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A Study of Convergence and Semi-Continuity of Functions in Metric Spaces

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Dedication

We dedicate this humble work to :
Our dear parents, our first source of strength in life, who have never failed to support us and have made countless sacrifices throughout our journey. Mom and Dad, your love is beyond words, and our greatest wish is to make you proud, just as you have always lifted us up with your kindness and belief in us.

To our loyal friends and relatives who have always stood by our side with advice and support we will never forget your presence through every stage. Especially our university friends, with whom we shared the most beautiful moments during these five wonderful years.

We also dedicate this work to everyone who believed in us and encouraged us with kind words our deepest gratitude goes to you. To everyone who crossed our path and brought a spark of hope or a smile, thank you from the bottom of our hearts. This work is the fruit of your efforts alongside ours, and we hope it lives up to the love and pride you deserve. You are all part of this achievement, and your place in it will never be forgotten. Thank you to everyone who contributed to this journey, even in the smallest of ways.

Acknowledgments

All praise is due to Allah, by whose grace all good deeds are accomplished and through whose assistance the impossible becomes possible. To Him belongs all gratitude, first and last, for the guidance and support He has granted me to complete this academic work.

I would like to express my deepest gratitude to my esteemed supervisor, Professor **Safia Meftah** , whose invaluable guidance, insightful advice, and unwavering support have been instrumental throughout this research journey. May Allah reward him abundantly for his efforts.

My sincere appreciation extends to all the distinguished professors of the Mathematics Department at Echahid Hamma Lakhdar University for their profound knowledge, continuous encouragement, and academic support throughout my studies.

Finally, I wish to acknowledge everyone who contributed, directly or indirectly, to the realization of this work. To all who offered help, support, or encouragement - you have my heartfelt gratitude and lasting remembrance.

Our final prayer is that all praise belongs to Allah, the Lord of all worlds.

إهْدَاء

«اَلْحَمْدُ لِلّٰهِ رَبِّ الْعَالَمِين، الذَّي بِنِعْمَتِهِ تَتِمُّ الصَّالِحَاتُ، وَبِهِ تُسْتَفْتَحُ الْبِدَايَاتُ وَتُخْتَمُ الْغَايَاتُ»

بِاسْمِ اللهِ، وَعَلى هَدْيِهِ أَمْضِي، وَبِهِ أَسْتَعِينْ، أُهْدِي ثَمَرَةَ هَذَا الْجُهْدِ الْمُتَوَاضِـع:

إِلَى وَالِدَيَّ الْكَرِيمَين، مَنْ سَقَيَانِي حُبًّا، وَغَمَرَانِي لُطْفًا، فَكُنْتُ بِفَضْلِهِمَا بَعْدَ اَللّٰهِ مَا أَنَا عَلَيه، وَإِنْ جَفَّ مِدَادِي فَلَنْ يَفِي بِحَقِّهِمَا، وَإِنْ طَالَ دُعَائِي فَلَنْ يَبْلُغَ بَعْضَ جَمِيلَهُمَا. **إِلَى إِخْوَتِي وَأَخَوَاتِي**، عِقْدِ اَلْمَوَدَّةِ وَنَبْضِ الصُّحْبَةِ، الذِّينَ شَدُّوا أَزْرِي إِذَا فَتَرَ، وَكَانُوا لِي ظِلًّا إِذَا اَشْتَدَّ الْحَرُّ، وَنُورًا إِذَا ادْلَهَمَّ الدَّرْبُ.

إِلَى صديقاتي الوفيات زَادُ الْطَّرِيقِ وَبَهْجَةِ الْمَسِيرِ، الذَّينَ طَيَّبُوا الرِّحْلَةَ بِصِدْقِهِمْ، وَزَيَّنُوهَا بِوَفَائِهِمْ.

إِلَى أُسْتَاذِي الْمُشْرِفْ **الدكتورة صفية مفتاح** ، مَنَارَ اَلْعِلْمِ، الذِّي أَفَاضَ مِنْ عِلْمِهِ، وَأَجَادَ فِي تَوْجِيهِهِ، فَلَهُ مِنِّي شُكْرٌ لاَ يَنْقَضِي، جَزَاهُ اللّٰهُ عَنِّي خَيْرَ الْجَزَاءِ.

«رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنعَمتَ عَلَيَّ وَعَلَى وَالِدَيَّ، وَأَنْ أَعْمَلَ صَالِحاً تَرْضَاهُ»

هاله رجاء - مسغوني صبرينة

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General Conclusion

This thesis has undertaken a systematic and rigorous study of two fundamental concepts in mathematical analysis — convergence of function sequences and semi-continuity of functions — within the abstract framework of metric spaces. The progression from the theoretical foundations of Chapter I to the concrete applications of Chapter IV reflects the natural movement of mathematical inquiry: from abstraction to illustration, and from definition to consequence.

In Chapter I, we established the foundational language of metric spaces, introducing the metric axioms, the topology of open and closed balls, the notions of Cauchy sequences and completeness, and the formal definition of continuity. The normed spaces provided the most important class of examples, and the induced metric structure connected the geometric and analytic perspectives. These foundations proved indispensable throughout the subsequent chapters.

Chapter II addressed the central question of how a sequence of functions can converge to a limit function, and what properties of the approximating functions are inherited by the limit. The fundamental distinction between pointwise and uniform convergence was established with full precision. The key result — that uniform convergence preserves continuity while pointwise convergence does not — was proved via Theorem 2.3.1 and illustrated through numerous examples. Dini's theorems provided important sufficient conditions under which monotone pointwise convergence becomes uniform, bridging the gap between the two modes of convergence.

Chapter III developed the theory of semi-continuity, which emerged naturally as the appropriate framework for functions that are "almost continuous" in one direction. Upper and lower semi-continuous functions were characterized through multiple equivalent formulations: via ε-δ definitions, via sequences, via open and closed sets, and via epigraphs and hypographs. The algebraic stability of semi-continuity under sums, scalar multiplication, and composition was established, and the behavior of semi-continuity under limiting processes was analyzed in detail. The Weierstrass-type theorem for semi-continuous functions on compact sets, guaranteeing the attainment of extrema, was proved and interpreted as the culmination of the theory.

Chapter IV consolidated all these results through a systematic collection of applications and illustrative examples. The convergence of classical function sequences — geometric series, Bernstein polynomials, Volterra integral equations, and Picard iterates — demonstrated the power of uniform convergence in preserving continuity and semi-continuity. The Banach Contraction Principle provided a constructive method for finding fixed points in complete metric spaces, connecting the Cauchy sequence theory of Chapter I to the uniform convergence of Chapter II. The explicit construction of semi-continuous envelopes illustrated the regularization procedure that transforms an arbitrary function into its best semi-continuous approximant.

The central research questions posed in the introduction of this thesis have been fully answered:

- **What is the relationship between different types of convergence and semi-continuity?**
  Uniform convergence preserves both upper and lower semi-continuity, and hence continuity. Pointwise convergence does not preserve semi-continuity in general, but monotone pointwise convergence preserves semi-continuity in the direction of the monotonicity: a decreasing limit of upper semi-continuous functions is upper semi-continuous, and an increasing limit of lower semi-continuous functions is lower semi-continuous.

- **Under what conditions does the limit function preserve continuity in a metric space?**
  The limit function is continuous whenever the convergence is uniform and the approximating functions are continuous. When the convergence is only pointwise, additional conditions — such as monotonicity combined with continuity of the limit (Dini's theorem), or equicontinuity of the sequence (Arzelà-Ascoli theorem) — are required to guarantee continuity of the limit.

The theory developed in this thesis opens several directions for further investigation. The extension of semi-continuity to general topological spaces, where sequences must be replaced by nets or filters, provides a natural generalization. In infinite-dimensional normed spaces, the notion of weak convergence plays the role of pointwise convergence, and the weak lower semi-continuity of convex functionals is the cornerstone of the direct method in the calculus of variations. The theory of Γ-convergence, due to De Giorgi, provides a variational generalization of pointwise convergence that is tailor-made for the study of limit problems in optimization and homogenization. Set-valued analysis extends semi-continuity to multifunctions, with applications in optimal control, game theory, and mathematical economics.

In conclusion, the study of convergence and semi-continuity in metric spaces is not merely a classical chapter of mathematical analysis, but a living and evolving theory whose fundamental results continue to find new applications in modern mathematics. The framework established in this thesis provides the reader with the essential tools to engage with these developments.

"In mathematics the art of proposing a question must be held of higher value than solving it."

— Georg Cantor

Notations

1. (X, d) : A metric space X equipped with a metric d.
2. $d(x, y)$: The distance between two points x and y in a metric space.
3. $B(a, r)$: The open ball with center a and radius r.
4. $\overline{B}(a, r)$: The closed ball with center a and radius r.
5. $S(a, r)$: The sphere with center a and radius r.
6. $\overline{A}$: The closure of a set A.
7. ∂A : The boundary of a set A.
8. $d(x, A)$: The distance from a point x to a set A.
9. (x_n) : A sequence in a metric space.
10. $x_n \to x$: The sequence (x_n) converges to x.
11. $\limsup_{n\to\infty} f(x_n)$: Limit superior of a sequence.
12. $\liminf_{n\to\infty} f(x_n)$: Limit inferior of a sequence.
13. $C([a, b], \mathbb{R})$: The space of continuous real-valued functions on $[a, b]$.
14. $\|f\|_\infty$: The supremum norm (uniform convergence norm).
15. $\|f\|_p$: The L^p-norm for $1 \leq p \leq \infty$.
16. $L^p(X, \mu)$: The space of p-integrable functions on a measure space.
17. $\operatorname{ess\,sup}$: Essential supremum.
18. $f_n \to f$ pointwise : Pointwise convergence of a sequence of functions.
19. $f_n \to f$ uniformly : Uniform convergence of a sequence of functions.
20. f is usc at x_0 : f is upper semi-continuous at x_0.
21. f is lsc at x_0 : f is lower semi-continuous at x_0.
22. $\operatorname{epi}(f)$: The epigraph of f, $\{(x, t) : f(x) \leq t\}$.

23. $\text{hyp}(f)$: The hypograph of f, $\{(x,t) : t \leq f(x)\}$.
24. $\mathbf{1}_A$: The indicator function of a set A.
25. $\limsup_{x\to x_0} f(x)$: Limit superior of f as x approaches x_0.
26. $\liminf_{x\to x_0} f(x)$: Limit inferior of f as x approaches x_0.
27. $f^{-1}(U)$: The preimage of a set U under f.
28. $\|x\|$: The norm of a vector x in a normed space.
29. $d(x,y) = \|x-y\|$: The metric induced by a norm.
30. $\inf_{x\in X} f(x)$: The infimum of f over X.
31. $\sup_{x\in X} f(x)$: The supremum of f over X.
32. $f_n \searrow f$: A decreasing sequence of functions converging to f.
33. $f_n \nearrow f$: An increasing sequence of functions converging to f.

Chapter 1

Theoretical Foundations of Metric Spaces and Functions

This chapter establishes the fundamental structures required to analyze functional behavior.By moving from Euclidean geometry to abstract Metric Spaces, we develop a generalized language for distance and proximity.We begin by defining the metric axioms, followed by the topological properties of sets, and conclude with the formal definition of continuous mappings.These concepts provide the necessary framework for studying convergence and semi-continuity in the subsequent chapters.

1.1 Metric Spaces: Definition and Examples

Definition 1.1.1 *[12, 19] A* ***distance*** *(or* ***metric****) Let X be a non-empty set. A funtion $d : X \times X \to \mathbb{R}$ is called a metric on X if for all $x, y, z \in X$, the following properties hold:*

1. ***Separation****: For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.*
2. ***Symmetry****: For all $x, y \in X$, $d(y, x) = d(x, y)$.*
3. ***Triangle inequality****: For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.*

1.1.1 Properties of Metric Spaces

Let (E, d) be a metric space. Then the distance d satisfies the following two properties:

1. The distance d is positive: $d(x, y) \geq 0$, $\forall x, y \in E$.
2. For all $x, y, z \in E$:
$$|d(x, z) - d(z, y)| \leq d(x, y).$$

1.1.2 Examples

Example 1.1.1 *[12, 19] The fundamental example of a metric space is the space $\mathbb{R}$ with the distance defined by $d(x,y) = |x-y|$. This distance is called the* ***usual distance*** *on $\mathbb{R}$.*

It is easy to verify that d is a metric. Indeed, for all $x, y, z \in \mathbb{R}$:

1. $d(x,y) = 0 \iff |x-y| = 0 \iff x = y$.
2. $d(x,y) = |x-y| = |y-x| = d(y,x)$.
3. $d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x,z) + d(z,y)$.

Example 1.1.2 *[12, 19] Let $d : E \times E \to \mathbb{R}$ be a mapping defined by*

$$\forall x, y \in E : \quad d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Let us show that d is a metric. Let $x, y, z \in E$. We have

1. *If $x = y$ then $d(x,y) = 0$ (by definition), and if $x \neq y$ then $d(x,y) = 1 \neq 0$. Hence*
$$d(x,y) = 0 \iff x = y.$$
2. *We have*
$$d(y,x) = \begin{cases} 1 & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases} = d(x,y).$$
3. - *If $x = y$ then*
$$0 = d(x,y) \leq d(x,z) + d(z,y).$$
   - *If $x \neq y$ then $x \neq z$ or $y \neq z$. Hence*
$$1 = d(x,y) \leq d(x,z) + d(z,y).$$

In both cases, we have $d(x,y) \leq d(x,z) + d(z,y)$.

Therefore, d is a metric on E. It is called the ***discrete metric****.*

Example 1.1.3 *[18, 19] (Distances on $C([a,b];\mathbb{R})$) Note that $C([a,b],\mathbb{R})$ is the space of continuous functions on the bounded interval $[a,b]$ with values in $\mathbb{R}$. We define the following three distances on this space:*

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx \tag{1.1}$$

$$d_2(f,g) = \left(\int_a^b |f(x) - g(x)|^2\, dx \right)^{1/2} \tag{1.2}$$

$$d_\infty(f,g) = \max_{x \in [a,b]} |f(x) - g(x)| \tag{1.3}$$

- *for all $f, g \in C([a,b], \mathbb{R})$. We verify that d_1 is a metric on $C([a,b], \mathbb{R})$. The other distances are left to the student.*

  1. ***Separation:***

$$\begin{aligned} d_1(f,g) = 0 &\iff \int_a^b |f(x) - g(x)|\, dx = 0 \\ &\iff |f(x) - g(x)| = 0, \quad \forall x \in [a,b] \\ &\iff f(x) = g(x), \quad \forall x \in [a,b] \\ &\iff f = g. \end{aligned}$$

  2. ***Symmetry:***

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx = \int_a^b |g(x) - f(x)|\, dx = d_1(g,f).$$

  3. ***Triangle inequality:***

$$\begin{aligned} d_1(f,g) &= \int_a^b |f(x) - g(x)|\, dx \\ &= \int_a^b |f(x) - h(x) + h(x) - g(x)|\, dx \\ &\leq \int_a^b \left(|f(x) - h(x)| + |h(x) - g(x)|\right)\, dx \\ &= \int_a^b |f(x) - h(x)|\, dx + \int_a^b |h(x) - g(x)|\, dx \\ &= d_1(f,h) + d_1(h,g). \end{aligned}$$

  Therefore, d_1 is a metric on $C([a,b], \mathbb{R})$.

- *for all $f, g \in C([a,b], \mathbb{R})$. We verify that d_2 is a metric on $C([a,b], \mathbb{R})$. The other distances are left to the student. Let $f, g, h \in C([a,b], \mathbb{R})$.*

1. *We have:*

$$\begin{aligned} d_2(f,g)=0 &\iff \int_a^b |f(x)-g(x)|^2\,dx = 0 \\ &\iff |f(x)-g(x)|=0, \quad \forall x\in[a,b] \\ &\iff f(x)=g(x), \quad \forall x\in[a,b] \\ &\iff f=g. \end{aligned}$$

2. *Symmetry is obvious:*

$$d_2(f,g)=\left(\int_a^b |f(x)-g(x)|^2\,dx\right)^{1/2} = \left(\int_a^b |g(x)-f(x)|^2\,dx\right)^{1/2} = d_2(g,f).$$

3. *For the triangle inequality, we apply the Cauchy-Schwarz inequality:*
Cauchy-Schwarz Inequality:

$$\int_a^b f(x)g(x)\,dx \le \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}\left(\int_a^b |g(x)|^2\,dx\right)^{1/2}, \quad \forall f,g\in C([a,b],\mathbb{R}). \tag{1.10}$$

We have:

$$\begin{aligned} d_2(f,g)^2 &= \int_a^b |f(x)-g(x)|^2\,dx \\ &= \int_a^b |(f(x)-h(x))+(h(x)-g(x))|^2\,dx \\ &\le \int_a^b \left(|f(x)-h(x)|+|h(x)-g(x)|\right)^2\,dx \\ &= \int_a^b |f(x)-h(x)|^2\,dx + \int_a^b |h(x)-g(x)|^2\,dx \\ &\quad + 2\int_a^b |f(x)-h(x)|\,|h(x)-g(x)|\,dx \\ &\overset{(1.10)}{\le} \int_a^b |f-h|^2\,dx + \int_a^b |h-g|^2\,dx \\ &\quad + 2\left(\int_a^b |f-h|^2\,dx\right)^{1/2}\left(\int_a^b |h-g|^2\,dx\right)^{1/2} \\ &= \left(\left(\int_a^b |f-h|^2\,dx\right)^{1/2} + \left(\int_a^b |h-g|^2\,dx\right)^{1/2}\right)^2 \\ &= \left(d_2(f,h)+d_2(h,g)\right)^2. \end{aligned}$$

Taking the square root of both sides, we obtain:

$$d_2(f,g)\le d_2(f,h)+d_2(h,g).$$

Therefore, d_2 is a metric on $C([a,b],\mathbb{R})$.

- *for all $f, g \in C([a,b],\mathbb{R})$. We verify that d_∞ is a metric on $C([a,b],\mathbb{R})$. The other distances are left to the student.*

  1. ***Separation:***

$$\begin{aligned} d_\infty(f,g) = 0 &\iff \max_{x\in[a,b]} |f(x) - g(x)| = 0 \\ &\iff |f(x) - g(x)| = 0, \quad \forall x \in [a,b] \\ &\iff f(x) = g(x), \quad \forall x \in [a,b] \\ &\iff f = g. \end{aligned}$$

  2. ***Symmetry:***

$$d_\infty(f,g) = \max_{x\in[a,b]} |f(x) - g(x)| = \max_{x\in[a,b]} |g(x) - f(x)| = d_\infty(g,f).$$

  3. ***Triangle inequality:** For all $x \in [a,b]$, we have*

$$|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)| \le d_\infty(f,h) + d_\infty(h,g).$$

  Taking the maximum over $x \in [a,b]$ on the left-hand side, we obtain

$$d_\infty(f,g) = \max_{x\in[a,b]} |f(x) - g(x)| \le d_\infty(f,h) + d_\infty(h,g).$$

Therefore, d_∞ is a metric on $C([a,b],\mathbb{R})$.

1.2 Topology in Metric Spaces

1.2.1 Open and Closed Balls

[12, 19] We define certain mathematical notions, inspired by Euclidean geometry of the plane and space.

Let (E,d) be a metric space, $a \in E$, and $r \ge 0$.

1. The **open ball** with center a and radius r is the set:

$$B(a,r) := \{x \in E : d(a,x) < r\}.$$

2. The **closed ball** with center a and radius r is the set:

$$\overline{B}(a,r) := \{x \in E : d(a,x) \le r\}.$$

3. The **sphere** with center a and radius r is the set:

$$S(a,r) := \{x \in E : d(a,x) = r\}.$$

Example 1.2.1. [12, 19] Let $E = \mathbb{R}$, equipped with the usual distance $d(x, y) = |x - y|$. We have shown that d is a metric on $\mathbb{R}$ and we have:

$$\begin{aligned} B(a, r) &:= \{x \in \mathbb{R} : d(a, x) < r\} \\ &= \{x \in \mathbb{R} : |x - a| < r\} \\ &= \{x \in \mathbb{R} : -r < x - a < r\} \\ &= \{x \in \mathbb{R} : a - r < x < a + r\} \\ &= (a - r, a + r). \end{aligned}$$

Similarly, we obtain the closed ball:

$$\overline{B}(a, r) = [a - r, a + r].$$

For the sphere, we have:

$$S(a, r) = \overline{B}(a, r) \setminus B(a, r) = \{a - r, a + r\}.$$

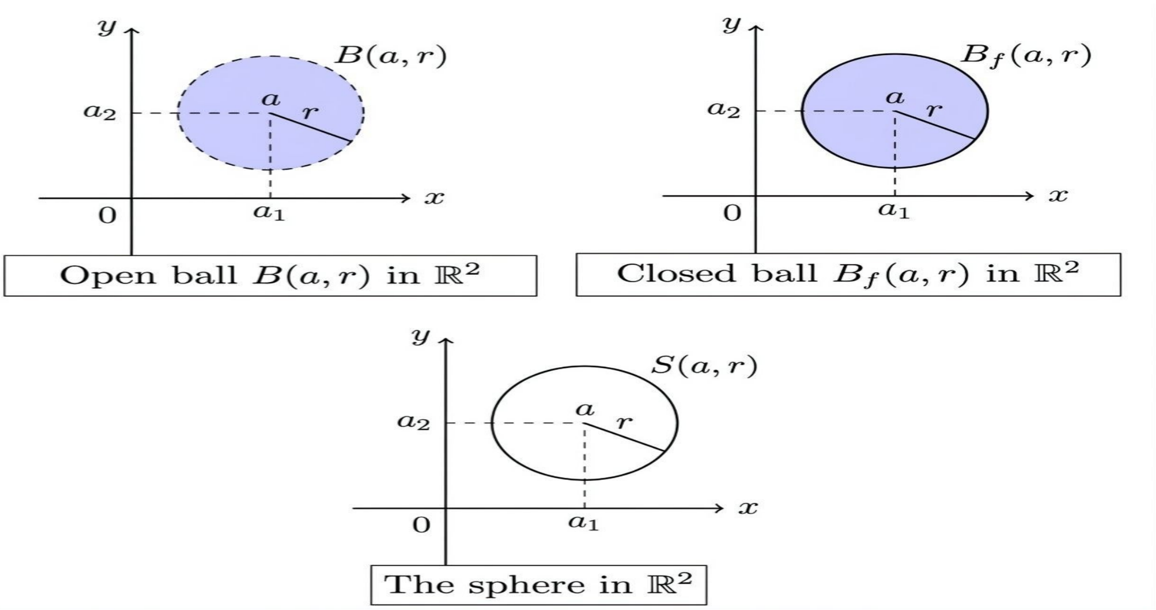
Example 1.2.2. [12, 19] Let $E = \mathbb{R}^2$ with the Euclidean distance :

$$d_2(x, y) := \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)^{1/2},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Then, for $a = (a_1, a_2) \in \mathbb{R}^2$, we have:

$$\begin{aligned} B(a, r) &:= \{x \in \mathbb{R}^2 : d_2(a, x) < r\} \\ &= \{x \in \mathbb{R}^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2\}. \end{aligned}$$

