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Thème

A Study of some Integral Equations

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Contents

List of symbols

$\mathbb{K}$ being $\mathbb{C}, \mathbb{R}$.

$\mathbb{C}, \mathbb{R}$: complex field and reel field.

$D(0,1)$: unit disc .

∂D : edge of set D.

$C([a,b])$: space of all continuous functions.

$C^m([a,b])$: space of all continuously differentible functions.

$< .;. >$: inner product.

$L^p(\omega)$: lebesgue space.

$||.||_p$: the norm of $L^p(\omega)$.

$(Au(x))$: linear operator on u.

(a,b) is closed ,open or half open interval.

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Introduction

The subject of Integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ODE(ordinary differential equations) and PDE(partial differential equations) can be transformed into problems of solving some approximate integral equations. Integral equations were first encountered in the theory of Fourier Integral. In 1826, another integral equations was obtained by Abel. Actual development of the theory of integral equations began with the works of the Italian Mathematician Volterra 1896 and the Swedish Mathematician Fredholm 1900.

Thus, the aim of the present work is to study an integral equations involving logarithmic kernel. Therefore, we approach in the first chapter recalls and basic concepts. In the second chapter, we talk about generality of integral equations. In the third and forth chapter,we study The singular integral equation

$$\int_B \phi(t) \log\left|\frac{t-x}{t+x}\right| dt = f(x), \quad x \in B \tag{1}$$

where B is a single or a finite union of finite intervals ,This type of integral equation is usually solved by converting the weak singularity (logarithmic type) in the kernel to a Cauchy-type singularity. For an integral equation with weakly singular kernel, the associated integral is defined in the standard Riemann sense, which requires artificial methods for its numerical evaluation. Noting this difficulty with the integrals with strong singularities, Recently, Chakrabarti has developed a direct function theoretic method for this integral with $B = [a, b]$ to obtain its solution [4]. We have applied the method of Chakrabarti to the integral equation (1), where B consists of **Case(i)** two disjoint finite intervals $(0, a) \cup (b, c)$ and **Case(ii)** a finite union of n disjoint intervals $\cup_{i=1}^{n} (a_i, b_i)$. As an application, we have applied the solution of (1) where B consists of two disjoint finite intervals $(0, a) \cup (b, c)$ to the problem of water wave scattering by a partially immersed vertical barrier with a gap present in deep water.

Chapter 1

Recalls and Basic Concepts

1.1 Functional Spaces

Definition 1.1.1 ***(Vector space) [13]***

A vector space X is a non-empty set with two operations-addition $(u, v \in X \mapsto u + v \in X)$ and multiplication to scalars $(u \in X, \alpha \in \mathbf{K} \mapsto \alpha u \in X)$ such that the following axioms are satisfied:

$u + v = v + u,$

$u + (v + w) = (u + v) + w$

$\alpha(u + v) = \alpha u + \alpha v,$

$(\alpha + \beta)u = \alpha u + \beta u,$

$(\alpha\beta)u = \alpha(\beta u),$

$1u = u$

For every $u \in X$ there is $\overline{u} \in X$ such that $u + \overline{u} = 0$

There is an element $\mathbf{0}$ in X such that $u + \mathbf{0} = u, 0u = 0$ for all $u \in X$.

Definition 1.1.2 ***(Dimension of vector space)***

- The elements of X (called also vectors) $u_1, \ldots, u_n$ of a vector space X are linearly dependent if there are scalars $\alpha_1, \ldots, \alpha_n$ not all of which are zero such that $\alpha_1 u_1 + \ldots + \alpha_n u_n = 0$; otherwise $u_1, \ldots, u_n$ are called linearly independent.

- The dimension of X is n $(\dim X = n)$ if there are n linearly independent elements in X and every set of $n + 1$ elements is linearly dependent.

- The dimension of X is infinite ($\dim X = \infty$) if for any natural number n, there are n linearly independent elements in X.

Definition 1.1.3 ***(vector subspace)***

A subspace X_0 of a vector space X is a non-empty subset of X which itself is a vector space with respect to the operations of X (thus $u, v \in X_0 \Rightarrow u + v \in X_0;\ u \in X_0, \alpha \in \mathbf{K} \Rightarrow \alpha u \in X_0$).

Definition 1.1.4 ***(inner product)***

Let X be a real linear space. A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is an inner product if, for all $x, y, z \in X, \alpha \in \mathbb{R}$,

1. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

2. $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$

3. $\langle x, y\rangle = \langle y, x\rangle$

4. $\langle x, x\rangle > 0$ if $x \neq 0$

Example 1.1

Continuously differentiable functions on $[0,1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt + \int_0^1 f'(t)\overline{g'(t)}dt$$

Definition 1.1.5 ***(normed space)***

A normed space X is a vector space which is equipped with a norm $\|\cdot\| = \|\cdot\|_X$, a function from X into $\mathbb{R}_+$, such that

$\|u\| = 0$ if and only if $u = 0$;

$\|\alpha u\| = |\alpha| \|u\| \forall \alpha \in \mathbf{K}, u \in X$;

$\|u+v\| \leq \|u\| + \|v\| \forall u, v \in X$.

Example 1.2

The applction $\begin{array}{c} \|u\| : \mathbb{R}^2 \longrightarrow \mathbb{R}_+ \\ (x, y) \to \|u\| = \sqrt{x^2+y^2} \end{array}$ is normed space .

Definition 1.1.6 ***(converges in normed space)***

- A sequence $(u_n) \subset X$ converges to $u \in X$ (one writes $u_n \to u$ or $\lim u_n = u$) if $\|u_n - u\| \to 0$ as $n \to \infty$.

- A sequence $(u_n) \subset X$ is a Cauchy sequence if $\|u_m - u_n\| \to 0$ as $m, n \to \infty$. Every convergent sequence $(u_n) \subset X$ is Cauchy but the inverse is not true in general.

Definition 1.1.7 ***(complete space)***

A normed space X is called complete if every Cauchy sequence of its elemets converges to an element of X.

Definition 1.1.8 ***(Hilbert space)***

A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product

Definition 1.1.9 ***(Banach space)***

A complete normed space is called Banach space.

Definition 1.1.10 ***(compact)***

U set on normed space X, we call U compact if and only if

$$\forall V_j, j \in J(open); U \subset \cup_{j\in J} V_j, \exists V_{j(k)}, j(k) = 1, 2, ..., n : U \subset \cup_{k=1}^{n} V_j(k)$$

Definition 1.1.11 ***(relative compact)***

we call subset S on normed space X relative compact if the adjacent $\overline{S}$ is compact.

Definition 1.1.12 ***(ball centre)***

For $u_0 \in X$ and $r > 0$, the set $B(u_0, r) := \{u \in X : \|u - u_0\| \leq r\}$ is called (closed) ball of X with the centre u_0 and radius r.

Basics[16]

A set $S \subset X$ is called:

- **bounded** if it is contained in a ball of X;
- **open** if for any $u_0 \in S$ there is an $r > 0$ such that $B(u_0, r) \subset S$;
- **closed** if $(u_n) \subset S, u_n \to u$ implies $u \in S$;
- **compact** if S is closed and relatively compact.
- **The closure** $\bar{S}$ of a set $S \subset X$ is the smallest closed set containing $S.A$ set $S \subset X$ is said to be dense in X if $\bar{S} = X$. A relatively compact, set is bounded; in finite dimensional spaces, also the inverse is true.

Theorem 1.1.1 ***(Arzelà -Ascoli)***

A set $S \subset C[0,1]$ is relatively compact in $C([0,1])$ if and only if the following two conditions are fulfilled:

(i) the functions $u \in S$ are uniformly bounded, i.e., there is a constant c such that $|u(x)| \leq c$ for all $x \in [0,1], u \in S$;

(ii) the functions $u \in S$ are equicontinuous, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x_1, x_2 \in [0,1], |x_1 - x_2| \leq \delta$ implies $|u(x_1) - u(x_2)| \leq \varepsilon$ for all $u \in S$.

Definition 1.1.13 ***(space*** $C([a,b])$ ***)***

$C([a,b])$ consists of all continuous functions $u : [a,b] \to \mathrm{K}$,

$$I\|u\|_{C([a,b])} = \|u\|_{\infty} = \max_{a \leq x \leq b} |u(x)|.$$

Definition 1.1.14 ***(space*** $BC([a,b])$ ***)***

$BC(a,b)$ consists of all bounded continuous functions $u : (a,b) \to K$,

$$\|u\|_{BC([a,b])} = \|u\|_{\infty} = \sup_{a<x<b} |u(x)|.$$

Definition 1.1.15 ***(space*** $C^m[a,b]$ ***)***

$C^m[a,b]$ consists of all m times continuously differentiable functions $u : [a,b] \to \mathbf{K}$,

$$\|u\|_{C^m[a,b]} = \sum_{k=0}^{m} \left\| u^{(k)} \right\|_{\infty}$$

Definition 1.1.16 ***(space*** $L^p(\Omega)$ ***)***

$L^p(\Omega), 1 \leq p < \infty$, consists of all (equivalence classes of) measurable functions $u : \Omega \to \mathrm{K}$ such that $\|u\|_p < \infty$

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p}$$

$L^{\infty}(\Omega)$ consists of all measurable functions $u : (\Omega) \to \mathbf{K}$ such that $\|u\|_{\infty} < \infty$,where

$$\|u\|_{L^{\infty}\Omega} = \|u\|_{\infty} = \sup_{x\in\Omega} |u(x)|.$$

Remark 1.1.1

- All these spaces are infinite dimensional and Banach space.
- The space $C([0,1])$ is a closed subspace of $BC(0,1)$; both are closed subspaces of $L^{\infty}(0,1)$.

Theorem 1.1.2 ***(Lebesgue dominated convergence theory)***

Let (f_n) be a sequence of functions of L^1. We suppose that

(i) $f_n(x) \to f(x) \quad p.p \quad in \quad \Omega$

(ii)there exists a function $g \in L^1$ such that for each n , $\quad |f_n(x)| \leq g(x) \quad p.p \quad in \quad \Omega$

then

$$f \in L^1 \quad and \quad \|f_n - f\|_{L^1} \to 0$$

1.2 Operators Notions

1.2.1 Linear Operators

Definition 1.2.1 ***(Linear operators)***

Let X and Y be two vector spaces. Operator $A : X \to Y$ is a function defined on X and with values in Y; operator A is called linear if

$$A(u+v) = Au + Av, \quad A(\alpha u) = \alpha Av$$

for all $u, v \in X$ and $\alpha \in \mathbf{K}$.

