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On Some Classes of Nonlinear Boundary Value Problems of Fractional Integro-Differential Equations

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Presented by:

Ouissam Chihi
Zineb Zidane

Defended on:06/06/2026, Before a jury composed of:

Mr. Abdellatif Ghendir Aoun	Prof.	University El Oued	**President**
Mr. Meneceur Bekkar	MCA	University El Oued	**Examiner**
Mr. Said Beloul	Prof.	University El Oued	**Provisor**

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿ يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ ﴾

صدق الله العلي العظيم

سورة المجادلة، آية ١١

شكر وتقدير

الحمد لله الذي وهبنا نعمة العقل وأمدنا بالقوة والإرادة حتى أتممنا هذا العمل المتواضع.

نتقدم في مستهل هذا العمل بخالص الشكر وعظيم التقدير لمشرفنا الفاضل، على ما قدمه من توجيهٍ سديد ومتابعةٍ دقيقة، وما لاحظه من ملاحظاتٍ بنّاءة طوال مراحل البحث والكتابة.

كما نتوجه بجزيل الشكر إلى من علّمونا حروفًا من ذهب، وكلماتٍ من درر، إلى من صاغوا لنا من علمهم حروفًا، ومن فكرهم مناراتٍ تنير لنا مسيرة العلم والنجاح،
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كنتم القدوة في الخلق والعلم والمرجع عند الحيرة،
فلكم منا خالص العرفان والامتنان، ونسأل الله أن يجزيكم عنا خير الجزاء، وأن يرفع قدركم في الدنيا والآخرة.

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بسم الله الرحمن الرحيم ﴿وَمَا تَوْفِيقِي إِلَّا بِاللَّهِ عَلَيْهِ تَوَكَّلْتُ وَإِلَيْهِ أُنِيبُ﴾ صدق الله العظيم.

إلى من كان دعاؤهما نورًا أنار دربي، وحبهما سندًا رافقني في كل خطوة، إلى من كان لجهودهما وصبرهما الفضل فيما وصلت إليه، إلى والديّ العزيزين، حفظهما الله وأدامهما تاجًا فوق رأسي

إلى عائلتي الكريمة، التي كانت لي مصدر دعم وتشجيع في مختلف مراحل حياتي الدراسية

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التعب.

زينب

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List of Symbols

Symbol	Meaning
J	Closed interval from 0 to 1
$\mathbb{N}$	The space of natural numbers
$\mathbb{R}$	The space of real numbers
$\mathbb{Q}$	The space of rational numbers
$\mathbb{C}$	The space of complex numbers
$\|\cdot\|_2$	Euclidean norm
$n!$	Factorial
$B(0,1)$	Open ball with center 0 and radius 1
Re	Real part of a complex number
$*$	Convolution
Γ	Gamma function
I^α	Riemann-Liouville fractional integral operator
${}^{RL}D^\alpha$	Riemann-Liouville fractional derivative operator
${}^{c}D^\alpha$	Caputo fractional derivative operator
${}^{C}D^\alpha_{a+}$, ${}^{C}D^\alpha_{b-}$	Left and right side of Caputo fractional derivative
${}^{RL}D^\alpha_{a+}$, ${}^{RL}D^\alpha_{b-}$	Left and right side of Riemann-Liouville fractional derivative
$E = C(J, \mathbb{R})$	The space of continuous functions from J to $\mathbb{R}$
$\|\cdot\|_E$	The norm defined on E (Supremum norm)

Introduction

Fractional calculus is considered a generalization of classical differential operators from integer orders to non-integer orders. The origins of this expansion date back to the 1695 scientific correspondence between Leibniz and L'Hopital regarding the definition of the operator D^n when $n = 1/2$. This inquiry led to the development of the Riemann-Liouville and Caputo operators, which provide a more accurate mathematical formulation for characterizing complex physical and engineering models.

Fractional calculus moves from the theoretical side to application through the formulation of differential equations using the Riemann-Liouville and Caputo operators. These operators are used in the study of simple physical problems such as thermal diffusion, pendulum motion, and mechanical systems; these are models where the fractional derivative emerges as a tool to describe change with greater precision than the ordinary derivative.

In this context, the importance of fixed-point theory emerges as a fundamental tool for studying qualitative properties, namely the existence and uniqueness of solutions for these problems. Banach's contraction principle is applied to prove uniqueness, while Schauder, Krasnoselskii, and Schaefer theories are employed to prove the existence of solutions in Banach spaces, which allows for understanding the mathematical behavior of these models and ensuring the stability of their solutions.

To explore these concepts systematically, this memory is organized as follows:

In the first chapter, we present some fundamental concepts used in our study, such as Banach spaces, completeness, and compactness. We also devoted an important space to introducing the field of fractional calculus, where we presented the definitions of the fractional integral and derivative in the senses of Riemann-Liouville and Caputo, with the most important properties for each of them. We concluded the chapter by presenting the definitions and concepts related to the fixed-point theorems used in the proofs, which are the theorems of: Banach, Schauder, Schaefer, and Krasnoselskii.

In the second chapter, we addressed the analysis of a mathematical problem involving the Caputo derivative with initial conditions. The study begins by transforming this differential problem into an equivalent integral equation, followed by proving the existence and uniqueness of the solution through the application of the Banach and Krasnoselskii theorems. The study was not limited to existence only, but also included the analysis of the stability of the solution according to the Hyers-Ulam concept. In conclusion, the theoretical results were supported by presenting numerical examples that illustrate how to apply these results mathematically.

The third chapter was devoted to studying a more general problem, where the Caputo derivative and the Riemann-Liouville operator were combined together, with the addition of boundary conditions to the problem. We formulated the integral equation for this mixed problem, and studied the existence and uniqueness of the solutions by relying on the Banach and Schauder theorems. The chapter also included a study of the stability of the solution in the Ulam-Hyers concept, and the chapter ended by presenting a numerical application that embodies the theoretical side studied in this part of the dissertation.

Chapter 1

Theoretical Background

In this chapter, we will recall some basic definitions and theorems around Banach spaces, normed spaces, convergent and Cauchy sequences, completeness, Compactness, compact operators, Arzelà-Ascoli theorem, Fractional Calculus, Gamma function, Riemann-Liouville and Caputo fractional derivatives, and Fixed Point Theorems.

1.1 Banach Spaces

1.1.1 Normed Space

Definition 1.1 [5]

Let E be a vector space. A **norm** on E is a function $\|\cdot\| : E \to \mathbb{R}^+$ such that for all $x, y \in E$ and $\forall \lambda \in \mathbb{R}$:

1. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.
2. $\|\lambda x\| = |\lambda| \|x\|$.
3. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

We say that $(E, \|\cdot\|)$ is a **normed space**.

Example 1.1

In the space $\mathbb{R}^n$ (or $\mathbb{C}^n$), the norm of any element $x = (x_1, x_2, \ldots, x_n)$ is:

$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

is called the **Euclidean norm**.

1.1.2 Convergent Sequence

Definition 1.2

Let $(E, \|\cdot\|)$ be a normed space. A sequence (x_n) is said to be **convergent** to a limit $\ell \in E$ if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$:

$$\|x_n - \ell\| < \varepsilon.$$

Example 1.2

Let $x_n = \frac{1}{n}$ in the normed space $(\mathbb{R}, |\cdot|)$. (x_n) converges to $\ell = 0$ because:

Let $\epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \implies |x_n - 0| < \epsilon$.
Since $|x_n - 0| = \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}$, we choose $n_0 = \lceil \frac{1}{\epsilon} \rceil + 1$.

$$\forall n \geq n_0 : |x_n - 0| = \frac{1}{n} \leq \frac{1}{n_0} < \epsilon$$

1.1.3 Cauchy Sequence

Definition 1.3

Let $(E, \|\cdot\|)$ be a normed space. A sequence (x_n) is said to be **Cauchy sequence** in $(E, \|\cdot\|)$ iff:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n, m \geq n_0 \implies \|x_n - x_m\| < \varepsilon.$$

Example 1.3

Let $E = C([0,1])$ and $f_n(t) = \frac{t}{n}$.

$$\|f_n - f_m\|_\infty = \max_{0 \leq t \leq 1} \left| \frac{t}{n} - \frac{t}{m} \right| = \left| \frac{1}{n} - \frac{1}{m} \right| \xrightarrow[n,m \to +\infty]{} 0.$$

Remark:

In any normed space (E,$\|\cdot\|$), every convergent sequence is a Cauchy sequence. However, the converse is not generally true.

Proof. (Counter example)

Let $E = (\mathbb{Q}, |\cdot|)$ and $(x_n)_{n \in \mathbb{N}}$ be the sequence in $\mathbb{Q}$ defined by:

$$x_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

For every $n \in \mathbb{N}$, x_n is a sum of rational numbers. Thus, $x_n \in \mathbb{Q}$.

1. The Cauchy Property:

   this sequence is a **Cauchy sequence** because for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m > N$:

$$|x_n - x_m| < \epsilon.$$

This means the terms of the sequence become arbitrarily close to each other as n and m increase.

2. Failure of Convergence:

in the larger space of real numbers $\mathbb{R}$, this sequence converges to Euler's number e:

$$\lim_{n\to\infty} x_n = e.$$

However, since e is an irrational number ($e \notin \mathbb{Q}$), the limit does not exist within the space $\mathbb{Q}$.

Conclusion:

(x_n) is a **Cauchy sequence** in $\mathbb{Q}$, but it is **not a convergent sequence** in $\mathbb{Q}$. This proves that the converse of the theorem (every convergent sequence is Cauchy) is not generally true. □

1.1.4 Completeness

Definition 1.4 Complete Metric Space

A metric space (E, d) is said to be complete if every Cauchy sequence (x_n) in E converges to an element of E.

$$(x_n \text{ Cauchy in } E) \implies (\exists x \in E : x_n \to x \text{ as } n \to \infty).$$

Example 1.4

Consider the real numbers $\mathbb{R}$ with the usual absolute value distance $|\cdot|$. Every Cauchy sequence (x_n) in $\mathbb{R}$ converges to a real number. Hence,

$$(\mathbb{R}, |\cdot|) \text{ is a complete metric space.}$$

Definition 1.5 Banach Space / Complete Normed Space

A normed vector space $(E, \|\cdot\|)$ is called a *Banach space* (or complete normed space) if every Cauchy sequence (x_n) in E converges with respect to the norm to an element of E.

$$((x_n) \text{ Cauchy in } E) \implies (\exists x \in E : \|x_n - x\| \to 0 \text{ as } n \to \infty).$$

Example 1.5

Let $C([a, b])$ be the space of continuous functions on the closed interval $[a, b]$, equipped with the sup norm:

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|.$$

Every Cauchy sequence of continuous functions with respect to this norm converges to a continuous function on $[a, b]$. Therefore,

$$(C([a, b]), \|\cdot\|_\infty) \text{ is a Banach space.}$$

1.2 Compactness

1.2.1 Compact sets

Definition 1.6

A subset K of a normed space E is said to be **compact** if every sequence in K has a subsequence that converges to a limit in K.

Example 1.6

In $\mathbb{R}^n$, consider the set

$$K = \{x \in \mathbb{R}^n : \|x\| \leq 1\},$$

which is the closed unit ball.

This set is compact because it is closed and bounded in $\mathbb{R}^n$.

Remark:

$$A \text{ is relatively compact} \quad \Longleftrightarrow \overline{A} \text{ is compact.}$$

In other words, a set is relatively compact if and only if its closure is compact.

1.2.2 Compact operators

Definition 1.7 [7]

An operator $T : E \to F$ (where E and F are normed spaces) is called **compact** if any of the following equivalent conditions hold:

1. The image of every bounded set in E is relatively compact in F.
2. The image of the unit ball $B(0, 1)$ is relatively compact in F.
3. For every bounded sequence (x_n) in E, there exists a subsequence (x_{n_k}) such that $(T(x_{n_k}))$ converges in F.

Example 1.7

Consider the operator $T : \mathbb{R} \to \mathbb{R}$ defined by: $T(x) = 3x$.
T is compact.

Proof.
Let $A \subset \mathbb{R}$ be a bounded set. Since A is bounded, there exists a constant $M > 0$ such that:

$$|x| \leq M, \qquad \forall x \in A.$$

For every $x \in A$, we have:

$$|T(x)| = |3x|$$

thus,

$$|T(x)| \leq 3M.$$

Hence, the set

$$T(A) = \{3x : x \in A\}$$

is bounded in $\mathbb{R}$.

Since every bounded set in $\mathbb{R}$ is relatively compact(endowed with the usual topology), it follows that:

$$T(A)$$

is relatively compact.

Therefore, the operator T is **compact**.

□

Arzelà-Ascoli theorem

Definition 1.8 Equicontinuous family

Let E and F be normed spaces. A family of mappings $\mathcal{F} = \{f_i : E \to F, i \in I\}$ is said to be **equicontinuous at a point** $x_0 \in E$ if:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in E, \forall i \in I \ : \ \|x - x_0\|_E < \delta \implies \|f_i(x) - f_i(x_0)\|_F < \epsilon.$$

If the family is equicontinuous at every point of E, we say it is **equicontinuous on** E.

Example 1.8

Consider the family $\mathcal{F} = \{f_a(x) = ax : a \in [0, 1]\}$ defined on $[0, 1]$. This family is equicontinuous since:

$$|f_a(x) - f_a(y)| = |a||x - y| \leq |x - y|,$$

which shows that all functions in the family share the same Lipschitz constant.

