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THEME

ON A NEW NORM AND THE REFINEMENT OF SOME INEQUALITIES BETWEEN OPERATOR NORM AND THE NUMERICAL RADIUS

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Dedication

I would love to dedicate this modest work to:

My beloved mother and father who have always been supporting me and making massive sacrifices for me. Mom and dad I am nothing without both you, words cannot describe how much I love you. My wish is just to repay you a small part of what you have done to me, and make you proud of me someday.

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Abstract

Our main target in this thesis is to refine some important inequalities between the usual operator norm and the numerical radius or obtain new inequalities. For that reason, we define a new norm $\|\cdot\|_{\alpha,\beta}$ on $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $\alpha, \beta \in \mathbb{R}_+^*$. We study some properties of this norm, also we see its applications on the inequalities between the usual operator norm and the numerical radius.

And we obtain new inequalities which are, for $T, S, R \in \mathcal{B}(\mathcal{H})$, then:

• If $|T|S = S|T|$, then:

$$\omega(TS) \leq \frac{1}{\sqrt{2}}\|S\|\|T\|_{1,1}$$

• If $TS = ST$ and ${T^*}^2 S^2 = S^2 {T^*}^2$, then:

$$\omega(TS)^2 \leq \frac{1}{4}\|S\|^2\|T\|_{1,1}^2 + \frac{1}{2}\|S^2\|\omega(T^2)$$

• If $TS = S^*T$, then:

$$\omega(TS)^2 \leq \|S\|^2\left(\tfrac{1}{4}\|T\|_{1,1}^2 + \tfrac{1}{2}\omega(T^2)\right)$$

Where $\|\cdot\|$ and $\omega(\cdot)$ are the usual operator norm and the numerical radius respectively, and $\|\cdot\|_{1,1}$ is the new norm when $\alpha = \beta = 1$, and it is given as follows $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Moreover, we get a refinement for some existing inequalities, and here are the inequalities

• If $TS = ST$, then:

$$\omega(TS)^2 \leq \frac{1}{4}(\|S\|^2 + \|S^2\|)\|T\|_{1,1}^2$$

• If $TS = ST$ and $TR = RT$, then:

$$\omega(TS \pm RT)^2 \leq \frac{1}{4}(\|S\| + \|R\|)^2\|T\|_{1,1}^2 + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\|$$

Note that these inequalities improve some existing inequalities in some cases, and we study some of them thoroughly down below.

Key words: Complex Hilbert space, bounded linear operators, spectrum, numerical range, usual operator norm, spectral radius, numerical radius.

ملخص

هدفنا الرئيسي في هذه المذكرة هو تحسين بعض المتراجحات المهمة بين النظيم الاعتيادي للمؤثرات الخطية ونصف القطر الرقمي أو ايجاد متراجحات جديدة. ولهذا السبب عرفنا نظيما جديدا $\|\cdot\|_{\alpha,\beta}$ على $\mathcal{B}(\mathcal{H})$، حيث $\mathcal{B}(\mathcal{H})$ هو جبر كل المؤثرات الخطية المحدودة على فضاء هلبرتي مركب $\mathcal{H}$ و $\alpha, \beta \in \mathbb{R}_+^*$. درسنا خواص هذا النظيم، وكذلك درسنا تطبيقاته على المتراجحات بين النظيم الاعتيادي للمؤثرات الخطية و نصف القطر الرقمي، وحصلنا على بعض المتراجحات الجديدة، وهي:

- إذا كان $|T|S = S|T|$، فإن:

$$\omega(TS) \leq \frac{1}{\sqrt{2}}\|S\|\|T\|_{1,1}$$

- إذا كان $TS = ST$ و $T^{*2}S^2 = S^2T^{*2}$، فإن:

$$\omega(TS)^2 \leq \frac{1}{4}\|S\|^2\|T\|_{1,1}^2 + \frac{1}{2}\|S^2\|\omega(T^2)$$

- إذا كان $TS = S^*T$، فإن:

$$\omega(TS)^2 \leq \|S\|^2 \left(\tfrac{1}{4}\|T\|_{1,1}^2 + \tfrac{1}{2}\omega(T^2)\right)$$

من أجل $T, S \in \mathcal{B}(\mathcal{H})$ ،حيث $\|\cdot\|$ و $\omega(\cdot)$ هما النظيم الاعتيادي للمؤثرات الخطية و نصف القطر الرقمي على التوالي، و $\|\cdot\|_{1,1}$ هو النظيم الجديد في حالة $\alpha = \beta = 1$ وفي هذه الحالة يكون معرفا كالتالي $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.
بالاضافة الى ذلك، تحصلنا على تحسين لمتراجحتين، وهما:

- إذا كان $TS = ST$، فإن:

$$\omega(TS)^2 \leq \frac{1}{4}(\|S\|^2 + \|S^2\|)\|T\|_{1,1}^2$$

- إذا كان $TS = ST$ و $TR = RT$، فإن:

$$\omega(TS \pm RT)^2 \leq \frac{1}{4}(\|S\| + \|R\|)^2\|T\|_{1,1}^2 + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\|$$

من أجل $R \in \mathcal{B}(\mathcal{H})$، وللاشارة فإن هاتين المتراجحتين يحسنان بعض المتراجحات في بعض الحالات، ولقد درسنا بعضا منها بالتفصيل في الاسفل.

كلمات مفتاحية: فضاء هلبرتي مركب، مؤثر خطي محدود، طيف مؤثر خطي، الصورة الرقمية لمؤثر خطي، نظيم مؤثرات الخطية الاعتيادي، نصف القطر الطيفي، نصف القطر الرقمي .

Contents

Notations

$\mathbb{N}$: The set of natural numbers $\{1, 2, 3 \cdots\}$.

$\mathbb{R}$: The set of real numbers.

$\mathbb{R}_+$: The set of positive real numbers.

$\mathbb{C}$: The set of complex numbers.

$\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$.

$\mathcal{H}$: Complex Hilbert space.

$\langle .,. \rangle$: The inner product of $\mathcal{H}$.

$\overline{M}$: The closure of M.

M° : The interior of M.

∂M : The boundary of M.

$M^\perp$: The orthogonal complement of M.

$\oplus$: The sign of direct sum.

$\mathcal{B}(\mathcal{H})$: Banach algebra of all bounded linear operators on Hilbert space $\mathcal{H}$.

$\mathcal{I}(\mathcal{H})$: The set of invertible operators in $\mathcal{B}(\mathcal{H})$.

T : A bounded linear operator defined on $\mathcal{H}$ ($T \in \mathcal{B}(\mathcal{H})$).

$\|T\|$: The norm of T.

T^{-1} : The inverse operator of T.

T^* : The adjoint operator of T.

$\Re(T)$: The real part of T.

$\Im(T)$: The imaginary part of T.

$|T|$: The absolute value of T.

$R(T)$: The range of T.

$N(T)$: The kernel of T.

$\sigma(T)$: The spectrum of T.

$\sigma_r(T)$: The residual spectrum of T.

$\sigma_p(T)$: The point spectrum of T.

$\sigma_c(T)$: The continuous spectrum of T.

$\sigma_{ap}(T)$: The approximate point spectrum of T.

$\rho(T)$: The resolvent of T.

$r(T)$: Spectral radius of T.

$W(T)$: The numerical range of T.

$\omega(T)$: The numerical radius of T.

Introduction

Operator Theory is a crucial part of modern (pure and applied) Mathematics. It belongs to a larger domain which is Functional Analysis. It is also indispensable to Physics, in particular, Quantum Mechanics as well as some parts of Engineering and Statistics.

This theory has developed considerably during the last decades, and it has got the attention of many scientists and researchers due to its significance and its wide applications. In the following, we will take a small overview about where this theory has come from and some of the people who have brought it to this point.

Early, quadratic forms played a major role. This led to the notion of the numerical range of an operator. Let $\mathcal{H}$ be a non trivial complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Let $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the numerical range of T was introduced by Toeplitz in [33] as

$$W(T) = \{\langle Tx, x \rangle; x \in \mathcal{H} \text{ with } \|x\| = 1\}$$

The Toeplitz-Hausdorff Theorem states that: $W(T)$ is convex subset of $\mathbb{C}$. This theorem has many proofs (for example see [16], [19] and [15]). This concept has been studied extensively in the last few decades. As pointed out by many authors (e.g. see [4] and [5]), this concept is very useful for studying matrices and operators, and has a lot of applications to other subjects.

Where the numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$, that is given by

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle|$$

It is well-known that the numerical radius is an equivalent norm to the usual norm on $\mathcal{B}(\mathcal{H})$ as we will see later. Moreover, this concept was the subject of a wide literature carrying out many inequalities involving it. Some developments toward this subject have been done in [20], [21] and [9] by many researchers such as F. Kittaneh and S.S. Dragomir and others, and there are many recent results and refinements for some important inequalities (see [2] and [32]).
Note that the literature of inequalities between the numerical radius the usual norm on $\mathcal{B}(\mathcal{H})$ is vital, since it has many applications in Physics and Engineering, so any improvement to these inequalities

will be very useful in term of giving a good estimation to some important values and factors.

While the spectrum of a bounded linear operator is a generalization of the set of eigenvalues of a matrix, and the spectrum of $T \in \mathcal{B}(\mathcal{H})$ defined as follows

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ is\ not\ invertible\}$$

The spectrum set has many interesting properties (see [25] and [34]), also there many classes of spectrum set, we will see them later on.
The study of spectrum and related properties is known as Spectral Theory, which has huge applications, most notably the Mathematical Formulation of Quantum Mechanics.

Another important value is the spectral radius of an operator $T \in \mathcal{B}(\mathcal{H})$, this is given by

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

Notice that the spectral radius, the usual operator norm and the numerical radius are very important values in the study of linear operators and their properties.

In this thesis, we will study the properties and some classifications of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$, as well as the spectrum and the spectral radius of an operator. In addition, we will study the numerical range and the numerical radius in detained, and we will include the latest results and inequalities in this context. Lastly, we will define a new norm on $\mathcal{B}(\mathcal{H})$, and we will study its properties and its applications, especially on the inequalities between the numerical radius and the usual norm on $\mathcal{B}(\mathcal{H})$.

Next, we will explain how the work is structured, and we will mention what we have considered in each chapter. This thesis is divided into four chapter.

In Chapter 1, we give a selection of known properties in Hilbert spaces. Also we include some significant theorems such as orthogonal decomposition theorem and Riesz's representation theorem. After that we move to provide some definitions and basic properties in $\mathcal{B}(\mathcal{H})$ that are needed in other chapters.

While Chapter 2 is designed to go deeper into the study of bounded linear operators. First, we define the adjoint of an operator, and we study some of its properties. Then, we offer some bounded linear operators classifications, and we go through their properties. Finally, we provide the definition of the square root and the absolute value of an operator, and we give some of their properties.

As for Chapter 3, we present the most important concepts in the whole thesis. First, we study the spectrum of an operator and its properties in detailed, and we provide the proof of every single property and theorem, then we consider the spectral radius with its known properties. The same we present the numerical range and the numerical radius with their properties, also we provide a wide range of inequalities including brand new inequalities.

Chapter 4 is the most important chapter in this thesis, since we make a small contribution in the literature of the inequalities between the usual operator norm and the numerical radius. First we define a new norm that is equivalent to the usual norm, and we study some of its properties. Then, we obtain inequalities involving the new norm, the usual operator norm and the numerical radius, and we obtain some new inequalities involving the usual norm on $\mathcal{B}(\mathcal{H})$ and the numerical radius, also we get a refinement for some inequalities in particular case.

Chapter 1

Preliminaries

In this chapter, we present vital definitions and properties in Hilbert spaces that we need throughout this thesis in order to make it self-contained, also in this chapter we tend to omit the proof of most of theorems and properties.

1.1 Inner product and Hilbert spaces

Definition 1.1. *Let X be a vector space over $\mathbb{K}$. X is said to be a normed space if there exists a map $\|.\|$ that is defined from X to $\mathbb{R}_+$ satisfies the following :*

(1) $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for any $x \in X$ and $\lambda \in \mathbb{K}$.

(3) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

And we denote $(X,\|.\|)$ is a normed space.

Definition 1.2. *Let $(X,\|.\|)$ be a normed space. Then $(X,\|.\|)$ is said to be a Banach space if every Cauchy sequence has a limit in X i.e.*
if $\|x_n - x_m\| \longrightarrow 0$ as $n \longrightarrow \infty$ and $m \longrightarrow \infty$, there exists $x \in X$ such that $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 1.3. *Let X be a vector space over $\mathbb{K}$. X is said to be an inner product space if there exists a map $\langle .,. \rangle$: $X \times X \longrightarrow \mathbb{K}$ holds the following :*

(1) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in X$.

(2) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$.

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for any $x, y \in X$.

(4) $\langle x, x\rangle > 0$ for all $x \in X \setminus \{0\}$.

And we denote $(X,\langle .,. \rangle)$ is an inner product space, and the map $\langle .,. \rangle$ is called an inner product.

Remark 1.1. *It comes directly from the definition of inner product that :*

(1) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ *for all* $x, y, z \in X$.

(2) $\langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$.

Theorem 1.1. *(Cauchy-Schwarz inequality)*
Let $(X,\langle .,.\rangle)$ *be an inner product space. Then we have :*

$$|\langle x, y\rangle| \leq |\langle x, x\rangle|^{1/2}|\langle y, y\rangle|^{1/2} \quad \text{for all } x, y \in X$$

Moreover if $|\langle x, y\rangle| = |\langle x, x\rangle|^{1/2}|\langle y, y\rangle|^{1/2}$*, then* x *and* y *are linearly dependent.*

Theorem 1.2. *Any inner product space* $(X,\langle .,.\rangle)$ *is a normed space where* $\|x\| = |\langle x, x\rangle|^{1/2}$*, and this norm is called the associated norm of the inner product.*

Remark 1.2. *We can write Cauchy-Schwarz inequality that way*

$$|\langle x, y\rangle| \leq \|x\|\|y\| \quad \text{for all } x, y \in X$$

Lemma 1.1. *Let* $(X,\langle .,.\rangle)$ *be an inner product space and suppose that* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are convergent sequences in* X*, with* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$*. Then* $\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x, y\rangle$

Theorem 1.3. *(Buzano's inequality)*
Let $(X,\langle .,.\rangle)$ *be an inner product space and* $x, y, e \in X$ *with* $\|e\|$*=1. Then we have :*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|)$$

Proof. *Let* x, y, e *be as in the theorem*

$$\begin{aligned}
2|\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle| &= |\langle x, 2\overline{\langle e, y\rangle}e\rangle| - |\langle x, y\rangle| \\
&\leqslant |\langle x, 2\overline{\langle e, y\rangle}e\rangle - \langle x, y\rangle| \\
&= |\langle x, 2\overline{\langle e, y\rangle}e - y\rangle| \quad \text{(use Cauchy-Schwarz inequality)} \\
&\leqslant \|x\|\|2\overline{\langle e, y\rangle}e - y\|
\end{aligned}$$

So we have

$$2|\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle| \leqslant \|x\|\|2\overline{\langle e, y\rangle}e - y\| \tag{1.1}$$

Let us calculate $\|2\overline{\langle e, y\rangle}e - y\|$

$$\begin{aligned}
\|2\overline{\langle e, y\rangle}e - y\|^2 &= \langle 2\overline{\langle e, y\rangle}e - y, 2\overline{\langle e, y\rangle}e - y\rangle \\
&= 4|\langle e, y\rangle|^2\|e\|^2 + \|y\|^2 - 2\langle e, y\rangle\overline{\langle e, y\rangle} - 2\langle e, y\rangle\langle y, e\rangle \quad (\text{using } \|e\|^2 = 1 \text{ and } \langle y, e\rangle = \overline{\langle e, y\rangle}) \\
&= 4|\langle e, y\rangle|^2 + \|y\|^2 - 4|\langle e, y\rangle|^2 \\
&= \|y\|^2
\end{aligned}$$

So $\|2\overline{\langle e, y\rangle}e - y\| = \|y\|$
Using (1.1) *we get*

$$2|\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle| \leqslant \|x\|\|y\|$$

Hence

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|)$$

Remark 1.3. *(1) Let* $x, y, z \in X$*. From Buzano's inequality we obtain*

$$|\langle x, y\rangle\langle y, z\rangle| \leq \frac{1}{2}(\|x\|\|z\| + |\langle x, z\rangle|)\|y\|^2 \tag{1.2}$$

(2) Buzano's inequality is a generalization of Cauchy-Schwarz inequality, just put $z = x$ *in* (1.2) *to obtain Cauchy-Schwarz inequality.*

Proposition 1.1. *Let $(X,\langle .,. \rangle)$ be an inner product space. Then we have :*
(1) $\forall x, y \in X : \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ *(Parallelogram law)*
(2) if X is a real vector space, then $\forall x, y \in X : \langle x, y \rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$
(3) if X is a complex vector space, then

$$\forall x, y \in X : \langle x, y \rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\} \qquad \text{(Polarization identity)}$$

Where $\|x\| = |\langle x, x\rangle|^{1/2}$ *and i is the imaginary unit of complex numbers* $(i^2 = -1)$.

Definition 1.4. *Let $\mathcal{H}$ be a vector space over $\mathbb{C}$. $\mathcal{H}$ is said to be a Hilbert space if it is an inner product space and $\mathcal{H}$ with associated norm is a Banach space, and we denote $(\mathcal{H},\langle .,. \rangle)$ is a complex Hilbert space.*

Example 1.1. *(1) The euclidean inner product in $\mathbb{C}^n$ $(n \in \mathbb{N})$ is defined as follows :*

$$\forall x, y \in \mathbb{C}^n : \qquad \langle x, y \rangle = \sum_{k=1}^{k=n} x_k \overline{y_k}$$

$\mathbb{C}^n$ with its euclidean inner product $(\mathbb{C}^n,\langle .,. \rangle)$ is a Hilbert space.
(2) Consider the following vector space over $\mathbb{C}$, $\ell^2(\mathbb{C}) = \left\{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{C}; \quad \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$.
This space is a Hilbert space when it is endowed with the following inner product :

$$\forall x, y \in \ell^2 : \qquad \langle x, y \rangle_{\ell^2} = \sum_{n=1}^{+\infty} x_n \overline{y_n}$$

1.2 Orthogonal Decomposition of Hilbert space and Riesz's representation theorem

From now on we consider $(\mathcal{H},\langle .,. \rangle)$ a complex Hilbert space, and $\|.\|$ is the associated norm of the inner product.

Definition 1.5. *M is said to be a closed linear subspace of $\mathcal{H}$ if it is a linear subspace of $\mathcal{H}$ and $\overline{M} = M$.*

Corollary 1.1. *Let M be a linear subspace of $\mathcal{H}$. Then $\overline{M}$ is a linear subspace of $\mathcal{H}$ as well.*

Definition 1.6. *Let M and F be two linear subspaces of $\mathcal{H}$. Then $\mathcal{H}$ is said to be the direct sum of M and F, and we write $\mathcal{H} = M \oplus F$, if $\mathcal{H} = M + F$ and $M \cap F = \{0\}$.*

Lemma 1.2. *Let M and F be two linear subspaces of $\mathcal{H}$. Then $\mathcal{H} = M \oplus F$ if and only if for every $x \in \mathcal{H}$ there exist unique vectors $y \in M$ and $z \in F$ such that $x = y + z$.*

Definition 1.7. *The vectors $x, y \in \mathcal{H}$ are said to be orthogonal if $\langle x, y \rangle = 0$, and we write $x \perp y$.*

Corollary 1.2. *if $x, y \in \mathcal{H} \setminus \{0\}$ are orthogonal. Then they are linearly independent.*

Theorem 1.4. *(Pythagoras's theorem)*
If $x, y \in \mathcal{H}$ are orthogonal, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Definition 1.8. *Let M be subset of $\mathcal{H}$. The orthogonal complement of M is the set*

$$M^{\perp} = \{x \in \mathcal{H} : \langle x, y \rangle = 0 \quad \forall y \in M\}$$

Proposition 1.2. *Let M and F be two subsets of $\mathcal{H}$ then :*
(1) $\mathcal{H}^{\perp} = \{0\}$ and $\{0\}^{\perp} = \mathcal{H}$.
(2) $M^{\perp}$ is a closed linear subspace of $\mathcal{H}$.

(3) If $0 \in M$ *then* $M \cap M^{\perp} = \{0\}$*, otherwise* $M \cap M^{\perp} = \emptyset$.

(4) If $M \subset F$ *then* $F^{\perp} \subset M^{\perp}$.

(5) $M \subset (M^{\perp})^{\perp}$.

Theorem 1.5. *(Orthogonal Decomposition)*
Let M *be a closed linear subspace of* $\mathcal{H}$*. Then* $\mathcal{H} = M \oplus M^{\perp}$ *is the direct sum of* M *and* $M^{\perp}$ *i.e. for every* $x \in \mathcal{H}$ *there exist a unique* $y \in M$ *and a unique* $z \in M^{\perp}$ *such that* $x = y + z$.

Corollary 1.3. *Let* M *be a linear subspace of* $\mathcal{H}$*. Then :*

(1) $\mathcal{H} = \overline{M} \oplus M^{\perp}$.

(2) $(M^{\perp})^{\perp} = \overline{M}$.

Definition 1.9. *Let* f *be a mapping from* $\mathcal{H}$ *to* $\mathbb{C}$*. Then* f *is said to be a linear functional if it satisfies:*

$$\forall x, y \in \mathcal{H},\ \forall \lambda \in \mathbb{C} : \qquad f(\lambda x + y) = \lambda f(x) + f(y)$$

Definition 1.10. *Let* f *be a linear functional from* $\mathcal{H}$ *to* $\mathbb{C}$*. Then* f *is said to be a bounded linear functional if it satisfies:*

$$\exists c > 0, \quad \forall x \in \mathcal{H} : \qquad |f(x)| \leqslant c\|x\|$$

Definition 1.11. *The set of all bounded linear functionals from* $\mathcal{H}$ *to* $\mathbb{C}$ *is a vector space over* $\mathbb{C}$ *denoted by* $\mathcal{H}'$*, where the addition and the external product are defined as follows :*

Let $f, g \in \mathcal{H}'$ *and* $\lambda \in \mathbb{C}$ *:*

(1) $\forall x \in \mathcal{H} \quad (f+g)(x) = f(x) + g(x)$

(2) $\forall x \in \mathcal{H} \quad (\lambda f)(x) = \lambda f(x)$

Moreover $\mathcal{H}'$ *is called the topological dual of* $\mathcal{H}$*, and* $\mathcal{H}'$ *is a normed space when we equip it with the following norm :*

Let $f \in \mathcal{H}'$*,then* $\qquad \|f\|_{\mathcal{H}'} = \sup\{|f(x)| : \|x\| = 1\}$

Theorem 1.6. $(\mathcal{H}', \|\|_{\mathcal{H}'})$ *is a Banach space.*

Theorem 1.7. *(Riesz's representation theorem)*
Let y *be any arbitrary fixed in* $\mathcal{H}$*, we define* $f(x)$ *by*

$$f(x) = \langle x, y \rangle \qquad \forall x \in \mathcal{H}$$

Then $f \in \mathcal{H}'$ *such that* $\|f\|_{\mathcal{H}'} = \|y\|$.

Conversely, for any $f \in \mathcal{H}'$*, there exists a unique* $y \in \mathcal{H}$ *such that*

$$f(x) = \langle x, y \rangle \qquad \forall x \in \mathcal{H}$$

1.3 Definitions and basic properties in $\mathcal{B}(\mathcal{H})$

Definition 1.12. *A mapping* T *from* $\mathcal{H}$ *to* $\mathcal{H}$ *is said to be a linear operator if it satisfies the following:*

(1) Additive : $T(x+y) = Tx + Ty \quad \forall x, y \in \mathcal{H}$

(2) Homogeneous : $T(\lambda x) = \lambda Tx \quad \forall x \in \mathcal{H}$ *and* $\forall \lambda \in \mathbb{C}$

Definition 1.13. *A linear operator* T *on* $\mathcal{H}$ *is said to be bounded if it satisfies:*

$$\exists c > 0 \ \ \forall x \in \mathcal{H} : \qquad \|Tx\| \leqslant c\|x\|$$

Definition 1.14. *The set of all bounded linear operators on $\mathcal{H}$ is an unitary algebra over $\mathbb{C}$ denoted by $\mathcal{B}(\mathcal{H})$, where the addition, the external product and the product are defined as follows :*
Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$:
(1) $\forall x \in \mathcal{H} \quad (T+S)x = Tx + Sx$
(2) $\forall x \in \mathcal{H} \quad (\lambda T)x = \lambda Tx$
(3) $\forall x \in \mathcal{H} \quad (TS)x = T(Sx)$
And the unitary element is I the identity($Ix = x, \ \forall x \in \mathcal{H}$), moreover $\mathcal{B}(\mathcal{H})$ is a Banach algebra when we equip it with the following norm :
Let $T \in \mathcal{B}(\mathcal{H})$,then $\quad \|T\| = \inf\{c > 0 : \|Tx\| \leq c\|x\| \quad \text{for all } x \in \mathcal{H}\}$

Theorem 1.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*
$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} = \sup\{\frac{\|Tx\|}{\|x\|} : x \in \mathcal{H} \setminus \{0\}\} = \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}$

Corollary 1.4. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then :*
(1) $\quad \forall x \in \mathcal{H} \quad \|Tx\| \leq \|T\|\|x\|$
(2) $\quad \|TS\| \leq \|T\|\|S\|$
(3) $\quad \|T^n\| \leq \|T\|^n$

Definition 1.15. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*
(1) The range of T is the set

$$R(T) = \{Tx : x \in \mathcal{H}\}$$

(1) The kernel of T is the set

$$N(T) = \{x \in \mathcal{H} : Tx = 0\}$$

Proposition 1.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*
(1) $R(T)$ is a linear subspace of $\mathcal{H}$.
(2) $N(T)$ is a closed linear subspace of $\mathcal{H}$.

Proposition 1.4. *(Generalized polarization identity)*
Let $T \in \mathcal{B}(\mathcal{H})$. Then
$\forall x, y \in X \ : \ \langle Tx, y\rangle = \frac{1}{4}\{\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle\}$

Theorem 1.9. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\forall x \in \mathcal{H} \qquad \langle Tx, x\rangle = 0 \quad \Longrightarrow \quad T = 0$$

Proof. *Let $x, y \in \mathcal{H}$ suppose that $\langle Tx, x\rangle = 0$ for any $x \in \mathcal{H}$, then*
Since $\langle Tx, y\rangle = \frac{1}{4}\{\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle\}$

$$\Longrightarrow \langle Tx, y\rangle = 0 \quad \forall x, y \in \mathcal{H}$$

Setting $y = Tx$ we get

$$\|Tx\|^2 = \langle Tx, Tx\rangle = 0 \qquad \forall x \in \mathcal{H}$$

$$\Longrightarrow Tx = 0 \qquad \forall x \in \mathcal{H}$$

Therefore $T = 0$.

Remark 1.4. *Theorem 1.9 is not correct in real Hilbert spaces.*
Considering the following real Hilbert space $(\mathbb{R}^2, \langle, \rangle)$, and let $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ note that $\langle Tx, x\rangle = 0$ for all $x \in \mathbb{R}^2$ but $T \neq 0$.

Corollary 1.5. *Let $T, S \in \mathcal{B}(\mathcal{H})$. If $\langle Tx, x\rangle = \langle Sx, x\rangle$ for all $x \in \mathcal{H}$, then $T = S$.*

Definition 1.16. *Let $T \in \mathcal{B}(\mathcal{H})$ and $M \subset \mathcal{H}$ a linear subspace of $\mathcal{H}$. Then $T_{|M}$ is the restriction of T on M such that $T_{|M} : M \longrightarrow \mathcal{H}$ by $T_{|M}x = Tx$ for all $x \in M$.*

Definition 1.17. *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ be a sequence of operators and $T \in \mathcal{B}(\mathcal{H})$. Then*
(1) $(T_n)_{n\in\mathbb{N}}$ is said to be strongly convergent to T if

$$\forall x \in \mathcal{H} \qquad \lim_{n\to\infty} \|T_n x - Tx\| = 0$$

And we denote $T_n \xrightarrow{s} T$
(2) $(T_n)_{n\in\mathbb{N}}$ is said to be uniformly convergent to T if

$$\lim_{n\to\infty} \|T_n - T\| = 0$$

And we denote $T_n \longrightarrow T$

Proposition 1.5. *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ be a sequence of operators and $T \in \mathcal{B}(\mathcal{H})$. Then*

$$(T_n)_{n\in\mathbb{N}} \text{ is uniformly convergent to} T \implies (T_n)_{n\in\mathbb{N}} \text{ is strongly convergent to} T$$

Briefly

$$T_n \longrightarrow T \Longrightarrow T_n \xrightarrow{s} T$$

Definition 1.18. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is said to be invertible if there exists $S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$, in which case S is the inverse of T and it is denoted by T^{-1}, and $\mathcal{I}(\mathcal{H})$ denotes to the set of all invertible operators in $\mathcal{B}(\mathcal{H})$.*

Lemma 1.3. *Let $T, S \in \mathcal{I}(\mathcal{H})$, and $n \in \mathbb{N}$. Then :*
(1) The inverse of T^{-1} is T.
(2) The inverse of ST is $T^{-1}S^{-1}$.
(3) The inverse of T^n is T^{-n} $(T^{-n} = (T^{-1})^n)$.

Corollary 1.6. *(Banach's Isomorphism Theorem)*
Let $T \in \mathcal{B}(\mathcal{H})$. If T is bijective, then T invertible.

Remark 1.5. *The result of Banach's Isomorphism Theorem is that in $\mathcal{B}(\mathcal{H})$, when T^{-1} exists, then it is bounded.*

Theorem 1.10. *(Neumann series)*
Let $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\| < 1$. Then $I - T$ is invertible and the inverse given by

$$(I-T)^{-1} = \textstyle\sum_{n=0}^{\infty} T^n \qquad \text{and} \qquad \|(I-T)^{-1}\| \leq \frac{1}{1 - \|T\|}$$

Lemma 1.4. *Let $\mathcal{I}(\mathcal{H})$ be the set of all invertible operators in $\mathcal{B}(\mathcal{H})$. Then $\mathcal{I}(\mathcal{H})$ is an open set in $\mathcal{B}(\mathcal{H})$.*

Proof. *Let $T \in \mathcal{I}(\mathcal{H})$ and let $\epsilon = \|T^{-1}\|^{-1}$, let's prove that $B(T, \epsilon) \subset \mathcal{I}(\mathcal{H})$ where*

$$B(T, \epsilon) = \{S \in \mathcal{B}(\mathcal{H}) : \|T - S\| \ < \epsilon\}$$

Let $S \in B(T, \epsilon) \iff \|T - S\| \ < \epsilon$,then $\|(T-S)T^{-1}\| \leq \|T - S\| \ \|T^{-1}\| < \epsilon\|T^{-1}\| = \|T^{-1}\|^{-1}\|T^{-1}\| = 1$
Then $\|(T-S)T^{-1}\| < 1 \implies I - (T-S)T^{-1}$ is invertible (using Theorem 1.10).
But $I - (T-S)T^{-1} = I - I + ST^{-1} = ST^{-1}$, which means that ST^{-1} is invertible.
We have $S = ST^{-1}T$, since ST^{-1} and T are invertible $\implies S$ is invertible $\implies S \in \mathcal{I}(\mathcal{H})$.
Therefore $B(T, \epsilon) \subset \mathcal{I}(\mathcal{H})$, then $\mathcal{I}(\mathcal{H})$ is open.

Corollary 1.7. *In the proof of Lemma 1.4, we proved if* $T \in \mathcal{I}(\mathcal{H})$ *and* $S \in \mathcal{B}(\mathcal{H})$ *such that* $\|T - S\| \leq \|T^{-1}\|^{-1}$, *then* $S \in \mathcal{I}(\mathcal{H})$.

Theorem 1.11. $\mathcal{F} : \mathcal{I}(\mathcal{H}) \longrightarrow \mathcal{I}(\mathcal{H})$ *defined by* $\mathcal{F}(T) = T^{-1}$ *is continuous.*

Proof. *Let* $(T_n) \subset \mathcal{I}(\mathcal{H})$ *and* $T \in \mathcal{I}(\mathcal{H})$ *such that* $\lim\limits_{n\to\infty} \|T_n - T\| = 0$. *Since* $T_n \underset{n\to\infty}{\longrightarrow} T$, *then :*
$\exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \|T_n - T\| < \dfrac{1}{2\|T^{-1}\|} \Longrightarrow \forall n \geq n_0 \quad \|I - T^{-1}T_n\| = \|T^{-1}(T - T_n)\| \leq \|T^{-1}\|\|T_n - T\| < \dfrac{1}{2}$.
$\Longrightarrow \forall n \geq n_0 \quad \|I - T^{-1}T_n\| < \dfrac{1}{2} \Longrightarrow \forall n \geq n_0 \quad T^{-1}T_n$ *is invertible and* $\|(T^{-1}T_n)^{-1}\| \leq \dfrac{1}{1 - \|I - T^{-1}T_n\|} < 2$
(by Theorem 1.10), then $\forall n \geq n_0 \quad \|(T^{-1}T_n)^{-1}\| < 2$.
$\forall n \geq n_0 \quad \|T_n^{-1}\| = \|T_n^{-1}TT^{-1}\| \leq \|T_n^{-1}T\|\|T^{-1}\| = \|(T^{-1}T_n)^{-1}\|\|T^{-1}\| < 2\|T^{-1}\|$
Then, $\forall n \geq n_0 \quad \|T_n^{-1}\| < 2\|T^{-1}\| \Longrightarrow \sup\limits_{n\in\mathbb{N}}\|T_n^{-1}\| \leq \max\{\|T_1^{-1}\|, \|T_2^{-1}\|, \cdots, \|T_{n_0}^{-1}\|, 2\|T^{-1}\|\} < \infty$.
Then, $\sup\limits_{n\in\mathbb{N}}\|T_n^{-1}\| < \infty \Longleftrightarrow \exists c > 0 \quad \forall n \in \mathbb{N} : \|T_n^{-1}\| \leq c$.
Since $\|T_n^{-1} - T^{-1}\| = \|T_n^{-1}(T_n - T)T^{-1}\| \leq \|T_n^{-1}\|\|T_n - T\|\|T^{-1}\| \leq c\|T_n - T\|\|T^{-1}\| \underset{n\to\infty}{\longrightarrow} 0$.
$\Longrightarrow \lim\limits_{n\to\infty} \|T_n^{-1} - T^{-1}\| = 0 \Longleftrightarrow \mathcal{F}(T_n) \underset{n\to\infty}{\longrightarrow} \mathcal{F}(T)$.
Therefore $\mathcal{F}$ *is continuous.*

Lemma 1.5. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be an invertible operator. Then:*

$$\forall x \in \mathcal{H} : \qquad \|Tx\| \geq \|T^{-1}\|^{-1}\|x\|$$

Proof. $\forall y \in \mathcal{H} \quad \|T^{-1}y\| \leq \|T^{-1}\|\|y\|$ *, let* $x \in \mathcal{H}$ *setting* $y = Tx$*,and since* T *is invertible* $\|T^{-1}\| \neq 0$*, we get :*
$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\| \Longrightarrow \|Tx\| \geq \|T^{-1}\|^{-1}\|x\| \qquad$ *for all* $x \in \mathcal{H}$.

Lemma 1.6. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be an operator. Assume that there exists* $\alpha > 0$ *such that* $\|Tx\| \geq \alpha\|x\|$ *for all* $x \in \mathcal{H}$*, then* $R(T)$ *is closed.*

Proof. *Let* $(y_n)_{n\in\mathbb{N}} \subset R(T)$ *and* $y \in \mathcal{H}$ *such that* $y_n \underset{n\to\infty}{\longrightarrow} y$*, since* $(y_n)_{n\in\mathbb{N}} \subset R(T)$*, there exists* $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that* $\forall n \in \mathbb{N} \quad y_n = Tx_n$ *, based on the assumption, we have:*
$\forall n, m \in \mathbb{N} : \ \|y_n - y_m\| = \|Tx_n - Tx_m\| \geq \alpha\|x_n - x_m\|$*, since* $(y_n)_{n\in\mathbb{N}}$ *is convergent, then* $\lim\limits_{n,m\to\infty} \|y_n - y_m\| = 0$
$\Longrightarrow \lim\limits_{n,m\to\infty} \|x_n - x_m\| = 0 \Longleftrightarrow (x_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence.*
Since $(\mathcal{H}, \|\cdot\|)$ *is a Banach space, then there exists* $x \in \mathcal{H}$ *such that* $x_n \underset{n\to\infty}{\longrightarrow} x$
We know that $\forall n \in \mathbb{N} \quad y_n = Tx_n$*, using the boundedness of* T*, we obtain :*
$y = \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} Tx_n = T(\lim\limits_{n\to\infty} x_n) = Tx \Longrightarrow y = Tx$ *, then* $y \in R(T)$.
Hence $\overline{R(T)} = R(T) \Longleftrightarrow R(T)$ *is closed.*

Theorem 1.12. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then the following are equivalent :*
(1) T *is invertible.*
(2) $R(T)$ *is dense in* $\mathcal{H}$ *and there exists* $\alpha > 0$ *such that* $\|Tx\| \geq \alpha\|x\|$ *for all* $x \in \mathcal{H}$.