Assume now that X and Y are normed spaces.

Proposition 1.2.1 ***(Continuous linear operators)***

An operator $A : X \to Y$ is said to be continuous if $\|u_n - u\|_X \to 0$ implies $\|Au_n - Au\|_Y \to 0$.

Proposition 1.2.2 ***(Bounded linear operators)***

A linear operator $A : X \to Y$ occurs to be continuous if and only if it is be bounded, i.e., if there is a constant c such that $\|Au\|_Y \leq c\|u\|X$ for all $u \in X$.

The smallest constant c in this inequality is called the norm of A,

$$\|A\| = \sup\{\|Au\|_Y : u \in X, \|u\|_X = 1\}.$$

Definition 1.2.2 ***convergent of sequence operators***

A sequence of bounded linear operators $A_n : X \to Y$ is said to be pointwise convergent (or strongly convergent) if the sequence $(A_n u)$ is convergent in Y for any $u \in X$.

Theorem 1.2.1 ***(Banach-Steinhaus)***

Let X and Y be Banach spaces. A sequence of bounded linear operators $A_n : X \to Y$ converges pointwise if and only if the following two conditions are fulfilled:

(i) there is a constant c such that $\|A_n\| \leq c$ for all n;

(ii) there is a dense set $S \subset X$ such that the sequence $(A_n u)$ is convergent in Y for every $u \in S$.

Under conditions (i) and (ii), the limit operator $A : X \to Y, Au = \lim A_n u$, is linear and bounded.

1.2.2 Inverse Operators

Let X and Y be Banach spaces and $A : X \to Y$ a linear operator. Introduce the subspaces

$$\mathcal{N}(A) = \{u \in X : Au = 0\} \subset X \text{ (the null space of } A\text{)},$$

$$\mathcal{R}(A) = \{f \in Y : f = Au, x \in X\} \subset Y \text{ (the range of } A\text{)}.$$

If $\mathcal{N}(A) = \{0\}$ then the inverse operator $A^{-1} : \mathcal{R}(A) \subset Y \to X$ exists on $\mathcal{R}(A)$, i.e. ,

$A^{-1}Au = u \quad \forall u \in X, AA^{-1}f = f \quad \forall f \in \mathcal{R}(A)$; clearly also A^{-1} is linear.

Theorem 1.2.2 ***(Banach 1)***

Let X and Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator with $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = Y$. Then the inverse operator $A^{-1} : Y \to X$ is linear and bounded.

Theorem 1.2.3 ***(Banach 2)***

Let X and Y be Banach spaces and $A : X \to Y$ a bounded linear operator having the inverse A^{-1} : $Y \to X$. Assume that the bounded linear operator $B : X \to Y$ satisfies the condition

$$\|B\| \left\|A^{-1}\right\| < 1.$$

Then $A + B : X \to Y$ has the inverse $(A + B)^{-1} : Y \to X$ (defined on whole Y) and

$$\left\|(A+B)^{-1}\right\| \leq \frac{\left\|A^{-1}\right\|}{1 - \|B\| \left\|A^{-1}\right\|}.$$

1.2.3 integral linear operator

Definition 1.2.3 *[9]*

A integral linear operator A is an operator which admits a formulation of the form

$$(Au) = \int_a^b K(x,t)u(t)dt$$

The function K being called the kernel of the operator A

Remark 1.2.1

If K is a continuous function of $[a;b] \times [a;b]$, the operator A is called the integral operator with continuous Kernel K.

1.2.4 Weakly singular kernels

Consider the integral operator T defined by its kernel function $K(x, y)$ via the formula

$$(Tu)(x) = \int_0^1 K(x,y)u(y)dy, \quad 0 \leq x \leq 1,$$

where u is taken from some set of functions, for example, from $C([0,1])$. In the literature, the weak singularity of the kernel K and of the corresponding operator T may have different senses. A tight understanding is that K has the form

$$K(x,y) = a(x,y)|x-y|^{-\nu}$$

where a is a continuous function on $[0,1]\times[0,1]$ and $0<\nu<1$. This kernel has the property

$$\sup_{0\le x\le 1}\int_0^1 |K(x,y)|dy < \infty \tag{1.1}$$

often used to define the weak singularity in the wide sense: a kernel K is weakly singular if it is absolutely integrable w.r.t. y and satisfies (1.1). The kernels we will consider in the sequel are somewhere in the middle of these two extremal understandings of the weak singularity: we assume that K is continuous on $\operatorname{diag}([0,1]\times[0,1])$ and

$$|K(x,y)| \le c_K\left(1+|x-y|^{-\nu}\right) \text{ for } (x,y)\in \operatorname{diag}([0,1]\times[0,1])$$

where $\nu<1$. Here diag means the diagonal of $\mathbb{R}^2$:

$$\operatorname{diag} = \operatorname{diag}\left(\mathbb{R}^2\right) = \left\{(x,y)\in\mathbb{R}^2 : x=y\right\}.$$

For instance, the kernels

$$K(x,y) = a(x,y)\log|x-y|, \quad K(x,y) = a(x,y)|x-y|^{-\nu}\log^k|x-y|$$

with $a\in C([0,1]\times[0,1])$ and many others are weakly singular in this sense.

1.2.5 Adjoint Operators

Definition 1.2.4 *[9]*

Let H be a Hilbert space and let A bounded linear operators defined to H at value in H, then there exists a denoted A^* bounded linear operator defined from H to H by

$$<Au,v>=<u,A^*v> \quad for\ all \quad u,v\in H$$

and we have

$$\|A\| = \|A^*\|$$

Proposition 1.2.3

If A an integral operator defined from a kernel K continue on $[a,b]\times[a,b]$ by the following formula

$$\forall x\in[a,b],(Au)x=\int_a^b K(x,t)u(t)dt$$

then, the operator A admits a unique adjoint operator A^* for the usual inner product of L^2. For all $x\in[a,b]$,

$$\left(A^*\psi\right)x=\int_a^b K(t,x)v(t)dt$$

Corollary 1.2.1

If K is the kernel integral operator A, and A^* is the integral operator of the kernel K^*, then we have

$$K^*(t,x)=K(x,t)$$

Definition 1.2.5 ***(Auto-adjoint operator)*** *[9]*

If H is a Hilbert space , we say that the operator A is auto-adjoint if $A^*=A$,and only if :

$$\forall u,v\in H\ \ \langle Au,v\rangle=\langle u,Av\rangle.$$

Corollary 1.2.2

The integral operator A with kernel K is auto-adjoint if and only if the kernel K is symmetrical :

$$K(x,t)=K(t,x),\quad \forall x,t\in[a,b].$$

1.2.6 Compact Linear Operators

Definition 1.2.6 ***(Compact linear operators) [16]***

Let X,Y be Banach spaces. A linear operator $T:X\to Y$ is said to be compact if it maps bounded subsets of X into relatively compact subsets of Y.

Theorem 1.2.4

Equivalently, $T : X \to Y$ is compact if for every bounded sequence $(u_n) \subset X$, the sequence (Tu_n) contains a subsequence that converges in Y.

Corollary 1.2.3

Let X, Y, U, V be Banach spaces.

(i) Compact linear operators are bounded.

(ii) A linear bounded finite dimensional operator (i.e., a bounded linear operator with finite dimensional range) is compact.

(iii) For compact linear operators $T_1, T_2 : X \to Y$, $\alpha_1, \alpha_2 \in \mathbf{K}$, the operator $\alpha_1 T_1 + \alpha_2 T_2 : X \to Y$ is compact.

(iv) For a compact linear operator $T : X \to Y$ and bounded linear operators $A : U \to X$ and $B : Y \to V$, the operators $TA : U \to Y$ and $BT : X \to V$ are compact.

Theorem 1.2.5

Let $T_{\mathrm{n}} : X \to Y, n = 1, 2, \ldots$, be compact linear operators, $T : X \to Y$ a bounded linear operator, and let $\|T_n - T\| \to 0$ as $n \to \infty$. Then $T : X \to Y$ is compact.

Theorem 1.2.6

Let $T : X \to Y$ be a compact linear operator and let the bounded linear operators $B_n : Y \to V$ converge point wise to $B : Y \to V$ as $n \to \infty$. Then

$$\|B_n T - BT\| \to 0 \text{ as } n \to \infty.$$

(Similar claim about $\|TA_n - TA\|_{U \to Y}$ is wrong.) Denote by $I = I_X$ the identity operator in X, i.e., $Iu = u$ for every $u \in X$.

Theorem 1.2.7 ***(Fredholm altertative)***

Let $T : X \to X$ be a compact linear operator and let

$$\mathcal{N}(I - T) = \{\mathbf{0}\}$$

Then $I - T$ has the bounded inverse $(I - T)^{-1} : X \to X$.

Theorem 1.2.8

Let X and Y be Banach spaces such that $Y \subset X$, Y is dense in X and $\|u\|_X \le c\|u\|_Y$ for every $u \in Y$. Let $T : X \to X$ be a compact linear operator that maps Y into Y, and let also $T : Y \to Y$ be compact. Assume that the equation $u = Tu + f$ with given $f \in Y$ has a solution u $\in X$. Then $u \in Y$.

The only claim $u \in Y$ of Theorem 1.2.8 will be trivial if we add the assumption that $\mathcal{N}(I - T) = \{0\}$, since then by Theorem 1.2.7 equation $u = Tu + f$ is uniquely solvable in X as well as in Y. Actually this additional assumption is acceptable for our needs in the sequel so far as we do not treat eigenvalue problems.

Examples of compact linear integral operators

With the help of Theorem 1.1.1 it easy to see that the Fredholm integral operator

$$T : C([0,1]) \to C([0,1]), \quad (Tu)(x) = \int_0^1 K(x,y)u(y)dy$$

is compact provided that its kernel $K(x,y)$ is continuous on the set $A = [0,1] \times [0,1]$. Similarly, the Volterra integral operator

$$T : C([0,1]) \to C([0,1]), \quad (Tu)(x) = \int_0^x K(x,y)u(y)dy,$$

is compact provided that the kernel $K(x,y)$ is continuous on the set $B = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}$.