This is the open disk with center a and radius r, excluding its boundary.as shown in the figure:



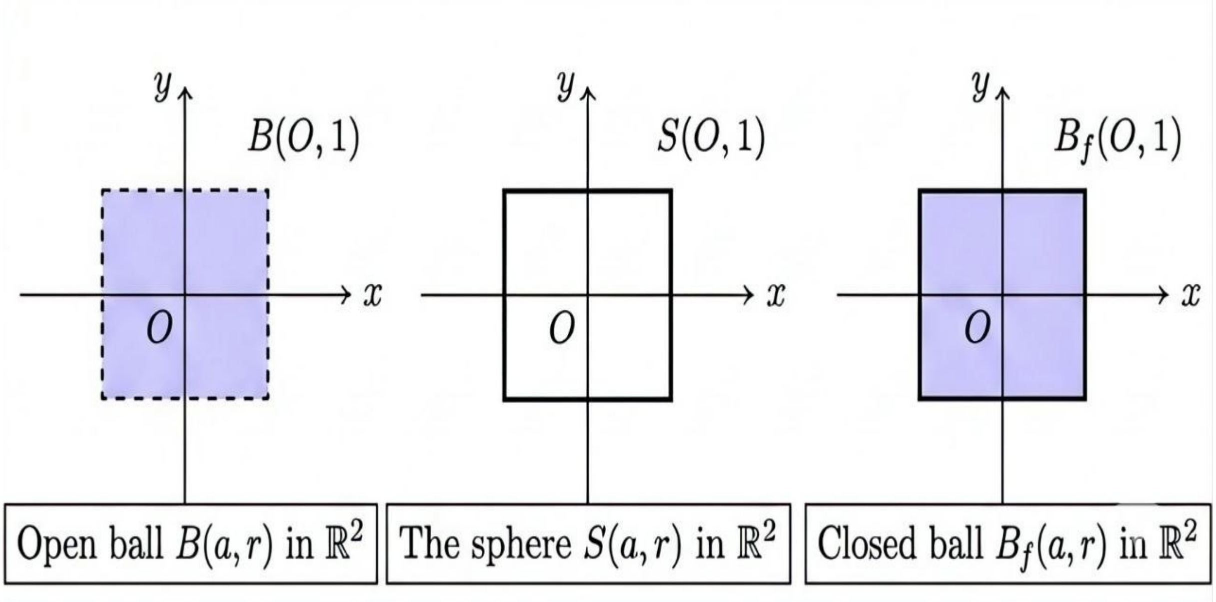
Example 1.2.3. [12, 19] Let $E = \mathbb{R}^2$ with the distance d_∞ defined by:

$$d_\infty(x, y) := \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Let us compute the unit ball (with center $O = (0, 0)$ and radius 1):

$$\begin{aligned} B(O,1) &:= \{X = (x,y) \in \mathbb{R}^2 : \max\{|x-0|, |y-0|\} < 1\} \\ &= \{(x,y) \in \mathbb{R}^2 : |x| < 1 \wedge |y| < 1\}. \end{aligned}$$

This is the square centered at O with side length 2, as shown in the figure:



Example 1.2.4. [12, 19] Let $E = \mathbb{R}^2$ with the distance d_1 defined by:

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

Let us compute the unit open ball. We have:

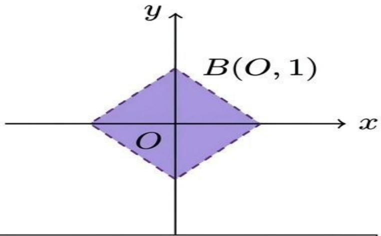
$$\begin{aligned} B(O,1) &:= \{(x,y) : |x-0| + |y-0| < 1\} \\ &= \{(x,y) : |x| + |y| < 1\}. \end{aligned}$$

This can be expressed as:

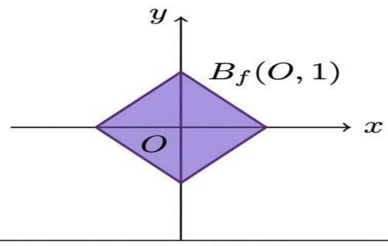
$$B(O,1) = \{(x,y) : x+y < 1 \wedge x-y < 1 \wedge -x+y < 1 \wedge -x-y < 1\}.$$

(Here, we distinguish the four cases: $x, y \geq 0$; $x \geq 0, y \leq 0$; $x \leq 0, y \geq 0$; and $x, y \leq 0$.)

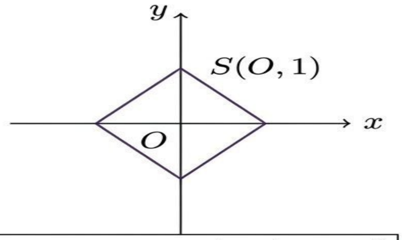
Thus, we obtain a diamond (rhombus). The following figures show the open ball, the closed ball, and the sphere.as shown in the figure:



Open ball $B(a,r)$ in $\mathbb{R}^2$



Closed ball $B_f(a,r)$ in $\mathbb{R}^2$



The sphere $S(a,r)$ in $\mathbb{R}^2$

Example 1.2.1 *Let (E, d) be a metric space. Find the open ball, closed ball, and sphere (with center a and radius $r > 0$) with respect to the discrete distance:*

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Let $a \in E$ and $r > 0$.

1. *For the distance* $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

   - ***Open ball:***

$$\begin{aligned} B(a,r) &:= \{x \in E : d(a,x) < r\} \\ &= \begin{cases} \{x \in E : d(a,x) = 0\} & \text{if } 0 < r \leq 1 \\ \\ \{x \in E : d(a,x) = 0 \vee d(a,x) = 1\} & \text{if } r > 1 \end{cases} \\ &= \begin{cases} \{a\} & \text{if } 0 < r \leq 1 \\ E & \text{if } r > 1 \end{cases} \end{aligned}$$

   - ***Closed ball:***

$$\begin{aligned} \overline{B}(a,r) &:= \{x \in E : d(a,x) \leq r\} \\ &= \begin{cases} \{x \in E : d(a,x) = 0\} & \text{if } 0 < r < 1 \\ \\ \{x \in E : d(a,x) = 0 \vee d(a,x) = 1\} & \text{if } r \geq 1 \end{cases} \\ &= \begin{cases} \{a\} & \text{if } 0 < r < 1 \\ E & \text{if } r \geq 1 \end{cases} \end{aligned}$$

- ***Sphere:***

$$\begin{aligned} S(a,r) &:= \overline{B}(a,r) \setminus B(a,r) \\ &= \begin{cases} \emptyset & \textit{if } r \neq 1 \\ E \setminus \{a\} & \textit{if } r = 1 \end{cases} \end{aligned}$$

1.2.2 Convergence of Sequences

[12, 19] Recall the definition of a sequence in E. It is a mapping from $\mathbb{N}$ to E, which associates to each n an element $u_n \in E$. The natural definition of convergence of a sequence in a metric space (E, d) is as follows:

Definition 1.2.1 *[12, 19] Let (E, d) be a metric space, $(u_n)_{n\in\mathbb{N}}$ a sequence in E, and $u \in E$. We say that the sequence (u_n)* ***converges*** *to u if*

$$\lim_{n\to\infty} d(u_n, u) = 0,$$

and u is called a ***limit*** *of (u_n).*

Properties of Convergence

- Since $d(u_n, u) \in \mathbb{R}$, it is well known that

$$\begin{aligned} \lim_{n\to\infty} d(u_n, u) = 0 &\iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} : n \geq N \implies d(u_n, u) \leq \varepsilon \\ &\iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} : n \geq N \implies u_n \in B(u, \varepsilon). \end{aligned}$$

- The limit u of the sequence (u_n) is unique. Indeed, if there exists another limit $v \in E$, then

$$0 \leq d(u, v) \leq d(u, u_n) + d(u_n, v)$$

  and taking the limit as $n \to \infty$, we obtain $0 \leq d(u, v) \leq 0$. Consequently, $d(u, v) = 0$ and thus $u = v$.

- A stationary sequence is convergent in any metric space. Indeed, if (u_n) is stationary, then there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N : u_n = u \quad \text{(is constant)}.$$

  Hence $d(u_n, u) = 0$ for all $n \geq N$. Therefore, $\lim\limits_{n\to\infty} d(u_n, u) = 0$.

Subsequence and Adherence Value

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (E, d).

1. A **subsequence** (or extracted sequence) of (u_n) is the sequence $(u_{\phi(n)})_{n\in\mathbb{N}}$, where $\phi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing mapping.

2. We say that $u \in E$ is an **adherence value** (or cluster point) of the sequence (u_n) if there exists a subsequence of (u_n) converging to u.

The following proposition gives a characterization between the closure of a set via sequences.

Properties Characterization of the Closure

Let A be a non-empty subset of a metric space (E, d). Then

$$\overline{A} = \{x \in E : \exists (x_n)_{n\in\mathbb{N}} \subset A \text{ such that } \lim_{n\to\infty} x_n = x\}.$$

In other words,

$$x \in \overline{A} \iff \exists (x_n)_{n\in\mathbb{N}} \subset A \text{ such that } \lim_{n\to\infty} x_n = x.$$

Proof

We have

$$\begin{aligned} x \in \overline{A} &\overset{\text{def}}{\iff} d(x, A) := \inf_{y\in A} d(x, y) = 0 \\ &\overset{\text{characterization of infimum}}{\iff} \forall \varepsilon > 0,\ \exists y \in A : d(x, y) < \varepsilon \\ &\iff \forall n \in \mathbb{N}^*,\ \exists y_n \in A : d(x, y_n) < \frac{1}{n} \\ &\iff (y_n)_{n\in\mathbb{N}} \subset A \text{ and } x = \lim_{n\to\infty} y_n. \end{aligned}$$

Corollary 1.2.1 *[12, 19]*

Let (E, d) be a metric space and $A \subseteq E$. Then

$$A \text{ is closed} \iff \big(\text{every convergent sequence } (a_n)_{n\in\mathbb{N}} \subset A \text{ has its limit in } A\big).$$

1.3 Cauchy Sequences and Complete Metric Spaces

Let (E, d) be a metric space. If the sequence $(u_n)_{n\in\mathbb{N}}$ in E converges to some limit ℓ, then $d(u_n, \ell) \to 0$ as $n \to \infty$, and by the triangle inequality, $d(u_p, u_q) \to 0$ as $p, q \to \infty$. That is, the terms u_p and u_q approach each other as p and q tend to $+\infty$. In this case, the sequence is said to be Cauchy. Thus, we give the following definition:

Definition 1.3.1 *[3, 19] A sequence $(u_n)_{n\in\mathbb{N}}$ in a metric space (E, d) is called a **Cauchy sequence** if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall p, q \in \mathbb{N} : q > p \geq N \implies d(u_p, u_q) \leq \varepsilon.$$

In other words,

$$\lim_{p,q\to\infty} d(u_p, u_q) = 0.$$

Remark 1.3.1 *[3, 19]*

Every convergent sequence is Cauchy (this is not difficult to verify). Indeed, if (u_n) *converges to* ℓ*, then for any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$*,* $d(u_n, \ell) < \varepsilon/2$*. Then for all* $p, q \geq N$*, we have*

$$d(u_p, u_q) \leq d(u_p, \ell) + d(\ell, u_q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (u_n) *is Cauchy. However, the converse is not always true in a metric space.*

For this, we give the following definition :

Definition 1.3.2 *Complete Metric Space [3, 19] A metric space* (E, d) *is said to be* ***complete*** *if every Cauchy sequence in this space is convergent.*

1.3.1 Properties Cauchy Sequences are Bounded

[3, 19]

Every Cauchy sequence in a metric space (E, d) is bounded. That is,

$$\exists a \in E,\ \exists M > 0 : d(a, u_n) \leq M, \quad \forall n \in \mathbb{N}.$$

Proof

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (E, d). By definition, for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $p, q \geq N$,

$$d(u_p, u_q) \leq 1.$$

In particular, for all $n \geq N$,

$$d(u_n, u_N) \leq 1.$$

Now, let

$$M = \max\{d(u_1, u_N), d(u_2, u_N), \ldots, d(u_{N-1}, u_N), 1\} + 1.$$

Then for all $n \in \mathbb{N}$:

- If $n < N$, we have $d(u_n, u_N) \leq M - 1 < M$.
- If $n \geq N$, we have $d(u_n, u_N) \leq 1 < M$.

Thus, taking $a = u_N$, we obtain

$$d(a, u_n) \leq M, \quad \forall n \in \mathbb{N}.$$

Hence, the sequence (u_n) is bounded.

1.3.2 Examples

Example 1.3.1 *[3, 19]*
The space $\mathbb{R}$ with the usual distance $d(x,y) = |x-y|$ is complete.

Proof Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. By the previous proposition, it is bounded. Therefore, there exists a subsequence $(u_{k_n})_{n\in\mathbb{N}}$ converging to $\ell \in \mathbb{R}$ (by the Bolzano-Weierstrass theorem). Let us show that the sequence converges to ℓ. Let $\varepsilon > 0$. Then there exist $N_1, N_2, N_3 \in \mathbb{N}$ such that:

$$\begin{cases} q > p \geq N_1 \implies d(u_p, u_q) \leq \varepsilon/2 & \text{since } (u_n) \text{ is Cauchy} \\ \\ n \geq N_2 \implies d(u_{k_n}, \ell) \leq \varepsilon/2 & \text{since } u_{k_n} \to \ell \\ \\ n \geq N_3 \implies k_n > N_1 & \text{since } k_n \to +\infty \text{ (definition of subsequence)} \end{cases}$$

Hence, for $N = \max\{N_1, N_2, N_3\}$, we have for all $n \geq N$:

$$d(u_n, \ell) \leq d(u_n, u_{k_n}) + d(u_{k_n}, \ell) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This gives the convergence of (u_n). Therefore, $(\mathbb{R}, d)$ is a complete space.

Example 1.3.2 *[3, 19] The space $\mathbb{R}$ equipped with the distance $d(x,y) = |\arctan x - \arctan y|$ is not complete.*

Proof Consider the sequence defined by $u_n = n$. Let us show that (u_n) is Cauchy, but not convergent.

Let $\varepsilon > 0$ and $p, q \in \mathbb{N}$ with $q > p$. We have

$$\begin{aligned} d(u_p, u_q) &= |\arctan p - \arctan q| \\ &= \left|\arctan \frac{p-q}{1+pq}\right| \\ &\leq \arctan \frac{q}{pq} = \arctan \frac{1}{p}. \end{aligned}$$

Moreover,

$$\arctan \frac{1}{p} \leq \varepsilon \iff \frac{1}{p} \leq \arctan \varepsilon \iff p \geq \frac{1}{\arctan \varepsilon}.$$

Hence, choosing $N := \left[\frac{1}{\arctan \varepsilon}\right] + 1 > \frac{1}{\arctan \varepsilon}$, we obtain

$$q > p \geq N \implies d(u_p, u_q) \leq \frac{1}{p} \leq \varepsilon.$$

Thus, (u_n) is Cauchy.

It remains to show that (u_n) is not convergent. Suppose, for contradiction, that there exists $a \in \mathbb{R}$ such that $\lim_{n\to\infty} u_n = a$. Then $\lim_{n\to\infty} d(n,a) = 0$. Hence,

$$d(n,a) = \left|\arctan \frac{n-a}{1+an}\right| \to 0 \quad \text{as} \quad n \to \infty,$$

and therefore $\frac{n-a}{1+an} \to 0$ as $n \to \infty$. However,

$$\frac{n-a}{1+an} \to \begin{cases} \frac{1}{a} & \text{if } a \neq 0 \\ +\infty & \text{if } a = 0 \end{cases}$$

as $n \to \infty$. This is a contradiction. Therefore, the sequence is not convergent and the space $(\mathbb{R}, d)$ is not complete.

1.4 Continuity of Functions

1.4.1 Continuity at a Point

Let (E,d) and (F,δ) be two metric spaces, $f : E \to F$ a mapping, and $a \in E$. We say that $\ell \in F$ is the **limit** of $f(x)$ as $x \to a$ if

$$\delta(\ell, f(x)) \to 0 \quad \text{as} \quad d(x,a) \to 0.$$

In other words,

$$\forall \varepsilon > 0,\ \exists r > 0 : 0 < d(a,x) < r \implies \delta(\ell, f(x)) < \varepsilon.$$

The following proposition gives a characterization of the limit of f at a point a by sequences.

Proposition 1.4.1 *[12, 19]*
Let (E,d) and (F,δ) be two metric spaces, $f : E \to F$ a mapping, $a \in E$, and $\ell \in F$. The following statements are equivalent:

1. $\lim\limits_{x\to a} f(x) = \ell$.
2. *For every non-stationary sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to a, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to ℓ.*

Proof We prove the equivalence in two steps.
(1) $\implies$ (2): Assume that $\lim\limits_{x\to a} f(x) = \ell$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E such that $x_n \to a$ and $x_n \neq a$ for all n (non-stationary). For any $\varepsilon > 0$, by definition of the limit, there exists $r > 0$ such that

$$0 < d(a,x) < r \implies \delta(\ell, f(x)) < \varepsilon.$$

Since $x_n \to a$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(a, x_n) < r$. Moreover, since $x_n \neq a$, we have $0 < d(a, x_n) < r$ for $n \geq N$. Hence, for all $n \geq N$, $\delta(\ell, f(x_n)) < \varepsilon$. Therefore, $f(x_n) \to \ell$.
(2) $\implies$ (1): We prove the contrapositive. Suppose that $\lim\limits_{x \to a} f(x) \neq \ell$. Then there exists $\varepsilon > 0$ such that for every $r > 0$, there exists $x \in E$ with $0 < d(a, x) < r$ and $\delta(\ell, f(x)) \geq \varepsilon$. In particular, for $r = \frac{1}{n}$, there exists $x_n \in E$ such that $0 < d(a, x_n) < \frac{1}{n}$ and $\delta(\ell, f(x_n)) \geq \varepsilon$. Then $x_n \to a$ and $x_n \neq a$ (non-stationary), but $f(x_n)$ does not converge to ℓ (since $\delta(\ell, f(x_n)) \geq \varepsilon$ for all n). This contradicts (2). Hence, $\lim\limits_{x \to a} f(x) = \ell$.

1.4.2 Continuity at a Point

[12, 19] Let (E, d) and (F, δ) be two metric spaces, $f : E \to F$ a mapping, and $a \in E$. We say that f is **continuous at** a if

$$\delta(f(x), f(a)) \to 0 \quad \text{as} \quad d(x, a) \to 0.$$

In other words,

$$\forall \varepsilon > 0,\ \exists r > 0 : d(a, x) < r \implies \delta(f(a), f(x)) < \varepsilon.$$

Remark 1.4.1 (Equivalent Formulations) *[12, 19]*
The previous definition is equivalent to:

$$\forall \varepsilon > 0,\ \exists r > 0 : x \in B_d(a, r) \implies f(x) \in B_\delta(f(a), \varepsilon).$$

or

$$\forall \varepsilon > 0,\ \exists r > 0 : B_d(a, r) \subset f^{-1}\big(B_\delta(f(a), \varepsilon)\big).$$

Proposition 1.4.2 (Sequential Characterization of Continuity) *[12, 19] Let (E, d) and (F, δ) be two metric spaces, $f : E \to F$ a mapping, and $x \in E$. The following statements are equivalent:*

1. *f is continuous at x.*
2. *For every sequence $(x_n)_{n \in \mathbb{N}}$ in E converging to x, the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$.*

Proof
We prove the equivalence in two steps.
(1) $\implies$ (2): Suppose that f is continuous at x. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in E such that $x_n \to x$ as $n \to \infty$. Let $\varepsilon > 0$. By continuity of f at x, there exists $r > 0$ such that

$$d(x, y) < r \implies \delta(f(x), f(y)) < \varepsilon.$$

Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x, x_n) < r$. Consequently, for all $n \geq N$, $\delta(f(x), f(x_n)) < \varepsilon$. Hence $f(x_n) \to f(x)$.
(2) $\implies$ (1): We prove the contrapositive. Suppose that f is not continuous at x. Then there exists $\varepsilon > 0$ such that for every $r > 0$, there exists $y \in E$ with $d(x, y) < r$ and $\delta(f(x), f(y)) \geq \varepsilon$. In particular, for $r = \frac{1}{n}$ (with $n \in \mathbb{N}^*$), there exists $x_n \in E$ such that

$$d(x, x_n) < \frac{1}{n} \quad \text{and} \quad \delta(f(x), f(x_n)) \geq \varepsilon.$$

Then $x_n \to x$ (since $d(x, x_n) < 1/n \to 0$), but $(f(x_n))$ does not converge to $f(x)$ because $\delta(f(x), f(x_n)) \geq \varepsilon$ for all n. Thus, condition (2) fails. Therefore, (2) implies (1).

Definition 1.4.1 (Global Continuity) *[12, 19]*
*Let (E, d) and (F, δ) be two metric spaces and $f : E \to F$ a mapping. We say that f is **continuous on** E (or simply **continuous**) if it is continuous at every point $a \in E$.*

The following proposition gives a very important characterization of global continuity.

Proposition 1.4.3 (Characterization of Continuity via Open Sets) *[12, 19]*
The mapping $f : (E, d) \to (F, \delta)$ is continuous if and only if the preimage of every open set in F is open in E.

Proof We prove both implications.
($\Rightarrow$) Suppose that f is continuous on E. Let O be an open set in (F, δ). Then $O = \bigcup_{i \in I} B_i$ where each B_i is an open ball in (F, δ). Let us first show that $f^{-1}(B_i)$ is open in (E, d). Let $x \in f^{-1}(B_i)$. Then $f(x) \in B_i$. Therefore, by (1.13), there exists $\varepsilon > 0$ such that

$$B_\delta(f(x), \varepsilon) \subset B_i.$$

By continuity of f at x, there exists $r > 0$ such that

$$B_d(x, r) \subset f^{-1}\big(B_\delta(f(x), \varepsilon)\big) \subset f^{-1}(B_i).$$

We have shown that

$$\forall x \in f^{-1}(B_i),\ \exists r > 0 : B_d(x, r) \subset f^{-1}(B_i).$$

Hence $f^{-1}(B_i)$ is open in (E, d). Consequently,

$$f^{-1}(O) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

is also open in (E, d).

($\Leftarrow$) Suppose that the preimage of every open set in F is open in E. Let us show that f is continuous at every $a \in E$. Let $\varepsilon > 0$. Then $f^{-1}\big(B_\delta(f(a), \varepsilon)\big)$ is open in (E, d) and $a \in f^{-1}\big(B_\delta(f(a), \varepsilon)\big)$. By the definition of an open set, there exists $r > 0$ such that

$$B_d(a, r) \subset f^{-1}\big(B_\delta(f(a), \varepsilon)\big).$$

This implies continuity at a.