Definition 1.9 Uniform Boundedness

A set S of functions from E to F is **uniformly bounded** if there exists $M > 0$ such that:

$$\|f(x)\| \leq M, \quad \forall f \in S, \forall x \in E.$$

Example 1.9

Consider the family of functions:

$$\mathcal{F} = \left\{ f_n(x) = \frac{\sin(nx)}{n} \ ; \ n \geq 1 \right\}$$

defined on $[0, 1]$.

Since

$$\left| \frac{\sin(nx)}{n} \right| \leq \frac{1}{n} \leq 1,$$

the family $\mathcal{F}$ is uniformly bounded on $[0, 1]$.

Theorem 1.1 [7] Arzela-Ascoli

Let E (compact) and F be normed spaces. A set H of continuous functions from E to F is **relatively compact** if and only if:

1. H is equicontinuous.
2. H is uniformly bounded.

Example 1.10

The family of functions defined on $[0, 1]$:

$$f_n(x) = \frac{\cos x}{n}, \quad n \geq 1.$$

1. Uniform boundedness:
Since

$$|\cos x| \leq 1,$$

then

$$|f_n(x)| = \left|\frac{\cos x}{n}\right| \leq \frac{1}{n} \leq 1.$$

Thus, the family is uniformly bounded.
2. Equicontinuity :
We have:

$$|f_n(x) - f_n(y)| = \left|\frac{\cos x - \cos y}{n}\right|.$$

Since f_n is differentiable on $[0, 1]$, by the **Mean Value Theorem**, for any $x, y \in [0, 1]$, there exists $c \in [0, 1]$ such that:

$$|f_n(x) - f_n(y)| = |f_n'(c)| \cdot |x - y| = \left|\frac{-\sin c}{n}\right| \cdot |x - y|.$$

As $|\sin c| \leq 1$, $\forall c \in [0, 1]$ and $n \geq 1$, we obtain:

$$|f_n(x) - f_n(y)| \leq |x - y|.$$

The obtained bound is independent of n, which proves the equicontinuity.

Consequently, by the **Arzelà–Ascoli theorem**, the family $\{f_n\}$ is **relatively compact** on $[0, 1]$.

1.3 Fractional Differential Calculus

1.3.1 Gamma function

Definition 1.10 [16]

$\Gamma(z)$ is defined by the following integral:

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt, \quad z \in \mathbb{C},\ (\mathrm{Re}(z) > 0).$$

Properties :

1. Recursive property:
$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$
2. Relation with factorial:
$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}.$$
3. Basic value:
$$\Gamma(1) = 1.$$
4. Reflection formula:
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad 0 < x < 1.$$
5. The Gamma function is continuous and well-defined on $(0, \infty)$.

1.3.2 Fractional Order Integration [16]

Let f be a continuous function on the interval $[a, b]$. We have:

$$I_a^{(1)} f(x) = \int_a^x f(t)dt.$$

And:

$$\begin{aligned} I_a^{(2)} f(x) &= \int_a^x I_a^{(1)} f(t)dt \\ &= \int_a^x \left(\int_a^t f(u)du \right) dt \\ &= \int_a^x dt \int_a^t f(u)du. \end{aligned}$$

By changing the order of integration, we obtain:

$$I_a^{(2)} f(x) = \int_a^x (x-t) f(t)dt.$$

By integrating n times:

$$I_a^{(n)} f(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n f(x_n).$$

We get:

$$I_a^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt \tag{1.1}$$

and that is for every $n \in \mathbb{N}$.
This formula is called the **Cauchy formula** using the Gamma function

$$\Gamma(n) = (n-1)!.$$

Riemann and Liouville realized that the right-hand side of equation (1.1) could have meaning even when n takes non-integer values.

Thus, the fractional integral of order α is defined as follows:

Riemann–Liouville Fractional Integral

Definition 1.11 [13]

Let $\Omega = [a, b]$, $\alpha \in \mathbb{R}_+$ and f be a function defined on $[a, b]$. The **left-sided Riemann–Liouville fractional integral** of order α is defined by:

$$I_{a+}^{(\alpha)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \tag{1.2}$$

where $x \in [a, b]$.

As for the **Right-sided Riemann-Liouville fractional integral** of order α, it is written as:

$$I_{b-}^{(\alpha)} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \tag{1.3}$$

where $x \in [a, b]$.

Remark:

We can write the fractional integral in the form:

$$I_a^\alpha f(x) = \int_a^x g(x-t) f(t) dt = (g * f)(x), \tag{1.4}$$

where

$$g(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}. \tag{1.5}$$

Properties:

In all the following, we assume that: $\alpha, \beta > 0, n = \lceil \alpha \rceil + 1, f \in C[a, b]$.

- **Linearity:**

$$I_a^\alpha [c_1 f(x) + c_2 g(x)] = c_1 I_a^\alpha f(x) + c_2 I_a^\alpha g(x). \tag{1.6}$$

Proof. By the definition of the fractional integral:

$$\begin{aligned} I_a^\alpha[c_1 f(x) + c_2 g(x)] &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}[c_1 f(t) + c_2 g(t)]dt \\ &= \frac{c_1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt + \frac{c_2}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt \\ &= c_1 I_a^\alpha f(x) + c_2 I_a^\alpha g(x). \end{aligned}$$

□

- **Power Rule:**[4]

$$I_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}. \tag{1.7}$$

Proof. Using the integral definition:

$$I_a^\alpha (x-a)^\beta = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}(t-a)^\beta dt.$$

Let $t = a + s(x-a)$, then $dt = (x-a)ds$. For the limits: if $t = a \to s = 0$, and if $t = x \to s = 1$. Substituting these:

$$\begin{aligned} I_a^\alpha (x-a)^\beta &= \frac{1}{\Gamma(\alpha)} \int_0^1 [x - (a + s(x-a))]^{\alpha-1} [s(x-a)]^\beta (x-a) ds \\ &= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^\beta ds. \end{aligned}$$

The integral $\int_0^1 s^\beta (1-s)^{\alpha-1} ds$ is the Beta function $B(\beta+1, \alpha)$.

Since $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we get:

$$I_a^\alpha (x-a)^\beta = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}.$$

□

- **Semigroup Property:**[16]

$$I_a^\alpha I_a^\beta f(x) = I_a^{\alpha+\beta} f(x). \tag{1.8}$$

Proof. Using the definition twice and applying Dirichlet's formula to switch the order of integration:

$$\begin{aligned} I_a^\alpha I_a^\beta f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} \left(\int_a^t (t-u)^{\beta-1} f(u)du \right) dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(u) \left(\int_u^x (x-t)^{\alpha-1} (t-u)^{\beta-1} dt \right) du. \end{aligned}$$

The inner integral simplifies to $(x-u)^{\alpha+\beta-1} B(\alpha, \beta)$. Thus,

$$I_a^\alpha I_a^\beta f(x) = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-u)^{\alpha+\beta-1} f(u)du = I_a^{\alpha+\beta} f(x).$$

□

- **Identity Operator:**[6]

$$\lim_{\alpha \to 0^+} I_a^\alpha f(x) = f(x). \tag{1.9}$$

Proof. Using the fundamental theorem of calculus and the derivative relation:

$$I_a^0 f(x) = \frac{d}{dx} I_a^1 f(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

□

- **Derivative Relation:**[4]

$$\frac{d^n}{dx^n} I_a^\alpha f(x) = I_a^{\alpha-n} f(x), \quad \alpha > n. \tag{1.10}$$

Proof. Using the semigroup property of fractional integrals, we can write:

$$I_a^\alpha f(x) = I_a^n (I_a^{\alpha-n} f(x)).$$

Applying the n-th order derivative operator $\frac{d^n}{dx^n}$ to both sides:

$$\frac{d^n}{dx^n} I_a^\alpha f(x) = \frac{d^n}{dx^n} \left[I_a^n (I_a^{\alpha-n} f(x)) \right].$$

By the Fundamental Theorem of Calculus, the n-th derivative cancels the n-th integral:

$$\frac{d^n}{dx^n} I_a^\alpha f(x) = I_a^{\alpha-n} f(x).$$

□

1.3.3 Fractional Order Differentiation

Riemann–Liouville Fractional Derivative

Definition 1.12 [13]

Let $\alpha > 0$ and let $n = \lceil \alpha \rceil + 1$. The Riemann–Liouville fractional derivative $D_a^\alpha f$ of order α is defined by:

$$({}^{RL}D_a^\alpha f)(x) = \frac{d^n}{dx^n}\left(I_a^{n-\alpha} f\right)(x),$$

that is,

$$({}^{RL}D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1} f(t)\,dt. \tag{1.11}$$

Properties:

In all the following, we assume that: $\alpha > 0, n = \lceil \alpha \rceil + 1, f \in C[a,b]$.

- **Derivative of Constant:**[13]

$$^{RL}D_a^\alpha C = \frac{C(x-a)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{1.12}$$

Proof. For $0 < \alpha < 1$:

$$\begin{aligned} ^{RL}D_a^\alpha C &= \frac{d}{dx} I_a^{1-\alpha} C = \frac{d}{dx}\left(\frac{C}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha} dt\right) \\ &= \frac{d}{dx}\left(\frac{C(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}\right) = \frac{C(1-\alpha)(x-a)^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \\ &= \frac{C(x-a)^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned}$$

□

- **Left Inverse Property:**[13]

$$^{RL}D_a^\alpha I_a^\alpha f(x) = f(x). \tag{1.13}$$

Proof.

$$\begin{aligned} ^{RL}D_a^\alpha I_a^\alpha f(x) &= \frac{d^n}{dx^n} I_a^{n-\alpha} I_a^\alpha f(x) \\ &= \frac{d^n}{dx^n} I_a^n f(x) \quad \text{(by semigroup property)} \\ &= f(x). \end{aligned}$$

□

- **Fundamental Theorem Relation:**[4]

$$I_a^{\alpha\,RL}D_a^\alpha f(x) = f(x) - \sum_{k=1}^{n} \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} [{}^{RL}D_a^{\alpha-k} f(t)]_{t=a}. \tag{1.14}$$

Proof. Let $\Phi(x) = I_a^{n-\alpha} f(x)$. By the definition of the Riemann-Liouville derivative:

$${}^{RL}D_a^\alpha f(x) = \frac{d^n}{dx^n}\Phi(x).$$

Applying the fractional integral I_a^α and using the semigroup property $I_a^\alpha = I_a^n I_a^{\alpha-n}$:

$$\begin{aligned} I_a^{\alpha\,RL}D_a^\alpha f(x) &= I_a^{\alpha-n}\left(I_a^n \frac{d^n}{dx^n}\Phi(x)\right) \\ &= I_a^{\alpha-n}\left(\Phi(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}\Phi^{(k)}(a)\right). \end{aligned}$$

Substituting $\Phi(x)$ back and noting that $I_a^{\alpha-n}\Phi(x) = I_a^{\alpha-n} I_a^{n-\alpha} f(x) = f(x)$, and evaluating the terms in the summation using the power rule for fractional integrals, we obtain:

$$I_a^{\alpha\,RL}D_a^\alpha f(x) = f(x) - \sum_{k=1}^{n} \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} [{}^{RL}D_a^{\alpha-k} f(t)]_{t=a}.$$

□

Example 1.11

Calculate the RL fractional derivative of $f(x) = (x-a)^2$ with order $\alpha = 0.5$:

$${}^{RL}D_a^{0.5}(x-a)^2 = \frac{\Gamma(2+1)}{\Gamma(2-0.5+1)}(x-a)^{2-0.5} = \frac{2}{\Gamma(2.5)}(x-a)^{1.5}.$$

Caputo Fractional Derivative

Definition 1.13 [4]

Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$. The Caputo fractional derivative of order α is defined by:

$$({}^{C}D_a^\alpha f)(x) = I_a^{n-\alpha}\left(f^{(n)}\right)(x),$$

or equivalently,

$$({}^{C}D_{a+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt. \tag{1.15}$$

Properties:

In all the following, we assume that: $\alpha > 0, n = \lceil \alpha \rceil + 1, f \in C[a,b]$.