1.2.7 Contraction Operators

Definition 1.2.7 *[9]*

If X is Banach space and T is bounded operator , we call contraction operator T if exist constant L such as $0 < L < 1$ and

$$\forall u_1, u_2 \in X, \|Tu_1 - Tu_2\| \le L\|u_1 - u_2\|$$

Theorem 1.2.9 ***(Principle Banach Contraction)***

If X is Banach space and T is contraction operator then $Tu = u$ admit unique solution u in X , this solution is fixed point of this operator.

Chapter 2

Generality in the Integral Equations

2.1 Concept of the integral equations

Definition 2.1.1 ***(integral equation)***

An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign.

The standard type of integral equation is of the form

$$\Phi(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t,u(t))dt \tag{2.1}$$

where:

$f(x)$, $\Phi(x)$ and $K(x,t,u(x))$ three functions are given ,$K(x,t,u(x))$ is called the kernel.

$\alpha(x)$ and $\beta(x)$ are the limits of integration.

$u(x)$ it the unknown function appears under the integral sign.

λ a no null constant parameter in to $\mathbb{R}$ or $\mathbb{C}$.

Remark 2.1.1

- if $f(x) = 0$, in integral equation 2.1, then the resulting equation is called a homogeneous integral equation, otherwise it is called non-homogeneous integral equation.
- if the kernel $K(x,t,u(x)) = K(x,t)u(x)$ then the integral equation 2.1 are classified as linear integral equation, otherwise it is called non-linear integral equation..
- the function $\Phi(x)$ will determine type space of the integral equation.
- $\alpha(x), \beta(x)$ will determine type of the integral equation.

Example 2.1

$u(x) = \lambda \int_5^{x^2} K(x,t)u^2(t)dt$	$homogeneous\ integral\ equation$
$u(x) = f(x) + \lambda \int_a^b K(x,t)\sin(u(x))dt$	$non-homogeneous\ integral\ equation$
$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$	$linear\ integral\ equation$
$u(x) = f(x) + \lambda \int_a^{x+1} K(x,t)\ln(u(t))dt$	$nonlinear\ integral\ equation$

.

2.2 Solution of an Integral equation

A solution of a differential or integral equation arises in any of the following two types [17]

1. Exact solution:

   The solution is called exact if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions.

2. Series solution:

   For concrete problems, sometimes we cannot obtain exact solutions. In this case we determine the solution in a series form that may converge to exact solution if such a solution exists. Other series may not give exact solution, and in this case the obtained series can be used for numerical purposes. The more terms that we determine the higher accuracy level that we can achieve. A solution of an integral equation is a function $u(x)$ that satisfies the given equation. In other words, the obtained solution $u(x)$ must satisfy both sides of the examined equation. The following examples will be examined to explain the meaning of a solution.

Example 2.2

Show that $u(x) = sinh(x)$ is a solution of the Volterra integral equation

$$u(x) = x + \int_0^x (x-t)u(t)dt$$

Substituting $u(x) = sinh(x)$ in the right hand side (RHS) of the integral equation yields

$$\begin{aligned} RHS &= x + \int_0^x (x-t)sinh(t)dt \\ &= x + (sinh(t) - t)|_0^X \\ &= sinh(x) = u(x) = LHS. \end{aligned}$$

2.3 Classifications of integral equations

Integral equations appear in many types[17][15][11][7]. The types depend mainly on the limits of integration and the kernel of the equation. In this text we will be concerned on the following types of integral equations.

2.3.1 Linear Integral Equations

Definition 2.3.1 ***(Volterra integral equations)***

The most standard form of linear Volterra integral equation is given by

$$\Phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \tag{2.2}$$

where the limits of integration a (constants) and x and unknown function $u(x)$ appears linearly under the integral sign.

1. If $\Phi(x) = 0$, then equation 2.2 simply becomes

$$f(x) = \lambda \int_a^x K(x,t)u(t)dt \tag{2.3}$$

   and this equation is known as the linear Volterra integral equation of the first space.

2. if $\Phi(x) = \phi = constant \neq 0$, then equation 2.2 becomes

$$\phi u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \tag{2.4}$$

   which is known as the linear Volterra integral equation of the second space.

3. if the function $\Phi(x) \neq 0$, then equation 2.2 is known as the Volterra linear integral equation of the third space.

For example

$$x^2 u(x) = e^x + 2\int_0^x (t-x)^2 u(t)dt$$

volterra 3^{ed} space.

Definition 2.3.2 ***(Fredholm integral equations)***

The most standard form of linear Fredholm integral equation is given by

$$\Phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{2.5}$$

where the limits of integration a and b are constants and the unknown function $u(x)$ appears linearly under the integral sign.

1. If $\Phi(x) = 0$, then equation 2.2 simply becomes

$$f(x) = \lambda \int_a^b K(x,t)u(t)dt \tag{2.6}$$

   and this equation is known as the linear Fredholm integral equation of the first space.

2. if $\Phi(x) = \phi = constant \neq 0$, then equation 2.2 becomes

$$\phi u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{2.7}$$

   Which is known as the linear Fredholm integral equation of the second space.

3. If the function $\Phi(x) \neq 0$, then equation 2.2 is known as the Fredholm linear integral equation of the third space.

For example

$$u(x) = \sin x + \int_{-1}^{1} (t-x)u(t)dt$$

fredholm 2^{ed} space.

Definition 2.3.3 ***(wiener-Hoph integral equations)***

We call wiener-Hoph integral equation the equation of form

$$\Phi(x)u(x) = f(x) + \lambda \int_a^\infty K(x-t)u(t)dt \tag{2.8}$$

For example

$$e^x u(x) = 2^x + 2\int_0^\infty (t-x)^2 u(t)dt$$

Definition 2.3.4 ***(Renwal integral equations)***

We call Renwal integral equation the equation of form

$$\Phi(x)u(x) = f(x) + \lambda \int_a^x K(x-t)u(t)dt \tag{2.9}$$

Definition 2.3.5 ***(Abel integral equations)***

We call linear Abel integral equation the equation of form

$$f(x) = \int_a^x \frac{u(t)}{(x-t)^\alpha}dt \qquad 0 < \alpha = constant < 1 \tag{2.10}$$

For example

$$x^3 = \int_2^x \frac{u(t)}{\sqrt{x-t}}dt$$

2.3.2 non-Linear Integral Equations

Definition 2.3.6 ***(Volterra Integral Equations)***

We called nonlinear Volterra integral equation of the first space given by form

$$f(x) = \lambda \int_a^x K(x,t,u(t))dt \tag{2.11}$$

and we called nonlinear Volterra integral equation of the second space given by form

$$\phi u(x) = f(x) + \lambda \int_a^x K(x,t,u(t))dt \qquad \phi = constant \neq 0 \tag{2.12}$$

We called nonlinear Volterra integral equation of the third space given by form

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t,u(t))dt \qquad \phi(x) \neq 0 \tag{2.13}$$

For example

$$\frac{1}{x} = \int_1^x x\ln(u(t))dt$$

Definition 2.3.7 ***(Fredholm Integral Equations)***

We called nonlinear Fredholm integral equation of the first space given by form

$$f(x) = \lambda \int_a^b K(x,t,u(t))dt \tag{2.14}$$

and we called nonlinear Fredholm integral equation of the second space given by form

$$\phi u(x) = f(x) + \lambda \int_a^b K(x,t,u(t))dt \qquad \phi = constant \neq 0 \tag{2.15}$$

We called nonlinear Fredholm integral equation of the third space given by form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t,u(t))dt \qquad \phi(x) \neq 0 \tag{2.16}$$

Remark 2.3.1

- If $f(x) = 0$, in integral equation , then the resulting equation is called a homogeneous integral equation, otherwise it is called non-homogeneous integral equation.

Definition 2.3.8 ***(Integro-differential equations)***

The nonlinear integro-differential equation appears in the form:

$$u^{(n)}(t) = f(t) + \lambda \int_G K(t,x,u(x))dx \ , \qquad x,y \in G$$

and the standard form of the nonlinear integro-differential equation of the first kind is given by

$$\int_G K_1(t,x,u(x))dx + \int_G K_2\left(t,x,u^{(n)}(x)\right)dx = f(t) \ , \quad K_2\left(t,x,u^{(n)}(x)\right) \neq 0,$$

Where $u^{(n)}$ indicates the nth derivative of $u(x)$, the kernels K, K_1, K_2 and the function $f(x)$ are given real valued functions. The Volterra-Fredholm integro-differential equation arise in the same manner as Volterra-Fredholm integral equation with one or more of ordinary derivatives in addition to the integral operators.

