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Thème

The Maximum Principle of Pontryagin

Présenté par:
NAOURA Abdelbasset
MITTOU Fatma Zohra

Soutenu publiquement devant le jury composé de

Dr.Beloul Said	Président	Univ. El Oued
Dr.Boukaf Samira	Rapporteur	Univ. El Oued
Dr.Touati MohammedSaid	Examinateur	Univ. El Oued

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NOTATIONS

$(\Omega, \mathcal{F}, P)$	probability space.
$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$	filtered probability space .
$\{F_t\}_{t\geq 0}$	filtration .
R^d	real space of dimension d.
N	the set of natural numbers.
I_d	identity matrix $d \times d$.
$R^n \otimes R^d$	the set of real matrices $n \times d$.
$B^j(t)$	j^{th}component of $B(t)$.
$\sigma^j(t, X)$	j^{th}column of $\sigma(t, X)$.
U	set of admissible controls. .
A	borelian of R^d.
$H(t, X_t, u_t, p_t)$	Hamiltonian.
g_x	the gradient of g to x.

ABSTRACT

In this work, we study a stochastic optimal control problem which consists in minimizing a cost function given by $J(U) = E(g(X_T))$, where X_T is the solution taken at terminal time T of a system governed by the next stochastic differential equation:

$$\begin{cases} dX_t = b\,(t, X_t, u_t)\, dt + \sigma\,(t, X_t)\, dB_t \\ X(0) = x \end{cases}$$

Our goal is to establish the necessary conditions (Principle of the stochastic maximum) optimality for diffusions with coefficients $b(t, ., u_t)$ and $\sigma(t, .)$ differentiable.

RÉSUMÉ

Dans ce travail, nous étudions un probléme de controle optimal stochastique qui consiste à minimiser une fonction coût donnée par $J(U) = E(g(X_T))$, où X_T est la solution prise au temps terminal T d'un système gouverné par l'équation di¤érentielle stochastique suivante :

$$\begin{cases} dX_t = b\left(t, X_t, u_t\right) dt + \sigma\left(t, X_t\right) dB_t \\ X(0) = x \end{cases}$$

Notre objectif est d'établir des conditions nécessaires (Principe du maximum sto- chastique) d'optimalité pour des diffusions avec des coefficients $b(t, ., u_t)$ et $\sigma(t, .)$ différentiables.

Contents

INTRODUCTION

Pontryagin's maximum principle is used in optimal control theory to find the best possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls. It states that it is necessary for any optimal control along with the optimal state trajectory to solve the so-called Hamiltonian system, which is a two-point boundary value problem, plus a maximum condition of the control Hamiltonian. These necessary conditions become sufficient under certain convexity conditions on the objective and constraint functions.

The maximum principle was formulated in 1956 by the Russian mathematician Lev Pontryagin and his students, and its initial application was to the maximization of the terminal speed of a rocket. The result was derived using ideas from the classical calculus of variations. After a slight perturbation of the optimal control, one considers the first-order term of a Taylor expansion with respect to the perturbation; sending the perturbation to zero leads to a variational inequality from which the maximum principle follows.

Our work is divided into three chapters in:

Chapter 1: The aim of this chapter is to study the stochastic differential equations in the regular case, this is the case where the coefficients of drift b and diffusion σ are Lipschitzienne. We start by presenting the main results of the controls stochastic in general.

First, we will list all the mathematical tools that will allow us to better understand the problem which is the principle of the maximum in the regular case. The fundamental Gronwall lemma and some inequalities that we will use throughout this work. The aim of the second section is to introduce the notion of stochastic differential equations; SDE for short as well as to demonstrate the existence and uniqueness theorem.

Chapter 2: This chapter will be organized as follows: In the first section, we will talk about stochastic control problems in the standard form. In the second section, we propose to recall the two essential methods in the study of optimal control systems: The Dynamic programming principle and The maximum principle.

Chapter 3:

In this chapter, we propose to study the mathematical structure for stochastic control problems in the case of differentiable coefficients.

This is formulated as follows:

The first section deals with the study of some properties of controlled equations.

The second section aims to study the linearization of the solution.

The third section is entitled to study the maximum principle.

CHAPTER

I

STOCHASTIC DIFFERENTIAL EQUATIONS

I.1 inequalities

Gronwalls Lemma

Let be ϕ and f two continuous functions on $[0, T]$ non-negative, c_0 a positive constant.

if:

$$\phi(t) \leq c_0 + \int_0^t f\phi(s)ds; 0 \leq t \leq T$$

Then:

$$\phi(t) \leq c_0 e^{\int_0^t f ds}; 0 \leq t \leq T$$

Burkholder-Davis-Gundy inequality

For $m > 0$, it exists C_m as for any stopping time τ we have:

$$E\left[\sup_{t\leq\tau}\left|\int_0^t f(s)dB_s\right|^m\right] \leq C_m E\left[\left(\int_0^t |f(s)|^2 ds\right)^{\frac{m}{2}}\right]$$

