}^{n} x_i\overline{y}_i = \alpha\langle x, y\rangle.$

4. $\langle y, x\rangle = x^T y = \sum_{i=1}^{n} y_i\overline{x}_i = \overline{\sum_{i=1}^{n} x_i\overline{y}_i} = \overline{\langle x, y\rangle}.$

Example 1.4. *Consider $\mathbb{P}_1$, with a standard basis $\mathfrak{B} = \{1, t\}$. We will show that the following defines a valid inner product on this space:*

$$\langle p, q\rangle := [q]_{\mathfrak{B}}^T[p]_{\mathfrak{B}}.$$

* *Let $p(t) = a_0 + a_1t$ be arbitrary. Since $\langle p, p\rangle = [p]_{\mathfrak{B}}^T[p]_{\mathfrak{B}} = a_0^2 + a_1^2$, this quantity is clearly greater than or equal to zero. It is zero iff both a_0 and a_1 are 0, in which case p is the zero polynomial (which is the 0 of the vector space).*

- *Let $p(t) = a_0 + a_1t$, $q(t) = b_0 + b_1t$, $z(t) = c_0 + c_1t$ be arbitrary vectors. Then*

$$\begin{aligned}
\langle p+z, q\rangle &= [b_0, b_1]\begin{pmatrix}(a_0+c_0)\\(a_1+c_1)\end{pmatrix}\\
&= [b_0, b_1]\begin{pmatrix}a_0\\a_1\end{pmatrix} + [b_0, b_1]\begin{pmatrix}c_0\\c_1\end{pmatrix}\\
&= \langle p, q\rangle + \langle z, q\rangle.
\end{aligned}$$

- *Let $p(t)$, $q(t)$ arbitrary as before, and $c \in \mathbb{R}$. Then*

$$\begin{aligned}
\langle cp, q\rangle &= [b_0, b_1]\begin{pmatrix}(ca_0)\\(ca_1)\end{pmatrix}\\
&= c[b_0, b_1]\begin{pmatrix}a_0\\a_1\end{pmatrix}\\
&= c\langle p, q\rangle.
\end{aligned}$$

- *Let $p(t)$, $q(t)$ arbitrary as before, then $\langle p, q\rangle = \langle q, p\rangle$ easily follows from the fact that for two vectors a, b in $\mathbb{R}^2$, $b^Ta = a^Tb$.*

Definition 1.12. *Given an inner product on a vector space V, we say that two vectors $v, u \in V$ are orthogonal if their inner product is zero.*

Important: if you change the inner product on a vector space, you may not preserve orthogonality!

For example, if $V = \mathbb{P}_1$, and given $p, q \in V$, if they are orthogonal in the inner product defined in Example 1.4, it does not necessarily mean they are orthogonal in, say, the $L_2[a, b]$ inner product. Indeed, $\mathbb{P}_1$ equipped with the Example 1.4 inner product is a different inner product space than $\mathbb{P}_1$ equipped with the $L_2[a, b]$ inner product.

1.5 Triangle inequality and its variation

The triangle inequality is one of the most important identities in calculus. If you haven't seen it in the past or don't know the proof, then this is document is for you! I will be using it throughout the course and it comes up enough that I think you guys should see the proof.

Recall: Given a real number$x$ we know intuitively that $|x|$ is the size of x. More formally, we can say that

$$|x| = \sqrt{x^2}.$$

When we say that $|x| < y$ we mean that the size of x is less than y independent of the sign of x. So $|x| \leq y$ is a shorter but equivalent way of saying

$$x \leq y \text{ and } -x \leq y.$$

With that bit of notation out of the way we can now state the triangle inequality.

Theorem 1.1 (Triangle inequality). *If a, b are real numbers then,*

$$|a + b| \leq |a| + |b|.$$

Proof. Since $x \leq |x|$ for all x we have $ab \leq |ab|$. Thus we have

$$a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2.$$

Since both sides are perfect squares the above is equivalent to,

$$(a + b)^2 \leq (|a| + |b|)^2.$$

By taking square roots of both sides we the triangle inequality,

$$|a+b| \leq |a| + |b|.$$

□

Sometimes the following corollary is useful.

Reverse triangle inequality

Corollary 1.3. *If a, b are real numbers then,*

$$\Big||a| - |a|\Big| \leq |a-b|.$$

Proof. Note that by the triangle inequality we have,

$$\begin{array}{rcccl} |a| & = & |(a-b)+b| & \leq & |a-b| + |b| \\ |b| & = & |(b-a)+a| & \leq & |b-a| + |a|. \end{array}$$

Or equivalently by rearranging,

$$|a| - |b| \leq |a-b| \text{ and } |b| - |a| \leq |a-b|.$$

By the remark above this is equivalent to the reverse triangle inequality,

$$\Big||a| - |a|\Big| \leq |a-b|.$$

□

1.6 Cauchy -Schwarz inequality and its signification

The Cauchy -Schwarz inequality is an elementary inequality and at the same time a powerful inequality, which can be stated as follows:

Theorem 1.2. *Let $(a_1, a_2, ..., a_n)$ and $b_1, b_2, ..., b_n)$ be two sequences of real numbers, then*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2. \tag{1.1}$$

with equality if, and only, if the sequences $(a_1, a_2, ..., a_n)$ *and* $b_1, b_2, ..., b_n)$ *are proportional, i.e., there is a constant* λ *such that* $a_k = \lambda b_k$ *for each* $k \in \{1, 2, ..., n\}$.

As is known to us, this classical inequality plays an important role in different branches of modern mathematics including Hilbert spaces theory, probability and statistics, classical real and complex analysis, numerical analysis, qualitative theory of differential equations and their applications (see [1-12]). In this paper we show some different proofs of the Cauchy -Schwarz inequality.

Some different proofs of the Cauchy -Schwarz inequality

Proof. Expanding out the brackets and collecting together identical terms we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 &= \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{i=1}^{n} b_i^2 \sum_{j=1}^{n} a_j^2 - 2\sum_{i=1}^{n} a_ib_i \sum_{j=1}^{n} b_ja_j \\ &= 2\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - 2\left(\sum_{i=1}^{n} a_ib_i\right).\end{aligned}$$

because the left-hand side of the equation is a sum of the squares of real numbers it is greater than or equal to zero, thus

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

□

Proof. Consider the following quadratic polynomial

$$f(x) = \left(\sum_{i=1}^{n} a_i^2\right)x^2 - 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \left(\sum_{i=1}^{n} b_i^2\right) = \sum_{i=1}^{n}(a_ix - b_i)^2.$$

Since $f(x) \geq 0$ for any $x \in \mathbb{R}$, it follows that the discriminant of $f(x)$ is negative, i.e.,

$$\left(\sum_{i=1}^{n} a_ib_i\right) - \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0.$$

The inequality (1.2) is proved. □

Proof. When $\sum_{i=1}^{n} a_i^2 = 0$ or $\sum_{i=1}^{n} b_i^2 = 0$, (1.2) is an identity. We can now assume that

$$A_n = \sum_{i=1}^{n} a_i^2 \neq 0, B_n = \sum_{i=1}^{n} b_i^2 \neq 0, x_i = \frac{a_i}{\sqrt{A_n}}, y_i = \frac{ab_i}{\sqrt{B_n}} \text{ for } (i = 1, 2, ..., n),$$

then

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = 1.$$

The inequality (1.2) is equivalent to

$$x_1y_1 + x_2y_2 + ... + x_ny_n \leq 1,$$

that is

$$2(x_1y_1 + x_2y_2 + ... + x_ny_n) \leq x_1^2 + x_2^2 + ... + x_n^2 + y_1^2 + y_2^2 + ... + y_n^2.$$