- **Derivative of a Constant:**[6]

$$ {}^C D_a^\alpha C = 0. \tag{1.16}$$

Proof. By the definition of the Caputo fractional derivative for $n-1 < \alpha \leq n$:

$$ {}^C D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt.$$

If $f(x) = C$ (a constant), its n-th order integer derivative is zero:

$$f^{(n)}(t) = \frac{d^n}{dt^n} C = 0, \quad \text{for } n \geq 1.$$

Substituting this into the integral expression:

$$ {}^C D_a^\alpha C = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} (0) dt = 0.$$

□

- **Linearity:**

$$ {}^C D_a^\alpha [c_1 f(x) + c_2 g(x)] = c_1 {}^C D_a^\alpha f(x) + c_2 {}^C D_a^\alpha g(x). \tag{1.17}$$

Proof. Using the linearity of the integer-order derivative and the fractional integral operator $I_a^{n-\alpha}$:

$$\begin{aligned} {}^C D_a^\alpha [c_1 f(x) + c_2 g(x)] &= I_a^{n-\alpha} \left[\frac{d^n}{dx^n} (c_1 f(x) + c_2 g(x)) \right] \\ &= I_a^{n-\alpha} [c_1 f^{(n)}(x) + c_2 g^{(n)}(x)] \\ &= c_1 I_a^{n-\alpha} f^{(n)}(x) + c_2 I_a^{n-\alpha} g^{(n)}(x) \\ &= c_1^C D_a^\alpha f(x) + c_2^C D_a^\alpha g(x). \end{aligned}$$

□

- **Power Rule**[6]

$$ {}^C D_a^\alpha (x-a)^p = \begin{cases} 0, & p \in \mathbb{N}_0, p < \lceil \alpha \rceil \\ \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} (x-a)^{p-\alpha}, & p > \lceil \alpha \rceil - 1. \end{cases} \tag{1.18}$$

Proof. For the case $p > n - 1$, where $n = \lceil \alpha \rceil$:

$$\frac{d^n}{dt^n}(t-a)^p = \frac{\Gamma(p+1)}{\Gamma(p-n+1)}(t-a)^{p-n}.$$

Applying the fractional integral $I_a^{n-\alpha}$:

$$\begin{aligned} {}^C D_a^\alpha (x-a)^p &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)} I_a^{n-\alpha}(x-a)^{p-n} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)} \frac{\Gamma(p-n+1)}{\Gamma(n-\alpha+p-n+1)}(x-a)^{n-\alpha+p-n} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-a)^{p-\alpha}. \end{aligned}$$

If $p < n$, the n-th derivative is zero, thus the Caputo derivative is zero. □

- **Left Inverse Property:**[13]

$$^C D_a^\alpha I_a^\alpha f(x) = f(x). \tag{1.19}$$

Proof.

$$\begin{aligned} {}^C D_a^\alpha I_a^\alpha f(x) &= \frac{d^n}{dx^n} I_a^{n-\alpha} I_a^\alpha f(x) \\ &= \frac{d^n}{dx^n} I_a^n f(x) \\ &= f(x). \end{aligned}$$

□

- **Fundamental Theorem Relation:**[6]

$$I_a^\alpha ({}^C D_a^\alpha f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{1.20}$$

Proof. By definition, ${}^C D_a^\alpha f(x) = I_a^{n-\alpha} f^{(n)}(x)$. Applying I_a^α and using the semigroup property ($I_a^\alpha I_a^{n-\alpha} = I_a^n$):

$$I_a^\alpha ({}^C D_a^\alpha f(x)) = I_a^n f^{(n)}(x).$$

According to the Fundamental Theorem of Calculus for n-fold integration:

$$I_a^n f^{(n)}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

□

Example 1.12

Let us compute the Caputo fractional derivative of $f(x) = (x-a)^3$ with order $\alpha = 0.5$.

Given $p = 3$ and $\alpha = 0.5$, since $p > \lceil \alpha \rceil - 1$, we apply the power rule for the Caputo derivative:

$$^{C}D_a^{\alpha}(x-a)^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-a)^{p-\alpha}.$$

substituting the values:

$$\begin{aligned} ^{C}D_a^{0.5}(x-a)^3 &= \frac{\Gamma(3+1)}{\Gamma(3-0.5+1)}(x-a)^{3-0.5} \\ &= \frac{\Gamma(4)}{\Gamma(3.5)}(x-a)^{2.5} \\ &= \frac{6}{\Gamma(3.5)}(x-a)^{2.5}. \end{aligned}$$

1.3.4 Relation Between Riemann–Liouville and Caputo Derivatives

Theorem 1.2 [5, 6]

Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. If $f \in C^n([a,b])$, then the Caputo and Riemann–Liouville derivatives are related by:

$$^{C}D_{a+}^{\alpha}f(x) = {}^{RL}D_{a+}^{\alpha}\left(f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right).$$

Equivalently, one may express the Riemann–Liouville derivative in terms of the Caputo derivative:

$$^{RL}D_{a+}^{\alpha}f(x) = {}^{C}D_{a+}^{\alpha}f(x) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}.$$

1.4 Fixed Point Theorems

Definition 1.14 [17] Contraction Mapping

In a Banach space X, an operator $T : X \to X$ is said to be a **contraction** if there exists a constant $k \in [0, 1)$ such that:

$$\|T(x) - T(y)\| \leq k\|x - y\|, \quad \forall x, y \in X. \tag{1.21}$$

Definition 1.15 [5] Convex set

Let V be a vector space, and let the set A be a subset of V. We say that A is **convex** if:

$$\forall u, v \in A, \forall \lambda \in [0, 1] : \lambda u + (1 - \lambda)v \in A. \tag{1.22}$$

1.4.1 Banach Fixed Point Theorem

Theorem 1.3 [17]

Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be a contraction mapping, i.e.,there exists a constant $k \in (0, 1)$ such that:

$$\|T(x) - T(y)\| \leq k\|x - y\|, \quad \forall x, y \in X.$$

Then there exists a unique element $x^* \in X$ such that:

$$T(x^*) = x^*.$$

Moreover, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to x^*.

1.4.2 Schauder Fixed Point Theorem

Theorem 1.4 [7]

Let X be a Banach space and $C \subset X$ be a nonempty, closed, and convex subset. If $T : C \to C$ is a continuous compact map, then T has at least one fixed point in C.

1.4.3 Schaefer Fixed Point Theorem

Theorem 1.5 [7]

Let X be a Banach space and $T : X \to X$ be a continuous compact map. If the set

$$\mathcal{S} = \{x \in X \mid x = \lambda Tx, \text{ for some } \lambda \in [0, 1]\} \tag{1.23}$$

is bounded, then T has at least one fixed point.

1.4.4 Krasnoselskii Fixed Point Theorem

Theorem 1.6 [17]

Let M be a closed, bounded, and convex subset of a Banach space E. Let T_1, T_2 be two operators such that:

1. $T_1x + T_2y \in M$ for all $x, y \in M$.
2. T_1 is a contraction mapping.
3. T_2 is continuous and compact (completely continuous).

Then there exists at least one fixed point $x \in M$ such that $x = T_1x + T_2x$.

Chapter 2

Existence results for Caputo fractional initial value problems

This chapter is devoted to the existence of solutions for fractional initial value problem (FIVP) involving Caputo fractional differential equations.By transforming the differential system into an equivalent integral equation,we establish the primary existence results using fixed-point theorems. The approach and findings developed here draw directly from the works of [2], [8], and [9].

2.1 Equivalent Integral Form of the FIVP

We consider the following nonlinear fractional integro-differential initial value problem (FIVP):

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)) + \int_0^t k(t, s, x(s))ds + \int_0^1 h(t, s, x(s))ds, \quad t \in J = [0, 1], \\ x(0) = 0, \\ x'(0) = \lambda \int_0^1 g(s, x(s))ds, \end{cases} \tag{2.1}$$

where:

- ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α, with $1 < \alpha \leq 2$.
- $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ are continous functions.
- $k, h : J \times J \times \mathbb{R} \to \mathbb{R}$ are continous Volterra and Fredholm integral kernels, respectively.
- $\lambda \in \mathbb{R}$ is a real constant.

Proposition 1 [9]

Let $J = [0, 1]$ and let $E = C(J, \mathbb{R})$ be the space of all continuous functions defined on the interval J. This space is a Banach space when endowed with the supremum norm (also known as the uniform norm) defined by:

$$\|x\|_E = \sup_{t \in J} |x(t)|.$$

Lemma 2.1
A function $x \in E$ is a solution to the FIVP (2.1) if and only if it satisfies the integral equation:

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} [\Psi x] (s)ds + \lambda t \int_0^1 g(s, x(s))ds, \tag{2.2}$$

where the operator Ψ is defined as:

$$[\Psi x](t) = f(t, x(t)) + \int_0^t k(t, s, x(s))ds + \int_0^1 h(t, s, x(s))ds. \tag{2.3}$$

Proof. Assume x satisfies (2.1). Integrating both sides with the Riemann-Liouville operator I^α:

$$x(t) = I^\alpha \left[f(s, x(s)) + \int_0^s k(s, \tau, x(\tau))d\tau + \int_0^1 h(s, \tau, x(\tau))d\tau \right] + c_0 + c_1 t.$$

Applying the first condition $x(0) = 0$:

$$x(0) = I^\alpha \Psi x(0) + c_0 + c_1(0) = 0 \implies c_0 = 0.$$

To find c_1, we differentiate $x(t)$ with respect to t:

$$x'(t) = \frac{d}{dt} I^\alpha \Psi x(t) + c_1 = I^{\alpha-1} \Psi x(t) + c_1.$$

Evaluating at $t = 0$, and noting that $\alpha - 1 > 0$, the integral term vanishes:

$$x'(0) = 0 + c_1 \implies c_1 = x'(0).$$

Substituting the given integral condition $x'(0) = \lambda \int_0^1 g(s, x(s))ds$, we obtain:

$$c_1 = \lambda \int_0^1 g(s, x(s))ds.$$

Thus, substituting c_0 and c_1 into the solution expression yields (2.2).

Conversely, Assume that $x(t)$ satisfies the integral equation (2.2). We shall show that x is a solution of the FIVP (2.1).

First, applying the Caputo fractional derivative ${}^C D^\alpha$ to both sides of (2.2), we get:

$${}^C D^\alpha x(t) = {}^C D^\alpha \left(I^\alpha [\Psi x](t) + \lambda t \int_0^1 g(s, x(s))ds \right).$$

By the Left Inverse Property (1.19), we have:

$${}^C D^\alpha I^\alpha [\Psi x](t) = [\Psi x](t).$$

Since $\lambda \int_0^1 g(s, x(s))ds$ is a constant, and using the Power Rule (1.18) for Caputo derivative where ${}^C D^\alpha t = 0$ for $1 < \alpha \leq 2$, it follows that:

$${}^C D^\alpha \left(\lambda t \int_0^1 g(s, x(s))ds \right) = 0.$$

Consequently, we obtain:

$${}^C D^\alpha x(t) = [\Psi x](t).$$

Next, we verify the initial conditions.

For $x(0)$, by substituting $t = 0$ into (2.2):

$$x(0) = \frac{1}{\Gamma(\alpha)} \int_0^0 (0 - s)^{\alpha-1} [\Psi x](s) ds + \lambda(0) \int_0^1 g(s, x(s)) ds = 0.$$

For $x'(0)$, we differentiate(2.2) with respect to t :

$$\begin{aligned} x'(t) &= \frac{d}{dt} \left[I^\alpha [\Psi x](t) + \lambda t \int_0^1 g(s, x(s)) ds \right] \\ &= \frac{d}{dt} I^\alpha [\Psi(x)](t) + \lambda \int_0^1 g(s, x(s)) ds. \end{aligned}$$

By applying the Derivative Relation (1.10), we obtain:

$$x'(t) = I^{\alpha-1} [\Psi x](t) + \lambda \int_0^1 g(s, x(s)) ds.$$

Evaluating at $t = 0$, and noting that since $\alpha > 1$, we have $\alpha - 1 > 0$, thus the integral term vanishes:

$$x'(0) = 0 + \lambda \int_0^1 g(s, x(s)) ds = \lambda \int_0^1 g(s, x(s)) ds.$$

Thus, $x(t)$ satisfies all conditions of the FIVP (2.1), which completes the proof of the equivalence.

□

2.2 Existence Results via Banach Principle

In this section, we apply the Banach fixed point theorem (1.4.1) to prove the existence and uniqueness of the solution for the problem (2.1).

First, for $x \in E$, we define an operator $\mathcal{T}$ by:

$$(\mathcal{T}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} [\Psi x](s) ds + \lambda t \int_0^1 g(s, x(s)) ds. \tag{2.4}$$

We begin by noting that the operator $\mathcal{T}$ is **well-defined**. For any $x \in E$, the continuity of f, g, k and h implies that the image $\mathcal{T}x$ is a continuous function on J.
Consequently, $\mathcal{T}x \in C(J, \mathbb{R})$, which ensures that $\mathcal{T}(E) \subset E$.

Now, Assume the following hypotheses hold:

(H1): There exists $L_f > 0$ such that $|f(t, u) - f(t, v)| \leq L_f |u - v|, \quad \forall t \in J, \forall u, v \in \mathbb{R}$.

(H2): There exist $L_k, L_h > 0$ such that:

$$|k(t, s, u) - k(t, s, v)| \leq L_k |u - v|, \quad |h(t, s, u) - h(t, s, v)| \leq L_h |u - v|.$$

(H3): There exists $L_g > 0$ such that $|g(s, u) - g(s, v)| \leq L_g |u - v|$.

(H4): There exist positive constants M_f, M_g, M_k and M_h such that:

$$\sup_{t \in J} |f(t, 0)| = M_f, \ \sup_{s \in J} |g(s, 0)| = M_g, \ \sup_{t,s \in J} |k(t, s, 0)| = M_k, \ \sup_{t,s \in J} |h(t, s, 0)| = M_h.$$

(H5): The constant λ and the Lipschitz constant L_g satisfy the condition: $|\lambda|L_g < 1$.

Theorem 2.1

With the assumptions (H1)-(H3), If the following condition is satisfied:

$$\Omega := \frac{L_f + L_k + L_h}{\Gamma(\alpha+1)} + |\lambda|L_g < 1, \tag{2.5}$$

then the FIVP (2.1) has a unique solution on J.