Proof. *(1)* $\Longrightarrow$ *(2) Since* T *is invertible, then* $R(T) = \mathcal{H}$ *which means that* $R(T)$ *is dense in* $\mathcal{H}$.
And from Lemma 1.5 we have that: $\qquad \forall x \in \mathcal{H} : \qquad \|Tx\| \geq \|T^{-1}\|^{-1}\|x\|$
(2) $\Longrightarrow$ *(1) Since* $R(T)$ *is dense in* $\mathcal{H} \Longleftrightarrow \overline{R(T)} = \mathcal{H}$*, and since there exists* $\alpha > 0$ *such that* $\|Tx\| \geq \alpha\|x\|$ *for all* $x \in \mathcal{H}$*, and using Lemma 1.6* $\Longrightarrow R(T)$ *is closed, then* $R(T) = \mathcal{H}$.
On the other hand, let $x \in N(T) \Longleftrightarrow Tx = 0 \Longrightarrow 0 = \|Tx\| \geq \alpha\|x\| \Longrightarrow \|x\| = 0 \Longleftrightarrow x = 0$
Then $N(T) \subset \{0\} \Longrightarrow N(T) = \{0\}$
Hence T *is bijective, using Corollary 1.6 we conclude that* T *is invertible.*

Corollary 1.8. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then* T *is not invertible if and only if* $R(T)$ *is not dense in* $\mathcal{H}$ *or there exists* $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *with* $\|x_n\| = 1$ *for all* $n \in \mathbb{N}$ *such that* $\lim\limits_{n\to\infty} Tx_n = 0$.

Chapter 2

Linear operators on Hilbert spaces

This chapter is devoted to introduce definitions and basic properties of linear operators acting on Hilbert space, as well as well-know theorems and results with their proof. Note that we purposely omit the proof of some properties and theorems when they are straightforward or when their proofs need another concepts and knowledge.

2.1 The adjoint of a linear operator

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a unique $T^* \in \mathcal{B}(\mathcal{H})$ such that*

$$\forall x, y \in \mathcal{H} \qquad \langle Tx, y\rangle = \langle x, T^*y\rangle$$

Proof. *Consider $f(x) = \langle Tx, y\rangle \quad \forall x \in \mathcal{H}$. First of all let's prove that $f \in \mathcal{H}'$ for every arbitrary $y \in \mathcal{H}$.*
f is linear :
Let $x, z \in \mathcal{H}$ and $\lambda \in \mathbb{C}$: $\quad f(\lambda x + z) = \langle T(\lambda x + z), y\rangle = \langle \lambda Tx + Tz, y\rangle = \lambda\langle Tx, y\rangle + \langle Tz, y\rangle = \lambda f(x) + f(z)$
f is bounded for every arbitrary $y \in \mathcal{H}$:
Let $x \in \mathcal{H}$, using the boundedness of T and Cauchy-Schwarz inequality, we obtain:

$$|f(x)| = |\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|y\|\|x\|$$

Therefore

$$\exists c = \|T\|\|y\| \geq 0 \quad \forall x \in \mathcal{H} \qquad |f(x)| \leq c\|x\|$$

Then $f \in \mathcal{H}'$ for every arbitrary $y \in \mathcal{H}$.
Applying Riesz's representation theorem on f, we get
For every arbitrary $y \in \mathcal{H}$, there exists a unique $z \in \mathcal{H}$ such that $f(x) = \langle Tx, y\rangle = \langle x, z\rangle$
We set for every $y \in \mathcal{H} \quad T^*y = z$ (the uniqueness of the element z for every y enables us to say T^* is well-defined)
Hence

$$\forall x, y \in \mathcal{H} \qquad \langle Tx, y\rangle = \langle x, T^*y\rangle$$

Now we need to prove that $T^* \in \mathcal{B}(\mathcal{H})$
Let $\lambda \in \mathbb{C}$ and $y_1, y_2 \in \mathcal{H}$, then $\langle Tx, y_1\rangle = \langle x, T^*y_1\rangle$ and $\langle Tx, y_2\rangle = \langle x, T^*y_2\rangle \qquad$ for every $x \in \mathcal{H}$

$$\langle Tx, \lambda y_1 + y_2\rangle = \langle x, T^*(\lambda y_1 + y_2)\rangle \qquad \text{for every } x \in \mathcal{H}$$

$\langle Tx, \lambda y_1 + y_2\rangle = \overline{\lambda}\langle Tx, y_1\rangle + \langle Tx, y_2\rangle = \overline{\lambda}\langle x, T^*y_1\rangle + \langle x, T^*y_2\rangle = \langle x, \lambda T^*y_1 + T^*y_2\rangle$

Then $\forall x \in \mathcal{H} \qquad \langle x, \lambda T^*y_1 + T^*y_2\rangle = \langle x, T^*(\lambda y_1 + y_2)\rangle$, *which gives us*

$\forall x \in \mathcal{H} \qquad \langle x, \lambda T^*y_1 + T^*y_2 - T^*(\lambda y_1 + y_2)\rangle = 0$

Hence $\quad T^*(\lambda y_1 + y_2) = \lambda T^*y_1 + T^*y_2$, *then* T^* *is linear.*

Let's prove that T^* *is bounded, using the boundedness of* T *and Cauchy-Schwarz inequality we get*

$\forall y \in \mathcal{H} \qquad \|T^*y\|^2 = \langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle \leq \|TT^*y\|\|y\| \leq \|T\|\|T^*y\|\|y\|$

Thus $\quad \forall y \in \mathcal{H} \qquad \|T^*y\| \leq \|T\|\|y\|$, *therefore* T^* *is bounded.*

Then $T^* \in \mathcal{B}(\mathcal{H})$. *We just still need to prove the uniqueness of* T^*.

Assume that there exists $S \in \mathcal{B}(\mathcal{H})$ *such that* $\forall x, y \in \mathcal{H} \qquad \langle Tx, y\rangle = \langle x, Sy\rangle$, *then:*

$$\forall x, y \in \mathcal{H}: \quad \langle x, T^*y\rangle = \langle x, Sy\rangle$$
$$\forall x, y \in \mathcal{H}: \quad \langle x, T^*y - Sy\rangle = 0$$
$$\forall y \in \mathcal{H}: \quad T^*y = Sy$$

Hence $T^* = S$, *so* T^* *is unique.*

Definition 2.1. *Let* $T \in \mathcal{B}(\mathcal{H})$. *The operator* T^* *constructed in Theorem 2.1 is called the adjoint of* T.

Example 2.1. *(1) Let* $S \in \mathcal{B}(\ell^2(\mathbb{C}))$ *be an operator defined as follows :*

$$S(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$$

S *is called unilateral shift, and its adjoint is*

$$S^*(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$$

(2) Let $I \in \mathcal{B}(\mathcal{H})$ *be the identity operator on* $\mathcal{H}$. *Then we have* $I^* = I$.

(3) Let $\mathcal{M}_n(\mathbb{C})$ *be the vector space of square matrices* $n \times n$ *on* $\mathbb{C}$. *If* $A = [a_{i,j}] \in \mathcal{B}(\mathbb{C}^n)$, *then* $A^* = [\overline{a_{j,i}}]$

Proposition 2.1. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *and* $\lambda \in \mathbb{C}$. *Then :*

(1) $(T + S)^* = T^* + S^*$

(2) $(\lambda T)^* = \overline{\lambda}T^*$

(3) $(TS)^* = S^*T^*$

(4) $(T^*)^* = T$

(5) $\|T^*\| = \|T\|$

(6) $\|T^*T\| = \|T\|^2$

Proof. *(1),(2), (3) and (4) are straightforward.*

(5) $\forall x \in \mathcal{H} \qquad \|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle \leq \|TT^*x\|\|x\| \leq \|T\|\|T^*x\|\|x\|$

$$\Longrightarrow \quad \forall x \in \mathcal{H} \setminus \{0\} \quad \frac{\|T^*x\|}{\|x\|} \leq \|T\| \qquad \Longrightarrow \qquad \|T^*\| = \sup_{x \neq 0}\{\frac{\|T^*x\|}{\|x\|}\} \leq \|T\|$$

$$\text{Then} \qquad \|T^*\| \leq \|T\| \tag{2.1}$$

Using 2.1 we get $\qquad \|(T^*)^*\| \leq \|T^*\|$, *but we know that* $(T^*)^* = T$ *from property (4)*

$$\text{Then} \qquad \|T\| \leq \|T^*\| \tag{2.2}$$

From 2.1 and 2.2 we obtain $\|T^*\| = \|T\|$.

(6) Let $x \in \mathcal{H}$ *such that* $\|x\| = 1$

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\| = \|T^*Tx\|$$

$$\Longrightarrow \quad \|T\|^2 = (\sup_{\|x\|=1} \|Tx\|)^2 = \sup_{\|x\|=1} \|Tx\|^2 \leq \sup_{\|x\|=1} \|T^*Tx\| = \|T^*T\|$$

$$Then \qquad \|T\|^2 \leq \|T^*T\| \tag{2.3}$$

Using property (5), we obtain $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$

$$Then \qquad \|T^*T\| \leq \|T\|^2 \tag{2.4}$$

From 2.3 and 2.4 we obtain $\|T^*T\| = \|T\|^2$.

Corollary 2.1. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then :*

1. *The function* $f : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ *by* $f(T) = T^*$ *is bounded (continuous) antilinear.(this comes from properties (1),(2) and (5)).*

2. $T = 0$ *in and only if* $T^*T = 0$*.(it comes form property (6)).*

3. *Let* $n \in \mathbb{N}$*, then* $(T^n)^* = (T^*)^n$*.*

Lemma 2.1. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then :*

(1) $N(T) = (R(T^*))^{\perp}$.

(2) $N(T^*) = (R(T))^{\perp}$.

(3) $N(T^*) = \{0\}$ *if and only if* $R(T)$ *dense in* $\mathcal{H}$*.*

Proof. *(1) Let* $x \in N(T) \Longleftrightarrow Tx = 0 \Longrightarrow \forall y \in \mathcal{H}\text{:} \quad \langle Tx, y\rangle = \langle x, T^*y\rangle = 0$

$$\Longrightarrow \forall y \in R(T^*) : \quad \langle x, y\rangle = 0$$
$$\Longrightarrow x \in (R(T^*))^{\perp}$$
$$\Longrightarrow N(T) \subset (R(T^*))^{\perp}$$

Conversely, let $x \in (R(T^*))^{\perp} \Longrightarrow \quad \forall y \in R(T^*) : \quad \langle x, y\rangle = 0 \quad \Longrightarrow \forall y \in \mathcal{H} : \quad \langle Tx, y\rangle = \langle x, T^*y\rangle = 0$

$$\Longrightarrow \langle Tx, Tx\rangle = 0$$
$$\Longrightarrow Tx = 0$$
$$\Longrightarrow x \in N(T)$$
$$\Longrightarrow (R(T^*))^{\perp} \subset N(T)$$

Therefore $N(T) = (R(T^*))^{\perp}$.

(2) From (1), we have $N(T^*) = (R((T^*)^*))^{\perp}$*, and since* $(T^*)^* = T$*, hence* $N(T^*) = (R(T))^{\perp}$.

(3) From (2), we have $N(T^*) = \{0\} \Longleftrightarrow (R(T))^{\perp} = \{0\} \Longleftrightarrow \overline{R(T)} = \mathcal{H}$ *(since* $\overline{R(T)} \oplus (R(T))^{\perp} = \mathcal{H}$*).*

Then $N(T^*) = \{0\}$ *if and only if* $R(T)$ *dense in* $\mathcal{H}$*.*

Lemma 2.2. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be an invertible operator, then* $(T^{-1})^* = (T^*)^{-1}$*.*

Proof. *We have* $TT^{-1} = T^{-1}T = I \Longrightarrow (TT^{-1})^* = (T^{-1}T)^* = I^*$

$$\Longrightarrow (T^{-1})^*T^* = T^*(T^{-1})^* = I$$
$$\Longrightarrow (T^*)^{-1} = (T^{-1})^*$$

Corollary 2.2. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then the following are equivalent :*

(1) T *is invertible.*

(2) $N(T^*) = \{0\}$ *and there exists* $\alpha > 0$ *such that* $\|Tx\| \geq \alpha\|x\|$ *for all* $x \in \mathcal{H}$*.*

Proof. *By Lemma 2.1 and Theorem 1.12.*

2.2 Some classes of linear operators in $\mathcal{B}(\mathcal{H})$

Definition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is called:*

- ***Self-adjoint operator :*** $T^* = T$.
- ***Normal operator :*** $TT^* = T^*T$.
- ***Positive operator :*** $\forall x \in \mathcal{H}$: $\langle Tx, x\rangle \geq 0$, *and we denote* $T \geq 0$.*(*T *is strictly positive operator if* $\forall x \in \mathcal{H} \setminus \{0\}$: $\langle Tx, x\rangle > 0$, *and we denote* $T > 0$*)*
- ***Unitary operator :*** $TT^* = T^*T = I$.
- ***Isometry operator :*** $T^*T = I$.
- ***Projection operator :*** $T^2 = T$.
- ***Orthogonal projection operator :*** $T^2 = T = T^*$.
- ***Quasinormal operator :*** $T(T^*T) = (T^*T)T$.
- ***Hyponormal operator :*** $T^*T \geq TT^*$.*(*$T \geq S$ *if and only if* $T - S \geq 0$*)*

Theorem 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then :*

(1) T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.

(2) T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$.

(3) T is unitary if and only if $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in \mathcal{H}$.

(4) T is hyponormal if and only if $\|Tx\| \geq \|T^*x\|$ for all $x \in \mathcal{H}$.

Proof. *(1) Assume that T is self-adjoint, then $\forall x \in \mathcal{H}$ $\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle$.*

Then $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$, therefore $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.

Conversely, suppose that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$, then

$\langle Tx, x\rangle = \langle x, Tx\rangle = \langle T^*x, x\rangle \implies \langle (T - T^*)x, x\rangle = 0$ *for all $x \in \mathcal{H}$.*

By Theorem 1.9, we get $T - T^* = 0$, hence T is self-adjoint.

(2) T *is normal*
$$\begin{aligned} \iff T^*T - TT^* = 0 &\iff \forall x \in \mathcal{H}: \quad \langle (T^*T - TT^*)x, x\rangle = 0 \\ &\iff \forall x \in \mathcal{H}: \quad \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle \\ &\iff \forall x \in \mathcal{H}: \quad \langle Tx, Tx\rangle = \langle T^*x, T^*x\rangle \\ &\iff \forall x \in \mathcal{H}: \quad \|Tx\|^2 = \|T^*x\|^2 \\ &\iff \forall x \in \mathcal{H}: \quad \|Tx\| = \|T^*x\| \end{aligned}$$

(3) Assume that T is unitary $\iff TT^* = T^*T = I$, then T is normal $\iff \|Tx\| = \|T^*x\|$ $\forall x \in \mathcal{H}$.

So we just need to prove that $\forall x \in \mathcal{H}$: $\|Tx\| = \|x\|$.

$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle x, x\rangle = \|x\|^2$, *therefore* $\|Tx\| = \|T^*x\| = \|x\|$ *for all* $x \in \mathcal{H}$.

Conversely, suppose that $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in \mathcal{H}$, then $TT^* = T^*T$ (using (2)).

So we just need to prove that $T^*T = I$, we have that $\|Tx\| = \|x\|$ for all $x \in \mathcal{H}$. Then:

$\|Tx\|^2 = \langle Tx, Tx\rangle = \|x\|^2 = \langle x, x\rangle \implies \langle T^*Tx, x\rangle = \langle x, x\rangle \implies \langle (T^*T - I)x, x\rangle = 0$ *for all* $x \in \mathcal{H} \implies T^*T = I$.

Thus T is unitary.

(4) T *is hyponormal*
$$\begin{aligned} \iff T^*T - TT^* \geq 0 &\iff \forall x \in \mathcal{H}: \quad \langle (T^*T - TT^*)x, x\rangle \geq 0 \\ &\iff \forall x \in \mathcal{H}: \quad \langle T^*Tx, x\rangle \geq \langle TT^*x, x\rangle \\ &\iff \forall x \in \mathcal{H}: \quad \langle Tx, Tx\rangle \geq \langle T^*x, T^*x\rangle \\ &\iff \forall x \in \mathcal{H}: \quad \|Tx\|^2 \geq \|T^*x\|^2 \\ &\iff \forall x \in \mathcal{H}: \quad \|Tx\| \geq \|T^*x\| \end{aligned}$$

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent :*

(1) T is isometry.

(2) $\langle Tx, Ty\rangle = \langle x, y\rangle$ *for all* $x, y \in \mathcal{H}$.

(3) $\|Tx\| = \|x\|$ *for all* $x \in \mathcal{H}$.

Proof. *Proceeding the same as the proof of Theorem 2.2.*

Proposition 2.2. *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $\mathcal{S}$ *be the set of all self-adjoint operators in* $\mathcal{B}(\mathcal{H})$*. Then :*

(1) Let $\alpha, \beta \in \mathbb{R}$ *and* $T_1, T_2 \in \mathcal{S}$*, then* $\alpha T_1 + \beta T_2 \in \mathcal{S}$.

(2) If $T \in \mathcal{S}$*, and* p *is a polynomial of real coefficients, then* $p(T) \in \mathcal{S}$.

(3) If $T \in \mathcal{S}$ *and* T *is invertible, then* $T^{-1} \in \mathcal{S}$.

(4) $TT^*, T^*T \in \mathcal{S}$.

(5) $\mathcal{S}$ *is a closed subset of* $\mathcal{B}(\mathcal{H})$.

Proof. *(1),(2),(3) and (4) are easy to prove.*

(5) Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ *and* $T \in \mathcal{B}(\mathcal{H})$ *such that* $T_n \underset{n\to\infty}{\longrightarrow} T$*, then*

Using the fact that $\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \Longrightarrow T_n^* \underset{n\to\infty}{\longrightarrow} T^*$

But $\forall n \in \mathbb{N}\quad T_n^* = T_n \Longrightarrow T_n \underset{n\to\infty}{\longrightarrow} T^*$*, from the uniqueness of the limit* $T^* = T$.

Then $\mathcal{S}$ *is closed.*

Theorem 2.3. *If* $T \in \mathcal{B}(\mathcal{H})$ *is self-adjoint, then* $\|T\| = \sup\limits_{\|x\|=1} |\langle Tx, x\rangle|$.

Proof. *Let* $\|x\| = 1$*, we know that* $|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$.

Then

$$\sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\| \tag{2.5}$$

Setting $\beta_T = \sup\limits_{\|x\|=1} |\langle Tx, x\rangle|$*, then* $\forall x \in \mathcal{H}:\quad |\langle Tx, x\rangle| \leq \beta_T\|x\|^2$.

Since T *is self-adjoint, we have* $\Re\langle Tx, y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle)$ *for all* $x, y \in \mathcal{H}$*. Then:*

$$\begin{aligned}
\forall x, y \in \mathcal{H}:\quad |\Re\langle Tx, y\rangle| &\leq \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|)\\
&\leq \frac{1}{4}\beta_T(\|x+y\|^2 + \|x-y\|^2)\\
&= \frac{1}{2}\beta_T(\|x\|^2 + \|y\|^2)
\end{aligned}$$

Set $y = \frac{1}{\|Tx\|}Tx$ *for all* $\|x\| = 1$ *such that* $\|Tx\| \neq 0$*, we obtain:*

$$\begin{aligned}
\frac{1}{\|Tx\|}|\Re\langle Tx, Tx\rangle| &\leq \beta_T\\
\|Tx\| &\leq \beta_T\\
\sup_{\|x\|=1}\|Tx\| &\leq \beta_T
\end{aligned}$$

Then

$$\|T\| \leq \sup_{\|x\|=1} |\langle Tx, x\rangle| \tag{2.6}$$

Combining 2.7 and 2.8, we get that $\|T\| = \sup\limits_{\|x\|=1} |\langle Tx, x\rangle|$.

Theorem 2.4. *(Cartesian form)*

Let $T \in \mathcal{B}(\mathcal{H})$*. Then there exist self-adjoint operators* R *and* S *such that* $T = R + iS$ *where* $R = \frac{1}{2}(T + T^*)$ *and* $S = \frac{1}{2i}(T - T^*)$*.* R *is called the real part of* T *and it is denoted by* $\Re(T)$*, while* S *is called the imaginary part of* T *and it is denoted by* $\Im(T)$.

Proposition 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then :*

T is positive $\Longrightarrow$ T is self-adjoint $\Longrightarrow$ T is normal $\Longrightarrow$ T is quasinormal $\Longrightarrow$ T is hyponormal

Proof. *All these implications are clear :*

T is positive $\Longrightarrow$ T is self-adjoint $\Longrightarrow$ T is normal $\Longrightarrow$ T is quasinormal

So we just need to prove that T is quasinormal $\Longrightarrow$ T is hyponormal.

Let T be a quasinormal operator $\Longleftrightarrow$ $(T^*T)T = T(T^*T)$, then $(T^*T)T = (TT^*)T$.

Thus $\forall x \in \mathcal{H}$ $\quad (T^*T - TT^*)Tx = 0 \Longrightarrow \forall x \in R(T) \quad (T^*T - TT^*)x = 0$.

Let's prove that $\forall x \in \overline{R(T)} \quad (T^*T - TT^*)x = 0$, let $x \in \overline{R(T)} \Longleftrightarrow \exists (x_n)_{n\in\mathbb{N}} \subset R(T)$ such that $x_n \underset{n\to\infty}{\longrightarrow} x$

$\forall n \in \mathbb{N}: \quad (T^*T - TT^*)x_n = 0$, since T and T^* are bounded, we obtain :

$\lim\limits_{n\to\infty} (T^*T - TT^*)x_n = 0 \Longrightarrow (T^*T - TT^*) \lim\limits_{n\to\infty} x_n = 0 \Longrightarrow (T^*T - TT^*)x = 0.$

Then $\forall x \in \overline{R(T)}: \quad (T^*T - TT^*)x = 0$, therefore

$$(T^*T - TT^*)_{|\overline{R(T)}} = 0 \tag{2.7}$$

From theorem of orthogonal decomposition, we have $\mathcal{H} = \overline{R(T)} \oplus (R(T))^{\perp}$, also we know that $(R(T))^{\perp} = N(T^*)$.

Then $\mathcal{H} = \overline{R(T)} \oplus N(T^*)$, let $x \in N(T^*) \Longleftrightarrow T^*x = 0$. Then:

$\forall x \in N(T^*)$: $\quad \langle (T^*T - TT^*)x, x\rangle = \langle T^*Tx - TT^*x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$, then

$$(T^*T - TT^*)_{|N(T^*)} \geq 0 \tag{2.8}$$

So $\forall x \in \mathcal{H}$, there exist $x_1 \in \overline{R(T)}$ and $x_2 \in N(T^*)$ such that $x = x_1 + x_2$. Using 2.7, we get:

$$\begin{aligned}
\langle (T^*T - TT^*)x, x\rangle &= \langle (T^*T - TT^*)(x_1 + x_2), x_1 + x_2\rangle \\
&= \langle (T^*T - TT^*)x_1, x_1 + x_2\rangle + \langle (T^*T - TT^*)x_2, x_1 + x_2\rangle \\
&= \langle (T^*T - TT^*)x_2, x_1\rangle + \langle (T^*T - TT^*)x_2, x_2\rangle \\
&= \langle x_2, (T^*T - TT^*)x_1\rangle + \langle (T^*T - TT^*)x_2, x_2\rangle \\
&= \langle (T^*T - TT^*)x_2, x_2\rangle \geq 0
\end{aligned}$$

Therefore $T^*T - TT^* \geq 0$, thus T is hyponormal.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator and $n \in \mathbb{N}$, then $\|T^n\| = \|T\|^n$.*

Proof. *For all $n \in \mathbb{N}$: $\quad \|T^n\| \leq \|T\|^n$ is always held.*

So we need to prove that $\|T\|^n \leq \|T^n\|$ for all $n \in \mathbb{N}$, in case T is normal.

By induction. First, for $n = 2$, let $x \in \mathcal{H}$, then:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\|$$

On the other hand, we have:

$$\|T^2x\|^2 = \langle T^2x, T^2x\rangle = \langle T^*TTx, Tx\rangle = \langle TT^*Tx, Tx\rangle = \langle T^*Tx, T^*Tx\rangle = \|T^*Tx\|^2$$

Then $\|T^2x\| = \|T^*Tx\|$ for all $x \in \mathcal{H}$, therefore

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \leq \|T^2\|\|x\|^2$$

Thus $\|T\|^2 \leq \|T^2\|$.

Second, suppose that $\|T\|^n \leq \|T^n\|$ is true, and let's prove that $\|T\|^{n+1} \leq \|T^{n+1}\|$ is also true. Let $x \in \mathcal{H}$:

$$\|T^nx\|^2 = \langle T^nx, T^nx\rangle = \langle T^*T^nx, T^{n-1}x\rangle \leq \|T^*T^nx\|\|T^{n-1}x\|$$

On the other hand we have $\|T^*T^n x\| = \|T^{n+1}x\|$ *for all* $x \in \mathcal{H}$*, since:*

$$\|T^*T^n x\|^2 = \langle T^*T^n x, T^*T^n x\rangle = \langle TT^*T^n x, T^n x\rangle = \langle T^*T^{n+1}x, T^n x\rangle = \langle T^{n+1}x, T^{n+1}x\rangle = \|T^{n+1}x\|^2$$

Therefore, for all $x \in \mathcal{H}$ *we have:*

$$\|T^n x\|^2 \le \|T^{n+1}x\|\|T^{n-1}x\|$$

Thus

$$\|T^n\|^2 \le \|T^{n+1}\|\|T^{n-1}\|$$

So that

$$\|T^n\|^2 \le \|T^{n+1}\|\|T\|^{n-1}$$

But we have that $\|T\|^{2n} \le \|T^n\|^2$*, then*

$$\|T\|^{2n} \le \|T^{n+1}\|\|T\|^{n-1}$$

As a result

$$\|T\|^{n+1} \le \|T^{n+1}\|$$

Hence $\|T^n\| = \|T\|^n$ *for all* $n \in \mathbb{N}$*.*

Theorem 2.6. *(Fuglede's theorem) Let* $T \in \mathcal{B}(\mathcal{H})$ *be normal and* $S \in \mathcal{B}(\mathcal{H})$*. Then :*

$$TS = ST \Longrightarrow T^*S = ST^*$$

Proposition 2.4. *Let* $T \in \mathcal{H}$ *is a normal operator. If there exists* $\alpha > 0$ $\|Tx\| \ge \alpha\|x\|$ *for all* $x \in \mathcal{H}$*. Then* $N(T^*) = \{0\}$*.*

Proof. T *is normal* $\Longleftrightarrow \forall x \in \mathcal{H} \quad \|Tx\| = \|T^*x\|$*, let* $x \in N(T^*) \Longleftrightarrow T^*x = 0$
Then $0 = \|T^*x\| = \|Tx\| \ge \alpha\|x\| \Longrightarrow \|x\| = 0 \Longleftrightarrow x = 0 \Longrightarrow \quad N(T^*) = \{0\}$*.*

Corollary 2.4. *Let* $T \in \mathcal{H}$ *is a normal operator. Then the following are equivalent :*
(1) T *is invertible.*
(2) There exists $\alpha > 0$ *such that* $\|Tx\| \ge \alpha\|x\|$ *for all* $x \in \mathcal{H}$*.*

Proposition 2.5. *Let* $T, S \in \mathcal{B}(\mathcal{H})$*. Then :*
(1) T *is unitary if and only if* T *is an isometry with* $R(T) = \mathcal{H}$*.*
(2) If T *is unitary, then* $\|T\| = 1$*.*
(3) If T *and* S *are unitary, then* TS *is also unitary.*

Proof. *(1) If* T *is unitary* $\Longrightarrow$ T *is an invertible isometry, and since* T *is invertible, then* $R(T) = \mathcal{H}$*.*
Now suppose that T *is an isometry with* $R(T) = \mathcal{H}$*, since* T *is an isometry* $\Longrightarrow N(T) = \{0\}$*.*
And we have that $R(T) = \mathcal{H} \Longrightarrow T$ *is bijective* $\Longrightarrow T$ *is invertible.*
We have that $T^*T = I \Longrightarrow T^*TT^{-1} = T^{-1} \Longrightarrow T^{-1} = T^*$*.*
Then $T^*T = T^{-1}T = TT^{-1} = TT^* = I \Longrightarrow T^*T = TT^* = I$*.*
Therefore T *is unitary.*
(2) T *is unitary* $\Longleftrightarrow \forall x \in \mathcal{H} \quad \|Tx\| = \|x\| \Longrightarrow \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \|x\| = 1$*.*
Hence $\|T\| = 1$*.*
(3) Let T, S *be two unitary operators, then* $(TS)^*TS = S^*(T^*T)S = S^*S = I$*.*
The same $TS(TS)^* = T(SS^*)T^* = TT^* = I$*, then* $(TS)^*TS = TS(TS)^* = I$*.*
Hence TS *is unitary.*

Proposition 2.6. *Let $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection. Then :*
(1) P is a positive operator.
(2) $R(P)$ is a closed linear subspace of $\mathcal{H}$.

Proof. *(1) Let $x \in \mathcal{H}$ $\quad \langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, P^*x\rangle = \langle Px, Px\rangle = \|Px\|^2 \geq 0$.*
Then P is positive.
(2) Let $y \in \overline{R(P)}$, there exists $(y_n)_{n\in\mathbb{N}} \subset R(P)$ such that $\lim\limits_{n\to\infty} y_n = y$, since $(y_n)_{n\in\mathbb{N}} \subset R(P)$.
$\Longrightarrow \exists (x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $\forall n \in \mathbb{N} \quad y_n = Px_n$, we have that
$y = \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} Px_n = \lim\limits_{n\to\infty} P^2x_n = \lim\limits_{n\to\infty} PPx_n = \lim\limits_{n\to\infty} Py_n = Py \Longrightarrow y \in R(P)$.
Hence $R(P)$ is closed.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a strictly positive operator with closed range $R(T)$, then T is invertible.*

Proof. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $T > 0 \Longleftrightarrow \forall x \in \mathcal{H}\setminus\{0\} \quad \langle Tx, x\rangle > 0$.*
Assume that $\{0\} \subsetneq N(T) \Longrightarrow \exists x_0 \in \mathcal{H}\setminus\{0\}$: $x_0 \in N(T) \Longrightarrow Tx_0 = 0 \Longrightarrow \langle Tx_0, x_0\rangle = 0$.
But this contradicts $T > 0$, then $N(T) = \{0\}$.
On the other hand, since $T > 0 \Longrightarrow T$ is self-adjoint $\Longrightarrow N(T) = N(T^*) = \{0\}$.
We know that $N(T^*) = (R(T))^\perp = \{0\} \Longrightarrow \overline{R(T)} = \mathcal{H}$.
Since $R(T)$ is closed $\Longrightarrow R(T) = \mathcal{H}$.
Hence T is bijective $\Longrightarrow T$ is invertible. (using Banach's Isomorphism Theorem)

Remark 2.1. *If $T \in \mathcal{B}(\mathcal{H})$ is a strictly positive operator, this doesn't imply that T is invertible.*
As counterexample, considering $T : \ell^2(\mathbb{C}) \longrightarrow \ell^2(\mathbb{C})$ such that $T(x_n) = (x_n/n)$, it is clear that $T \in \mathcal{B}(\ell^2(\mathbb{C}))$.

$$\text{Let } x \in \ell^2(\mathbb{C}): \quad \langle Tx, x\rangle = \sum_{n=1}^{\infty} \frac{x_n}{n}\overline{x_n} = \sum_{n=1}^{\infty} \frac{|x_n|^2}{n} \Longrightarrow \langle Tx, x\rangle > 0 \ \text{ for all } x \in \ell^2(\mathbb{C})\setminus\{0\}$$

Then T is strictly positive, now let's prove that T is not invertible.
Let $(x_m)_{m\in\mathbb{N}} \subset \ell^2(\mathbb{C})$ such that $x_m = (x_{n,m})$ where $x_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$
(we can write it this way $\forall m \in \mathbb{N} \quad x_m = (0, 0, \cdots, \underset{\text{component } m}{1}, 0, 0, \cdots)$), it is obvious that $(x_m)_{m\in\mathbb{N}} \subset \ell^2(\mathbb{C})$.
And, $\forall m \in \mathbb{N} \quad \|x_m\| = 1$, we have that $\forall m \in \mathbb{N} \quad \|Tx_m\|^2 = \langle Tx_m, Tx_m\rangle = \dfrac{1}{m^2}$.
Then $\lim\limits_{m\to\infty} \|Tx_m\| = 0$, depending on Corollary 1.8, we obtain that T is not invertible.

Corollary 2.5. *If $\mathcal{H}$ is finite-dimensional and $T \in \mathcal{B}(\mathcal{H})$ is strictly positive, then T is invertible.*

Corollary 2.6. *Let $T, S \in \mathcal{B}(\mathcal{H})$, and $\alpha \geq 0$. Then :*
(1) T^*T and TT^* are positive operators.
(2) If T is positive, then αT is positive too.
(3) If T and S are positive, then $T + S$ is positive as well.

Proposition 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator and $n \in \mathbb{N}$. Then :*
(1) T^n is positive.
(2) Let p be a polynomial of positive coefficients, then $p(T)$ is positive.
(3) If T is invertible, then T^{-1} is also positive.

Proof. *(1) Since T is positive $\Longrightarrow T$ is self-adjoint, if n is even, then there exists $k \in \mathbb{N}$ such that $n = 2k$.*
Let $x \in \mathcal{H} \qquad \langle T^nx, x\rangle = \langle T^{2k}x, x\rangle = \langle T^kx, (T^k)^*x\rangle = \langle T^kx, (T^*)^kx\rangle = \langle T^kx, T^kx\rangle = \|T^kx\|^2 \geq 0$.