Or equivalently

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2 \geq 0,$$

which is evidently true. The desired conclusion follows. □

Proof. Let $A = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$, $B = \sqrt{b_1^2 + b_2^2 + ... + b_n^2}$.
By the arithmetic-geometric means inequality, we have

$$\sum_{i=1}^{n} \frac{a_ib_i}{AB} \leq \sum_{i=1}^{n} \frac{1}{2}\left(\frac{a_i^2}{A^2} + \frac{b_i^2}{B^2}\right) = 1,$$

so that

$$\sum_{i=1}^{n} a_ib_i \leq AB = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}.\sqrt{b_1^2 + b_2^2 + ... + b_n^2}.$$

Thus

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

□

Proof. Construct the vectors $\alpha = (a_1, a_2, ..., a_n)$, $\beta = (b_1, b_2, ..., b_n)$. Then for arbitrary real numbers t, one has the following identities for scalar product:

$$\begin{aligned}
(\alpha + \beta t).(\alpha + \beta t) \quad &= \quad \alpha.\alpha + 2(\alpha.\beta)t + (\beta.\beta)t^2 \\
&\Longleftrightarrow \quad |\alpha|^2 + 2(\alpha.\beta)t + |\beta|^2t^2 \\
&= \quad |\alpha + t\beta|^2 \geq 0.
\end{aligned}$$

Thus

$$(\alpha.\beta)^2 - |\alpha|^2|\beta|^2 \ \leq \ 0.$$

Using the expressions

$$\alpha.\beta = a_1b_1 + a_2b_2 + ... + a_nb_n, \quad |\alpha|^2 = \sum_{i=1}^{n} a_i^2, \quad |\beta|^2 = \sum_{i=1}^{n} b_i^2,$$

we obtain

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 - \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \ \leq \ 0.$$

□

Proof. Construct the vectors $\alpha = (a_1, a_2, ..., a_n)$, $\beta = (b_1, b_2, ..., b_n)$. From the formula for scalar product:

$$\alpha.\beta \ = \ |\alpha||\beta|\cos(\alpha, \beta),$$

we deduce that

$$\alpha.\beta \ \leq \ |\alpha||\beta|.$$

$$\alpha.\beta = a_1b_1 + a_2b_2 + ... + a_nb_n, \quad |\alpha|^2 = \sum_{i=1}^{n} a_i^2, \quad |\beta|^2 = \sum_{i=1}^{n} b_i^2,$$

we get the desired inequality (1.2). □

Proof. Since the function $f(x) = x^2$ is convex on $(-\infty, +\infty)$, it follows from the Jensen's inequality that

$$(p_1x_1 + p_2x_2 + ... + p_nx_n)^2 \leq p_1x_1^2 + p_2x_2^2 + ... + p_nx_n^2. \tag{1.2}$$

where $x_i \in \mathbb{R}$, $p_i > 0 (i = 1, 2, ..., n)$, $p_1 + p_2 + ... + p_n = 1$.

Case I If $b_i \neq 0$ for $i = 1, 2, ..., n$, we apply $x_i = \frac{a_i}{b_i}$ and $p_i = \dfrac{b_i^2}{b_1^2 + b_2^2 + ... + b_n^2}$ to the inequality 1.2 to obtain that

$$\left(\frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{b_1^2 + b_2^2 + ... + b_n^2}\right)^2 \leq \frac{a_1^2 + a_2^2 + ... + a_n^2}{b_1^2 + b_2^2 + ... + b_n^2},$$

that is

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2).$$

Case II If there exists $b_{i_1} = b_{i_2} = ... = b_{i_k} = 0$, one has

$$\begin{aligned}
\left(\sum_{i=1}^{n} a_i b_i\right)^2 &= \left(\sum_{i \neq i_1, ..., i_k, 1 \leq i \leq n} a_i b_i\right)^2 \\
&\leq \left(\sum_{i \neq i_1, ..., i_k, 1 \leq i \leq n} a_i^2\right)\left(\sum_{i \neq i_1, ..., i_k, 1 \leq i \leq n} b_i^2\right) \\
&\leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).
\end{aligned}$$

This completes the proof of inequality (1.1). □

Proof. Define a sequence $\{S_n\}$ by

$$Sn = (a_1b_1 + a_2b_2 + ... + a_nb_n)^2 - (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2).$$

Then

$$\begin{aligned}
S_{n+1} - S_n &= (a_1b_1 + a_2b_2 + ... + a_{n+1}b_{n+1})^2 - (a_1^2 + a_2^2 + ... + a_{n+1}^2)(b_1^2 + b_2^2 + ... + b_{n+1}^2) \\
&- (a_1b_1 + a_2b_2 + ... + a_nb_n)^2 + (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2),
\end{aligned}$$

which can be simplified to

$$S_{n+1} - S_n = -\left[(a_1b_{n+1} - b_1a_{n+1})^2 + (a_2b_{n+1} - b_2a_{n+1})^2 + ... + (a_nb_{n+1} - b_na_{n+1})^2\right],$$

so

$$S_{n+1} \leq S_n (n \in \mathbb{N}).$$

We thus have

$$S_n \leq S_{n-1} \leq ... \leq S_1 = 0,$$

which implies the inequality (1.2). □

Chapter 2

Quadruple inequality

Introduction

In this chapter, we represent the important theorems, which give us a mean for the relationship between the Cauchy -Schwarz and triangular inequalities. In this chapter, we use [7], [15], [8], [1], [10], [30].

2.1 Relating Cauchy -Schwarz and Triangle Inequality

The Cauchy -Schwarz inequality states that in any inner product space $(V\langle .,.\rangle)$.

$$\langle u, v\rangle \leq \|u\|\|v\|. \tag{2.1}$$

for all $u, v \in V$, where $\|u\| = \sqrt{\langle u, u\rangle}$.

Indeed, by using the inequality (2.1).
For all $u \in V$ we have

$$\langle u, u\rangle \leq \|u\|^2.$$

So,

$$\|u\| \geq \sqrt{\langle u, u\rangle}. \tag{2.2}$$

And, for all $u \in V$ so $-u \in V$ we have

$$\begin{aligned} \langle -u, u \rangle &= -\langle u, u \rangle \\ &\leq -\|u\|\|u\| \\ &= -\|u\|^2. \end{aligned}$$

Then, $\langle u, u \rangle \geq \|u\|^2$. So,

$$\|u\| \geq \sqrt{\langle u, u \rangle}. \tag{2.3}$$

By using (2.2) and (2.3), then

$$\|u\| = \sqrt{\langle u, u \rangle}.$$

In any Metric space (E, d) the triangle inequality

$$\overline{p,z} \leq \overline{p,y} + \overline{y,z}. \tag{2.4}$$

is true for all $p, y, z \in E$, where we use the short notation $\overline{p,z} := d(p; z)$.

Definition 2.1. *Let (E, d) be a Metric space. Consider the inequality be:*

$$f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}) \leq L_f \overline{q,p} f'(\overline{y,z}). \tag{2.5}$$

for $y, z, q, p \in E$, a differentiable function $f : [0, \infty) \to \mathbb{R}$ with derivative f', and a constant $L_f \in [0, \infty)$.

We call (2.5) a quadruple inequality, since it creates a relationship between the six distances among four points. See figure 3.2.