Proof. Let $x, y \in E$. For each $t \in J$:

$$\begin{aligned}|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|[\Psi x](s) - [\Psi y](s)|ds + |\lambda|t\int_0^1 |g(s,x(s)) - g(s,y(s))|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Bigg(L_f|x(s)-y(s)| + \int_0^s L_k|x(\tau)-y(\tau)|d\tau \\ &+ \int_0^1 L_h|x(\tau)-y(\tau)|d\tau\Bigg)ds + |\lambda|L_g\|x-y\|.\end{aligned}$$

Since $\|x-y\| = \sup_{t\in J}|x(t)-y(t)|$, we have:

$$\begin{aligned}|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\leq \frac{(L_f + L_k + L_h)\|x-y\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + |\lambda|L_g\|x-y\| \\ &\leq \left(\frac{(L_f + L_k + L_h)t^\alpha}{\Gamma(\alpha+1)} + |\lambda|L_g\right)\|x-y\|.\end{aligned}$$

Taking the supremum over $t \in J$:

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \left(\frac{L_f + L_k + L_h}{\Gamma(\alpha+1)} + |\lambda|L_g\right)\|x-y\| = \Omega\|x-y\|.$$

Since $\Omega < 1$, the operator $\mathcal{T}$ is a contraction , E is a Banach space and $\mathcal{T}$ maps E into itself. By the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point on E. □

2.3 Existence Results via Krasnoselskii's Fixed Point Theorem

To establish the existence of at least one solution for the fractional system (2.1), we partition the fixed-point operator $\mathcal{T}$ defined in (2.2) into two distinct sub-operators $\mathcal{A}$ and $\mathcal{S}$ as follows:

$$(\mathcal{A}x)(t) = \lambda t\int_0^1 g(s,x(s))ds, \quad t \in J$$

and

$$(\mathcal{S}x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,x(s)) + \int_0^s k(s,\tau,x(\tau))d\tau + \int_0^1 h(s,\tau,x(\tau))d\tau\right]ds.$$

1. Stability of the Ball $\mathcal{B}_r$

We seek to determine a radius $r > 0$ such that for any pair $x, y \in \mathcal{B}_r = \{z \in E : \|z\| \leq r\}$, the condition $\mathcal{A}x + \mathcal{S}y \in \mathcal{B}_r$ is satisfied.

Using the Lipschitz conditions (H1)-(H3) and (H4), we obtain :

$$\begin{aligned}
|g(s, x(s))| &\leq |g(s, x(s)) - g(s, 0)| + |g(s, 0)| \leq L_g\|x\| + M_g \leq L_g r + M_g, \\
|[\Psi y](s)| &\leq |f(s, y(s))| + \int_0^s |k(s, \tau, y(\tau))| d\tau + \int_0^1 |h(s, \tau, y(\tau))| d\tau \\
&\leq (L_f r + M_f) + s(L_k r + M_k) + (L_h r + M_h) \\
&\leq (L_f + L_k + L_h) r + (M_f + M_k + M_h).
\end{aligned}$$

Let $L_\Psi = L_f + L_k + L_h$ and $M^* = M_f + M_k + M_h$. Then:

$$\begin{aligned}
|(\mathcal{A}x)(t) + (\mathcal{S}y)(t)| &\leq |\lambda||t| \int_0^1 |g(s, x(s))| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |[\Psi y](s)| ds \\
&\leq |\lambda|(L_g r + M_g) + \frac{L_\Psi r + M^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\leq |\lambda| L_g r + |\lambda| M_g + \frac{L_\Psi r + M^*}{\Gamma(\alpha+1)} t^\alpha.
\end{aligned}$$

By taking the supremum over $t \in [0, 1]$, the condition $\|\mathcal{A}x + \mathcal{S}y\| \leq r$ becomes:

$$|\lambda| M_g + \frac{M^*}{\Gamma(\alpha+1)} + \left(|\lambda| L_g + \frac{L_\Psi}{\Gamma(\alpha+1)} \right) r \leq r.$$

Then, we have:

$$|\lambda| M_g + \frac{M^*}{\Gamma(\alpha+1)} \leq r - \left(|\lambda| L_g + \frac{L_\Psi}{\Gamma(\alpha+1)} \right) r.$$

Thus:

$$|\lambda| M_g + \frac{M^*}{\Gamma(\alpha+1)} \leq r \left[1 - \left(|\lambda| L_g + \frac{L_\Psi}{\Gamma(\alpha+1)} \right) \right],$$

$$r \geq \frac{|\lambda| M_g + \frac{M^*}{\Gamma(\alpha+1)}}{1 - \left(|\lambda| L_g + \frac{L_\Psi}{\Gamma(\alpha+1)} \right)}.$$

This condition ensures the stability of the solution within the ball $\mathcal{B}_r$.

- **$\mathcal{B}_r$ is convex:**
  For all $x, y \in \mathcal{B}_r$, and for all $\alpha \in [0, 1]$, let $z = \alpha x + (1-\alpha) y$. We have:

  $$\begin{aligned}
  \|z\| &= \|\alpha x + (1-\alpha) y\| \\
  &\leq \alpha \|x\| + (1-\alpha) \|y\|.
  \end{aligned}$$

  Since $\|x\| \leq r$ and $\|y\| \leq r$, then: $\|z\| \leq \alpha r + (1-\alpha) r = r$. Thus, $z \in \mathcal{B}_r$.

- **$\mathcal{B}_r$ is closed:**
  Let the complement of $\mathcal{B}_r$ be $U = \{z \in E : \|z\| > r\}$. We show that U is an open set.
  Let $x_0 \in U$, then there exists $\epsilon > 0$ such that $\|x_0\| = r + \epsilon$. We choose $\delta = \epsilon$. For any $y \in B(x_0, \delta)$, we have:

$$\begin{aligned}\|y\| &= \|x_0 - (x_0 - y)\| \\ &\geq \|x_0\| - \|x_0 - y\|.\end{aligned}$$

  Since $\|x_0\| = r + \epsilon$ and $\|x_0 - y\| < \delta = \epsilon$: $\|y\| > (r + \epsilon) - \epsilon = r$.
  Thus, $\|y\| > r \implies y \in U$.

  Therefore, U is an open set; consequently, its complement $\mathcal{B}_r$ is closed.

2. Contraction of Operator $\mathcal{A}$

For any $x, y \in \mathcal{B}_r$ and for each $t \in J$, we evaluate the difference:

$$\begin{aligned}|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| &= \left|\lambda t \int_0^1 g(s, x(s))ds - \lambda t \int_0^1 g(s, y(s))ds\right| \\ &\leq |\lambda||t| \int_0^1 |g(s, x(s)) - g(s, y(s))|ds \\ &\leq |\lambda| L_g \|x - y\| \int_0^1 ds = |\lambda| L_g \|x - y\|.\end{aligned}$$

Taking the supremum over $t \in [0, 1]$, we obtain $\|\mathcal{A}x - \mathcal{A}y\| \leq |\lambda| L_g \|x - y\|$. By the assumption (H5): $|\lambda| L_g < 1$, the operator $\mathcal{A}$ is a contraction.

3. Compactness and Continuity of Operator $\mathcal{S}$

The complete continuity of $\mathcal{S}$ is established by proving its continuity and compactness through the Arzelà-Ascoli theorem.

Continuity: Since f, k, and h are continuous, the composite operator Ψ is continuous. Consequently, $\mathcal{S}$ being an integral of a continuous function is also continuous.

Uniform Boundedness: For all $x \in \mathcal{B}_r$, we have already shown:

$$\|\mathcal{S}x\| \leq \frac{L_\Psi r + M^*}{\Gamma(\alpha + 1)}.$$

This constant bound proves that the set $\mathcal{S}(\mathcal{B}_r)$ is uniformly bounded.

Equicontinuity:
Let $t_1, t_2 \in J$ with $t_1 < t_2$. Let $M_r = \sup_{x \in \mathcal{B}_r} \|\Psi x\|$. We compute:

$$\begin{aligned}|(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)| &= \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_2} (t_2 - s)^{\alpha-1} \Psi x(s)ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} \Psi x(s)ds\right| \\ &\leq \frac{M_r}{\Gamma(\alpha)} \left|\int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds\right| \\ &= \frac{M_r}{\Gamma(\alpha)} \left|\int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1})ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds\right|\end{aligned}$$

$$= \frac{M_r}{\Gamma(\alpha+1)} \left|(t_1^\alpha - t_2^q + (t_2 - t_1)^\alpha) + (t_2 - t_1)^\alpha\right|$$
$$= \frac{M_r}{\Gamma(\alpha+1)} \left|t_1^\alpha - t_2^\alpha + 2(t_2 - t_1)^\alpha\right|.$$

The right-hand side of the inequality is independent of $x \in \mathcal{B}_r$ and, by the continuity of t^α and $(t_2 - t_1)^\alpha$, approaches zero as $t_1 \to t_2$.
By the Arzelà-Ascoli theorem, $\mathcal{S}(\mathcal{B}_r)$ is relatively compact. Hence, $\mathcal{S}$ is completely continuous.

Conclusion

We have demonstrated that:
1. $\mathcal{B}_r$ is a closed, convex, and non-empty subset of the Banach space $C(J, \mathbb{R})$.
2. $\mathcal{A}x + \mathcal{S}y \in \mathcal{B}_r$ for all $x, y \in \mathcal{B}_r$.
3. $\mathcal{A}$ is a contraction mapping.
4. $\mathcal{S}$ is continuous and compact.

All hypotheses of Krasnoselskii's fixed point theorem are satisfied. Therefore, the operator $\mathcal{T} = \mathcal{A} + \mathcal{S}$ has at least one fixed point $x^* \in \mathcal{B}_r$, which is the solution to the fractional integro-differential equation (2.1).

2.4 Hyers-Ulam Stability

In this section, we investigate the Hyers–Ulam stability of the fractional integro–differential equation given by problem (2.1).

Definition 2.1 [1]
The fractional initial value problem (2.1) is said to be *Hyers–Ulam stable* if there exists a constant $C > 0$ such that for every $\varepsilon > 0$ and for any function $y \in C(J, \mathbb{R})$ satisfying

$$\left|{}^C D^\alpha y(t) - f(t, y(t)) - \int_0^t k(t, s, y(s))ds - \int_0^1 h(t, s, y(s))ds\right| \le \varepsilon,$$

for all $t \in J = [0, 1]$, there exists an exact solution x of problem (2.1) such that

$$\|y - x\| \le C\varepsilon.$$

Stability Result

Theorem 2.2
Assume that the functions f, k, h, g satisfy the following Lipschitz conditions:

$$|f(t, x) - f(t, y)| \le L_f |x - y|,$$

$$|k(t, s, x) - k(t, s, y)| \le L_k |x - y|,$$

$$|h(t, s, x) - h(t, s, y)| \le L_h |x - y|,$$

$$|g(t,x) - g(t,y)| \leq L_g |x-y|.$$

If the constant

$$L = L_f + L_k + L_h + |\lambda| L_g$$

satisfies $L < 1$, then the initial value problem (2.1) is Hyers–Ulam stable.

Proof. Let y be an approximate solution satisfying

$$\left| {}^C D^\alpha y(t) - f(t, y(t)) - \int_0^t k(t,s,y(s))ds - \int_0^1 h(t,s,y(s))ds \right| \leq \varepsilon.$$

Applying the fractional integral operator I^α on both sides, we obtain:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Psi y(s) ds + \lambda t \int_0^1 g(s, y(s)) ds + E(t),$$

where $|E(t)| \leq C_1 \varepsilon$.
Let $x(t)$ be the exact solution of equation (2.1). Then

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Psi x(s) ds + \lambda t \int_0^1 g(s, x(s)) ds.$$

Subtracting the two expressions gives

$$|y(t) - x(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\Psi y(s) - \Psi x(s)| ds + |\lambda| \left| \int_0^1 [g(s,y(s)) - g(s,x(s))] ds \right| + C_1 \varepsilon.$$

Using the Lipschitz conditions, we obtain:

$$|y(t) - x(t)| \leq L \int_0^t |y(s) - x(s)| ds + C_1 \varepsilon.$$

Applying Gronwall's inequality yields

$$|y(t) - x(t)| \leq C\varepsilon,$$

for some constant $C > 0$. Therefore

$$\|y - x\| \leq C\varepsilon.$$

Hence, the fractional initial value problem (2.1) is Hyers–Ulam stable.

□

2.5 Applications

Example 1:

Consider the following fractional integro-differential initial value problem:

$$\begin{cases} {}^c D^{1.5} x(t) = \frac{1}{10} e^t \sin(x(t)) + \int_0^t \frac{s}{10} x(s) ds + \int_0^1 \frac{1}{10}(x(s)+1) ds, \\ x(0) = 0, \\ x'(0) = \frac{1}{4} \int_0^1 x(s) ds, \end{cases} \tag{2.6}$$

where $t \in [0, 1]$.

Method 1: Existence results by Banach Principle

a) Verification of Lipschitz conditions ($H1 - H3$)

We have $\alpha = 1.5$, $\lambda = \frac{1}{4}$ and $g(s, x(s)) = x(s)$.