In this case T^n *is positive.*
If n *is odd, then there exists* $k \geq 0$ *such that* $n = 2k+1$*. Let* $x \in \mathcal{H}$
$\langle T^n x, x\rangle = \langle T^{2k+1}x, x\rangle = \langle T^{k+1}x, T^k x\rangle = \langle TT^k x, T^k x\rangle \geq 0$ *(because* T *is positive).*
Overall T^n *is positive for all* $n \in \mathbb{N}$*.*
(2) Since $p(T) = \alpha_0 I + \alpha_1 T + \cdots + \alpha_n T^n$*, where* $\alpha_i \geq 0$ *for all* $0 \leq i \leq n$*, by Corollary 2.6 and (1), we obtain that* $p(T)$ *is positive.*
(3) Assume that T *is invertible, and we have* $\forall x \in \mathcal{H} \quad \langle Tx, x\rangle \geq 0$*, then* $\forall x \in \mathcal{H} \quad \langle TT^{-1}x, T^{-1}x\rangle \geq 0$
Thus $\forall x \in \mathcal{H} \quad \langle x, T^{-1}x\rangle \geq 0 \Longrightarrow \forall x \in \mathcal{H} \quad \langle T^{-1}x, x\rangle \geq 0$*, therefore* T^{-1} *is positive.*

Theorem 2.8. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *be two positive operators. Then* ST *is positive if and only if* S *commutes* T *i.e.* $ST = TS$*.*

Proof. *Assume that* ST *is positive, then* $\forall x \in \mathcal{H} \quad \langle STx, x\rangle \geq 0$*, then:*

$$\forall x \in \mathcal{H}: \quad \langle STx, x\rangle = \langle x, (ST)^* x\rangle = \langle x, TSx\rangle = \langle TSx, x\rangle \Longrightarrow \forall x \in \mathcal{H}: \quad \langle (ST - TS)x, x\rangle = 0.$$

Thus $ST - TS = 0$*, hence* $ST = TS$*.*
Now suppose that $ST = TS$*, if* $S = 0$*, it clear that* $0 = ST \geq 0$*.*
Let $S \neq 0$*, then:*

$$\begin{cases} S_1 = \dfrac{S}{\|S\|} \\ S_{n+1} = S_n - S_n^2, \quad \forall n \in \mathbb{N} \end{cases}$$

Note that S_n *is written as a polynomial of real coefficients of* S*, then* S_n *is self adjoint for all* $n \in \mathbb{N}$*.*
Let's prove by induction that $0 \leq S_n \leq I$ *for all* $n \in \mathbb{N}$*. First, for* $n = 1$*, we have:*
$S_1 = \dfrac{S}{\|S\|}$*, since* $S \geq 0$*, then* $S_1 \geq 0$*. On the hand, let* $x \in \mathcal{H}$*:*

$$\langle Sx, x\rangle \leq \|Sx\|\|x\| \leq \|S\|\|x\|^2 = \|S\|\langle x, x\rangle \Longrightarrow \langle \frac{S}{\|S\|}x, x\rangle \leq \langle x, x\rangle$$

Thus $S_1 \leq I$*, hence* $0 \leq S_1 \leq I$*.*
Second assume that $0 \leq S_n \leq I$ *is held, and let's prove that* $0 \leq S_{n+1} \leq I$*.*
Note that $S_{n+1} = S_n^2(I - S_n) + S_n(I - S_n)^2$*, since* S_n *and* $I - S_n$ *are written as a polynomial of* S*.*
Then $S_n(I - S_n) = (I - S_n)S_n$*, let* $x \in \mathcal{H}$*:*

$$\langle S_n^2(I - S_n)x, x\rangle = \langle S_n(I - S_n)x, S_n x\rangle = \langle (I - S_n)S_n x, S_n x\rangle \geq 0 \quad (\textit{since } I - S_n \geq 0)$$

And $\quad \langle S_n(I - S_n)^2 x, x\rangle = \langle (I - S_n)S_n(I - S_n)x, x\rangle = \langle S_n(I - S_n)x, (I - S_n)x\rangle \geq 0$ *(since* $S_n \geq 0$*)*
Thus $S_{n+1} \geq 0$*.*
We have $I - S_{n+1} = I - S_n + S_n^2$*, but* $I - S_n \geq 0$ *and* $S_n^2 \geq 0$*, therefore* $S_{n+1} \leq I$*.*
Then $0 \leq S_n \leq I$ *for all* $n \in \mathbb{N}$*.*
Obverse that $S_n = S_n^2 + S_{n+1}$*, then* $S_1 = S_1^2 + S_2 = S_1^2 + S_2^2 + S_3 = \cdots = S_1^2 + S_2^2 + \cdots + S_n^2 + S_{n+1}$
Thus

$$S_1^2 + S_2^2 + \cdots + S_n^2 = S_1 - S_{n+1} \leq S_1 \quad (\textit{since } S_{n+1} \geq 0)$$

So $\quad \forall x \in \mathcal{H}$ *and* $n \in \mathbb{N}$ *:* $\sum\limits_{i=1}^{i=n} \|S_i x\|^2 = \sum\limits_{i=1}^{i=n} \langle S_i x, S_i x\rangle = \sum\limits_{i=1}^{i=n} \langle S_i^2 x, x\rangle \leq \langle S_1 x, x\rangle$
Which means that $\forall x \in \mathcal{H}$ *:* $\quad \sum\limits_{i=1}^{i=\infty} \|S_i x\|^2 \leq \langle S_1 x, x\rangle \Longleftrightarrow (\sum\limits_{i=1}^{i=n} \|S_i x\|^2)_{n \in \mathbb{N}}$ *is convergent for all* $x \in \mathcal{H}$*.*
Therefore $\quad \lim\limits_{n \to \infty} \|S_n x\| = 0$ *for all* $x \in \mathcal{H}$*.*

Since we have, $\forall x \in \mathcal{H}$ *and* $n \in \mathbb{N}$ *:* $\sum_{i=1}^{i=n} S_i^2 x = S_1 x - S_{n+1}x \Longrightarrow \|\sum_{i=1}^{i=n} S_i^2 x - S_1 x\| = \|S_{n+1}x\|$
Then

$$\forall x \in \mathcal{H} : \quad \lim_{n\to\infty} \|\sum_{i=1}^{i=n} S_i^2 x - S_1 x\| = 0 \Longleftrightarrow \forall x \in \mathcal{H} : \quad \lim_{n\to\infty} \sum_{i=1}^{i=n} S_i^2 x = S_1 x$$

Since S_n *is written as a polynomial of* S*, and* T *commutes with* S*, therefore* T *commutes with* S_n *for all* $n \in \mathbb{N}$*.*
And using the fact that $S = \|S\|S_1$*, we obtain :*

$$\begin{aligned}\forall x \in \mathcal{H} : \quad \langle STx, x\rangle &= \|S\|\langle S_1 Tx, x\rangle = \|S\|\langle TS_1 x, x\rangle \\ &= \|S\|\langle T \lim_{n\to\infty} \sum_{i=1}^{i=n} S_i^2 x, x\rangle \\ &= \|S\| \lim_{n\to\infty} \sum_{i=1}^{i=n} \langle TS_i^2 x, x\rangle \\ &= \|S\| \lim_{n\to\infty} \sum_{i=1}^{i=n} \langle S_i T S_i x, x\rangle \\ &= \|S\| \lim_{n\to\infty} \sum_{i=1}^{i=n} \langle TS_i x, S_i x\rangle \geq 0 \quad (T \geq 0)\end{aligned}$$

Thus $\forall x \in \mathcal{H}$ *:* $\langle STx, x\rangle \geq 0$*, hence* $ST \geq 0$*.*

Corollary 2.7. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $0 \leq S \leq T$ *and* $TS = ST$*, then* $0 \leq S^n \leq T^n$ *for all* $n \in \mathbb{N}$*.*

Proof. *By Preposition 2.7,* $S^n \geq 0$ *and* $T^n \geq 0$ *for all* $n \in \mathbb{N}$*.*
By induction let's prove $S^n \leq T^n$*. First, for* $n = 2$*, then:*
Since $TS = ST$*, then* $T^2 - S^2 = (T+S)(T-S)$*. We have that* $T + S \geq 0$ *and* $T - S \geq 0$ *(since* $0 \leq S \leq T$*), and it is easy to see that* $(T+S)(T-S) = (T-S)(T+S)$*.*
By Theorem 2.8, $T^2 - S^2 = (T+S)(T-S) \geq 0 \Longrightarrow S^2 \leq T^2$*.*
Second, assume that $S^n \leq T^n$ *is correct, and let's prove that* $S^{n+1} \leq T^{n+1}$ *is correct as well.*
Notice that

$$T^{n+1} - S^{n+1} = T(T^n - S^n) + (T-S)S^n$$

We have that $T \geq 0$ *and* $T^n - S^n \geq 0$*, and using the fact that* $TS = ST$*, we get:*

$$T(T^n - S^n) = T^{n+1} - TS^n = T^{n+1} - S^n T = (T^n - S^n)T$$

Which means that $T(T^n - S^n) = (T^n - S^n)T$*, thus* $T(T^n - S^n) \geq 0$ *(by Theorem 2.8)*
The same $T - S \geq 0$ *and* $S^n \geq 0$*, and* $(T-S)S^n = S^n(T-S)$*, then* $(T-S)S^n \geq 0$*.*
Therefore $T^{n+1} - S^{n+1} \geq 0$*, hence* $0 \leq S^n \leq T^n$ *for all* $n \in \mathbb{N}$*.*

Theorem 2.9. *(Generalized Schwarz inequality)*
Let $T \in \mathcal{B}(\mathcal{H})$ *be a positive operator. Then:*

$$\forall x, y \in \mathcal{H} \qquad |\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle \tag{2.9}$$

Proof. *Let* $t \in \mathbb{R}$ *and* $x, y \in \mathcal{H}$*, since* T *is positive we have :*

$$0 \leq \langle T(x+ty), x+ty\rangle = t^2\langle Ty, y\rangle + t(\langle Tx, y\rangle + \langle Ty, x\rangle) + \langle Tx, x\rangle$$

But $\langle Ty, x\rangle = \langle y, T^*x\rangle = \langle y, Tx\rangle = \overline{\langle Tx, y\rangle} \Longrightarrow \langle Tx, y\rangle + \langle Ty, x\rangle = 2\Re\langle Tx, y\rangle$*.*

$$\text{Then} \quad \forall x, y \in \mathcal{H} \quad \forall t \in \mathbb{R} \qquad t^2\langle Ty, y\rangle + 2t\Re\langle Tx, y\rangle + \langle Tx, x\rangle \geq 0.$$

$$Since \quad t^2\langle Ty,y\rangle + 2t\Re\langle Tx,y\rangle + \langle Tx,x\rangle \le t^2\langle Ty,y\rangle + 2t|\langle Tx,y\rangle| + \langle Tx,x\rangle.$$

$$Thus \quad \forall x,y\in\mathcal{H}, \quad \forall t\in\mathbb{R}: \qquad t^2\langle Ty,y\rangle + 2t|\langle Tx,y\rangle| + \langle Tx,x\rangle \ge 0.$$

$$Therefore \quad \forall x,y\in\mathcal{H}: \qquad \Delta = 4|\langle Tx,y\rangle|^2 - 4\langle Tx,x\rangle\langle Ty,y\rangle \le 0.$$

Hence $\forall x,y\in\mathcal{H} \qquad |\langle Tx,y\rangle|^2 \le \langle Tx,x\rangle\langle Ty,y\rangle.$

2.3 Square root and the absolute value of a linear operator

Definition 2.3. *Let $T\in\mathcal{B}(\mathcal{H})$. A square root of T is an operator $S\in\mathcal{B}(\mathcal{H})$ such that $T=S^2$.*

Remark 2.2. *There are operators do not have any square root, and others might have many or infinite square roots. For instance $A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ does not have any square root, and $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ has infinite square roots of the form $\begin{pmatrix}a & b\\ c & -a\end{pmatrix}$ such that $a^2+bc=1$, and I.*

Definition 2.4. *Let $T\in\mathcal{B}(\mathcal{H})$. Then the commutant of T is denoted by $\{T\}$ such that :*

$$\{T\} = \{S\in\mathcal{B}(\mathcal{H}) : TS = ST\}$$

Corollary 2.8. *(1) $\{T\}\subset\{T^n\}$ for all $n\in\mathbb{N}$.*
(2) If $\lambda\in\mathbb{C}\setminus\{0\}$, then $\{\lambda T\}=\{T\}$.
(3) $\{T\}\subset\{p(T)\}$ where $p(T)$ is a polynomial of T.

Definition 2.5. *Let $(T_n)_{n\in\mathbb{N}}\subset\mathcal{B}(\mathcal{H})$ be a sequence of self-adjoint operators, then $(T_n)_{n\in\mathbb{N}}$ is said to be bounded monotone increasing, if there exists $T\in\mathcal{B}(\mathcal{H})$ such that $T_1\le T_2\le\cdots\le T_n\le\cdots\le T$.*

Theorem 2.10. *If $(T_n)_{n\in\mathbb{N}}\subset\mathcal{B}(\mathcal{H})$ a sequence of positive operators is bounded monotone increasing, then there exists a positive operator $T\in\mathcal{B}(\mathcal{H})$ such that $T_n\xrightarrow{s}T$.*

Proof. *Since $(T_n)_{n\in\mathbb{N}}$ is bounded monotone increasing, then there exists positive $S\in\mathcal{B}(\mathcal{H})$ such that*

$$0\le T_1\le T_2\le\cdots\le T_n\le\cdots\le S$$

Let $x\in\mathcal{H}$ we have that $(T_nx)_{n\in\mathbb{N}}\subset\mathcal{H}$, since $(\mathcal{H},\|.\|)$ is a Banach space, so to prove that $(T_nx)_{n\in\mathbb{N}}$ is convergent, it is enough to prove that $(T_nx)_{n\in\mathbb{N}}$ is a Cauchy sequence.
Let $n,m\in\mathbb{N}$ such that $m\le n$, then $T_n-T_m\ge 0$, by Theorem 2.9, we obtain:

$$\|(T_n-T_m)x\|^4 = \langle (T_n-T_m)x,(T_n-T_m)x\rangle^2 \le \langle (T_n-T_m)x,x\rangle\langle (T_n-T_m)(T_n-T_m)x,(T_n-T_m)x\rangle$$

We have that $T_n-T_m\le T_n\le S$, then:

$$\langle (T_n-T_m)(T_n-T_m)x,(T_n-T_m)x\rangle \le \langle S(T_n-T_m)x,(T_n-T_m)x\rangle \le \|S(T_n-T_m)x\|\|(T_n-T_m)x\| \le \|S\|\|(T_n-T_m)x\|^2$$

Then $\quad\langle (T_n-T_m)(T_n-T_m)x,(T_n-T_m)x\rangle \le \|S\|\|T_n-T_m)x\|^2$, *therefore:*

$$\|(T_n-T_m)x\|^4 \le \|S\|\langle (T_n-T_m)x,x\rangle\|T_n-T_m)x\|^2 \Longrightarrow \|T_nx-T_mx\|^2 \le \|S\|\langle T_nx-T_mx,x\rangle$$

On the other hand, we have that $(\langle T_nx,x\rangle)_{n\in\mathbb{N}}\subset\mathbb{R}$, since $T_1\le T_2\le\cdots\le T_n$, then:
$\forall n\in\mathbb{N} \quad \langle T_1x,x\rangle\le\langle T_2x,x\rangle\le\cdots\le\langle T_nx,x\rangle$, *thus $(\langle T_nx,x\rangle)_{n\in\mathbb{N}}$ is increasing sequence for all $x\in\mathcal{H}$.*

Since $\forall n \in \mathbb{N} \quad T_n \leq S$*, then* $\forall n \in \mathbb{N} \quad \langle T_n x, x\rangle \leq \langle Sx, x\rangle$*, therefore* $(\langle T_n x, x\rangle)_{n\in\mathbb{N}}$ *is bounded from above for all* $x \in \mathcal{H}$*. Which means that* $(\langle T_n x, x\rangle)_{n\in\mathbb{N}}$ *is convergent for all* $x \in \mathcal{H}$*. Then:*

$$\lim_{n,m\to\infty} \|T_n x - T_m x\|^2 \leq \|S\|(\lim_{n\to\infty}\langle T_n x, x\rangle - \lim_{m\to\infty}\langle T_m x, x\rangle) = 0$$

Then $\lim_{n,m\to\infty} \|T_n x - T_m x\| = 0 \iff (T_n x)_{n\in\mathbb{N}}$ *is a Cauchy sequence* $\iff (T_n x)_{n\in\mathbb{N}}$ *is convergent for all* $x \in \mathcal{H}$*.*
We set for all $x \in \mathcal{H} \quad Tx = \lim_{n\to\infty} T_n x$*, now let's prove that* $T \in \mathcal{B}(\mathcal{H})$ *:*
Let $x, y \in \mathcal{H}$ *and* $\lambda \in \mathbb{C}$*:*

$$T(\lambda x + y) = \lim_{n\to\infty} T_n(\lambda x + y) = \lim_{n\to\infty} \lambda T_n x + T_n y = \lambda \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = \lambda Tx + Ty$$

Hence T *is linear. Let's prove that* T *is bounded:*
Let $x, y \in \mathcal{H}$ *and* $\forall n \in \mathbb{N}$ *:* $\quad |\langle T_n x, y\rangle|^2 \leq \langle T_n x, x\rangle\langle T_n y, y\rangle$ *(by Theorem 2.9).*
Since $\forall n \in \mathbb{N} \quad T_n \leq S \implies \langle T_n x, x\rangle \leq \langle Sx, x\rangle \leq \|S\|\|x\|^2$ *the same* $\langle T_n y, y\rangle \leq \|S\|\|y\|^2$*.*
Then $|\langle T_n x, y\rangle|^2 \leq \|S\|^2\|x\|^2\|y\|^2$*, setting* $y = T_n x$*, we get :*
$\forall x \in \mathcal{H}$ *and* $\forall n \in \mathbb{N} \quad \|T_n x\|^4 = \langle T_n x, T_n x\rangle^2 \leq \|S\|^2\|x\|^2\|T_n x\|^2$
Thus $\forall x \in \mathcal{H}$ *and* $\forall n \in \mathbb{N} \quad \|T_n x\| \leq \|S\|\|x\| \implies \forall x \in \mathcal{H} \quad \|Tx\| \leq \|S\|\|x\|$
Then T *is bounded, hence* $T \in \mathcal{B}(\mathcal{H})$*.*
We just still need to prove that T *is positive, let* $x \in \mathcal{H}$ *:*

$$\forall n \in \mathbb{N}: \quad T_n \geq 0 \implies \forall n \in \mathbb{N} \quad \langle T_n x, x\rangle \geq 0 \implies \lim_{n\to\infty}\langle T_n x, x\rangle \geq 0 \implies \langle \lim_{n\to\infty} T_n x, x\rangle \geq 0 \implies \langle Tx, x\rangle \geq 0$$

Therefore T *is positive.*

Corollary 2.9. *If* $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ *is bounded monotone increasing, then there exists* $T \in \mathcal{B}(\mathcal{H})$ *such that* $T_n \xrightarrow{s} T$*.*

Proof. *Since* $(T_n)_{n\in\mathbb{N}}$ *is bounded monotone increasing, then* $T_1 \leq T_2 \leq \cdots \leq T_n \leq \cdots \leq S$*.*
Setting $S_n = T_n - T_1$ *for all* $n \in \mathbb{N}$*, note that* $S_n \geq 0$ *and* $S - T_1 \geq 0$*.*
Then we have $0 \leq S_1 \leq S_2 \leq \cdots \leq S_n \leq \cdots \leq S - I$*, by applying Theorem 2.10 on* $(S_n)_{n\in\mathbb{N}}$*, there exists* $S^{'} \in \mathcal{B}(\mathcal{H})$ *positive such that* $S_n \xrightarrow{s} S^{'}$*, and since* $T_n = S_n + T_1$ *for all* $n \in \mathbb{N} \implies T_n \xrightarrow{s} S^{'} + T_1$*.*
Thus $\quad \exists T \in \mathcal{B}(\mathcal{H})$*:* $T = S^{'} + T_1$ *such that* $T_n \xrightarrow{s} T$*.*

Theorem 2.11. *For any positive operator* $T \in \mathcal{B}(\mathcal{H})$*, there exists a unique positive square* $\sqrt{T} \in \mathcal{B}(\mathcal{H})$ *such that* $\{T\} \subset \{\sqrt{T}\}$*. In addition,* $\sqrt{T}$ *is a limit of a polynomial of* T*.*

Proof. *First of all, if* $T = 0 \implies S = 0$ *verifies* $S^2 = T$*,* $S = 0$ *is the unique positive square root of* $T = 0$*, because if* $\exists S^{'} \in \mathcal{B}(\mathcal{H})$ *a square root of* T *such that* $S^{'} \neq 0 \implies \|S^{'}x\|^2 = \langle S^{'}x, S^{'}x\rangle = \langle (S^{'})^2 x, x\rangle = 0$*,* $\forall x \in \mathcal{H}$*.*
Then $S^{'} = 0 \implies \sqrt{T} = 0$ *is the unique positive square root and* $\{T\} = \{0\} = \{\sqrt{T}\}$*.*
Assume that $T \neq 0$*, we know that* $T \leq \|T\|I$*, then* $\frac{1}{\|T\|}T \leq I$*, set* $T^{'} = \frac{1}{\|T\|}T$*, thus* $T^{'} \leq I$*.*
Let S_n *be defined as follows :*

$$S_0 = 0 \quad \text{and } \forall n \geq 0 \quad S_{n+1} = S_n + \frac{1}{2}(T^{'} - S_n^2)$$

Note that S_n *is written as a polynomial of* $T^{'}$*, and its coefficients are real, for example* $S_1 = \frac{1}{2}T^{'}$ *and* $S_2 = T^{'} - \frac{1}{8}(T^{'})^2$ *and so on, and it can be easily proved by induction. By Preposition 2.2* S_n *is self-adjoint for all* $n \in \mathbb{N}$*, moreover* $\{T^{'}\} \subset \{S_n\}$ *for all* $n \in \mathbb{N}$ *by Corollary 2.8.*
We have that :

$$S_0 \leq S_1 \leq \cdots \leq S_n \leq \cdots \leq I \tag{2.10}$$

Since $\forall n \geq 0$: $\quad I - S_{n+1} = I - S_n - \frac{1}{2}T' + \frac{1}{2}S_n^2 = \frac{1}{2}(I - T') + \frac{1}{2}(S_n^2 + I - 2S_n)$

Then $\forall n \geq 0$: $\quad I - S_{n+1} = \frac{1}{2}(I - T') + \frac{1}{2}(S_n - I)^2$

Since $T' \leq I$, *and* $(S_n - I)^2 \geq 0$ *for all* $n \geq 0$, *therefore* $\forall n \geq 0$: $\quad I - S_{n+1} \geq 0$

Since $S_0 = 0$, *we obtain that* $\forall n \geq 0$: $\quad S_n \leq I$

Now we just need to prove that, $\forall n \geq 0$: $\quad S_n \leq S_{n+1}$

By induction. First, for $n = 0$, *we have* $S_0 = 0$ *and* $S_1 = \frac{1}{2}T' \implies S_0 \leq S_1$

Second, assume that $S_n \leq S_{n+1}$ *is true, and let's prove that* $S_{n+1} \leq S_{n+2}$ *is held.*

$$\begin{aligned}S_{n+2} - S_{n+1} =& S_{n+1} + \frac{1}{2}T' - \frac{1}{2}S_{n+1}^2 - S_n - \frac{1}{2}T' + \frac{1}{2}S_n^2\\ =& \frac{1}{2}S_n^2 - S_n + S_{n+1} - \frac{1}{2}S_{n+1}^2\\ =& \frac{1}{2}(I + S_n^2 - 2S_n) - \frac{1}{2}(I + S_{n+1}^2 - 2S_{n+1})\\ =& \frac{1}{2}(I - S_n)^2 - \frac{1}{2}(I - S_{n+1})^2\end{aligned}$$

We have $I - S_n - (I - S_{n+1}) = S_{n+1} - S_n \geq 0$, *then* $I - S_{n+1} \leq I - S_n$.

Therefore $(I - S_{n+1})^2 \leq (I - S_n)^2$ *(by Corollary 2.8), so that* $\frac{1}{2}(I - S_n)^2 - \frac{1}{2}(I - S_{n+1})^2 \geq 0$.

Hence $S_{n+2} - S_{n+1} \geq 0$, *which means that* $S_{n+2} \geq S_{n+1}$.

Then 2.10 is held. By applying Theorem 2.10 on $(S_n)_{n\in\mathbb{N}}$, $\exists S \in \mathcal{B}(\mathcal{H})$ *a positive operator such that* $S_n \xrightarrow{s} S$.

Since $S_{n+1} = S_n + \frac{1}{2}(T' - S_n^2)$, *then* $S = S + \frac{1}{2}(T' - S^2)$, *thus* $S^2 = T'$.

Now let's prove that $\{T'\} \subset \{S\}$, *let* $R \in \{T'\}$ *and* $x \in \mathcal{H}$. *Using the fact that* R *is continuous and* $\{T'\} \subset \{S_n\}$, *we obtain :*

$$RSx = R(\lim_{n\to\infty} S_n x) = \lim_{n\to\infty} RS_n x = \lim_{n\to\infty} S_n R x = SRx, \text{ hence } RS = SR$$

Then $\{T'\} \subset \{S\}$.

Now let's prove the uniqueness of S *that satisfies* $S^2 = T'$, *assume that there exists a positive operator* $S' \in \mathcal{B}(\mathcal{H})$ *such that* $(S')^2 = T'$. *Then:*

$$T'S' = S'S'S' = S'T' \implies S' \in \{T'\} \implies S' \in \{S\} \quad (since \{T'\} \subset \{S\})$$

Then $(S - S')(S + S') = S^2 - (S')^2 - SS' + S'S = 0$ *(since* $S^2 = (S')^2 = T'$ *and* $S'S = SS'$*)*

Thus $(S - S')x = 0$ *for all* $x \in R(S + S')$. *Let's prove that* $(S - S')x = 0$ *for all* $x \in \overline{R(S + S')}$.

Let $x \in \overline{R(S + S')}$, *then there exists* $(x_n)_{n\in\mathbb{N}} \subset R(S + S')$ *such that* $\lim_{n\to\infty} x_n = x$, *then:*

$$\forall n \in \mathbb{N} : \quad (S - S')x_n = 0 \Longrightarrow \lim_{n\to\infty}(S - S')x_n = 0 \Longrightarrow (S - S')\lim_{n\to\infty} x_n = 0 \Longrightarrow (S - S')x = 0.$$

Then $(S - S')x = 0$ *for all* $x \in \overline{R(S + S')}$. *By orthogonal decomposition we have* $\mathcal{H} = \overline{R(S + S')} \oplus R(S + S')^{\perp}$.

And $R(S + S')^{\perp} = N((S + S')^*) = N(S + S')$ *(since* $S + S' \geq 0$*), then* $\mathcal{H} = \overline{R(S + S')} \oplus N(S + S')$.

Let $x \in N(S + S') \iff Sx + S'x = 0$, *and since* $\langle Sx, x\rangle, \langle S'x, x\rangle \geq 0$, *and* $\langle Sx + S'x, x\rangle = \langle Sx, x\rangle + \langle S'x, x\rangle = 0$.

Then $\langle Sx, x\rangle = \langle S'x, x\rangle = 0$, *and since* $S \geq 0$, *then* $\exists R \in \mathcal{B}(\mathcal{H})$ *a positive operator such that* $R^2 = S$. *Then:*

$$\langle Sx, x\rangle = \langle R^2 x, x\rangle = \langle Rx, Rx\rangle = \|Rx\|^2 = 0 \Longrightarrow Rx = 0 \Longrightarrow Sx = RRx = 0 \Longrightarrow Sx = 0.$$

Since $Sx + S'x = 0 \implies S'x = 0$ *as well, then* $Sx = S'x = 0$ *for all* $x \in N(S + S')$.

Therefore $(S - S')x = Sx - S'x = 0$ *for all* $x \in N(S + S')$.

Thus $(S - S')x = 0$ *for all* $x \in \mathcal{H}$, *then* $S = S'$. *Hence* S *is unique.*

We get that there is a unique positive operator $S^2 = T'$ *and* $\{T'\} \subset \{S\}$, *and since* $T' = \frac{1}{\|T\|}T$.

Then $T = \|T\|S^2 = (\sqrt{\|T\|}S)^2$ and $\{T\} = \{T^{'}\} \subset \{S\} = \{\sqrt{\|T\|}S\}$. (check Corollary 2.8)
The uniqueness of S gives the uniqueness of $\sqrt{\|T\|}S$. then $\sqrt{T} = \sqrt{\|T\|}S$ is the unique positive square root of T, and $\{T\} \subset \{\sqrt{T}\}$.
Then for all $T \geq 0$, there exists a unique positive square root $\sqrt{T}$, and $\{T\} \subset \{\sqrt{T}\}$, and $\sqrt{T}$ is a limit of a polynomial of T as we have seen in the proof.

Remark 2.3. *(1) Theorem 2.11 does not deny the existence of non-positive square roots for positive operators, as it is shown in Remark 2.2, $I \geq 0$, but it has infinite square roots, however the only positive one is I.*
(2) When we say a square root of a positive operator we mean the unique positive square root as defined in Theorem 2.11.

Proposition 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then :*
(1) $\|\sqrt{T}\| = \sqrt{\|T\|}$.
(2) $N(\sqrt{T}) = N(T)$.
(3) $\overline{R(\sqrt{T})} = \overline{R(T)}$.

Proof. *(1) Let $x \in \mathcal{H}$:* $\|\sqrt{T}x\|^2 = \langle \sqrt{T}x, \sqrt{T}x\rangle = \langle Tx, x\rangle \implies \sup\limits_{\|x\|=1} \|\sqrt{T}x\|^2 = \sup\limits_{\|x\|=1} \langle Tx, x\rangle$
Since $T \geq 0$, then T is self-adjoint, by Theorem 2.3 $\|T\| = \sup\limits_{\|x\|=1} \langle Tx, x\rangle$. *($\langle Tx, x\rangle \geq 0$)*
Then $\|\sqrt{T}\|^2 = \|T\|$, hence $\|\sqrt{T}\| = \sqrt{\|T\|}$.
(2) $N(T) \subset N(\sqrt{T})$, let $x \in N(T) \iff Tx = 0$, and we have $\|\sqrt{T}x\|^2 = \langle Tx, x\rangle = 0$.
Then $\|\sqrt{T}x\| = 0 \iff \sqrt{T}x = 0 \iff x \in N(\sqrt{T})$, therefore $N(T) \subset N(\sqrt{T})$.
$N(\sqrt{T}) \subset N(T)$, let $x \in N(\sqrt{T}) \iff \sqrt{T}x = 0 \implies \sqrt{T}\sqrt{T}x = 0 \implies Tx = 0$.
Then $x \in N(T)$, thus $N(\sqrt{T}) \subset N(T)$. Hence $N(\sqrt{T}) = N(T)$.
(3) Since $N(\sqrt{T}) = N(T)$, and we have $N(\sqrt{T}) = R(\sqrt{T})^{\perp}$ and $N(T) = R(T)^{\perp}$ (T and $\sqrt{T}$ are self-adjoint)
Then $R(\sqrt{T})^{\perp} = R(T)^{\perp} \implies (R(\sqrt{T})^{\perp})^{\perp} = (R(T)^{\perp})^{\perp} \implies \overline{R(\sqrt{T})} = \overline{R(T)}$ by Corollary 1.3.

Corollary 2.10. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two positive operators, then :*

$$TS = ST \iff \sqrt{T}\sqrt{S} = \sqrt{S}\sqrt{T}$$

Proof. • *Assume that $\sqrt{T}\sqrt{S} = \sqrt{S}\sqrt{T}$, then:*

$$TS = \sqrt{T}\sqrt{T}\sqrt{S}\sqrt{S} = \sqrt{T}\sqrt{S}\sqrt{T}\sqrt{S} = \sqrt{S}\sqrt{T}\sqrt{S}\sqrt{T} = \sqrt{S}\sqrt{S}\sqrt{T}\sqrt{T} = ST$$

Thus $TS = ST$.
• *Suppose that $TS = ST$. Since $TS = ST$ and $\{T\} \subset \{\sqrt{T}\}$, then $\sqrt{T}S = S\sqrt{T}$. And since $\sqrt{T}S = S\sqrt{T}$ and $\{S\} \subset \{\sqrt{S}\}$, hence $\sqrt{T}\sqrt{S} = \sqrt{S}\sqrt{T}$.*

Proposition 2.9. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two positive operators such that $TS = ST$. Then :*
(1) $\sqrt{TS} = \sqrt{T}\sqrt{S}$.
(2) If $S \leq T \implies \sqrt{S} \leq \sqrt{T}$.
(3) $\sqrt{T+S} \leq \sqrt{T} + \sqrt{S}$.

Proof. *(1) Since $T \geq 0$ and $S \geq 0$ and $TS = ST \implies TS \geq 0$ (by Theorem 2.8), then TS has a unique positive square root (by Theorem 2.11). Using Corollary 2.10, we obtain :*

$$(\sqrt{T}\sqrt{S})^2 = \sqrt{T}\sqrt{S}\sqrt{T}\sqrt{S} = \sqrt{T}\sqrt{T}\sqrt{S}\sqrt{S} = TS \implies (\sqrt{T}\sqrt{S})^2 = TS$$

Since $\sqrt{T}, \sqrt{S} \geq 0$ and $\sqrt{T}\sqrt{S} = \sqrt{S}\sqrt{T}$, then $\sqrt{T}\sqrt{S} \geq 0$, therefore $\sqrt{TS} = \sqrt{T}\sqrt{S}$ (by the unique of the positive square root of TS).

(2) If $S \leq T$, then $T - S \geq 0$,let $x \in \mathcal{H}$:

$$\langle(\sqrt{T} - \sqrt{S})(\sqrt{T} + \sqrt{S})x, (\sqrt{T} + \sqrt{S})x\rangle = \langle(T - S)x, (\sqrt{T} + \sqrt{S})x\rangle = \langle(\sqrt{T} + \sqrt{S})(T - S)x, x\rangle$$

Since $\sqrt{T}, \sqrt{S} \geq 0$, then $\sqrt{T} + \sqrt{S} \geq 0$, and we have that $T - S \geq 0$, and since

$$(\sqrt{T} + \sqrt{S})(T - S) = (T - S)(\sqrt{T} + \sqrt{S})$$

Then by Theorem 2.8, we get $(\sqrt{T} + \sqrt{S})(T - S) \geq 0$, therefore

$$\langle(\sqrt{T} - \sqrt{S})(\sqrt{T} + \sqrt{S})x, (\sqrt{T} + \sqrt{S})x\rangle \geq 0 \text{ for all } x \in \mathcal{H}$$

Thus $\langle(\sqrt{T} - \sqrt{S})y, y\rangle \geq 0$ for all $y \in R(\sqrt{T} + \sqrt{S})$.
Let $y \in \overline{R(\sqrt{T} + \sqrt{S})}$, $\exists (y_n)_{n\in\mathbb{N}} \subset R(\sqrt{T} + \sqrt{S})$ such that $\lim_{n\to\infty} y_n = y$, then:

$$\forall n \in \mathbb{N} : \langle(\sqrt{T} - \sqrt{S})y_n, y_n\rangle \geq 0 \Longrightarrow \lim_{n\to\infty}\langle(\sqrt{T} - \sqrt{S})y_n, y_n\rangle \geq 0 \Longrightarrow \langle \lim_{n\to\infty}(\sqrt{T} - \sqrt{S})y_n, \lim_{n\to\infty} y_n\rangle \geq 0$$

Therefore $\langle(\sqrt{T} - \sqrt{S})y, y\rangle \geq 0$ for all $y \in \overline{R(\sqrt{T} + \sqrt{S})}$, thus

$$\sqrt{T} - \sqrt{S}_{|\overline{R(\sqrt{T}+\sqrt{S})}} \geq 0 \tag{2.11}$$

Using orthogonal decomposition theorem, we get $\mathcal{H} = \overline{R(\sqrt{T} + \sqrt{S})} \oplus (R(\sqrt{T} + \sqrt{S}))^{\perp}$.
But $(R(\sqrt{T} + \sqrt{S}))^{\perp} = N((\sqrt{T} + \sqrt{S})^*) = N(\sqrt{T} + \sqrt{S})$, then $\mathcal{H} = \overline{R(\sqrt{T} + \sqrt{S})} \oplus N(\sqrt{T} + \sqrt{S})$.
Let $x \in N(\sqrt{T} + \sqrt{S}) \iff \sqrt{T}x + \sqrt{S}x = 0 \iff \sqrt{S}x = -\sqrt{T}x$, then:

$$\langle(\sqrt{T} - \sqrt{S})x, x\rangle = \langle 2\sqrt{T}x, x\rangle = 2\langle\sqrt{T}x, x\rangle \geq 0$$

And $$\langle(\sqrt{T} - \sqrt{S})x, x\rangle = \langle -2\sqrt{S}x, x\rangle = -2\langle\sqrt{S}x, x\rangle \leq 0$$

Thus $\langle(\sqrt{T} - \sqrt{S})x, x\rangle = 0$ for all $x \in N(\sqrt{T} + \sqrt{S})$.
Let $x \in \mathcal{H}$, there exist $x_1 \in \overline{R(\sqrt{T} + \sqrt{S})}$ and $x_2 \in N(\sqrt{T} + \sqrt{S})$ such that $x = x_1 + x_2$, then:

$$\begin{aligned}\langle(\sqrt{T} - \sqrt{S})x, x\rangle =& \langle(\sqrt{T} - \sqrt{S})(x_1 + x_2), x_1 + x_2\rangle \\ =& \langle(\sqrt{T} - \sqrt{S})x_1, x_1\rangle + \langle(\sqrt{T} - \sqrt{S})x_2, x_2\rangle + \langle(\sqrt{T} - \sqrt{S})x_1, x_2\rangle + \langle(\sqrt{T} - \sqrt{S})x_2, x_1\rangle \\ =& \langle(\sqrt{T} - \sqrt{S})x_1, x_1\rangle + \langle(\sqrt{T} - \sqrt{S})x_1, x_2\rangle + \langle(\sqrt{T} - \sqrt{S})x_2, x_1\rangle\end{aligned}$$

If $x_1 \in R(\sqrt{T} + \sqrt{S})$, there exists $y \in \mathcal{H}$ such that $x_1 = (\sqrt{T} + \sqrt{S})y$, recall that $x_2 \in N(\sqrt{T} + \sqrt{S})$, and using the fact that $(\sqrt{T} + \sqrt{S})(\sqrt{T} - \sqrt{S}) = (\sqrt{T} - \sqrt{S})(\sqrt{T} + \sqrt{S})$, we obtain :

$$\begin{aligned}\langle(\sqrt{T} - \sqrt{S})x_1, x_2\rangle =& \langle(\sqrt{T} - \sqrt{S})(\sqrt{T} + \sqrt{S})y, x_2\rangle \\ =& \langle(\sqrt{T} + \sqrt{S})(\sqrt{T} - \sqrt{S})y, x_2\rangle \\ =& \langle(\sqrt{T} - \sqrt{S})y, (\sqrt{T} + \sqrt{S})x_2\rangle = 0\end{aligned}$$

And since $\langle(\sqrt{T} - \sqrt{S})x_2, x_1\rangle = \overline{\langle(\sqrt{T} - \sqrt{S})x_1, x_2\rangle}$, then:

$$\langle(\sqrt{T} - \sqrt{S})x_1, x_2\rangle = \langle(\sqrt{T} - \sqrt{S})x_2, x_1\rangle = 0 \qquad \forall x_1 \in R(\sqrt{T} + \sqrt{S})$$

And we can easily extend it to $\overline{R(\sqrt{T} + \sqrt{S})}$, and we get:

$$\langle(\sqrt{T} - \sqrt{S})x_1, x_2\rangle = \langle(\sqrt{T} - \sqrt{S})x_2, x_1\rangle = 0 \qquad \forall x_1 \in \overline{R(\sqrt{T} + \sqrt{S})}$$

Then $$\langle(\sqrt{T} - \sqrt{S})x, x\rangle = \langle(\sqrt{T} - \sqrt{S})x_1, x_1\rangle \geq 0 \text{ (using 2.11)}$$

Thus $\sqrt{T} - \sqrt{S} \geq 0$, hence $\sqrt{S} \leq \sqrt{T}$.
(3) We have that $\sqrt{T}\sqrt{S} = \sqrt{S}\sqrt{T}$, then:

$$T+S-(\sqrt{T}+\sqrt{S})^2 = T+S-(T+S+2\sqrt{T}\sqrt{S}) = -2\sqrt{T}\sqrt{S}$$

Since $\sqrt{T}\sqrt{S} \geq 0$, *then* $-2\sqrt{T}\sqrt{S} \leq 0$, *which means that* $\quad T+S-(\sqrt{T}+\sqrt{S})^2 \leq 0$.
Thus $T+S \leq (\sqrt{T}+\sqrt{S})^2$, *since* $(T+S)(\sqrt{T}+\sqrt{S})^2 = (\sqrt{T}+\sqrt{S})^2(T+S)$, *and using (2) we get:*

$$\sqrt{T+S} \leq \sqrt{(\sqrt{T}+\sqrt{S})^2}$$

Since $\sqrt{T}+\sqrt{S} \geq 0$, *then* $\sqrt{T+S} \leq \sqrt{T}+\sqrt{S}$.