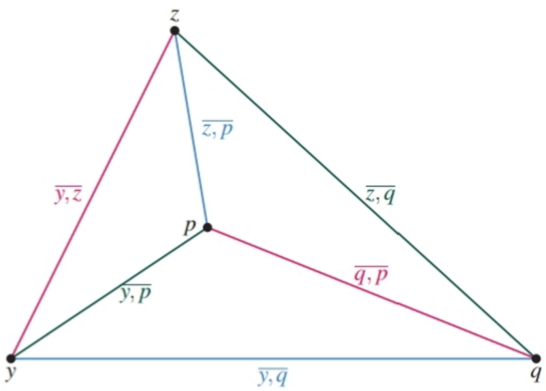

Figure 2.1: Four points and their six distances.

Proposition 2.1. *For $x, y, p, q \in E$, and for $f_1(x) = x$, $f_1'(x) = 1$ and $L_f = 2$ we obtain:*

$$\overline{y,q} - \overline{y,p} - \overline{z,q} + \overline{z,p} \leq 2\overline{q,p}. \tag{2.6}$$

Proof. For all $y, p, q \in E$, we have:

$$\overline{y,q} \leq \overline{y,p} + \overline{p,q}.$$

And for all $z, p, q \in E$, we have:

$$\overline{z,p} \leq \overline{z,q} + \overline{q,p}.$$

So,

$$\overline{y,q} + \overline{z,p} \leq \overline{y,p} + \overline{p,q} + \overline{z,q} + \overline{q,p},$$

that implies,

$$\overline{y,q} - \overline{y,p} - \overline{z,q} + \overline{z,p} \leq 2\overline{q,p}.$$

□

Remark 2.1. • *1. The triangle inequality (2.4) implies (2.6). Indeed:*

For all $y, p, q \in E$, we have:

$$\overline{y,q} \leq \overline{y,p} + \overline{p,q}. \tag{2.7}$$

For all $y, p, q \in E$, we have:

$$\overline{y,p} \leq \overline{y,q} + \overline{q,p}. \tag{2.8}$$

For all $z, p, q \in E$, we have:

$$\overline{z,p} \leq \overline{z,q} + \overline{q,p}. \tag{2.9}$$

We add (2.7), (2.8) and (2.9) side by side, we get

$$\begin{aligned}
\overline{y,q} + \overline{y,p} + \overline{z,p} &\leq \overline{y,p} + \overline{p,q} + \overline{y,q} + \overline{q,p} + \overline{z,q} + \overline{q,p} \\
\overline{y,q} - \overline{y,p} - \overline{z,q} + \overline{z,p} &\leq \overline{y,q} - \overline{y,p} + 2\overline{p,q} + \overline{q,p} \\
&\leq -\overline{y,p} + \overline{y,p} + \overline{p,q} + 2\overline{p,q} - \overline{p,q} \\
&\qquad \textit{by using the triangly inequality} \\
&\leq 2\overline{p,q}.
\end{aligned}$$

2. The triangle inequality, (2.6) implies (2.4) by setting $z = q$.

Indeed:

By setting $z = q$, we have

$$\overline{y,q} - \overline{y,p} + \overline{q,p} \leq 2\overline{q,p}.$$

so

$$\overline{y,q} - \overline{y,p} \leq \overline{q,p}.$$

then

$$\overline{y,q} \leq \overline{y,p} + \overline{q,p}.$$

since the distance is Symmetric, we have

$$\overline{y,q} \leq \overline{y,p} + \overline{p,q}..$$

- *Next, let us evaluate (2.5) with $f = f_2 := (x \mapsto x^2)$, $f_2'(x) = 2x$ and $L_f = 1$. We get*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\overline{q,p} \quad \overline{y,z}. \tag{2.10}$$

Indeed:
for $f_2(x) = x^2$, $f_2'(x) = 2x$ and $L_f = 1$, we substituted in (2.5), we have

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\overline{q,p} \quad \overline{y,z}.$$

If we assume that the Metric space (E, d) is induced by an inner product space $(V\langle .,.\rangle)$,i.e.,$E = V$ and $d(q,p) = \|q - p\|$, then (2.10) becomes

$$2\langle q - p, z - y\rangle \leq 2\|q - p\|\|z - y\|.$$

Definition 2.2 (Convex Map)**.** *Let f be a function. We call that f is convex if and only if*

$$\forall \lambda \in]0,1[: \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Lemma 2.1. *Let f be a derivable function, so:*

* *f convex $\Leftrightarrow$ f' non decreasing.*

* *f concave $\Leftrightarrow$ f' decreasing.*

Definition 2.3 (Set S)**.** *Let $f :]0, +\infty[\longrightarrow \mathbb{R}$, and let S be the set of f verified:*

1. *f non decreasing and derivable.*

2. *f convex.*

3. *f' concave.*

In the following, the function $f \in S$.

Theorem 2.1. *Let $f \in S$. Assume f is not constant. Then the quadruple constant of f fulfills $L_f \in [1, 2]$.*

Proof. The poof of the previous theorem is in [4]. See page 821. □

Remark 2.2. *Note that $L_{f_2} = 1$ and $f = f_2$ in (2.5) yields the Cauchy -Schwarz inequality. Hence, we can interpret the quadruple constant L_f as a factor describing the distortion of the Cauchy -Schwarz inequality induced by applying f instead of f_2.*

Definition 2.4 (The set S_0)**.** *Let $f :]0, +\infty[\longrightarrow \mathbb{R}$, and let S_0 be the set of functions in S with $f(0) = 0$.*

Theorem 2.2. *Let $(V, \langle ., . \rangle)$ be an inner product space with induced Metric d. Let $y, z, q, p \in V$. Let $f \in S_0$. Then*

$$f(\overline{y,q}) - f(\overline{y,p})f(\overline{z,q}) + f(\overline{z,p}) \leq f(\overline{q,p}) + f(\overline{y,z}). \tag{2.11}$$

Proof. The poof of the previous theorem is in [4]. See page 821. □

Remark 2.3. *This can be viewed as a generalization of the parallelogram law in inner product spaces, $\|u + v\|^2 + \|u - v\|^2 \leq 2\|u\|^2 + 2\|v\|^2$ for $u, v \in V$, as it implies*

$$f(\|u + v\|) + f(\|u - v\|) \leq 2f(\|u\|) + 2f(\|v\|). \tag{2.12}$$

for $f \in S_0$.

Corollary 2.1 (In inner product spaces)**.** *Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product space with induced norm $\|.\|$. Let $u, v \in V$.*

(i) Let $f \in S_0$. Then

$$f(\|u\|) - f(\|v\|) \leq \|u - v\| f'(\|u + v\|). \tag{2.13}$$

(ii) Let $\alpha \in [1, 2]$. Then

$$\|u\|^\alpha - \|v\|^\alpha \leq \alpha 2^{1-\alpha} \|u - v\| \|u + v\|^{\alpha - 1}. \tag{2.14}$$

(i) Let $f \in S_0$. *Then*

$$f(\|u+v\|) + ff(\|u-v\|) \leq 2f(\|u\|) + 2f(\|v\|). \tag{2.15}$$

(ii) Let $\alpha \in [1,2]$. *Then*

$$\|u+v\|^\alpha + \|u-v\|^\alpha \leq 2\|u\|^\alpha + 2\|v\|^\alpha. \tag{2.16}$$

Proof. For the first two parts, set $y = 0$, $z = u+v$, $q = u$, $p = v$; for the last two parts, set $y = 0$, $z = v$, $q = u+v$, $p = u$. Then apply Theorem 2.3, Theorem 2.2, and Proposition 2.2. □