1. $|f(t, x(t)) - f(t, y(t))| \leq \frac{e^t}{10}|\sin(x(t)) - \sin(y(t))| \leq \frac{e}{10}|x - y|$. Thus, $L_f = \frac{e}{10}$.
2. $|K(t, s, x(s)) - K(t, s, y(s))| = |\frac{s}{10}x(s) - \frac{s}{10}y(s)| = \frac{s}{10}|x - y|$. Since $t \in J$, then $L_k = \frac{1}{10} = 0.1$.
3. $|h(t, s, x) - h(t, s, y)| = |\frac{1}{10}(x + 1) - \frac{1}{10}(y + 1)| = \frac{1}{10}|x - y|$. Then, $L_h = 0.1$.
4. $|g(s, x(s)) - g(s, y(s))| = |x - y|$. Then $L_g = 1$.

Consequently, the total Lipschitz constant L_Ψ for the operator Ψ is:

$$L_\Psi = L_f + L_k + L_h = \frac{e}{10} + 0.1 + 0.1 \approx 0.4718.$$

b) Uniqueness Conditions (Theorem 2.1)

To apply the Banach Fixed Point Theorem, we evaluate the condition $\Omega < 1$ using formula (2.5):

$$\Omega = \frac{L_\Psi}{\Gamma(\alpha + 1)} + |\lambda| L_g.$$

With $\Gamma(1.5 + 1) = \Gamma(2.5) \approx 1.3293$, we substitute the values:

$$\Omega = \frac{0.4718}{1.3} + |\frac{1}{4}| \times 1 \approx 0.3549 + 0.25 \approx 0.6049.$$

Conclusion: Since $\Omega \approx 0.60 < 1$, all the hypotheses of Theorem (2.1) are satisfied. Therefore, the fractional integro-differential initial value problem has a unique solution on $[0, 1]$.

Method 2: Existence results by Krasnoselskii's Fixed Point Theorem

Consider the operator $\mathcal{T} = \mathcal{A} + \mathcal{S}$ defined on the ball $\mathcal{B}_r$ by:

$$(\mathcal{A}x)(t) = \frac{1}{4} t \int_0^1 x(s) ds$$

and

$$(\mathcal{S}x)(t) = \frac{1}{\Gamma(1.5)} \int_0^t (t - s)^{\alpha - 1} \left[\frac{1}{10} e^s \sin(x(s)) + \int_0^s \frac{\tau}{10} x(\tau) d\tau + \int_0^1 \frac{1}{10}(x(\tau) + 1) d\tau \right] ds,$$

where $t \in [0, 1]$.

a) Contraction of operator $\mathcal{A}$:

For all $x, y \in \mathcal{B}_r$, we have:

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \frac{1}{4} t \int_0^1 |x(s) - y(s)| ds \leq 0.25 \|x - y\|.$$

Since $0.25 < 1$, the operator $\mathcal{A}$ is a contraction.

b) Compactness and Continuity of $\mathcal{S}$:

Continuity: The operator $\mathcal{S}$ is continuous since the functions f, h, k are continuous.
Uniform Boundedness: from hypothesis $(H5)$, we have $M_f = 0, M_k = 0$, $M_h = 0.1, M_g = 0$ which implies $M^* = 0.1$. For all $x \in \mathcal{B}_r$:

$$\|\mathcal{S}x\| \leq \frac{L_\Psi r + M^*}{\Gamma(\alpha+1)} = \frac{0.4718r + 0.1}{\Gamma(2.5)} \approx 0.355r + 0.075.$$

Thus, $\mathcal{S}(\mathcal{B}_r)$ is uniformly bounded.
Equicontinuity:
Let $t_1, t_2 \in J$ with $t_1 < t_2$, and let $M_r = \sup_{x \in \mathcal{B}_r} \|\Psi x\|$. We have:

$$\begin{aligned} |(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)| &\leq \frac{M_r}{\Gamma(\alpha+1)} [t_2^\alpha - t_1^\alpha + 2(t_2 - t_1)^\alpha] \\ &= \frac{0.4718r + 0.1}{1.329} \left[t_2^{1.5} - t_1^{1.5} + 2(t_2 - t_1)^{1.5}\right]. \end{aligned}$$

The right-hand side of the inequality is independent of $x \in \mathcal{B}_r$ and approaches zero as $t_1 \to t_2$. Therefore, by the Arzelà-Ascoli theorem, the operator $\mathcal{S}$ is completely continuous.

c) Stability of $\mathcal{B}_r$:

To show that $\mathcal{A}x + \mathcal{S}y \in \mathcal{B}_r$, we must have $\|\mathcal{A}x + \mathcal{S}y\| \leq r$. Using the constants from our example:

$$r \geq \frac{|\lambda| M_g + \frac{M^*}{\Gamma(\alpha+1)}}{1 - \left(|\lambda| L_g + \frac{L_\Psi}{\Gamma(\alpha+1)}\right)} = \frac{0.075}{1 - (0.25 + 0.355)} \approx 0.189.$$

The condition is satisfied for any $r \geq 0.189$.
Conclusion: By Krasnoselskii's fixed point theorem, there exists at least one solution for the problem in $\mathcal{B}_r$.

Study of Hyers-Ulam Stability

Let $y(t)$ be an approximate solution satisfying :

$$\left| {}^C D^\alpha y(t) - f(t, y(t)) - \int_0^t k(t, s, y(s)) ds - \int_0^1 h(t, s, y(s)) ds \right| \leq \varepsilon.$$

By applying the fractional integral operator, we get:

$y(t) = \Psi y(t) + E(t)$ where $|E(t)| \leq C_1 \epsilon$.

For the exact solution $x(t) = \Psi x(t)$, we have:

$$\|y - x\| \leq \Omega \|y - x\| + C_1 \varepsilon \implies (1 - \Omega)\|y - x\| \leq C_1 \varepsilon$$

$$\|y - x\| \leq \frac{C_1}{1 - 0.6049} \epsilon \approx (2.531 C_1)\epsilon.$$

Thus, Example 1 is Hyers-Ulam stable with $C = 2.531 C_1$.

Example 2:

Consider the second problem:

$$\begin{cases} {}^{c}D^{1.5}x(t) = \dfrac{1}{20}\displaystyle\int_0^t \sqrt{t+s}\,\dfrac{|x(s)|}{1+|x(s)|}ds + \dfrac{1}{20}\displaystyle\int_0^1 \dfrac{s^2}{s^2+|x(s)|+2}ds, \quad t \in [0,1], \\ x(0) = 0, \\ x'(0) = \dfrac{1}{10}\displaystyle\int_0^1 s^3 x(s)ds. \end{cases} \tag{2.7}$$

a) Verification of Lipschitz Conditions

We have $\alpha = 1.5$, $\lambda = \frac{1}{10}$ and $g(s, x(s)) = s^3 x(s)$.

1. Using the inequality $\left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| \leq |x-y|$, we get:

   $|K(t,s,x(s)) - K(t,s,y(s))| \leq \frac{\sqrt{t+s}}{20}|x-y|$,

   $\sup_{\substack{t\in J\\ s\in J}} \frac{\sqrt{t+s}}{20} = \frac{\sqrt{2}}{20}$. Thus, $L_k = \frac{\sqrt{2}}{20} \approx 0.0707$.

2. Using the Lipschitz property for rational functions, we have:

   $|h(t,s,x) - h(t,s,y)| \leq \frac{1}{20}\frac{s^2}{(s^2+2)^2}|x-y|$,

   since $s \in J$ then $L_h = \frac{1}{20(3)^2} = \frac{1}{180} \approx 0.0055$.

3. $|g(s,x(s)) - g(s,y(s))| = s^3|x-y| \leq |x-y|$. Thus, $L_g = 1$.

The total Lipschitz constant L_Ψ is:

$$L_\Psi = L_f + L_k + L_h = 0 + 0.0707 + 0.0055 = 0.0762.$$

b) Uniqueness Condition (Theorem 2.1)

$$\begin{aligned} \Omega &= \frac{L_\Psi}{\Gamma(\alpha+1)} + |\lambda| L_g \\ &= \frac{0.0762}{1.3} + 0.1 \times 1 \approx 0.1573. \end{aligned}$$

Conclusion: Since $\Omega \approx 0.1573 < 1$, the uniqueness condition is satisfied, and the problem has a unique solution on $[0,1]$.

c) Study of Hyers-Ulam Stability

To establish the stability results for the rational functions problem, let y(t) be an approximate solution satisfying the inequality

$$\left|{}^{C}D^{\alpha}y(t) - f(t,y(t)) - \int_0^t k(t,s,y(s))ds - \int_0^1 h(t,s,y(s))ds\right| \leq \varepsilon.$$

For the unique exact solution x(t), we establish the following integral inequality:

$$|y(t)-x(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\Psi y(s)-\Psi x(s)| ds + |\lambda| \left| \int_0^1 [g(s,y(s)) - g(s,x(s))] ds \right| + C_1 \varepsilon.$$

By substituting the specific Lipschitz constants: $L_\Psi = 0.0762$, $L_g = 1$ and $\lambda = 0.1$, the inequality becomes:

$$\begin{aligned} \|y-x\| &\leq \left(\frac{0.0762}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + 0.1 \int_0^1 (1) ds \right) \|y-x\| + C_1 \varepsilon \\ &\leq \left(\frac{0.0762}{\Gamma(2.5)} + 0.1 \right) \|y-x\| + C_1 \varepsilon. \end{aligned}$$

Substituting $\Gamma(2.5) \approx 1.3293$, we obtain:

$$\begin{aligned} \|y-x\| &\leq \left(\frac{0.0762}{1.3293} + 0.1 \right) \|y-x\| + C_1 \varepsilon \\ &\leq (0.0573 + 0.1) \|y-x\| + C_1 \varepsilon \\ &\leq 0.1573 \|y-x\| + C_1 \varepsilon. \end{aligned}$$

After simplifying the previous inequality, we obtain:

$$(1 - 0.1573)\|y-x\| \leq C_1 \varepsilon \implies \|y-x\| \leq \frac{C_1}{0.8427} \varepsilon.$$

Setting $C = \frac{1}{0.8427} C_1 \approx 1.1866 C_1$, we find that $C > 0$.

Conclusion: The fractional integro-differential problem is **Hyers-Ulam stable** on $[0,1]$.

Chapter 3

Study the existence of solutions of boundary value problems

This chapter is devoted to the study of the existence and uniqueness of solutions for a mixed system of fractional differential equations involving Riemann-Liouville and Caputo derivatives, which represents a class of fractional Langevin equations. By transforming the problem into an equivalent integral form, we establish our main results using fixed point theorems. The uniqueness of the solution is proved via the Banach Fixed Point Theorem, while the existence is established using the Schauder Fixed Point Theorem. These results provide the necessary qualitative analysis for the proposed fractional boundary value problem, which extends and generalizes earlier models of this type studied in ([10],[12],[14],[18]).

3.1 Problem Formulation

Consider the following boundary value problem:

$$\begin{cases} {}^{RL}D^{\beta}\left({}^{C}D^{\alpha}x(t)+\lambda x(t)\right)=f\left(t,x(t),\int_0^t K(t,s,x(s))ds\right), \quad t\in J=[0,1], \\ x(0)=0, \\ x(1)=\mu\int_0^1 x(s)ds, \\ {}^{C}D^{\alpha}x(0)=0, \end{cases} \tag{3.1}$$

where:

- ${}^{C}D^{\alpha}$ denotes the **Caputo fractional derivative** of order α, with $0<\alpha\leq 1$ (${}^{C}D^{\alpha}$ is continous).
- ${}^{RL}D^{\beta}$ denotes the **Riemann-Liouville fractional derivative** of order β,with $1<\beta\leq 2$.
- $f: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
- $K: J\times J\times\mathbb{R}\to\mathbb{R}$ is a continuous Volterra integral kernel.
- $\lambda,\mu\in\mathbb{R}$ are real constants such that $\mu\neq\beta+\alpha$.