For example

$$u'(x) = e^x + \int_0^1 x\sin(u(t))dt$$

Definition 2.3.9 ***(Hammerstein integral equations)***

We call Hammerstein integral equation the equation of form

$$\phi(x)u(x)+\lambda\int_a^b K(x,t)F(t,u(t))dt=f(x) \tag{2.17}$$

For example

$$2u(x)+\int_0^2 \sqrt{(x-t)}\cos(u(t)+t)dt=e^{-2x}$$

Definition 2.3.10 ***(Hammerstein-Volterra integral equations)***

We call Hammerstein-Volterra integral equation the equation of form

$$\phi(x)u(x)+\lambda\int_a^x K(x,t)F(t,u(t))dt=f(x) \tag{2.18}$$

For example

$$xu(x)+2\int_{-2}^x \frac{x-t}{2}\sin(u(t)-t)dt=2x^2$$

Definition 2.3.11 ***(Abel integral equations)***

We call Abel integral equation the equation of form

$$u(x)=\int_{-\infty}^x (x-t)^{\alpha-1}g(u(t))dt \tag{2.19}$$

$0<\alpha<1$ and $g:[0,\infty)\to[0,\infty)$ $g(0)=0$ and $g(x)>0$

For example

$$u(x)=\int_{-\infty}^x \frac{1}{\sqrt{x-t}}e^{u(t)}dt$$

Definition 2.3.12 ***(Lalesco integral equations)***

We call Lalesco integral equation the equation of form

$$u(x)=f(x)+\int_0^x [K_1(x,t)u(t)+K_2(x,t)u^2(t)+K_3(x,t)u^3(t)+...+K_n(x,t)u^n(t)]dt \tag{2.20}$$

Definition 2.3.13 ***(Bratu integral equations)***

We call nonlinear Bratu integral equation the equation of form

$$u(x) = \lambda \int_a^b G(x,t)e^{u(t)}dt \tag{2.21}$$

b positive number and the $G(x,t)$ is function of Green

$$G(x;t) = \begin{cases} \dfrac{(b-x)(t-a)}{b-a}, t \leq x \\ \dfrac{(b-t)(x-a)}{b-a}, t \geq x \end{cases}$$

For example

$$u(x) = \int_1^3 G(x,t)e^{u(t)}dt$$

with

$$G(x,t) = \begin{cases} \dfrac{(3-x)(t-1)}{2};\ t \leq x \\ \dfrac{(3-t)(x-1)}{2};\ x \leq t \end{cases}$$

2.3.3 Singular Integral Equations

Definition 2.3.14 ***(Singular integral equations)***

A singular integral equation is defined as an integral with the infinite limits or when the kernel of the integral becomes unbounded at a certain point in the interval. As for examples,

$$u(x) = f(x) + \lambda \int_{-\infty}^{+\infty} u(t)dt$$

$$f(x) = \int_0^x \frac{u(t)}{(x-t)^\alpha}dt, 0 < \alpha < 1$$

are classified as the singular integral equations.

Definition 2.3.15 ***(The Weakly Singular Volterra Equations)***

The weakly-singular Volterra-type integral equation of the second space are given by

$$u(x) = f(x) + \lambda \int_0^x \frac{\beta}{\sqrt{(x-t)}}u(t)dt, \quad x \in [0,T] \tag{2.22}$$

and

$$u(x) = f(x) + \lambda \int_0^x \frac{\beta}{[g(x)-g(t)]^\alpha}u(t)dt, \quad 0 < \alpha < 1, x \in [0,T] \tag{2.23}$$

Definition 2.3.16 ***(The Weakly Singular Fredholm Equations)***

The weakly-singular Fredholm-type integral equation of the second space are given by

$$u(x) = f(x) + \lambda \int_a^b M(x,t)K(x,t)u(t)dt \quad , x \in [0,T] \tag{2.24}$$

where $K(x,t)$ the Weakly Singular kernel

$$K(x;t) = \begin{cases} (x-t)^{-\alpha}, 0 < \alpha < 1 \\ \\ \log(x-t) \end{cases}$$

for example

$$u(x) = e^{-x} + \int_0^1 \sqrt{x-t}\log(x-t)u(t)dt$$

Remark 2.3.2

Where the limit of integration $b = x$ then we call (2.24) Weakly Singular Volterra equation.

Definition 2.3.17 ***(Carleman integral equations)***

We call Carleman integral equation the equation of form

$$p(t)\frac{1}{\pi i}\int_{-1}^{1}\frac{u(t)}{(t-x)}dt + \frac{1}{\pi i}\int_{-1}^{1}\frac{q(t)}{(t-x)}u(t)dt = f(x) \tag{2.25}$$

where p and q are two function continue

Definition 2.3.18 ***(Cauchy singularity)***

Let D be a bounded and connected domain in a complex plane, then the Cauchy integral is given by the formula

$$\frac{1}{\pi i}\int_{\partial D}\frac{u(t)}{(t-x)}dt = f(x) \quad t \in \mathbb{C} \tag{2.26}$$

For example

$$\frac{1}{i\pi}\int_{\partial(D(0;1))}\frac{u(t)}{t-x}dt = x^2$$

2.3.4 Mixed Integral Equations

Definition 2.3.19 *(Volterra-Fredholm Integral Equations)*

The Volterra-Fredholm integral equation second space appear in two forms, namely

$$u(x) = f(x) + {}_1\int_0^x K_1(x,t)u(t)dt + {}_2\int_a^b K_2(x,t)u(t)dt \tag{2.27}$$

and the mixed form

$$u(x) = f(x) + \int_0^x \int_a^b K(r,t)u(t)dtdr \tag{2.28}$$

where $f(x)$ and $K(x,t)$ are analytic functions.For example

$$u(x) = \sin x + \int_0^x \int_0^1 r^2 u(t)dtdr$$

2.4 Existence and Uniqueness a Integral Equations

2.4.1 Existence and Uniqueness for Nonlinear Fredholm integral equations

Banach's fixed point theorem can be used to prove the following result.

Theorem 2.4.1 *[18]*

Suppose that $k(t,x,u)$ is defined and continuous on the set $[a,b] \times [a,b] \times \mathbb{R}$ and that it satisfies a Lipschitz condition of the form

$$|k(t,x,u_1) - k(t,x,u_2)| < C|u_1 - u_2|,$$

suppose further that $f \in C[a,b]$. Then the nonlinear Urysohn integral equation

$$u(t) = \lambda \int_a^b k(t,x,u(x))dx + f(t)$$

has a unique solution on the interval $[a,b]$ whenever $|\lambda| < 1/(C(b-a))$.

Proof. See, [18].

2.4.2 Existence and Uniqueness Nonlinear Volterra integral equations

The purpose of this section is to study the existence of fixed point for a nonlinear integral operator in the framework of Banach space [8] $X := C([a,b],\mathbb{R}^n)$. we intend to prove the existence and uniqueness of the solutions of the following nonhomogeneous nonlinear Volterra integral equation

$$u(x) = f(x) + \varphi\left(\int_0^x K(x,t,u(t))dt\right) \equiv Tu, \quad u \in X \tag{2.29}$$

where $x,t \in [a,b]$ $,-\infty < a < b < \infty$, $f:[a,b] \to \mathbb{R}^n$is a mapping, and K is a continuous function on the domain $D := \{(x,t,u) : x \in [a,b], t \in [a,x), u \in X\}$. In this study, we will use an iterative method to prove that (2.29) has the mentioned cases under some appropriate conditions.

Basic Concepts

In this section, we recall basic result which we will need in this study.

Consider the nonhomogeneous nonlinear Volterra integral equation (2.29). We consider the complete metric space (X,d), which $d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$, for all $f,g \in X$ and assume that φ is a bounded linear transformation on X.

Note 1

Since the linear mapping $\varphi : X \to X$is called bounded, if there exists $M > 0$ such that

$$||\varphi x|| \leq M||x|| \quad ; \quad for \;\; all \quad x \in X$$

In this case, we define

$$||\varphi|| = \sup\left\{\frac{||\varphi x||}{||x||}; x \neq 0, x \in X\right\}$$

Thus, φ is bounded if and only if $||\varphi|| < \infty$.

Note 2

As φ is a bounded linear mapping on X, then $\varphi(x) = \lambda x$, where λ does not depend on $x \in X$.

Definition 2.4.1

Let S denote the class of those functions $\alpha : [0,\infty) \to [0,1)$ satisfying the condition

$$\lim_{s\to t^+} \sup \alpha(s) < 1, \forall t \in [0,\infty)$$

Definition 2.4.2

Let $\mathscr{B}$ denoted the class of those functions $\phi : [0,\infty) \to [0,\infty)$ which satisfies the following conditions:

(i) φ is increasing,

(ii) for each $x > 0$, $\phi(x) < x$,

(iii) $\alpha(x) = \phi(x)/x \in S, x \neq 0$.

For example, $\phi(t) = \mu t$, where $0 \leq \mu < 1$, $\phi(t) = \dfrac{t}{t+1}$ and $\phi(t) = \ln(1+t)$ are in $\mathscr{B}$.

Existence and Uniqueness of the Solution

We will study the existence and uniqueness of the nonlinear functional-integral equation (2.29) on X.

Theorem 2.4.2

Consider the integral equation (2.29) such that

(i) $\varphi : X \to X$ is a bounded linear transformation,

(ii) $K : D \to \mathbb{R}^n$ and $f : [a,b] \to \mathbb{R}^n$ are continuous,

(iii) there exists a integrable function $p : [a,b] \times [a,b] \to \mathbb{R}$ such that

$$|K(x,t,u) - K(x,t,v)| \leq p(x,t)\phi(|u-v|) \tag{2.30}$$

for each $x,t \in [a,b]$ and $u,v \in \mathbb{R}^n$.

(iv) $\sup\limits_{x\in[a,b]} \int\limits_a^b p^2(x,t)\,dt \leq \frac{1}{||\varphi||^2}(b-a)$. Then, the integral equation (2.29) has a unique fixed point u in X.