2.2 Parametrization

Proposition 2.2 (4-point parametrization). *(i) Let* (E,d) *be a Metric space and* $y, z, q, p \in E$. *Set* $a := \overline{z,p}$, $c := \overline{y,p}$, $b := \overline{q,p}$, $s := \cos(\widehat{ypq})$, $r := \cos(\widehat{zpq})$, $t := \cos(\widehat{ypz})$, *with all angles measured as in an isometric 3-point embedding in the Euclidean plane. Then,*

$$\overline{y,z}^2 = a^2 + c^2 - 2tac, \quad \overline{y,q}^2 = c^2 + b^2 - 2scb, \quad \overline{z,q}^2 = a^2 + b^2 - 2rab.$$

Furthermore, $a, b, c \in [0, +\infty)$, $r, s, t \in [-1, 1]$, *and*

$$\begin{array}{rcl} -tac & \leq & b^2 - rab - scb + \sqrt{a^2 - 2rab + b^2}\sqrt{c^2 - 2scb + b^2}, \\ -rab & \leq & c^2 - tac - scb + \sqrt{a^2 - 2tac + c^2}\sqrt{c^2 - 2scb + b^2}, \\ -scb & \leq & a^2 - rab - tac + \sqrt{a^2 - 2rab + b^2}\sqrt{a^2 - 2tac + c^2}. \end{array} \tag{2.17}$$

(ii) For all $a, b, c \in [0, +\infty)$, $r, s, t \in [-1, 1]$ *that fulfill (2.17), there is a Metric space* (E, d) *with a quadruple of points* $y, z, q, p \in E$ *such that*

$$\begin{array}{lll} a := \overline{z,p}, & c := \overline{y,p}, & b := \overline{q,p} \\ \overline{y,z}^2 = a^2 - 2tac + c^2 & \overline{y,q}^2 = c^2 - 2scb + b^2 & \overline{z,q}^2 = a^2 - 2rab + b^2. \end{array}$$

Proof. (i) The inequalities (2.17) are due to the triangle inequality,

$$\overline{y,z} \leq \overline{y,q} + \overline{z,q}, \qquad \overline{z,q} \leq \overline{y,z} + \overline{y,q}, \qquad \overline{y,q} \leq \overline{y,z} + \overline{z,q}.$$

Indeed,

$$
\begin{aligned}
\overline{y,z}^2 &\leq (\overline{y,q} + \overline{z,q})^2 \\
\overline{y,z}^2 &\leq \overline{y,q}^2 + \overline{z,q}^2 + 2\overline{y,q}\quad \overline{z,q}.
\end{aligned}
$$

and, we have

$$
\begin{aligned}
\overline{y,z}^2 &= (\overline{z,p} - \overline{y,p})^2 \\
&= \overline{z,p}^2 + \overline{y,p}^2 - 2\overline{y,p}.
\end{aligned}
$$

As $t = \dfrac{\overline{y,p}}{\overline{z,p}}$ so $t\,\overline{z,p}\quad \overline{y,p} = \overline{y,p}^2$ i.e: $tac = c^2$ and the previous equality become $\overline{y,z}^2 = a^2 + c^2 - 2tac$. Then

$$
\begin{aligned}
a^2 + c^2 - 2tac &\leq c^2 + b^2 - 2scb + a^2 + b^2 - 2rab + 2\sqrt{c^2 + b^2 - 2scb}\sqrt{a^2 + b^2 - 2rab} \\
a^2 + c^2 - 2tac &\leq c^2 + 2b^2 - 2scb + a^2 - 2rab + 2\sqrt{c^2 + b^2 - 2scb}\sqrt{a^2 + b^2 - 2rab} \\
-2tac &\leq 2b^2 - 2scb - 2rab + 2\sqrt{c^2 + b^2 - 2scb}\sqrt{a^2 + b^2 - 2rab} \\
-tac &\leq b^2 - scb - rab + \sqrt{c^2 + b^2 - 2scb}\sqrt{a^2 + b^2 - 2rab}.
\end{aligned}
$$

We can proved the other inequality by the same way.

(ii) Define a four point set E with elements named y, z, q, p. Define $d : E \times E \to [0, +\infty)$ with the equations given in the lemma, extended by symmetry and $d(x, x) = 0$ for all $x \in E$. By construction, d is a semi metric [8] (vanishing distance for non-identical points allowed) so that identifying points with distance 0 yields a Metric space.

□

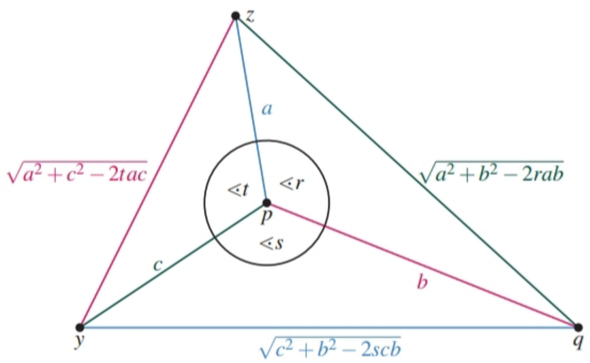

Figure 2.2: A 4-point parametrization. We denote $\sphericalangle x := arccos(x)$

With this parametrization and

$$a^2 - c^2 - (a^2 - 2rab + b^2) + (c^2 - 2scb + b^2) = 2b(ra - sc). \tag{2.18}$$

(2.10) can be expressed as

$$b(ra - sc) \leq b\sqrt{a^2 - 2tac + c^2}, \tag{2.19}$$

and (3) becomes

$$f(a) - f(c) - f_{\searrow}(a^2 - 2rab + b^2) + f_{\searrow}(c^2 - 2scb + b^2) \leq L_f b f'(a^2 - 2tac + c^2), \qquad (2.20)$$

where we use the shorthand $f_{\searrow}(x) := f(\sqrt{x})$. Thus, Theorem 1 is equivalent to showing that (2.19) implies (2.20) for all $a, b, c \in [0, +\infty)$, $r, s, t \in [-1, 1]$ that fulfill (2.17). We will prove a stronger but simpler looking result in section A :

Theorem 2.3. *Let (E, d) be a Metric space. Let $y, z, q, p \in E$. Assume*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\overline{q,p}\,\overline{y,z}.$$

Let $f : [0, \infty) \to \mathbb{R}$ be differentiable. Assume is non-decreasing, convex and has a concave derivative f'. Then

$$f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}) \leq L_f \overline{q,p} f'(\overline{y,z}).$$

Definition 2.5 (The set S_0^k). *Let $C^k([0, +\infty))$ denote the space of k-times continuously differentiable functions $[0, +\infty) \longrightarrow \mathbb{R}$, where derivatives at 0 are taken as right derivatives. Denote $S_0 := \{f \in S : f(0) = 0\}$. Denote $S^k := S \cap C^k([0, +\infty))$ and $S_0^k := S_0 \cap C^k([0, +\infty))$ for $k \in \mathbb{N} \cup \infty$.*

Since,

Theorem 2.4. *Let $a, b, c \geq 0$, $r, s \in [-1, 1]$, and $f \in S_0^3$, where a, b, c, r, s are defined in parametrization. Then*

$$f(a) - f(c) - f_{\searrow}(a^2 - 2rab + b^2) + f_{\searrow}(c^2 - 2scb + b^2) \leq 2bf'(\max(ra{-}sc, |a{-}c|)). \qquad (2.21)$$

Proof. The poof of the previous theorem is in [4]. See page 826. ∎

Theorem 2.5. *Let $f \in S$. Let (E, d) be a Metric space. Let $y, z, q, p \in E$. Assume, there is $L \in [2, \infty)$ such that*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq L\overline{q,p}\,\overline{y,z}. \qquad (2.22)$$

Then

$$f(y,q) - f(y,p) - f(z,q) + f(z,p) \leq L\overline{q,p} f'(y,z). \qquad (2.23)$$

Proof. The poof of the previous theorem is in [4]. See page 827. ∎

Proof. Let (2.10) hold, then using the proposition assume 2.2, we have:

$$f(y,q)-f(y,p)-f(z,q)+f(z,p)\leq 2d(p,q)f'(d(y,z)).$$

Using the parametrization, this translates into:

$$f(a)-f(c)-f_{\curlywedge}(a^2-2rab+b^2)+f_{\curlywedge}(c^2-2scb+b^2)\leq 2bf'_{\curlywedge}(a^2-2tac+c^2).$$

This equivalent formulation reduces the proof of theorem 1 to proving this inequality for all values of a,b,c,r,s,t that satisfy the Metric constraints.