Theorem 2.12. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *be two positive operators and* $X \in \mathcal{B}(\mathcal{H})$ *such that* $TX = XS$ *and* $\|T\| = \|S\|$. *Then :*

$$\sqrt{T}X = X\sqrt{S}$$

Proof. *From Theorem 2.11, we have* $\sqrt{T} = \lim\limits_{n\to\infty} S_n(T)$ *such that:*

$$S_n(T) = \alpha_1(\|T\|)T + \alpha_2(\|T\|)T^2 + \cdots + \alpha_n(\|T\|)T^n$$

We mean by $\alpha_i(\|T\|)$ *that* α_i *is dependent of* $\|T\|$, *and the same* $\sqrt{S} = \lim\limits_{n\to\infty} S_n(S)$.
Then using $TX = XS$, $\|T\| = \|S\|$ *and continuity of* X, *we get:*

$$\begin{aligned}
\sqrt{T}X = \lim_{n\to\infty} S_n(T)X &= \lim_{n\to\infty}(\alpha_1(\|T\|)TX + \alpha_2(\|T\|)T^2X + \cdots + \alpha_n(\|T\|)T^nX) \\
&= \lim_{n\to\infty}(\alpha_1(\|T\|)XS + \alpha_2(\|T\|)XS^2 + \cdots + \alpha_n(\|T\|)XS^n) \\
&= \lim_{n\to\infty} X(\alpha_1(\|S\||)S + \alpha_2(\|S\|)S^2 + \cdots + \alpha_n(\|S\|)S^n) \\
&= X\lim_{n\to\infty}(\alpha_1(\|S\||)S + \alpha_2(\|S\|)S^2 + \cdots + \alpha_n(\|S\|)S^n) \\
&= X\sqrt{S}
\end{aligned}$$

Hence $\sqrt{T}X = X\sqrt{S}$.

Corollary 2.11. *(1) Let* $T, S \in \mathcal{B}(\mathcal{H})$ *be two positive operators and* $X \in \mathcal{B}(\mathcal{H})$ *such that* $TX = SX$ *and* $\|T\| = \|S\|$. *Then :*
$$\sqrt{T}X = \sqrt{S}X$$
(2) Let $T, S \in \mathcal{B}(\mathcal{H})$ *be two positive operators and* $X \in \mathcal{B}(\mathcal{H})$ *such that* $XT = XS$ *and* $\|T\| = \|S\|$. *Then :*
$$X\sqrt{T} = X\sqrt{S}$$

Corollary 2.12. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a positive operator, then the following are equivalent :*

(1) $T \leq I$ *(2)* $T^2 \leq T$ *(3)* $\|T\| \leq 1$

Proof. *(1)*$\Longrightarrow$*(2) : Since* $T \leq I$, *then* $T, I-T \geq 0$ *and* $T(I-T) = (I-T)T$, *therefore* $T(I-T) \geq 0$.
Then $T - T^2 \geq 0$, *hence* $T^2 \leq T$.
(2)$\Longrightarrow$*(3) : Since* $T^2 \leq T$, *then* $\forall x \in \mathcal{H}$: $\langle T^2x, x\rangle \leq \langle Tx, x\rangle \Longrightarrow \forall x \in \mathcal{H}$: $\langle Tx, Tx\rangle \leq \langle Tx, x\rangle$.
Thus $\forall x \in \mathcal{H}$: $\|Tx\|^2 \leq \langle Tx, x\rangle \leq \|Tx\|\|x\|$, *then* $\forall x \in \mathcal{H}$: $\|Tx\| \leq \|x\|$, *hence* $\|T\| \leq 1$.
(3)$\Longrightarrow$*(1) : Since* $\|T\| \leq 1$, *then* $\sup\limits_{\|x\|=1} \langle Tx, x\rangle \leq 1 \Longrightarrow \forall \|x\| = 1 \quad \langle Tx, x\rangle \leq 1$. *Therefore:*

$$\forall x \in \mathcal{H}\setminus\{0\}: \quad \langle T\frac{x}{\|x\|}, \frac{x}{\|x\|}\rangle \leq 1 \Longrightarrow \forall x \in \mathcal{H}\setminus\{0\}: \quad \frac{1}{\|x\|^2}\langle Tx, x\rangle \leq 1 \Longrightarrow \forall x \in \mathcal{H}\setminus\{0\}: \quad \langle Tx, x\rangle \leq \|x\|^2$$

Thus $\forall x \in \mathcal{H}$: $\langle Tx, x\rangle \leq \langle x, x\rangle$, *hence* $T \leq I$.

Remark 2.4. *We can prove Generalized Schwarz inequality anther way. Let* $T \geq 0$ *and* $x, y \in \mathcal{H}$, *then:*

$$|\langle Tx, y\rangle|^2 = |\langle \sqrt{T}^2x, y\rangle|^2 = |\langle \sqrt{T}x, \sqrt{T}y\rangle|^2 \leq \|\sqrt{T}x\|^2\|\sqrt{T}y\|^2$$

Since $\|\sqrt{T}x\|^2 = \langle\sqrt{T}x, \sqrt{T}x\rangle = \langle Tx, x\rangle$, *the same* $\|\sqrt{T}y\|^2 = \langle Ty, y\rangle$, *thus:*

$$\forall x, y \in \mathcal{H}: \qquad |\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle$$

Theorem 2.13. *[22] Let* $T, S \in \mathcal{B}(\mathcal{H})$ *be two positive operators. Then :*

$$\|T + S\| \leq \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|\sqrt{T}\sqrt{S}\|^2}\right) \tag{2.12}$$

Remark 2.5. (2.12) *is an improvement of the triangle inequality in the positive case.*
Since $\|\sqrt{T}\sqrt{S}\|^2 \leq \|\sqrt{T}\|^2\|\sqrt{S}\|^2 = \|T\|\|S\|$, *then :*

$$\begin{aligned}\frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|\sqrt{T}\sqrt{S}\|^2}\right) &\leq \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{\|T\|^2 - \|S\|^2 - 2\|T\|\|S\| + 4\|T\|\|S\|}\right) \\ &= \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| + \|S\|)^2}\right) = \|T\| + \|S\|\end{aligned}$$

Thus $\quad \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|\sqrt{T}\sqrt{S}\|^2}\right) \leq \|T\| + \|S\|$

Corollary 2.13. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *be two positive operators. Then :*

$$\|T + S\| \leq \max\{\|T\|, \|S\|\} + \|\sqrt{T}\sqrt{S}\| \tag{2.13}$$

Proof. *By* (2.12), *we have that :*

$$\begin{aligned}\|T + S\| &\leq \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|\sqrt{T}\sqrt{S}\|^2}\right) \\ &\leq \frac{1}{2}\left(\|T\| + \|S\| + |\|T\| - \|S\|| + 2\|\sqrt{T}\sqrt{S}\|\right)\end{aligned}$$

Since $\max\{\|T\|, \|S\|\} = \frac{1}{2}(\|T\| + \|S\| + |\|T\| - \|S\||)$, *then :*

$$\|T + S\| \leq \frac{1}{2}\left(2\max\{\|T\|, \|S\|\} + 2\|\sqrt{T}\sqrt{S}\|\right)$$

Therefore $\qquad \|T + S\| \leq \max\{\|T\|, \|S\|\} + \|\sqrt{T}\sqrt{S}\|.$

Definition 2.6. *Let* $T \in \mathcal{B}(\mathcal{H})$, *the absolute value of* T *is the unique positive square root of the positive operator* T^*T, *and we denote it by* $|T|$, *that is* $|T| = \sqrt{T^*T}$.

Proposition 2.10. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then :*

(1) $\||T|\| = \|T\|$ *(2)* $|T| = T$ *if and only if* $T \geq 0$ *(3)* $|T| = |T^*|$ *if and only if* T *is normal*

Proof. *(1)* $\||T|\| = \|\sqrt{T^*T}\| = \sqrt{\|T^*T\|} = \sqrt{\|T\|^2} = \|T\|$. *(we used* $\|\sqrt{T}\| = \sqrt{\|T\|}$ *and* $\|T^*T\| = \|T\|^2$*).*
(2) First assume that $|T| = T$, *since* $\sqrt{T^*T} = T$ *and* $\sqrt{T^*T} \geq 0$, *then* $T \geq 0$.
Now suppose that $T \geq 0$, *then:*

$$T^* = T \implies T^*T = T^2 \implies \sqrt{T^*T} = T \iff |T| = T.$$

(3) Assume that $|T| = |T^*|$, *then:*

$$\sqrt{T^*T} = \sqrt{TT^*} \implies \sqrt{T^*T}^2 = \sqrt{TT^*}^2 \implies T^*T = TT^* \iff T \text{ is normal.}$$

Suppose that T *is normal, then* $T^*T = TT^* \implies \sqrt{T^*T} = \sqrt{TT^*} \iff |T| = |T^*|$.

Remark 2.6. *If* T *is a self-adjoint operator, this doesn't imply that* $|T| = T$.

Considering the following self-adjoint matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T^*T = T^2 = I \implies |T| = \sqrt{T^*T} = I$

So $|T|$ *equals* I *not* T.

Proposition 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\||T||T^*|\| = \|T^2\|$.*

Proof. *Since $T|T|^2 = TT^*T = |T^*|^2T$, then $T|T|^2 = |T^*|^2T$, also $\||T|^2\| = \||T^*|^2\| = \|T\|^2$, then By Theorem 2.12, we obtain $|T^*|T = T|T|$.*

- *Let $x \in \mathcal{H}$ then :*

$$\begin{aligned}\||T||T^*|Tx\|^2 = \langle |T||T^*|Tx, |T||T^*|Tx\rangle &= \langle |T|^2|T^*|Tx, |T^*|Tx\rangle \\ &= \langle T^*T|T^*|Tx, |T^*|Tx\rangle \\ &= \langle T|T^*|Tx, T|T^*|Tx\rangle \\ &= \langle TT|T|x, TT|T|x\rangle \\ &= \|T^2|T|x\|^2 \\ &\leq \|T^2\|^2\||T|x\|^2\end{aligned}$$

Since $\||T|x\| = \|Tx\|$, then $\||T||T^*|Tx\| \leq \|T^2\|\|Tx\|$ for all $x \in \mathcal{H}$. Therefore:

$$\||T||T^*|x\| \leq \|T^2\|\|x\| \qquad \forall x \in R(T)$$

And it can be extended as usual to $\overline{R(T)}$, so $\||T||T^*|x\| \leq \|T^2\|\|x\|$ for all $x \in \overline{R(T)}$.
Since $\mathcal{H} = \overline{R(T)} \oplus N(T^*)$. Let $x \in N(T^*) \iff T^*x = 0 \implies TT^*x = 0$

$$\implies \sqrt{TT^*}x = 0 \implies |T^*|x = 0$$

Then $|T||T^*|x = 0$ for all $x \in N(T^*)$.
$\forall\ x \in \mathcal{H}$, there exist $x_1 \in \overline{R(T)}$ and $x_2 \in N(T^*)$ such that $x = x_1 + x_2$ and $\|x\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$, then:

$$\||T||T^*|x\| = \||T||T^*|(x_1 + x_2)\| = \||T||T^*|x_1\| \leq \|T^2\|\|x_1\| \leq \|T^2\|\sqrt{\|x_1\|^2 + \|x_2\|^2} = \|T^2\|\|x\|$$

Thus $\||T||T^*|x\| \leq \|T^2\|\|x\|$ for all $x \in \mathcal{H}$, hence $\||T||T^*|\| \leq \|T^2\|$.

- *Let $x \in \mathcal{H}$ then :*

$$\begin{aligned}\|T^2T^*x\|^2 = \langle T^2T^*x, T^2T^*x\rangle &= \langle T^*TTT^*x, TT^*x\rangle \\ &= \langle |T|^2|T^*|^2x, |T^*|^2x\rangle \\ &= \langle |T||T^*||T^*|x, |T||T^*||T^*|x\rangle \\ &= \||T||T^*||T^*|x\|^2 \\ &\leq \||T||T^*|\|^2\||T^*|x\|^2\end{aligned}$$

Since $\||T^*|x\| = \|T^*x\|$, then $\|T^2T^*x\| \leq \||T||T^*|\|\|T^*x\|$ for all $x \in \mathcal{H}$. Therefore:

$$\|T^2x\| \leq \||T||T^*|\|\|x\| \qquad \forall x \in R(T^*)$$

And it can be extended to $\overline{R(T^*)}$, then $\|T^2x\| \leq \||T||T^*|\|\|x\|$ for all $x \in \overline{R(T^*)}$.
Let $x \in N(T) \iff Tx = 0 \implies T^2x = 0$.
We have that $\mathcal{H} = \overline{R(T^*)} \oplus N(T)$, then $\forall x \in \mathcal{H}$, $\exists x_1 \in \overline{R(T^*)}$ and $\exists x_2 \in N(T)$ such that $x = x_1 + x_2$, then:

$$\|T^2x\| = \|T^2(x_1 + x_2)\| = \|T^2x_1\| \leq \||T||T^*|\|\|x_1\| \leq \||T||T^*|\|\sqrt{\|x_1\|^2 + \|x_2\|^2} = \||T||T^*|\|\|x\|$$

Then $\|T^2x\| \leq \||T||T^*|\|\|x\|$ for all $x \in \mathcal{H}$, therefore $\|T^2\| \leq \||T||T^*|\|$.
Hence $\||T||T^*|\| = \|T^2\|$.

Theorem 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\forall x, y \in \mathcal{H} \qquad |\langle Tx, y\rangle|^2 \leq \langle |T|x, x\rangle\langle |T^*|y, y\rangle \tag{2.14}$$

Proof. *Since* $|T^*| \geq 0$*, then:*

$$\forall x, y \in \mathcal{H}: \quad |\langle |T^*|x, y\rangle|^2 \leq \langle |T^*|x, x\rangle\langle |T^*|y, y\rangle \text{ (Generalized Schwarz inequality)}$$

Since $T|T|^2 = TT^*T = |T^*|^2T$*, and* $\||T|^2\| = \||T^*|^2\| = \|T\|^2$*, by Theorem 2.12, we get:*

$$T\sqrt{|T|^2} = \sqrt{|T^*|^2}T \iff T|T| = |T^*|T$$

Set $x = Tx$*, and for all* $x, y \in \mathcal{H}$ *we obtain:*

$$\begin{aligned}
|\langle |T^*|Tx, y\rangle|^2 \leq& \langle |T^*|Tx, Tx\rangle\langle |T^*|y, y\rangle \\
|\langle T|T|x, y\rangle|^2 \leq& \langle T^*T|T|x, x\rangle\langle |T^*|y, y\rangle \\
=& \langle |T|^2|T|x, x\rangle\langle |T^*|y, y\rangle \\
=& \langle |T||T|x, |T|x\rangle\langle |T^*|y, y\rangle
\end{aligned}$$

Then, $\forall x, y \in \mathcal{H}$*:* $\quad |\langle T|T|x, y\rangle|^2 \leq \langle |T||T|x, |T|x\rangle\langle |T^*|y, y\rangle$
Thus $\forall x \in R(|T|)$ *and* $\forall y \in \mathcal{H}$*:* $\quad |\langle Tx, y\rangle|^2 \leq \langle |T|x, x\rangle\langle |T^*|y, y\rangle$
We can extend it to $\overline{R(|T|)}$ *as follows, let* $x \in \overline{R(|T|)}$*,* $\exists (x_n) \subset R(|T|)$ *such that* $\lim\limits_{n\to\infty} x_n = x$*, then:*

$$\forall n \in \mathbb{N}: \qquad |\langle Tx_n, y\rangle|^2 \leq \langle |T|x_n, x_n\rangle\langle |T^*|y, y\rangle \implies \lim_{n\to\infty} |\langle Tx_n, y\rangle|^2 \leq \lim_{n\to\infty} \langle |T|x_n, x_n\rangle\langle |T^*|y, y\rangle$$

Then, $\forall x \in \overline{R(|T|)}$ *and* $\forall y \in \mathcal{H}$*, we have:*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|x, x\rangle\langle |T^*|y, y\rangle$$

Let $x \in N(|T|) \iff |T|x = 0$*, then:*

$$\||T|x\|^2 = \langle |T|x, |T|x\rangle = \langle |T|^2x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \implies \||T|x\| = \|Tx\|$$

Then $\|Tx\| = 0 \implies Tx = 0$*, therefore for all* $x \in N(|T|)$*:* $\; Tx = 0$*.*
By orthogonal decomposition, $\mathcal{H} = \overline{R(|T|)} \oplus (R(|T|))^\perp$*, but* $(R(|T|))^\perp = N(|T|^*) = N(|T|)$*.*
Then $\mathcal{H} = \overline{R(|T|)} \oplus N(|T|)$*, so that* $\forall x \in \mathcal{H}$*, there exist* $x_1 \in \overline{R(|T|)}$ *and* $x_2 \in N(|T|)$ *where* $x = x_1 + x_2$*, then:*

$$|\langle Tx, y\rangle|^2 = |\langle T(x_1 + x_2), y\rangle|^2 = |\langle Tx_1, y\rangle + \langle Tx_2, y\rangle|^2 = |\langle Tx_1, y\rangle|^2 \leq \langle |T|x_1, x_1\rangle\langle |T^*|y, y\rangle$$

But $\langle |T|x_2, x_2\rangle = \langle |T|x_2, x_1\rangle = \langle |T|x_1, x_2\rangle = 0$*.*
Thus

$$\Big(\langle |T|x_2, x_2\rangle + \langle |T|x_2, x_1\rangle + \langle |T|x_1, x_2\rangle\Big)\langle |T^*|y, y\rangle = 0$$

As a result

$$\langle |T|x_1, x_1\rangle\langle |T^*|y, y\rangle = \langle |T|x_1, x_1\rangle\langle |T^*|y, y\rangle + \Big(\langle |T|x_2, x_2\rangle + \langle |T|x_2, x_1\rangle + \langle |T|x_1, x_2\rangle\Big)\langle |T^*|y, y\rangle = \langle |T|x, x\rangle\langle |T^*|y, y\rangle$$

Therefore $\forall y \in \mathcal{H}$ *and* $\forall x \in \mathcal{H}$*:*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|x, x\rangle\langle |T^*|y, y\rangle$$

As desired.

Remark 2.7. (2.14) *generalizes* (2.9)*, because if* $T \geq 0$*, then* $|T| = |T^*| = T$*.*

Chapter 3

Spectrum and numerical range of a linear operator

The purpose of this chapter is to provide the main concepts and facts of the spectrum, the numerical range, spectral radius and numerical radius of an operator. Moreover, we offer the latest results in the literature, also we provide well-know techniques that are used in this domain.

3.1 Spectrum of a linear operator

Definition 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$. The spectrum of T is denoted by $\sigma(T)$, and it is defined as follows :*

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

And the resolvent of T is denoted by $\rho(T)$, and $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Example 3.1. $\sigma(\alpha I) = \{\alpha\}$.

Remark 3.1. *It is clear that $\sigma(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) \neq \{0\} \text{ or } R(T - \lambda I) \neq \mathcal{H}\}$.*

Definition 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$, then the resolvent function of T is the map $R : \rho(T) \longrightarrow \mathcal{I}(\mathcal{H})$ defined by $R(\lambda) = (T - \lambda I)^{-1}$.*

Proposition 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$, and let R be the resolvent function of T. Then :*
(1) $R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu) \quad$ for all $\lambda, \mu \in \rho(T)$.
(2) $R(\lambda)R(\mu) = R(\mu)R(\lambda) \quad$ for all $\lambda, \mu \in \rho(T)$.
(3) R is continuous.

Proof. *(1)*
$$\begin{aligned} R(\lambda) - R(\mu) &= (T - \lambda I)^{-1} - (T - \mu I)^{-1} \\ &= (T - \lambda I)^{-1}(T - \mu I)(T - \mu I)^{-1} - (T - \lambda I)^{-1}(T - \lambda I)(T - \mu I)^{-1} \\ &= (T - \lambda I)^{-1}((T - \mu I) - (T - \lambda I))(T - \mu I)^{-1} \\ &= (T - \lambda I)^{-1}(\lambda - \mu)I(T - \mu I)^{-1} \\ &= (\lambda - \mu)R(\lambda)R(\mu) \end{aligned}$$

(2) If $\mu = \lambda$, then $R(\lambda)R(\mu) = R(\mu)R(\lambda)$. Assume that $\mu \neq \lambda$, from (1) we have :

$$R(\lambda)R(\mu) = \frac{R(\lambda) - R(\mu)}{\lambda - \mu} = \frac{R(\mu) - R(\lambda)}{\mu - \lambda} = R(\mu)R(\lambda)$$

(3) From Theorem 1.11, we have that $\mathcal{F} : \mathcal{I}(\mathcal{H}) \longrightarrow \mathcal{I}(\mathcal{H})$ defined by $\mathcal{F}(T) = T^{-1}$ is continuous.
Let $G : \rho(T) \longrightarrow \mathcal{I}(\mathcal{H})$ defined by $G(\lambda) = T - \lambda I$, it is clear that G is continuous, since:

$$\|G(\lambda) - G(\mu)\| = \|\lambda I - \mu I\| = |\lambda - \mu|$$

And since $R = \mathcal{F} \circ G$, then R is continuous.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then :*
(1) If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$.
(2) $\sigma(T)$ is a closed set.

Proof. *(1) If $|\lambda| > \|T\|$, then $\|\lambda^{-1}T\| < 1$, by Theorem 1.10 $I - \lambda^{-1}T$ is invertible, then $T - \lambda I$ is invertible. Hence $\lambda \in \rho(T)$.*
(2) Define $G : \mathbb{C} \longrightarrow \mathcal{B}(\mathcal{H})$ by $G(\lambda) = T - \lambda I$, and let $\mathcal{N}$ be the set of non-invertible operators.
Then $\sigma(T) = \{\lambda \in \mathbb{C} : G(\lambda) \in \mathcal{N}\} = G^{-1}(\mathcal{N})$, by Lemma 1.4 $\mathcal{N}$ is a closed subset of $\mathcal{B}(\mathcal{H})$, and we have that G is continuous, then $\sigma(T)$ is closed of $\mathbb{C}$.

Corollary 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then :*
(1) $\sigma(T)$ is compact.
(2) $\rho(T)$ is an open non-empty set.

Proposition 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T)$ is non-empty.*

Proof. *Suppose that $\sigma(T) = \emptyset$, then $\rho(T) = \mathbb{C}$. Let $f \in \mathcal{B}(\mathcal{H})'$.*
First, let's prove that $f \circ R : \mathbb{C} \longrightarrow \mathbb{C}$ is bounded :
By Proposition 3.1, R is continuous, then $\|R\|$ is also continuous, by Bolzano–Weierstrass theorem, we get:

$$\sup_{|\lambda| \leq \|T\|} \|R(\lambda)\| < \infty$$

When $|\lambda| > \|T\|$, using Theorem 1.10, then:

$$\|(I - \lambda^{-1}T)^{-1}\| \leq \frac{|\lambda|}{|\lambda| - \|T\|} \Longrightarrow \|\lambda(T - \lambda I)^{-1}\| \leq \frac{|\lambda|}{|\lambda| - \|T\|} \Longrightarrow \|(T - \lambda I)^{-1}\| \leq \frac{1}{|\lambda| - \|T\|} \underset{|\lambda| \to \infty}{\longrightarrow} 0$$

$$\Longrightarrow \lim_{|\lambda| \to \infty} \|(T - \lambda I)^{-1}\| = 0$$

$$\Longrightarrow \sup_{|\lambda| > \|T\|} \|R(\lambda)\| < \infty$$

Therefore

$$\sup_{|\lambda| \in \mathbb{C}} \|R(\lambda)\| < \infty$$

Then

$$\sup_{|\lambda| \in \mathbb{C}} \|(f \circ R)(\lambda)\| \leq \|f\| \sup_{|\lambda| \in \mathbb{C}} \|R(\lambda)\| < \infty$$

Thus $f \circ R$ is bounded.
Second, let's prove that $f \circ R : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic :
Let $\lambda, \mu \in \mathbb{C}$ such that $\lambda \neq \mu$, by Proposition 3.1, we get :

$$\frac{R(\lambda) - R(\mu)}{\lambda - \mu} - R(\lambda)^2 = (R(\mu) - R(\lambda))R(\lambda)$$

Set $g = f \circ R$, and let $g' : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $g'(\lambda) = f(R(\lambda)^2)$, then :

$$|\frac{g(\lambda) - g(\mu)}{\lambda - \mu} - g'(\lambda)| = |f(\frac{R(\lambda) - R(\mu)}{\lambda - \mu} - R(\lambda)^2)| = |f((R(\mu) - R(\lambda))R(\lambda))|$$

Then $$|\frac{g(\lambda) - g(\mu)}{\lambda - \mu} - g'(\lambda)| \leq \|f\|\|R(\lambda)\|\|R(\mu) - R(\lambda)\| \underset{\mu \to \lambda}{\longrightarrow} 0 \text{ (since } R \text{ is continuous)}$$

$$\implies \quad \lim_{\mu\to\lambda} \frac{g(\lambda)-g(\mu)}{\lambda-\mu} = g'(\lambda)$$

Therefore $f \circ R$ is analytic.
Thus for all $f \in \mathcal{B}(\mathcal{H})'$, $f \circ R$ is bounded and analytic over all $\mathbb{C}$, then by the Liouville theorem $f \circ R$ is constant for all $f \in \mathcal{B}(\mathcal{H})'$, but we have that:

$$\|R(\lambda)\| \underset{|\lambda|\to\infty}{\longrightarrow} 0 \implies f(R(\lambda)) \underset{|\lambda|\to\infty}{\longrightarrow} 0 \ (f \text{ is continuous})$$

Then $f \circ R = 0$ for all $f \in \mathcal{B}(\mathcal{H})'$, so $R = 0$ (it is well-known result that comes from Hahn–Banach theorem).
Thus $(T - \lambda I)^{-1} = 0$ for all $\lambda \in \mathbb{C}$, which is a contradiction because 0 is not invertible i.e. $0 \notin \mathcal{I}(\mathcal{H})$.
Hence $\sigma(T) \neq \emptyset$.

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$, then $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$.*

Proof. $\lambda \in \rho(T) \iff T - \lambda I$ *is invertible* $\iff (T - \lambda I)^*$ *is invertible*

$$\iff T^* - \overline{\lambda} I \text{ is invertible}$$
$$\iff \overline{\lambda} \in \rho(T^*)$$

Therefore $\quad \lambda \notin \rho(T) \iff \overline{\lambda} \notin \rho(T^*)$
Thus $\quad \sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$.

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then :*
(1) If p is a polynomial then $\sigma(p(T)) = p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.
(2) If T is invertible then $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

Proof. *(1) • Let $\mu \in \mathbb{C}$ and $q(z) = p(z) - \mu$, it is obvious that q is a polynomial, so it has factorization of the form $q(z) = c(z - \mu_1)\cdots(z - \mu_n)$ where $c, \mu_1, \cdots, \mu_n \in \mathbb{C}$ with $c \neq 0$, then :*

$$\begin{aligned}
\text{Let } \mu \in \sigma(p(T)) \implies p(T) - \mu I \text{ is not invertible} &\implies q(T) \text{ is not invertible} \\
&\implies c(T - \mu_1 I)\cdots(T - \mu_n I) \text{ is not invertible} \\
&\implies \exists k \text{ such that } 1 \le k \le n \text{ and } (T - \mu_k I) \text{ is not invertible} \\
&\implies \mu_k \in \sigma(T) \\
&\implies \mu = p(\mu_k) \\
&\implies \mu \in p(\sigma(T))
\end{aligned}$$

Then $\sigma(p(T)) \subset p(\sigma(T))$.
• Let $\mu \in p(\sigma(T))$, then $\exists \lambda \in \sigma(T)$ such that $\mu = p(\lambda)$, so $p(\lambda) - \mu = 0$.
Thus $\exists k \in \{1, 2, \cdots, n\}$ such that $\lambda = \mu_k$. Since $(T - \mu_k I)$ commutes with $(T - \mu_j I)$ for all $1 \le j \le n$, then:

$$\begin{aligned}
p(T) - \mu I =& c(T - \mu_1 I)\cdots(T - \mu_k I)\cdots(T - \mu_n I) \\
=& (T - \mu_k I)(c(T - \mu_1 I)\cdots(T - \mu_n I)) \\
=& (c(T - \mu_1 I)\cdots(T - \mu_n I))(T - \mu_k I)
\end{aligned}$$

If $\mu \in \rho(p(T))$, then $p(T) - \mu I$ is invertible, then we have the following :

$$(T - \mu_k I)(c(T - \mu_1 I)\cdots(T - \mu_n I))(p(T) - \mu I)^{-1} = (p(T) - \mu I)^{-1}(c(T - \mu_1 I)\cdots(T - \mu_n I))(T - \mu_k I) = I$$

Which means that $T - \mu_k I$ has right and left inverses, therefore $T - \mu_k I$ is injective and surjective, so that $T - \mu_k I$ is bijective, then $T - \mu_k I$ is invertible, thus $\mu_k \in \rho(T)$, and this contradict $\mu_k \in \sigma(T)$, then $\mu \notin \rho(p(T))$. Thus $\mu \in \sigma(p(T))$, hence $p(\sigma(T)) \subset \sigma(p(T))$. Overall $p(\sigma(T)) = \sigma(p(T))$.
(2) Since T is invertible, then $0 \notin \sigma(T)$, therefore λ^{-1} is defined when $\lambda \in \sigma(T)$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$ $\quad T^{-1} - \lambda^{-1}I = -\lambda^{-1}T^{-1}(T - \lambda I)$, *and* $\lambda^{-1}T^{-1}$ *is invertible for all* $\lambda \in \mathbb{C} \setminus \{0\}$.

Then $\quad \lambda \in \sigma(T) \iff T - \lambda I$ *is not invertible.*

$\iff -\lambda^{-1}T^{-1}(T - \lambda I)$ *is not invertible.*

$\iff T^{-1} - \lambda^{-1}I$ *is not invertible.*

$\iff \lambda^{-1} \in \sigma(T^{-1})$.

Hence $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

Corollary 3.2. *Let* $T \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$ *and* $n \in \mathbb{N}$, *then :*
(1) $\sigma(\alpha T) = \alpha\sigma(T) = \{\alpha\lambda : \lambda \in \sigma(T)\}$.
(2) $\sigma(T^n) = (\sigma(T))^n = \{\lambda^n : \lambda \in \sigma(T)\}$.

Theorem 3.3. *Let* $T, S \in \mathcal{B}(\mathcal{H})$. *Then* $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.

Proof. *Let* $\lambda \in \rho(TS) \setminus \{0\} \iff TS - \lambda I$ *is invertible.*
Set $R = (TS - \lambda I)^{-1}$, *then:*

$$(TS - \lambda I)R = R(TS - \lambda I) = I \implies TSR - \lambda R = RTS - \lambda R = I$$

Therefore $\quad TSR = RTS = I + \lambda R$.

$$\begin{aligned}(ST - \lambda I)(SRT - I) &= STSRT - ST - \lambda SRT + \lambda I \\ &= S(I + \lambda R)T - ST - \lambda SRT + \lambda I \\ &= ST + \lambda SRT - ST - \lambda SRT + \lambda I \\ &= \lambda I\end{aligned}$$

Since $\lambda \neq 0$, *then :* $\quad (ST - \lambda I)\frac{1}{\lambda}(SRT - I) = I.$

$$\begin{aligned}(SRT - I)(ST - \lambda I) &= SRTST - \lambda SRT - ST + \lambda I \\ &= S(I + \lambda R)T - \lambda SRT - ST + \lambda I \\ &= ST + \lambda SRT - \lambda SRT - ST + \lambda I \\ &= \lambda I\end{aligned}$$

Since $\lambda \neq 0$, *then :* $\quad \frac{1}{\lambda}(SRT - I)(ST - \lambda I) = I$
Therefore $ST - \lambda I$ *is invertible, then* $\lambda \in \rho(ST) \setminus \{0\}$.
Hence $\rho(TS) \setminus \{0\} = \rho(ST) \setminus \{0\} \iff \sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.

Theorem 3.4. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be self-adjoint, then* $\sigma(T) \subset \mathbb{R}$.

Proof. *Let* $\lambda \in \mathbb{C}$ *such that* $\Im\lambda \neq 0$, *and let* $x \in \mathcal{H}$. *Since* T *is self-adjoint* $\langle Tx, x\rangle \in \mathbb{R}$, *then :*

$$-\Im\lambda\|x\|^2 = \Im\langle(T - \lambda I)x, x\rangle \implies |\Im\lambda|\|x\|^2 = |\Im\langle(T - \lambda I)x, x\rangle| \leq |\langle(T - \lambda I)x, x\rangle| \leq \|(T - \lambda I)x\|\|x\|$$

Then $\quad |\Im\lambda|\|x\| \leq \|(T - \lambda I)x\|$.
By Corollary 2.4 (since $|\Im\lambda| \neq 0$*), then* $T - \lambda I$ *is invertible, thus* $\lambda \in \rho(T)$.
Therefore $\forall\lambda \in \mathbb{C}$ *such that* $\Im\lambda \neq 0$ *we have* $\lambda \in \rho(T)$.
Thus if $\lambda \in \sigma(T)$, *then* $\Im\lambda = 0$, *which means that* $\sigma(T) \subset \mathbb{R}$.

Corollary 3.3. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be positive. Then :*
(1) $\sigma(T) \subset \mathbb{R}_+$.
(2) $\sigma(\sqrt{T}) = \sqrt{\sigma(T)} = \{\sqrt{\lambda} : \lambda \in \sigma(T)\}$

Proof. *(1) Since T is positive, then T is self-adjoint, therefore $\sigma(T) \subset \mathbb{R}$ by Theorem 3.4.*
Let $\lambda < 0$, and $x \in \mathcal{H}$. Recall that $\langle Tx, x\rangle \geq 0$, then:

$$-\lambda\|x\|^2 = -\lambda\langle x, x\rangle \leq \langle Tx, x\rangle - \lambda\langle x, x\rangle = \langle (T-\lambda I)x, x\rangle \leq \|(T-\lambda I)x\|\|x\|$$

Thus $\quad -\lambda\|x\| \leq \|(T-\lambda I)x\|$. *By Corollary 2.4, $T-\lambda I$ is invertible (since $-\lambda > 0$), hence $\lambda \in \rho(T)$. Therefore for all $\lambda < 0$, then $\lambda \in \rho(T) \iff \lambda \notin \sigma(T)$, thus if $\lambda \in \sigma(T)$, then $\lambda \geq 0$, hence $\sigma(T) \subset \mathbb{R}_+$.*
(2) First of all, the set $\sqrt{\sigma(T)}$ is well-defined since $\sigma(T) \subset \mathbb{R}_+$. Then using Corollary 3.2, we get:

$$\sigma(T) = \sigma(\sqrt{T}^2) = \sigma(\sqrt{T})^2 \implies \sigma(T) = \{\lambda^2 : \lambda \in \sigma(\sqrt{T})\}$$

Thus $\sigma(\sqrt{T}) = \sqrt{\sigma(T)} = \{\sqrt{\lambda} : \lambda \in \sigma(T)\}$.

Definition 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then:*
(1) The point spectrum of T is the set:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : N(T-\lambda I) \neq \{0\}\}$$

$\lambda \in \sigma_p(T)$ is called eigenvalue of T, and the $x \in \mathcal{H} \setminus \{0\}$ that verifies $Tx = \lambda x$ is called eigenvector of λ.
(2) The continuous spectrum of T is the set:

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : N(T-\lambda I) = \{0\} \text{ and } \overline{R(T-\lambda I)} = \mathcal{H}\}$$

(3) The residual spectrum of T is the set:

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : N(T-\lambda I) = \{0\} \text{ and } \overline{R(T-\lambda I)} \subsetneq \mathcal{H}\}$$

(4) The approximate point spectrum of T is the set:

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists (x_n) \subset \mathcal{H} : \|x_n\| = 1 \text{ and } \lim_{n\to\infty}(T-\lambda I)x_n = 0\}$$

Remark 3.2. *(1) $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$, where $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are mutually disjoint, i.e.*

$$\sigma_p(T) \cap \sigma_c(T) = \sigma_p(T) \cap \sigma_r(T) = \sigma_c(T) \cap \sigma_r(T) = \emptyset$$

(2) By Corollary 1.8, $\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$, also $\sigma_p(T) \subset \sigma_{ap}(T)$.