Proof[8]

Consider the iterative scheme

$$u_{n+1}(x) = f(x) + \varphi\left(\int_a^x K(x,t,u_n(t))\,dt\right) \equiv Tu_n\,,\ n = 0,1,\ldots. \tag{2.31}$$

$$\begin{aligned} |Tu_n(x) - Tu_{n-1}(x)| &= \left|\varphi\left(\int_a^x K(x,t,u_n(t))dt\right) - \varphi\left(\int_a^x K(x,t,u_{n-1}(t))dt\right)\right| \\ &\le \left|\varphi\left(\int_a^x K(x,t,u_n(t)) - K(x,t,u_{n-1}(t))dt\right)\right| \\ &\le \|\varphi\| \left|\int_a^x K(x,t,u_n(t)) - K(x,t,u_{n-1}(t))dt\right| \\ &\le \|\varphi\| \int_a^x |K(x,t,u_n(t)) - K(x,t,u_{n-1}(t))| dt \\ &\le \|\varphi\| \int_a^b p(x,t)\phi(|u_n(t) - u_{n-1}(t)|)dt \\ &\le \|\varphi\| \left(\int_a^b p^2(x,t)dt\right)^{\frac{1}{2}} \left(\phi^2(|u_n(t) - u_{n-1}(t)|)dt\right)^{\frac{1}{2}} \end{aligned} \tag{2.32}$$

As the function ϕ is increasing then

$$\phi(|u_n(t) - u_{n-1}(t)|) \le \phi(d(u_n, u_{n-1})), \tag{2.33}$$

so, we obtain

$$d^2(u_{n+1}, u_n) \le \|\varphi\|^2 \left(\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt\right)\left(\int_a^b \phi^2(d(u_n, u_{n-1}))dt\right) \le \phi^2(d(u_n, u_{n-1})). \tag{2.34}$$

Therefore,

$$du_n + 1, u_n) \le \phi(d(u_n, u_{n-1})) = \frac{\phi(d(u_n, u_{n-1}))}{d(u_n, u_{n-1})} d(u_n, u_{n-1}) = \alpha(d(u_n, u_{n-1}))d(u_n, u_{n-1}), \tag{2.35}$$

and so the sequence $d(u_{n+1}, u_n)$ is nonincreasing and bounded below. Thus, there exists $\tau \ge 0$ such that $\lim_{n\to\infty} d(u_{n+1}, u_n) = \tau$. Since $\limsup_{s\to\tau} \alpha(s) < 1$ and $\alpha(\tau) < 1$, then there exist $r \in [0,1)$ and $\epsilon > 0$ such that $\alpha(s) < r$ for all $s \in [\tau, \tau + \epsilon]$. We can take $v \in \mathbb{N}$ such that $\tau \le d(u_{n+1}, u_n) \le \tau + \epsilon$ for all $n \in \mathbb{N}$ with $n \ge v$. On the other hand, we have

$$d(u_{n+2}, u_{n+1}) \le \alpha(d(u_{n+1}, u_n))d(u_{n+1}, u_n) \le rd(u_{n+1}, u_n) \tag{2.36}$$

for all $n \in \mathbb{N}$ with $n \ge v$. It follows that

$$\sum_{n=1}^{\infty} d(u_{n+1}, u_n) \le \sum_{n=1}^{v} d(u_{n+1}, u_n) + \sum_{n=1}^{\infty} r^n d(u_{v+1}, u_v) \le \infty, \tag{2.37}$$

and hence, $\{u_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there exists a $u \in X$ such that $\lim_{n\to\infty} u_n = u$. Now, by taking the limit of both sides of (2.31), we have

$$\begin{aligned} u = \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \left(f(x) + \varphi\left(\int_a^x K(x,t,u_n(t))dt\right)\right) \\ &= f(x) + \varphi\left(\int_a^x K\left(x,t,\lim_{n\to\infty} u_n(t)\right)dt\right) \\ &= f(x) + \varphi\left(\int_a^x K(x,t,u(t))dt\right). \end{aligned} \tag{2.38}$$

So, there exists a solution $u \in X$ such that $Tu = u$. It is clear that the fixed point of T is unique. □

Example 2.3

Consider the following linear Volterra integral equation:

$$u(x) = \ln(x^2) + \int_0^x txu(t)dt, \ \ (x, t \in [0, 1])$$

We have

$$\begin{aligned} |K(x,t,u) - K(x,t,v)| &= |(tx)u - (tx - v| = |(tx)(u - v)| \\ &\leq |tx||u - v| = \left| \left(\frac{|tx|}{\mu} \right) (\mu|u - v|) \right| \end{aligned}$$

where $\frac{\sqrt{3}}{3} \leq \mu < 1$. Now, we put $p(x,t) = \dfrac{(tx)}{\mu}$ and $\phi(t) = \mu t$. Because $\sup\limits_{x\in[0,1]} \int\limits_0^1 p^2(x,t)dt = \dfrac{1}{3\mu^2} \leq 1$, then by applying the result obtained in (2.4.2), we deduce that integral equation has a unique solution in Banach space $C([0,1],\mathbb{R})$.

Example 2.4

Consider the following nonlinear Volterra integral equation:

$$u(x) = \sin\left(\frac{1}{1+x}\right) + \frac{x}{9}\int_0^x \frac{\cos(x^2 t)}{(1+xt)^2} \arctan(u(t))dt, \ \ (x, t \in [0,1])$$

We write

$$\begin{aligned} |K(x,t,u) - K(x,t,v)| &= \left| \frac{x\cos(x^2 t)}{9(1+xt)^2} (\arctan(u) - \arctan(v)) \right| \\ &\leq \left| \frac{x\cos(x^2 t)}{(1+xt)^2} \right| \left| \frac{u - v}{9} \right| \end{aligned}$$

Take $p(x,t) = \dfrac{x\cos(x^2 t)}{(1+xt)^2}$ and $\varphi(t) = \dfrac{t}{9}$. Since $\sup\limits_{x\in[0,1]} \int\limits_0^1 p^2(x,t)dt \leq 1$, then integral equation has a unique solution in $C([0,1],\mathbb{R})$.

2.5 Volterra integral equations and linear differential equations

There is a fundamental relationship between Volterra integral equations and ordinary differential equations. In fact, the solution of any differential equation of the type

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_n(x)y = F(x) \tag{2.39}$$

with continuous coefficients, together with the initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, ..., y^{(n-1)}(0) = c_{n-1} \tag{2.40}$$

can be reduced to the solution of a certain Volterra integral equation of the second kind

$$u(x)+\int_0^x K(x,t)u(t)dt=f(x) \tag{2.41}$$

In order to achieve this, let us make the transformation

$$\frac{d^n y}{dx^n}=u(x) \tag{2.42}$$

Hence integrating with respect to x from 0 to x

$$\frac{d^{n-1}y}{dx^{n-1}}=\int_0^x u(t)dt+c_{n-1}$$

Thus, the successive integrals are

$$\begin{aligned}
\frac{d^{n-2}y}{dx^{n-2}} &= \int_0^x\int_0^x u(t)dtdt+c_{n-1}x+c_{n-2}\\
\frac{d^{n-3}y}{dx^{n-3}} &= \int_0^x\int_0^x\int_0^x u(t)dtdtdt+c_{n1}\frac{x^2}{2!}+c_{n-2}x+c_{n-3}\\
\ldots\ldots &= \ldots\ldots
\end{aligned}$$

Proceeding in this manner we obtain

$$\begin{aligned}
y(x) = & \underbrace{\int_0^x\int_0^x\int_0^x\ldots\int_0^x}_{n-foldintegration} u(t)dtdtdt\ldots dt\\
& +c_{n-1}\frac{x^{n-1}}{(n-1)!}+c_{n-2}\frac{x^{n-2}}{(n-2)!}+\ldots+c_1x+c_0\\
= & \int_0^x\frac{(x-t)^{n-1}}{(n-1)!}u(t)dt+c_{n-1}\frac{x^{n-1}}{(n-1)!}+c_{n-2}\frac{x^{n-2}}{(n-2)!}+\ldots+c_1x+c_0
\end{aligned} \tag{2.43}$$

Returning to the differential equation (2.39), we see that it can be written as

$$\begin{aligned}
u(x)+ & \int_0^x\left\{a_1(x)+a_2(x)(x-t)+a_3(x)\frac{(x-t)^2}{2!}+\ldots+a_n(x)\frac{(x-t)^{n-1}}{(n-1)!}\right\}u(t)dt\\
= & F(x)-c_{n-1}a_1(x)-(c_{n-1}+c_{n-2})a_2(x)-\ldots\\
& -(c_{n-1}\frac{(x)^{n-1}}{(n-1)!}+\ldots+c_1x+c_0)a_n(x)
\end{aligned}$$

and this reduces to

$$u(x)+\int_0^x K(x,t)u(t)dt=f(x)$$

where

$$K(x,t)=\sum_{\upsilon=1}^{n}a(x)\frac{(x-t)^{\upsilon-1}}{(\upsilon-1)!} \tag{2.44}$$

and

$$f(x)=F(x)-c_{n-1}a_1(x)-(c_{n-1}+c_{n-2})a_2(x)-\ldots-\left(c_{n-1}\frac{(x)^{n-1}}{(n-1)!}+\ldots+c_1x+c_0\right)a_n(x). \tag{2.45}$$

Conversely, the solution, i.e. equation (2.41) with K and f given by equation (2.44) and (2.45) and substituting values for $u(x)$ in the last equation of equation (2.43), we obtain the (unique) solution of equation (2.39) which satisfies the initial conditions (equation (2.40). If the leading coefficient in equation (2.39) is not unity but $a_0(x)$, equation (2.41) becomes

$$a_0(x)u(x) + \int_0^x K(x,t)u(t)dt = f(x) \tag{2.46}$$

where K and f are still given by equation (2.44) and (2.45), respectively.

Remark 2.5.1

If $a_0(x) \neq 0$ in the interval considered, nothing is changed; however, if $a_0(x)$ vanishes at some point, we see that an equation of the type (2.46) is equivalent to a singular differential equation, at least, when $K(x,t)$ is a polynomial in t.

Example 2.5

Reduce the initial value problem

$$y''(x) + 4y(x) = sinx; \quad at \quad x = 0, y(0) = 0, y'(0) = 0$$

to Volterra integral equation of the second space .