Since theorem 2.4. This result is stronger than theorem 2.3 because it allows the right-hand side to depend on different values of f' rather than just $\overline{y,z}$.

Lemma 2.2 (Elimination of r)**.** *Let $f \in S_0^3$.*

(i) For all $a,b,c \in [0,+\infty)$, $s \in [-1, \min(1, 2\frac{a}{c}-1)]$, we have

$$f(a)-f(c)-f(|a-b|)+f_{\curlywedge}(c^2-2scb+b^2)\leq 2bS'(a-sc). \tag{2.24}$$

(ii) For all $a,b,c \in [0,+\infty)$ with $a \geq c$, we have

$$f(a)-f(c)-f_{\curlywedge}((a-b)^2-4bc)+f_{\curlywedge}(b+c)\leq 2bf'(a-c). \tag{2.25}$$

According to lemma 2.2, we show that the problem reduces to considering only the parameters a,b,c,s simplifying the proof.

We define the function:

$$F_f(a,b,c,s,r)=f(a)-f(c)-f_{\curlywedge}(a^2-2rab+b^2)+f_{\curlywedge}(c^2-2scb+b^2)-2b\left(\max\left(ra-sc,|a-c|\right)\right).$$

The strategy is:

1. Show f is convex in r.

2. Check the inequality at the extreme value of r, since a convex function attains its maximum at the boundary.

3. Use explicit computations of F_f for special values of r, s to confirm that the inequality holds.

The proof is completed by checking cases for different values of a, b, c:

- If $a \geq c$, a direct bound follows from lemma 2.2.

- If $a \leq c$, the inequality follows from another case in lemma 2.2.

Thus, theorem 2.4 is proven.

Now, we apply theorem 2.4 back to theorem 2.3:

1. The assumption $d(y,q)^2 - d(y,p)^2 - d(z,q)^2 + d(z,p)^2 \leq 2d(q,p)d(y,z)$ allows us to set:

$$L = 2.$$

2. Theorem 2.4 gives an inequality with a more flexible bound.

3. The key idea is that since the bound in theorem 2.4 never exceeds $2d(q,p)d(y,z)$, theorem 2.3 follows as a special case.

□

As a, b, c be a lengths so they are greater then 0. and as s, r be a cos of angles so they are in $[-1; 1]$.
From Theorem 2.4, we obtain a slightly stronger result than Theorem 1 by relaxing (5):

Theorem 2.6. *Let $f \in S$. Let (E, d) be a Metric space. Let $y, z, q, p \in E$. Assume, there is $L \in [2, +\infty)$ such that*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq L\overline{q,p}\,\overline{y,z}. \tag{2.26}$$

Then

$$f(y,q) - f(y,p) - f(z,q) + f(z,p) \leq L\overline{q,p}f'(y,z). \tag{2.27}$$

Proof. Proof that Theorem 2.4 implies Theorem 2.6. Let $f \in S_0^3$. Using (2.18), the Metric version of (2.21) is

$$f(\overline{y,q})-f(\overline{y,p})-f(\overline{z,q})+f(\overline{z,p}) = f(a)-f(c)-f_{\curlywedge}(a^2-2rab+b^2)+f_{\curlywedge}(c^2-2scb+b^2). \tag{2.28}$$

By using the theorem (2.4):

$$f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}) \leq 2bf'(\max(ra{-}sc, |a{-}c|)).$$

Assume that (2.18) holds. then

$$f(\overline{y,q})-f(\overline{y,p})-f(\overline{z,q})+f(\overline{z,p}) \leq 2bf'\left(\max\left(\frac{a^2-c^2-(a^2-2rab+b^2)+(c^2-2scb+b^2)}{2b}, |a-c|\right)\right).$$

So

$$\begin{aligned} &f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}) \\ &\leq 2\overline{q,p}f'\left(\max\left(\frac{\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2}{2\overline{q,p}}, |\overline{z,p} - \overline{y,p}|\right)\right). \end{aligned} \tag{2.29}$$

Assume that (2.26) holds, and by using the triangle inequality. the previous inequality become:

$$f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}) \leq 2\overline{q,p}f'\left(\max\left(\frac{L\overline{y,z}}{2}, \overline{y,z}\right)\right).$$

By using Lemma 2.5 (ii) and $\frac{L}{2} \geq 1$, we obtain (2.27).

□

2.3 Nondecreasing, convex functions with concave derivative

The following lemma illustrates that all functions $f \in S^3$ are between a nondecreasing linear function and a parabola that opens upward. It can be shown via Taylor approximations.

Definition 2.6 (Nondecreasing Function). *Let $f \in S^3$. Let $x, y \in [0, +\infty)$. Then f is nondecreasing function if*

$$x \leq y \Longrightarrow f(x) \leq f(y).$$

Lemma 2.3 (Polynomial bounds). *Let $f \in S^3$. Let $x, y \in [0, +\infty)$. Then*

(i)

$$f(x) + yf'(x) \leq f(x + y) \leq f(x) + yf'(x) + \frac{1}{2}y^2 f''(x).$$

(ii)

$$f'(x) \leq f'(x + y) \leq f'(x) + yf''(x).$$

In the next lemma, we provide useful bounds for the proof of the main results.

Lemma 2.4 (Difference bound). *Let $f \in S$.*

(i) Let $x, y \in [0, +\infty)$. Assume $x \geq y$. Then

$$\frac{x-y}{2}(f'(x)+f'(y)) \leq f(x)-f(y) \leq (x-y)f'\left(\frac{x+y}{2}\right).$$

(ii) Let $x, y \in [0, +\infty)$. Then

$$f(x+y)-f(|x-y|) \leq 2\min(x,y)f'(\max(x,y)).$$

Proof. (i) For the lower bound, as f' is concave,

$$\begin{aligned} f(x)-f(y) &= \int_y^x f'(u)\,du \\ &\geq (x-y)\int_0^1 (1-t)f'(y)+tf'(x)\,dt \\ &= \frac{x-y}{2}(f'(y)+f'(x)). \end{aligned}$$

For the upper bound, concavity of f' implies the existence of an affine linear function h with $h(u) \geq f'(u)$ for all $u \in [0, +\infty)$ and

$$h\left(\frac{x+y}{2}\right) = f'\left(\frac{x+y}{2}\right). \tag{2.30}$$

Thus,

$$\begin{aligned} f(x)-f(y) &\leq \int_y^x h(u)\,du \\ &= \frac{x-y}{2}(h(y)+h(x)) \\ &= (x-y)h\left(\frac{x+y}{2}\right). \end{aligned}$$

(ii) Follows directly from the upper bound in (i)

□

The following lemma is a consequence of f' being concave.