Proposition 2 [3]

Let $E = C(J, \mathbb{R})$ be the Banach space of all continuous functions defined on J endowed with the supremum norm:

$$\|x\|_E = \sup_{t \in J} |x(t)|.$$

Lemma 3.1
Let $h \in C(J, \mathbb{R})$. The linear fractional boundary value problem:

$$\begin{cases} {}^{RL}D^{\beta}\left({}^{C}D^{\alpha}x(t) + \lambda x(t)\right) = h(t), \quad t \in J, \\ x(0) = 0, \quad x(1) = \mu \int_0^1 x(s)ds, \quad {}^{C}D^{\alpha}x(0) = 0. \end{cases} \tag{3.2}$$

In view of (3.2), then x is a solution of the following integral equation:

$$x(t) = I^{\alpha+\beta}h(t) - \lambda I^{\alpha}x(t) + \frac{t^{\beta+\alpha-1}}{\Delta}\Phi(x, h), \tag{3.3}$$

where the non-resonance constant Δ and the functional $\Phi(x, h)$ are defined as:

$$\Delta = 1 - \frac{\mu}{\beta + \alpha} \neq 0,$$
$$\Phi(x, h) = \mu I^{\alpha+\beta+1}h(1) - I^{\alpha+\beta}h(1) - \mu\lambda I^{\alpha+1}x(1) + \lambda I^{\alpha}x(1).$$

Proof.
Consider the differential equation in (3.2). To solve it, we apply the Riemann-Liouville fractional integral I^{β} to both sides. We obtain:

$${}^{C}D^{\alpha}x(t) + \lambda x(t) = I^{\beta}h(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2}, \tag{3.4}$$

where c_1, c_2 are arbitrary constants in $\mathbb{R}$. We must now examine the boundary condition ${}^{C}D^{\alpha}x(0) = 0$. In equation (3.4), as t approaches 0^+, the term $t^{\beta-2}$ becomes unbounded since $\beta - 2 \in]-1, 0]$. Specifically, for any $1 < \beta \leq 2$:

$$\lim_{t \to 0^+} t^{\beta-2} = +\infty.$$

According to the continuity of ${}^{C}D^{\alpha}x(t)$ at $t = 0$, we must eliminate this singularity. Therefore, we set $c_2 = 0$. The equation reduces to:

$${}^{C}D^{\alpha}x(t) = I^{\beta}h(t) - \lambda x(t) + c_1 t^{\beta-1}. \tag{3.5}$$

Applying the Caputo fractional integral I^{α} to both sides of (3.5) :

$$I^{\alpha}({}^{C}D^{\alpha}x(t)) = I^{\alpha}I^{\beta}h(t) - \lambda I^{\alpha}x(t) + c_1 I^{\alpha}t^{\beta-1}.$$

Since $x(0) = 0$, the left-hand side becomes $x(t)$. Using Semigroup Property (1.8) $I^{\alpha}I^{\beta} = I^{\alpha+\beta}$ and the power rule for the integral (1.7) of $t^{\beta-1}$, we obtain the general form:

$$x(t) = I^{\alpha+\beta}h(t) - \lambda I^{\alpha}x(t) + c_1 \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}. \tag{3.6}$$

To determine the constant c_1, we impose the condition $x(1) = \mu \int_0^1 x(s)ds$. Evaluating (3.6) at $t = 1$:

$$x(1) = I^{\alpha+\beta}h(1) - \lambda I^{\alpha}x(1) + c_1\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}. \tag{3.7}$$

Now, we compute the integral part by integrating (3.6) over $[0, 1]$:

$$\begin{aligned}\int_0^1 x(s)ds &= \int_0^1 I^{\alpha+\beta}h(s)ds - \lambda\int_0^1 I^{\alpha}x(s)ds + \frac{c_1\Gamma(\beta)}{\Gamma(\beta+\alpha)}\int_0^1 s^{\beta+\alpha-1}ds\\ &= I^{\alpha+\beta+1}h(1) - \lambda I^{\alpha+1}x(1) + \frac{c_1\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left[\frac{s^{\beta+\alpha}}{\beta+\alpha}\right]_0^1\\ &= I^{\alpha+\beta+1}h(1) - \lambda I^{\alpha+1}x(1) + \frac{c_1\Gamma(\beta)}{\Gamma(\beta+\alpha+1)}. \end{aligned} \tag{3.8}$$

Substitute (3.7) and (3.8) into the boundary condition $x(1) - \mu\int_0^1 x(s)ds = 0$:

$$\left(I^{\alpha+\beta}h(1) - \lambda I^{\alpha}x(1) + c_1\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\right) - \mu\left(I^{\alpha+\beta+1}h(1) - \lambda I^{\alpha+1}x(1) + \frac{c_1\Gamma(\beta)}{\Gamma(\beta+\alpha+1)}\right) = 0.$$

Grouping the terms involving c_1:

$$c_1\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(1 - \frac{\mu}{\beta+\alpha}\right) = \mu I^{\alpha+\beta+1}h(1) - I^{\alpha+\beta}h(1) - \mu\lambda I^{\alpha+1}x(1) + \lambda I^{\alpha}x(1).$$

Let $\Delta = 1 - \frac{\mu}{\beta+\alpha}$ and the right-hand side be $\Phi(x, h)$. We find:

$$c_1 = \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Delta}\Phi(x, h).$$

Substituting this back into (3.6) yields the final integral representation (3.3).

□

3.2 Existence of a Unique Solution

(Banach Fixed Point Theorem)

In this section, we apply the Banach fixed point theorem (1.4.1) to prove the existence and uniqueness of the solution for the problem (3.1).

First, we define the operator $T : E \to E$ as follows:

$$(Tx)(t) = I^{\alpha+\beta}f(t, x, \mathcal{K}x) - \lambda I^{\alpha}x(t) + \frac{t^{\beta+\alpha-1}}{\Delta}\Phi(x, f). \tag{3.9}$$

Hypotheses:

(H1): The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2): There exist constants $L_1, L_2 > 0$ such that for any $t \in J$ and $x, y, u, v \in \mathbb{R}$:

$$|f(t, x, u) - f(t, y, v)| \leq L_1|x - y| + L_2|u - v|.$$

(H3): There exists $L_k > 0$ such that for any $t, s \in J$ and $x, y \in \mathbb{R}$:

$$|K(t, s, x) - K(t, s, y)| \leq L_k|x - y|.$$

Combining (H2) and (H3), we define the composite Lipschitz constant $L = L_1 + L_2L_k$.

Theorem 3.1
Suppose that (H1)-(H3) hold. If the condition $\Omega < 1$ is satisfied, where

$$\Omega = \frac{L}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left[\frac{L(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}\right],$$

then the problem (3.1) has a unique solution on J.

Proof.
Let $x, y \in E$. For any $t \in J$, we have:

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \Bigg| \left(I^{\alpha+\beta} f(t, x, \mathcal{K}x) - \lambda I^{\alpha} x(t) + \frac{t^{\beta+\alpha-1}}{\Delta}\Phi(x, f) \right) \\ &\quad - \left(I^{\alpha+\beta} f(t, y, \mathcal{K}y) - \lambda I^{\alpha} y(t) + \frac{t^{\beta+\alpha-1}}{\Delta}\Phi(y, f) \right) \Bigg| \quad (3.10) \\ &\leq I^{\alpha+\beta}|f(x, \mathcal{K}x) - f(y, \mathcal{K}y)| + |\lambda| I^{\alpha}|x - y| \\ &\quad + \frac{1}{|\Delta|}|\Phi(x, f) - \Phi(y, f)|. \quad (3.11) \end{aligned}$$

From the Lipschitz conditions, we have $|f(x, \mathcal{K}x) - f(y, \mathcal{K}y)| \leq L\|x - y\|$. Thus:

$$I^{\alpha+\beta}|f(x)-f(y)| \leq I^{\alpha+\beta}L\|x-y\| = \frac{Lt^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|x-y\| \leq \frac{L}{\Gamma(\alpha+\beta+1)}\|x-y\|. \quad (3.12)$$

For the boundary term Φ:

$$\begin{aligned} |\Phi(x) - \Phi(y)| &\leq |\mu| I^{\alpha+\beta+1}|f(x) - f(y)|(1) + I^{\alpha+\beta}|f(x) - f(y)|(1) \\ &\quad + |\mu\lambda| I^{\alpha+1}|x(1) - y(1)| + |\lambda| I^{\alpha}|x(1) - y(1)| \\ &\leq \left(\frac{|\mu|L}{\Gamma(\alpha+\beta+2)} + \frac{L}{\Gamma(\alpha+\beta+1)} + \frac{|\mu\lambda|}{\Gamma(\alpha+2)} + \frac{|\lambda|}{\Gamma(\alpha+1)} \right) \|x - y\| \quad (3.13) \\ &\leq \left(\frac{L(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|(|\mu|+\alpha+1)}{\Gamma(\alpha+2)} \right) \|x - y\|. \quad (3.14) \end{aligned}$$

and

$$|\lambda| I^{\alpha}|x(t) - y(t)| \leq \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\|x - y\| ds \leq \frac{|\lambda|}{\Gamma(\alpha+1)}\|x - y\|. \quad (3.15)$$

Substituting (3.12), (3.13) and (3.15) back into (3.10) and taking the supremum over $t \in J$:

$$\begin{aligned}|Tx(t) - Ty(t)| \leq &\left(\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|}{\Gamma(\alpha+1)}\right)\|x-y\| \\ &+ \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)|\Delta|}\left[\frac{L(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}\right]\|x-y\|.\end{aligned}$$

Then :

$$\|Tx - Ty\|_E \leq \Omega\|x-y\|_E.$$

Since $\Omega < 1$, the operator T is a contraction mapping, E is a Banach space and T maps E into itself. By the Banach fixed point theorem, there exists a unique fixed point $x \in E$, which is the unique solution of (3.1). □

3.3 Existence of at least one solution

(Schauder Fixed Point Theorem)

Theorem 3.2
Assume that the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition:

(H4): There exists a continuous non-decreasing function $\psi : [0,\infty[\to]0,\infty[$ and a positive continuous function $p \in C(J,\mathbb{R}^+)$ such that for all $t \in J$ and $(x,u) \in \mathbb{R}^2$:

$$|f(t,x,u)| \leq p(t)\psi(\|x\|).$$

If there exists a positive constant $r > 0$ such that:

$$\frac{r}{\frac{\|p\|\psi(r)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|r}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left[\frac{\|p\|\psi(r)(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|r(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}\right]} \geq 1, \tag{3.16}$$

then the problem (3.1) possesses at least one solution on the interval J.

Proof.
The proof is constructed to verify the three core hypotheses of Schauder's Fixed Point Theorem:
continuity, uniform boundedness and equicontinuity of the operator T.
Let us define a bounded, closed and convex set $B_r = \{x \in E : \|x\| \leq r\}$, where r is chosen to satisfy (3.16).

1. $T(B_r) \subseteq B_r$.
Let $x \in B_r$.
Then

$$\|x\| \leq r.$$

Using assumption $(H4)$, we have:

$$|f(t, x(t), (Kx)(t))| \leq p(t)\psi(\|x\|) \leq \|p\|\psi(r).$$

For any $t \in J$, from the definition of the operator T, we obtain:

$$\begin{aligned}|(Tx)(t)| &\leq I^{\alpha+\beta}|f(t,x,Kx)| + |\lambda| I^{\alpha}|x(t)| + \frac{t^{\alpha+\beta-1}}{|\Delta|}|\Phi(x,f)| \\ &\leq \frac{\|p\|\psi(r)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}|\Phi(x,f)|.\end{aligned}$$

Now, using the estimate of $\Phi(x, f)$, we get:

$$|\Phi(x,f)| \leq \frac{\|p\|\psi(r)(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}.$$

Hence

$$\begin{aligned}|(Tx)(t)| \leq& \frac{\|p\|\psi(r)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r}{\Gamma(\alpha+1)} \\ &+ \frac{1}{|\Delta|}\left[\frac{\|p\|\psi(r)(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}\right].\end{aligned}$$

By condition (3.16), we obtain:

$$\|Tx\| \leq r.$$

Therefore,

$$Tx \in B_r,$$

for every $x \in B_r$. Thus,

$$T(B_r) \subseteq B_r.$$

2. **Continuity of the Operator T:**

The continuity of T is a direct consequence of the continuity of the function f and the kernel K, and we can check that by the following method:

Let $\{x_n\}$ be a sequence in B_r such that $x_n \to x$ as $n \to \infty$. By the continuity of f, we observe that:

$$f(s, x_n(s), (\mathcal{K}x_n)(s)) \to f(s, x(s), (\mathcal{K}x)(s)), \forall s \in J.$$

Since the functions are bounded on B_r, we can apply the Lebesgue Dominated Convergence Theorem to the integral terms of the operator T.

This ensures that:

$$\|Tx_n - Tx\| \to 0 \quad \text{as} \quad n \to \infty,$$

thus establishing that T is a continuous operator on B_r.

3. **Uniform Boundedness:**

We demonstrated that T maps B_r into itself, we evaluate the norm of Tx for any $x \in B_r$.

Utilizing the condition **(H4)**, we have:

$$\begin{aligned}|(Tx)(t)| &\leq I^{\alpha+\beta}|f(t,x,\mathcal{K}x)| + |\lambda| I^{\alpha}|x(t)| + \frac{t^{\beta+\alpha-1}}{|\Delta|}|\Phi(x,f)| \\ &\leq \frac{\|p\|\psi(r)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}|\Phi(x,f)|.\end{aligned}$$

By substituting the bound for $|\Phi(x,f)|$ and following the condition (3.16), it follows that $\|Tx\| \leq r$. This confirms that $T(B_r) \subset B_r$, meaning the operator is uniformly bounded on B_r.

4. **Equicontinuity and Compactness:**

Let $x \in B_r$ and let $t_1, t_2 \in J$ with $0 \leq t_1 < t_2 \leq 1$.

From the definition of the operator T, we have:

$$\begin{aligned}|(Tx)(t_2) - (Tx)(t_1)| \leq &\left| I^{\alpha+\beta} f(t_2,x,Kx) - I^{\alpha+\beta} f(t_1,x,Kx)\right| \\ &+ |\lambda|\,|I^{\alpha}x(t_2) - I^{\alpha}x(t_1)| \\ &+ \left|\frac{t_2^{\alpha+\beta-1} - t_1^{\alpha+\beta-1}}{\Delta}\Phi(x,f)\right|.\end{aligned}$$

We estimate each term separately.