Solution

Let us consider

$$\begin{aligned} y''(x) &= u(x) \\ y'(x) &= \int_0^x u(t)dt \\ y(x) &= \int_0^x \int_0^x u(t)dt^2 \\ &= \int_0^x (x-t)u(t)dt \end{aligned}$$

Then the given ODE becomes

$$u(x) + 4\int_0^x (x-t)u(t)dt = sinx$$

Chapter 3

Solution of Singular Integral Equations Involving Logarithmically Singular Kernel

3.1 Solution of the Integral Equation (1)

In this section, we present a method of solution [4] of

$$\int_B \phi(t)\log\left|\frac{t-x}{t+x}\right| dt = f(x), \quad x \in B \tag{3.1}$$

Case(i)

Consider that

$$B = (0,a)\cup(b,c) \tag{3.2}$$

and

$$\phi(t) \sim O(|t-p|)^{-1/2} \text{ as } t \longrightarrow p \tag{3.3}$$

p being a, b, or c. We introduce a sectionally analytic function $F(z)$ in the complex $(z = x+iy)$ plane, cut along the segment $(-b,-c)\cup(-a,0)\cup(0,a)\cup(b,c)$ on the real axis as given by

$$F(z) = \int_B \phi(t)\ln\left(\frac{t-z}{t+z}\right) dt, \quad (z = x+iy, x \in B). \tag{3.4}$$

If we use the following results, valid for the limiting values of $\Phi(z) = \log(z) \equiv \log(x+iy)$:

$$\Phi^{\pm}(x) \sim \begin{cases} \ln|x| & \text{for } x > 0 \\ \ln|x| \pm \pi i & \text{for } x < 0 \end{cases} \tag{3.5}$$

we find the following limiting values of $F(z)$:

$$F^{\pm}(x) = \begin{cases} \int_B \phi(t)\ln\left|\frac{t-x}{t+x}\right| dt \mp \pi i \int_0^x \phi(t)dt, & x \in (0,a), \\ \int_B \phi(t)\ln\left|\frac{t-x}{t+x}\right| dt \mp \pi i \int_0^a \phi(t)dt \mp \pi i \int_b^x \phi(t)dt, & x \in (b,c), \\ \int_B \phi(t)\ln\left|\frac{t-x}{t+x}\right| dt \pm \pi i \int_0^{-x} \phi(t)dt, & x \in (-a,0), \\ \int_B \phi(t)\ln\left|\frac{t-x}{t+x}\right| dt \pm \pi i \int_0^{-x} \phi(t)dt, & x \in (-c,-b). \end{cases} \tag{3.6}$$

Equations (3.6) give rise to the following Plemelj-type alternative formulae:

$$F^{+}(x) - F^{-}(x) = r(x), \tag{3.7}$$

where

$$r(x)=\begin{cases}-2\pi i\int_0^x\phi(t)dt, & x\in(0,a),\\ -2\pi i\int_0^a\phi(t)dt-2\pi i\int_b^x\phi(t)dt, & x\in(b,c),\\ 2\pi i\int_0^{-x}\phi(t)dt, & x\in(-a,0),\\ 2\pi i\int_0^a\phi(t)dt+2\pi i\int_b^{-x}\phi(t)dt, & x\in(-c,-b),\end{cases}\tag{3.8}$$

where $B'=(-c,-b)\cup(-a,0)$. By using the relation (3.8), the singular integral equation (3.1) can be expressed in the form of a Riemann-Hilbert problem as given by

$$F^+(x)+F^-(x)=2h(x),\quad x\in\left(B\cup B'\right),\tag{3.9}$$

where

$$h(x)=\begin{cases}f(x), & x\in B\\ -f(-x), & x\in B'\end{cases}\tag{3.10}$$

Again noting the relation (3.6), we can write

$$r'(x)=\begin{cases}-2\pi i\phi(x), & x\in B\\ -2\pi i\phi(-x), & x\in B'\end{cases}\tag{3.11}$$

Hence,

$$\phi(x)=-\frac{1}{2\pi i}r'(x),\quad x\in B.\tag{3.12}$$

Using the relation (3.12) in (3.4), we get the solution of special Riemann-Hilbert problem (3.7) as

$$F(z)=-\frac{1}{2\pi i}\int_B r'(t)\ln\left(\frac{t-z}{t+z}\right)dt.\tag{3.13}$$

using the above idea, we can solve the Riemann-Hilbert problem (3.9) in the following manner. Let

$$F(z)=F_0(z)F_2(z),\tag{3.14}$$

where $F_0(z)$ is the solution of the corresponding homogeneous problem (3.9), that is,

$$F_0^+(x)-F_0^-(x)=0,\tag{3.15}$$

Thus, (3.9) becomes

$$F_2^+(x)-F_2^-(x)=\frac{2h(x)}{F_0^+(x)},\quad x\in B\cup B'.\tag{3.16}$$

This is a Riemann-Hilbert problem which is similar to (3.7). Noting (3.7), (3.11), and (3.13), the solution of this Riemann-Hilbert problem is given by

$$F_2(z) = \frac{F(z)}{F_0(z)} = -\frac{1}{2\pi i}\int_B \left(\left(\frac{d}{dt}\frac{2f(t)}{F_0^+(t)}\right)\ln\left(\frac{t-z}{t+z}\right)\right)dt + E(z), \tag{3.17}$$

where $E(z)$ is an entire function. We can choose

$$F_0(z) = \left[z^2\left(z^2-a^2\right)\left(z^2-b^2\right)\left(z^2-c^2\right)\right]^{-1/2} \tag{3.18}$$

with

$$F_0^{\pm}(x) = \begin{cases} \pm\frac{i}{\rho(x)}, & x \in (0,a) \\ \mp\frac{i}{\rho(x)}, & x \in (b,c) \end{cases} \tag{3.19}$$

where

$$\rho(x) = \left|x^2\left(x^2-a^2\right)\left(x^2-b^2\right)\left(x^2-c^2\right)\right|^{1/2} \tag{3.20}$$

Knowing $F(z)$ from (3.17),(3.18), and (3.19), we can find the limiting values $F^{\pm}(x)$ of $F(z)$ using the formula (3.5). Substituting $F^{\pm}(x)$ in (3.7) and noting the relation (3.8), the solution of the integral equation (3.1) can be obtained as

$$\phi(x) = \begin{cases} \frac{d}{dx}\left(\frac{\Psi(x)}{\rho(x)}\right), & x \in (0,a) \\ -\frac{d}{dx}\left(\frac{\Psi(x)}{\rho(x)}\right), & x \in (b,c) \end{cases} \tag{3.21}$$

where

$$\begin{aligned}\psi(x) = &-\frac{1}{\pi^2}\int_0^a \frac{d}{dt}(f(t)\rho(t))\ln\left|\frac{t-x}{t+x}\right|dt + \frac{1}{\pi^2}\int_b^c \frac{d}{dt}(f(t)\rho(t))\ln\left|\frac{t-x}{t+x}\right|dt \\ &-\pi\left(Ax^4+Bx^3+Cx^2+Dx+E\right)\end{aligned} \tag{3.22}$$

and A, B, C, D, and E are arbitrary constants. It is observed here that the original integral equation (3.1) can be solved by the aid of formula (3.21) if and only if the following consistency conditions are satisfied:

$$(i)\psi(0) = 0,\ \ (ii)\psi'(0) = 0,\ \ (iii)\psi(a) = 0,\ \ (iv)\psi(b) = 0,\ \ (v)\psi(c) = 0 \tag{3.23}$$

We can determine the arbitrary constants A, B, C, D, and E by using the relation (3.23). Thus, $\phi(x)$ can be determined completely by using the relation (3.21). It is observed that the solution as obtained by the relations (3.21) and (3.22) of (3.1) neither involves any strong singularity nor any arbitrary constant.

Case(ii)

Consider that

$$\mathbf{B} = B_1 = \cup_{i=1}^{n}(a_i, b_i) \tag{3.24}$$

In this section, we proceed to solve the integral equation

$$\int_{B_1} \phi_1(t) \ln\left|\frac{t-x}{t+x}\right| dt = f_1(x), \quad x \in B_1, \tag{3.25}$$

where B_1 consists of n finite disjoint intervals $B_1 = \cup_{i=1}^{n}(a_i, b_i)$ and $\phi_1(t) \sim O\left(\left|t-p_i\right|\right)^{-1/2}$ as $t \to p_i, p_i$ being $a_i, b_i, i = 1, \ldots, n$.

In this case, we introduce the following sectionally analytic function

$$F_1(z) = \int_{B_1} \phi_1(t) \ln\left(\frac{t-z}{t+z}\right) dt, \quad \left(z = x+iy, x \in B_1\right), \tag{3.26}$$

cut along $\cup_{i=1}^{n}(-b_i, -a_i) \cup_{i=1}^{n}(a_i, b_i)$. Using the formula (3.5), we get following limiting values of $F(z)$:

$$F_1^{\pm} = \begin{cases} \int_{B_1} \phi_1(t) \ln\left|\frac{t-x}{t+x}\right| dt \mp \pi i \int_{a_1}^{x} \phi_1(t) dt, & x \in (a_1, b_1), \\ \int_{B_1} \phi_1(t) \ln\left|\frac{t-x}{t+x}\right| dt \mp \pi i \int_{a_1}^{b_1} \phi_1(t) dt \mp \pi i \int_{a_2}^{x} \phi_1(t) dt, & x \in (a_2, b_2), \\ \int_{B_1} \phi_1(t) \ln\left|\frac{t-x}{t+x}\right| dt \mp \pi i \int_{a_1}^{b_1} \phi_1(t) dt \mp \pi i \int_{a_{n-1}}^{b_{n-1}} \phi_1(t) dt \mp \pi i \int_{a_n}^{x} \phi_1(t) dt, & x \in (a_n, b_n), \\ \int_{B_1} \phi_1(t) \ln\left|\frac{t-x}{t+x}\right| dt \pm \pi i \int_{a_1}^{-x} \phi_1(t) dt, & x \in (-b_1, -a_1,), \\ \int_{B_1} \phi_1(t) \ln\left|\frac{t-x}{t+x}\right| dt \pm \pi i \int_{a_1}^{b_1} \phi_1(t) dt \pm \pi i \int_{a_2}^{-x} \phi_1(t) dt, & x \in (-b_2, -a_2), \\ \int_{B_1} \phi_1(t) \ln\left|\frac{t-x}{t+x}\right| dt \pm \pi i \int_{a_1}^{b_1} \phi_1(t) dt \pm \pi i \int_{a_{n-1}}^{b_{n-1}} \phi_1(t) dt \pm \pi i \int_{a_n}^{x} \phi_1(t) dt, & x \in (-b_n, -a_n). \end{cases} \tag{3.27}$$