Definition 1.4.2 (Homeomorphism) *[12, 19]*
*Let (E, d) and (F, δ) be two metric spaces and $f : E \to F$ a mapping. We say that f is a **homeomorphism** if:*

1. *f is bijective.*
2. *f and f^{-1} are continuous (where f^{-1} is the inverse mapping of f).*

*If such a mapping exists, we say that the spaces (E, d) and (F, δ) are **homeomorphic**.*

Definition 1.4.3 (Uniform Continuity and Lipschitz Functions) *[12, 19]*
Let (E, d) and (F, δ) be two metric spaces and $f : E \to F$ a mapping.

1. *We say that f is **uniformly continuous** if*
$$\forall \varepsilon > 0,\ \exists \eta > 0 : d(x, y) \leq \eta \implies \delta(f(x), f(y)) \leq \varepsilon.$$
2. *We say that f is **Lipschitz** (or **Lipschitz continuous**) if*
$$\exists L > 0,\ \forall x, y \in E : \delta(f(x), f(y)) \leq L\, d(x, y).$$

Proposition 1.4.4 (Relations between Types of Continuity) *[12, 19]*
Let $f : (E, d) \to (F, \delta)$ be a mapping. Then

1. *f is Lipschitz $\implies$ f is uniformly continuous.*
2. *f is uniformly continuous $\implies$ f is continuous.*

Proof
We prove both implications.

1. Suppose that f is Lipschitz. Then there exists $L > 0$ such that for all $x, y \in E$,
$$\delta(f(x), f(y)) \leq L\, d(x, y).$$
Let $\varepsilon > 0$. Choose $\eta = \frac{\varepsilon}{L} > 0$. Then for any $x, y \in E$ with $d(x, y) \leq \eta$, we have
$$\delta(f(x), f(y)) \leq L\, d(x, y) \leq L \cdot \frac{\varepsilon}{L} = \varepsilon.$$
Hence f is uniformly continuous.

2. Suppose that f is uniformly continuous. Let $a \in E$ and $\varepsilon > 0$. By uniform continuity, there exists $\eta > 0$ such that for all $x, y \in E$,
$$d(x,y) \leq \eta \implies \delta(f(x), f(y)) \leq \varepsilon.$$
In particular, for $y = a$, we have for all $x \in E$ with $d(x,a) \leq \eta$,
$$\delta(f(x), f(a)) \leq \varepsilon.$$
This is exactly the definition of continuity of f at a. Since $a \in E$ was arbitrary, f is continuous on E.

1.5 Normed Spaces

Definition 1.5.1 *[12, 19]*

*Let X be a vector space over $\mathbb{F}$. A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for all $x, y \in X$ and $\alpha \in \mathbb{F}$:*

(i) $\|x\| \geq 0$;

(ii) $\|x\| = 0$ *if and only if* $x = 0$;

(iii) $\|\alpha x\| = |\alpha|\|x\|$;

(iv) $\|x + y\| \leq \|x\| + \|y\|$ *(Triangle Inequality).*

Example 1.5.1 *(Standard Norm on $\mathbb{F}^n$) [12, 19]*

The function $\|\cdot\| : \mathbb{F}^n \to \mathbb{R}$ defined by
$$\|(x_1, \ldots, x_n)\| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}$$
*is a norm on $\mathbb{F}^n$ called the **standard norm** on $\mathbb{F}^n$.*

Proof We verify the four properties of a norm.

(i) Positivity: For any $(x_1, \ldots, x_n) \in \mathbb{F}^n$, we have $|x_j|^2 \geq 0$ for all j. Hence $\sum_{j=1}^{n} |x_j|^2 \geq 0$, and taking the square root gives $\|(x_1, \ldots, x_n)\| \geq 0$.

(ii) Definiteness:
$$\begin{aligned}
\|(x_1, \ldots, x_n)\| = 0 &\iff \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}} = 0 \\
&\iff \sum_{j=1}^{n} |x_j|^2 = 0 \\
&\iff |x_j|^2 = 0 \quad \forall j = 1, \ldots, n \\
&\iff x_j = 0 \quad \forall j = 1, \ldots, n \\
&\iff (x_1, \ldots, x_n) = (0, \ldots, 0).
\end{aligned}$$

(iii) Homogeneity: For any $\alpha \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$,

$$\begin{aligned}
\|\alpha(x_1, \ldots, x_n)\| &= \|(\alpha x_1, \ldots, \alpha x_n)\| \\
&= \left(\sum_{j=1}^{n} |\alpha x_j|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^{n} |\alpha|^2 |x_j|^2 \right)^{\frac{1}{2}} \\
&= \left(|\alpha|^2 \sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \\
&= |\alpha| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \\
&= |\alpha| \|(x_1, \ldots, x_n)\|.
\end{aligned}$$

(iv) Triangle Inequality: For any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{F}^n$,

$$\|(x_1, \ldots, x_n) + (y_1, \ldots, y_n)\| = \left(\sum_{j=1}^{n} |x_j + y_j|^2 \right)^{\frac{1}{2}}.$$

By the Cauchy-Schwarz inequality,

$$\sum_{j=1}^{n} |x_j + y_j|^2 \leq \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |y_j|^2 \right)^{\frac{1}{2}}.$$

Taking the square root of both sides yields

$$\|(x_1, \ldots, x_n) + (y_1, \ldots, y_n)\| \leq \|(x_1, \ldots, x_n)\| + \|(y_1, \ldots, y_n)\|.$$

Example 1.5.2 *(Standard Norm on $C_{\mathbb{F}}(M)$) [12, 19]*
Let M be a compact metric space and let $C_{\mathbb{F}}(M)$ be the vector space of continuous, $\mathbb{F}$-valued functions defined on M. Then the function $\|\cdot\| : C_{\mathbb{F}}(M) \to \mathbb{R}$ defined by

$$\|f\| = \sup\{|f(x)| : x \in M\}$$

*is a norm on $C_{\mathbb{F}}(M)$ called the **standard norm** (or **supremum norm**) on $C_{\mathbb{F}}(M)$.*

[12, 19] We verify the four properties of a norm.

(i) Positivity: For any $f \in C_{\mathbb{F}}(M)$, we have $|f(x)| \geq 0$ for all $x \in M$. Hence $\sup_{x \in M} |f(x)| \geq 0$, i.e., $\|f\| \geq 0$.

(ii) Definiteness:

$$\begin{aligned}\|f\| = 0 &\iff \sup_{x\in M} |f(x)| = 0 \\ &\iff |f(x)| = 0 \quad \forall x \in M \\ &\iff f(x) = 0 \quad \forall x \in M \\ &\iff f = 0 \text{ (the zero function).}\end{aligned}$$

(iii) Homogeneity: For any $\alpha \in \mathbb{F}$ and $f \in C_{\mathbb{F}}(M)$,

$$\begin{aligned}\|\alpha f\| &= \sup_{x\in M} |\alpha f(x)| \\ &= \sup_{x\in M} (|\alpha||f(x)|) \\ &= |\alpha| \sup_{x\in M} |f(x)| \\ &= |\alpha|\|f\|.\end{aligned}$$

(iv) Triangle Inequality: For any $f, g \in C_{\mathbb{F}}(M)$ and for every $x \in M$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|.$$

Taking the supremum over $x \in M$ on the left-hand side, we obtain

$$\|f + g\| = \sup_{x\in M} |f(x) + g(x)| \leq \|f\| + \|g\|.$$

Example 1.5.3 *(Standard Norms on L^p Spaces) [12, 19]*
Let (X, Σ, μ) be a measure space.

(a) If $1 \leq p < \infty$ then

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$$

*is a norm on $L^p(X)$ called the **standard norm** on $L^p(X)$.*

(b)

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| : x \in X\}$$

*is a norm on $L^\infty(X)$ called the **standard norm** on $L^\infty(X)$.*

Proof for (a). We verify the four properties of a norm for $1 \leq p < \infty$.

1. **Positivity:** For any $f \in L^p(X)$, we have $|f(x)|^p \geq 0$ for almost every $x \in X$. Hence $\int_X |f|^p \, d\mu \geq 0$, and taking the p-th root gives $\|f\|_p \geq 0$.

2. **Definiteness:**

$$\begin{aligned}
\|f\|_p = 0 &\iff \left(\int_X |f|^p \, d\mu\right)^{1/p} = 0 \\
&\iff \int_X |f|^p \, d\mu = 0 \\
&\iff |f|^p = 0 \text{ almost everywhere} \\
&\iff f = 0 \text{ almost everywhere} \\
&\iff f = 0 \text{ in } L^p(X).
\end{aligned}$$

3. **Homogeneity:** For any $\alpha \in \mathbb{F}$ and $f \in L^p(X)$,

$$\begin{aligned}
\|\alpha f\|_p &= \left(\int_X |\alpha f|^p \, d\mu\right)^{1/p} \\
&= \left(\int_X |\alpha|^p |f|^p \, d\mu\right)^{1/p} \\
&= \left(|\alpha|^p \int_X |f|^p \, d\mu\right)^{1/p} \\
&= |\alpha| \left(\int_X |f|^p \, d\mu\right)^{1/p} \\
&= |\alpha| \|f\|_p.
\end{aligned}$$

4. **Triangle Inequality (Minkowski Inequality):** For any $f, g \in L^p(X)$,

$$\|f+g\|_p = \left(\int_X |f+g|^p \, d\mu\right)^{1/p} \leq \left(\int_X |f|^p \, d\mu\right)^{1/p} + \left(\int_X |g|^p \, d\mu\right)^{1/p} = \|f\|_p + \|g\|_p.$$

This is Minkowski's inequality, which holds for all $1 \leq p < \infty$.

Thus, $\| \cdot \|_p$ is a norm on $L^p(X)$.

Proof for (b). We verify the four properties of a norm for $p = \infty$.

1. **Positivity:** For any $f \in L^\infty(X)$, we have $\operatorname{ess\,sup} |f(x)| \geq 0$ by definition. Hence $\|f\|_\infty \geq 0$.

2. **Definiteness:**

$$\begin{aligned}
\|f\|_\infty = 0 &\iff \operatorname{ess\,sup} |f(x)| = 0 \\
&\iff |f(x)| = 0 \text{ almost everywhere} \\
&\iff f = 0 \text{ almost everywhere} \\
&\iff f = 0 \text{ in } L^\infty(X).
\end{aligned}$$

3. **Homogeneity:** For any $\alpha \in \mathbb{F}$ and $f \in L^\infty(X)$,

$$\begin{aligned}\|\alpha f\|_\infty &= \operatorname{ess\,sup} |\alpha f(x)| \\ &= \operatorname{ess\,sup} (|\alpha||f(x)|) \\ &= |\alpha| \operatorname{ess\,sup} |f(x)| \\ &= |\alpha|\|f\|_\infty.\end{aligned}$$

4. **Triangle Inequality:** For any $f, g \in L^\infty(X)$ and for almost every $x \in X$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty.$$

Taking the essential supremum on the left-hand side, we obtain

$$\|f + g\|_\infty = \operatorname{ess\,sup} |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty.$$

Example (Spaces of Bounded Functions) 1.3.4 [12, 19]

Let X be a non-empty set and $B(X, \mathbb{K})$ the set of bounded mappings from X to $\mathbb{K}$. This is a vector subspace of $\mathcal{F}(X, \mathbb{K})$, the vector space formed by all mappings from X to $\mathbb{K}$). If $f \in B(X, \mathbb{K})$, set

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Since f is bounded, $\|f\|_\infty \in \mathbb{R}_+$. We obtain a norm $\|\cdot\|_\infty$ on $B(X, \mathbb{K})$. Let us verify, for example, the triangle inequality. Let $f, g \in B(X, \mathbb{K})$. For every $x \in X$, the triangle inequality in $\mathbb{K}$ gives:

$$|(f + g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty.$$

Hence, by taking the supremum,

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

This norm is called the **norm of uniform convergence**.

Lemma (Norm Induces a Metric) 1.3.4 [12, 19]

Let X be a vector space with norm $\|\cdot\|$. If $d : X \times X \to \mathbb{R}$ is defined by

$$d(x, y) = \|x - y\|,$$

then (X, d) is a metric space.

Proof:

We verify the four properties of a metric.

1. **Positivity:** For any $x, y \in X$, we have $\|x - y\| \geq 0$ by property (i) of the norm. Hence $d(x, y) \geq 0$.

2. **Identity of indiscernibles:**

$$\begin{aligned} d(x,y)=0 &\iff \|x-y\|=0 \\ &\iff x-y=0 \quad \text{(by property (ii) of the norm)} \\ &\iff x=y. \end{aligned}$$

3. **Symmetry:** For any $x, y \in X$,

$$\begin{aligned} d(y,x) &= \|y-x\| \\ &= \|-(x-y)\| \\ &= |-1|\cdot\|x-y\| \quad \text{(by property (iii) of the norm)} \\ &= \|x-y\| \\ &= d(x,y). \end{aligned}$$

4. **Triangle inequality:** For any $x, y, z \in X$,

$$\begin{aligned} d(x,z) &= \|x-z\| \\ &= \|(x-y)+(y-z)\| \\ &\leq \|x-y\|+\|y-z\| \quad \text{(by property (iv) of the norm)} \\ &= d(x,y)+d(y,z). \end{aligned}$$

Thus, d satisfies all the properties of a metric, and (X, d) is a metric space.

Notation 1.3.4 [12, 19]
Let X be a vector space with norm $\|\cdot\|$. The metric $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \|x-y\|$$

is called the **metric associated with** the norm $\|\cdot\|$.

In this case, (X, d) is the metric space induced by the normed vector space $(X, \|\cdot\|)$.

Chapter 2

Types of Convergence of Function Sequences

The focus of this chapter is to analyze how a sequence of functions f_n approaches a limit function f. We will define and compare two fundamental modes of convergence: Pointwise Convergence and Uniform Convergence. The primary objective is to determine which type of convergence is strong enough to preserve properties like continuity. By establishing these criteria, we provide the necessary analytical tools to bridge the gap between function sequences and the study of semi-continuity in the following chapters.

2.1 Simple Convergence

Let X be a non-empty set, $(E, \|\cdot\|)$ a real normed vector space , and (f_n) a sequence of mappings from X to E. as is most often the case in applications, we rather say that (f_n) is a sequence of functions (real or complex). Can we talk about "convergence of the sequence (f_n)" in the same way we talk about convergence of a numerical sequence? We will see that the term "convergence" can be given several meanings; we say that there are several "modes of convergence" of the sequence (f_n).

The first mode of convergence, natural but not very useful, is "simple convergence". Here is what it is about. For each fixed $x \in X$, $\big(f_n(x)\big)$ is a sequence in E, and we can require that this sequence converges (in E) for every $x \in X$.

This leads to the following definition:

Definition 2.1.1 *[12, 19]*

We say that the sequence (f_n) ***converges simply*** *(or* ***pointwise****) to a function $f : X \to E$ if*

$$\forall x \in X, \quad \lim_{n\to\infty} f_n(x) = f(x).$$

In other words,

$$\forall x \in X,\ \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ \|f_n(x) - f(x)\| < \varepsilon. \tag{1}$$

It is important to note that the integer N here depends on both x and ε. Moreover, if (f_n) converges simply to f and also to $g : X \to E$, then $g = f$ (uniqueness of the possible simple limit). Indeed, for every $x \in X$, the sequence $\big(f_n(x)\big)$ converges to both $f(x)$ and $g(x)$, hence $f(x) = g(x)$ (by uniqueness of the limit of a sequence in E). [12, 19]

Example 2.1.1 (Simple Convergence of $f_n(x) = x^n$) [12, 19]

1. Let $X := \mathbb{R}$, $E := \mathbb{R}$ and $f_n(x) := x^n$ for all $(n, x) \in \mathbb{N} \times \mathbb{R}$. Fix $x \in \mathbb{R}$. The real sequence (x^n) converges if and only if $x \in]-1, 1]$. We then say that the **domain of simple convergence** of the sequence (f_n) is $]-1, 1]$. Moreover, (x^n) converges to 0 if $|x| < 1$ and to 1 if $x := 1$. Therefore, we say that the sequence (f_n) converges simply on $]-1, 1]$ to the function $f :]-1, 1] \to \mathbb{R}$ defined by $f(1) := 1$ and $f(x) := 0$ if $x \neq 1$.

   For every n, let g_n be the restriction of f_n to $]-1, 1]$. The sequence (g_n) thus converges simply to f. Note that the function f is not continuous (from the left) at 1, even though the g_n are continuous. Therefore, simple convergence is not a strong enough mode of convergence to preserve continuity. **Uniform convergence** will correct this defect.

   Let $x \in]-1, 1[$ and $\varepsilon > 0$. Let $N \in \mathbb{N}$. The implication

   $$n \geq N \implies |x^n| < \varepsilon$$

   holds if and only if $N \ln |x| < \ln \varepsilon$, which is equivalent to $N > \dfrac{\ln \varepsilon}{\ln |x|}$ since $\ln |x| < 0$. The fact that N depends on ε and x is clearly visible here.

Example 2.1.2 (Simple Convergence to the Exponential Function) [12, 19]

Let $X := \mathbb{C}$, $E := \mathbb{C}$ and $f_n(z) := \left(1 + \frac{z}{n}\right)^n$ for all $z \in \mathbb{C}$ and $n \in \mathbb{N}^*$. We will prove the following formula concerning the exponential function $\exp(z)$ of z:

$$\exp z = \lim_{n \to +\infty} \left(1 + \frac{z}{n}\right)^n. \tag{2}$$

Thus, the function $\exp : \mathbb{C} \to \mathbb{C}$ is the simple limit of the sequence $(f_n)_{n \geq 1}$.

Example 2.1.3 (Simple Convergence to the Square Root Function) [12, 19]

Define a sequence of functions $\big(f_n : [0, 1] \to \mathbb{R}\big)$ by $f_0 := 0$ and, for all $n \in \mathbb{N}$:

$$f_{n+1}(x) := f_n(x) + \frac{1}{2}\big(x - f_n(x)^2\big) \quad \text{for all } x \in I := [0, 1]. \tag{3}$$

Let us study the simple convergence of the sequence (f_n). Fix $x \in I$ and let $\phi : I \to \mathbb{R}$ be the function $t \mapsto t + \frac{1}{2}(x - t^2)$. It increases from $\frac{x}{2}$ to $\frac{x+1}{2}$, hence

$\phi(I) \subset I$. Set $u_n := f_n(x)$ for all n. Thus $u_0 := 0$ and $u_{n+1} := \phi(u_n)$ for all $n \in \mathbb{N}$. It follows that (u_n) is an increasing sequence in I, therefore it converges to some $\ell \in I$.

Since ϕ is continuous, ℓ is a fixed point of ϕ, from which we immediately obtain $\ell := \sqrt{x}$. Thus the sequence $\big(f_n(x)\big)$ converges to $\sqrt{x}$. Conclusion: the sequence (f_n) converges simply to the function $x \mapsto \sqrt{x}$.

2.2 Uniform Convergence

Definition 2.2.1 [12, 19]

We say that the sequence of mappings $(f_n : X \to E)$ **converges uniformly** to a mapping $f : X \to E$ if the following condition holds:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ \forall x \in X,\ \|f_n(x) - f(x)\| \leq \varepsilon. \tag{4}$$

Compared to condition (1), it is essential to note that $(\forall x \in X)$ has changed position: in condition (4), N must be independent of x. This comparison between conditions (1) and (4) shows, in any case, that if (f_n) converges uniformly to f, then (f_n) converges simply to f: uniform convergence implies simple convergence.

One can interpret this definition using the "norm of uniform convergence", which generalizes Example 6 on page 14. For any mapping $f : X \to E$, set:

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|. \tag{5}$$

This supremum may be equal to $+\infty$, and it is finite if and only if f is bounded. Let $B(X, E)$ denote the set of bounded mappings from X to E; .

Proposition and Definition 2.2.2 [12, 19]

The mapping $f \mapsto \|f\|_\infty$ from $B(X, E)$ to $\mathbb{R}$ is a norm on $B(X, E)$. It is called the **uniform convergence norm** (or **supremum norm**).

Proof

We verify the three properties of a norm:

1. **Positivity**: Clearly $\|f\|_\infty \geq 0$. Moreover, $\|f\|_\infty = 0$ implies $\sup_{x \in X} \|f(x)\| = 0$, hence $f(x) = 0$ for all $x \in X$, i.e., $f = 0$.

2. **Homogeneity**: For any $\lambda \in \mathbb{R}$, $\|\lambda f\|_\infty = \sup_{x \in X} \|\lambda f(x)\| = |\lambda| \sup_{x \in X} \|f(x)\| = |\lambda| \|f\|_\infty$.

3. **Triangle inequality**: For any $f, g \in B(X, E)$ and any $x \in X$, we have $\|f(x) + g(x)\| \leq \|f(x)\| + \|g(x)\| \leq \|f\|_\infty + \|g\|_\infty$. Taking the supremum over $x \in X$ yields $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Thus, $\|\cdot\|_\infty$ is a norm on $B(X, E)$.
Remark 2.2.2 [12, 19]

Uniform convergence of (f_n) to f is equivalent to:

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|f_n - f\|_\infty = \sup_{x\in X} \|f_n(x) - f(x)\| \leq \varepsilon.$$

Corollary 2.2.2 [12, 19]

The sequence $(f_n : X \to E)$ converges uniformly to $f : X \to E$ if and only if

$$\lim_{n\to+\infty} \|f_n - f\|_\infty = 0.$$

2.2.1 Simple Convergence and Uniform Convergence

In particular, if each f_n is bounded, the sequence (f_n) converges uniformly if and only if it is a convergent sequence in the normed vector space $\big(B(X, E), \|\cdot\|_\infty\big)$. For the necessity, note that the limit f of (f_n) is bounded because, if n is sufficiently large, $f_n - f$ and f_n are bounded, hence f is bounded. This justifies the terminology ”norm of uniform convergence”. [12, 19]
Clarification [12, 19]

If each f_n is bounded and $f_n \to f$ uniformly, then there exists N such that $\|f_n - f\|_\infty \leq 1$ for all $n \geq N$. Hence, for all $x \in X$,

$$\|f(x)\| \leq \|f(x) - f_N(x)\| + \|f_N(x)\| \leq 1 + \|f_N\|_\infty < +\infty.$$

Thus f is bounded. Therefore, uniform convergence preserves boundedness, and the space $B(X, E)$ equipped with $\|\cdot\|_\infty$ is a natural setting for studying uniform convergence.