Estimate of the first term

By assumption (H_4), we have:

$$|f(t,x(t),(Kx)(t))| \leq p(t)\psi(\|x\|).$$

Since $x \in B_r$, then $\|x\| \leq r$. Since ψ is non-decreasing, it follows that:

$$\psi(\|x\|) \leq \psi(r).$$

Moreover,

$$p(t) \leq \|p\|, \quad \forall t \in J.$$

Hence,

$$|f(t,x(t),(Kx)(t))| \leq \|p\|\psi(r).$$

Therefore,

$$\begin{aligned}&\left| I^{\alpha+\beta} f(t_2,x,Kx) - I^{\alpha+\beta} f(t_1,x,Kx)\right| \\ =&\frac{1}{\Gamma(\alpha+\beta)}\left| \int_0^{t_2}(t_2-s)^{\alpha+\beta-1} f(s,x,Kx)\,ds - \int_0^{t_1}(t_1-s)^{\alpha+\beta-1} f(s,x,Kx)\,ds\right|.\end{aligned}$$

Splitting the integral over $[0,t_1]$ and $[t_1,t_2]$, we obtain:

$$
\begin{aligned}
=&\frac{1}{\Gamma(\alpha+\beta)}\bigg|\int_0^{t_1}(t_2-s)^{\alpha+\beta-1}f(s,x,Kx)\,ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}f(s,x,Kx)\,ds\\
&\quad-\int_0^{t_1}(t_1-s)^{\alpha+\beta-1}f(s,x,Kx)\,ds\bigg|\\
\leq&\frac{\|p\|\psi(r)}{\Gamma(\alpha+\beta)}\int_0^{t_1}\left|(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}\right|ds+\frac{\|p\|\psi(r)}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds.
\end{aligned}
$$

Now, for the second integral,

$$\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds=\frac{(t_2-t_1)^{\alpha+\beta}}{\alpha+\beta}.$$

Thus,

$$\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds=\frac{(t_2-t_1)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Gamma(\alpha+\beta).$$

Similarly, the first integral tends to zero as $t_2\to t_1$ by continuity of the kernel. Consequently,

$$\left|I^{\alpha+\beta}f(t_2,x,Kx)-I^{\alpha+\beta}f(t_1,x,Kx)\right|\to 0,$$

as $t_2\to t_1$, uniformly for $x\in B_r$.

Estimate of the second term
Since $\|x\|\leq r$, we have:

$$
\begin{aligned}
|I^\alpha x(t_2)-I^\alpha x(t_1)|\leq&\frac{r}{\Gamma(\alpha)}\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|ds\\
&+\frac{r}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds.
\end{aligned}
$$

Evaluating the second integral gives

$$\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds=\frac{(t_2-t_1)^\alpha}{\alpha}.$$

Hence,

$$|I^\alpha x(t_2)-I^\alpha x(t_1)|\to 0$$

as $t_2\to t_1$.
Therefore,

$$|\lambda||I^\alpha x(t_2)-I^\alpha x(t_1)|\to 0.$$

Estimate of the third term
Since $\Phi(x,f)$ is bounded on B_r, there exists $M>0$ such that

$$|\Phi(x,f)|\leq M.$$

Hence,

$$\left|\frac{t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}}{\Delta}\Phi(x,f)\right| \leq \frac{M}{|\Delta|}|t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}|.$$

Since the function

$$t \mapsto t^{\alpha+\beta-1}$$

is continuous on J (continuous on a compact set, hence uniformly continuous), we obtain:

$$|t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}| \to 0$$

as $t_2 \to t_1$.
Thus,

$$\left|\frac{t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}}{\Delta}\Phi(x,f)\right| \to 0.$$

Combining all previous estimates, we deduce that:

$$|(Tx)(t_2)-(Tx)(t_1)| \to 0 \quad \text{as } t_2 \to t_1,$$

$$\sup_{x\in B_r}|(Tx)(t_2)-(Tx)(t_1)| \to 0 \quad \text{as } t_2 \to t_1.$$

Therefore, the family $T(B_r)$ is equicontinuous on J.

Since we have already proved that $T(B_r)$ is uniformly bounded, the Arzelà-Ascoli theorem implies that $T(B_r)$ is relatively compact in E.

Hence, the operator T is compact.
Since T is continuous and maps B_r into itself, all conditions of Schauder Fixed Point Theorem are satisfied.
Consequently, T admits at least one fixed point in B_r, which is a solution of problem (3.1).

□

3.4 Hyers-Ulam Stability

Definition 3.1 [1]
The fractional differential equation (3.1) is said to be Ulam-Hyers stable if there exists a real constant $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in E$ of the following inequality:

$$|{}^{RL}D^{\beta}\left({}^{C}D^{\alpha}z(t)+\lambda z(t)\right) - f(t,z(t),(\mathcal{K}z)(t))| \leq \epsilon, \quad t \in J, \tag{3.17}$$

there exists a unique solution $x \in E$ of problem (3.1) such that:

$$\|z-x\| \leq C_f\epsilon, \quad \forall t \in J.$$

Remark:

A function $z \in E$ is a solution of the inequality (3.17) if and only if there exists a function $h \in E$ (dependent on z) such that:

1. $|h(t)| \leq \epsilon, \quad \forall t \in J.$
2. ${}^{RL}D^{\beta}\left({}^{C}D^{\alpha}z(t) + \lambda z(t)\right) = f(t, z(t), (\mathcal{K}z)(t)) + h(t), \quad \forall t \in J.$

Main Stability Theorem

Theorem 3.3
Assume that the hypotheses **(H1)-(H3)** hold. If the condition $\Omega < 1$ (as defined in (3.1)) is satisfied, then the nonlinear problem (3.1) is Ulam-Hyers stable.

Proof.

Let $\epsilon > 0$ and let $z \in E$ be a function satisfying the inequality (3.17).
Let $x \in E$ be the unique solution of the problem (3.1).
From Lemma 3.1, the solution $z(t)$ can be represented in its integral form as:

$$\begin{aligned} z(t) = I^{\alpha+\beta} f(t, z, \mathcal{K}z) - \lambda I^{\alpha} z(t) + \frac{t^{\beta+\alpha-1}}{\Delta}\Phi(z, f) \\ + I^{\alpha+\beta} h(t) + \frac{t^{\beta+\alpha-1}}{\Delta}\left[\mu I^{\alpha+\beta+1}h(1) - I^{\alpha+\beta}h(1)\right]. \end{aligned}$$

Applying the triangle inequality to the difference $|z(t)-(Tz)(t)|$ and using the property $|h(t)| \leq \epsilon$, we derive the following bound:

$$\begin{aligned} |z(t) - (Tz)(t)| &\leq I^{\alpha+\beta}|h(t)| + \frac{1}{|\Delta|}\left(|\mu| I^{\alpha+\beta+1}|h(1)| + I^{\alpha+\beta}|h(1)|\right) \\ &\leq \frac{\epsilon}{\Gamma(\alpha+\beta+1)} + \frac{1}{|\Delta|}\left(\frac{|\mu|\epsilon}{\Gamma(\alpha+\beta+2)} + \frac{\epsilon}{\Gamma(\alpha+\beta+1)}\right). \end{aligned}$$

Let Λ denote the constant factor of ϵ in (**??**), so that $|z(t) - (Tz)(t)| \leq \Lambda\epsilon$.

Now, we evaluate the norm of the difference between z and x:

$$\begin{aligned} |z(t) - x(t)| &= |z(t) - (Tx)(t)| \\ &\leq |z(t) - (Tz)(t)| + |(Tz)(t) - (Tx)(t)| \\ &\leq \Lambda\epsilon + \Omega\|z - x\|. \end{aligned}$$

By taking the supremum over $t \in J$, we obtain:

$$\|z - x\| \leq \Lambda\epsilon + \Omega\|z - x\|.$$

Rearranging the terms, we get:

$$\|z - x\|(1 - \Omega) \leq \Lambda\epsilon.$$

Since $\Omega < 1$, we can divide by $(1 - \Omega)$ to yield:

$$\|z - x\| \leq \frac{\Lambda}{1 - \Omega}\epsilon.$$

By setting $C_f = \frac{\Lambda}{1-\Omega}$, the requirement $\|z - x\| \leq C_f\epsilon$ is satisfied. This demonstrates that the problem is Ulam-Hyers stable, concluding the proof. □

3.5 Applications

Example 3.1

Consider the following nonlinear fractional boundary value problem:

$$\begin{cases} {}^{RL}D^{\frac{3}{2}}\left({}^{C}D^{\frac{1}{2}}x(t)+\frac{1}{10}x(t)\right)=\frac{1}{20}\left(x(t)+(Kx)(t)\right), \quad t\in[0,1], \\ x(0)=0, \\ x(1)=\dfrac{1}{4}\displaystyle\int_0^1 x(s)\,ds, \\ {}^{C}D^{\frac{1}{2}}x(0)=0, \end{cases}$$

where the Volterra integral operator is defined by:

$$(Kx)(t)=\int_0^t \frac{x(s)}{20}\,ds.$$

From the problem, we identify the following constants:

$$\alpha=\frac{1}{2}, \qquad \beta=\frac{3}{2}, \qquad \alpha+\beta=2,$$

$$\lambda=\frac{1}{10}=0.1, \qquad \mu=\frac{1}{4}=0.25.$$

The interval is

$$J=[0,1].$$

Method 1 : Existence and uniqueness by Banach Fixed Point Theorem

1. **Verification of Lipschitz conditions**
The nonlinear function is given by:

$$f(t,x,u)=\frac{1}{20}(x+u).$$

For any $x,y,u,v\in\mathbb{R}$, we have:

$$\begin{aligned} |f(t,x,u)-f(t,y,v)| &= \left|\frac{1}{20}(x+u)-\frac{1}{20}(y+v)\right| \\ &= \frac{1}{20}|(x-y)+(u-v)| \\ &\le \frac{1}{20}|x-y|+\frac{1}{20}|u-v|. \end{aligned}$$

Hence,

$$L_1=\frac{1}{20}, \qquad L_2=\frac{1}{20}.$$

Now consider the kernel

$$K(t,s,x)=\frac{x}{20}.$$

Then,

$$|K(t,s,x) - K(t,s,y)| = \left|\frac{x}{20} - \frac{y}{20}\right| = \frac{1}{20}|x-y|.$$

Therefore,

$$L_k = \frac{1}{20}.$$

Thus, the global Lipschitz constant becomes

$$L = L_1 + L_2 L_k.$$

Substituting the values:

$$L = \frac{1}{20} + \frac{1}{20} \cdot \frac{1}{20}.$$

Hence,

$$L = \frac{21}{400} = 0.0525.$$

2. Computation of Δ

We compute

$$\Delta = 1 - \frac{\mu}{\alpha + \beta}.$$

Thus,

$$\Delta = 1 - \frac{0.25}{2}.$$

Hence,

$$\Delta = 0.875.$$

Since

$$\Delta \neq 0,$$

the integral operator is well defined.

3. Computation of the contraction constant Ω

Using Theorem 3.1, we have:

$$\Omega = \frac{L}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left[\frac{L(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}\right].$$

Since

$$\Gamma(3) = 2, \qquad \Gamma(1.5) \approx 0.886, \qquad \Gamma(2.5) \approx 1.329,$$

we obtain:

$$\Omega = \frac{0.0525}{2} + \frac{0.1}{0.886} + \frac{1}{0.875}\left[\frac{0.0525(1.25)}{2} + \frac{0.1(1.75)}{1.329}\right].$$

Now,

$$\frac{0.0525}{2} = 0.02625,$$

and

$$\frac{0.1}{0.886} \approx 0.1128.$$

Also,

$$\frac{0.0525(1.25)}{2} \approx 0.0328,$$

and

$$\frac{0.175}{1.329} \approx 0.1317.$$

Therefore,

$$\Omega \approx 0.02625 + 0.1128 + 1.143(0.1645).$$

Hence,

$$\Omega \approx 0.13905 + 0.188.$$

Thus,

$$\Omega \approx 0.327.$$

Since

$$\Omega < 1,$$

the operator T is a contraction mapping.

Therefore, by Banach Fixed Point Theorem, the problem admits a unique solution on $[0, 1]$.

Method 2 : Existence by Schauder Fixed Point Theorem

We now verify assumption $(H4)$.

Since

$$f(t, x, u) = \frac{1}{20}(x + u),$$

we have:

$$|f(t, x, u)| \leq \frac{1}{20}(|x| + |u|).$$

Choose

$$p(t) = \frac{1}{20}, \qquad \psi(r) = 2r.$$

Then,

$$|f(t, x, u)| \leq p(t)\psi(r).$$

Thus assumption $(H4)$ is satisfied.
Choose

$$r = 1.$$

Then,

$$\|p\| = \frac{1}{20} = 0.05,$$

and

$$\psi(1) = 2.$$

Hence,

$$\|p\|\psi(r) = 0.1.$$

Now we verify condition (3.16):

$$\frac{\|p\|\psi(r)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|r}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left[\frac{\|p\|\psi(r)(|\mu|+1)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|r(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}\right].$$

Substituting the values:

$$= \frac{0.1}{2} + \frac{0.1}{0.886} + \frac{1}{0.875}\left[\frac{0.1(1.25)}{2} + \frac{0.1(1.75)}{1.329}\right].$$

Thus,

$$= 0.05 + 0.1128 + 1.143(0.1941).$$

Hence,

$$= 0.1628 + 0.2218.$$

Therefore,

$$\approx 0.3846 < 1.$$

Hence all assumptions of Schauder Fixed Point Theorem are satisfied.
Consequently, the problem possesses at least one solution on $[0, 1]$.