Corollary 3.4. *If $\mathcal{H}$ is a finite-dimensional Hilbert space, then $\sigma(T) = \sigma_p(T)$ for all $T \in \mathcal{B}(\mathcal{H})$.*

Proof. *Let $T \in \mathcal{B}(\mathcal{H})$, by the dimensional theorem, we have $dim(\mathcal{H}) = dim(N(T)) + dim(R(T))$.*
Therefore $\quad$ *T is inverible $\iff N(T) = \{0\}$.*
Then if $T-\lambda I$ is not inverible $\iff N(T-\lambda I) \neq \{0\}$, hence $\sigma(T) = \sigma_p(T)$.

Proposition 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal, if λ and μ are disjoint ($\lambda \neq \mu$) eigenvalues i.e. $\exists x, y \in \mathcal{B}(\mathcal{H}) \setminus \{0\} : Tx = \lambda x$ and $Ty = \mu y$, then $\langle x, y\rangle = 0$.*

Proof.

$$\begin{aligned}\|T^*y - \overline{\mu}y\|^2 = \langle T^*y - \overline{\mu}y, T^*y - \overline{\mu}y\rangle &= \langle T^*y, T^*y\rangle + |\mu|^2\langle y, y\rangle - \mu\langle T^*y, y\rangle - \overline{\mu}\langle y, T^*y\rangle \\ &= \langle TT^*y, y\rangle + \|\mu y\|^2 - \langle \mu y, Ty\rangle - \langle Ty, \mu y\rangle \\ &= \langle T^*Ty, y\rangle + \|Ty\|^2 - \langle Ty, Ty\rangle - \langle Ty, Ty\rangle \\ &= \langle Ty, Ty\rangle + \langle Ty, Ty\rangle - 2\langle Ty, Ty\rangle = 0\end{aligned}$$

Therefore $T^*y = \overline{\mu}y$. Since $\lambda \neq \mu$, then:

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\mu}y\rangle = \mu\langle x, y\rangle \implies \lambda\langle x, y\rangle = \mu\langle x, y\rangle \implies (\lambda-\mu)\langle x, y\rangle = 0$$

Hence $\langle x, y\rangle = 0$.

Theorem 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma_{ap}(T)$ is closed and $\partial\sigma(T) \subset \sigma_{ap}(T)$.*

Proof. • *First, let's prove that $\sigma_{ap}(T)$ is closed:*
Let $\lambda \notin \sigma_{ap}(T)$, then $\forall (x_n) \subset \mathcal{H}$ such that $\|x_n\| = 1$ $\lim_{n\to\infty}(T-\lambda I)x_n \neq 0$.
Then $\exists \alpha > 0$ such that $\forall x \in \mathcal{H}$ and $\|x\| = 1$: $\|(T-\lambda I)x\| \geq \alpha$.
Thus $\forall x \in \mathcal{H}$ $\|(T-\lambda I)x\| \geq \alpha\|x\|$. Set $\epsilon = \frac{\alpha}{2}$ and let $\mu \in \mathbb{C}$ such that $|\lambda-\mu| < \epsilon$, then:

$$\|(T-\lambda I)x\| = \|(T-\mu I)x + \mu x - \lambda x\| \leq \|(T-\mu I)x\| + \|\lambda x - \mu x\| = \|(T-\mu I)x\| + |\lambda-\mu|\|x\|$$

Thus $\|(T-\lambda I)x\| \leq \|(T-\mu I)x\| + |\lambda-\mu|\|x\|$. Therefore:

$$\alpha\|x\| - |\lambda-\mu|\|x\| \leq \|(T-\lambda I)x\| - |\lambda-\mu|\|x\| \leq \|(T-\mu I)x\| \Longrightarrow (\alpha - |\lambda-\mu|)\|x\| \leq \|(T-\mu I)x\|$$

Since $|\lambda-\mu| < \epsilon = \frac{\alpha}{2}$, then:

$$\frac{\alpha}{2}\|x\| \leq \|(T-\mu I)x\| \Longrightarrow \forall (x_n) \subset \mathcal{H} \text{ where } \|x_n\| = 1 : \quad \lim_{n\to\infty}(T-\lambda I)x_n \neq 0$$

Then $\mu \notin \sigma_{ap}(T)$, thus $\mathbb{C} \setminus \sigma_{ap}(T)$ is open, hence $\sigma_{ap}(T)$ is closed .
• *Second, prove that $\partial\sigma(T) \subset \sigma_{ap}(T)$:*
We have that $\partial\sigma(T) = \overline{\sigma(T)} \cap \overline{\rho(T)}$, since $\sigma(T)$ is closed, then $\partial\sigma(T) = \sigma(T) \cap \overline{\rho(T)}$.
Let $\lambda \in \partial\sigma(T)$, then $\exists (\lambda_n) \subset \rho(T)$ such that $\lambda_n \underset{n\to\infty}{\longrightarrow} \lambda$, then:
$\|(T-\lambda_n I)^{-1}\| \underset{n\to\infty}{\longrightarrow} \infty$, *if not, $\exists c > 0$ such that $\|(T-\lambda_n I)^{-1}\| \leq c$, for n sufficiently large, we have :*

$$\|T-\lambda I - T + \lambda_n I\| = |\lambda_n - \lambda| \leq c^{-1} \leq \|(T-\lambda_n I)^{-1}\|^{-1} \Longrightarrow \|T-\lambda I - T + \lambda_n I\| \leq \|(T-\lambda_n I)^{-1}\|^{-1}$$

By Corollary 1.7, $T-\lambda I$ is invertible, and this contradicts $\lambda \in \partial\sigma(T) = \sigma(T) \cap \overline{\rho(T)}$.
Therefore $\|(T-\lambda_n I)^{-1}\| \underset{n\to\infty}{\longrightarrow} \infty$.
We have that $\|(T-\lambda_n I)^{-1}\| = \sup_{\|x\|=1}\|(T-\lambda_n I)^{-1}x\|$, then:

$$\forall n \in \mathbb{N},\ \exists (x_n) \subset \mathcal{H} \text{ where } \|x_n\| = 1 \text{ and } \|(T-\lambda_n I)^{-1}x_n\| > \|(T-\lambda_n I)^{-1}\| - \frac{1}{n}$$

Set $\alpha_n = \|(T-\lambda_n I)^{-1}x_n\| > \|(T-\lambda_n I)^{-1}\| - \frac{1}{n}$, then $\alpha_n \underset{n\to\infty}{\longrightarrow} \infty$.
Put $y_n = \alpha_n^{-1}(T-\lambda_n I)^{-1}x_n$, using that fact that $\|y_n\| = 1$ we get:

$$\begin{aligned} &(T-\lambda I)y_n = (T-\lambda_n I)y_n + (\lambda_n-\lambda)y_n = \alpha_n^{-1}x_n + (\lambda_n-\lambda)y_n \\ \Longrightarrow \quad &\|(T-\lambda I)y_n\| \leq \|\alpha_n^{-1}x_n\| + \|(\lambda_n-\lambda)y_n\| = \alpha_n^{-1} + |\lambda_n-\lambda| \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

Therefore $\|(T-\lambda I)y_n\| \underset{n\to\infty}{\longrightarrow} 0$, thus $\lambda \in \sigma_{ap}(T)$.
Hence $\partial\sigma(T) \subset \sigma_{ap}(T)$.

Corollary 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$, then $\sigma_{ap}(T)$ is a compact non-empty set.*

Proof. *Since $\sigma_{ap}(T) \subset \sigma(T)$, then $\sigma_{ap}(T)$ is closed in compact, then $\sigma_{ap}(T)$ is compact as well.*
On the other hand, we have that $\partial\sigma(T) \subset \sigma_{ap}(T)$, we have that $\partial\sigma(T) \neq \emptyset$.
Because $\partial\sigma(T) = \sigma(T) \setminus \sigma(T)^\circ$, and since $\sigma(T) \neq \emptyset$ nor $\mathbb{C}$ and we know that $\mathbb{C}$ is connected, then $\sigma(T)^\circ \subsetneq \sigma(T)$.
Thus $\sigma(T) \setminus \sigma(T)^\circ \neq \emptyset$, therefore $\partial\sigma(T) \neq \emptyset$.
Hence $\sigma_{ap}(T) \neq \emptyset$.

3.2 Spectral radius

Definition 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$. The spectral radius of T is denoted by $r(T)$, and it is defined as follows :*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

Needless to say that $\sup_{\lambda \in \sigma(T)} |\lambda|$ exists and finite. (remember that $\sigma(T)$ is a compact non-empty set)

Corollary 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$, then $r(T) \leq \|T\|$.*

Proof. *If $|\lambda| > \|T\| \Longrightarrow \lambda \in \rho(T)$, then $\forall \lambda \in \sigma(T)$: $|\lambda| \leq \|T\|$.*
Thus $\sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\|$, therefore $r(T) \leq \|T\|$.

Proposition 3.4. *Let $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$. Then :*
(1) $r(\alpha T) = |\alpha| r(T)$.
(2) $r(T^n) = r(T)^n$.
(3) $r(TS) = r(ST)$.
(4) $r(T^*) = r(T)$.
(5) If $T \geq 0$, then $r(\sqrt{T}) = \sqrt{r(T)}$.

Proof. *(1) By Corollary 3.2, we have that $\sigma(\alpha T) = \alpha\sigma(T)$, then:*

$$r(\alpha T) = \sup_{\lambda \in \sigma(\alpha T)} |\lambda| = \sup_{\lambda \in \sigma(T)} |\alpha\lambda| = |\alpha| \sup_{\lambda \in \sigma(T)} |\lambda| = |\alpha| r(T)$$

Hence $r(\alpha T) = |\alpha| r(T)$.
(2) By Corollary 3.2, we have that $\sigma(T^n) = (\sigma(T))^n$, then:

$$r(T^n) = \sup_{\lambda \in \sigma(T^n)} |\lambda| = \sup_{\lambda \in \sigma(T)} |\lambda|^n = (\sup_{\lambda \in \sigma(T)} |\lambda|)^n = r(T)^n$$

Therefore $r(T^n) = r(T)^n$.
(3) By Theorem 3.3, we have that $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$, then:

$$\sup_{\lambda \in \sigma(TS) \cup \{0\}} |\lambda| = \sup_{\lambda \in \sigma(ST) \cup \{0\}} |\lambda|$$

But $\sup_{\lambda \in \sigma(TS) \cup \{0\}} |\lambda| = \sup_{\lambda \in \sigma(TS)} |\lambda|$, and the same $\sup_{\lambda \in \sigma(ST) \cup \{0\}} |\lambda| = \sup_{\lambda \in \sigma(ST)} |\lambda|$.

$$\Longrightarrow \sup_{\lambda \in \sigma(TS)} |\lambda| = \sup_{\lambda \in \sigma(ST)} |\lambda| \Longrightarrow r(TS) = r(ST)$$

As desired.
(4) By Lemma 3.1, we have that $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$, then:

$$r(T^*) = \sup_{\lambda \in \sigma(T^*)} |\lambda| = \sup_{\lambda \in \sigma(T)} |\overline{\lambda}| = \sup_{\lambda \in \sigma(T)} |\lambda| = r(T)$$

Hence $r(T^*) = r(T)$.
(5) By Corollary 3.3, we have that $\sigma(\sqrt{T}) = \sqrt{\sigma(T)}$, then:

$$r(\sqrt{T}) = \sup_{\lambda \in \sigma(\sqrt{T})} \lambda = \sup_{\lambda \in \sigma(T)} \sqrt{\lambda} = \sqrt{\sup_{\lambda \in \sigma(T)} \lambda} = \sqrt{r(T)}$$

Therefore $r(\sqrt{T}) = \sqrt{r(T)}$.

Remark 3.3. $T = 0 \implies r(T) = 0$. *But the converse is not true, that is if* $r(T) = 0$, *this does not imply that* $T = 0$. *Consider* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, *we have that* $\sigma(T) = \sigma_p(T) = \{0\}$, *then* $r(T) = 0$, *but* $T \neq 0$.

Lemma 3.2. *Let* $T \in \mathcal{B}(\mathcal{H})$, *then* $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ *exists and equals* $\inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}$.

Proof. *Set* $\alpha = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}$. *Then* $\forall \epsilon > 0$, *there exists* $m \in \mathbb{N}$ *such that* $\|T^m\|^{\frac{1}{m}} \leq \alpha + \epsilon$.
Any $n \in \mathbb{N}$ *can be written as* $n = pm + q$ *where* $0 \leq q < m$ *and* $p \in \mathbb{N}$. *Then :*

$$\|T^n\|^{\frac{1}{n}} = \|T^{pm+q}\|^{\frac{1}{n}} \leq \|T^m\|^{\frac{p}{n}} \|T\|^{\frac{q}{n}} \leq (\alpha + \epsilon)^{\frac{pm}{n}} \|T\|^{\frac{q}{n}}$$

Since $\frac{pm}{n} \underset{n\to\infty}{\longrightarrow} 1$, *and* $\frac{q}{n} \underset{n\to\infty}{\longrightarrow} 0$. *It follows that :* $\limsup_n \|T^n\|^{\frac{1}{n}} \leq \alpha + \epsilon$
As ϵ *is arbitrary, we have :* $\limsup_n \|T^n\|^{\frac{1}{n}} \leq \alpha$
Since, $\forall n \in \mathbb{N}$ $\alpha \leq \|T^n\|^{\frac{1}{n}} \implies \alpha \leq \liminf_n \|T^n\|^{\frac{1}{n}}$, *then* $\liminf_n \|T^n\|^{\frac{1}{n}} = \limsup_n \|T^n\|^{\frac{1}{n}} = \alpha$.
Therefore $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ *exists, and* $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}$.

Theorem 3.6. *(Gelfand's formula)*
Let $T \in \mathcal{B}(\mathcal{H})$. *Then :* $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}$.

Proof. • $\forall n \in \mathbb{N}$ $r(T)^n = r(T^n) \leq \|T^n\|$, *then* $\forall n \in \mathbb{N}$ $r(T) \leq \|T^n\|^{\frac{1}{n}}$, *thus* $r(T) \leq \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

• *Let* $|\lambda| > r(T) \implies \lambda \in \rho(T)$. *Recall that if* $|\lambda| > \|T\|$, *then* $R(\lambda) = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} T^n$.
Let $f \in \mathcal{B}(\mathcal{H})'$, *by the continuity of* f, *we have* $f(R(\lambda)) = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} f(T^n)$ *for all* $|\lambda| > \|T\|$.
$\lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} f(T^n)$ *is a Laurent expansion of* $f(R(\lambda))$ *about the origin when* $\lambda \in \rho(T)$ *where* $|\lambda| > \|T\|$. *But we see that* $f \circ R$ *is analytic in the proof of Proposition 3.2 on* $\rho(T)$, *so* $f(R(\lambda))$ *has a unique Laurent expansion about the origin for every* $\lambda \in \rho(T)$, *and so is for* $|\lambda| > r(T)$. *We have that:*

$$f(R(\lambda)) = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} f(T^n) \text{ for all } |\lambda| > \|T\| \geq r(T)$$

By the uniqueness of Laurent expansion, we get that:

$$f(R(\lambda)) = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} f(T^n) \text{ for all } |\lambda| > r(T)$$

Thus $f(\lambda^{-n} T^n) = \lambda^{-n} f(T^n) \underset{n\to\infty}{\longrightarrow} 0$ *for all* $|\lambda| > r(T)$ *and* $f \in \mathcal{B}(\mathcal{H})'$.
Hence $(\lambda^{-n} T^n)$ *is weakly convergent to* 0 *for all* $|\lambda| > r(T)$, *thus* $(\lambda^{-n} T^n)$ *is bounded for all* $|\lambda| > r(T)$, *then:*

$$\begin{aligned}
&\exists c > 0,\ \forall n \in \mathbb{N} : \quad |\lambda^{-n}| \|T^n\| = \|\lambda^{-n} T^n\| \leq c \\
&\implies \forall n \in \mathbb{N} : \quad |\lambda^{-1}| \|T^n\|^{\frac{1}{n}} \leq c^{\frac{1}{n}} \\
&\implies |\lambda^{-1}| \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq 1 \\
&\implies \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq |\lambda| \text{ for all } |\lambda| > r(T) \\
&\implies \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \lim_{|\lambda|\to r(T)} |\lambda| = r(T)
\end{aligned}$$

Thus $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq r(T)$, *hence* $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}$.

Corollary 3.7. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be normal. Then* $r(T) = \|T\|$.

Proof. *By Theorem 2.5, we have* $\forall n \in \mathbb{N}$: $\|T^n\| = \|T\|^n \implies \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \|T\|$.
Since $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \implies r(T) = \|T\|$.

Theorem 3.7. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that T commutes with S i.e. $TS = ST$. Then :*

(1) $\quad r(TS) \leq r(T)r(S)$.

(2) $\quad r(T+S) \leq r(T) + r(S)$.

Proof. *(1)* $r(TS) = \lim_{n\to\infty} \|(TS)^n\|^{\frac{1}{n}}$, *since* $TS = ST$, *then* $(TS)^n = T^nS^n$, *therefore:*

$$r(TS) = \lim_{n\to\infty} \|T^nS^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}\|S^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \lim_{n\to\infty} \|S^n\|^{\frac{1}{n}} = r(T)r(S)$$

Hence $r(TS) \leq r(T)r(S)$.

(2) Let $\epsilon > 0$, *then there exists* $m \in \mathbb{N}$ *large enough such that :*

$$\|T^n\|^{\frac{1}{n}} \leq r(T) + \epsilon \text{ and } \|S^n\|^{\frac{1}{n}} \leq r(S) + \epsilon \qquad \text{for all } n \geq m.$$

Set $A = r(T) + \epsilon$ *and* $B = r(S) + \epsilon$, *then* $\|T^n\| \leq A^n$ *and* $\|S^n\| \leq B^n$ *for all* $n \geq m$.

For $n \geq 2m$, *and using the fact that* $TS = ST$, *we have the following :*

$$\begin{aligned}\|(T+S)^n\| = \|\sum_{k=0}^{n} C_k^n T^k S^{n-k}\| &\leq \sum_{k=0}^{n} C_k^n \|T^k\|\|S^{n-k}\| \\ &= \sum_{k=0}^{m-1} C_k^n \|T^k\|\|S^{n-k}\| + \sum_{k=m}^{n-m} C_k^n \|T^k\|\|S^{n-k}\| + \sum_{k=n-m+1}^{n} C_k^n \|T^k\|\|S^{n-k}\|\end{aligned}$$

When $0 \leq k \leq m-1$, *then* $n-k \geq m$ *(recall that* $n \geq 2m$*), thus* $\|S^{n-k}\| \leq B^{n-k}$ *for all* $0 \leq k \leq m-1$.

When $m \leq k \leq n-m$, *then* $k \geq m$ *and* $-n+m \leq -k \leq -m$, *so* $m \leq n-k \leq n-m$, *thus* $n-k \geq m$ *for all* $m \leq k \leq n-m$, *therefore* $\|T^k\| \leq A^k$ *and* $\|S^{n-k}\| \leq B^{n-k}$ *for all* $m \leq k \leq n-m$.

When $n-m+1 \leq k \leq n$, *then* $k \geq m$ *(recall that* $n \geq 2m$*), thus* $\|T^k\| \leq A^k$ *for all* $n-m+1 \leq k \leq n$.*Then:*

$$\begin{aligned}\|(T+S)^n\| &\leq \sum_{k=0}^{m-1} C_k^n \|T^k\| B^{n-k} + \sum_{k=m}^{n-m} C_k^n A^k B^{n-k} + \sum_{k=n-m+1}^{n} C_k^n A^k \|S^{n-k}\| \\ &\leq \sum_{k=0}^{m-1} C_k^n \|T^k\| B^{n-k} + \sum_{k=0}^{n} C_k^n A^k B^{n-k} + \sum_{k=0}^{m-1} C_k^n A^{n-k} \|S^k\| \\ &= p(n)B^n + (A+B)^n + q(n)A^n\end{aligned}$$

Where $p(n)$ *and* $q(n)$ *are polynomial in* n *of degree* $m-1$ *given by:*

$$p(n) = \sum_{k=0}^{m-1} C_k^n \|T^k\| B^{-k} \text{ and } q(n) = \sum_{k=0}^{m-1} C_k^n A^{-k} \|S^k\|. \text{ For example}$$

$p(n) = 1 + \|T\|B^{-1}n + 2^{-1}\|T^2\|B^{-2}n(n-1) + \cdots + m!^{-1}\|T^{m-1}\|B^{-m+1}n(n-1)\cdots(n-m+2)$.*Then:*

$$\lim_{n\to\infty} \|(T+S)^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} (p(n)B^n + (A+B)^n + q(n)A^n)^{\frac{1}{n}} = (A+B) \lim_{n\to\infty} (1 + p(n)(\frac{B}{A+B})^n + q(n)(\frac{A}{A+B})^n)^{\frac{1}{n}}$$

Let's prove that $\quad \lim_{n\to\infty} (1 + p(n)(\frac{B}{A+B})^n + q(n)(\frac{A}{A+B})^n)^{\frac{1}{n}} \leq 1.$

Since $(\frac{B}{A+B})^n, (\frac{A}{A+B})^n \leq 1$, *then:*

$$(1 + p(n)(\tfrac{B}{A+B})^n + q(n)(\tfrac{A}{A+B})^n)^{\frac{1}{n}} \leq (1 + p(n) + q(n))^{\frac{1}{n}}$$

Since $1 + p(n) + q(n) = \alpha_0 + \alpha_1 n + \cdots + \alpha_{m-1}n^{m-1}$, *then:*

$$\begin{aligned}(1 + p(n) + q(n))^{\frac{1}{n}} = \exp(\tfrac{1}{n}ln(1 + p(n) + q(n))) = \exp(\tfrac{1}{n}ln(\alpha_0 + \alpha_1 n + \cdots + \alpha_{m-1}n^{m-1})) \underset{n\to\infty}{\longrightarrow} 1 \\ \Longrightarrow (1 + p(n) + q(n))^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} 1 \Longrightarrow \lim_{n\to\infty} (1 + p(n)(\tfrac{B}{A+B})^n + q(n)(\tfrac{A}{A+B})^n)^{\frac{1}{n}} \leq 1\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} \|(T+S)^n\|^{\frac{1}{n}} \leq (A+B) \lim_{n\to\infty} (1 + p(n)(\tfrac{B}{A+B})^n + q(n)(\tfrac{A}{A+B})^n)^{\frac{1}{n}} \leq A+B \\ \Longrightarrow \lim_{n\to\infty} \|(T+S)^n\|^{\frac{1}{n}} \leq A+B \Longrightarrow r(T+S) \leq r(T) + r(S) + 2\epsilon \quad \text{for all } \epsilon > 0\end{aligned}$$

Hence $r(T+S) \leq r(T) + r(S)$ *as required.*

3.3 Numerical range of a linear operator

Definition 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of T is denoted by $W(T)$, and it is defined as follows :*

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$$

Proposition 3.5. *Let $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha, \beta \in \mathbb{C}$ and $U \in \mathcal{B}(\mathcal{H})$ be unitary. Then :*
(1) $W(\alpha T + \beta I) = \alpha W(T) + \beta = \{\alpha z + \beta : z \in W(T)\}$.
(2) $W(T + S) \subset W(T) + W(S)$.
(3) $W(T^*) = \{\overline{z} : z \in W(T)\}$.
(4) $W(\Re(T)) = \Re W(T) = \{\Re z : z \in W(T)\}$ and $W(\Im(T)) = \Im W(T) = \{\Im z : z \in W(T)\}$.
(5) $W(U^*TU) = W(T)$.

Proof. *(1) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:*

$$\langle(\alpha T + \beta I)x, x\rangle = \alpha\langle Tx, x\rangle + \beta\langle x, x\rangle = \alpha\langle Tx, x\rangle + \beta \Longrightarrow W(\alpha T + \beta I) = \alpha W(T) + \beta$$

(2) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle(T + S)x, x\rangle = \langle Tx, x\rangle + \langle Sx, x\rangle \in W(T) + W(S) \Longrightarrow W(T + S) \subset W(T) + W(S)$$

(3) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \Longrightarrow W(T^*) = \{\overline{z} : z \in W(T)\}$$

(4) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, we know that $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2i}(T - T^*)$, then :

$$\langle\Re(T)x, x\rangle = \langle\frac{1}{2}(T + T^*)x, x\rangle = \frac{1}{2}\langle Tx, x\rangle + \frac{1}{2}\langle T^*x, x\rangle = \frac{1}{2}(\langle Tx, x\rangle + \overline{\langle Tx, x\rangle}) = \Re(\langle Tx, x\rangle)$$

Thus $W(\Re(T)) = \Re W(T)$, and proceeding the same the way, we obtain $W(\Im(T)) = \Im W(T)$.
(5) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, and since U is unitary, we have $\|Ux\| = \|x\| = 1$, then :

$$\langle U^*TUx, x\rangle = \langle TUx, Ux\rangle \in W(T) \Longrightarrow W(U^*TU) \subset W(T)$$

Since U is surjective, then for all $\|x\| = 1$, $\exists y \in \mathcal{H}$ such that $x = Uy$, we have $\|y\| = \|Uy\| = \|x\| = 1$, then:

$$\langle Tx, x\rangle = \langle TUy, Uy\rangle = \langle U^*TUy, y\rangle \in W(U^*TU) \Longrightarrow W(T) \subset W(U^*TU)$$

Hence $W(U^*TU) = W(T)$.

Corollary 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$, then:*
1) $W(T) = \{\alpha\}$ if and only if $T = \alpha I$.
2) If $\mathcal{H}$ is a finite dimensional space, then $\omega(T)$ is compact.

Proof. *1) If $T = \alpha I$, it is obvious that $W(T) = \{\alpha\}$.*
Assume that $W(T) = \{\alpha\} \Longrightarrow \forall x \in \mathcal{H}$ where $\|x\| = 1$: $\quad \langle Tx, x\rangle = \alpha$.
Then $\quad \forall x \in \mathcal{H} \quad \langle Tx, x\rangle = \alpha\|x\|^2 = \alpha\langle x, x\rangle \Longrightarrow \forall x \in \mathcal{H} \quad \langle(T - \alpha I)x, x\rangle = 0$.
Thus $\quad T = \alpha I$ as desired.
2) Since $\mathcal{H}$ is a finite dimensional space, then the unit sphere $\mathrm{S}_{\mathcal{H}}$ is compact. ($\mathrm{S}_{\mathcal{H}} = \{x \in \mathcal{H} : \quad \|x\| = 1\}$)
Define $f : \mathrm{S}_{\mathcal{H}} \longrightarrow \mathbb{C}$ by $f(x) = \langle Tx, x\rangle$, then f is continuous, since: $\forall x, y \in \mathrm{S}_{\mathcal{H}}$:

$$\begin{aligned}|f(x) - f(y)| = |\langle Tx, x\rangle - \langle Ty, y\rangle + \langle Ty, x\rangle - \langle Ty, x\rangle| &= |\langle Tx - Ty, x\rangle + \langle Ty, x - y\rangle| \\ &\leq |\langle Tx - Ty, x\rangle| + |\langle Ty, x - y\rangle| \\ &\leq \|T\|\|x - y\|\|x\| + \|T\|\|x - y\|\|y\| = 2\|T\|\|x - y\|\end{aligned}$$

Therefore f is continuous, and since $W(T) = f(\mathcal{S}_{\mathcal{H}})$, then $W(T)$ is compact.

Remark 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, then $W(T) \subset \mathbb{R}$.(by Theorem 2.2)*

Example 3.2. *Let $T=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $W(T) = [0,1]$*

Theorem 3.8. *(Toeplitz-Hausdorff Theorem)*
The numerical range $W(T)$ of any operator T is a convex set in $\mathbb{C}$.

Proof. *Let $\xi = \langle Tx, x\rangle$ and $\eta = \langle Ty, y\rangle$ for any $\|x\| = \|y\| = 1$. We want to prove that the segment joining ξ and η is in $W(T)$. If $\xi = \eta$, then the segment is ξ and $\xi \in W(T)$.*
Suppose that $\xi \neq \eta$, choose complex numbers a, b such that $a\xi + b = 1$ and $a\eta + b = 0$. Indeed $a = \frac{1}{\xi-\eta}$ and $b = \frac{-\eta}{\xi-\eta}$ are the desired numbers. By Proposition 3.3, we have $W(aT + bI) = aW(T) + b$.
Consequently $\{0,1\} \in W(aT + bI)$ (since $a\xi + b = 1$ and $a\eta + b = 0$ where $\xi, \eta \in W(T)$).
Then we have the following:

$$\forall t \in [0,1] : \qquad t\xi + (1-t)\eta \in W(T) \Longleftrightarrow t \in W(aT + bI)$$

- *Let $t\xi + (1-t)\eta \in W(T)$, then $\exists z \in \mathcal{H}$ where $\|z\| = 1$ such that $\langle Tz, z\rangle = t\xi + (1-t)\eta$, therefore:*

$$\begin{aligned} a\langle Tz, z\rangle + b = ta\xi + (1-t)a\eta + b = t(a\xi + b) + (1-t)(a\eta + b) = t \\ \Longrightarrow t = a\langle Tz, z\rangle + b \end{aligned}$$

Hence $t \in W(aT + bI)$.
- *Let $t \in W(aT + bI)$, then $\exists z \in \mathcal{H}$ where $\|z\| = 1$ such that $t = a\langle Tz, z\rangle + b$, therefore:*

$$\begin{aligned} t = \tfrac{1}{\xi-\eta}\langle Tz, z\rangle - \tfrac{\eta}{\xi-\eta} \Longrightarrow \langle Tz, z\rangle = t(\xi - \eta) + \eta = t\xi + (1-t)\eta \\ \Longrightarrow t\xi + (1-t)\eta = \langle Tz, z\rangle \end{aligned}$$

Hence $t\xi + (1-t)\eta \in W(T)$.
Then to prove that $t\xi + (1-t)\eta \in W(T)$ for all $t \in [0,1]$, it is enough to prove that $[0,1] \subset W(aT + bI)$.
Set $S = aT + bI$, then:

$$\langle Sx, x\rangle = a\langle Tx, x\rangle + b = a\xi + b = 1$$

Therefore $\langle Sx, x\rangle = 1$, and the same $\langle Sy, y\rangle = 0$. Define $g : \mathbb{R} \longrightarrow \mathbb{C}$, by:

$$g(\theta) = \langle Sy, x\rangle e^{-i\theta} + \langle Sx, y\rangle e^{i\theta}$$

It is obvious that g is continuous. Let $\theta \in \mathbb{R}$, and using $e^{-i(\theta+\pi)} = -e^{-i\theta}$ and $e^{i(\theta+\pi)} = -e^{i\theta}$, we get :

$$g(\theta + \pi) = \langle Sy, x\rangle e^{-i(\theta+\pi)} + \langle Sx, y\rangle e^{i(\theta+\pi)} = -\langle Sy, x\rangle e^{-i\theta} - \langle Sx, y\rangle e^{i\theta} = -g(\theta)$$

Then $g(\theta + \pi) = -g(\theta)$ for all $\theta \in \mathbb{R}$, which means that $g(\pi) = -g(0)$, then $\Im g(\pi) = -\Im g(0)$.
Since $\Im g$ is continuous, and the sign of $\Im g(0)$ and $\Im g(\pi)$ are different (because $\Im g(\pi) = -\Im g(0)$), by intermediate value theorem, there exists $\theta_0 \in [0, \pi]$ such that $\Im g(\theta_0) = 0$.
Put $\hat{x} = e^{i\theta_0}x$. $\hat{x}$ and y are linearly independent, if not, $\exists \alpha \in \mathbb{C}$ such that $y = \alpha\hat{x}$ where $|\alpha| = 1$ (since $\|y\| = 1$).
We have $0 = \langle Sy, y\rangle = |\alpha|^2\langle S\hat{x}, \hat{x}\rangle = |\alpha|^2\langle Sx, x\rangle = 1$, contradiction.
Then, $\hat{x}$ and y are linearly independent. Therefore $\|(1-t)y + t\hat{x}\| \neq 0$ for all $t \in [0,1]$. Put

$$\varphi(t) = \frac{(1-t)y + t\hat{x}}{\|(1-t)y + t\hat{x}\|} \qquad \textit{for all } t \in [0,1]$$

Clearly $\|\varphi(t)\| = 1$ *for all* $t \in [0,1]$. *Furthermore* $\langle S\varphi(0), \varphi(0)\rangle = 0$ *and* $\langle S\varphi(1), \varphi(1)\rangle = 1$.
Define $f(t) = \langle S\varphi(t), \varphi(t)\rangle$ *for all* $t \in [0,1]$. *Set* $\beta = \|(1-t)y + t\hat{x}\|$, *using the fact that* $g(\theta_0) \in \mathbb{R}$, *we get:*

$$\begin{aligned} f(t) = \langle S\varphi(t), \varphi(t)\rangle &= \frac{(1-t)^2}{\beta^2}\langle Sy, y\rangle + \frac{t^2}{\beta^2}\langle S\hat{x}, \hat{x}\rangle + \frac{t(1-t)}{\beta^2}(\langle Sy, \hat{x}\rangle + \langle S\hat{x}, y\rangle) \\ &= \frac{t^2}{\beta^2} + \frac{t(1-t)}{\beta^2}(e^{-i\theta_0}\langle Sy, x\rangle + e^{i\theta_0}\langle Sx, y\rangle) \\ &= \frac{t^2}{\beta^2} + \frac{t(1-t)}{\beta^2} g(\theta_0) \in \mathbb{R} \qquad \textit{for all } t \in [0,1] \end{aligned}$$

Thus f *is a real-valued function. Moreover* f *is clearly continuous and* $f([0,1]) \subset W(S)$. *(since* $\|\varphi(t)\| = 1$*)*
Since $f(0) = 0$, $f(1) = 1$ *and* f *is continuous, then* $[0,1] \subset f([0,1])$ *(by intermediate value theorem).*
And since $f([0,1]) \subset W(S)$, *thus* $[0,1] \subset W(S)$.
Hence $t\xi + (1-t)\eta \in W(T)$ *for all* $t \in [0,1]$, *therefore* $W(T)$ *is convex.*

Proposition 3.6. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then:*
(1) $\sigma_p(T), \sigma_r(T) \subset W(T)$.
(2) $\sigma_{ap}(T) \subset \overline{W(T)}$.

Proof. *(1) Let* $\lambda \in \sigma_p(T)$, *then there exists* $x \in \mathcal{H}$ *where* $\|x\| = 1$ *such that* $Tx = \lambda x$, *then:*

$$\lambda = \lambda\|x\|^2 = \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle \implies \lambda = \langle Tx, x\rangle \implies \lambda \in W(T)$$

Therefore $\sigma_p(T) \subset W(T)$.
Let $\lambda \in \sigma_r(T)$, *then* $\overline{R(T-\lambda I)} \neq \mathcal{H}$, *since* $\mathcal{H} = \overline{R(T-\lambda I)} \oplus (R(T-\lambda I))^{\perp}$, *thus* $(R(T-\lambda I))^{\perp} \neq \{0\}$.
Since $N(T^* - \overline{\lambda} I) = (R(T-\lambda I))^{\perp} \neq \{0\}$, *then* $\overline{\lambda} \in \sigma_p(T^*)$, *thus* $\overline{\lambda} \in W(T^*)$ *(*$\sigma_p(T^*) \subset W(T^*)$*).*
Hence $\lambda \in W(T)$ *(by Proposition 3.3), therefore* $\sigma_r(T) \subset W(T)$.
(2) Let $\lambda \in \sigma_{ap}(T)$, *then* $\exists (x_n) \subset \mathcal{H}$ *such that* $\|x_n\| = 1$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty}(T-\lambda I)x_n = 0$, *then:*

$$|\langle Tx_n, x_n\rangle - \lambda| = |\langle Tx_n, x_n\rangle - \lambda\langle x_n, x_n\rangle| = |\langle (T-\lambda I)x_n, x_n\rangle| \le \|(T-\lambda I)x_n\|\|x_n\| = \|(T-\lambda I)x_n\| \underset{n\to\infty}{\longrightarrow} 0$$

Thus $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda$. *Since* $(\langle Tx_n, x_n\rangle) \subset W(T)$, *then* $\lambda \in \overline{W(T)}$.
Hence $\sigma_{ap}(T) \subset \overline{W(T)}$.

Corollary 3.9. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then* $\sigma(T) \subset \overline{W(T)}$.

Proof. *Since* $\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$, *and we have that* $\sigma_{ap}(T) \subset \overline{W(T)}$, *and* $\sigma_r(T) \subset W(T)$.
Then $\sigma(T) \subset \overline{W(T)}$.
Another way, we have that $\partial\sigma(T) \subset \sigma_{ap}(T) \subset \overline{W(T)}$, *and since* $\overline{W(T)}$ *is convex (the closure of convex is convex), then* $\sigma(T) \subset \overline{W(T)}$.