Example 2.2.1 [12, 19]

1. Let us revisit Example 1 from the previous page, with $X := [0, 1[$ and $f_n(x) := x^n$ for all $(n, x) \in \mathbb{N} \times X$. The sequence (f_n) converges pointwise to the zero function. The convergence of (f_n) to 0 is not uniform because

$$\|f_n - 0\|_\infty = \sup_{x\in X} x^n = 1 \quad \text{for all } n.$$

Consider a fixed real number $a \in]0, 1[$, and let $X_a := [0, a]$. We have

$$\sup_{x\in X_a} x^n = a^n \quad \text{for all } n,$$

which shows that the convergence of (f_n) to 0 is uniform on X_a. In conclusion, the convergence of (f_n) to 0 is uniform on X_a for every a, but is not uniform on X, which is nevertheless the union of the X_a. The notion of uniform convergence is therefore relative to the set on which we consider it.

Example 2.2.2 [12, 19]

Let us show that the sequence of functions (f_n) defined on $\mathbb{R}$ by $f_n(x) := \cos^n x \sin x$ converges uniformly to 0.

Pointwise convergence is obvious. Indeed, let $x \in \mathbb{R}$. If $\sin x = 0$, then $f_n(x) = 0$ for all n. If $\sin x \neq 0$, then $|\cos x| < 1$, hence $\lim_{n\to+\infty} \cos^n x = 0$. In both cases, the sequence $\big(f_n(x)\big)$ converges to 0.

For **uniform convergence**, we need to work a little more. Let $n \in \mathbb{N}^*$, and compute $\|f_n\|_\infty$ (it would suffice to bound this norm). The function $|f_n|$ is even, π-periodic, and continuous. Therefore, it attains its absolute maximum at a point $a_n \in \left]0, \frac{\pi}{2}\right[$, at which f_n' vanishes.

Since

$$f_n'(x) = \cos^{n-1} x \left(\cos^2 x - n \sin^2 x\right),$$

we have $\cos^2 a_n - n(1 - \cos^2 a_n) = 0$, hence

$$\cos a_n = \sqrt{\frac{n}{n+1}}, \quad \text{and then} \quad \sin a_n = \frac{1}{\sqrt{n+1}}.$$

Under these conditions, we obtain:

$$\|f_n\|_\infty = \cos^n a_n \sin a_n = \left(1 + \frac{1}{n}\right)^{-n/2} \frac{1}{\sqrt{n+1}} \sim \frac{1}{\sqrt{en}}.$$

When n tends to $+\infty$, $\|f_n\|_\infty$ therefore tends to 0. In fact, $\|f_n\|_\infty \sim \frac{1}{\sqrt{en}}$. The previous corollary proves, in any case, that the sequence of functions (f_n) converges uniformly to 0.

Example 2.2.3 [12, 19]

Define a sequence of functions $\big(f_n : [0,1] \to \mathbb{R}\big)$ by $f_0 := 0$ and, for all $n \in \mathbb{N}$:

$$f_{n+1}(x) := f_n(x) + \frac{1}{2}\big(x - f_n(x)^2\big) \quad \text{for all } x \in I := [0,1].$$

Let us show that the sequence (f_n) converges uniformly to the function $f : x \mapsto \sqrt{x}$.

Solution. We show that (f_n) converges to f in $\big(C(I,\mathbb{R}), \|\cdot\|_\infty\big)$, i.e., that the real sequence $\big(\|f_n - f\|_\infty\big)$ converges to 0. If $n \in \mathbb{N}$ and $x \in I$, formula (3) on page 476 gives:

$$\sqrt{x} - f_{n+1}(x) = \sqrt{x} - f_n(x) - \frac{\big(x - f_n(x)^2\big)}{2} = \big(\sqrt{x} - f_n(x)\big)\left(1 - \frac{\sqrt{x} + f_n(x)}{2}\right).$$

The expression in brackets is positive and bounded above by $1-\sqrt{x}$. Since $\sqrt{x}-f_0(x) := \sqrt{x}$, we deduce, by induction on n, the inequality:

$$0 \leq \sqrt{x} - f_n(x) \leq \sqrt{x}\big(1-\sqrt{x}\big)^n.$$

Fix $n \in \mathbb{N}$. An immediate calculation shows that on I, the function $t \mapsto t(1-t)^n$ attains its absolute maximum at $t := \frac{1}{n+1}$, and this absolute maximum equals:

$$M_n := \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^n.$$

The above inequality shows that $\|f-f_n\|_\infty \leq M_n$. When n tends to $+\infty$, M_n tends to 0; in fact, $M_n \sim \frac{1}{en}$. Consequently, $\|f-f_n\|_\infty$ tends to 0, which yields the conclusion.

Theorem 2.2.1 (Uniform Cauchy Criterion) [12, 19]
Suppose that E is complete. For a sequence of mappings $(f_n : X \to E)$ to converge uniformly, it is necessary and sufficient that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the inequalities $p \geq N$ and $q \geq N$ imply:

$$\forall x \in X, \quad \|f_p(x) - f_q(x)\| \leq \varepsilon. \tag{6}$$

Proof
Suppose that (f_n) converges uniformly to $f : X \to E$, and let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\|f_n(x) - f(x)\| \leq \varepsilon$ for all $n \geq N$ and all $x \in X$. Let p, q be two integers such that $p \geq N$ and $q \geq N$. For every $x \in X$, we have $\|f_p(x) - f(x)\| \leq \varepsilon$ and $\|f_q(x) - f(x)\| \leq \varepsilon$, hence $\|f_p(x) - f_q(x)\| \leq 2\varepsilon$ by the triangle inequality. Thus the sequence (f_n) satisfies the uniform Cauchy criterion (i.e., the condition stated in the theorem).

Conversely, suppose that (f_n) satisfies the uniform Cauchy criterion. Let x be a fixed point of X. The sequence $\big(f_n(x)\big)$ in E is Cauchy. Since E is complete, this sequence converges; denote its limit by $f(x)$. We thus define a mapping $f : X \to E$, which is the pointwise limit of the sequence (f_n). It remains to show that the convergence of (f_n) to f is uniform.

Let $\varepsilon > 0$. By hypothesis, there exists $N \in \mathbb{N}$ such that condition (6) holds whenever $p \geq N$ and $q \geq N$. Fix an integer $p \geq N$ and a point $x \in X$. In inequality (6), let $q \to +\infty$, so that $f_q(x) \to f(x)$. By continuity of the norm, we obtain $\|f_p(x) - f(x)\| \leq \varepsilon$. This inequality holds for every $p \geq N$ and every $x \in X$; therefore, the uniform convergence of (f_n) to f is established.

With the norm $\|\cdot\|_\infty$, the uniform Cauchy criterion translates as follows: for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that the inequalities $p \geq N$ and $q \geq N$ imply $\|f_p - f_q\|_\infty \leq \varepsilon$. If the f_n are bounded, this means that (f_n) is a Cauchy sequence in the normed vector space $\big(B(X,E), \|\cdot\|_\infty\big)$.

Corollary 2.2.1 (Completeness of $B(X, E)$) [12, 19]

If the normed vector space E is complete, then the normed vector space $\big(B(X, E), \|\cdot\|_\infty\big)$ is also complete.

Proof

Let (f_n) be a Cauchy sequence in $\big(B(X, E), \|\cdot\|_\infty\big)$. Since each f_n is bounded, the remark above shows that (f_n) satisfies the uniform Cauchy criterion. Because E is complete, the Uniform Cauchy Criterion (Theorem 3) shows that the sequence (f_n) converges uniformly to a mapping $f : X \to E$, which is also bounded. What precedes then shows that (f_n) converges to f in the normed vector space $\big(B(X, E), \|\cdot\|_\infty\big)$.

2.2.2 Uniform Convergence and Continuity

In this section, (X, d) is a non-empty metric space and $(E, \|\cdot\|)$ is a normed vector space. The interest of uniform convergence is that it preserves continuity, as we shall see.

Theorem 2.3.1 (Uniform Convergence Preserves Continuity) [12, 19]

Let $(f_n : X \to E)$ be a sequence of mappings converging uniformly to $f : X \to E$. If each f_n is continuous at a point $a \in X$, then f is continuous at a.

Proof

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty \leq \varepsilon$ whenever $n \geq N$. In particular, $\|f_N - f\|_\infty \leq \varepsilon$, i.e., $\|f_N(x) - f(x)\| \leq \varepsilon$ for every $x \in X$. Since f_N is continuous at a, there exists $\alpha > 0$ such that

$$d(x, a) \leq \alpha \implies \|f_N(x) - f_N(a)\| \leq \varepsilon.$$

Let $x \in X$ such that $d(x, a) \leq \alpha$. By the triangle inequality, we have:

$$\begin{aligned}\|f(x) - f(a)\| &\leq \|f(x) - f_N(x)\| + \|f_N(x) - f_N(a)\| + \|f_N(a) - f(a)\| \\ &\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

Hence f is continuous at a. It should be observed that it is the fact that N does not depend on x which allowed us, in the right-hand side, to bound the two extreme terms by ε.

Theorem 2.3.2 (Uniform Convergence on Compacts Preserves Continuity) [12, 19]

Let $(f_n : X \to E)$ be a sequence of mappings converging to $f : X \to E$ uniformly on every compact subset of X. If each f_n is continuous at a point $a \in X$, then f is continuous at a.

This theorem follows immediately from the previous theorem and the following lemma:

Lemma 2.3.2 (Continuity on Compacts) [12, 19]

Let $f : X \to Y$ be a mapping between two metric spaces and let a be a point in X. If the restriction of f to every compact subset of X containing a is continuous at a, then f is continuous at a.

Proof

Let (a_n) be a sequence in X converging to a. It suffices to show that $\big(f(a_n)\big)$ converges to $f(a)$. Set

$$Z := \{a_n \mid n \in \mathbb{N}\} \cup \{a\}.$$

If we show that Z is compact, then we are done, because then, by hypothesis, the restriction g of f to Z will be continuous, and therefore $\big(g(a_n)\big) = \big(f(a_n)\big)$ will converge to $g(a) = f(a)$.

So let $(u_k)_{k\geq 0}$ be a sequence in Z. We need to show that this sequence has an adherence value ℓ in Z. If a is an adherence value of (u_k), then $\ell := a$ works. If a is not an adherence value of (u_k), there exists a real number $r > 0$ such that the set $\{k \in \mathbb{N} \mid u_k \in B(a, r)\}$ is finite. Hence there exists $A \in \mathbb{N}$ such that for all $k \geq A$, $u_k \notin B(a, r)$. On the other hand, since the sequence (a_n) converges to a, there exists $N \in \mathbb{N}$ such that for all $n > N$, $a_n \in B(a, r)$. For every integer $k \geq A$, u_k therefore belongs to the finite set $S := \{a_0, \dots, a_N\}$. Since S is compact, the subsequence $(u_k)_{k\geq A}$ has an adherence value $\ell \in S \subset Z$, and then $\ell \in Z$ works.

Theorem 2.3.2 (First Dini Theorem) [12, 19]

Let X be a compact (non-empty) metric space and let $(f_n : X \to \mathbb{R})$ be a sequence of continuous functions converging pointwise to a continuous function f. Suppose that for every $x \in X$, the sequence $\big(f_n(x)\big)$ is decreasing. Then (f_n) converges uniformly to f.

Proof

Since f is continuous, we can reduce to the case where $f = 0$ by replacing $(f_n)_n$ with $(f_n - f)_n$. Assume therefore that $f = 0$. Let $\varepsilon > 0$. For every $n \in \mathbb{N}$,

$$F_n := f_n^{-1}\big([\varepsilon, +\infty)\big)$$

is closed, hence compact, in X. The sequence $(F_n)_n$ is decreasing with respect to inclusion because $n \geq p$ implies $f_n \leq f_p$. Moreover, the intersection of the F_n is empty because if x were a point in this intersection, then $f_n(x) \geq \varepsilon$ for all n, hence $0 = \lim_{n\to+\infty} f_n(x) \geq \varepsilon$, which is absurd. By Proposition 49 on page 435, there exists $N \in \mathbb{N}$ such that $F_N = \emptyset$. Consider then an integer $n \geq N$ and a point $x \in X$. First, $x \notin F_n$, i.e., $f_n(x) < \varepsilon$. Next, since the sequence $\big(f_k(x)\big)_k$ converges to 0 while decreasing, all its terms are positive, so $0 \leq f_n(x) < \varepsilon$. Finally, $|f_n(x)| < \varepsilon$ whenever $n \geq N$ and $x \in X$, which proves the uniform convergence of $(f_n)_n$ to 0.

Note that the continuity hypothesis on f is necessary, as shown by Example 1 on page 476, with $X := [0, 1]$.

Theorem 2.3.2 (Second Dini Theorem) [12, 19]

Let (f_n) be a sequence of increasing functions from $I := [a, b]$ to $\mathbb{R}$ $(a < b)$ converging pointwise to a function f. If f is continuous, then the convergence of (f_n) to f is uniform.

Proof

The function f is increasing: if $a \leq x \leq y \leq b$, then $f_n(x) \leq f_n(y)$ for all n, hence $f(x) \leq f(y)$ by letting $n \to +\infty$. Since f is continuous and increasing, $f(I)$ is the interval $J := \big[f(a), f(b)\big]$. Now let $\varepsilon > 0$. Choose $p \in \mathbb{N}^*$ such that $\big(f(b) - f(a)\big)/p \leq \varepsilon$. Partition J into p equal intervals using the subdivision points

$$f(a) = y_0 < y_1 < \cdots < y_p := f(b).$$

For every $k \in \{0, \ldots, p\}$, there exists $x_k \in I$ such that $y_k := f(x_k)$, and we have $a = x_0 < x_1 < \cdots < x_p = b$ because f is increasing.

For each $k \in \{0, \ldots, p\}$, the sequence $\big(f_n(x_k)\big)_{n \geq 0}$ converges to y_k. Hence there exists N such that $|f_n(x_k) - y_k| \leq \varepsilon$ for all $n \geq N$ and all $k \in \{0, \ldots, p\}$.

Now let $x \in I$. There exists $k \in \{0, \ldots, p-1\}$ such that $x_k \leq x \leq x_{k+1}$, so that $y_k \leq f(x) \leq y_{k+1}$. Consider an integer $n \geq N$. If $h \in \{k, k+1\}$, we have $|f(x) - y_h| \leq y_{k+1} - y_k \leq \varepsilon$ and $|f_n(x_h) - y_h| \leq \varepsilon$, hence $|f_n(x_h) - f(x)| \leq 2\varepsilon$. Thus $f_n(x_k)$ and $f_n(x_{k+1})$ belong to $\big[f(x) - 2\varepsilon, f(x) + 2\varepsilon\big]$. Since $f_n(x_k) \leq f_n(x) \leq f_n(x_{k+1})$, it follows that $f_n(x) \in \big[f(x) - 2\varepsilon, f(x) + 2\varepsilon\big]$, i.e., $|f_n(x) - f(x)| \leq 2\varepsilon$. This inequality holds for every $x \in I$ and every integer $n \geq N$, which proves that (f_n) converges uniformly to f.

Example 2.2.1 *(Application of Dini's Second Theorem) [12, 19]*

Let $I := [0, 1]$. Consider again the sequence of functions (f_n) defined in Example 3 on page 23. We have seen that $f_n(I) \subset I$ for all n. Each f_n is differentiable, and

$$f'_{n+1} = \frac{1}{2} + f'_n(1 - f_n).$$

Since $f_0 := 0$, f_n is increasing ($f'_n \geq 0$) for all $n \in \mathbb{N}$. By the Second Dini Theorem, the sequence (f_n) converges uniformly to $x \mapsto \sqrt{x}$.

2.3 Conclusion

In this chapter, we have established the fundamental distinction between pointwise and uniform convergence. We demonstrated that while pointwise convergence is the most intuitive mode of convergence, it is often insufficient to preserve the essential structural property of continuity. Conversely, uniform convergence provides the necessary strength to ensure that the limit function inherits the continuity of the sequence. However, in many mathematical scenarios, particularly when dealing with limits of functions that are not uniformly convergent, we encounter functions that are "almost" continuous. This realization leads us to explore a broader class of functions—those that are semi-continuous—which will be the core focus of our next chapter.

Chapter 3

Semi-Continuity of Functions

Continuity is a powerful property, but it is often too restrictive for solving advanced problems in mathematical analysis, optimization, and variational calculus. This chapter introduces the theory of Semi-Continuity, a directional generalization of continuity that allows for certain types of ”jumps” in function values. We begin by providing formal definitions and fundamental properties of Lower and Upper Semi-Continuous functions. We will then examine their algebraic stability and their behavior under different types of convergence. A key objective of this chapter is to explore Semi-Continuous Envelopes and the Interplay between convergence and semi-continuity, providing a theoretical bridge that connects the sequence limits studied in Chapter II with the stability requirements of modern functional analysis.

3.1 Definitions and Properties

Let us fix the framework. Let (X, d) be a metric space and let $f : X \longrightarrow \mathbb{R}$ be a real-valued function. Let $x_0 \in X$ be a point in X.

Definition 3.1.1 *(Upper Semi-Continuity) [19] The function f is said to be* ***upper semi-continuous (usc)*** *at the point x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$\forall x \in X, \quad d(x, x_0) < \delta \quad \Longrightarrow \quad f(x) < f(x_0) + \varepsilon.$$

In other words, the values of f near x_0 do not exceed $f(x_0)$ by more than ε.

Definition 3.1.2 *(Lower Semi-Continuity) [18] The function f is said to be* ***lower semi-continuous (lsc)*** *at the point x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$\forall x \in X, \quad d(x, x_0) < \delta \quad \Longrightarrow \quad f(x_0) - \varepsilon < f(x).$$

This means that $f(x)$ does not fall below $f(x_0)$ by more than ε near x_0.

Remark 3.1.1 *[10, 13] We can rewrite the definition of upper semi-continuity in an equivalent way:*

$$\limsup_{x \to x_0} f(x) \le f(x_0).$$

Similarly, lower semi-continuity is equivalent to:

$$f(x_0) \le \liminf_{x \to x_0} f(x).$$

These characterizations using the limit superior and limit inferior are often more convenient for proofs.

Proposition 3.1.1 *[19] A function f is continuous at the point x_0 if and only if it is both upper semi-continuous and lower semi-continuous at x_0.*

Proof If f is continuous at x_0, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $d(x, x_0) < \delta$. Then we have $f(x) < f(x_0) + \varepsilon$ (usc) and $f(x_0) - \varepsilon < f(x)$ (lsc). Conversely, if both semi-continuities hold, for a given $\varepsilon > 0$, choose δ_1 for usc and δ_2 for lsc; then with $\delta = \min(\delta_1, \delta_2)$ we obtain $|f(x) - f(x_0)| < \varepsilon$.

3.2 Fundamental Examples

Let us illustrate these definitions with typical examples.

Example 3.2.1 *(Characteristic function of an open set) [3, 18] Let $A \subset X$ be a non-empty subset. Consider the indicator function:*

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

We show that $\mathbf{1}_A$ is lower semi-continuous at every point of X if and only if A is open. Indeed, suppose A is open. Let $x_0 \in X$. Two cases:
Case 1: *$x_0 \in A$. Then $\mathbf{1}_A(x_0) = 1$. By openness of A, there exists $\delta > 0$ such that the ball $B(x_0, \delta) \subset A$. Then for every x in this ball, $\mathbf{1}_A(x) = 1$, so $\mathbf{1}_A(x_0) - \varepsilon = 1 - \varepsilon < 1 = \mathbf{1}_A(x)$ for every $\varepsilon > 0$. The lsc condition holds.*
Case 2: *$x_0 \notin A$. Then $\mathbf{1}_A(x_0) = 0$. For every $\varepsilon > 0$, we need to find δ such that $d(x, x_0) < \delta$ implies $0 - \varepsilon < \mathbf{1}_A(x)$. Since $-\varepsilon < 0$ and $\mathbf{1}_A(x) \ge 0$, the inequality is always true. So lsc is satisfied.*

Conversely, if $\mathbf{1}_A$ is lsc at every point, take $x_0 \in A$. Then $\mathbf{1}_A(x_0) = 1$. By lsc, for $\varepsilon = 1/2$, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $1 - 1/2 < \mathbf{1}_A(x)$, i.e., $1/2 < \mathbf{1}_A(x)$. Since $\mathbf{1}_A$ takes only the values 0 or 1, we must have $\mathbf{1}_A(x) = 1$, so $x \in A$. Thus $B(x_0, \delta) \subset A$, proving that A is open.

Example 3.2.2 *(Characteristic function of a closed set) [16] Symmetrically, the indicator function $\mathbf{1}_A$ is upper semi-continuous at every point if and only if A is closed. The proof is analogous by considering the sets $\{x : \mathbf{1}_A(x) \ge a\}$.*

Example 3.2.3 *(An usc function that is not continuous) [3, 19] Consider $f : \mathbb{R} \to \mathbb{R}$ defined by:*

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

At $x_0 = 0$, we have $f(0) = 1$. For every $\varepsilon > 0$, take $\delta = 1$. If $|x - 0| < \delta$, then $f(x) \leq 1 = f(0) < f(0) + \varepsilon$. Hence f is usc at 0. However, it is not lsc at 0 because $f(0) - \varepsilon = 1 - \varepsilon$ is not always less than $f(x)$ for x near 0 (take $x \neq 0$ gives $f(x) = 0$, and $1 - \varepsilon > 0$ for $\varepsilon < 1$). Thus f is not continuous at 0.

3.3 Characterizations of Semi-Continuity

In this section, we establish several equivalent characterizations of upper and lower semi-continuity. These characterizations are useful in practice and reveal the deep connection between semi-continuity and the topology of the underlying metric space.

Theorem 3.4.1 (Characterization by open sets) [18, 19] Let $f : X \to \mathbb{R}$ be a function. Then:

(i) f is upper semi-continuous at every point of X if and only if for every $a \in \mathbb{R}$, the set $f^{-1}(-\infty, a) = \{x \in X : f(x) < a\}$ is open in X.

(ii) f is lower semi-continuous at every point of X if and only if for every $a \in \mathbb{R}$, the set $f^{-1}(a, \infty) = \{x \in X : f(x) > a\}$ is open in X.

Proof. We prove part (i); part (ii) is analogous.

($\Rightarrow$) Suppose f is usc at every point. Let $a \in \mathbb{R}$ and take $x_0 \in f^{-1}(-\infty, a)$. Then $f(x_0) < a$. Set $\varepsilon = a - f(x_0) > 0$. By usc at x_0, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $f(x) < f(x_0) + \varepsilon = a$. Hence $x \in f^{-1}(-\infty, a)$. Thus $f^{-1}(-\infty, a)$ is open.

($\Leftarrow$) Conversely, assume $f^{-1}(-\infty, a)$ is open for every $a \in \mathbb{R}$. Fix $x_0 \in X$ and let $\varepsilon > 0$. Take $a = f(x_0) + \varepsilon$. Then $x_0 \in f^{-1}(-\infty, a)$. Since this set is open, there exists $\delta > 0$ such that $B(x_0, \delta) \subset f^{-1}(-\infty, a)$. For any x with $d(x, x_0) < \delta$, we have $f(x) < a = f(x_0) + \varepsilon$. Hence f is usc at x_0. $\square$

Remark 3.4.1 [16] An equivalent formulation using closed sets is often more convenient: f is usc if and only if for every $a \in \mathbb{R}$, the set $\{x : f(x) \geq a\}$ is closed. Similarly, f is lsc if and only if for every $a \in \mathbb{R}$, the set $\{x : f(x) \leq a\}$ is closed.