This gives the following Plemelj-type alternative formulae for the integral equation (3.25):

$$F_1^{+}(x) - F_1^{-}(x) = r_1(x) \tag{3.28}$$

where

$$r_1(x) = \begin{cases} -2\pi i \int_{a_1}^{x} \phi_1(t)dt, & x \in (a_1, b_1) \\ -2\pi i \int_{a_1}^{b_1} \phi_1(t)dt - 2\pi i \int_{a_2}^{x} \phi_1(t)dt, & x \in (a_2, b_2) \\ -2\pi i \cup_{j=1}^{(n-1)} \int_{a_f}^{b_j} \phi_1(t)dt - 2\pi i \int_{a_n}^{x} \phi_1(t)dt, & x \in (a_n, b_n) \\ 2\pi i \int_{a_1}^{-x} \phi_1(t)dt, & x \in (-b_1, -a_1) \\ 2\pi i \int_{a_1}^{b_1} \phi_1(t)dt + 2\pi i \int_{a_2}^{-x} \phi_1(t)dt, & x \in (-b_2, -a_2) \\ 2\pi i \bigcup_{j=1}^{b_j} \int_{a_1}^{b_j} \phi_1(t)dt - 2\pi i \int_{a_n}^{-x} \phi_1(t)dt, & x \in (-b_n, -a_n) \end{cases} \tag{3.29}$$

$$F_1^+(x) + F_1^-(x) = \begin{cases} 2f_1(x), & x \in B_1, \\ -2f_1(-x), & x \in B_1' \equiv \cup_{i=1}^{n} (-b_i - a_i). \end{cases} \tag{3.30}$$

Hence, from the relation (3.29), we get

$$\phi_1(x) = -\frac{1}{2\pi i} r_1'(x), \quad x \in B_1. \tag{3.31}$$

Thus, noting the relation (3.31), the equation (3.26) becomes

$$F_1(z) = -\frac{1}{2\pi i} \int_{B_1} r_1'(t) \ln\left(\frac{t-z}{t+z}\right) dt \tag{3.32}$$

using the above idea, we can solve the Riemann-Hilbert problem (3.30) in the following form:

$$\frac{F_1(z)}{F_1^{(0)}(z)} = -\frac{1}{2\pi i} \int_{B_1} \left(\left(\frac{d}{dt} \frac{2f_1(t)}{F_1^{(0)+}(t)} \right) \ln\left(\frac{t-z}{t+z}\right) \right) dt + E_1(z) \tag{3.33}$$

where $E_1(z)$ is an entire function, and we can choose $F_1^{(0)}(z)$, the solution of the homogeneous equation of (3.30) as

$$F_1^{(0)}(z) = \prod_{i=1}^{n} \left[\left(z^2 - a_i^2\right)\left(z^2 - b_i^2\right) \right]^{-1/2} \tag{3.34}$$

with

$$F_1^{(0)\pm}(x) = \begin{cases} \pm\frac{(-1)^n i}{\rho_1(x)}, & x \in (a_1, b_1) \\ \mp\frac{(-1)^n i}{\rho_1(x)}, & x \in (a_2, b_2) \\ \pm\frac{(-1)^n i}{\rho_1(x)}, & x \in (a_3, b_3) \\ \vdots & \\ \mp\frac{i}{\rho_1(x)}, & x \in (a_n, b_n) \end{cases} \tag{3.35}$$

where

$$\rho_1(x) = \prod_{i=1}^{n} \left[\left(x^2 - a_i^2\right)\left(x^2 - b_i^2\right)\right]^{1/2}. \tag{3.36}$$

Using (3.34) in (3.33), $F_1(z)$ is known. Noting the relation (3.5), one can find the limiting values $F_1^{\pm}(x)$ of $F_1(z)$. Using the limiting values $F_1^{\pm}(x)$, we obtain, from (3.28) and (3.29), the solution $\phi_1(x)$ of the integral equation (3.25) as

$$\phi_1(x) = \begin{cases} \frac{d}{dx}\left(\frac{\psi_1(x)}{\rho_1(x)}\right), & x \in (a_1, b_1) \\ \frac{d}{dx}\left(\frac{\psi_2(x)}{\rho_1(x)}\right), & x \in (a_2, b_2) \\ \vdots \\ \frac{d}{dx}\left(\frac{\psi_n(x)}{\rho_1(x)}\right), & x \in (a_n, b_n) \end{cases} \tag{3.37}$$

where

$$\begin{aligned} \psi_1(x) &= \frac{1}{\pi^2} \sum_{i=1}^{n} \int_{a_i}^{b_1} (-1)^i \frac{d}{dt}\left(f_1(t)\rho_1(t)\right) \ln\left|\frac{t-x}{t+x}\right| dt - \pi(-1)^n \sum_{i=1}^{2n+1} A_i x^{(2n+1-i)}, \\ \psi_2(x) &= \frac{1}{\pi^2} \sum_{i=1}^{n} \int_{a_t}^{b_1} (-1)^{i+1} \frac{d}{dt}\left(f_1(t)\rho_1(t)\right) \ln\left|\frac{t-x}{t+x}\right| dt - \pi(-1)^n \sum_{i=1}^{2n+1} A_i x^{(2n+1-i)}, \\ &\vdots \\ \psi_n(x) &= \frac{1}{\pi^2(-1)^n} \sum_{i=1}^{n} \int_{a_t}^{b_1} (-1)^{i+1} \frac{d}{dt}\left(f_1(t)\rho_1(t)\right) \ln\left|\frac{t-x}{t+x}\right| dt + \pi \sum_{i=1}^{2n+1} A_i x^{(2n+1-i)}. \end{aligned} \tag{3.38}$$

We observe that the original integral equation (3.25) can be solved by using the above relations if and only if the following consistency conditions are satisfied:

$$\psi_1(a_i) = \psi_1(b_i) = 0, \quad i = 1, \ldots, n. \tag{3.39}$$

From these $2n$ conditions, we find the arbitrary constant $\prod_{i=2}^{2n+1} A_i$, and from the order condition, the arbitrary constant A_1 can be determined, and finally the solution of the singular integral equation (3.25) can be obtained from (3.37).

In the next section, we give the genesis of the integral equation (3.1) with $B = (0, a) \cup (b, c)$ and use its solution to the problem of scattering of water waves by a partially immersed vertical barrier with gap.

Chapter 4

Genesis of the Integral Equation (3.1) Connection with a Water Wave Problem

4.1 The Detailed Analysis and the Analytical Solution

In this section, we show the connection of integral equation (3.1) with the problem of scattering of surface water waves by a thin partially immersed vertical barrier with a gap [4]

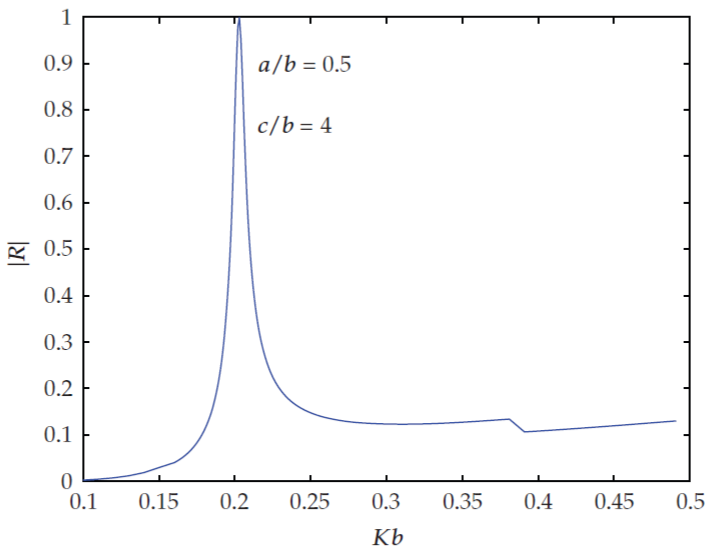


Figure 1

in deep water occupying the position $x = 0, 0 < y < a$, and $b < y < c$, y-axis being directed downwards and x-axis along the mean free surface. We assume the motion to be two dimensional, irrotational, and time harmonic. The velocity potential is represented by $\mathrm{Re}(\phi(x, y)\exp(-i\sigma t))$, σ being circular frequency. Here, $\phi(x, y)$ satisfies the following boundary value problem:

$$\begin{aligned} \nabla^2\phi = 0 \quad &\text{in } y \geq 0 \\ K\Phi + \phi_y = 0 \quad &\text{on } y = 0 \end{aligned} \tag{4.1}$$

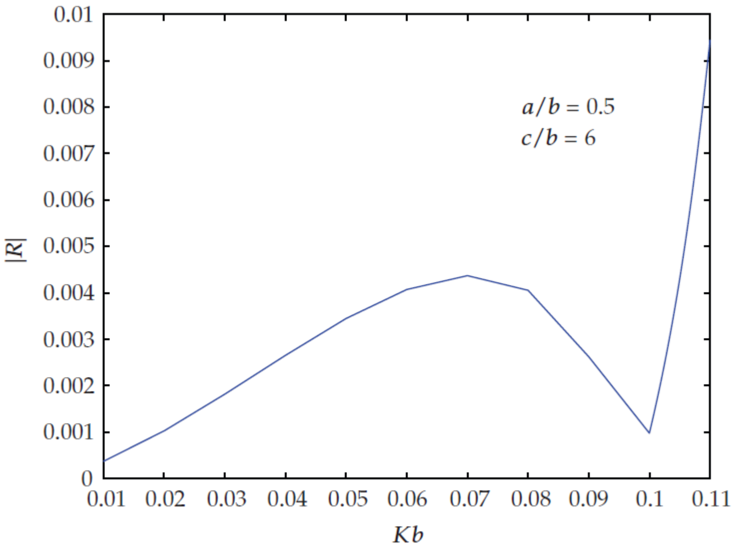
where $K = \sigma^2/g$, g being the acceleration of gravity,

$$\phi_x = 0 \quad \text{on } x = 0, \quad y \in B \equiv (0, a) \cup (b, c), \tag{4.2}$$