Theorem 3.9. *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $\alpha, \beta \in \mathbb{C}$ *such that* $\alpha \neq 0$. *Then :*
$W(T)$ *is a line segment i.e.* $W(T) = \{\alpha t + \beta : t \in \mathbb{R}\}$ *if and only if* $T = \alpha S + \beta I$, *where* S *is self-adjoint.*

Proof. • *Assume that* $W(T)$ *is a line segment i.e.* $W(T) = \{\alpha t + \beta : t \in \mathbb{R}\}$.
Since $\alpha^{-1}(\alpha t + \beta) - \alpha^{-1}\beta$ *is real for all* $t \in \mathbb{R}$. *Set* $S = \alpha^{-1}T - \alpha^{-1}\beta I$, *then for any* $\|x\| = 1$, *we have :*

$$\langle Sx, x\rangle = \alpha^{-1}\langle Tx, x\rangle - \alpha^{-1}\beta = \alpha^{-1}(\alpha t + \beta) - \alpha^{-1}\beta, \textit{ for some } t \in \mathbb{R}, \implies \langle Sx, x\rangle \in \mathbb{R} \textit{ for all } \|x\| = 1$$

Then S *is self-adjoint. Since* $S = \alpha^{-1}T - \alpha^{-1}\beta I$, *thus* $T = \alpha S + \beta I$.
• *Now suppose that* $T = \alpha S + \beta I$ *for some self-adjoint operator* S.
Let $\|x\| = 1$, *since* S *is self-adjoint, then* $\langle Sx, x\rangle \in \mathbb{R}$, *so that:*
$\langle Tx, x\rangle = \alpha\langle Sx, x\rangle + \beta \in \{\alpha t + \beta : t \in \mathbb{R}\}$, *hence* $W(T) = \{\alpha t + \beta : t \in \mathbb{R}\}$.

Corollary 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$. If $W(T)$ is a line segment, then T is normal.*

Proof. *By Theorem 3.9, there exist $\alpha, \beta \in \mathbb{C}$ and self-adjoint operator S such that $T = \alpha S + \beta I$. Then $T^* = \overline{\alpha} S + \overline{\beta} I$, therefore:*

$$TT^* = (\alpha S + \beta I)(\overline{\alpha} S + \overline{\beta} I) = |\alpha|^2 S + |\beta|^2 I + \alpha\overline{\beta} S + \beta\overline{\alpha} S$$
$$T^*T = (\overline{\alpha} S + \overline{\beta} I)(\alpha S + \beta I) = |\alpha|^2 S + |\beta|^2 I + \alpha\overline{\beta} S + \beta\overline{\alpha} S$$

Hence $TT^* = T^*T$.

3.4 Numerical radius

Definition 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical radius of T is denoted by $\omega(T)$, and it is defined as follows :*

$$\omega(T) = \sup_{\lambda \in W(T)} |\lambda| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Proposition 3.7. *The numerical radius ω defines a norm on $\mathcal{B}(\mathcal{H})$.*

Proof. • *If $T = 0$, it is clear that $\omega(T) = 0$. Now assume that $\omega(T) = 0$, then:*

$$\forall x \in \mathcal{H} \text{ where } \|x\| = 1 : \langle Tx, x\rangle = 0 \Longrightarrow \forall x \in \mathcal{H} \quad \langle Tx, x\rangle = 0.$$

Then $T = 0$ (by Theorem 1.9). Hence $T = 0 \Longleftrightarrow \omega(T) = 0$.

• *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$. Using the fact that $W(\alpha T) = \alpha W(T)$, we get:*

$$\omega(\alpha T) = \sup_{\lambda \in W(\alpha T)} |\lambda| = \sup_{\lambda \in W(T)} |\alpha\lambda| = |\alpha| \sup_{\lambda \in W(T)} |\lambda| = |\alpha|\omega(T)$$

Thus $\omega(\alpha T) = |\alpha|\omega(T)$.

• *Let $T, S \in \mathcal{B}(\mathcal{H})$. Using the fact that $W(T + S) \subset W(T) + W(S)$, we obtain:*

$$\omega(T+S) = \sup_{\lambda \in W(T+S)} |\lambda| \leq \sup_{\lambda \in W(T)+W(S)} |\lambda| = \sup_{\lambda_1 \in W(T), \lambda_2 \in W(S)} |\lambda_1 + \lambda_2| \leq \sup_{\lambda_1 \in W(T)} |\lambda_1| + \sup_{\lambda_2 \in W(S)} |\lambda_2|$$

Then $\quad \omega(T + S) \leq \omega(T) + \omega(S)$.

Therefore ω is a norm on $\mathcal{B}(\mathcal{H})$.

Proposition 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{3.1}$$

Proof. • *Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:*

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$$

Thus $\sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\|$, therefore $\omega(T) \leq \|T\|$.

• *Let $x, y \in \mathcal{H}$, then:*

$$|\langle T\frac{x}{\|x\|}, \frac{x}{\|x\|}\rangle| \leq \omega(T) \Longrightarrow |\langle Tx, x\rangle| \leq \omega(T)\|x\|^2$$

We have that $\langle Tx, y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle)$.

Therefore

$$\begin{aligned}|\langle Tx, y\rangle| &\leq \frac{1}{4}(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| + |\langle T(x+iy), x+iy\rangle| + |\langle T(x-iy), x-iy\rangle|)\\ &\leq \frac{\omega(T)}{4}(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2)\\ &= \omega(T)(\|x\|^2 + \|y\|^2)\end{aligned}$$

Thus $|\langle Tx, y\rangle| \leq \omega(T)(\|x\|^2 + \|y\|^2)$ *for all* $x, y \in \mathcal{H}$*, then:*

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| \leq \sup_{\|x\|=\|y\|=1} \omega(T)(\|x\|^2 + \|y\|^2) = 2\omega(T)$$

Hence $\|T\| \leq 2\omega(T)$.

Remark 3.5. *By* (3.1)*, we deduce that usual norm on* $\mathcal{B}(\mathcal{H})$ *and the numerical radius* ω *are equivalent norms.*

Corollary 3.11. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then:* $\quad r(T) \leq \omega(T)$.

Proof. *By Corollary 3.9, we have* $\sigma(T) \subset \overline{W(T)}$*. Then :*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \leq \sup_{\lambda \in \overline{W(T)}} |\lambda| = \sup_{\lambda \in W(T)} |\lambda| = \omega(T)$$

Thus $r(T) \leq \omega(T)$.

Corollary 3.12. *Let* $T \in \mathcal{B}(\mathcal{H})$*. If* $r(T) = \|T\|$*, then* $\omega(T) = \|T\|$.

Proof. *Since* $r(T) = \|T\|$*. And we have that* $r(T) \leq \omega(T) \leq \|T\|$*, therefore* $\omega(T) = r(T) = \|T\|$.

Remark 3.6. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be normal. Then* $\omega(T) = r(T) = \|T\|$*.* ($r(T) = \|T\|$ *by Corollary 3.7*)

Theorem 3.10. *Let* $T \in \mathcal{B}(\mathcal{H})$*. If* $\omega(T) = \|T\|$*, then* $r(T) = \|T\|$.

Proof. *Since* $\omega(T) = \|T\|$*, then* $\exists (x_n) \subset \mathcal{H}$ *where* $\|x_n\| = 1$ *such that* $\lim_{n\to\infty} |\langle Tx_n, x_n\rangle| = \|T\|$.
Set $\lambda = \lim_{n\to\infty} \langle Tx_n, x_n\rangle$*, so* $|\lambda| = \|T\|$*, then:*

$$|\langle Tx_n, x_n\rangle| \leq \|Tx_n\|\|x_n\| = \|Tx_n\| \leq \|T\|$$

Thus $\quad \lim_{n\to\infty} \|Tx_n\| = \|T\|$*, therefore:*

$$\begin{aligned} \|(T-\lambda I)x_n\|^2 = \langle (T-\lambda I)x_n, (T-\lambda I)x_n\rangle &= \|Tx_n\|^2 + |\lambda|^2 - \overline{\lambda}\langle Tx_n, x_n\rangle - \lambda\langle x_n, Tx_n\rangle \\ \Longrightarrow \|(T-\lambda I)x_n\|^2 &\underset{n\to\infty}{\longrightarrow} 2\|T\|^2 - 2|\lambda|^2 = 0 \\ \Longrightarrow \lim_{n\to\infty} \|(T-\lambda I)x_n\| &= 0 \end{aligned}$$

Therefore $\lambda \in \sigma_{ap}(T)$*, then* $\lambda \in \sigma(T)$*, and since* $r(T) \leq \|T\|$ *and* $|\lambda| = \|T\|$*, hence* $r(T) = \|T\|$ *as required.*

Proposition 3.9. *Let* $T \in \mathcal{B}(\mathcal{H})$*. If* $\lambda \in W(T)$ *such that* $|\lambda| = \|T\|$*, then* $\lambda \in \sigma_p(T)$.

Proof. *Since* $\lambda \in W(T)$*, then* $\exists x \in \mathcal{H}$ *where* $\|x\| = 1$ *and* $\lambda = \langle Tx, x\rangle$*. Since* $\|T\| = |\langle Tx, x\rangle|$*, then:*

$$|\lambda| = |\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\| = |\langle Tx, x\rangle| \Longrightarrow |\langle Tx, x\rangle| = \|Tx\|\|x\|$$

Then Tx *and* x *are linearly dependent i.e.* $\exists \mu \in \mathbb{C}$ *such that* $Tx = \mu x$*, then:*

$$\lambda = \langle Tx, x\rangle = \langle \mu x, x\rangle = \mu \Longrightarrow \lambda = \mu \Longrightarrow Tx = \lambda x$$

Hence $\lambda \in \sigma_p(T)$ *as desired.*

Lemma 3.3. *[29] Let* $T \in \mathcal{B}(\mathcal{H})$*,* $x \in \mathcal{H}$ *such that* $\|x\| = 1$ *and* $n \in \mathbb{N}$*. Then :*

$$1 - \langle T^n x, x\rangle = \frac{1}{n}\sum_{j=1}^{n} \|x_j\|^2 \left(1 - u_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle\right)$$

Where $u_j = e^{\frac{2\pi j i}{n}}$ *and* $x_j = (\prod_{\substack{k=1 \\ k\neq j}}^{n} (1 - u_k T))x$ *for all* $j \in \{1, \cdots, n\}$.

Proof. *We have these two well-known polynomial identities :*

$$1 - z^n = \prod_{k=1}^{n}(1 - u_k z) \quad \text{and} \quad 1 = \frac{1}{n}\sum_{j=1}^{n}\prod_{\substack{k=1\\k\neq j}}^{n}(1 - u_k z)$$

Since these identities remain valid when we replace z by T, then we have :

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\|x_j\|^2(1 - u_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle) &= \frac{1}{n}\sum_{j=1}^{n}\langle (I - u_jT)x_j, x_j\rangle \\ &= \frac{1}{n}\sum_{j=1}^{n}\langle (I - T^n)x, x_j\rangle \\ &= \langle (I - T^n)x, \frac{1}{n}\sum_{j=1}^{n}\prod_{\substack{k=1\\k\neq j}}^{n}(1 - u_kT)x\rangle \\ &= \langle (I - T^n)x, x\rangle = 1 - \langle T^n x, x\rangle\end{aligned}$$

As desired.

Theorem 3.11. *[29](Power inequality) Let $T \in \mathcal{B}(\mathcal{H})$, and $n \in \mathbb{N}$. Then :*

$$\omega(T^n) \leq (\omega(T))^n$$

Proof. *By the homogeneity of ω, it suffices to prove that if $\omega(T) \leq 1$, then $\omega(T^n) \leq 1$. Since:*

$$\omega(\frac{1}{\omega(T)}T) = \frac{\omega(T)}{\omega(T)} = 1 \Longrightarrow \omega((\frac{1}{\omega(T)}T)^n) = \omega(\frac{1}{(\omega(T))^n}T^n) = \frac{1}{(\omega(T))^n}\omega(T^n) \leq 1$$

Then $\quad \omega(T^n) \leq (\omega(T))^n$.
Assume that $\omega(T) \leq 1$. Let $x \in \mathcal{H}$ where $\|x\| = 1$ and $z \in \mathbb{C}$ such that $|z| \leq 1$, then :

$$\Re(1 - z\langle Tx, x\rangle) = 1 - \Re(z\langle Tx, x\rangle) \geq 1 - |z\langle Tx, x\rangle| \geq 1 - |z|\omega(T) \geq 1 - |z| \geq 0$$

Then $\Re(1 - z\langle Tx, x\rangle) \geq 0$. If $\Re(1 - z^n\langle T^n x, x\rangle) \geq 0$, then $\omega(T^n) \leq 1$, since:

$$\Re(1 - z^n\langle T^n x, x\rangle) \geq 0 \Longrightarrow \Re(z^n\langle T^n x, x\rangle) \leq 1$$

Choose $z^n = \frac{\overline{\langle T^n x, x\rangle}}{|\langle T^n x, x\rangle|}$, note that $|z| = 1$, then:

$$\Re(\frac{\overline{\langle T^n x, x\rangle}}{|\langle T^n x, x\rangle|}\langle T^n x, x\rangle) \leq 1 \Longrightarrow |\langle T^n x, x\rangle| \leq 1 \Longrightarrow \omega(T^n) \leq 1$$

Therefore to prove $\omega(T^n) \leq 1$, it is enough to prove $\Re(1 - z^n\langle T^n x, x\rangle) \geq 0$ for all $|z| \leq 1$.
By Lemma 3.3, we have $1 - z^n\langle T^n x, x\rangle = 1 - \langle (zT)^n x, x\rangle = \frac{1}{n}\sum_{j=1}^{n}\|x_j\|^2(1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle)$. Then:

$$\begin{aligned}\Re(1 - z^n\langle T^n x, x\rangle) &= \Re(\frac{1}{n}\sum_{j=1}^{n}\|x_j\|^2(1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle)) \\ &= \frac{1}{n}\sum_{j=1}^{n}\|x_j\|^2\Re(1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle)\end{aligned}$$

Since $|zu_j| = |e^{\frac{2\pi ji}{n}}z| = |z| \leq 1$ for all $j \in \{1, \cdots, n\}$, and we have that $\Re(1 - z\langle Tx, x\rangle) \geq 0$.
Then $\Re(1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle) \geq 0$ for all $j \in \{1, \cdots, n\}$, therefore $\Re(1 - z^n\langle T^n x, x\rangle) \geq 0$.
Hence $\omega(T^n) \leq 1$.

Proposition 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$. If $R(T) \perp R(T^*)$, then $\omega(T) = \frac{1}{2}\|T\|$.*

Proof. *We have that $\mathcal{H} = N(T) \oplus (N(T))^\perp$, and since $(N(T))^\perp = \overline{R(T^*)}$, then $\mathcal{H} = N(T) \oplus \overline{R(T^*)}$.*
Since $R(T) \perp R(T^*)$, we can easily prove that $R(T) \perp \overline{R(T^*)}$.
Let $x \in \mathcal{H}$ where $\|x\| = 1$, then $\exists x_1 \in N(T)$ and $\exists x_2 \in \overline{R(T^*)}$ such that $x = x_1 + x_2$ and $\|x_1\|^2 + \|x_2\|^2 = 1$.
Using the fact that $x_1 \in N(T)$ and $R(T) \perp \overline{R(T^*)}$, we get:

$$\begin{aligned}\langle Tx, x\rangle = \langle T(x_1 + x_2), x_1 + x_2\rangle &= \langle Tx_2, x_1\rangle \\ &\leq \|T\|\|x_1\|\|x_2\| \\ &\leq \frac{\|T\|}{2}(\|x_1\|^2 + \|x_2\|^2) \\ &= \frac{1}{2}\|T\|\end{aligned}$$

Thus $\omega(T) \leq \frac{1}{2}\|T\|$, and we always have $\frac{1}{2}\|T\| \leq \omega(T)$ by (3.1).
Hence $\omega(T) = \frac{1}{2}\|T\|$.

Theorem 3.12. *[11] Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{3.2}$$

Proof. *Let $\|x\| = 1$, using Theorem 2.14, we get:*

$$\begin{aligned}|\langle Tx, x\rangle| &\leq \sqrt{\langle |T|x, x\rangle}\sqrt{\langle |T^*|x, x\rangle} \\ &\leq \frac{1}{2}(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle) \\ &= \frac{1}{2}\langle (|T| + |T^*|)x, x\rangle \\ &\leq \frac{1}{2}\||T| + |T^*|\|\end{aligned}$$

Thus $\omega(T) \leq \frac{1}{2}\||T| + |T^*|\|$.

Remark 3.7. (3.2) *refines the right side of* (3.1)*, since:*

$$\frac{1}{2}\||T| + |T^*|\| \leq \frac{1}{2}(\||T|\| + \||T^*|\|) = \frac{1}{2}(\|T\| + \|T\|) = \|T\|$$

Lemma 3.4. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be positive operators. Then :*

$$r(TS) = \|\sqrt{T}\sqrt{S}\|^2$$

Proof. *By the commutativity of the spectral radius r, and using the fact that the spectral radius of any self-adjoint operator is equal its the norm (by Corollary 3.7), we obtain :*

$$r(TS) = r(\sqrt{T}\sqrt{T}\sqrt{S}\sqrt{S}) = r(\sqrt{T}\sqrt{S}\sqrt{S}\sqrt{T}) = r(\sqrt{T}\sqrt{S}(\sqrt{T}\sqrt{S})^*) = \|\sqrt{T}\sqrt{S}(\sqrt{T}\sqrt{S})^*\| = \|\sqrt{T}\sqrt{S}\|^2$$

Hence $r(TS) = \|\sqrt{T}\sqrt{S}\|^2$.

Corollary 3.13. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be positive operators. Then :*

$$\|\sqrt{T}\sqrt{S}\| \leq \sqrt{\|TS\|}$$

Proof. *By Lemma 3.4, we have $\|\sqrt{T}\sqrt{S}\| = \sqrt{r(TS)} \leq \sqrt{\|TS\|}$ as desired.*

Proposition 3.11. *[20] Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(T) \leq \frac{1}{2}(\|T\| + \sqrt{\|T^2\|}) \tag{3.3}$$

Proof. *By (3.2), we have:*

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\|$$

And by (2.13), we have:

$$\||T| + |T^*|\| \leq \max\{\||T|\|, \||T^*|\|\} + \|\sqrt{|T|}\sqrt{|T^*|}\|$$

Since $\||T|\| = \||T^*|\| = \|T\|$, then $\max\{\||T|\|, \||T^*|\|\} = \|T\|$.
By Corollary 3.13, we have $\|\sqrt{|T|}\sqrt{|T^*|}\| \leq \sqrt{\||T||T^*|\|}$,and by Proposition 2.11 $\||T||T^*|\| = \|T^2\|$.
Then $\|\sqrt{|T|}\sqrt{|T^*|}\| \leq \sqrt{\|T^2\|}$, *therefore:*

$$\||T| + |T^*|\| \leq \|T\| + \sqrt{\|T^2\|}$$

Hence $\omega(T) \leq \frac{1}{2}(\|T\| + \sqrt{\|T^2\|})$.

Corollary 3.14. *Let $T \in \mathcal{B}(\mathcal{H})$. Then :*
(1) If $T^2 = 0$, then $\omega(T) = \frac{1}{2}\|T\|$.
(2) If $\omega(T) = \|T\| \implies \|T^2\| = \|T\|^2$.

Proof. *(1) By (3.3), we have:*

$$\omega(T) \leq \frac{1}{2}(\|T\| + \sqrt{\|T^2\|})$$

Since $T^2 = 0$, then $\omega(T) \leq \frac{1}{2}\|T\|$, and $\frac{1}{2}\|T\| \leq \omega(T)$ is always true.
Hence $\omega(T) = \frac{1}{2}\|T\|$.
(2) Since $\omega(T) \leq \frac{1}{2}(\|T\| + \sqrt{\|T^2\|}) \leq \|T\|$, and we have that $\omega(T) = \|T\|$.
Then $\frac{1}{2}(\|T\| + \sqrt{\|T^2\|}) = \|T\| \implies \sqrt{\|T^2\|} = \|T\|$.
Therefore $\|T^2\| = \|T\|^2$.

Theorem 3.13. *[21] Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\| \tag{3.4}$$

Proof. • *Let $x \in \mathcal{H}$ where $\|x\| = 1$. Using Theorem 2.14, we get:*

$$\begin{aligned} |\langle Tx, x\rangle|^2 \leq& \langle |T|x, x\rangle \langle |T^*|x, x\rangle \\ \leq& \frac{1}{2}((\langle |T|x, x\rangle)^2 + (\langle |T^*|x, x\rangle)^2) \\ \leq& \frac{1}{2}(\||T|x\|^2 + \||T^*|x\|^2) \\ =& \frac{1}{2}(\langle |T|^2x, x\rangle + \langle |T^*|^2x, x\rangle) \\ =& \frac{1}{2}\langle (T^*T + TT^*)x, x\rangle \\ \leq& \frac{1}{2}\|T^*T + TT^*\| \end{aligned}$$

Hence

$$\omega(T)^2 \leq \frac{1}{2}\|T^*T + TT^*\|$$

• *$T = \Re(T) + i\Im(T)$ such that $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2i}(T - T^*)$, and by simple calculation, we get that $T^*T + TT^* = 2(\Re(T)^2 + \Im(T)^2)$. Let $x \in \mathcal{H}$ where $\|x\| = 1$, since $\Re(T)$ and $\Im(T)$ are self-adjoint, and using the convexity of the square function, we obtain:*

$$|\langle Tx, x\rangle|^2 = \langle \Re(T)x, x\rangle^2 + \langle \Im(T)x, x\rangle^2 \geq \frac{1}{2}(\langle \Re(T)x, x\rangle + \langle \Im(T)x, x\rangle)^2 \geq \frac{1}{2}(\langle \Re(T) \pm \Im(T)x, x\rangle)^2$$
$$\Longrightarrow \omega(T)^2 = \sup_{\|x\|=1} |\langle Tx, x\rangle|^2 \geq \frac{1}{2} \sup_{\|x\|=1} (\langle \Re(T) \pm \Im(T)x, x\rangle)^2 = \frac{1}{2}\|\Re(T) \pm \Im(T)\|^2 = \frac{1}{2}\|(\Re(T) \pm \Im(T))^2\|$$

Therefore

$$\begin{aligned} 2(\omega(T))^2 \geq& \frac{1}{2}\|(\Re(T) + \Im(T))^2\| + \frac{1}{2}\|(\Re(T) - \Im(T))^2\| \\ \geq& \frac{1}{2}\|(\Re(T) + \Im(T))^2 + (\Re(T) - \Im(T))^2\| \\ =& \|\Re(T)^2 + \Im(T)^2\| = \frac{1}{2}\|T^*T + TT^*\| \end{aligned}$$

Thus

$$\frac{1}{4}\|T^*T + TT^*\| \leq (\omega(T))^2$$

Remark 3.8. *(3.4) improves (3.1), since:*

$$\frac{1}{2}\|T^*T + TT^*\| \leq \frac{1}{2}(\|T^*T\| + \|TT^*\|) = \frac{1}{2}(\|T\|^2 + \|T\|^2) = \|T\|^2$$
$$\|Tx\|^2 = \langle T^*Tx, x\rangle \leq \langle T^*Tx, x\rangle + \langle TT^*x, x\rangle = \langle (T^*T + TT^*)x, x\rangle$$

Then $\|T\|^2 \leq \|T^*T + TT^*\|$.

Theorem 3.14. *[9] Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(T)^2 \leq \frac{1}{2}(w(T^2) + \|T\|^2) \tag{3.5}$$

Proof. *Let $x \in \mathcal{H}$ such that $\|x\| = 1$. Using Theorem 1.3, we obtain:*

$$\begin{aligned} |\langle Tx, x\rangle|^2 = |\langle Tx, x\rangle\langle x, T^*x\rangle| &\leq \frac{1}{2}(\|Tx\|\|T^*x\| + \langle Tx, T^*x\rangle) \\ &= \frac{1}{2}(\|Tx\|\|T^*x\| + \langle T^2x, x\rangle) \\ &\leq \frac{1}{2}(\|T\|^2 + \omega(T^2)) \end{aligned}$$

Therefore

$$\omega(T)^2 \leq \frac{1}{2}(w(T^2) + \|T\|^2)$$

Theorem 3.15. *[1] Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(T)^2 \leq \frac{1}{2}\omega(T^2) + \frac{1}{4}\|T^*T + TT^*\| \tag{3.6}$$

Proof. *Let $x \in \mathcal{H}$ such that $\|x\| = 1$. Using Theorem 1.3, we obtain:*

$$\begin{aligned} |\langle Tx, x\rangle|^2 = |\langle Tx, x\rangle\langle x, T^*x\rangle| &\leq \frac{1}{2}(\|Tx\|\|T^*x\| + \langle Tx, T^*x\rangle) \\ &= \frac{1}{2}(\|Tx\|\|T^*x\| + \langle T^2x, x\rangle) \\ &\leq \frac{1}{2}\left(\tfrac{1}{2}(\|Tx\|^2 + \|T^*x\|^2) + \langle T^2x, x\rangle\right) \\ &= \frac{1}{4}(\langle T^*Tx, x\rangle + \langle TT^*x, x\rangle) + \frac{1}{2}\langle T^2x, x\rangle \\ &\leq \frac{1}{2}\omega(T^2) + \frac{1}{4}\|T^*T + TT^*\| \end{aligned}$$

Thus

$$\omega(T)^2 \leq \frac{1}{2}\omega(T^2) + \frac{1}{4}\|T^*T + TT^*\|$$

Remark 3.9. *The inequality* (3.6) *improves both* (3.5) *and the right side of* (3.4). *Since:*

$$\frac{1}{2}\|T^*T + TT^*\| \leq \frac{1}{2}(\|T^*T\| + \|TT^*\|) = \|T\|^2$$

$$\frac{1}{2}\omega(T^2) \leq \frac{1}{2}\omega(T)^2 \leq \frac{1}{4}\|T^*T + TT^*\|$$

Theorem 3.16. *[2] Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right) \tag{3.7}$$

Proof. *Let $x \in \mathcal{H}$ where $\|x\| = 1$, using Theorem 2.14, Corollary 2.13 and Lemma 3.4, we obtain:*

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \langle |T|x, x\rangle^{\frac{1}{2}} \langle |T^*|x, x\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2}(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle) \\
&= \frac{1}{2}\langle (|T| + |T^*|)x, x\rangle \\
&\leq \frac{1}{2}\||T| + |T^*|\| \\
&\leq \frac{1}{2}(\|T\| + \|\sqrt{|T|}\sqrt{|T^*|}\|) \\
&= \frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right)
\end{aligned}$$

Thus

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right)$$

Remark 3.10. (3.7) *refines* (3.3), *since:*

$$\sqrt{r(|T||T^*|)} \leq \sqrt{\||T||T^*|\|} = \sqrt{\|T^2\|}$$

Corollary 3.15. *Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

(1) If $r(|T||T^*|) = 0$, then $\omega(T) = \frac{1}{2}\|T\|$.

(2) If $\omega(T) = \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right)$, then $r(|T||T^*|) = \|T^2\|$.

Proof. *(1) By* (3.7), *we have:*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right)$$

And since $r(|T||T^*|) = 0$, then $\omega(T) \leq \frac{1}{2}\|T\|$. And $\frac{1}{2}\|T\| \leq \omega(T)$ is always held.

Thus $\quad \omega(T) = \frac{1}{2}\|T\|$.

(2) We have that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right) \leq \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right)$$

Since $\omega(T) = \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right)$, then $\frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right) = \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right)$.

Therefore $\quad r(|T||T^*|) = \|T^2\|$.

Proposition 3.12. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then:*

(1) $\omega(TS) \leq 4\omega(T)\omega(S)$.

(2) If $TS = ST$, then $\omega(TS) \leq 2\omega(T)\omega(S)$.

(3) $\omega(TS) \leq \frac{1}{2}\|TT^* + S^*S\|$.

Proof. *(1)* $\omega(TS) \leq \|TS\| \leq \|T\|\|S\| \leq 2\omega(T)2\omega(S) = 4\omega(T)\omega(S).$

Hence $\omega(TS) \leq 4\omega(T)\omega(S).$

(2) By the homogeneity of w*, it suffices to prove if* $\omega(T) = 1$ *and* $\omega(S) = 1$*, then* $\omega(TS) \leq 2$*.*

Assume $T \neq 0$ *and* $S \neq 0$*, it is clear in case one of them equals* 0*, then:*

$$\omega(\tfrac{1}{\omega(T)}T) = 1 \text{ and } \omega(\tfrac{1}{\omega(S)}S) = 1 \Longrightarrow \omega(\tfrac{1}{\omega(T)\omega(S)}TS) \leq 2 \Longrightarrow \omega(TS) \leq 2\omega(T)\omega(S)$$

Assume that $\omega(T) = 1$ *and* $\omega(S) = 1$*, since* $TS = ST$*, then* $TS = \frac{1}{4}\Big((T+S)^2 - (T-S)^2\Big)$*. Therefore:*

$$\begin{aligned}\omega(TS) = \frac{1}{4}\omega((T+S)^2 - (T-S)^2) \leq& \frac{1}{4}\Big(\omega((T+S)^2) + \omega((T-S)^2)\Big)\\ \leq& \frac{1}{4}\Big(\omega((T+S))^2 + \omega((T-S))^2\Big)\\ \leq& \frac{1}{4}\Big((\omega(T)+\omega(S))^2 + (\omega(T)+\omega(S))^2\Big) = 2\end{aligned}$$

Thus $\omega(TS) \leq 2$*. Hence* $\omega(TS) \leq 2\omega(T)\omega(S)$*.*

(3) Let $x \in \mathcal{H}$ *such that* $\|x\| = 1$*, then:*

$$\begin{aligned}|\langle TSx, x\rangle| = |\langle Sx, T^*x\rangle| &\leq \|T^*x\|\|Sx\|\\ &\leq \frac{1}{2}(\|T^*x\|^2 + \|Sx\|^2)\\ &= \frac{1}{2}(\langle TT^*x, x\rangle + \langle S^*Sx, x\rangle)\\ &= \frac{1}{2}\langle (TT^* + S^*S)x, x\rangle\\ &\leq \frac{1}{2}\|TT^* + S^*S\|\end{aligned}$$

Thus

$$\omega(TS) \leq \frac{1}{2}\|TT^* + S^*S\|$$

Theorem 3.17. *[18] Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $T^*S = ST^*$*. Then:*

$$\omega(TS) \leq \|T\|\omega(S) \tag{3.8}$$

Corollary 3.16. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$*. Then:*

(1) If T *is normal, then* $\omega(TS) \leq \omega(T)\omega(S)$*.*

(2) If T *is unitary, then* $\omega(TS) \leq \omega(S)$*.*

Proof. *(1) Since* T *is normal and* $TS = ST \Longrightarrow T^*S = ST^*$ *(by Theorem 2.6).*

Then by (3.8)*, we have:*

$$\omega(TS) \leq \|T\|\omega(S)$$

Since T *is normal, then* $\omega(T) = \|T\|$*, hence*

$$\omega(TS) \leq \omega(T)\omega(S)$$

(2) Since T *is unitary* $\Longrightarrow$ T *is normal, then* $\omega(TS) \leq \omega(T)\omega(S)$*.*

And we have that $\omega(T) = \|T\| = 1$*. Thus* $\omega(TS) \leq \omega(S)$*.*

Lemma 3.5. *[23] Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $|T|S = S^*|T|$*. Then:*

$$|\langle TSx, y\rangle| \leq r(S)\|\sqrt{|T|}x\|\|\sqrt{|T^*|}y\| \quad \textit{for all } x, y \in \mathcal{H}$$

Proof. *By induction, let's prove that:*

$$|\langle TSx, y\rangle|^{2^n} \leq \langle |T|S^{2^n}x, x\rangle \langle |T|x, x\rangle^{2^{n-1}-1} \langle |T^*|y, y\rangle^{2^{n-1}} \text{ for all } n \in \mathbb{N} \text{ and } x, y \in \mathcal{H}$$

For $n = 1$, let's prove that $|\langle TSx, y\rangle|^2 \leq \langle |T|S^2x, x\rangle\langle |T^*|y, y\rangle$. Using Theorem 2.14 and $S^*|T| = |T|S$, we get:

$$\begin{aligned}
|\langle TSx, y\rangle|^2 &\leq \langle |T|Sx, Sx\rangle\langle |T^*|y, y\rangle = \langle S^*|T|Sx, x\rangle\langle |T^*|y, y\rangle \\
|\langle TSx, y\rangle|^2 &\leq \langle |T|S^2x, x\rangle\langle |T^*|y, y\rangle
\end{aligned}$$

Assume that $|\langle TSx, y\rangle|^{2^n} \leq \langle |T|S^{2^n}x, x\rangle\langle |T|x, x\rangle^{2^{n-1}-1}\langle |T^*|y, y\rangle^{2^{n-1}}$ is true.
And let's prove that $|\langle TSx, y\rangle|^{2^{n+1}} \leq \langle |T|S^{2^{n+1}}x, x\rangle\langle |T|x, x\rangle^{2^n-1}\langle |T^*|y, y\rangle^{2^n}$ is also true. Then:

$$\begin{aligned}
|\langle TSx, y\rangle|^{2^{n+1}} = (|\langle TSx, y\rangle|^{2^n})^2 &\leq (\langle |T|S^{2^n}x, x\rangle\langle |T|x, x\rangle^{2^{n-1}-1}\langle |T^*|y, y\rangle^{2^{n-1}})^2 \\
&= \langle |T|S^{2^n}x, x\rangle^2\langle |T|x, x\rangle^{2^n-2}\langle |T^*|y, y\rangle^{2^n}
\end{aligned}$$

By Theorem 2.14, we have:

$$\langle |T|S^{2^n}x, x\rangle^2 \leq \langle |T|S^{2^n}x, S^{2^n}x\rangle\langle |T|x, x\rangle$$

Therefore

$$\begin{aligned}
|\langle TSx, y\rangle|^{2^{n+1}} &\leq \langle |T|S^{2^n}x, S^{2^n}x\rangle\langle |T|x, x\rangle\langle |T|x, x\rangle^{2^n-2}\langle |T^*|y, y\rangle^{2^n} \\
&= \langle (S^{2^n})^*|T|S^{2^n}x, x\rangle\langle |T|x, x\rangle^{2^n-1}\langle |T^*|y, y\rangle^{2^n} \\
&= \langle |T|S^{2^n}S^{2^n}x, x\rangle\langle |T|x, x\rangle^{2^n-1}\langle |T^*|y, y\rangle^{2^n} \\
&= \langle |T|S^{2^{n+1}}x, x\rangle\langle |T|x, x\rangle^{2^n-1}\langle |T^*|y, y\rangle^{2^n}
\end{aligned}$$

Hence

$$|\langle TSx, y\rangle|^{2^{n+1}} \leq \langle |T|S^{2^{n+1}}x, x\rangle\langle |T|x, x\rangle^{2^n-1}\langle |T^*|y, y\rangle^{2^n}$$

Then $|\langle TSx, y\rangle|^{2^n} \leq \langle |T|S^{2^n}x, x\rangle\langle |T|x, x\rangle^{2^{n-1}-1}\langle |T^*|y, y\rangle^{2^{n-1}}$ *for all $n \in \mathbb{N}$ and $x, y \in \mathcal{H}$. Therefore:*

$$\begin{aligned}
|\langle TSx, y\rangle|^{2^n} &\leq \||T|\|\|S^{2^n}\|\|x\|^2\langle |T|x, x\rangle^{2^{n-1}-1}\langle |T^*|y, y\rangle^{2^{n-1}} \\
|\langle TSx, y\rangle| &\leq \||T|\|^{2^{-n}}\|S^{2^n}\|^{2^{-n}}\|x\|^{2^{1-n}}\langle |T|x, x\rangle^{2^{-1}-2^{-n}}\langle |T^*|y, y\rangle^{2^{-1}}
\end{aligned}$$

Since

$$\lim_{n\to\infty} \||T|\|^{2^{-n}}\|S^{2^n}\|^{2^{-n}}\|x\|^{2^{1-n}}\langle |T|x, x\rangle^{2^{-1}-2^{-n}}\langle |T^*|y, y\rangle^{2^{-1}} = r(S)\sqrt{\langle |T|x, x\rangle}\sqrt{\langle |T^*|y, y\rangle}$$

Then

$$|\langle TSx, y\rangle| \leq r(S)\sqrt{\langle |T|x, x\rangle}\sqrt{\langle |T^*|y, y\rangle} = r(S)\|\sqrt{|T|}x\|\|\sqrt{|T^*|}y\|$$

Hence

$$|\langle TSx, y\rangle| \leq r(S)\|\sqrt{|T|}x\|\|\sqrt{|T^*|}y\| \quad \text{for all } x, y \in \mathcal{H}.$$

Theorem 3.18. *[2] Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $|T|S = S^*|T|$. Then:*

$$\omega(TS) \leq \frac{r(S)}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right) \tag{3.9}$$