Theorem 3.4.2 (Characterization by sequential limits) [10, 19] Let $f : X \to \mathbb{R}$ be a function and let $x_0 \in X$. Then:

(i) f is upper semi-continuous at x_0 if and only if for every sequence (x_n) in X with $x_n \to x_0$, we have $\limsup_{n\to\infty} f(x_n) \leq f(x_0)$.

(ii) f is lower semi-continuous at x_0 if and only if for every sequence (x_n) in X with $x_n \to x_0$, we have $f(x_0) \le \liminf_{n\to\infty} f(x_n)$.

Proof. We prove (i); (ii) is similar by considering $-f$.

($\Rightarrow$) Assume f is usc at x_0 and let $x_n \to x_0$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $f(x) < f(x_0) + \varepsilon$. Since $x_n \to x_0$, there exists N such that for all $n \ge N$, $d(x_n, x_0) < \delta$. Then $f(x_n) < f(x_0) + \varepsilon$ for all $n \ge N$. Hence $\limsup_{n\to\infty} f(x_n) \le f(x_0) + \varepsilon$. Since ε is arbitrary, we obtain $\limsup f(x_n) \le f(x_0)$.

($\Leftarrow$) Conversely, suppose that for every sequence $x_n \to x_0$, we have $\limsup f(x_n) \le f(x_0)$. Assume, for contradiction, that f is not usc at x_0. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, we can find x with $d(x, x_0) < \delta$ and $f(x) \ge f(x_0)+\varepsilon_0$. Taking $\delta = 1/n$, we construct a sequence x_n with $d(x_n, x_0) < 1/n$ and $f(x_n) \ge f(x_0)+\varepsilon_0$. Then $x_n \to x_0$ but $\limsup f(x_n) \ge f(x_0) + \varepsilon_0 > f(x_0)$, a contradiction. □

Theorem 3.4.3 (Characterization by epigraph and hypograph) [13, 18] Let $f : X \to \mathbb{R}$ be a function. Define:

- The **epigraph** of f is $\text{epi}(f) = \{(x, t) \in X \times \mathbb{R} : f(x) \le t\}$.
- The **hypograph** of f is $\text{hyp}(f) = \{(x, t) \in X \times \mathbb{R} : t \le f(x)\}$.

Then f is lower semi-continuous at every point if and only if $\text{epi}(f)$ is closed in $X \times \mathbb{R}$. Dually, f is upper semi-continuous at every point if and only if $\text{hyp}(f)$ is closed in $X \times \mathbb{R}$.

Proof. We prove the statement for lsc and the epigraph; the usc case is dual.

($\Rightarrow$) Suppose f is lsc. Let $(x_n, t_n) \in \text{epi}(f)$ with $(x_n, t_n) \to (x, t)$ in $X \times \mathbb{R}$. Then $f(x_n) \le t_n$. Since f is lsc, we have $f(x) \le \liminf f(x_n) \le \liminf t_n = t$. Hence $(x, t) \in \text{epi}(f)$. So $\text{epi}(f)$ is closed.

($\Leftarrow$) Conversely, assume $\text{epi}(f)$ is closed. We show that f is lsc. Suppose not; then there exists x_0 and $\varepsilon_0 > 0$ such that for every $\delta > 0$, there is x with $d(x, x_0) < \delta$ and $f(x) \le f(x_0) - \varepsilon_0$. Take $\delta = 1/n$ to get $x_n \to x_0$ with $f(x_n) \le f(x_0) - \varepsilon_0$. Then $(x_n, f(x_0) - \varepsilon_0) \in \text{epi}(f)$ and converges to $(x_0, f(x_0) - \varepsilon_0)$. Since $\text{epi}(f)$ is closed, we have $(x_0, f(x_0) - \varepsilon_0) \in \text{epi}(f)$, i.e., $f(x_0) \le f(x_0) - \varepsilon_0$, a contradiction. □

Theorem 3.10.1 (Characterization by nets) [13] Let $f : X \to \mathbb{R}$ be a function. Then f is upper semi-continuous at x_0 if and only if for every net (x_α) in X converging to x_0, we have $\limsup_\alpha f(x_\alpha) \le f(x_0)$. This generalizes the sequential characterization to non-metrizable spaces.

Remark 3.10.1 [16] In metric spaces, sequences suffice. In general topological spaces, nets (or filters) are required.

Theorem 3.10.2 (Characterization by neighborhood bases) [19] f is upper semi-continuous at x_0 if and only if for every $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $\sup_{x\in U} f(x) \le f(x_0) + \varepsilon$.

Proof. This is just a reformulation of Definition 3.2.1. □

Theorem 3.10.3 (Semi-continuity of distance functions) [12, 19] Let A be a non-empty subset of a metric space X. Define the distance function $d_A(x) = \inf_{a\in A} d(x,a)$. Then d_A is Lipschitz continuous (hence both usc and lsc). More generally, for any x_0, the function $x \mapsto d(x,x_0)$ is continuous.
Proof. The triangle inequality gives $|d_A(x) - d_A(y)| \leq d(x,y)$, so d_A is Lipschitz with constant 1. □
Theorem 3.10.4 (Baire's theorem on semicontinuous functions) [2, 18] Let $f : X \to \mathbb{R}$ be a function. Then f is lower semi-continuous if and only if it is the pointwise supremum of a family of continuous functions. Dually, f is upper semi-continuous if and only if it is the pointwise infimum of a family of continuous functions.
Proof Sketch. If f is lsc, define for each $x_0 \in X$ and each $n \in \mathbb{N}$ a continuous function $\varphi_{x_0,n}(x) = f(x_0) - nd(x,x_0)$. Then $\sup_{x_0,n} \varphi_{x_0,n}(x) = f(x)$. Conversely, the supremum of continuous functions is lsc. □

3.4 Algebraic Properties of Semi-Continuous Functions

In this section, we study how semi-continuity behaves under algebraic operations. These properties are essential for constructing new semi-continuous functions from existing ones.
Theorem 3.5.1 (Sum of semi-continuous functions) [3, 18]Let $f, g : X \to \mathbb{R}$ be two functions.

(i) If f and g are upper semi-continuous at x_0, then $f+g$ is upper semi-continuous at x_0.

(ii) If f and g are lower semi-continuous at x_0, then $f+g$ is lower semi-continuous at x_0.

Proof. We prove (i); the proof of (ii) is similar. Let $\varepsilon > 0$. Since f is usc at x_0, there exists $\delta_1 > 0$ such that $d(x,x_0) < \delta_1$ implies $f(x) < f(x_0) + \varepsilon/2$. Since g is usc at x_0, there exists $\delta_2 > 0$ such that $d(x,x_0) < \delta_2$ implies $g(x) < g(x_0) + \varepsilon/2$. Take $\delta = \min(\delta_1, \delta_2)$. Then for $d(x,x_0) < \delta$, we have:

$$f(x) + g(x) < f(x_0) + g(x_0) + \varepsilon.$$

Hence $f+g$ is usc at x_0. □
Theorem 3.5.2 (Multiplication by a scalar) [18, 19] Let $f : X \to \mathbb{R}$ be a function and let $\lambda \in \mathbb{R}$.

(i) If $\lambda \geq 0$ and f is upper semi-continuous at x_0, then λf is upper semi-continuous at x_0.

(ii) If $\lambda \geq 0$ and f is lower semi-continuous at x_0, then λf is lower semi-continuous at x_0.

(iii) If $\lambda < 0$, then upper semi-continuity becomes lower semi-continuity and vice versa.

Proof. For $\lambda = 0$, the statement is trivial. For $\lambda > 0$, the proof follows directly from the definition. For $\lambda < 0$, note that $\lambda f = -|\lambda| f$, and the sign reversal flips the inequalities. □

Theorem 3.5.3 (Maximum and minimum) [3, 18] Let $f, g : X \to \mathbb{R}$ be two functions.

(i) If f and g are upper semi-continuous at x_0, then $\max\{f, g\}$ is upper semi-continuous at x_0.

(ii) If f and g are lower semi-continuous at x_0, then $\min\{f, g\}$ is lower semi-continuous at x_0.

Proof. We prove (i). Let $\varepsilon > 0$. Since f is usc at x_0, there exists $\delta_1 > 0$ such that $d(x, x_0) < \delta_1$ implies $f(x) < f(x_0) + \varepsilon$. Since g is usc at x_0, there exists $\delta_2 > 0$ such that $d(x, x_0) < \delta_2$ implies $g(x) < g(x_0) + \varepsilon$. Take $\delta = \min(\delta_1, \delta_2)$. Then for $d(x, x_0) < \delta$:

$$\max\{f(x), g(x)\} < \max\{f(x_0), g(x_0)\} + \varepsilon.$$

Thus $\max\{f, g\}$ is usc at x_0. □

Remark 3.5.1 [18] Note that $\min\{f, g\}$ is not necessarily usc even if f and g are usc. Similarly, $\max\{f, g\}$ is not necessarily lsc even if f and g are lsc. However, if both f and g are usc, then $\min\{f, g\}$ is usc only under additional conditions (e.g., if the functions are non-negative or if we consider the infimum over a family).

3.5 Stability Under Limits

One of the most important aspects of semi-continuity is its behavior under limiting processes. The following results show that semi-continuity is preserved under certain types of convergence.

Theorem 3.6.1 (Infimum of a family of usc functions) [18] Let $\{f_i\}_{i \in I}$ be a family of functions from X to $\mathbb{R}$. Define $f(x) = \inf_{i \in I} f_i(x)$. If each f_i is upper semi-continuous at x_0, then f is also upper semi-continuous at x_0.

Proof. [18] Let $\varepsilon > 0$. By definition of the infimum, there exists $i_0 \in I$ such that $f_{i_0}(x_0) < f(x_0) + \varepsilon$. Since f_{i_0} is usc at x_0, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $f_{i_0}(x) < f_{i_0}(x_0) + \varepsilon$. Then:

$$f(x) \leq f_{i_0}(x) < f_{i_0}(x_0) + \varepsilon < f(x_0) + 2\varepsilon.$$

Since ε is arbitrary, we obtain $\limsup_{x \to x_0} f(x) \leq f(x_0)$. Hence f is usc at x_0. □

Theorem 3.6.2 (Supremum of a family of lsc functions) [18] Let $\{f_i\}_{i \in I}$ be a family of functions from X to $\mathbb{R}$. Define $f(x) = \sup_{i \in I} f_i(x)$. If each f_i is lower semi-continuous at x_0, then f is also lower semi-continuous at x_0.

Proof. This follows from Theorem 3.6.1 by considering $-f_i$. □

Theorem 3.6.3 (Limit of a monotone sequence) [10, 18] Let (f_n) be a sequence of functions from X to $\mathbb{R}$.

(i) If each f_n is upper semi-continuous and $f_{n+1}(x) \leq f_n(x)$ for all x (decreasing sequence), then $f(x) = \lim_{n \to \infty} f_n(x)$ is upper semi-continuous.

(ii) If each f_n is lower semi-continuous and $f_{n+1}(x) \geq f_n(x)$ for all x (increasing sequence), then $f(x) = \lim_{n \to \infty} f_n(x)$ is lower semi-continuous.

Proof. For (i), note that $f(x) = \inf_n f_n(x)$ because the sequence is decreasing. By Theorem 3.6.1, f is usc. For (ii), $f(x) = \sup_n f_n(x)$ and by Theorem 3.6.2, f is lsc. □

3.6 Composition of Semi-Continuous Functions

We now study how semi-continuity behaves under composition with continuous functions.

Theorem 3.7.1 (Composition with a continuous function) [19] Let X and Y be metric spaces. Suppose $f : X \to Y$ is continuous at $x_0 \in X$, and $g : Y \to \mathbb{R}$ is upper semi-continuous at $f(x_0)$. Then the composition $g \circ f : X \to \mathbb{R}$ is upper semi-continuous at x_0.

Proof. Let $\varepsilon > 0$. Since g is usc at $f(x_0)$, there exists $\eta > 0$ such that for all $y \in Y$ with $d_Y(y, f(x_0)) < \eta$, we have $g(y) < g(f(x_0)) + \varepsilon$. Since f is continuous at x_0, there exists $\delta > 0$ such that for all $x \in X$ with $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \eta$. Then for such x:

$$g(f(x)) < g(f(x_0)) + \varepsilon = (g \circ f)(x_0) + \varepsilon.$$

Hence $g \circ f$ is usc at x_0. □

Corollary 3.7.1 [18] If $f : X \to \mathbb{R}$ is upper semi-continuous at x_0 and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and non-decreasing, then $\varphi \circ f$ is upper semi-continuous at x_0.

Proof. Since φ is continuous, the result follows directly from Theorem 3.7.1. The monotonicity is not needed for semi-continuity but is useful for preserving properties like convexity. □

Remark 3.7.1 [19] If g is lower semi-continuous and f is continuous, then $g \circ f$ is lower semi-continuous. The proof is identical with the inequalities reversed.

Warning 3.7.1 [19] If f is only semi-continuous (not continuous) and g is continuous, the composition $g \circ f$ may fail to be semi-continuous. For example, take

$f(x) = 0$ for $x \le 0$ and $f(x) = 1$ for $x > 0$ (which is usc but not continuous), and $g(y) = -y$ (continuous). Then $g \circ f$ is not usc.

3.7 Application in Optimization and Variational Analysis

One of the most important applications of semi-continuity is in optimization theory. The following theorems show that semi-continuous functions attain their minima or maxima on compact sets.

Theorem 3.9.1 (Minimum of lsc function on compact set) [18, 7] Let X be a compact metric space and let $f : X \to \mathbb{R}$ be lower semi-continuous on X. Then f attains its minimum on X, i.e., there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$.

Proof. Let $m = \inf_{x\in X} f(x)$. By definition of the infimum, there exists a sequence (x_n) in X such that $f(x_n) \to m$. Since X is compact, (x_n) has a convergent subsequence (x_{n_k}) with $x_{n_k} \to x_0 \in X$. By lower semi-continuity of f at x_0, we have:

$$f(x_0) \le \liminf_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = m.$$

But $m \le f(x_0)$ because m is the infimum. Hence $f(x_0) = m$. □

Theorem 3.9.2 (Maximum of usc function on compact set) [18] Let X be a compact metric space and let $f : X \to \mathbb{R}$ be upper semi-continuous on X. Then f attains its maximum on X, i.e., there exists $x_0 \in X$ such that $f(x_0) = \sup_{x\in X} f(x)$.

Proof. Apply Theorem 3.9.1 to $-f$, which is lower semi-continuous. □

Example 3.9.1 (Necessity of compactness) [19] Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-x}$. This function is continuous (hence lsc) but $\mathbb{R}$ is not compact. The infimum is 0 but $f(x) > 0$ for all x, so the minimum is not attained.

Example 3.9.2 (Necessity of semi-continuity) [3, 18] Consider $f : [0, 1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

This function is not lsc at $x = 1$ (since $f(1) = 0$ but values near 1 are close to 1). The infimum is 0 but is it attained? $f(1) = 0$, so yes, it is attained. However, consider a slight modification:

$$g(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ -1 & \text{if } x = 1. \end{cases}$$

Then $\inf g(x) = -1$ attained at $x = 1$, so that's fine. But if we consider $h(x) = x$ on $(0, 1)$ and extend arbitrarily, compactness fails. The key is: for lsc functions on compact sets, the minimum is always attained.

We now explicitly answer the research questions posed in the general introduction of this thesis. [3, 19]

Research Question 1: "What is the relationship between different types of convergence of functions and semi-continuity?" [18]
Answer: The relationship can be summarized as follows:

(i) **Pointwise convergence** alone does NOT preserve semi-continuity. The limit of a pointwise convergent sequence of continuous (hence semi-continuous) functions may fail to be semi-continuous (e.g., Dirichlet function). [19]

(ii) **Uniform convergence** DOES preserve semi-continuity. If $f_n \to f$ uniformly and each f_n is usc (resp. lsc), then f is usc (resp. lsc). [19]

(iii) **Monotone convergence** preserves semi-continuity in the direction of the monotonicity: [18]

- Decreasing sequence of usc functions → usc limit.
- Increasing sequence of lsc functions → lsc limit.

(iv) **Infimum of usc functions** is usc. [18]

(v) **Supremum of lsc functions** is lsc. [18]

Research Question 2: "Under what conditions does the limit function preserve continuity in a metric space?" [3, 19]
Answer: The limit function $f = \lim f_n$ preserves continuity under the following conditions:

(i) **Uniform convergence:** If $f_n \to f$ uniformly and each f_n is continuous, then f is continuous. [19]

(ii) **Monotone convergence with continuous limit:** If f_n is a monotone sequence of continuous functions converging pointwise to f, then f may not be continuous (Example 3.8.1). However, if in addition the convergence is uniform on compact sets or if f is known to be continuous, then the convergence is actually uniform on compact sets (Dini's theorem). [19]

(iii) **Equicontinuity:** If $\{f_n\}$ is equicontinuous and $f_n \to f$ pointwise, then $f_n \to f$ uniformly on compact sets and f is continuous (Arzelà-Ascoli theorem). [18]

3.8 Semi-Continuous Envelopes

Given an arbitrary function, we can associate with it two important semi-continuous functions: its lower semi-continuous envelope and its upper semi-continuous envelope.

Definition 3.11.1 (Lower semi-continuous envelope) [18] Let $f : X \to \mathbb{R}$ be any function. The **lower semi-continuous envelope** (or lsc regularization) of f is defined as:

$$\operatorname{lsc}(f)(x) = \sup\{g(x) : g \text{ is lower semi-continuous and } g \leq f\}.$$

Equivalently:

$$\operatorname{lsc}(f)(x) = \liminf_{y \to x} f(y).$$

Definition 3.11.2 (Upper semi-continuous envelope) [18] The **upper semi-continuous envelope** of f is defined as:

$$\operatorname{usc}(f)(x) = \inf\{h(x) : h \text{ is upper semi-continuous and } f \leq h\}.$$

Equivalently:

$$\operatorname{usc}(f)(x) = \limsup_{y \to x} f(y).$$

Theorem 3.11.1 (Properties of envelopes) [18, 10] Let $f : X \to \mathbb{R}$ be any function. Then:

(i) $\operatorname{lsc}(f)$ is the greatest lower semi-continuous function below f.

(ii) $\operatorname{usc}(f)$ is the least upper semi-continuous function above f.

(iii) $\operatorname{lsc}(f)(x) \leq f(x) \leq \operatorname{usc}(f)(x)$ for all x.

(iv) f is lower semi-continuous at x if and only if $\operatorname{lsc}(f)(x) = f(x)$.

(v) f is upper semi-continuous at x if and only if $\operatorname{usc}(f)(x) = f(x)$.

(vi) f is continuous at x if and only if $\operatorname{lsc}(f)(x) = f(x) = \operatorname{usc}(f)(x)$.

Proof. We prove (i) and (iv); the others are analogous. For (i), note that $\operatorname{lsc}(f)$ is lsc because it is the supremum of lsc functions. For any lsc $g \leq f$, we have $g \leq \operatorname{lsc}(f)$ by definition. Hence $\operatorname{lsc}(f)$ is the greatest. For (iv), if f is lsc at x, then $\liminf_{y \to x} f(y) \geq f(x)$, but also $\liminf_{y \to x} f(y) \leq f(x)$ trivially, so equality holds. Conversely, if $\operatorname{lsc}(f)(x) = f(x)$, then $\liminf_{y \to x} f(y) = f(x)$, which implies f is lsc at x. □

Example 3.11.1 [18] Consider $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\operatorname{lsc}(f)(0) = -1$ and $\operatorname{usc}(f)(0) = 1$. The function is not continuous at 0.

3.9 Applications to the Study of Convergence

We now return to the central questions of this thesis: the relationship between convergence of functions and semi-continuity.

Theorem 3.12.1 (Pointwise limit of continuous functions) [18] Let (f_n) be a sequence of continuous functions from X to $\mathbb{R}$ converging pointwise to f. Then:

(i) f is not necessarily continuous (as seen in Example 3.8.1).

(ii) f is always of Baire class 1 (i.e., the pointwise limit of continuous functions). [2]

(iii) f may fail to be semi-continuous. For example, define $f_n(x) = \max(0, 1 - n|x|)$ on $\mathbb{R}$. These are continuous, and their pointwise limit is $f(0) = 1$, $f(x) = 0$ for $x \neq 0$, which is usc but not lsc. To get a function that is neither usc nor lsc, consider a sequence whose limit is the Dirichlet function. [18]

Theorem 3.12.2 (Semi-continuity of the limit under monotone convergence) [18] Let (f_n) be a sequence of functions on X.

(i) If f_n is upper semi-continuous and $f_n \searrow f$ pointwise (decreasing to f), then f is upper semi-continuous.

(ii) If f_n is lower semi-continuous and $f_n \nearrow f$ pointwise (increasing to f), then f is lower semi-continuous.

Proof. This follows from Theorems 3.8.1 and 3.8.2. $\square$

Theorem 3.12.3 (Semi-continuity of the limit under uniform convergence) [19] If $f_n \to f$ uniformly on X and each f_n is semi-continuous (usc or lsc), then f has the same type of semi-continuity.

Proof. This is Theorem 3.8.3. $\square$

3.10 Interplay between Convergence and Semi-continuity

[18, 19]

We now connect the notions of convergence studied in Chapter II with semi-continuity.

Theorem 3.8.1 (Pointwise limit of usc functions) [18] Let (f_n) be a sequence of upper semi-continuous functions on X. Suppose $f_n(x) \to f(x)$ pointwise for all $x \in X$, and assume the convergence is decreasing, i.e., $f_{n+1}(x) \leq f_n(x)$ for all x. Then f is upper semi-continuous.

Proof. As noted earlier, $f(x) = \inf_n f_n(x)$. By Theorem 3.6.1, the infimum of a family of usc functions is usc. Hence f is usc. $\square$

Theorem 3.8.2 (Pointwise limit of lsc functions) [18] Let (f_n) be a sequence of lower semi-continuous functions on X. Suppose $f_n(x) \to f(x)$ pointwise for all $x \in X$, and assume the convergence is increasing, i.e., $f_{n+1}(x) \geq f_n(x)$ for all x. Then f is lower semi-continuous.
Proof. This follows from Theorem 3.6.2 since $f(x) = \sup_n f_n(x)$. $\square$
Theorem 3.8.3 (Uniform limit preserves semi-continuity) [19] Let (f_n) be a sequence of functions from X to $\mathbb{R}$ that converges uniformly to f on X. Then:

(i) If each f_n is upper semi-continuous, then f is upper semi-continuous.

(ii) If each f_n is lower semi-continuous, then f is lower semi-continuous.