Lemma 2.5 (Concave derivative)**.** *Let $f \in S$.*

(i) Let $x, y \in [0, +\infty)$. Then

$$f'(x+y) \leq f'(x)+f'(y) \leq 2f'\left(\frac{x+y}{2}\right).$$

(ii) Let $a, x \in [0, +\infty)$. Then

$$f'(ax) \geq af'(x) \text{ for } a \leq 1,$$

$$f'(ax) \leq af'(x) \text{ for } a \geq 1.$$

(iii) Let $x, y \in [0, +\infty)$. Assume $y \geq x$. Then

$$xf'(y) \leq yf'(x).$$

In the proof of Theorem 2.3, we will first show the result for $f \in S^3$ and then approximate the remaining functions in S via smooth functions. The following lemma shows that this is possible.

Lemma 2.6 (Smooth approximation)**.** *Let $f \in S$. Then there is a sequence $(f_n)_{n\in\mathbb{N}} \subset S^\infty$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ and $f'(x) = \lim_{n\to\infty} f'_n(x)$.*

Proof. We will smooth f' by convolution with a mollifier. The convolution is executed in the group of positive real numbers under multiplication endowed with its Haar measure $(A) = \displaystyle\int_A \frac{1}{x} dx$ for $A \subset (0, +\infty)$ measurable.

For $n \in \mathbb{N}$, let $\varphi_n \in C^\infty((0, +\infty))$ be a sequence of nonnegative functions with support in $[\exp(\frac{-1}{n}), \exp(\frac{1}{n})]$ and

$$\int_0^\infty \frac{\varphi_n(x)}{x} dx = 1. \tag{2.31}$$

Let $f \in S$ with derivative f'. For $n \in \mathbb{N}$, $x \in [0, +\infty)$, we define

$$f_n(x) := f(0) + \int_0^x \int_0^\infty \frac{\varphi_n(z)}{z} f'\left(\frac{y}{z}\right) dz\, dy. \tag{2.32}$$

Then, for $y \in [0, +\infty)$,

$$f'_n(y) = \int_0^\infty \frac{\varphi_n(z)}{z} f'\left(\frac{y}{z}\right) dz = \int_{\mathbb{R}\varphi_n(e^t) f'\left(e^{\log(y)-t}\right) dt} .(2.33)$$

Thus, $s \mapsto f'_n(ye^s)$ is the convolution of $t \mapsto \varphi_n(e^t)$ with $t \mapsto f'(e^t)$. Using standard results on convolutions, the mollified function has following properties:

(i) f'_n is infinitely differentiable on $(0, +\infty)$, because φ_n is,

(ii) f'_n is nonnegative, nondecreasing, and concave, because f' is and φ_n is nonnegative,

(iii) $\lim\limits_{n\to\infty} f'_n = f'(x)$. because f' is continuous.

Furthermore, f_n is convex, as ff'_n is nondecreasing and $\lim\limits_{n\to\infty} f_n = f(x)$ by dominated convergence. Thus, $(f_n)_{n\in\mathbb{N}} \subset S^\infty$, and the sequence has the desired point-wise limits. □

Theorem 2.7. *Let (E, d) be a Metric space. Let $y, z, q, p \in E$. Assume*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\overline{q,p}\overline{y,z}.$$

Let $f \in S$. Then

$$f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}) \leq 2\overline{q,p}f'(\overline{y,z}). \tag{2.34}$$

Chapter 3

Application

In this chapter, we represent the important corollaries of the relationship between the Cauchy -Schwarz inequality and triangular inequality. After that we give an example in Euclidean space. we use [25], [14], [24], [1].

3.1 Corollaries

With Theorem 2.3 and Theorem 2.2, we have shown new fundamental inequalities in Metric spaces and inner product spaces that are related to the Cauchy –Schwarz and the Triangle inequalities. In this section, we discuss some corollaries of these results.

3.1.1 Symmetries

Figure 6 illustrates the symmetries in quadruple inequalities. Sides of the same color contribute essentially in the same way in the inequality. In (2.5), the diagonals of Figure 3.1 come up in non-exchangeable terms. But they can be swapped in the assumption (2.10). Thus, if the conditions of Theorem 2.3 are fulfilled, we have, for $f \in S$,

$$f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}) \leq 2\min(\overline{q,p}f'(\overline{y,z}), \overline{y,z}f'(\overline{q,p})). \tag{3.1}$$

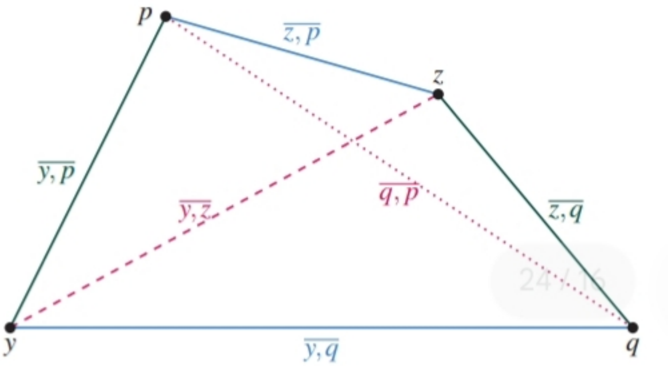

Figure 3.1: Four points as a quadrilateral. The sides of the quadrilateral show up on the left of the quadruple inequalities (the terms of opposite sides have the same sign); the diagonals form the right-hand side.

Furthermore, swapping y and z or q and p does not influence the right-hand side but changes the sign on the left-hand side. Thus, assuming

$$|\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2| \leq 2\overline{q,p}\,\overline{y,z}. \tag{3.2}$$

we get

$$|f(\overline{y,q}) - f(\overline{y,p}) - ff(\overline{z,q}) + ff(\overline{z,p})| \leq 2\min(\overline{q,p}f'(\overline{y,z}), \overline{y,z}f'(\overline{q,p})). \tag{3.3}$$

Further bounds of the right-hand side are shown in the next subsection.

3.1.2 Bounds for the right-hand side

Corollary 3.1. *Let $f \in S_0$. Let (E, d) be a Metric space. Let $y, z, q, p \in E$. Assume*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\overline{q,p} \quad \overline{y,z}. \tag{3.4}$$

Then the value

$$f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}). \tag{3.5}$$

is bounded from above by all of the following values:

(i) $2\min(\overline{q,p}\,\overline{y,z})f'(\max(\overline{q,p}\,\overline{y,z}))$,

(ii) $2\overline{q,p}^{\beta}\overline{y,z}^{1-\beta}f'(\overline{q,p}^{1-\beta}\overline{y,z}^{\beta})$. *for all* $\beta \in [0,1]$,

(iii) $2(\beta\overline{q,p} + (1-\beta)\overline{y,z})f'((1-\beta)\overline{q,p} + \beta\overline{y,z})$. *for all* $\beta \in [0,1]$,

(iv) $2\sqrt{\overline{q,p}\,\overline{y,z}}f'(\sqrt{\overline{q,p}})$,

(v) $(\overline{q,p} + \overline{y,z})f'\left(\frac{\overline{q,p} + \overline{y,z}}{2}\right)$,

(vi) $4f(\sqrt{\overline{q,p}\overline{y,z}})$,

(vii) $4f\left(\dfrac{\overline{q,p}+\overline{y,z}}{2}\right)$,

(viii) $2f(\overline{q,p})+2f(\overline{y,z})$.