Proof. *Let $x \in \mathcal{H}$ such that $\|x\| = 1$, Using Lemma 3.5, Corollary 2.13 and Lemma 3.4, we obtain:*

$$\begin{aligned}
|\langle TSx, x\rangle| &\leq r(S)\|\sqrt{|T|}x\|\|\sqrt{|T^*|}x\| \\
&\leq \frac{r(S)}{2}(\|\sqrt{|T|}x\|^2 + \|\sqrt{|T^*|}x\|^2)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{r(S)}{2}(\langle |T|x,x\rangle + \langle |T^*|x,x\rangle)\\
&= \frac{r(S)}{2}\langle (|T|+|T^*|)x,x\rangle\\
&\leq \frac{r(S)}{2}\||T|+|T^*|\|\\
&\leq \frac{r(S)}{2}(\|T\| + \|\sqrt{|T|}\sqrt{|T^*|}\|)\\
&= \frac{r(S)}{2}(\|T\| + \sqrt{r(|T||T^*|)})
\end{aligned}
$$

Hence

$$\omega(TS) \leq \frac{r(S)}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right)$$

Theorem 3.19. *[13] Let $T, S, R \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(TS \pm RT) \leq 2\sqrt{2}\max\{\|S\|, \|R\|\}\omega(T) \tag{3.10}$$

Proof. *Let $x \in \mathcal{H}$ where $\|x\| = 1$, using (3.4), we obtain:*

$$
\begin{aligned}
|\langle (TS \pm RT)x,x\rangle| &\leq |\langle TSx,x\rangle| + |\langle RTx,x\rangle|\\
&= |\langle Sx,T^*x\rangle| + |\langle Tx,R^*x\rangle|\\
&\leq \|Sx\|\|T^*x\| + \|Tx\|\|R^*x\|\\
&\leq \|R\|\|Tx\| + \|S\|\|T^*x\|\\
&\leq \max\{\|S\|,\|R\|\}(\|Tx\| + \|T^*x\|)\\
&\leq \max\{\|S\|,\|R\|\}\sqrt{2(\|Tx\|^2 + \|T^*x\|^2)}\\
&= \sqrt{2}\max\{\|S\|,\|R\|\}\sqrt{\langle T^*Tx,x\rangle + \langle TT^*x,x\rangle}\\
&= \sqrt{2}\max\{\|S\|,\|R\|\}\sqrt{\langle (T^*T+TT^*)x,x\rangle}\\
&\leq \sqrt{2}\max\{\|S\|,\|R\|\}\sqrt{\|T^*T+TT^*\|}\\
&\leq 2\sqrt{2}\max\{\|S\|,\|R\|\}\omega(T)
\end{aligned}
$$

Thus

$$\omega(TS \pm RT) \leq 2\sqrt{2}\max\{\|S\|, \|R\|\}\omega(T)$$

Remark 3.11. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(TS \pm ST) \leq 2\sqrt{2}\|T\|\omega(T) \tag{3.11}$$

Proposition 3.13. *[21] Let $T, S \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(TS \pm ST) \leq \frac{1}{2}\|T^*T + TT^* + S^*S + SS^*\| \tag{3.12}$$

Proof. *Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:*

$$
\begin{aligned}
|\langle (TS \pm ST)x,x\rangle| &\leq |\langle TSx,x\rangle| + |\langle STx,x\rangle|\\
&\leq \|T^*x\|\|Sx\| + \|Tx\|\|S^*x\|\\
&\leq \frac{1}{2}(\|T^*x\|^2 + \|Sx\|^2 + \|Tx\|^2 + \|S^*x\|^2)\\
&= \frac{1}{2}(\langle TT^*x,x\rangle + \langle S^*Sx,x\rangle + \langle T^*Tx,x\rangle + \langle SS^*x,x\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\langle (TT^* + S^*S + T^*T + SS^*)x, x\rangle \\
&\leq \frac{1}{2}\|T^*T + TT^* + S^*S + SS^*\|
\end{aligned}
$$

Therefore

$$\omega(TS \pm ST) \leq \frac{1}{2}\|T^*T + TT^* + S^*S + SS^*\|$$

Definition 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $c(T)$ is called the Crawford number of T, and it is defined as follows:*

$$c(T) = \inf_{\|x\|=1} |\langle Tx, x\rangle|$$

Lemma 3.6. *[3] Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

$$\|T^*T + TT^*\| \leq 4\left(\omega(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}\right)$$

Proof. *We have that $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2i}(T - T^*)$.*
By a simple calculation, we get that $\frac{1}{2}\|T^*T + TT^*\| = \|\Re(T)^2 + \Im(T)^2\|$, then we have:

$$
\begin{aligned}
\frac{1}{4}\|T^*T + TT^*\| &= \frac{1}{2}\|\Re(T)^2 + \Im(T)^2\| \\
&\leq \frac{1}{2}(\|\Re(T)\|^2 + \|\Im(T)\|^2) \\
&= \frac{1}{8}(\|T + T^*\|^2 + \|T - T^*\|^2)
\end{aligned}
$$

Since

$$
\begin{aligned}
&|\langle Tx, x\rangle|^2 = |\langle \Re(T)x, x\rangle|^2 + |\langle \Im(T)x, x\rangle|^2 \\
\implies &c(\Re(T))^2 + \|\Im(T)\|^2 \leq \omega(T)^2 \quad \text{and} \quad c(\Im(T))^2 + \|\Re(T)\|^2 \leq \omega(T)^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&c(\Re(T))^2 + c(\Im(T))^2 + \|\Re(T)\|^2 + \|\Im(T)\|^2 \leq 2\omega(T)^2 \\
\implies &\frac{1}{8}(\|T + T^*\|^2 + \|T - T^*\|^2) + \frac{1}{2}(c(\Re(T))^2 + c(\Im(T))^2) \leq \omega(T)^2 \\
\implies &\frac{1}{8}(\|T + T^*\|^2 + \|T - T^*\|^2) \leq \omega(T)^2 - \frac{1}{2}(c(\Re(T))^2 + c(\Im(T))^2)
\end{aligned}
$$

Since

$$\frac{1}{4}\|T^*T + TT^*\| \leq \frac{1}{8}(\|T + T^*\|^2 + \|T - T^*\|^2)$$

Hence

$$\|T^*T + TT^*\| \leq 4\left(\omega(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}\right)$$

Theorem 3.20. *[2] Let $T, S \in \mathcal{B}(\mathcal{H})$. Then:*

$$\omega(TS \pm ST) \leq 2\sqrt{2}\|S\|\sqrt{\omega(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}} \tag{3.13}$$

Proof. *Let $x \in \mathcal{H}$ such that $\|x\| = 1$, using Lemma 3.6, we get:*

$$
\begin{aligned}
|\langle (TS \pm ST)x, x\rangle| &\leq |\langle TSx, x\rangle| + |\langle STx, x\rangle| \\
&= |\langle Sx, T^*x\rangle| + |\langle Tx, S^*x\rangle| \\
&\leq \|Sx\|\|T^*x\| + \|Tx\|\|S^*x\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|S\|(\|T^*x\| + \|Tx\|)\\
&\leq \sqrt{2}\|S\|\sqrt{\|T^*x\|^2 + \|Tx\|^2}\\
&= \sqrt{2}\|S\|\sqrt{\langle TT^*x, x\rangle + \langle T^*Tx, x\rangle}\\
&\leq \sqrt{2}\|S\|\sqrt{\|TT^* + T^*T\|}\\
&\leq 2\sqrt{2}\|S\|\sqrt{\omega(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}}
\end{aligned}
$$

Thus

$$\omega(TS \pm ST) \leq 2\sqrt{2}\|S\|\sqrt{\omega(T)^2 - \frac{c(\Re(T))^2 + c(\Im(T))^2}{2}}$$

Remark 3.12. *Clearly* (3.13) *refines* (3.11).

Proposition 3.14. *Let $T, S, R \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ and $TR = RT$. Then:*

$$\omega(TS \pm RT) \leq 2(\omega(S) + \omega(R))\omega(T) \tag{3.14}$$

Proof. *Since $TS = ST$, by Proposition 3.12 $\omega(TS) \leq 2\omega(S)\omega(T)$, and the same $\omega(RT) \leq 2\omega(R)\omega(T)$. Then*

$$
\begin{aligned}
\omega(TS \pm RT) &\leq \omega(TS) + \omega(RT)\\
&\leq 2\omega(S)\omega(T) + 2\omega(R)\omega(T)\\
&= 2(\omega(S) + \omega(R))\omega(T)
\end{aligned}
$$

Hence

$$\omega(TS \pm RT) \leq 2(\omega(S) + \omega(R))\omega(T)$$

Remark 3.13. *Note that* (3.14) *could be a refinement to* (3.11) *in some cases, exactly when*

$$\omega(S) + \omega(R) \leq \sqrt{2}\max\{\|S\|, \|R\|\}$$

Chapter 4

A new norm and some inequalities

In this chapter, we define a new norm and we study its properties, as well as its applications on the inequalities between the usual norm on $\mathcal{B}(\mathcal{H})$ and the numerical radius.

4.1 A new norm and its properties

Theorem 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\|T\|_{\alpha,\beta} = \sup_{\|x\|=1} \sqrt{\alpha\|Tx\|^2 + \beta\|T^*x\|^2}$$

$\|.\|_{\alpha,\beta}$ defines a norm on $\mathcal{B}(\mathcal{H})$.

Proof. • *If $T = 0$, it is obvious that $\|T\|_{\alpha,\beta} = 0$.*
Conversely, assume that $\|T\|_{\alpha,\beta} = 0$, then:
$\forall x \in \mathcal{H}$ *where* $\|x\| = 1$: $\quad \alpha\|Tx\|^2 + \beta\|T^*x\|^2 = 0 \Longrightarrow \forall x \in \mathcal{H}$ *where* $\|x\| = 1$: $\quad \|Tx\| = 0$.
Thus $\|T\| = 0$, *hence* $T = 0$. *Therefore* $T = 0 \Longleftrightarrow \|T\|_{\alpha,\beta} = 0$.
• *Let $\lambda \in \mathbb{C}$ and $T \in \mathcal{B}(\mathcal{H})$, then:*

$$\begin{aligned}
\|\lambda T\|_{\alpha,\beta} &= \sup_{\|x\|=1} \sqrt{\alpha\|(\lambda T)x\|^2 + \beta\|(\lambda T)^*x\|^2} \\
&= \sup_{\|x\|=1} \sqrt{\alpha\|\lambda Tx\|^2 + \beta\|\overline{\lambda}T^*x\|^2} \\
&= \sup_{\|x\|=1} \sqrt{\alpha|\lambda|^2\|Tx\|^2 + \beta|\overline{\lambda}|^2\|T^*x\|^2} \\
&= \sup_{\|x\|=1} |\lambda|\sqrt{\alpha\|Tx\|^2 + \beta\|T^*x\|^2} \\
&= |\lambda|\|T\|_{\alpha,\beta}
\end{aligned}$$

Then $\|\lambda T\|_{\alpha,\beta} = |\lambda|\|T\|_{\alpha,\beta}$.

• *Let $T, S \in \mathcal{B}(\mathcal{H})$, then:*

$$\|T+S\|_{\alpha,\beta} = \sup_{\|x\|=1} \sqrt{\alpha\|Tx+Sx\|^2 + \beta\|T^*x+S^*x\|^2}$$
$$\leq \sup_{\|x\|=1} \sqrt{\alpha\|Tx\|^2 + \alpha\|Sx\|^2 + 2\alpha\|Tx\|\|Sx\| + \beta\|T^*x\|^2 + \beta\|S^*x\|^2 + 2\beta\|T^*x\|\|S^*x\|}$$

We have that $\alpha\|Tx\|\|Sx\| + \beta\|T^*x\|\|S^*x\| \leq \sqrt{\alpha\|Tx\|^2+\beta\|T^*x\|^2}\sqrt{\alpha\|Sx\|^2+\beta\|S^*x\|^2}$.

Since:

$$(\alpha\|Tx\|\|Sx\|+\beta\|T^*x\|\|S^*x\|)^2 = \alpha^2\|Tx\|^2\|Sx\|^2+\beta^2\|T^*x\|^2\|S^*x\|^2+2\alpha\beta\|Tx\|\|Sx\|\|T^*x\|\|S^*x\|$$
$$\leq \alpha^2\|Tx\|^2\|Sx\|^2+\beta^2\|T^*x\|^2\|S^*x\|^2+\alpha\beta\|Tx\|^2\|S^*x\|^2+\alpha\beta\|Sx\|^2\|T^*x\|^2$$
$$= (\alpha\|Tx\|^2+\beta\|T^*x\|^2)(\alpha\|Sx\|^2+\beta\|S^*x\|^2)$$

Therefore $\alpha\|Tx\|\|Sx\| + \beta\|T^*x\|\|S^*x\| \leq \sqrt{\alpha\|Tx\|^2+\beta\|T^*x\|^2}\sqrt{\alpha\|Sx\|^2+\beta\|S^*x\|^2}$.

Thus

$$\|T+S\|_{\alpha,\beta} \leq \sup_{\|x\|=1} \sqrt{\alpha\|Tx\|^2+\beta\|T^*x\|^2+\alpha\|Sx\|^2+\beta\|S^*x\|^2+2\sqrt{\alpha\|Tx\|^2+\beta\|T^*x\|^2}\sqrt{\alpha\|Sx\|^2+\beta\|S^*x\|^2}}$$
$$= \sup_{\|x\|=1} \sqrt{(\sqrt{\alpha\|Tx\|^2+\beta\|T^*x\|^2}+\sqrt{\alpha\|Sx\|^2+\beta\|S^*x\|^2})^2}$$
$$= \sup_{\|x\|=1} (\sqrt{\alpha\|Tx\|^2+\beta\|T^*x\|^2}+\sqrt{\alpha\|Sx\|^2+\beta\|S^*x\|^2})$$
$$\leq \sup_{\|x\|=1} \sqrt{\alpha\|Tx\|^2+\beta\|T^*x\|^2} + \sup_{\|x\|=1} \sqrt{\alpha\|Sx\|^2+\beta\|S^*x\|^2}$$
$$= \|T\|_{\alpha,\beta} + \|S\|_{\alpha,\beta}$$

Then $\|T+S\|_{\alpha,\beta} \leq \|T\|_{\alpha,\beta} + \|S\|_{\alpha,\beta}$.

Hence $\|.\|_{\alpha,\beta}$ *defines a norm on* $\mathcal{B}(\mathcal{H})$.

Remark 4.1. *If $\alpha = 0$ and $\beta \in \mathbb{R}_+^*$ (or $\beta = 0$ and $\alpha \in \mathbb{R}_+^*$), then $\|T\|_{\alpha,\beta} = \sqrt{\beta}\|T\|$ (or $\|T\|_{\alpha,\beta} = \sqrt{\alpha}\|T\|$), that is why we take $\alpha, \beta \in \mathbb{R}_+^*$.*

Corollary 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\|T\|_{\alpha,\beta} = \sqrt{\|\alpha T^*T + \beta TT^*\|}$$

Proof.

$$\|T\|_{\alpha,\beta} = \sup_{\|x\|=1} \sqrt{\alpha\|Tx\|^2 + \beta\|T^*x\|^2}$$
$$= \sup_{\|x\|=1} \sqrt{\alpha\langle T^*Tx, x\rangle + \beta\langle TT^*x, x\rangle}$$
$$= \sup_{\|x\|=1} \sqrt{\langle(\alpha T^*T + \beta TT^*)x, x\rangle}$$
$$= \sqrt{\sup_{\|x\|=1} \langle(\alpha T^*T + \beta TT^*)x, x\rangle}$$
$$= \sqrt{\|\alpha T^*T + \beta TT^*\|}$$

Lemma 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}_+^*$. Then we have:*

$$\max\{\sqrt{\alpha}, \sqrt{\beta}\}\|T\| \leq \|T\|_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\|T\| \tag{4.1}$$

$$\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}\omega(T) \leq \|T\|_{\alpha,\beta} \leq 2\max\{\sqrt{\alpha}, \sqrt{\beta}\}\omega(T) \tag{4.2}$$

Proof. • *Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:*

$$\begin{aligned}\sqrt{\alpha\|Tx\|^2 + \beta\|T^*x\|^2} \leq& \sqrt{\alpha\|T\|^2 + \beta\|T^*\|^2} \\ =& \sqrt{\alpha\|T\|^2 + \beta\|T\|^2} \\ =& \sqrt{\alpha+\beta}\|T\|\end{aligned}$$

Therefore $\|T\|_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\|T\|$.

First of all, we have that: $\sqrt{\alpha}\|T\| = \sup_{\|x\|=1} \sqrt{\alpha}\|Tx\| \leq \sup_{\|x\|=1} \sqrt{\alpha\|Tx\|^2 + \beta\|T^*x\|^2} = \|T\|_{\alpha,\beta}$.

And the same, we have that: $\sqrt{\beta}\|T\| \leq \|T\|_{\alpha,\beta}$.

Thus

$$\max\{\sqrt{\alpha}, \sqrt{\beta}\}\|T\| \leq \|T\|_{\alpha,\beta}$$

Hence

$$\max\{\sqrt{\alpha}, \sqrt{\beta}\}\|T\| \leq \|T\|_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\|T\|$$

• *Using Theorem 3.13 and Corollary 4.1, we obtain:*

$$\begin{aligned}\omega(T)^2 \leq& \frac{1}{2}\|T^*T + TT^*\| \\ =& \frac{1}{2}\sup_{\|x\|=1} \langle (T^*T + TT^*)x, x\rangle \\ =& \frac{1}{2}\sup_{\|x\|=1} (\frac{\alpha}{\alpha}\langle T^*Tx, x\rangle + \frac{\beta}{\beta}\langle TT^*x, x\rangle) \\ \leq& \frac{1}{2}\sup_{\|x\|=1} (\frac{\alpha}{\min\{\alpha,\beta\}}\langle T^*Tx, x\rangle + \frac{\beta}{\min\{\alpha,\beta\}}\langle TT^*x, x\rangle) \\ =& \frac{1}{2\min\{\alpha,\beta\}}\sup_{\|x\|=1} (\alpha\langle T^*Tx, x\rangle + \beta\langle TT^*x, x\rangle) \\ =& \frac{1}{2\min\{\alpha,\beta\}}\|\alpha T^*T + \beta TT^*\| \\ =& \frac{1}{2\min\{\alpha,\beta\}}\|T\|^2_{\alpha,\beta}\end{aligned}$$

Thus

$$\omega(T)^2 \leq \frac{1}{2\min\{\alpha,\beta\}}\|T\|^2_{\alpha,\beta}$$

Therefore

$$\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}\omega(T) \leq \|T\|_{\alpha,\beta}$$

On the other hand, we have:

$$\begin{aligned}\omega(T)^2 \geq& \frac{1}{4}\|T^*T + TT^*\| \\ =& \frac{1}{4}\sup_{\|x\|=1} \langle (T^*T + TT^*)x, x\rangle \\ =& \frac{1}{4}\sup_{\|x\|=1} (\frac{\alpha}{\alpha}\langle T^*Tx, x\rangle + \frac{\beta}{\beta}\langle TT^*x, x\rangle) \\ \geq& \frac{1}{4}\sup_{\|x\|=1} (\frac{\alpha}{\max\{\alpha,\beta\}}\langle T^*Tx, x\rangle + \frac{\beta}{\max\{\alpha,\beta\}}\langle TT^*x, x\rangle) \\ =& \frac{1}{4\max\{\alpha,\beta\}}\|\alpha T^*T + \beta TT^*\| \\ =& \frac{1}{4\max\{\alpha,\beta\}}\|T\|^2_{\alpha,\beta}\end{aligned}$$

Thus

$$\frac{1}{4\max\{\alpha,\beta\}}\|T\|^2_{\alpha,\beta} \le \omega(T)^2$$

Therefore

$$\|T\|_{\alpha,\beta} \le 2\max\{\sqrt{\alpha},\sqrt{\beta}\}\omega(T)$$

Hence

$$\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}\omega(T) \le \|T\|_{\alpha,\beta} \le 2\max\{\sqrt{\alpha},\sqrt{\beta}\}\omega(T)$$

Remark 4.2. *(4.1) and (4.2) mean that $\|.\|_{\alpha,\beta}$, $\|.\|$ and ω are equivalent norms.*

Corollary 4.2. *The norm $\|.\|_{\alpha,\beta}$ does not satisfy the power inequality in general i.e.*

$$\exists T \in \mathcal{B}(\mathcal{H}),\ \ \exists n \in \mathbb{N}: \qquad \|T^n\|_{\alpha,\beta} > \|T\|^n_{\alpha,\beta}$$

For instance, take $\mathcal{H} = \mathbb{C}^2$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, by simple calculation, we get that $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.
Also we find that $\|T\|_{\alpha,\beta} \le \sqrt{3(\alpha+\beta)}$ and $\|T^n\|_{\alpha,\beta} \ge \sqrt{(n^2+1)(\alpha+\beta)}$. When $3(\alpha+\beta) < 1$ and for $n \in \mathbb{N}$ large enough, we obtain that:

$$\sqrt{(n^2+1)(\alpha+\beta)} > (3(\alpha+\beta))^{\frac{n}{2}}$$

And this is because $\quad (3(\alpha+\beta))^{\frac{n}{2}} \underset{n\to\infty}{\longrightarrow} 0 \quad$ *and* $\quad \sqrt{(n^2+1)(\alpha+\beta)} \underset{n\to\infty}{\longrightarrow} \infty$.
Hence for n large enough, we have

$$\|T^n\|_{\alpha,\beta} > \|T\|^n_{\alpha,\beta}$$

Remark 4.3. *The norm $\|.\|_{\alpha,\beta}$ is not an algebra norm, that is*

$$\exists T, S \in \mathcal{B}(\mathcal{H}): \qquad \|TS\|_{\alpha,\beta} > \|T\|_{\alpha,\beta}\|S\|_{\alpha,\beta}$$

Proof. *If $\|.\|_{\alpha,\beta}$ is an algebra norm $\Longrightarrow$ $\|.\|_{\alpha,\beta}$ satisfy the power inequality.*
Since $\|.\|_{\alpha,\beta}$ does not satisfy the power inequality, then $\|.\|_{\alpha,\beta}$ is not an algebra norm.

Theorem 4.2. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}^*_+$. Then:*

$$\|TS\|_{\alpha,\beta} \le \frac{\sqrt{\alpha+\beta}}{\max\{\alpha,\beta\}}\|T\|_{\alpha,\beta}\|S\|_{\alpha,\beta} \tag{4.3}$$

Proof. *Using Lemma 4.1, we obtain:*

$$\begin{aligned}\|TS\|^2_{\alpha,\beta} &= \sup_{\|x\|=1}\left(\alpha\|TSx\|^2 + \beta\|(TS)^*x\|^2\right)\\ &\le \alpha\|TS\|^2 + \beta\|(TS)^*\|^2\\ &= (\alpha+\beta)\|TS\|^2\\ &\le (\alpha+\beta)\|T\|^2\|S\|^2\\ &\le \frac{(\alpha+\beta)}{\max\{\alpha,\beta\}^2}\|T\|^2_{\alpha,\beta}\|S\|^2_{\alpha,\beta}\end{aligned}$$

Therefore

$$\|TS\|_{\alpha,\beta} \le \frac{\sqrt{\alpha+\beta}}{\max\{\alpha,\beta\}}\|T\|_{\alpha,\beta}\|S\|_{\alpha,\beta}$$

Proposition 4.1. *Let $T, U \in \mathcal{B}(\mathcal{H})$ where U is unitary, and let $\alpha, \beta \in \mathbb{R}_+^*$, $n \in \mathbb{N}$. Then:*
1) $\|T^*\|_{\alpha,\beta} = \|T\|_{\beta,\alpha}$.
2) If T is normal, then $\|T^n\|_{\alpha,\beta} = \sqrt{\alpha+\beta}\|T\|^n$.
3) If T is normal, then $\|T^n\|_{\alpha,\beta} = (\alpha+\beta)^{\frac{1-n}{2}}\|T\|^n_{\alpha,\beta}$.
4) $\|U^*TU\|_{\alpha,\beta} = \|T\|_{\alpha,\beta}$.

Proof. *1)*
$$\begin{aligned}\|T^*\|_{\alpha,\beta} &= \sup_{\|x\|=1}\sqrt{\alpha\|T^*x\|^2 + \beta\|(T^*)^*x\|^2}\\ &= \sup_{\|x\|=1}\sqrt{\alpha\|T^*x\|^2 + \beta\|Tx\|^2}\\ &= \|T\|_{\beta,\alpha}\end{aligned}$$

2) If T is normal, then $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$, and $\|T^n\| = \|T\|^n$. Then:
$$\begin{aligned}\|T\|_{\alpha,\beta} &= \sup_{\|x\|=1}\sqrt{\alpha\|Tx\|^2 + \beta\|T^*x\|^2}\\ &= \sup_{\|x\|=1}\sqrt{\alpha\|Tx\|^2 + \beta\|Tx\|^2}\\ &= \sqrt{\alpha+\beta}\|T\|\end{aligned}$$

Thus
$$\|T\|_{\alpha,\beta} = \sqrt{\alpha+\beta}\|T\|$$

Then
$$\begin{aligned}\|T^n\|_{\alpha,\beta} &= \sqrt{\alpha+\beta}\|T^n\|\\ &= \sqrt{\alpha+\beta}\|T\|^n\end{aligned}$$

Hence
$$\|T^n\|_{\alpha,\beta} = \sqrt{\alpha+\beta}\|T\|^n$$

3) If T is normal, then $\|T\|_{\alpha,\beta} = \sqrt{\alpha+\beta}\|T\|$, and $\|T^n\|_{\alpha,\beta} = \sqrt{\alpha+\beta}\|T\|^n$.
Then $\|T^n\|_{\alpha,\beta} = \sqrt{\alpha+\beta}(\alpha+\beta)^{\frac{-n}{2}}\|T\|^n_{\alpha,\beta} = (\alpha+\beta)^{\frac{1-n}{2}}\|T\|^n_{\alpha,\beta}$.
Hence
$$\|T^n\|_{\alpha,\beta} = (\alpha+\beta)^{\frac{1-n}{2}}\|T\|^n_{\alpha,\beta}$$

4) Let $\|x\| = 1$, since U is unitary, then $\|Ux\| = \|x\| = 1$, also there exists a unique $y \in \mathcal{H}$ such that $Ux = y$, which means that $\|y\| = 1$. Then:

$$\begin{aligned}\|U^*TU\|_{\alpha,\beta} &= \sup_{\|x\|=1}\sqrt{\alpha\|U^*TUx\|^2 + \beta\|(U^*TU)^*x\|^2}\\ &= \sup_{\|x\|=1}\sqrt{\alpha\|U^*TUx\|^2 + \beta\|U^*T^*Ux\|^2}\\ &= \sup_{\|x\|=1}\sqrt{\alpha\langle U^*TUx, U^*TUx\rangle + \beta\langle U^*T^*Ux, U^*T^*Ux\rangle}\\ &= \sup_{\|x\|=1}\sqrt{\alpha\langle TUx, TUx\rangle + \beta\langle T^*Ux, T^*Ux\rangle}\\ &= \sup_{\|y\|=1}\sqrt{\alpha\|Ty\|^2 + \beta\|T^*y\|^2}\\ &= \|T\|_{\alpha,\beta}\end{aligned}$$

Hence $\|U^*TU\|_{\alpha,\beta} = \|T\|_{\alpha,\beta}$.

Remark 4.4. $\|T\|_{\alpha,\beta} = \sqrt{\alpha+\beta}\|T\|$ *does not implies that T is normal.*

As counterexample, let $\mathcal{H} = \mathbb{C}^3$ and consider $T = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$, *then* $T^* = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$.
By simple calculation, we get that $\|T\| = 1$ and $\|T\|_{\alpha,\beta} = \sqrt{\alpha+\beta}$, but $TT^* \neq T^*T$.

4.2 Inequalities

Theorem 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(T)^2 \leq \frac{1}{2}\omega(T^2) + \frac{1}{4\min\{\alpha,\beta\}}\|T\|_{\alpha,\beta}^2 \tag{4.4}$$

$$\omega(T) \leq \frac{1}{2}\left(\frac{1}{\max\{\sqrt{\alpha},\sqrt{\beta}\}}\|T\|_{\alpha,\beta} + \sqrt{r(|T||T^*|)}\right) \tag{4.5}$$

Proof. • *Using Theorem 3.15 and Corollary 4.1, we obtain:*

$$\begin{aligned}
\omega(T)^2 \leq& \frac{1}{2}\omega(T^2) + \frac{1}{4}\|T^*T + TT^*\| \\
=& \frac{1}{2}\omega(T^2) + \frac{1}{4}\sup_{\|x\|=1}\left(\langle T^*Tx, x\rangle + \langle TT^*x, x\rangle\right) \\
=& \frac{1}{2}\omega(T^2) + \frac{1}{4}\sup_{\|x\|=1}\left(\frac{\alpha}{\alpha}\langle T^*Tx, x\rangle + \frac{\beta}{\beta}\langle TT^*x, x\rangle\right) \\
\leq& \frac{1}{2}\omega(T^2) + \frac{1}{4\min\{\alpha,\beta\}}\sup_{\|x\|=1}\left(\alpha\langle T^*Tx, x\rangle + \beta\langle TT^*x, x\rangle\right) \\
=& \frac{1}{2}\omega(T^2) + \frac{1}{4\min\{\alpha,\beta\}}\|\alpha T^*T + \beta TT^*\| \\
=& \frac{1}{2}\omega(T^2) + \frac{1}{4\min\{\alpha,\beta\}}\|T\|_{\alpha,\beta}^2
\end{aligned}$$

As required.

• *Using Theorem 3.16 and Lemma 4.1, we get:*

$$\begin{aligned}
\omega(T) \leq& \frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right) \\
\leq& \frac{1}{2}\left(\frac{1}{\max\{\sqrt{\alpha},\sqrt{\beta}\}}\|T\|_{\alpha,\beta} + \sqrt{r(|T||T^*|)}\right)
\end{aligned}$$

As desired.

Remark 4.5. *It is obvious that (4.4) refines the left side of (4.2).*

Corollary 4.3. *If $T^2 = 0$, then $\omega(T) = \frac{1}{2}\|T\|_{1,1}$.*

Proof. *By (4.4), we have* $\omega(T)^2 \leq \frac{1}{2}\omega(T^2) + \frac{1}{4}\|T\|_{1,1}^2$.
Since $T^2 = 0$, then $\omega(T) \leq \frac{1}{2}\|T\|_{1,1}$.
On the other hand, by (4.2), we have that $\|T\|_{1,1} \leq 2\omega(T)$, *thus* $\frac{1}{2}\|T\|_{1,1} \leq \omega(T)$.
Therefore $\omega(T) = \frac{1}{2}\|T\|_{1,1}$.

Corollary 4.4. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS)^2 \leq \frac{1}{2}\omega((TS)^2) + \frac{\alpha+\beta}{4\max\{\alpha^2,\beta^2\}\min\{\alpha,\beta\}}\|T\|_{\alpha,\beta}\|S\|_{\alpha,\beta}$$

Proof. *Depending on (4.4) and (4.3), we have:*

$$\begin{aligned}
\omega(TS)^2 \leq& \frac{1}{2}\omega((TS)^2) + \frac{1}{4\min\{\alpha,\beta\}}\|TS\|_{\alpha,\beta}^2 \\
\leq& \frac{1}{2}\omega((TS)^2) + \frac{\alpha+\beta}{4\max\{\alpha^2,\beta^2\}\min\{\alpha,\beta\}}\|T\|_{\alpha,\beta}\|S\|_{\alpha,\beta}
\end{aligned}$$

As desired.

Lemma 4.2. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS) \leq \frac{1}{\max\{\alpha, \beta\}} \|T\|_{\alpha,\beta} \|S\|_{\alpha,\beta} \tag{4.6}$$

Proof. *Using Corollary 4.1, we get:*

$$\begin{aligned}\omega(TS) \leq& \|T\|\|S\| \\ \leq& \frac{1}{\max\{\sqrt{\alpha}, \sqrt{\beta}\}^2} \|T\|_{\alpha,\beta} \|S\|_{\alpha,\beta} \\ =& \frac{1}{\max\{\alpha, \beta\}} \|T\|_{\alpha,\beta} \|S\|_{\alpha,\beta}\end{aligned}$$

As required.

Remark 4.6. *When $\alpha = \beta = 1$ in (4.6), we get:*

$$\omega(TS) \leq \sqrt{\|T^*T + TT^*\|} \sqrt{\|S^*S + SS^*\|} \leq 4\omega(T)\omega(S)$$

Theorem 4.4. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that T commutes with S i.e. $TS = ST$, and let $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS)^2 \leq \frac{1}{4\min\{\alpha, \beta\}} (\|S\|^2 + \|S^2\|) \|T\|_{\alpha,\beta}^2 \tag{4.7}$$

Proof. *Let $\|x\| = 1$, using the fact that $TS = ST$, Theorem 1.3 and Corollary 4.1. We obtain:*

$$\begin{aligned}|\langle TSx, x\rangle|^2 = |\langle TSx, x\rangle\langle x, (TS)^*x\rangle| \leq& \frac{1}{2}(\|TSx\|\|(TS)^*x\| + |\langle TSx, (TS)^*x\rangle|) \\ =& \frac{1}{2}(\|STx\|\|S^*T^*x\| + |\langle S^2Tx, T^*x\rangle|) \\ \leq& \frac{1}{2}(\|S\|\|S^*\|\|Tx\|\|T^*x\| + \|S^2\|\|Tx\|\|T^*x\|) \\ \leq& \frac{1}{4}(\|S\|^2 + \|S^2\|)(\|Tx\|^2 + \|T^*x\|^2) \\ =& \frac{1}{4}(\|S\|^2 + \|S^2\|)(\frac{\alpha}{\alpha}\langle T^*Tx, x\rangle + \frac{\beta}{\beta}\langle TT^*x, x\rangle) \\ \leq& \frac{1}{4\min\{\alpha, \beta\}}(\|S\|^2 + \|S^2\|)\|\alpha T^*T + \beta TT^*\| \\ =& \frac{1}{4\min\{\alpha, \beta\}}(\|S\|^2 + \|S^2\|)\|T\|_{\alpha,\beta}^2\end{aligned}$$

As desired.

Remark 4.7. *We can replace T by S and S by T in (4.7) i.e.*

$$\omega(TS)^2 \leq \frac{1}{4\min\{\alpha, \beta\}} (\|T\|^2 + \|T^2\|) \|S\|_{\alpha,\beta}^2$$

Corollary 4.5. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$. Then:*

$$\omega(TS)^2 \leq \frac{1}{4} (\|S\|^2 + \|S^2\|) \|T\|_{1,1}^2 \tag{4.8}$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Remark 4.8. *It is clear that*

$$\frac{1}{4}(\|S\|^2 + \|S^2\|)\|T^*T + TT^*\| \leq \|T\|^2\|S\|^2$$

Also

$$\begin{aligned}\frac{1}{4}(\|S\|^2 + \|S^2\|)\|T^*T + TT^*\| \leq& (\|S\|^2 + \|S^2\|)\omega(T)^2 \\ =& 2\|S\|^2\omega(T)^2\end{aligned}$$

Thus

$$\frac{1}{4}(\|S\|^2 + \|S^2\|)\|T^*T + TT^*\| \leq 2\omega(T)^2\|S\|^2 \tag{4.9}$$

Hence (4.8) *is a refinement of* $\omega(TS) \leq 2\omega(T)\omega(S)$ *in case* $\omega(S) = \|S\|$. *Moreover,* (4.8) *may give better estimation then* $\omega(TS) \leq 2\omega(T)\omega(S)$ *when* $\omega(S) \neq \|S\|$.

Remark 4.9. *Let* $T \in \mathcal{B}(\mathcal{H})$*, then* T *is normal* $\Longrightarrow \|T^2\| = \|T\|^2$. *And the converse in not true in general.*

Proof. *By Theorem 2.5, if* T *is normal, then* $\|T^n\| = \|T\|^n$. *Conversely, let* $\mathcal{H} = \mathbb{C}^3$ *and consider* $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$*, then* $T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$*, and we have that* $\|T^2\| = \|T\|^2 = 1$*, but* T *is not normal.*

Corollary 4.6. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $\|S^2\| = \|S\|^2$. *Then* (4.8) *refines* $\omega(TS) \leq 2\omega(T)\omega(S)$.

Proof. *Since* $\|S^2\| = \|S\|^2$*, then* $\|S\| = \sqrt{\|S^2\|} \leq \sqrt{2\omega(S^2)}$*, thus* $\|S\| \leq \sqrt{2}\sqrt{\omega(S^2)}$.

$$\begin{aligned} \frac{1}{4}(\|S\|^2 + \|S^2\|)\|T^*T + TT^*\| &\leq 2\|S\|^2\omega(T)^2 \\ &\leq 4\omega(S^2)\omega(T)^2 \\ &\leq 4\omega(S)^2\omega(T)^2 \end{aligned}$$

Corollary 4.7. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $S^2 = 0$*, then:*

$$\omega(TS) \leq \frac{1}{2}\|S\|\|T\|_{1,1} \leq \|S\|\omega(T)$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Proof. *By* (4.8)*, we have that:*

$$\omega(TS)^2 \leq \frac{1}{4}(\|S\|^2 + \|S^2\|)\|T\|_{1,1}^2$$

Then

$$\omega(TS) \leq \frac{1}{2}\|S\|\|T\|_{1,1} \leq \|S\|\omega(T)$$

Remark 4.10. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $\sqrt{\|S\|^2 + \|S^2\|} \leq 2\omega(S)$. *Then* (4.8) *improves* $\omega(TS) \leq 2\omega(T)\omega(S)$.