Proof. We prove (i); the proof of (ii) is similar. Let $x_0 \in X$ and let $\varepsilon > 0$. By uniform convergence, there exists N such that for all $n \geq N$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon/3$. Fix such an n. Since f_n is usc at x_0, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $f_n(x) < f_n(x_0) + \varepsilon/3$. Then for $d(x, x_0) < \delta$:

$$f(x) \leq f_n(x) + \frac{\varepsilon}{3} < f_n(x_0) + \frac{2\varepsilon}{3} \leq f(x_0) + \varepsilon.$$

Hence f is usc at x_0. $\square$
Example 3.8.1 (Pointwise limit does not preserve semi-continuity) [3, 19] Consider $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Each f_n is continuous (hence both usc and lsc). The pointwise limit is:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

This function is usc at $x = 1$ (since $f(1) = 1$ and values near 1 are close to 0) but not lsc at $x = 1$. Thus pointwise convergence alone does not preserve semi-continuity; monotonicity or uniform convergence is required.
Theorem 3.14.1 (Weierstrass-type theorem) [7, 18] Let X be a compact metric space and let $f : X \to \mathbb{R}$ be lower semi-continuous. Then f attains its minimum on X. Similarly, if f is upper semi-continuous on a compact set, it attains its maximum.
Proof. Let $m = \inf_{x \in X} f(x)$. By definition of the infimum, there exists a sequence (x_n) in X such that $f(x_n) \to m$. Since X is compact, (x_n) has a convergent subsequence (x_{n_k}) with $x_{n_k} \to x_0 \in X$. By lower semi-continuity:

$$f(x_0) \leq \liminf_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(x_{n_k}) = m.$$

Since $m \leq f(x_0)$ by definition of the infimum, we have $f(x_0) = m$. $\square$
Example 3.14.1 (Application to distance functions) [19] Let A be a non-empty closed subset of a compact metric space X. The distance function $d_A(x) = \inf_{a \in A} d(x, a)$ is continuous (hence lsc). By Theorem 3.14.1, it attains its minimum.

This recovers the fact that a closed subset of a compact set is compact and that the distance from a point to a closed set is attained.

Theorem 3.14.2 (Lower semi-continuity of the infimum) [7, 9] Let $F : X \times Y \to \mathbb{R}$ be a function, where Y is compact. Define $f(x) = \inf_{y \in Y} F(x, y)$. If F is lower semi-continuous on $X \times Y$, then f is lower semi-continuous on X.

Proof. Let $x_n \to x_0$ in X. For each n, choose $y_n \in Y$ such that $F(x_n, y_n) \leq f(x_n) + 1/n$. Since Y is compact, (y_n) has a convergent subsequence $y_{n_k} \to y_0$. Then:

$$f(x_0) \leq F(x_0, y_0) \leq \liminf_{k\to\infty} F(x_{n_k}, y_{n_k}) \leq \liminf_{k\to\infty} \left(f(x_{n_k}) + \frac{1}{n_k} \right) = \liminf_{n\to\infty} f(x_n).$$

Thus f is lsc. $\square$

Chapter 4

Applications and Illustrative Examples

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

— David Hilbert

Introduction

The first three chapters of this thesis have developed, in a rigorous and progressive manner, the theoretical foundations necessary for a deep understanding of convergence and semi-continuity in metric spaces. Chapter I established the abstract framework of metric spaces, their topological properties, and the fundamental notions of continuity. Chapter II analyzed two principal modes of convergence of function sequences — pointwise and uniform — and established the key result that uniform convergence preserves continuity while pointwise convergence does not. Chapter III developed the theory of semi-continuity, providing characterizations via sequences, open sets, epigraphs, and envelopes, and established the essential connection between semi-continuity and optimization.

The purpose of this final chapter is to consolidate and extend these theoretical results through a systematic study of concrete applications and carefully constructed examples. Rather than merely illustrating definitions, each example in this chapter is designed to reveal a deeper mathematical principle, and each application demonstrates that the abstract theory developed in the preceding chapters is an indispensable tool in modern mathematical analysis.

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4.1 Convergence in Classical Function Spaces

We begin by examining, in detail and with explicit computations, the behavior of function sequences in the most natural and well-studied metric spaces. The goal is to make the abstract distinctions of Chapter II fully concrete, and to connect them directly to the semi-continuity theory of Chapter III.

4.1.1 Pointwise Convergence: Detailed Analysis

Example 4.1.1 *Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $n \in \mathbb{N}^*$. This is perhaps the most classical example in the theory of function sequences, and it illustrates with perfect clarity why pointwise convergence is insufficient to preserve continuity and semi-continuity.*

Step 1: Pointwise limit.
Fix $x \in [0,1]$.

- *If $0 \le x < 1$: since $|x| < 1$, the geometric sequence (x^n) satisfies $x^n \to 0$ as $n \to \infty$.*
- *If $x = 1$: $f_n(1) = 1^n = 1$ for all n, so $f_n(1) \to 1$.*

Therefore, the pointwise limit is:

$$f(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Step 2: Each f_n is continuous.
For every $n \in \mathbb{N}^*$, the function $f_n(x) = x^n$ is a polynomial, hence continuous on $[0,1]$. In particular, each f_n is both upper and lower semi-continuous at every point.

Step 3: The limit f is discontinuous at $x = 1$.
We have:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 0 = 0 \neq 1 = f(1).$$

Hence f is not continuous at $x = 1$.

Step 4: Semi-continuity of f at $x = 1$.
We analyze separately:

- *Upper semi-continuity: We need $\limsup_{x\to 1} f(x) \le f(1) = 1$. Since $f(x) = 0$ for all $x \in [0,1)$, we have $\limsup_{x\to 1^-} f(x) = 0 \le 1$.* ✓
  *Hence f is **upper semi-continuous** at $x = 1$.*
- *Lower semi-continuity: We need $\liminf_{x\to 1} f(x) \ge f(1) = 1$. But $\liminf_{x\to 1^-} f(x) = 0 < 1$.* $\times$
  *Hence f is **not lower semi-continuous** at $x = 1$.*

Step 5: Non-uniform convergence.
We compute:

$$\|f_n - f\|_\infty = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} x^n = 1 \quad \text{for all } n.$$

Since $\|f_n - f\|$

Example 4.1.2 *For* $n \in \mathbb{N}^*$*, define* $f_n : (-1,1) \to \mathbb{R}$ *by the partial sum of the geometric series:*

$$f_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Step 1: Pointwise limit.
For every fixed $x \in (-1,1)$*, since* $|x| < 1$*, we have* $x^{n+1} \to 0$ *as* $n \to \infty$*, so:*

$$f(x) = \lim_{n\to\infty} f_n(x) = \frac{1}{1-x}.$$

The pointwise limit is the function $f(x) = \frac{1}{1-x}$*, which is continuous on* $(-1,1)$*.*

Step 2: Uniform convergence on $[-a,a]$ ***for*** $0 < a < 1$***.***
For $x \in [-a,a]$ *with* $0 < a < 1$*:*

$$|f_n(x) - f(x)| = \left|\frac{x^{n+1}}{1-x}\right| \le \frac{a^{n+1}}{1-a} \to 0 \quad \text{as } n \to \infty.$$

Since the bound $\frac{a^{n+1}}{1-a}$ *is independent of* x*, the convergence is* ***uniform*** *on* $[-a,a]$*.*

Step 3: Non-uniform convergence on $(-1,1)$***.***
For any fixed n*, choose* $x_n = \left(\frac{1}{2}\right)^{1/(n+1)} \in (0,1)$*. Then* $x_n^{n+1} = \frac{1}{2}$*, and:*

$$|f_n(x_n) - f(x_n)| = \frac{x_n^{n+1}}{1-x_n} = \frac{1/2}{1-(1/2)^{1/(n+1)}} \to +\infty,$$

since $(1/2)^{1/(n+1)} \to 1^-$*. Hence:*

$$\sup_{x\in(-1,1)} |f_n(x) - f(x)| = +\infty,$$

confirming that the convergence is ***not uniform*** *on* $(-1,1)$*.*

Step 4: Semi-continuity of the limit.
The limit function $f(x) = \frac{1}{1-x}$ *is continuous on* $(-1,1)$*, hence both upper and lower semi-continuous at every point. The uniform convergence on compact subsets* $[-a,a]$*, combined with Theorem 2.3.1, is consistent with this observation.*

Example 4.1.3 *The* ***Bernstein polynomials*** *provide one of the most elegant proofs of the Weierstrass Approximation Theorem. For a continuous function* $f : [0,1] \to \mathbb{R}$*, the* n*-th Bernstein polynomial is defined by:*

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Theorem (Weierstrass, 1885): *$B_n(f) \to f$ uniformly on $[0,1]$ for every $f \in C([0,1],\mathbb{R})$.*

Key estimate:
Using the probabilistic interpretation (each $B_n(f)(x)$ is the expected value of f applied to a Binomial(n,x) random variable), one can show:

$$|B_n(f)(x) - f(x)| \leq \omega_f\left(\frac{1}{\sqrt{n}}\right),$$

where $\omega_f(\delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|$ is the modulus of continuity of f. Since f is uniformly continuous on $[0,1]$, we have $\omega_f(1/\sqrt{n}) \to 0$, giving uniform convergence.

Explicit example: $f(x) = \sqrt{x}$.
The first three Bernstein polynomials are:

$$\begin{aligned}
B_1(\sqrt{\cdot})(x) &= \sqrt{0}\cdot(1-x) + \sqrt{1}\cdot x = x,\\
B_2(\sqrt{\cdot})(x) &= 0\cdot(1-x)^2 + \frac{1}{\sqrt{2}}\cdot 2x(1-x) + 1\cdot x^2 = \sqrt{2}\,x(1-x) + x^2,\\
B_4(\sqrt{\cdot})(x) &= \sum_{k=0}^{4}\sqrt{\frac{k}{4}}\binom{4}{k}x^k(1-x)^{4-k}.
\end{aligned}$$

As $n \to \infty$, $B_n(\sqrt{\cdot}) \to \sqrt{\cdot}$ uniformly on $[0,1]$, and each $B_n(\sqrt{\cdot})$ is a polynomial (hence continuous, hence both usc and lsc), so the limit $\sqrt{\cdot}$ inherits continuity by Theorem 2.3.1.

Connection to semi-continuity:
Since $f(x) = \sqrt{x}$ is continuous on $[0,1]$, it is both usc and lsc everywhere. The Weierstrass theorem guarantees that every continuous function is the uniform limit of polynomials. Combined with Theorem 3.8.3, this means: every continuous function is simultaneously a uniform limit of usc and lsc functions. This is the strongest possible regularity statement.

4.1.2 The Role of Completeness in Uniform Convergence

Example 4.1.4 *We illustrate the Uniform Cauchy Criterion (Theorem 2.2.1) and the completeness of $\big(B([0,1],\mathbb{R}), \|\cdot\|_\infty\big)$.*

Define $f_n : [0,1] \to \mathbb{R}$ by:

$$f_n(x) = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1}}{n}x^n.$$

This is the n-th partial sum of the power series of $\ln(1+x)$.

Step 1: Uniform Cauchy criterion on $[0, 1]$.
For $p > q \geq N$*:*

$$|f_p(x) - f_q(x)| = \left| \sum_{k=q+1}^{p} \frac{(-1)^{k+1}}{k} x^k \right| \leq \sum_{k=q+1}^{p} \frac{x^k}{k} \leq \sum_{k=q+1}^{p} \frac{1}{k}.$$

Since the harmonic series tail $\sum_{k=N+1}^{\infty} \frac{1}{k}$ *can be made arbitrarily small... Actually, this bound is too crude. Using the alternating series estimate more carefully: since each* f_n *has alternating signs and decreasing coefficients for* $x \in [0, 1]$*:*

$$\|f_p - f_q\|_\infty \leq \frac{1}{q+1} \xrightarrow{q \to \infty} 0.$$

Hence (f_n) *is uniformly Cauchy on* $[0, 1]$.

Step 2: Identification of the limit.
By the completeness of $B([0, 1], \mathbb{R})$ *(Corollary 2.2.1),* (f_n) *converges uniformly to some bounded function* f*. By the theory of power series, this limit is:*

$$f(x) = \ln(1 + x), \quad x \in [0, 1].$$

Step 3: Continuity and semi-continuity.
Each f_n *is a polynomial, hence continuous. By Theorem 2.3.1, the uniform limit* $f(x) = \ln(1+x)$ *is continuous on* $[0, 1]$*, hence both usc and lsc. At* $x = 0$*:* $f(0) = 0$*; at* $x = 1$*:* $f(1) = \ln 2 \approx 0.693$*. The function is strictly increasing and continuous, confirming both semi-continuity properties.*

Proposition 4.1.1 *Let* (X, d) *be a metric space and let* (f_n) *be a sequence of lower semi-continuous functions from* X *to* $\mathbb{R}$*. If* (f_n) *converges uniformly to* $f : X \to \mathbb{R}$*, then* f *is lower semi-continuous.*

Let $x_0 \in X$ *and* $\varepsilon > 0$*. Since* (f_n) *converges uniformly to* f*, there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *and all* $x \in X$*:*

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

In particular, $f_N(x) > f(x) - \frac{\varepsilon}{3}$ *for all* $x \in X$*, and* $f_N(x_0) < f(x_0) + \frac{\varepsilon}{3}$*.*

Since f_N *is lower semi-continuous at* x_0*, there exists* $\delta > 0$ *such that for all* $x \in X$ *with* $d(x, x_0) < \delta$*:*

$$f_N(x) > f_N(x_0) - \frac{\varepsilon}{3}.$$

Therefore, for all $x \in X$ *with* $d(x, x_0) < \delta$*:*

$$\begin{aligned} f(x) &> f_N(x) - \frac{\varepsilon}{3} \\ &> f_N(x_0) - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\ &> f(x_0) - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\ &= f(x_0) - \varepsilon. \end{aligned}$$

Hence f *is lower semi-continuous at* x_0*. Since* x_0 *was arbitrary,* f *is lsc on* X*.*

Remark 4.1.1 *An entirely analogous argument shows that if each* f_n *is upper semi-continuous and* $f_n \to f$ *uniformly, then* f *is upper semi-continuous. This is Theorem 3.8.3 of Chapter III, and Proposition 4.1.1 provides its detailed proof for the lsc case. Together, these results show that uniform convergence preserves* ***both*** *types of semi-continuity, and hence, by Proposition 3.1.1, also preserves continuity — recovering Theorem 2.3.1 as a special case.*

4.1.3 Dini's Theorem: Verification and Counterexamples

Example 4.1.5 *We verify Dini's First Theorem (Theorem 2.3.2) in a concrete setting.*

Let $f_n : [0,1] \to \mathbb{R}$ *be defined by:*

$$f_n(x) = \frac{x^n}{n}, \quad n \in \mathbb{N}^*.$$

Checking the hypotheses of Dini's Theorem:

(i) $X = [0,1]$ *is compact.* ✓

(ii) *Each* f_n *is continuous. Indeed,* f_n *is a polynomial divided by a constant.* ✓

(iii) *The sequence is decreasing. We check:* $f_{n+1}(x) \leq f_n(x)$ *iff* $\frac{x^{n+1}}{n+1} \leq \frac{x^n}{n}$*, i.e.,* $nx^{n+1} \leq (n+1)x^n$*, i.e.,* $nx \leq n+1$*, i.e.,* $x \leq \frac{n+1}{n} = 1 + \frac{1}{n}$*. Since* $x \leq 1 \leq 1 + \frac{1}{n}$*, this holds for all* $x \in [0,1]$*.* ✓

(iv) *Pointwise limit. For any* $x \in [0,1]$*:* $f_n(x) = \frac{x^n}{n} \leq \frac{1}{n} \to 0$*. So* $f_n \to f \equiv 0$ *pointwise.* ✓

(v) *The limit* $f \equiv 0$ *is continuous.* ✓

Conclusion by Dini's Theorem: *All five hypotheses are satisfied, so* (f_n) *converges* ***uniformly*** *to* $f \equiv 0$ *on* $[0,1]$*.*

Direct verification:

$$\|f_n - 0\|_\infty = \sup_{x \in [0,1]} \frac{x^n}{n} = \frac{1}{n} \to 0. \checkmark$$

Example 4.1.6 (Failure of Dini when the limit is discontinuous) *Let $f_n : [0,1] \to \mathbb{R}$ be defined by:*

$$f_n(x) = \frac{1}{1+nx}, \quad n \in \mathbb{N}^*.$$

Checking Dini's hypotheses:

(i) $[0,1]$ *is compact.* ✓

(ii) *Each f_n is continuous.* ✓

(iii) *The sequence is decreasing: for $x > 0$, $\frac{1}{1+nx} > \frac{1}{1+(n+1)x}$.* ✓

(iv) *Pointwise limit:*

$$f(x) = \lim_{n\to\infty} \frac{1}{1+nx} = \begin{cases} 1 & x = 0, \\ 0 & x \in (0,1]. \end{cases}$$

(v) *The limit f is **not continuous** at $x = 0$.* $\times$

Hypothesis (v) fails. *Dini's Theorem does not apply, and indeed the convergence is **not uniform**:*

$$\|f_n - f\|_\infty \geq \left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \left| \frac{1}{2} - 0 \right| = \frac{1}{2} \not\to 0.$$

Semi-continuity of f:
The limit f is upper semi-continuous on $[0,1]$: at $x_0 = 0$, $\limsup_{x\to 0} f(x) = 0 \leq 1 = f(0)$, and at all $x_0 > 0$, f is continuous. This is consistent with Theorem 3.8.1: the decreasing pointwise limit of usc functions is usc. However, f is not lsc at $x_0 = 0$ since $\liminf_{x\to 0^+} f(x) = 0 < 1 = f(0)$.

Remark 4.1.2 *Examples 4.1.5 and 4.1.6 together demonstrate that the continuity of the limit function in Dini's First Theorem is an **essential hypothesis** that cannot be weakened to mere semi-continuity. The precise statement is:*

Hypothesis	***Can it be weakened?***
X compact	*No*
Each f_n continuous	*Yes: lsc suffices for lsc limit*
Sequence decreasing	*No*
Limit f continuous	*No (Example 4.1.6)*

4.2 Applications to Optimization Theory

One of the most significant applications of semi-continuity in mathematical analysis is the theory of optimization. The classical Extreme Value Theorem of Weierstrass states that a continuous function on a compact set attains both its minimum and maximum. The theory of semi-continuity generalizes this result by relaxing the continuity requirement to its minimal form: lower semi-continuity for the existence of a minimum, and upper semi-continuity for the existence of a maximum. This section develops these results in detail and illustrates them through a series of carefully chosen examples.

4.2.1 The Weierstrass Theorem for Semi-Continuous Functions

We begin by recalling the main theorem, whose proof was given in Chapter III (Theorem 3.9.1 and Theorem 3.9.2), and then examine its hypotheses through examples that show each condition is indispensable.

Theorem 4.2.1 (Weierstrass-type for semi-continuous functions) *Let (X, d) be a compact metric space.*

(i) *If $f : X \to \mathbb{R}$ is lower semi-continuous on X, then f is bounded below and attains its infimum: there exists $x^* \in X$ such that*

$$f(x^*) = \inf_{x \in X} f(x).$$

(ii) *If $f : X \to \mathbb{R}$ is upper semi-continuous on X, then f is bounded above and attains its supremum: there exists $x^* \in X$ such that*

$$f(x^*) = \sup_{x \in X} f(x).$$

Example 4.2.1 *Let $X = [0, 2]$ with the usual metric and define $f : [0, 2] \to \mathbb{R}$ by:*

$$f(x) = \begin{cases} x^2 - 2x + 2 & \text{if } x \in [0, 1), \\ \dfrac{1}{2} & \text{if } x = 1, \\ x^2 - 2x + 2 & \text{if } x \in (1, 2]. \end{cases}$$

Step 1: Analysis of f at $x_0 = 1$.
Note that $x^2 - 2x + 2 = (x - 1)^2 + 1$, so:

$$\lim_{x \to 1} f(x) = (1 - 1)^2 + 1 = 1, \quad f(1) = \frac{1}{2}.$$

Since $f(1) = \frac{1}{2} < 1 = \lim_{x \to 1} f(x)$, we have:

$$\liminf_{x \to 1} f(x) = 1 > \frac{1}{2} = f(1).$$

Hence f is ***not lower semi-continuous*** *at $x = 1$.*

Step 2: Is the infimum attained?
For $x \neq 1$: $f(x) = (x-1)^2 + 1 \geq 1$.
At $x = 1$: $f(1) = \frac{1}{2}$.
Therefore $\inf_{[0,2]} f = \frac{1}{2} = f(1)$, which is attained.

Step 3: Is f upper semi-continuous?
At $x_0 = 1$:

$$\limsup_{x \to 1} f(x) = 1 > \frac{1}{2} = f(1).$$

Hence f is ***not upper semi-continuous*** *at $x = 1$ either.*

Conclusion:
This example shows that Theorem 4.2.1 gives a sufficient condition, not a necessary one. Here f is neither lsc nor usc, yet the infimum is attained. However, one cannot rely on this in general: the next example shows a case where the infimum is ***not*** *attained precisely because lsc fails.*

Example 4.2.2 *Let $X = [0,1]$ and define $g : [0,1] \to \mathbb{R}$ by:*

$$g(x) = \begin{cases} x & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0. \end{cases}$$

Step 1: Semi-continuity at $x_0 = 0$.
$g(0) = 1$ and $\liminf_{x \to 0^+} g(x) = 0 < 1 = g(0)$.
Hence g is ***not lower semi-continuous*** *at $x = 0$.*

Step 2: Is the infimum attained?
$\inf_{[0,1]} g = \inf\{x : x \in (0,1]\} = 0$.
But $g(x) > 0$ for all $x \in (0,1]$, and $g(0) = 1 > 0$.
Hence the infimum 0 is ***not attained****.*

Step 3: Explanation.
The failure of lsc at $x = 0$ is precisely what prevents the infimum from being attained. Had we defined $g(0) = 0$ instead, g would be lsc on $[0,1]$ (the indicator of the open set $(0,1]$ is lsc, but here g would equal x everywhere, which is continuous), and the minimum would be attained.

Example 4.2.3 *We demonstrate that compactness of the domain is an essential hypothesis in Theorem 4.2.1.*

(a) Non-compact domain, continuous function:
Let $f : (0,1) \to \mathbb{R}$, $f(x) = x$. f is continuous (hence lsc), and $\inf_{(0,1)} f = 0$, but $f(x) > 0$ for all $x \in (0,1)$. The infimum is not attained. The domain $(0,1)$ is bounded but not closed, hence not compact.