Proof. We first apply Theorem 2.3 twice, to (y,z,q,p) and to (q,p,y,z), to obtain

$$|f(\overline{y,q})-f(\overline{y,p})-f(\overline{z,q})+f(\overline{z,p})|\leq 2\min(\overline{q,p}\overline{y,z})f'(\max(\overline{q,p}\overline{y,z})). \tag{3.6}$$

This shows (i). Let $a,b\in[0,+\infty)$ and $\beta\in[0,1]$. Next we use Lemma 2.5 (ii) and the weighted arithmetic–geoMetric mean inequality,

$$\begin{aligned}\min(a,b)f'(\max(a,b)) &\leq a^{\beta}b^{1-\beta}f'(a^{1-\beta}b^{\beta})\\ &\leq (\beta a+(1-\beta)b)f'((1-\beta)a+\beta b).\end{aligned}$$

Applying these inequalities to 3.6 shows (ii) and (iii), and their special cases (iv) and (v), where $\beta=\frac{1}{2}$. By Lemma 2.4 (i) with $y=0$, we have

$$xf'(x)\leq 2f(x). \tag{3.7}$$

for all $x\in[0,+\infty)$. Thus,

$$\begin{aligned}\min(a,b)f'(\max(a,b)) &\leq \sqrt{ab}f'(\sqrt{ab})\\ &\leq 2f(\sqrt{ab}),\end{aligned}$$

which yields (vi). The remaining parts (vii) and (viii), can be obtained using 3.7 and Jensen's inequality:

$$\begin{aligned}\min(a,b)f'(\max(a,b)) &\leq \frac{a+b}{2}f'\left(\frac{a+b}{2}\right)\\ &\leq 2f\left(\frac{a+b}{2}\right)\\ &\leq f(a)+f(b).\end{aligned}$$

□

3.1.3 Corollaries for special cases

We apply Theorem 2.3, Theorem 2.2, and Proposition 2.2 for a triple of points (a quadruple of points with two identical points) and for parallelograms in inner product spaces to demonstrate the main results.

Corollary 3.2. *Let (E,d) be a Metric space. Let $y,q,p \in E$.*

(i) Let $f \in S_0$. Then

$$f(\overline{y,q}) - f(\overline{y,p}) + f(\overline{q,p}) \leq 2\overline{q,p}f'(\overline{y,q}). \tag{3.8}$$

(ii) Let $\alpha \in [1,2]$. Then

$$\overline{y,q}^{\alpha} - \overline{y,p}^{\alpha} + \overline{q,p}^{\alpha} \leq \alpha 2^{2-\alpha}\overline{q,p}\overline{y,q}^{\alpha-1}. \tag{3.9}$$

Proof. By the triangle inequality, $|\overline{y,q} - \overline{q,p}| \leq \overline{y,p}$. After squaring this inequality, we obtain (5) with $z = q$, which is

$$\overline{y,q}^2 - \overline{y,p}^2 + \overline{q,p}^2 \leq 2\overline{q,p}\overline{y,q}. \tag{3.10}$$

Thus, Theorem 2.3 implies (3.8) and Proposition 2.2 implies 3.9. □

Remark 3.1. *Recall the parallelogram law: Let $(V, \langle \cdot , \cdot \rangle)$ be an inner product space with induced norm $\|.\|$. Let $u,v \in V$. Then*

$$\|u+v\|^2 + \|u-v\|^2 \leq 2\|u\|^2 + 2\|v\|^2. \tag{3.11}$$

which is also true with equality. Thus, we can say that Corollary 2.1 generalizes the parallelogram law.

3.2 Quadruple inequality in Euclidean space: $\mathbb{R}^2$

Example 3.1. *Let $p = (0,0)$, $q = (1,-1)$, $y = (-1,-1)$, $z = (0,1)$. We have:*

$$\overline{y,p} = \sqrt{(0-1)^2 + (0+1)^2} = \sqrt{1+1} = \sqrt{2}.$$

$$\overline{z,q} = \sqrt{(1-0)^2 + (-1-1)^2} = \sqrt{1+4} = \sqrt{5}.$$

$$\overline{y,q} = \sqrt{(1+1)^2 + (-1+1)^2} = \sqrt{4+0} = 2.$$

$$\overline{z,p} = \sqrt{(0-0)^2 + (0-1)^2} = \sqrt{0+1} = 1.$$

$$\overline{y,z} = \sqrt{(0+1)^2 + (1+1)^2} = \sqrt{1+4} = \sqrt{5}.$$

$$\overline{q,p} = \sqrt{(0-1)^2 + (0+1)^2} = \sqrt{1+1} = \sqrt{2}.$$

The points $p, q, y, z \in \mathbb{R}^2$ verify the Cauchy - Swartz inequality 2.10 for $f(x) = x^2$. because:

$$\begin{aligned} \overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 &= 2^2 - \sqrt{2}^2 - \sqrt{5}^2 + 1^2 = -2. \\ &\leq 2\overline{q,p}.\overline{y,z} = 2 * \sqrt{2} * \sqrt{5} = 2\sqrt{10}. \end{aligned}$$

And it verify the inequality 2.5. because:

$$-2 \leq 2.L_f * \sqrt{10}.$$

for all $L_f \in [0, +\infty[$.

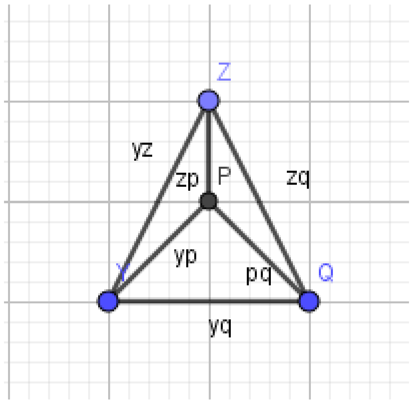

Figure 3.2: Example of quadruple inequality in $\mathbb{R}^2$

3.3 Statistics

[see [4]] Quadruple inequalities, such as (2.5), can be used to establish convergence rates for certain types of means in metric spaces. A key example is the Fréchet mean (or barycenter), which is defined as the minimizer of the expected squared distance to random variable Y in a metric space (E, d). If (Y) has a finite second moment, the Fréchet mean is given by:

$$m := \arg\min_{q \in Q} \mathbb{E}[\overline{Y,q}^2].$$

Similarly, the Fréchet median minimizes the expected distance:

$$m := \arg\min_{q \in Q} \mathbb{E}[\overline{Y,q}],$$

and more generally, the f-Fréchet mean is defined for a loss function $f : [0, \infty) \longrightarrow \mathbb{R}$ as:

$$m := \arg\min_{q \in Q} \mathbb{E}[f(\overline{Y,q})],$$

Given a sequence of independent random variables $Y_1, Y_2, ...$ with the same distribution as Y, a fundamental statistical problem is to bound the distance between the sample f-Fréchet mean:

$$\widehat{m}_n := \arg\min_{q \in Q} \frac{1}{n} \sum_{i=1}^{n} f(\overline{Y_i, q}),$$

and its population counterpart m. Quadruple inequalities, particularly (2.5), provide a way to derive such bounds without requiring the metric space to have finite diameter.

This approach is especially useful for robust statistics, where loss function like:

* The Huber loss: $f_{H,\delta}$,
* The Pseudo-Huber loss: $f_{pH,\delta}$,
* The Log-cosh loss: $x \longmapsto \ln(\cosh(x))$,

are employed. These functions intrepolate between the behaviour of the classical mean for $f(x) = x^2$ and the median for $f(x) = x$, making them valuable in application like robust estimation and image processing.