Ulam–Hyers Stability

We now study the Ulam–Hyers stability of the problem. From Method 1, we already proved that:

- the hypotheses $(H1) - (H3)$ are satisfied,
- the operator T is a contraction,
- and

$$\Omega \approx 0.327 < 1.$$

Therefore, all assumptions of Theorem 3.3 are fulfilled.

Let $z \in X$ satisfy the inequality

$$\left| {}^{RL}D^{\frac{3}{2}} \left({}^{C}D^{\frac{1}{2}} z(t) + \frac{1}{10} z(t) \right) - \frac{1}{20} \left(z(t) + (Kz)(t) \right) \right| \leq \varepsilon, \quad t \in [0,1].$$

According to Definition 3.1, there exists a function $h \in X$ such that:

1. $$|h(t)| \leq \varepsilon, \qquad \forall t \in [0,1].$$

2. $${}^{RL}D^{\frac{3}{2}} \left({}^{C}D^{\frac{1}{2}} z(t) + \frac{1}{10} z(t) \right) = \frac{1}{20} \left(z(t) + (Kz)(t) \right) + h(t).$$

Using Lemma 3.1, we obtain:

$$\begin{aligned} z(t) =& I^2 f(t, z, Kz) - \frac{1}{10} I^{\frac{1}{2}} z(t) + \frac{t}{\Delta} \Phi(z, f) \\ &+ I^2 h(t) + \frac{t}{\Delta} \left[\mu I^3 h(1) - I^2 h(1) \right]. \end{aligned}$$

Using the triangle inequality, we get:

$$|z(t) - Tz(t)| \leq I^2 |h(t)| + \frac{1}{|\Delta|} \left(|\mu| I^3 |h(1)| + I^2 |h(1)| \right).$$

Since

$$|h(t)| \leq \varepsilon,$$

then

$$I^2 |h(t)| \leq \frac{\varepsilon}{\Gamma(3)}.$$

Also,

$$I^3 |h(1)| \leq \frac{\varepsilon}{\Gamma(4)},$$

and

$$I^2 |h(1)| \leq \frac{\varepsilon}{\Gamma(3)}.$$

Therefore,

$$|z(t) - Tz(t)| \leq \frac{\varepsilon}{\Gamma(3)} + \frac{1}{0.875}\left(\frac{0.25\varepsilon}{\Gamma(4)} + \frac{\varepsilon}{\Gamma(3)}\right).$$

Since

$$\Gamma(3) = 2, \qquad \Gamma(4) = 6,$$

we obtain:

$$|z(t) - Tz(t)| \leq \frac{\varepsilon}{2} + 1.143\left(\frac{0.25\varepsilon}{6} + \frac{\varepsilon}{2}\right).$$

Hence,

$$|z(t) - Tz(t)| \leq 0.5\varepsilon + 1.143(0.5417\varepsilon).$$

Thus,

$$|z(t) - Tz(t)| \leq 0.5\varepsilon + 0.619\varepsilon.$$

Finally,

$$|z(t) - Tz(t)| \leq 1.119\varepsilon.$$

Let

$$\Lambda = 1.119.$$

Then

$$|z(t) - Tz(t)| \leq \Lambda\varepsilon.$$

Now let $x \in E$ be the unique solution of the problem. Then

$$|z(t) - x(t)| = |z(t) - Tx(t)|.$$

Using the triangle inequality,

$$|z(t) - x(t)| \leq |z(t) - Tz(t)| + |Tz(t) - Tx(t)|.$$

Thus,

$$|z(t) - x(t)| \leq \Lambda\varepsilon + \Omega\|z - x\|.$$

Taking the supremum over $t \in [0, 1]$, we obtain:

$$\|z - x\| \leq \Lambda\varepsilon + \Omega\|z - x\|.$$

Hence,

$$\|z - x\|(1 - \Omega) \leq \Lambda\varepsilon.$$

Since

$$\Omega \approx 0.327 < 1,$$

we get:

$$\|z-x\| \leq \frac{\Lambda}{1-\Omega}\varepsilon.$$

Substituting the values:

$$\|z-x\| \leq \frac{1.119}{1-0.327}\varepsilon.$$

Therefore,

$$\|z-x\| \leq 1.662\varepsilon.$$

Setting

$$C_f = 1.662,$$

we conclude that

$$\|z-x\| \leq C_f\varepsilon.$$

Hence, the fractional boundary value problem is Ulam–Hyers stable on $[0,1]$.

Example 3.2

Consider the following nonlinear fractional boundary value problem involving both Riemann-Liouville and Caputo derivatives:

$$\begin{cases} {}^{RL}D^{3/2}\left({}^{C}D^{1/2}x(t)+\frac{1}{20}x(t)\right) = \frac{e^{-t}}{(10+e^t)}\left(\frac{|x(t)|}{1+|x(t)|}+\sin(\mathcal{K}x(t))\right), t\in[0,1],\\ x(0)=0,\\ x(1)=\frac{1}{4}\int_0^1 x(s)\,ds,\\ {}^{C}D^{1/2}x(0)=0, \end{cases} \tag{3.18}$$

where the Volterra integral operator $\mathcal{K}$ is given by:

$$(\mathcal{K}x)(t) = \int_0^t \frac{e^{-(s-t)}}{10}x(s)\,ds.$$

From the problem (3.18), we extract the following constants values:

- Orders of derivatives: $\beta = 1.5$ and $\alpha = 0.5$, which implies $\alpha+\beta = 2$.
- System parameters: $\lambda = \frac{1}{20} = 0.05$ and $\mu = \frac{1}{4} = 0.25$.
- The interval is $J = [0,1]$.

Existence results by Banach fixed point theorem

1. Verification of Lipschitz Conditions:
The nonlinear function is

$$f(t,x,u) = \frac{e^{-t}}{10+e^t}\left(\frac{|x|}{1+|x|}+\sin(u)\right).$$

We calculate the Lipschitz constants L_1 and L_2 as follows:

$$\begin{aligned}|f(t,x,u)-f(t,y,v)| &\leq \frac{e^{-t}}{10+e^t}\left(\left|\frac{|x|}{1+|x|}-\frac{|y|}{1+|y|}\right|+|\sin(u)-\sin(v)|\right)\\ &\leq \frac{1}{11}(|x-y|+|u-v|).\end{aligned}$$

Thus,

$$L_1=\frac{1}{11},\qquad L_2=\frac{1}{11}.$$

For the kernel

$$K(t,s,x)=\frac{e^{-(s-t)}}{10}x,$$

the Lipschitz constant is

$$L_k=\frac{1}{10}.$$

The aggregate Lipschitz constant is therefore:

$$L=L_1+L_2L_k=\frac{1}{11}+\left(\frac{1}{11}\times\frac{1}{10}\right)=\frac{11}{110}=0.1.$$

2. Calculation of Δ:

$$\Delta=1-\frac{\mu}{\alpha+\beta}=1-\frac{0.25}{2}=0.875.$$

Since $\Delta=0.875\neq 0$, the integral operator is well-defined.

3. Computation of the Contraction Constant Ω:

Using the formula derived in Theorem 3.1:

$$\Omega=\frac{L}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\left[\frac{L(|\mu|+1)}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|(|\mu|+\alpha+1)}{\Gamma(\alpha+2)}\right].$$

Substituting the values

$$\Gamma(3)=2,\qquad \Gamma(1.5)\approx 0.886,\qquad \Gamma(2.5)\approx 1.329,$$

we obtain:

$$\begin{aligned}\Omega &\approx \frac{0.1}{2}+\frac{0.05}{0.886}+\frac{1}{0.875}\left[\frac{0.1(1.25)}{2}+\frac{0.05(1.75)}{1.329}\right]\\ &\approx 0.05+0.056+1.143\,[0.0625+0.0658]\\ &\approx 0.106+1.143(0.1283)\\ &\approx 0.106+0.1467\end{aligned}$$

$$\approx 0.2527.$$

Hence,

$$\Omega \approx 0.253.$$

Conclusion:

Since

$$\Omega \approx 0.253 < 1,$$

the operator T is a contraction.
Consequently, according to the Banach Fixed Point Theorem, the fractional problem (3.18) has a unique solution on $[0, 1]$.

Verification of Ulam-Hyers Stability:

Finally, we verify the Ulam-Hyers stability of problem (3.18) by applying Theorem 3.3. From the previous computations, we obtained:

$$\Omega \approx 0.253 < 1.$$

Since all hypotheses **(H1)-(H3)** are satisfied and the contraction condition holds, the assumptions of Theorem 3.3 are fulfilled.
Now, we compute the stability constant Λ.
Using the expression:

$$\Lambda = \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{|\Delta|}\left(\frac{|\mu|}{\Gamma(\alpha+\beta+2)} + \frac{1}{\Gamma(\alpha+\beta+1)}\right),$$

and substituting

$$\Gamma(3) = 2, \qquad \Gamma(4) = 6, \qquad \Delta = 0.875, \qquad \mu = 0.25,$$

we obtain:

$$\begin{aligned}\Lambda &= \frac{1}{2} + \frac{1}{0.875}\left(\frac{0.25}{6} + \frac{1}{2}\right)\\ &= 0.5 + 1.143\,(0.0417 + 0.5)\\ &= 0.5 + 1.143(0.5417)\\ &= 0.5 + 0.6192\\ &\approx 1.119.\end{aligned}$$

The Ulam-Hyers stability constant is then

$$C_f = \frac{\Lambda}{1-\Omega}.$$

Substituting the computed values:

$$\begin{aligned} C_f &= \frac{1.119}{1-0.253} \\ &= \frac{1.119}{0.747} \\ &\approx 1.498. \end{aligned}$$

Hence,

$$C_f \approx 1.50.$$

Therefore, for every approximate solution $z \in E$ satisfying

$$\left| {}^{RL}D^{3/2}\left({}^{C}D^{1/2}z(t) + \frac{1}{20}z(t)\right) - f(t, z(t), (\mathcal{K}z)(t))\right| \leq \epsilon,$$

there exists a unique exact solution $x \in E$ such that:

$$\|z - x\| \leq 1.50\,\epsilon.$$

Conclusion:
The fractional boundary value problem (3.18) is Ulam-Hyers stable on $[0, 1]$.

Future Research Perspectives

Based on the results obtained in this thesis, and building upon the study conducted in Chapters 2 and 3 regarding fractional integro-differential equations, we suggest the following research directions that could extend this work:

- Studying the Impact of Changing Boundary Conditions: The scope of the problems studied can be expanded by replacing the boundary conditions used in Chapter 3 with more complex or nonlinear ones, which may lead to new existence and uniqueness results reflecting different dynamic behaviors of the problem.
- Investigating Various Fractional Operators: This work opens the door to studying the same problem using other fractional derivative operators (such as the Hadamard or Caputo-Fabrizio operators) instead of the Caputo and Riemann-Liouville operators, as each operator allows for a mathematical description that varies according to the nature of the physical or engineering model being studied.
- Analyzing Different Types of Stability: Although Hyers-Ulam stability was studied in this work, researchers can investigate other types of stability, such as Hyers-Ulam-Rassias stability or stability in the exponential sense, to enrich the qualitative aspect of fractional problems.
- Expanding towards Fractional Systems: The individual models studied can be developed into systems of coupled fractional differential equations, which represents a significant mathematical challenge and a fundamental requirement for modeling many complex natural and biological phenomena.

These suggestions represent a first step towards deepening the mathematical understanding of these problems, and we hope they will serve as an incentive for further research in this promising field.

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الملخص

تطرقنا في هذه المذكرة إلى دراسة وجود، وحدانية، وإستقرار حلول بعض أصناف المعادلات التفاضلية غير الخطية ذات الرتب الكسرية بمفهومي كابوتو وريمان - ليوفيل وذلك بالاعتماد على نظريات النقطة الثابتة لبناخ، شودر وكراسنوسلسكي. أما فيما يخص الإستقرار، فقد تم تطبيق مفهوم إستقرار أولام - هايرز.

الكلمات المفتاحية: معادلات تفاتكاملية ذات رتب كسرية، التكامل الكسري لريمان - ليوفيل، المشتقة الكسرية لكابوتو وريمان - ليوفيل، النقطة الصامدة، إستقرار الحلول.

Abstract

In this memory, we address the existence, uniqueness, and stability of solutions for certain classes of non-linear fractional differential equations under the Caputo and Riemann-Liouville concepts. This study is conducted by applying the fixed-point theorems of Banach, Schauder, and Krasnoselskii. Regarding the stability analysis, the Ulam-Hyers stability concept has been applied.

Key words: Fractional integro-differential equations, Riemann-Liouville fractional integral, Caputo and Riemann-Liouville fractional derivative, fixed point, stability of solutions.

Résumé

Dans ce mémoire, nous abordons l'existence, l'unicité et la stabilité des solutions pour certaines classes d'équations différentielles non linéaires d'ordre fractionnaire au sens de Caputo et de Riemann-Liouville. Cette étude est menée en appliquant les théorèmes du point fixe de Banach, Schauder et Krasnoselskii. En ce qui concerne l'analyse de la stabilité, le concept de stabilité d'Ulam-Hyers a été appliqué.

Mots clés : Équations intégro-différentielles d'ordre fractionnaire, Intégrale fractionnaire de Riemann-Liouville, dérivée fractionnaire au sens de Caputo et de Riemann-Liouville, point fixe, stabilité des solutions.