(b) Unbounded domain, continuous function:
Let $f : [1, +\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. *f is continuous,* $\inf_{[1,+\infty)} f = 0$, *but* $f(x) > 0$ *for all* $x \geq 1$. *The infimum is not attained. The domain is closed but unbounded, hence not compact.*

(c) Compact domain, lsc function:
Let $f : [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} x & x \in (0, 1], \\ 0 & x = 0. \end{cases}$$

$f(0) = 0$ *and* $\liminf_{x\to 0^+} f(x) = 0 = f(0)$, *so f is lsc at* 0. *At all* $x > 0$, *f is continuous. Hence f is lsc on* $[0, 1]$. *By Theorem 4.2.1(i), the minimum is attained:* $\min_{[0,1]} f = f(0) = 0$. ✓

4.2.2 Optimization Problems with Semi-Continuous Objectives

Example 4.2.4 *Consider the optimization problem:*

$$\textit{Minimize } f(x, y) \textit{ over } K = [0, 1] \times [0, 1],$$

where $f : [0, 1]^2 \to \mathbb{R}$ *is defined by:*

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x + y} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Step 1: Lower semi-continuity of f.
For $(x, y) \neq (0, 0)$, *f is a ratio of continuous functions with positive denominator (for* $(x, y) \in [0, 1]^2 \setminus \{(0, 0)\}$ *with* $x + y > 0$*), hence continuous there.*
At $(0, 0)$*: along the path* $x = y = t \to 0^+$*:*

$$f(t, t) = \frac{2t^2}{2t} = t \to 0 = f(0, 0).$$

Along the path $y = 0$, $x = t \to 0^+$*:*

$$f(t, 0) = \frac{t^2}{t} = t \to 0 = f(0, 0).$$

More generally, writing $(x, y) = r(\cos\theta, \sin\theta)$ *with* $\theta \in [0, \pi/2]$*:*

$$f(r\cos\theta, r\sin\theta) = \frac{r^2}{\cos\theta + \sin\theta} = \frac{r}{\cos\theta + \sin\theta} \cdot r.$$

Since $\cos\theta + \sin\theta \geq 1$ *for* $\theta \in [0, \pi/2]$, *we have* $f \leq r \to 0$. *Hence* $\liminf_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$, *so f is lsc at* $(0, 0)$.

Step 2: $K = [0, 1]^2$ ***is compact.*** ✓

Step 3: By Theorem 4.2.1(i), the minimum is attained.
Since $f \geq 0$ and $f(0,0) = 0$, the minimum is:

$$\min_{(x,y)\in K} f(x,y) = f(0,0) = 0.$$

Step 4: Finding the maximum.
By the AM-QM inequality: $\frac{x^2+y^2}{x+y} \geq \frac{x+y}{2}$. On $[0,1]^2$: $f(x,y) \leq \frac{x^2+y^2}{x+y} \leq x+y \leq 2$. At $(x,y) = (1,1)$: $f(1,1) = \frac{2}{2} = 1$. At $(x,y) = (1,0)$: $f(1,0) = \frac{1}{1} = 1$. The maximum is 1, attained at $(1,0)$ and $(0,1)$ and $(1,1)$.

Example 4.2.5 *We consider a* ***minimax problem****, which arises naturally in game theory and convex analysis.*

Let $F : [0,1] \times [0,1] \to \mathbb{R}$ be defined by:

$$F(x,y) = x^2 - 2xy + y.$$

We study the functions:

$$f(x) = \max_{y\in[0,1]} F(x,y), \qquad g(y) = \min_{x\in[0,1]} F(x,y).$$

Computing $f(x) = \max_{y\in[0,1]} F(x,y)$:
For fixed $x \in [0,1]$, $F(x,y) = (1-2x)y + x^2$ is linear in y.

- *If $1 - 2x > 0$, i.e., $x < \frac{1}{2}$: maximum at $y = 1$, giving $f(x) = 1 - 2x + x^2 = (x-1)^2$.*
- *If $1 - 2x < 0$, i.e., $x > \frac{1}{2}$: maximum at $y = 0$, giving $f(x) = x^2$.*
- *If $x = \frac{1}{2}$: $F = \frac{1}{4}$ for all y, so $f\left(\frac{1}{2}\right) = \frac{1}{4}$.*

Therefore:

$$f(x) = \begin{cases} (x-1)^2 & 0 \leq x \leq \frac{1}{2}, \\ x^2 & \frac{1}{2} < x \leq 1. \end{cases}$$

Upper semi-continuity of f:
By Theorem 3.6.1, the infimum of a family of usc functions is usc. Dually, the supremum of a family of functions continuous in y (and hence usc in x) might lose usc. However, here f is the maximum over a compact set, so by Theorem 3.14.2 and continuity of F in x, f is lower semi-continuous.
Indeed: at $x = \frac{1}{2}$:

$$\lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2^-} (x-1)^2 = \frac{1}{4} = f\left(\tfrac{1}{2}\right),$$

$$\lim_{x\to 1/2^+} f(x) = \lim_{x\to 1/2^+} x^2 = \frac{1}{4} = f\left(\tfrac{1}{2}\right).$$

*So f is **continuous** at $x = \frac{1}{2}$, and hence both usc and lsc there. f is continuous everywhere.*

The minimax value:

$$\min_{x\in[0,1]} f(x) = \min_{x\in[0,1]} \max_{y\in[0,1]} F(x,y).$$

On $[0,1/2]$: $f(x) = (x-1)^2$, minimized at $x = 1/2$: $f(1/2) = 1/4$.
On $[1/2,1]$: $f(x) = x^2$, minimized at $x = 1/2$: $f(1/2) = 1/4$.
Hence $\min_x \max_y F(x,y) = \frac{1}{4}$, attained at $x^* = \frac{1}{2}$.

4.2.3 Lower Semi-Continuity and Existence of Minimizers

Proposition 4.2.1 *Let (X,d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous function that is **coercive**: $f(x) \to +\infty$ whenever $d(x,x_0) \to +\infty$ for some fixed $x_0 \in X$. Then f attains its infimum on X.*

Let $m = \inf_{x\in X} f(x) \in \mathbb{R} \cup \{-\infty\}$. Since f is coercive, there exists $R > 0$ such that $f(x) > m+1$ for all x with $d(x,x_0) > R$. Hence $m = \inf_{x\in\overline{B}(x_0,R)} f(x)$, where $\overline{B}(x_0,R)$ is the closed ball.

In general metric spaces, $\overline{B}(x_0,R)$ need not be compact. However, by the coercivity, there exists a minimizing sequence (x_n) with $f(x_n) \to m$ and $d(x_n,x_0) \leq R$ for all n. If X is a finite-dimensional normed space, the Bolzano-Weierstrass theorem gives a convergent subsequence $x_{n_k} \to x^*$. By lsc of f:

$$f(x^*) \leq \liminf_{k\to\infty} f(x_{n_k}) = m.$$

Since $m \leq f(x^*)$ by definition of infimum, $f(x^*) = m$.

Example 4.2.6 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:*

$$f(x,y) = (x-1)^2 + (y+2)^2 + |x-y|.$$

Lower semi-continuity: *f is the sum of continuous functions, hence continuous, hence lsc.* ✓
Coercivity: *As $(x,y) \to \infty$, $(x-1)^2 + (y+2)^2 \to +\infty$, so $f(x,y) \to +\infty$.* ✓
By Proposition 4.2.1, f attains its minimum on $\mathbb{R}^2$.

Finding the minimizer:
The function f is convex (sum of convex functions). Setting $\nabla f = 0$ away from $\{x = y\}$: for $x > y$,

$$\frac{\partial f}{\partial x} = 2(x-1) + 1 = 0 \implies x = \frac{1}{2},$$

$$\frac{\partial f}{\partial y} = 2(y+2) - 1 = 0 \implies y = -\frac{3}{2}.$$

Check: $x = \frac{1}{2} > -\frac{3}{2} = y$. ✓
The minimizer is $\left(\frac{1}{2}, -\frac{3}{2}\right)$, *with:*

$$f\left(\tfrac{1}{2}, -\tfrac{3}{2}\right) = \left(\tfrac{1}{2} - 1\right)^2 + \left(-\tfrac{3}{2} + 2\right)^2 + \left|\tfrac{1}{2} - \left(-\tfrac{3}{2}\right)\right| = \frac{1}{4} + \frac{1}{4} + 2 = \frac{5}{2}.$$

Example 4.2.7 *Let (X, d) be a metric space and $A \subset X$ be a non-empty closed subset. Define the* ***distance function*** *to A by:*

$$d_A(x) = \inf_{a \in A} d(x, a), \quad x \in X.$$

Claim: *d_A is Lipschitz continuous with constant 1, hence both lsc and usc on X.*

Proof:
For any $x, y \in X$ and $a \in A$:

$$d_A(x) \le d(x, a) \le d(x, y) + d(y, a).$$

Taking the infimum over $a \in A$:

$$d_A(x) \le d(x, y) + d_A(y).$$

Exchanging x and y: $d_A(y) \le d(x, y) + d_A(x)$.
Hence $|d_A(x) - d_A(y)| \le d(x, y)$, so d_A is Lipschitz with constant 1.

Application to optimization:
If X is compact, then by Theorem 4.2.1, d_A attains its minimum and maximum on X. The minimum is 0 (attained at any point of A, since A is non-empty and closed). The maximum is $\max_{x \in X} d_A(x) = \max_{x \in X} \inf_{a \in A} d(x, a)$, which is attained at some point $x^* \in X$ farthest from A.

Explicit example:
Let $X = [0, 1]$ with the usual metric and $A = \{0, 1\}$. Then:

$$d_A(x) = \min\{|x - 0|, |x - 1|\} = \min\{x, 1 - x\}.$$

This function is continuous (piecewise linear), lsc and usc. $\min_{[0,1]} d_A = 0$ (attained at $x = 0$ and $x = 1$).
$\max_{[0,1]} d_A = \frac{1}{2}$ *(attained at $x = \frac{1}{2}$).*

4.3 Fixed Point Theory in Complete Metric Spaces

Fixed point theory occupies a central position in modern mathematical analysis. The existence of a fixed point — a point x^* satisfying $T(x^*) = x^*$ for a given mapping T — underlies the solution theory for a vast range of equations, from algebraic *systems to integral and differential equations. The connection to the themes of this thesis is profound: the iterative sequence generated by repeated application of T is a Cauchy sequence in a complete metric space, and its convergence is a direct application of the theory developed in Chapter I. Moreover, the fixed point inherits the regularity properties of the iterates through the uniform convergence established in Chapter II.*

4.3.1 The Banach Contraction Principle

Theorem 4.3.1 (Banach Contraction Principle, 1922) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping, that is, there exists a constant $k \in [0, 1)$ such that:*

$$d(T(x), T(y)) \leq k\, d(x, y) \quad \forall\, x, y \in X.$$

Then the following conclusions hold:

(i) *T has a unique fixed point $x^* \in X$, i.e., $T(x^*) = x^*$.*

(ii) *For any initial point $x_0 \in X$, the iterative sequence defined by $x_{n+1} = T(x_n)$ converges to x^*.*

(iii) *The following error estimates hold:*

$$d(x_n, x^*) \leq \frac{k^n}{1-k}\, d(x_1, x_0) \quad \textit{(a priori estimate)},$$

$$d(x_n, x^*) \leq \frac{k}{1-k}\, d(x_n, x_{n-1}) \quad \textit{(a posteriori estimate)}.$$

Step 1: The iterative sequence is Cauchy.
Let $x_0 \in X$ be arbitrary and define $x_{n+1} = T(x_n)$. By the contraction condition applied repeatedly:

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq k\, d(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0).$$

For $p > q \geq N$, by the triangle inequality:

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x_{p-1}) + d(x_{p-1}, x_{p-2}) + \cdots + d(x_{q+1}, x_q) \\ &\leq (k^{p-1} + k^{p-2} + \cdots + k^q)\, d(x_1, x_0) \\ &= k^q \cdot \frac{1 - k^{p-q}}{1-k}\, d(x_1, x_0) \\ &\leq \frac{k^q}{1-k}\, d(x_1, x_0) \xrightarrow{q \to \infty} 0. \end{aligned}$$

Hence (x_n) is a Cauchy sequence in (X, d).

Step 2: Convergence to a fixed point.
Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. By the contraction property:

$$d(T(x^*), x^*) \leq d(T(x^*), T(x_n)) + d(T(x_n), x^*) \leq k\, d(x^*, x_n) + d(x_{n+1}, x^*).$$

As $n \to \infty$, both terms tend to 0, so $d(T(x^*), x^*) = 0$, i.e., $T(x^*) = x^*$.

Step 3: Uniqueness.
Suppose $T(y^*) = y^*$ for some $y^* \in X$. Then:

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq k\, d(x^*, y^*).$$

Since $k < 1$, this implies $d(x^*, y^*) = 0$, hence $x^* = y^*$.

Step 4: Error estimate.
For the a priori estimate, letting $p \to \infty$ in Step 1 gives $d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_1, x_0)$.

Example 4.3.1 *Let $X = \mathbb{R}$ with the usual metric and define $T : \mathbb{R} \to \mathbb{R}$ by:*

$$T(x) = \frac{x}{2} + \frac{3}{4}.$$

Step 1: T is a contraction.
For any $x, y \in \mathbb{R}$:

$$|T(x) - T(y)| = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y|.$$

So T is a contraction with constant $k = \frac{1}{2}$.

Step 2: Finding the fixed point.
Solving $T(x^*) = x^*$:

$$\frac{x^*}{2} + \frac{3}{4} = x^* \implies \frac{3}{4} = \frac{x^*}{2} \implies x^* = \frac{3}{2}.$$

Step 3: Iterative convergence from $x_0 = 0$.

$$\begin{aligned}
x_1 &= T(0) = \frac{3}{4},\\
x_2 &= T\left(\frac{3}{4}\right) = \frac{3}{8} + \frac{3}{4} = \frac{9}{8},\\
x_3 &= T\left(\frac{9}{8}\right) = \frac{9}{16} + \frac{3}{4} = \frac{21}{16},\\
x_4 &= T\left(\frac{21}{16}\right) = \frac{21}{32} + \frac{3}{4} = \frac{45}{32}.
\end{aligned}$$

One can verify that $x_n = \frac{3}{2} - \frac{3}{2^{n+1}}$, so $x_n \to \frac{3}{2} = x^*$ as $n \to \infty$.

Step 4: Error estimate.
$d(x_1, x_0) = |3/4 - 0| = 3/4$. *Hence:*

$$d(x_n, x^*) \leq \frac{(1/2)^n}{1 - 1/2} \cdot \frac{3}{4} = \frac{3}{2} \cdot \frac{1}{2^n} = \frac{3}{2^{n+1}}.$$

Indeed, $|x_n - x^*| = \frac{3}{2^{n+1}}$*, confirming the estimate is sharp in this case.*

Connection to semi-continuity:
The function T is continuous (linear), and by Theorem 2.3.1, the uniform convergence of the Lipschitz functions $f_n(x) = x_n$ (constant sequences) to x^* is trivial. More meaningfully, the mapping $x \mapsto T(x)$ is both usc and lsc (being continuous), and the fixed point x^* inherits this regularity.

Example 4.3.2 *We apply the Banach Contraction Principle to prove the existence and uniqueness of a solution to the equation:*

$$x = \cos(x), \quad x \in \left[0, \frac{\pi}{2}\right].$$

Let $T : [0, \pi/2] \to \mathbb{R}$ be defined by $T(x) = \cos(x)$.

Step 1: T maps $[0, \pi/2]$ into itself.
For $x \in [0, \pi/2]$: $T(x) = \cos(x) \in [0, 1] \subset [0, \pi/2]$. ✓

Step 2: T is a contraction on $[0, \pi/2]$.
By the Mean Value Theorem, for $x, y \in [0, \pi/2]$:

$$|T(x) - T(y)| = |\cos(x) - \cos(y)| = |\sin(\xi)||x - y|$$

for some ξ between x and y. Since $|\sin(\xi)| \leq \sin(\pi/2) = 1$, this gives $|T(x)-T(y)| \leq |x - y|$, which is not a strict contraction on all of $[0, \pi/2]$.

However, on $[0, \pi/2]$, $|\sin(\xi)| \leq \sin(\pi/2) = 1$, and we need a value strictly less than 1. Restricting to $[0.5, 1]$: $|\sin(\xi)| \leq \sin(1) \approx 0.841 < 1$. So T is a contraction with $k = \sin(1)$ on $[0.5, 1]$.

Step 3: The fixed point exists and is unique.
The equation $x = \cos(x)$ has the unique solution $x^* \approx 0.7391$ (the Dottie number), which lies in $(0.5, 1)$.

Step 4: Iterative computation from $x_0 = 1$.

n	$x_n = \cos(x_{n-1})$	$\lvert x_n - x^*\rvert$
0	1.000000	0.260787
1	0.540302	0.198893
2	0.857553	0.118358
3	0.654290	0.084905
4	0.793480	0.054285
5	0.701369	0.037826
6	0.763960	0.024765
7	0.722102	0.017093
8	0.750418	0.011223
9	0.731404	0.007791
10	0.744237	0.005042

The iterates converge to $x^* \approx 0.739085$*, the unique solution of* $\cos(x^*) = x^*$.

4.3.2 Application to Integral Equations

Example 4.3.3 *We apply the Banach Contraction Principle to prove existence and uniqueness of solution for the* ***Volterra integral equation of the second kind****:*

$$u(t) = g(t) + \lambda \int_0^t K(t,s)\,u(s)\,ds, \quad t \in [0,1],$$

where $g \in C([0,1],\mathbb{R})$*,* $\lambda \in \mathbb{R}$*, and* $K : [0,1]^2 \to \mathbb{R}$ *is continuous.*

Setting:
The complete metric space is $\big(C([0,1],\mathbb{R}), \|\cdot\|_\infty\big)$*, which is complete by Corollary 2.2.1. Define the operator* $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ *by:*

$$(Tu)(t) = g(t) + \lambda \int_0^t K(t,s)\,u(s)\,ds.$$

Step 1: T ***is well-defined.***
If $u \in C([0,1],\mathbb{R})$*, then since* K *and* u *are continuous, the function* (Tu) *is also continuous.* ✓

Step 2: Estimating $\|T^n u - T^n v\|_\infty$***.***
Let $M = \max_{[0,1]^2} |K(t,s)|$*. For* $u, v \in C([0,1],\mathbb{R})$*:*

$$|(Tu)(t) - (Tv)(t)| \le |\lambda| \int_0^t |K(t,s)|\,|u(s) - v(s)|\,ds \le |\lambda| M t\,\|u - v\|_\infty.$$

Applying T again:

$$|(T^2u)(t) - (T^2v)(t)| \leq |\lambda|^2 M^2 \frac{t^2}{2!} \|u - v\|_\infty.$$

By induction:

$$\|T^n u - T^n v\|_\infty \leq \frac{(|\lambda| M)^n}{n!} \|u - v\|_\infty.$$

Step 3: Contraction of T^n for large n.
Since $\frac{(|\lambda|M)^n}{n!} \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $\frac{(|\lambda|M)^N}{N!} < 1$. Hence T^N is a contraction on $C([0,1],\mathbb{R})$.

Step 4: Conclusion.
*By the Banach Contraction Principle applied to T^N, there exists a unique fixed point u^* of T^N: $T^N(u^*) = u^*$. Moreover, $T(u^*) = T(T^N(u^*)) = T^N(T(u^*))$, so $T(u^*)$ is also a fixed point of T^N. By uniqueness, $T(u^*) = u^*$, i.e., u^* is the unique solution of the Volterra equation, valid for **any** $\lambda \in \mathbb{R}$, without restriction on $|\lambda|$.*

Example 4.3.4 *Consider the **Fredholm integral equation of the second kind**:*

$$u(t) = g(t) + \lambda \int_0^1 K(t,s)\, u(s)\, ds, \quad t \in [0,1],$$

with $g(t) = t^2$, $K(t,s) = ts$, and $\lambda = \frac{1}{2}$.

Step 1: Setting up the operator.
$(Tu)(t) = t^2 + \frac{1}{2}\int_0^1 ts\, u(s)\, ds = t^2 + \frac{t}{2}\int_0^1 s\, u(s)\, ds.$

Let $c(u) = \int_0^1 s\, u(s)\, ds$ (a constant depending on u). Then $(Tu)(t) = t^2 + \frac{c(u)}{2}t$.

Step 2: Contracting estimate.
$M = \max_{[0,1]^2} |ts| = 1$ and $|\lambda| = \frac{1}{2}$. So $|\lambda|M = \frac{1}{2} < 1$:

$$\|Tu - Tv\|_\infty \leq \frac{1}{2}\int_0^1 |ts||u(s) - v(s)|\, ds \leq \frac{1}{2}\|u - v\|_\infty.$$

Hence T is a contraction with $k = \frac{1}{2}$.

Step 3: Finding the exact solution.
The fixed point satisfies $u(t) = t^2 + \frac{c}{2}t$ where $c = \int_0^1 s\, u(s)\, ds$. Substituting:

$$c = \int_0^1 s\left(s^2 + \frac{c}{2}s\right) ds = \int_0^1 s^3\, ds + \frac{c}{2}\int_0^1 s^2\, ds = \frac{1}{4} + \frac{c}{6}.$$

Solving: $c - \frac{c}{6} = \frac{1}{4}$, so $\frac{5c}{6} = \frac{1}{4}$, giving $c = \frac{3}{10}$.

The unique solution is:

$$\boxed{u^*(t) = t^2 + \frac{3}{20}t.}$$

Verification:

$$t^2 + \frac{1}{2}\int_0^1 ts\left(s^2 + \frac{3}{20}s\right) ds = t^2 + \frac{t}{2}\left(\frac{1}{4} + \frac{3}{60}\right) = t^2 + \frac{t}{2}\cdot\frac{3}{10} = t^2 + \frac{3t}{20}.\checkmark$$

4.3.3 Application to Differential Equations

Example 4.3.5 *The* ***Picard-Lindelöf Theorem*** *guarantees local existence and uniqueness of solutions to the initial value problem:*

$$\begin{cases} y'(t) = F(t, y(t)), \\ y(t_0) = y_0, \end{cases}$$

under the Lipschitz condition on F. We illustrate with:

$$y'(t) = y(t) - t^2 + 1, \quad y(0) = \frac{1}{2}.$$

Converting to integral form:
By integration from 0 to t:

$$y(t) = \frac{1}{2} + \int_0^t \big(y(s) - s^2 + 1\big)\, ds.$$

Define $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ by:

$$(Ty)(t) = \frac{1}{2} + \int_0^t (y(s) - s^2 + 1)\, ds.$$

Contraction:

$$|(Ty)(t) - (Tz)(t)| \leq \int_0^t |y(s) - z(s)|\, ds \leq t\, \|y - z\|_\infty \leq \|y - z\|_\infty.$$

This gives $k = 1$, which is not a contraction on $[0,1]$. We restrict to $[0,h]$ for $h < 1$: then $k = h < 1$.