By leveraging theorem 2.3 with inequality (2.5), one can obtain convergence rates for $d(\widehat{m}_n, m)$ in settings where alternative inequalities (e.g., [15]) would not apply.

3.3.1 Numerical Study

In this part, we cheek the quadruple inequality (2.5):

We put

$$\begin{aligned} f := f_1 : [0, +\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto x \end{aligned}$$

$$\begin{aligned} f := f_2 : [0, +\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto x^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} f := f_3 : [0, +\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$$

$$\begin{aligned} f := f_4 : [0, +\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \log(\cosh(x)) \end{aligned}$$

Firstly, we prove that $f \in S_0$:

For $f : f_1(x) = x$, so $f_1'(x) = 1$.

* f_1 nondecreasing, because:

$$\forall x, y \in [0, +\infty) : x \leq y \Longrightarrow f(x) = x \leq f(y) = y.$$

* f_1 convex, because :

$$\forall x, y \in [0, +\infty), \forall t \in (0, 1) : f_1(tx + (1-t)y) = tx + (1-t)y \leq tf_1(x) + (1-t)f_1(y).$$

* f_1' concave, because:

a)- $\forall a, x \in [0, +\infty)$

$$f_1'(ax) = 1 \geq af_1'(x) = a \qquad \text{for } a \leq 1.$$

And

$$f_1'(ax) = 1 \leq af_1'(x) = a \qquad \text{for } a \geq 1.$$

b)- $\forall x, y \in [0, +\infty)$, $y \geq x$

$$xf_1'(y) = x \leq y = yf_1'(x).$$

* We have $f_1(0) = 0$.

For f : $f_2(x) = x^{\frac{3}{2}}$ so $f_2'(x) = \dfrac{3}{2}\sqrt{x}$.

* f_2 nondecreasing, because:

$$\forall x \in [0, +\infty) : f_2'(x) = \frac{3}{2}\sqrt{x} \geq 0.$$

* f_2 convex, by using lemma 2.1 we have

$$\forall x \in [0, +\infty) : f_2^{(2)}(x) = \frac{3}{4\sqrt{x}} \geq 0.$$

so, f_2' nondecreasing, then f_2 convex.

* f_2' concave, by using lemma 2.1 we have

$$\forall x \in [0, +\infty) : f_2^{(3)}(x) = -\frac{3}{8\sqrt{x^3}} \leq 0.$$

so, $f_2^{(2)}$ decreasing, then f_2' concave.

* We have $f_2(0) = 0$.

For f : $f_3(x) = x^2$ so $f_2'(x) = 2x$.

* f_3 nondecreasing, because:

$$\forall x \in [0, +\infty) : f_2'(x) = 2x \geq 0.$$

* f_3 convex, by using lemma 2.1 we have

$$\forall x \in [0, +\infty) : f_3^{(2)}(x) = 2 \geq 0.$$

so, f_3' nondecreasing, then f_3 convex.

* f_3' concave, because:

a)- $\forall a, x \in [0, +\infty)$

$$f_3'(ax) = 2ax \geq af_3'(x) = 2ax \qquad \text{for } a \leq 1.$$

And

$$f_3'(ax) = 2ax \leq af_3'(x) = 2ax \qquad \text{for } a \geq 1.$$

b)- $\forall x, y \in [0, +\infty)$, $y \geq x$

$$xf_3'(y) = 2xy \leq yf_3'(x) = 2xy.$$

* We have $f_3(0) = 0$.

For f : $f_4(x) = \log(\cosh(x))$ so $f_4'(x) = \tanh(x)$.

* f_4 nondecreasing, because:

$$\forall x \in [0, +\infty) : f_4'(x) = \tanh(x) \geq 0.$$

* f_4 convex, by using lemma 2.1 we have

$$\forall x \in [0, +\infty) : f_4^{(2)}(x) = \frac{1}{\cosh^2(x)} \geq 0.$$

so, f_4' nondecreasing, then f_4 convex.

* f_2' concave, by using lemma 2.1 we have

$$\forall x \in [0, +\infty) : f_4^{(3)}(x) = -\frac{2\sinh(x)}{\cosh^3(x)} \leq 0.$$

so, $f_4^{(2)}$ decreasing, then f_4' concave.

* We have $f_4(0) = 0$.

And, we put

$$LHS = f(\overline{y,q}) - f(\overline{y,p}) - f(\overline{z,q}) + f(\overline{z,p}).$$

$$RHS = L_f \overline{q,p} * f'(\overline{y,z}).$$

1. For $L_f = 2$ we have

   * $f := f_1$ we have

   $$\overline{y,q} - \overline{y,p} - \overline{z,q} + \overline{z,p} \leq 2\overline{q,p}.$$

   * $f := f_2$ we have

   $$\overline{y,q}^{\frac{3}{2}} - \overline{y,p}^{\frac{3}{2}} - \overline{z,q}^{\frac{3}{2}} + \overline{z,p}^{\frac{3}{2}} \leq 3\overline{q,p} \quad \sqrt{\overline{y,z}}.$$

   * $f := f_3$ we have

   $$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 4\overline{q,p} \quad \overline{y,z}.$$

   * $f := f_4$ we have

   $$\log(\cosh(\overline{y,q})) - \log(\cosh(\overline{y,p})) - \log(\cosh(\overline{z,q})) + \log(\cosh(\overline{z,p})) \leq 2\overline{q,p}\tanh(\overline{y,z}).$$

   Then,

f	L H S	R H S	$\frac{\text{L H S}}{\text{R H S}}$
x	-0.650281539	2.828427125	-0.229909243
$x^{\frac{3}{2}}$	-1.197067231	6.344227581	-0.188686048
x^2	-2	$4 * \sqrt{10}$	$\frac{-\sqrt{10}}{20}$
$\log(\cosh(x))$	-0.24927923913	$\frac{2*\sqrt{2}*\exp 2\sqrt{5} - 2\sqrt{2}}{\exp 2\sqrt{5}+1} \simeq 2.76454$	-0.090170241

2. For $L_f = 1$ we have

   * $f := f_1$ we have
$$\overline{y,q} - \overline{y,p} - \overline{z,q} + \overline{z,p} \leq \overline{q,p}.$$

   * $f := f_2$ we have
$$\overline{y,q}^{\frac{3}{2}} - \overline{y,p}^{\frac{3}{2}} - \overline{z,q}^{\frac{3}{2}} + \overline{z,p}^{\frac{3}{2}} \leq \frac{3}{2}\overline{q,p} \quad \sqrt{\overline{y,z}}.$$

   * $f := f_3$ we have
$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\overline{q,p} \quad \overline{y,z}.$$

   * $f := f_4$ we have
$$\log(\cosh(\overline{y,q})) - \log(\cosh(\overline{y,p})) - \log(\cosh(\overline{z,q})) + \log(\cosh(\overline{z,p})) \leq \overline{q,p}\tanh(\overline{y,z}).$$

Then,

f	L H S	R H S	$\frac{\text{L H S}}{\text{R H S}}$
x	-0.650281539	1.414213563	-0.459818485
$x^{\frac{3}{2}}$	-1.197067231	3.171113791	-0.377372096
x^2	-2	$2 * \sqrt{10}$	$\frac{-\sqrt{10}}{10}$
$\log(\cosh(x))$	-0.24927923913	$\frac{\sqrt{2} * \exp 2\sqrt{5} - 2\sqrt{2}}{\exp 2\sqrt{5} + 1} \simeq 1.38227$	-0.180340482

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