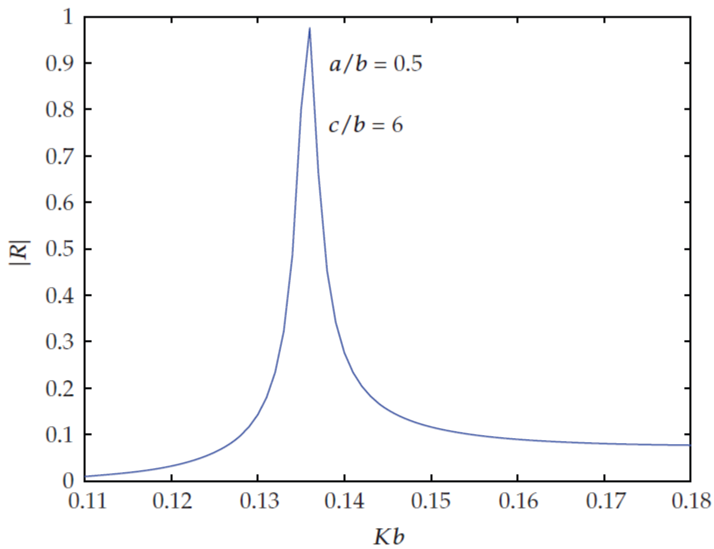
$$\begin{aligned} r^{1/2}\nabla\phi is\ bounded\ as r \to 0 \\ r = \left\{(x)^2 + (y - v)^2\right\}^{1/2}, \quad v = a, borc \end{aligned} \tag{4.3}$$

$$\nabla\phi \to 0 as y \to \infty, \tag{4.4}$$

$$\phi(x, y) = \begin{cases} R\exp(-Ky - iKx) + \exp(-Ky + iKx) & \text{as } x \longrightarrow -\infty \\ T\exp(-Ky + iKx) & \text{as } x \to \infty \end{cases} \tag{4.5}$$



(a)



(b) **Figure** 2

where R and T are reflection and transmission coefficients. A suitable representation of ϕ satisfying the equations (4.1),(4.4), and (4.5) is given by

$$\phi(x,y) = \begin{cases} \text{Rexp}(-Ky - iKx) + \exp(-Ky + iKx) + \int_0^\infty B(k)L(k,y)\exp(kx)\,dk, & x < 0, \\ T\exp(-Ky + iKx) + \int_0^\infty A(k)L(k,y)\exp(-kx)\,dk, & x > 0, \end{cases} \tag{4.6}$$

where,

$$L(k,y) = k\cos ky - K\sin ky. \tag{4.7}$$

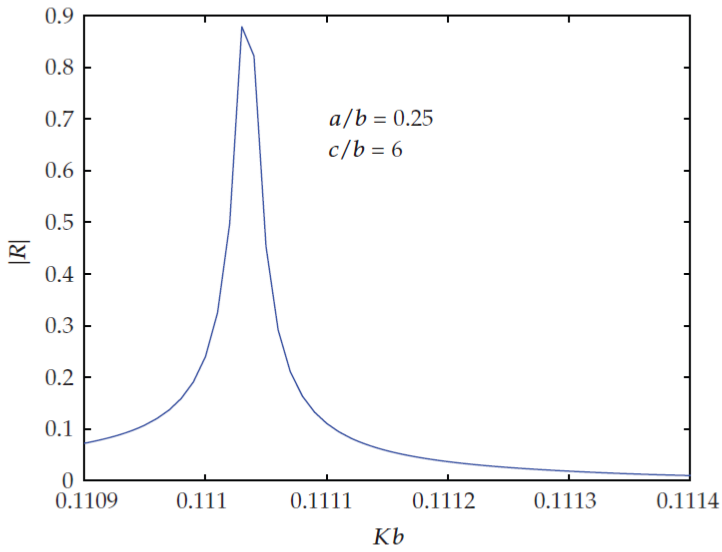


Figure 3

By Havelock's expansion theorem,

$$A(k) = -B(k), \quad R = 1 - T \tag{4.8}$$

Then, using the condition (4.2) along with the condition of continuity of ϕ across the gap $G : (a, b) \cup (c, \infty)$, along $x = 0$, we arrive at the following multiple integral equations:

$$\int_0^\infty kA(k)L(k, y)dk = iK(1 - R)\exp(-Ky), \quad y \in B \tag{4.9}$$

$$\int_0^\infty A(k)L(k, y)dk = R\exp(-Ky), \quad y \in G. \tag{4.10}$$

The Relation (4.9) can be recast in the form

$$\frac{d}{dy}\int_0^\infty \frac{A(k)}{k}L(k, y)dk = \begin{cases} i(1 - R)\exp(-Ky) + D_1, & 0 < y < a \\ i(1 - R)\exp(-Ky) + D_2, & b < y < c \end{cases} \tag{4.11}$$

where D_1 and D_2 are two arbitrary constants. Then, applying $d/dy + K$ to the equations (4.10) and (4.11), we obtain

$$\int_0^\infty F(k)\sin kydk = 0, \quad y \in G, \tag{4.12}$$

$$\int_0^\infty \frac{F(k)}{k} \sin kydk = \begin{cases} KD_1 y + C_1, & 0 < y < a \\ KD_2 y + C_2, & b < y < c \end{cases} \tag{4.13}$$

where

$$F(k) = -\left(k^2 + K^2\right) A(k) \tag{4.14}$$

It is observed from the relation (4.13) that we must choose $C_1 = 0$ for consistency. Setting

$$\int_0^\infty F(k) \sin kydk = g(y), \quad y \in B, \tag{4.15}$$

where $g(y)$ is an unknown function and using sine inversion formula, to the relations (4.12) and (4.15), we obtain from (4.13)

$$\int_B g(t) \log \left| \frac{y-t}{y+t} \right| = f(y), \quad y \in B, \tag{4.16}$$

with

$$f(y) = \begin{cases} -\pi KD_1 y, & 0 < y < a, \\ -\pi KD_2 y - \pi C_2, & b < y < c. \end{cases} \tag{4.17}$$

The solution of the integral equation (4.16) is given by the equations (3.21), (3.22), and (3.23) with $f(y)$ given by relation (4.17).

Thus, knowing $g(t)$, we can obtain $F(k)$ by using Fourier Sine inversion formulae to the relations (4.12) and (4.15). Hence, $A(k)$ can be obtained from equation (4.14). Now, $f(y)$ contains unknown constants D_1, D_2, and C_2. To obtain the unknowns D_1, D_2, C_2, and R, we substitute $A(k)$ in (4.10) and (4.11) to get the following four equations:

$$
\begin{aligned}
&\int_b^c \frac{\Psi}{\rho} e^{Kt} dt = 0 \\
&K\left[-\int_0^a \frac{\Psi}{\rho} e^{-Kt} dt + \int_b^c \frac{\Psi}{\rho} e^{-Kt} dt - \int_0^a \frac{\Psi}{\rho} e^{Kt} dt\right] = R \\
&-K\int_a^b \frac{\Psi}{\rho} e^{Kt} dt - K^2 D_1 \int_0^a x e^{Kx} + e^{Kb}\left(KbD_2 - D_2 + C_2\right) + D_1 = 0 \\
&K\int_a^b \frac{\Psi}{\rho} e^{-Kt} dt - K\int_c^\infty \frac{\Psi}{\rho} e^{-Kt} dt - \frac{K}{\pi}\int_0^a \left(-\pi K x D_1\right) e^{-Kx} \\
&-\frac{K}{\pi}\int_b^c \left(-\pi K x D_2 - \pi C_2\right) e^{-Kx} - D_1 = i(1-R)
\end{aligned}
\tag{4.18}
$$

where

$$
\begin{aligned}
\psi(x) = &-\frac{1}{\pi^2}\int_0^a \frac{d}{dt}(f(t)\rho(t)) \ln\left|\frac{t-x}{t+x}\right| dt + \frac{1}{\pi^2}\int_b^c \frac{d}{dt}(f(t)\rho(t)) \ln\left|\frac{t-x}{t+x}\right| dt \\
&-\pi\left(Ax^4 + Bx^3 + Cx^2 + Dx + E\right),
\end{aligned}
\tag{4.19}
$$

and A, B, C, D, and E are arbitrary constants and

$$
\rho(x) = \left|x^2\left(x^2-a^2\right)\left(x^2-b^2\right)\left(x^2-c^2\right)\right|^{1/2} \tag{4.20}
$$

Thus, the relations (4.18) can be solved to determine four unknowns D_1, D_2, C_2, and R. This completes the description of the analytical solution of the water wave problem.

Conclusion

A direct function theoretic method is applied here to solve the integral equations (3.1) and (3.25) without converting them to the integral equation with strong singularity. It may be noted here that the solution functions given in (3.21) and (3.37) do not involve any strong singularity. Also using the presently detailed solution of (3.1), the problem of scattering of water waves by a partially immersed barrier with a gap is examined fully.

Abstract

In this research paper,we have dealt with Solution and application of Singular Integral Equations Involving Logarithmically Kernel.This research consists of fort main chapters.The first chapter was about generalities of spaces and operator .The second one covers the Classification ,Existence , Uniqueness and converting of integral equations and the last two chapter tacked solution and application Singular Integral Equations Involving Logarithmically Kernel.Eventually we hope that we succeed in presenting this thesis throughly and simtly.

Key words: spaces, operators , Integral Equations, Existence and Uniqueness , Logarithmically Kernel, Water Wave Problem.

Résume

Dans ce document de recherche, nous avons traité de la solution et de l'application de l'intégrale singulière Équations impliquant un noyau logarithmique.Cette recherche consiste en fort main chapitres. Le premier chapitre portait sur les généralités des espaces et des opérateurs. Le second on couvre la classification, l'existence, l'unicité et la conversion des équations intégrales et les deux derniers chapitres ont abordé la solution et l'application des équations intégrales singulières Impliquant logarithmiquement le noyau. Nous espérons finalement réussir à présenter cette thèse de manière approfondie et simple.

mots clés: espaces, opérateurs, équations intégrales, existence et unicité, Kernel logarithmique, problème d'onde d'eau.

ملخص

في مذكرتنا هذه ، تعاملنا مع حل وتطبيق المعادلة التكاملية الاحادية ذات النواة اللوغرتمية ،هذه المذكرة تحتوي على اربعة فصول أساسية ، كان الفصل الأول عموميات حول الفضاءات والمؤثرات ، اما الفصل الثاني فتطرقنا فيه لتصنيف المعادلات التكاملية ، الوجود والوحدانية ، تحويل المعادلات التكاملية، أما في الفصلية الأخيرين فقد خصصناهما لحل المعادلة التكاملية الأحادية ذات النواة اللوغرتمية وتطبيقها في الطبيعة، في الختام نرجو أن نكون قد وفقنا في طرح وكتابة هذه المذكرة.

الكلمات المفتاحية: الفضاءات، المؤثرات، المعادلات التكاملية، الوجود والوحدانية، النواة اللغريتمية، معادلة الامواج.

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