Example 4.1. *1) Let* $\mathcal{H} = \mathbb{C}^2$*, and consider* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$*, by simple calculation we get the following:* $TS = ST$, $\omega(T) = \frac{3}{2}$, $\|T^*T + TT^*\| = 5$, $\omega(S) = 2$, $5.8 \leq \|S\|^2 \leq 5.84$, $\sqrt{17.8} \leq \|S^2\| \leq \sqrt{18}$ *and* $\|S\|^2 \neq \|S^2\|$.
By $\omega(TS) \leq 2\omega(T)\omega(S)$*, we get that* $\omega(TS) \leq 6$*, and by* (4.8) *we get that* $\omega(TS) \leq 3.55$.
2) Let $\mathcal{H} = \mathbb{C}^3$*, and consider* $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$*, by simple calculation we get the following:*
$TS = ST$, $\omega(T) = \omega(S) = 1$, $\|T^*T + TT^*\| = 2$, $\|S\| = 2$*, and* $\|S^2\| = 1$.
By the previous date, we get that $\sqrt{\|S\|^2 + \|S^2\|} > 2\omega(S)$*, and by* $\omega(TS) \leq 2\omega(T)\omega(S)$*, we get that* $\omega(TS) \leq 2$*, and by* (4.8) *we get that* $\omega(TS) \leq \sqrt{\frac{5}{2}} \approx 1.58$.
Hence (4.8) *gives a better estimation then* $\omega(TS) \leq 2\omega(T)\omega(S)$ *in this case even though* $\sqrt{\|S\|^2 + \|S^2\|} > 2\omega(S)$.

Therefore $\sqrt{\|S\|^2 + \|S^2\|} \leq 2\omega(S)$ *ensures that* (4.8) *gives a better estimation then* $\omega(TS) \leq 2\omega(T)\omega(S)$. *However, if it is not satisfied this doesn't mean that* (4.8) *will not give a better estimation then* $\omega(TS) \leq 2\omega(T)\omega(S)$.

Theorem 4.5. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $T^{*2}S^2 = S^2T^{*2}$, *and let* $\alpha, \beta \in \mathbb{R}_+^*$. *Then:*

$$\omega(TS)^2 \leq \frac{1}{4\min\{\alpha,\beta\}}\|S\|^2\|T\|_{\alpha,\beta}^2 + \frac{1}{2}\|S^2\|\omega(T^2) \tag{4.10}$$

Proof. *Since* $TS = ST$, *then* $T^2S^2 = S^2T^2$, *and we have that* $(T^2)^*S^2 = S^2(T^2)^*$, *by Theorem 3.17, we have*

$$\omega(T^2S^2) \leq \omega(T^2)\|S^2\|$$

Let $\|x\| = 1$, *Applying Theorem 1.3 and Corollary 4.1. We get:*

$$\begin{aligned}
|\langle TSx, x\rangle|^2 = |\langle TSx, x\rangle\langle x, (TS)^*x\rangle| \leq & \frac{1}{2}(\|TSx\|\|(TS)^*x\| + |\langle TSx, (TS)^*x\rangle|) \\
\leq & \frac{1}{2}\|S\|\|S^*\|\|Tx\|\|T^*x\| + \frac{1}{2}|\langle T^2S^2x, x\rangle| \\
\leq & \frac{1}{4}\|S\|^2(\|Tx\|^2 + \|T^*x\|^2) + \frac{1}{2}\omega(T^2S^2) \\
\leq & \frac{1}{4}\|S\|^2(\frac{\alpha}{\alpha}\langle T^*Tx, x\rangle + \frac{\beta}{\beta}\langle TT^*x, x\rangle) + \frac{1}{2}\|S^2\|\omega(T^2) \\
\leq & \frac{1}{4\min\{\alpha,\beta\}}\|S\|^2\|T\|_{\alpha,\beta}^2 + \frac{1}{2}\|S^2\|\omega(T^2)
\end{aligned}$$

As required.

Remark 4.11. *We can replace* T *by* S *and* S *by* T *in* (4.10) *i.e.*

$$\omega(TS)^2 \leq \frac{1}{4\min\{\alpha,\beta\}}\|T\|^2\|S\|_{\alpha,\beta}^2 + \frac{1}{2}\|T^2\|\omega(S^2)$$

Remark 4.12. *Let* $T, S \in \mathcal{B}(\mathcal{H})$, *then* $T^2S^2 = S^2T^2$ *does not imply that* $TS = ST$.

Proof. *Let* $\mathcal{H} = \mathbb{C}^3$, *consider* $T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. *After calculation we get that* $T^2S^2 = S^2T^2$, *but* $TS \neq ST$.

Remark 4.13. *By Remark 4.12,* $T^{*2}S^2 = S^2T^{*2}$ *does not imply that* $T^*S = ST^*$.

Corollary 4.8. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $T^{*2}S^2 = S^2T^{*2}$. *Then:*

$$\omega(TS)^2 \leq \frac{1}{4}\|S\|^2\|T\|_{1,1}^2 + \frac{1}{2}\|S^2\|\omega(T^2) \leq \omega(T)^2\|S\|^2 + \frac{1}{2}\omega(T^2)\|S^2\| \tag{4.11}$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Corollary 4.9. *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $T^{*2}S^2 = S^2T^{*2}$. *Then:*

$$\omega(TS) \leq \frac{\sqrt{3}}{\sqrt{2}}\omega(T)\|S\|$$

Proof. *Using* (4.11), *we get:*

$$\begin{aligned}
\omega(TS)^2 \leq & \omega(T)^2\|S\|^2 + \frac{1}{2}\omega(T^2)\|S^2\| \\
\leq & \omega(T)^2\|S\|^2 + \frac{1}{2}\omega(T)^2\|S\|^2 \\
= & \frac{3}{2}\omega(T)^2\|S\|^2
\end{aligned}$$

As desired.

Corollary 4.10. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ and $T^{*2}S^2 = S^2T^{*2}$. Then:*
1) If $\omega(S) = \|S\|$, then $\omega(TS) \leq \frac{\sqrt{3}}{\sqrt{2}}\omega(T)\omega(S)$.
2) If $\|S^2\| = \|S\|^2$, then $\omega(TS) \leq \sqrt{3}\omega(T)\omega(S)$.

Theorem 4.6. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $TS = S^*T$, and let $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS)^2 \leq \|S\|^2 \left(\tfrac{1}{4\min\{\alpha,\beta\}} \|T\|_{\alpha,\beta}^2 + \tfrac{1}{2}\omega(T^2) \right) \tag{4.12}$$

Proof. *Since $TS = S^*T$, then $S^*T^* = T^*S$. Let $\|x\| = 1$, using Theorem 1.3 and Corollary 4.1. We get:*

$$\begin{aligned}
|\langle TSx, x\rangle|^2 = |\langle TSx, x\rangle\langle x, (TS)^*x\rangle| &\leq \frac{1}{2}(\|TSx\|\|(TS)^*x\| + |\langle TSx, (TS)^*x\rangle|) \\
&= \frac{1}{2}(\|S^*Tx\|\|S^*T^*x\| + |\langle TSx, S^*T^*x\rangle|) \\
&\leq \frac{1}{2}(\|S^*\|\|S^*\|\|Tx\|\|T^*x\| + |\langle TSx, T^*Sx\rangle|) \\
&\leq \frac{1}{4}\|S\|^2(\|Tx\|^2 + \|T^*x\|^2) + \frac{1}{2}|\langle T^2Sx, Sx\rangle| \\
&= \frac{1}{4}\|S\|^2(\frac{\alpha}{\alpha}\langle T^*Tx, x\rangle + \frac{\beta}{\beta}\langle TT^*x, x\rangle) + \frac{1}{2}\|Sx\|^2|\langle T^2\frac{Sx}{\|Sx\|}, \frac{Sx}{\|Sx\|}\rangle| \\
&\leq \frac{1}{4\min\{\alpha,\beta\}}\|S\|^2\|\alpha T^*T + \beta TT^*\| + \frac{1}{2}\|S\|^2\omega(T^2) \\
&= \|S\|^2 \left(\tfrac{1}{4\min\{\alpha,\beta\}} \|T\|_{\alpha,\beta}^2 + \tfrac{1}{2}\omega(T^2) \right)
\end{aligned}$$

As required.

Corollary 4.11. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $TS = S^*T$. Then:*

$$\omega(TS)^2 \leq \|S\|^2 \left(\tfrac{1}{4}\|T\|_{1,1}^2 + \tfrac{1}{2}\omega(T^2) \right) \leq \frac{3}{2}\|S\|^2\omega(T)^2 \tag{4.13}$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Corollary 4.12. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $TS = S^*T$ and $T^2 = 0$. Then:*

$$\omega(TS) \leq \|S\|\omega(T)$$

Theorem 4.7. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $|T|S = S|T|$, and let $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS) \leq \frac{1}{\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}}\|S\|\|T\|_{\alpha,\beta} \tag{4.14}$$

Proof. *Let $\|x\| = 1$, since $|T|S = S|T|$, then $\sqrt{|T|}S = S\sqrt{|T|}$. Using Theorem 2.14 and Corollary 4.1, we get:*

$$\begin{aligned}
\langle TSx, x\rangle|^2 &\leq \langle |T|Sx, Sx\rangle\langle |T^*|x, x\rangle \\
&= \langle S|T|x, Sx\rangle\langle |T^*|x, x\rangle \\
&= \langle |T|x, S^*Sx\rangle\langle |T^*|x, x\rangle \\
&\leq \|S^*Sx\|\||T|x\|\||T^*|x\| \\
&\leq \frac{1}{2}\|S\|^2(\||T|x\|^2 + \||T^*|x\|^2) \\
&= \frac{1}{2}\|S\|^2(\langle T^*Tx, x\rangle + \langle TT^*x, x\rangle) \\
&\leq \frac{1}{2\min\{\alpha,\beta\}}\|S\|^2(\alpha\langle T^*Tx, x\rangle + \beta\langle TT^*x, x\rangle)
\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{2\min\{\alpha,\beta\}}\|S\|^2\|\alpha T^*T+\beta TT^*\|\\&=\frac{1}{2\min\{\alpha,\beta\}}\|S\|^2\|T\|^2_{\alpha,\beta}\end{aligned}$$

Hence $\omega(TS)\leq \frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}\|S\|\|T\|_{\alpha,\beta}$.

Corollary 4.13. *Let $T,S\in\mathcal{B}(\mathcal{H})$. Then:*

$$TS=ST \text{ and } T^*S=ST^* \quad\Longrightarrow\quad |T|S=S|T|$$

And the converse is not true in general.

Proof. • *Assume that $TS=ST$ and $T^*S=ST^*$, then:*
$|T|^2S=T^*TS=T^*ST=ST^*T=S|T|^2$, *then* $|T|^2S=S|T|^2$, *therefore* $|T|S=S|T|$.

• *Let $\mathcal{H}=\mathbb{C}^3$ and consider* $T=\begin{pmatrix}0&1&0\\0&0&0\\1&0&0\end{pmatrix}$, *then* $T^*=\begin{pmatrix}0&0&1\\1&0&0\\0&0&0\end{pmatrix}$ *and* $T^*T=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$.

Also we have that $(T^*T)^2=T^*T$, *thus* $|T|=\sqrt{T^*T}=T^*T$. *Let* $S=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$.

By simple calculation, we get that $|T|S=S|T|$, but $TS\neq ST$ and $T^*S\neq ST^*$.

Corollary 4.14. *Let $T,S\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S|T|$. Then:*

$$\omega(TS)\leq\frac{1}{\sqrt{2}}\|S\|\|T\|_{1,1}\leq\sqrt{2}\|S\|\omega(T) \tag{4.15}$$

Recall that $\|T\|_{1,1}=\sqrt{\|T^*T+TT^*\|}$, *also*

$$\frac{1}{\sqrt{2}}\|S\|\sqrt{\|T^*T+TT^*\|}\leq\frac{1}{\sqrt{2}}\|S\|\sqrt{\|T\|^2+\|T^2\|}$$

Corollary 4.15. *Let $T,S\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S|T|$. Then:*
1) If $\omega(S)=\|S\|$, then $\omega(TS)\leq\sqrt{2}\omega(T)\omega(S)$.
2) If $\|S^2\|=\|S\|^2$, then $\omega(TS)\leq 2\omega(T)\omega(S)$.

Corollary 4.16. *Let $T,S\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S|T|$ and $T^2=0$. Then:*

$$\omega(TS)\leq\frac{1}{\sqrt{2}}\|S\|\|T\|$$

Theorem 4.8. *Let $T,S\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S^*|T|$, and let $\alpha,\beta\in\mathbb{R}^*_+$. Then:*

$$\omega(TS)\leq\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}r(S)\|T\|_{\alpha,\beta} \tag{4.16}$$

Proof. *Let $\|x\|=1$, using Lemma 3.5 and Corollary 4.1, we obtain:*

$$\begin{aligned}|\langle TSx,x\rangle|^2&\leq r(S)^2\|\sqrt{|T|}x\|^2\|\sqrt{|T^*|}x\|^2\\&=r(S)^2\langle|T|x,x\rangle\langle|T^*|x,x\rangle\\&\leq\frac{1}{2}r(S)^2(\langle|T|x,x\rangle^2+\langle|T^*|x,x\rangle^2)\\&\leq\frac{1}{2}r(S)^2(\||T|x\|^2+\||T^*|x\|^2)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} r(S)^2 (\langle T^*Tx, x\rangle + \langle TT^*x, x\rangle) \\
&\le \frac{1}{2\min\{\alpha,\beta\}} r(S)^2 (\alpha\langle T^*Tx, x\rangle + \beta\langle TT^*x, x\rangle) \\
&\le \frac{1}{2\min\{\alpha,\beta\}} r(S)^2 \|\alpha T^*T + \beta TT^*\| \\
&= \frac{1}{2\min\{\alpha,\beta\}} r(S)^2 \|T\|_{\alpha,\beta}^2
\end{aligned}
$$

Thus $\quad \omega(TS) \le \frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}} r(S)\|T\|_{\alpha,\beta}$.

Corollary 4.17. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $|T|S = S^*|T|$. Then:*

$$\omega(TS) \le \frac{1}{\sqrt{2}} r(S)\|T\|_{1,1} \le \sqrt{2} r(S)\omega(T) \tag{4.17}$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Corollary 4.18. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $|T|S = S^*|T|$. Then:*
1) If $\omega(T) = \frac{1}{\sqrt{2}}\sqrt{T^*T + TT^*}$, then $\quad \omega(TS) \le r(S)\omega(T)$.
2) If $T^2 = 0$, then $\quad \omega(TS) \le \frac{1}{\sqrt{2}} r(S)\|T\|$.

Theorem 4.9. *Let $T, S, R \in \mathcal{B}(\mathcal{H})$, and $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS \pm RT) \le \frac{\sqrt{2}}{\min\{\sqrt{\alpha},\sqrt{\beta}\}} \max\{\|S\|, \|R\|\}\|T\|_{\alpha,\beta} \le \frac{\sqrt{2}}{\sqrt{\alpha\beta}} \max\{\|S\|_{\alpha,\beta}, \|R\|_{\alpha,\beta}\}\|T\|_{\alpha,\beta} \tag{4.18}$$

Proof. *Let $\|x\| = 1$, using Corollary 4.1 and (4.1), we obtain:*

$$
\begin{aligned}
|\langle (TS \pm RT)x, x\rangle| &\le |\langle TSx, x\rangle| + |\langle RTx, x\rangle| \\
&\le \|Sx\|\|T^*x\| + \|RTx\| \\
&\le \|S\|\|T^*x\| + \|R\|\|Tx\| \\
&\le \max\{\|S\|, \|R\|\}(\|T^*x\| + \|Tx\|) \\
&\le \sqrt{2}\max\{\|S\|, \|R\|\}\sqrt{\|Tx\|^2 + \|T^*x\|^2} \\
&= \sqrt{2}\max\{\|S\|, \|R\|\}\sqrt{\langle T^*Tx, x\rangle + \langle TT^*x, x\rangle} \\
&\le \frac{\sqrt{2}}{\min\{\sqrt{\alpha},\sqrt{\beta}\}} \max\{\|S\|, \|R\|\}\sqrt{\alpha\langle T^*Tx, x\rangle + \beta\langle TT^*x, x\rangle} \\
&= \frac{\sqrt{2}}{\min\{\sqrt{\alpha},\sqrt{\beta}\}} \max\{\|S\|, \|R\|\}\|T\|_{\alpha,\beta} \\
&= \frac{\sqrt{2}}{\max\{\sqrt{\alpha},\sqrt{\beta}\}\min\{\sqrt{\alpha},\sqrt{\beta}\}} \max\{\max\{\sqrt{\alpha},\sqrt{\beta}\}\|S\|, \max\{\sqrt{\alpha},\sqrt{\beta}\}\|R\|\}\|T\|_{\alpha,\beta} \\
&\le \frac{\sqrt{2}}{\sqrt{\alpha\beta}} \max\{\|S\|_{\alpha,\beta}, \|R\|_{\alpha,\beta}\}\|T\|_{\alpha,\beta}
\end{aligned}
$$

Corollary 4.19. *Let $T, S \in \mathcal{B}(\mathcal{H})$, then:*

$$\omega(TS \pm ST) \le \sqrt{2}\|S\|\|T\|_{1,1} \le 2\sqrt{2}\|S\|\omega(T)$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Theorem 4.10. *Let $T, S, R \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ and $TR = RT$, and let $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS \pm RT)^2 \le \frac{1}{4\min\{\alpha,\beta\}}(\|S\| + \|R\|)^2\|T\|_{\alpha,\beta}^2 + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \tag{4.19}$$

Proof. *Let* $\|x\| = 1$*, using* Theorem 1.3 *and* Corollary 4.1*. We obtain:*

$$
\begin{aligned}
|\langle (TS \pm RT)x, x\rangle|^2 \leq & (|\langle TSx, x\rangle| + |\langle RTx, x\rangle|)^2 \\
=& |\langle TSx, x\rangle|^2 + |\langle RTx, x\rangle|^2 + 2|\langle TSx, x\rangle||\langle RTx, x\rangle| \\
=& |\langle TSx, x\rangle\langle x, S^*T^*x\rangle| + |\langle RTx, x\rangle\langle x, T^*R^*x\rangle| + 2|\langle TSx, x\rangle\langle x, T^*R^*x\rangle| \\
\leq & \frac{1}{2}(\|TSx\|\|S^*T^*x\| + |\langle TSx, S^*T^*x\rangle|) + \frac{1}{2}(\|RTx\|\|T^*R^*x\| + |\langle RTx, T^*R^*x\rangle|) \\
& + \|TSx\|\|T^*R^*x\| + |\langle TSx, T^*R^*x\rangle| \\
=& \frac{1}{2}(\|STx\|\|S^*T^*x\| + |\langle T^2S^2x, x\rangle|) + \frac{1}{2}(\|RTx\|\|R^*T^*x\| + |\langle T^2R^2x, x\rangle|) \\
& + \|STx\|\|R^*T^*x\| + |\langle T^2Sx, R^*x\rangle| \\
\leq & \frac{1}{2}\|S\|^2\|Tx\|\|T^*x\| + \frac{1}{2}\|R\|^2\|Tx\|\|T^*x\| + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) \\
& + (\|S\|\|R\|\|Tx\|\|T^*x\| + \|S\|\|R\|\|T^2\|) \\
=& \frac{1}{2}(\|S\|^2 + \|R\|^2 + 2\|S\|\|R\|)\|Tx\|\|T^*x\| + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \\
\leq & \frac{1}{4}(\|S\| + \|R\|)^2(\|Tx\|^2 + \|T^*x\|^2) + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \\
\leq & \frac{1}{4}(\|S\| + \|R\|)^2(\frac{\alpha}{\alpha}\langle T^*Tx, x\rangle + \frac{\beta}{\beta}\langle TT^*x, x\rangle) + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \\
\leq & \frac{1}{4\min\{\alpha, \beta\}}(\|S\| + \|R\|)^2\|T\|^2_{\alpha,\beta} + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\|
\end{aligned}
$$

As desired.

Corollary 4.20. *Let* $T, S, R \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $TR = RT$*. Then:*

$$
\omega(TS \pm RT)^2 \leq \frac{1}{4}(\|S\| + \|R\|)^2\|T\|^2_{1,1} + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \tag{4.20}
$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$*. In addition, we have:*
Set $F(T, S, R) = \frac{1}{4}(\|S\| + \|R\|)^2\|T\|^2_{1,1} + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right)$*, then:*

$$
\begin{aligned}
F(T, S, R) + \|S\|\|R\|\|T^2\| \leq & (\|S\| + \|R\|)^2\omega(T)^2 + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \\
\leq & (\|S\| + \|R\|)^2\omega(T)^2 + \frac{1}{2}(\|T^2S^2\| + \|T^2R^2\|) + 2\|S\|\|R\|\omega(T^2) \\
\leq & (\|S\| + \|R\|)^2\omega(T)^2 + \frac{1}{2}(\|T^2\|\|S^2\| + \|T^2\|\|R^2\|) + 2\|S\|\|R\|\omega(T)^2 \\
\leq & (\|S\| + \|R\|)^2\omega(T)^2 + (\omega(T^2)\|S\|^2 + \omega(T^2)\|R\|^2) + 2\|S\|\|R\|\omega(T)^2 \\
\leq & (\|S\| + \|R\|)^2\omega(T)^2 + (\omega(T)^2\|S\|^2 + \omega(T)^2\|R\|^2) + 2\|S\|\|R\|\omega(T)^2 \\
=& ((\|S\| + \|R\|)^2 + \|S\|^2 + \|R\|^2 + 2\|S\|\|R\|)\omega(T)^2 \\
=& 2(\|S\| + \|R\|)^2\omega(T)^2
\end{aligned}
$$

Hence

$$
\frac{1}{4}(\|S\| + \|R\|)^2\|T\|^2_{1,1} + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \leq 2(\|S\| + \|R\|)^2\omega(T)^2
$$

This means that (4.20) *improves* $\omega(TS \pm RT) \leq 2\sqrt{2}\max\{\|S\|, \|R\|\}\omega(T)$*, also it refines* $\omega(TS \pm RT) \leq 2(\omega(S) + \omega(R))\omega(T)$ *in case* $\omega(S) = \|S\|$ *and* $\omega(R) = \|R\|$*.*

Remark 4.14. *Let $T, S, R \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ and $TR = RT$. Then:*

$$\omega(TS \pm RT) \leq \sqrt{2}(\|S\| + \|R\|)\omega(T)$$

Corollary 4.21. *Let $T, S, R \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ and $TR = RT$, also $\|S^2\| = \|S\|^2$ and $\|R^2\| = \|R\|^2$. Then (4.20) refines $\omega(TS \pm RT) \leq 2(\omega(S) + \omega(R))\omega(T)$.*

Proof. *Since $\|S^2\| = \|S\|^2$, then $\|S\| = \sqrt{\|S^2\|} \leq \sqrt{2\omega(S^2)} \leq \sqrt{2}\omega(S)$, thus $\|S\| \leq \sqrt{2}\omega(S)$ and $\|R\| \leq \sqrt{2}\omega(R)$. Set $F(T, S, R) = \frac{1}{4}(\|S\| + \|R\|)^2\|T\|_{1,1}^2 + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right)$, then we get:*

$$\begin{aligned} F(T,S,R) + \|S\|\|R\|\|T^2\| \leq& (\|S\| + \|R\|)^2\omega(T)^2 + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \\ \leq& (\|S\| + \|R\|)^2\omega(T)^2 + \frac{1}{2}\|T^2\|\|S^2\| + \frac{1}{2}\|T^2\|\|R^2\| + 2\|S\|\|R\|\omega(T^2) \\ \leq& (\|S\| + \|R\|)^2\omega(T)^2 + (\|S\|^2 + \|R\|^2 + 2\|S\|\|R\|)\omega(T^2) \\ \leq& 2(\|S\| + \|R\|)^2\omega(T)^2 \\ \leq& 4(\omega(S) + \omega(R))^2\omega(T)^2 \end{aligned}$$

Therefore

$$\frac{1}{4}(\|S\| + \|R\|)^2\|T\|_{1,1}^2 + \frac{1}{2}\left(\omega(T^2S^2) + \omega(T^2R^2)\right) + \|S\|\|R\|\|T^2\| \leq 4(\omega(S) + \omega(R))^2\omega(T)^2$$

Remark 4.15. *Let $T, S, R \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ and $TR = RT$. If $\|S\| + \|R\| \leq \sqrt{2}(\omega(S) + \omega(R))$, then (4.20) is a refinement of $\omega(TS \pm RT) \leq 2(\omega(S) + \omega(R))\omega(T)$.*

Example 4.2. *1) Let $\mathcal{H} = \mathbb{C}^2$, and consider $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$, after calculation we obtain the following: $TS = ST$, $TR = RT$, $\omega(T) = \frac{3}{2}$, $\|TT^* + T^*T\| = 5$, $\|T^2\| \leq \sqrt{5.84}$, $\omega(S) = 2$, $\sqrt{5.8} \leq \|S\| \leq \sqrt{5.84}$, $\omega(R) = \frac{5}{4}$, $\sqrt{1.6} \leq \|R\| \leq \sqrt{1.642}$, $\omega(T^2S^2) = 4$ and $\omega(T^2R^2) = \frac{5}{2}$.*
Then $2\sqrt{2}\max\{\|S\|, \|R\|\}\omega(T) \geq 3\sqrt{2}\sqrt{5.8} \approx 10.22$, on the other hand, we have $2(\omega(S) + \omega(R))\omega(T) = 9.75$.
Thus $\omega(TS \pm RT) \leq 9.75$. However, by (4.20), we get that $\omega(TS \pm RT) \leq 5.28$.
2) Let $\mathcal{H} = \mathbb{C}^3$, consider $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, after calculation we obtain the following: $TS = ST$, $TR = RT$, $\omega(T) = \frac{3}{2}$, $\|T^*T + TT^*\| = 5$, $\|T^2\| \leq \sqrt{5.84}$, $\omega(S) = 3$, $\sqrt{17.8} \leq \|S\| \leq \sqrt{18}$, $\omega(R) = \frac{1}{2}$, $\|R\| = 1$, $\omega(T^2S^2) \leq 6$ and $\omega(T^2R^2) = 0$.
Then $2\sqrt{2}\max\{\|S\|, \|R\|\}\omega(T) \geq 3\sqrt{2}\sqrt{17.8} \approx 17.9$, on the other hand, we have $2(\omega(S) + \omega(R))\omega(T) = 10.5$.
Thus $\omega(TS \pm RT) \leq 10.5$. Also we have that $\|S\| + \|R\| \geq 5.21$ and $\sqrt{2}(\omega(S) + \omega(R)) = \frac{7}{\sqrt{2}}$.
Hence $\|S\| + \|R\| > \sqrt{2}(\omega(S) + \omega(R))$. By (4.20), we get that $\omega(TS \pm RT) \leq 6.9$, see that (4.20) gives a better estimation even though $\|S\| + \|R\| > \sqrt{2}(\omega(S) + \omega(R))$.

Theorem 4.11. *Let $T, S, R \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ and $TR = RT$, and let $\alpha, \beta \in \mathbb{R}_+^*$. Then:*

$$\omega(TS \pm RT) \leq \frac{1}{2\min\{\sqrt{\alpha}, \sqrt{\beta}\}}(\sqrt{\|S\|^2 + \|S^2\|} + \sqrt{\|R\|^2 + \|R^2\|})\|T\|_{\alpha,\beta} \tag{4.21}$$

Proof. *Since $TS = ST$ and $T^*R^* = R^*T^*$. Using (4.7), we get:*

$$\omega(TS \pm RT) \leq \omega(TS) + \omega(RT)$$

$$=\omega(TS)+\omega(TR)$$
$$\leq\frac{1}{2\min\{\sqrt{\alpha},\sqrt{\beta}\}}\sqrt{\|S\|^2+\|S^2\|}\|T\|_{\alpha,\beta}+\frac{1}{2\min\{\sqrt{\alpha},\sqrt{\beta}\}}\sqrt{\|R\|^2+\|R^2\|}\|T\|_{\alpha,\beta}$$
$$=\frac{1}{2\min\{\sqrt{\alpha},\sqrt{\beta}\}}(\sqrt{\|R\|^2+\|R^2\|}+\sqrt{\|S\|^2+\|S^2\|})\|T\|_{\alpha,\beta}$$

As required.

Theorem 4.12. *Let $T,S,R\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S|T|$ and $|T^*|R=R|T^*|$, and let $\alpha,\beta\in\mathbb{R}_+^*$. Then:*

$$\omega(TS\pm RT)\leq\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}(\|S\|+\|R\|)\|T\|_{\alpha,\beta} \tag{4.22}$$

Proof. *We have that:* $|T^*|R=R|T^*|\iff|T^*|R^*=R^*|T^*|$

Since $|T|S=S|T|$ and $|T^*|R^*=R^*|T^*|$. Using (4.14) and (4.15), we obtain:

$$\begin{aligned}\omega(TS\pm RT)\leq&\omega(TS)+\omega(RT)\\=&\omega(TS)+\omega(T^*R^*)\\\leq&\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}\|S\|\|T\|_{\alpha,\beta}+\frac{1}{\sqrt{2}}\|R^*\|\sqrt{\|T^*T+TT^*\|}\\\leq&\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}\|S\|\|T\|_{\alpha,\beta}+\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}\|R\|\sqrt{\|\alpha T^*T+\beta TT^*\|}\\=&\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}(\|S\|+\|R\|)\|T\|_{\alpha,\beta}\end{aligned}$$

As desired.

Corollary 4.22. *Let $T,S,R\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S|T|$ and $|T^*|R=R|T^*|$. Then:*

$$\omega(TS\pm RT)\leq\frac{1}{\sqrt{2}}(\|S\|+\|R\|)\|T\|_{1,1} \tag{4.23}$$

Recall that $\|T\|_{1,1}=\sqrt{\|T^*T+TT^*\|}$, and note that (4.23) refines $\omega(TS\pm RT)\leq2\sqrt{2}\max\{\|S\|,\|R\|\}\omega(T)$ in this case. Since:

$$\frac{1}{\sqrt{2}}(\|S\|+\|R\|)\sqrt{\|T^*T+TT^*\|}\leq\sqrt{2}(\|S\|+\|R\|)\omega(T)\leq2\sqrt{2}\max\{\|S\|,\|R\|\}\omega(T)$$

Corollary 4.23. *Let $T,S\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S|T|$ and $|T^*|R=R|T^*|$. Then:*

1) If $\omega(S)=\|S\|$ and $\omega(R)=\|R\|$, then $\omega(TS\pm RT)\leq\sqrt{2}(\omega(S)+\omega(R))\omega(T)$.

2) If $\|S^2\|=\|S\|^2$ and $\|R^2\|=\|R\|^2$, then $\omega(TS\pm RT)\leq2(\omega(S)+\omega(R))\omega(T)$.

Theorem 4.13. *Let $T,S\in\mathcal{B}(\mathcal{H})$ such that $|T|S=S^*|T|$ and $|T^*|R=R^*|T^*|$, and let $\alpha,\beta\in\mathbb{R}_+^*$. Then:*

$$\omega(TS\pm RT)\leq\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}(r(S)+r(R))\|T\|_{\alpha,\beta} \tag{4.24}$$

Proof. *Since $|T|S=S^*|T|$ and $|T^*|R=R^*|T^*|$. Using (4.16) and (4.17), we get:*

$$\begin{aligned}\omega(TS\pm RT)\leq&\omega(TS)+\omega(RT)\\=&\omega(TS)+\omega(T^*R^*)\\\leq&\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}r(S)\|T\|_{\alpha,\beta}+\frac{1}{\sqrt{2}}r(R^*)\sqrt{\|T^*T+TT^*\|}\\\leq&\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}r(S)\|T\|_{\alpha,\beta}+\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}r(R)\sqrt{\|\alpha T^*T+\beta TT^*\|}\\=&\frac{1}{\sqrt{2}\min\{\sqrt{\alpha},\sqrt{\beta}\}}(r(S)+r(R))\|T\|_{\alpha,\beta}\end{aligned}$$

As required.

Corollary 4.24. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $|T|S = S^*|T|$ and $|T^*|R = R^*|T^*|$. Then:*

$$\omega(TS \pm RT) \leq \frac{1}{\sqrt{2}}(r(S) + r(R))\|T\|_{1,1} \tag{4.25}$$

Recall that $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Theorem 4.14. *Let $T, S \in \mathcal{B}(\mathcal{H})$, and $\alpha, \beta \in \mathbb{R}_+^*$. Then:*
1) If $TS = ST$ and $|T^*|R = R|T^*|$, then:

$$\omega(TS \pm RT) \leq \frac{1}{\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}}\left(\frac{\sqrt{\|S\|^2 + \|S^2\|}}{\sqrt{2}} + \|R\|\right)\|T\|_{\alpha,\beta} \tag{4.26}$$

2) If $TS = ST$ and $|T^*|R = R^*|T^*|$, then:

$$\omega(TS \pm RT) \leq \frac{1}{\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}}\left(\frac{\sqrt{\|S\|^2 + \|S^2\|}}{\sqrt{2}} + r(R)\right)\|T\|_{\alpha,\beta} \tag{4.27}$$

3) If $|T|S = S|T|$ and $|T^*|R = R^*|T^*|$, then:

$$\omega(TS \pm RT) \leq \frac{1}{\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}}(\|S\| + r(R))\|T\|_{\alpha,\beta} \tag{4.28}$$

Proof. *1) Since $|T|S = S^*|T|$ and $|T^*|R^* = R^*|T^*|$. Using (4.7) and (4.14), we obtain:*

$$\begin{aligned}
\omega(TS \pm RT) \leq& \omega(TS) + \omega(RT) \\
=& \omega(TS) + \omega(T^*R^*) \\
\leq& \frac{1}{2\min\{\sqrt{\alpha}, \sqrt{\beta}\}}\sqrt{\|S\|^2 + \|S^2\|}\|T\|_{\alpha,\beta} + \frac{1}{\sqrt{2}}\|R^*\|\sqrt{\|T^*T + TT^*\|} \\
\leq& \frac{1}{2\min\{\sqrt{\alpha}, \sqrt{\beta}\}}\sqrt{\|S\|^2 + \|S^2\|}\|T\|_{\alpha,\beta} + \frac{1}{\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}}\|R\|\sqrt{\|\alpha T^*T + \beta TT^*\|} \\
=& \frac{1}{\sqrt{2}\min\{\sqrt{\alpha}, \sqrt{\beta}\}}\left(\frac{\sqrt{\|S\|^2 + \|S^2\|}}{\sqrt{2}} + \|R\|\right)\|T\|_{\alpha,\beta}
\end{aligned}$$

As desired.

2) and 3) Proceeding the same as 1).

Corollary 4.25. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then:*
1) If $TS = ST$ and $|T^*|R = R|T^*|$, then:

$$\omega(TS \pm RT) \leq \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\|S\|^2 + \|S^2\|}}{\sqrt{2}} + \|R\|\right)\|T\|_{1,1} \tag{4.29}$$

2) If $TS = ST$ and $|T^*|R = R^*|T^*|$, then:

$$\omega(TS \pm RT) \leq \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\|S\|^2 + \|S^2\|}}{\sqrt{2}} + r(R)\right)\|T\|_{1,1} \tag{4.30}$$

3) If $|T|S = S|T|$ and $|T^*|R = R^*|T^*|$, then:

$$\omega(TS \pm RT) \leq \frac{1}{\sqrt{2}}(\|S\| + r(R))\|T\|_{1,1} \tag{4.31}$$

Where $\|T\|_{1,1} = \sqrt{\|T^*T + TT^*\|}$.

Conclusion

Here we summarize our work, and we discuss prospects for future research.

After presenting four chapters in this thesis, our conclusion is summed up as follows.

First, we give some preliminaries (important theorems and properties) on Hilbert spaces. Then, we study bounded linear operators in detailed, and we provide some vital linear operator classes, also we define the square root and the absolute value of an operator. After that we present the following notions, spectrum, spectral radius, numerical range and the numerical radius of an operator, and we study them thoroughly with their known properties. As we mentioned before, we provide some recent inequalities involving the usual operator norm and the numerical radius. Our work in this thesis is that we give easier proofs to many properties and theorems, which will help beginners to better understand them. More importantly, we define a new norm on $\mathcal{B}(\mathcal{H})$ that is equivalent to the usual operator norm, and we study some of its properties. Also we see its applications on the inequalities between the usual operator norm and the numerical radius, and we obtain new inequalities, in addition we get a refinement for some inequalities as we see in the last chapter.

However, we think that further results can be obtained out of this norm, since we have not studied all the applications and the consequences of the obtained inequalities, therefore there might be opportunities for improvements. Moreover, we have not covered all the applications of the new norm in all the types of inequalities involving the usual operator norm and the numerical radius, so that there may be chances to product new inequalities or get an improvement for others.

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