Picard iterates from $y_0(t) = \frac{1}{2}$***:***

$$\begin{aligned} y_1(t) &= \frac{1}{2} + \int_0^t \left(\frac{1}{2} - s^2 + 1\right) ds = \frac{1}{2} + \frac{3t}{2} - \frac{t^3}{3}, \\ y_2(t) &= \frac{1}{2} + \int_0^t \big(y_1(s) - s^2 + 1\big)\, ds \\ &= \frac{1}{2} + \int_0^t \left(\frac{1}{2} + \frac{3s}{2} - \frac{s^3}{3} - s^2 + 1\right) ds \\ &= \frac{1}{2} + \frac{3t}{2} + \frac{3t^2}{4} - \frac{t^3}{3} - \frac{t^4}{12}. \end{aligned}$$

Exact solution:
The ODE $y' - y = -t^2 + 1$ is linear with integrating factor e^{-t}:

$$y^*(t) = (t+1)^2 - \frac{1}{2}e^t.$$

The Picard iterates converge uniformly to y^* on $[0,h]$, confirming Theorem 4.3.1.

Remark 4.3.1 *The Banach Contraction Principle connects to semi-continuity theory in the following way. The operator $T : X \to X$ is continuous (being Lipschitz), hence both usc and lsc as a mapping. The iterative sequence (x_n) converges uniformly (in the sense that $d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_1, x_0)$ independently of any free parameter), and by Theorem 2.3.1, the limit x^* inherits the continuity of the approximants. In the function space setting (Examples 4.3.3–4.3.5), the fixed point $u^* \in C([0,1], \mathbb{R})$ is itself a continuous function, confirming that uniform convergence in the complete metric space $(C([0,1], \mathbb{R}), \|\cdot\|_\infty)$ preserves continuity.*

4.4 Semi-Continuous Envelopes: Explicit Constructions

Given an arbitrary function $f : X \to \mathbb{R}$ defined on a metric space (X, d), one of the most natural questions in analysis is how to associate with f the "best" semi-continuous function that approximates it from below or from above. This leads to the concept of semi-continuous envelopes, introduced in Definition 3.11.1 and Definition 3.11.2 of Chapter III. In this section, we construct these envelopes explicitly for a variety of concrete functions, illustrating both the regularization procedure and its role in connecting convergence theory to semi-continuity.

4.4.1 Definition and General Properties

We recall the definitions from Chapter III for convenience.

Definition 4.4.1 *Let (X, d) be a metric space and $f : X \to \mathbb{R}$ any function.*

(i) *The **lower semi-continuous envelope** (or lsc regularization) of f is the function $\overline{f} : X \to \mathbb{R}$ defined by:*

$$\overline{f}(x) = \liminf_{y \to x} f(y) = \lim_{\delta \to 0^+} \inf_{d(y,x) < \delta} f(y).$$

(ii) *The **upper semi-continuous envelope** of f is the function $\widehat{f} : X \to \mathbb{R}$ defined by:*

$$\widehat{f}(x) = \limsup_{y \to x} f(y) = \lim_{\delta \to 0^+} \sup_{d(y,x) < \delta} f(y).$$

Proposition 4.4.1 *Let $f : X \to \mathbb{R}$ be any function. Then:*

(i) *$\overline{f} \leq f \leq \widehat{f}$ pointwise on X.*

(ii) *$\overline{f}$ is the greatest lsc function that is everywhere $\leq f$.*

(iii) *$\widehat{f}$ is the least usc function that is everywhere $\geq f$.*

(iv) *f is lsc at x if and only if $\overline{f}(x) = f(x)$.*

(v) *f is usc at x if and only if $\widehat{f}(x) = f(x)$.*

(vi) *f is continuous at x if and only if $\overline{f}(x) = f(x) = \widehat{f}(x)$.*

4.4.2 Explicit Computations of Envelopes

Example 4.4.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be the generalized Heaviside function defined by:*

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Computing $\overline{f}$ (lsc envelope):
For $x < 0$: f is locally constant equal to 0, so $\overline{f}(x) = 0$.
For $x > 0$: f is locally constant equal to 1, so $\overline{f}(x) = 1$.
At $x = 0$:

$$\overline{f}(0) = \liminf_{y \to 0} f(y) = \min\left(\lim_{y \to 0^-} f(y), \lim_{y \to 0^+} f(y)\right) = \min(0, 1) = 0.$$

Hence:

$$\overline{f}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

This is the indicator function $\mathbf{1}_{(0,+\infty)}$, which is lsc (indicator of an open set, by Example 3.2.1).

Computing $\widehat{f}$ (usc envelope):
For $x < 0$: $\widehat{f}(x) = 0$.
For $x > 0$: $\widehat{f}(x) = 1$.
At $x = 0$:

$$\widehat{f}(0) = \limsup_{y \to 0} f(y) = \max\left(\lim_{y \to 0^-} f(y), \lim_{y \to 0^+} f(y)\right) = \max(0, 1) = 1.$$

Hence:

$$\widehat{f}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

This is the indicator function $\mathbf{1}_{[0,+\infty)}$, which is usc (indicator of a closed set, by Example 3.2.2).

Verification of Proposition 4.4.1:

- *At $x = 0$: $\overline{f}(0) = 0 \leq f(0) = \frac{1}{2} \leq \widehat{f}(0) = 1$. ✓*
- *f is neither lsc nor usc at $x = 0$: $\overline{f}(0) = 0 \neq \frac{1}{2} = f(0)$ and $\widehat{f}(0) = 1 \neq \frac{1}{2} = f(0)$. ✓*
- *At all $x \neq 0$: $\overline{f}(x) = f(x) = \widehat{f}(x)$, so f is continuous there. ✓*

Example 4.4.2 *The* ***Dirichlet function*** $D : [0,1] \to \mathbb{R}$ *is defined by:*

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

This is the most extreme example of a discontinuous function: it is discontinuous at every point of $[0,1]$.

Computing $\overline{D}$ ***(lsc envelope):***
Fix any $x_0 \in [0,1]$. *Since every interval* $(x_0 - \delta, x_0 + \delta)$ *contains irrational numbers, at which* $D = 0$*:*

$$\overline{D}(x_0) = \liminf_{y \to x_0} D(y) \leq \lim_{\text{irrationals}} D(y) = 0.$$

Since $D \geq 0$*, we have* $\overline{D}(x_0) \geq 0$. *Hence:*

$$\overline{D}(x) = 0 \quad \text{for all } x \in [0,1].$$

Computing $\widehat{D}$ ***(usc envelope):***
Since every interval contains rational numbers, at which $D = 1$*:*

$$\widehat{D}(x_0) = \limsup_{y \to x_0} D(y) \geq \lim_{\text{rationals}} D(y) = 1.$$

Since $D \leq 1$*, we have* $\widehat{D}(x_0) \leq 1$. *Hence:*

$$\widehat{D}(x) = 1 \quad \text{for all } x \in [0,1].$$

Interpretation:
The lsc envelope of D *is the zero function (continuous, hence lsc), and the usc envelope is the constant function* 1 *(continuous, hence usc). The gap* $\widehat{D} - \overline{D} = 1$ *is maximal everywhere, reflecting the total discontinuity of* D. *In particular:*

$$\overline{D}(x) = 0 \leq D(x) \leq 1 = \widehat{D}(x) \quad \forall x \in [0,1],$$

with equality on the left iff x *is irrational, and equality on the right iff* x *is rational. Hence* D *is neither lsc nor usc at any point of* $[0,1]$*, as expected.*

Example 4.4.3 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by:*

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Behavior near $x = 0$***:***
As $x \to 0$, $\sin(1/x)$ *oscillates between* -1 *and* 1 *infinitely often. More precisely,*

along the sequence $x_n^+ = \frac{1}{(2n+1/2)\pi}$ *we have* $\sin(1/x_n^+) = 1$*, and along* $x_n^- = \frac{1}{(2n-1/2)\pi}$ *we have* $\sin(1/x_n^-) = -1$*. Hence:*

$$\limsup_{x\to 0} f(x) = 1, \qquad \liminf_{x\to 0} f(x) = -1.$$

lsc envelope at $x = 0$***:***

$$\overline{f}(0) = \liminf_{x\to 0} f(x) = -1.$$

usc envelope at $x = 0$***:***

$$\widehat{f}(0) = \limsup_{x\to 0} f(x) = 1.$$

Semi-continuity of f ***at*** $x = 0$***:***

- *lsc condition:* $\overline{f}(0) = -1 \le 0 = f(0)$*. So lsc holds? NO — the lsc condition is* $\overline{f}(x_0) = f(x_0)$*, i.e.,* $-1 = 0$*? NO. Hence* f *is* ***not lsc*** *at* $x = 0$*.*
- *usc condition:* $\widehat{f}(0) = 1 \ge 0 = f(0)$*. But usc requires* $\widehat{f}(0) = f(0)$*, i.e.,* $1 = 0$*? NO. Hence* f *is* ***not usc*** *at* $x = 0$*.*

At all $x \neq 0$*,* f *is continuous (composition of continuous functions), so both lsc and usc there.*

Complete envelopes:

$$\overline{f}(x) = \begin{cases} -1 & x = 0, \\ \sin(1/x) & x \neq 0, \end{cases} \qquad \widehat{f}(x) = \begin{cases} 1 & x = 0, \\ \sin(1/x) & x \neq 0. \end{cases}$$

Example 4.4.4 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be the* ***floor function*** $f(x) = \lfloor x \rfloor$ *(greatest integer* $\le x$*).*

Structure of f***:***
f *is constant on each interval* $[n, n+1)$ *for* $n \in \mathbb{Z}$*, with* $f(x) = n$ *for* $x \in [n, n+1)$*. At each integer* $n \in \mathbb{Z}$*:*

$$\lim_{x\to n^-} f(x) = n - 1, \quad \lim_{x\to n^+} f(x) = n, \quad f(n) = n.$$

Semi-continuity at integer points $n \in \mathbb{Z}$***:***

- *lsc:* $\liminf_{x\to n} f(x) = \min(n-1, n) = n - 1 < n = f(n)$*. So* f *is* ***not lsc*** *at* n*.*
- *usc:* $\limsup_{x\to n} f(x) = \max(n-1, n) = n = f(n)$*. So* f *is* ***usc*** *at* n*.* ✓

lsc envelope:

$$\overline{f}(x) = \liminf_{y\to x} \lfloor y \rfloor = \begin{cases} n - 1 & \text{if } x = n \in \mathbb{Z}, \\ n & \text{if } x \in (n, n+1), \end{cases} = \lceil x \rceil - 1,$$

where $\lceil x \rceil$ denotes the ceiling function. More precisely: $\overline{f}(x) = \lfloor x \rfloor$ for $x \notin \mathbb{Z}$, and $\overline{f}(n) = n - 1$ for $n \in \mathbb{Z}$.

usc envelope:

$$\widehat{f}(x) = \limsup_{y \to x} \lfloor y \rfloor = \lfloor x \rfloor = f(x) \quad \forall x \in \mathbb{R}.$$

This confirms that f is usc everywhere (Proposition 4.4.1(v)).

Example 4.4.5 *We now connect envelope theory to the convergence results of Chapter II. Let (f_n) be the sequence of Example 4.1.1: $f_n(x) = x^n$ on $[0, 1]$, with pointwise limit:*

$$f(x) = \begin{cases} 0 & x \in [0, 1), \\ 1 & x = 1. \end{cases}$$

Envelopes of the limit f:

For $x \in [0, 1)$: f is continuous (equal to 0), so $\overline{f}(x) = f(x) = 0 = \widehat{f}(x)$.

At $x = 1$:

$$\overline{f}(1) = \liminf_{y \to 1} f(y) = 0, \quad \widehat{f}(1) = \limsup_{y \to 1} f(y) = 0.$$

Wait: $f(y) = 0$ for all $y \in [0, 1)$, so both $\liminf$ and $\limsup$ as $y \to 1^-$ equal 0. Hence $\overline{f}(1) = 0 \neq 1 = f(1)$ and $\widehat{f}(1) = 0 \neq 1 = f(1)$.

Interpretation:

Since $\widehat{f}(1) = 0 < 1 = f(1)$, the function f violates the usc condition at $x = 1$. Since $\overline{f}(1) = 0 < 1 = f(1)$, it also violates lsc. Hence the pointwise limit f is neither usc nor lsc at $x = 1$ from the left — or rather:

$$\overline{f}(x) = \begin{cases} 0 & x \in [0, 1], \end{cases} \qquad \widehat{f}(x) = \begin{cases} 0 & x \in [0, 1). \\ 1 & x = 1 \end{cases}$$

*So $\widehat{f} = f$, meaning f **is** usc at $x = 1$ (since $\limsup_{y \to 1} f(y) = 0 \leq 1 = f(1)$, the usc condition $\limsup \leq f(x_0)$ is satisfied). And $\overline{f}(1) = 0 < 1 = f(1)$, so f is not lsc.*

This confirms the analysis of Example 4.1.1: the pointwise limit of the continuous functions $f_n(x) = x^n$ is usc but not lsc at $x = 1$. The usc envelope equals f itself, while the lsc envelope $\overline{f}$ is the zero function on all of $[0, 1]$.

4.4.3 Envelopes and the Baire Characterization

Proposition 4.4.2 *Let (X, d) be a metric space and $f : X \to \mathbb{R}$ any bounded function. Then:*

(i) $\overline{f}(x) = \sup\{g(x) : g \leq f,\ g \text{ is continuous}\}$.

(ii) $\widehat{f}(x) = \inf\{h(x) : h \geq f,\ h \text{ is continuous}\}$.

In other words, $\overline{f}$ is the supremum of all continuous minorants of f, and $\widehat{f}$ is the infimum of all continuous majorants of f.

We prove (i); the proof of (ii) is analogous.
Let $\phi(x) = \sup\{g(x) : g \leq f,\ g \text{ continuous}\}$.

Step 1: $\phi \leq \overline{f}$**.**
For any continuous $g \leq f$:

$$g(x) = \lim_{y \to x} g(y) \leq \liminf_{y \to x} f(y) = \overline{f}(x).$$

Taking the supremum over all such g: $\phi(x) \leq \overline{f}(x)$.

Step 2: $\overline{f} \leq \phi$**.**
For any $x_0 \in X$ and $\varepsilon > 0$, by definition of $\overline{f}(x_0)$, there exists $\delta > 0$ such that $\inf_{d(y,x_0)<\delta} f(y) > \overline{f}(x_0) - \varepsilon$.

Define $g_\varepsilon(x) = \overline{f}(x_0) - \varepsilon - \frac{\overline{f}(x_0)-\varepsilon}{\delta} d(x, x_0)$ (a continuous function). One verifies that $g_\varepsilon \leq f$ on $B(x_0, \delta)$ and $g_\varepsilon(x_0) = \overline{f}(x_0) - \varepsilon$. A global continuous minorant can be constructed by multiplying by a Lipschitz cutoff. Hence $\phi(x_0) \geq \overline{f}(x_0) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\phi(x_0) \geq \overline{f}(x_0)$.

Example 4.4.6 *We illustrate Proposition 4.4.2 for the floor function $f(x) = \lfloor x \rfloor$ on $[0, 2]$.*

Continuous minorants of f:
Any continuous function $g \leq \lfloor x \rfloor$ satisfies: $g(x) \leq 0$ for $x \in [0, 1)$, $g(1) \leq 1$, $g(x) \leq 1$ for $x \in [1, 2)$, $g(2) \leq 2$. The supremum of all such g must satisfy:

$$\phi(x) = \sup\{g(x) : g \leq \lfloor \cdot \rfloor,\ g \text{ cont.}\}$$

At $x = 1$: any continuous g with $g(x) \leq 0$ for $x < 1$ must satisfy $g(1) \leq 0$ (by continuity). So $\phi(1) = 0$. For $x \in (1, 2)$: $\phi(x) = 1$.
Hence $\phi(x) = \overline{f}(x)$, confirming the proposition.

Continuous majorants of f:
Any continuous $h \geq \lfloor x \rfloor$ must satisfy $h(x) \geq 1$ for $x \geq 1$ and h must jump smoothly near integers. The infimum of all such majorants gives $\widehat{f}(x) = \lfloor x \rfloor = f(x)$, confirming the usc of f.

4.4.4 Stability of Envelopes Under Convergence

Proposition 4.4.3 *Let (f_n) be a sequence of functions from X to $\mathbb{R}$ converging pointwise to f. Then:*

(i) $\overline{f}(x) \leq \liminf_{n \to \infty} \overline{f_n}(x)$ *for all $x \in X$.*

(ii) $\limsup_{n \to \infty} \widehat{f}_n(x) \leq \widehat{f}(x)$ *for all $x \in X$.*

(iii) *If* $f_n \to f$ *uniformly, then* $\overline{f_n} \to \overline{f}$ *and* $\widehat{f_n} \to \widehat{f}$ *uniformly.*

We prove (iii). Suppose $\|f_n - f\|_\infty \leq \varepsilon_n \to 0$*. Then for any* $x, y \in X$*:*

$$f_n(y) \geq f(y) - \varepsilon_n \geq \liminf_{y \to x} f(y) - \varepsilon_n = \overline{f}(x) - \varepsilon_n.$$

Taking $\liminf_{y \to x}$*:*

$$\overline{f_n}(x) = \liminf_{y \to x} f_n(y) \geq \overline{f}(x) - \varepsilon_n.$$

Similarly, $\overline{f}(x) \geq \overline{f_n}(x) - \varepsilon_n$*. Hence* $|\overline{f_n}(x) - \overline{f}(x)| \leq \varepsilon_n$ *for all* $x \in X$*, giving uniform convergence of* $\overline{f_n}$ *to* $\overline{f}$*. The proof for* $\widehat{f_n}$ *is analogous.*

Example 4.4.7 *Let* $f_n(x) = x^n$ *on* $[0, 1]$*, with pointwise limit* f *as in Example 4.1.1. We compute the envelopes of* f_n *and verify that they do not converge uniformly to the envelopes of* f*, consistent with the non-uniform convergence of* (f_n)*.*

Since each $f_n(x) = x^n$ *is continuous on* $[0, 1]$*, we have* $\overline{f_n} = f_n = \widehat{f_n}$ *for all* n*. The lsc envelope of* f_n *is* f_n *itself.*

The lsc envelope of the limit f *is:*

$$\overline{f}(x) = \begin{cases} 0 & x \in [0, 1], \end{cases}$$

(the zero function on all of $[0, 1]$*, as computed in Example 4.4.5).*

Now:

$$\|\overline{f_n} - \overline{f}\|_\infty = \|f_n - 0\|_\infty = \sup_{x \in [0,1]} x^n = 1 \not\to 0.$$

This confirms Proposition 4.4.3(iii) in the contrapositive direction: since the envelopes do not converge uniformly, the original sequence cannot converge uniformly on $[0, 1]$*. This is fully consistent with Step 5 of Example 4.1.1.*

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- **ملخص:**

الهدف الرئيسي من هذه المذكرة هو دراسة العلاقة بين أنواع التقارب المختلفة لمتتاليات الدوال و حفظ خاصية نصف الاستمرارية في الفضاءات المترية. السؤال المركزي هو: تحت أي شروط تحتفظ الدالة النهاية بنصف استمرارية المتتالية؟
تنقسم المذكرة إلى أربعة فصول. الفصل الأول يقدم المفاهيم الأساسية للفضاءات المترية، الاستمرارية. الفصل الثاني يعرض أنواع التقارب: التقارب النقطي، المنتظم، و على المتراصات. الفصل الثالث يدرس بالتفصيل نصف الاستمرارية و الدوال نصف المستمرة السفلى و العليا، مع توصيفاتها و خصائصها. الفصل الرابع يحتوي على تمارين تطبيقية، أمثلة، و ملاحظات إضافية تدمج و تطبق ما تم دراسته في الفصول السابقة.
أبرز النتائج هي:

- **التقارب النقطي لا يحافظ على نصف الاستمرارية السفلى.**
- **التقارب المنتظم يحافظ على نصف الاستمرارية السفلى و العليا.**
- **النهايات النقطية الرتيبة تحافظ على نصف الاستمرارية السفلى.**

تخلص المذكرة إلى أن الرابط بين التقارب و نصف الاستمرارية يتحدد بنوع التقارب المستعمل، و أن التقارب المنتظم هو الضمان الأقوى لحفظ نصف الاستمرارية.

- **الكلمات المفتاحية:**

فضاءات مترية، تقارب، نصف استمرارية، تقارب منتظم، تقارب نقطي، دوال نصف مستمرة سفلياً، دوال نصف مستمرة علياً، تحليل دالي، أمثلية.

- **Résumé :**

L'objectif principal de ce mémoire est d'étudier la relation entre les différents types de convergence de suites de fonctions et la préservation de la semi-continuité dans les espaces métriques. La question centrale est la suivante : sous quelles conditions la fonction limite préserve-t-elle la semi-continuité de la suite ?
Le mémoire est divisé en quatre chapitres. Le premier chapitre présente les concepts fondamentaux des espaces métriques et de la continuité. Le deuxième chapitre expose les types de convergence : ponctuelle, uniforme et sur les compacts. Le troisième chapitre étudie en détail la semi-continuité et les fonctions semi-continues inférieurement et supérieurement, avec leurs caractérisations et propriétés. Le quatrième chapitre contient des exercices appliqués, des exemples et des remarques supplémentaires qui intègrent et appliquent ce qui a été étudié dans les chapitres précédents.
Les principaux résultats sont les suivants :

- La convergence ponctuelle **ne préserve pas** la semi-continuité inférieure.
- La convergence uniforme **préserve** la semi-continuité inférieure et supérieure.
- Les limites ponctuelles monotones préservent la semi-continuité inférieure.

Le mémoire conclut que le lien entre convergence et semi-continuité est déterminé par le type de convergence utilisé, et que la convergence uniforme est la garantie la plus forte pour préserver la semi-continuité.

- **Mots clés :**

Espaces métriques, convergence, semi-continuité, convergence uniforme, convergence ponctuelle, fonctions semi-continues inférieurement, fonctions semi-continues supérieurement, analyse fonctionnelle, optimisation.

- **Abstract:**

The main objective of this thesis is to study the relationship between different types of convergence of sequences of functions and the preservation of semicontinuity in metric spaces. The central question is: under what conditions does the limit function preserve the semicontinuity of the sequence?

The thesis is divided into four chapters. The first chapter presents the fundamental concepts of metric spaces and continuity. The second chapter introduces the types of convergence: pointwise, uniform, and compact convergence. The third chapter studies in detail semicontinuity and lower and upper semicontinuous functions, with their characterizations and properties. The fourth chapter contains applied exercises, examples, and additional remarks that integrate and apply what has been studied in the previous chapters.

The main results are:

— Pointwise convergence **does not preserve** lower semicontinuity.

— Uniform convergence **preserves** both lower and upper semicontinuity.

— Monotone pointwise limits preserve lower semicontinuity.

The thesis concludes that the link between convergence and semicontinuity is determined by the type of convergence used, and that uniform convergence is the strongest guarantee for preserving semicontinuity.

- **Keywords:**

Metric spaces, convergence, semicontinuity, uniform convergence, pointwise convergence, lower semicontinuous functions, upper semicontinuous functions, functional analysis, optimization.