Especially for $m = 2$ and $\tau = T$ we have:

$$E\left[\sup_{0\leq t\leq T}\left|\int_0^t f(s)dB_s\right|^2\right] \leq CE\left[\int_0^t |f(s)|^2 ds\right]$$

Cauchy-Schwarz inequality

Let be f, g two functions of Square-integrable, so we have:

$$E(fg) \leq \left(E\left(f^2\right) E\left(g^2\right)\right)^{\frac{1}{2}}$$

I.2 Brownian Motion

Definition 1. *(1-dimensional Brownian motion). A stochastic process $\{B_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called Brownian motion if it satisfies the conditions:*

1.$P(\omega : B_0(\omega) = x) = 1$

2. For any $0 \leq s < t$, the random variable $B_t - B_s$ is normally distributed with mean x and variance $t - s$, i.e. for $a < b$

$$P\left(B_t - B_s \in [a, b]\right) = (2\pi(t-s))^{-1/2} \int_a^b e^{-(y-x)^2/2(t-s)} dy$$

3. B_t has independent increments, i.e. for any $0 \leq t_1 < t_2 < \ldots < t_k$, the random variables

$$B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}},$$

are independent.

4. The sample paths of B_t are continuous for almost all $\omega \in \Omega$, i.e.

$$P(\omega : B(\cdot, \omega) \text{ is continuous }) = 1$$

I.3 Ito's Formula

I.3.1 Ito's Process

We call Ito process, a stochastic process $(X_t)_{t\geq 0}$ are written in the form:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s, \quad \forall 0 \leq t \leq T$$

where X_0 is $\mathcal{F}_0-$measurable, b and σ two progressively measurable processes verifying the conditions $P-p.s$

I.3.2 Ito's Formula

Theorem 1. *Suppose f of class C^2, then:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, \sigma_s^2 ds$$

in differential form:

$$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, d\langle X\rangle_t$$

Theorem 2. *Let be f a function defined on $\mathbf{R}_+ \times \mathbf{R}$ of the class C then:*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{df}{dt}(s, X_s)\, ds + \int_0^t \frac{df}{dx}(s, X_s)\, dX_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dx^2}(s, X_s)\, d\langle X_s\rangle$$

in differential form:

$$df(t, X_t) = \frac{df}{dt}(t, X_t)\, dt + \frac{df}{dx}(t, X_t)\, dX_t + \frac{1}{2}\frac{d^2 f}{dx^2}(t, X_t)\, d\langle X\rangle_t$$

Theorem 3. *Let X and Y be two Itô processes, Let f be a function of $\mathbf{R}^2 \to \mathbf{R}$ of the class C^2, we have:*

$$f(X_t, Y_t) = f(x, y) + \int_0^t \frac{df}{dx}(X_s, Y_s)\, dX_s + \int_0^t \frac{df}{dy}(X_s, Y_s)\, dY_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dx^2}(X_s, Y_s)\, d\langle X\rangle_s$$
$$+ \frac{1}{2}\int_0^t \frac{d^2 f}{dy^2}(X_s, Y_s)\, d\langle Y\rangle_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dxdy}(X_s, Y_s)\, d\langle X, Y\rangle_s.$$

I.4 Stochastic differential equations

The purpose of stochastic differential equations is to provide a mathematical model for a Perturbed Differential Equations by random noise.Leaving of an ordinary differential equation of the form:

$$X'(t) = b(X(t))$$

Or in differential form:

$$dX_l = b\left(X_1\right) dt$$

Such an equation is used to describe the evolution of a physical system. if we take into account the random disturbances, we add a noise term, which will be of the form σdB_t, where dB_t denotes a Brownian motion and σ is for the moment a constant which corresponds to the intensity of the noise. We arrive at a "stochastic" differential equation of the form:

$$dX_t = b\left(X_t\right) dt + \sigma dB_t$$

Or in integral form, the only one that has a mathematical meaning,

$$X_t = x + \int_0^t b\left(X_s\right) ds + \sigma B_l$$

We generalize this equation by allowing σdepending on the state at time t:

$$dX_t = b\left(X_t\right) dt + \sigma\left(X_t\right) dB_t$$

Or again in the integral form

$$X_l = x + \int_0^t b\left(X_s\right) ds + \int_0^t \sigma\left(X_s\right) dB_s$$

Noting that the meaning given to this equation depends on the theory of the integral stochastic which is a very nice mathematical tool. this concept was already used in stochastic control or in finance in particular.

We continue to generalize the equation while allowing b and σ depend on time t, we therefore place ourselves in a vector frame. It is represented in the following form:

$$dX_t = b\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dB_t \text{ for } 0 \leq t \leq T$$

I.4.1 Examples on some SDEs

Example 1. *Let $m = n = 1$ and suppose that g is a continuous function (it is not a random variable)*

The unique solution of:

$$\begin{cases} dX_t = g\left(X_t\right) dB_t \\ X(0) = 1 \end{cases}$$

and :

$$X(t) = e^{-\frac{1}{2}\int_0^t g^2 ds + \int_0^t g dB_s} \text{ for : } 0 \le t \le T$$

$$Y(t) = -\frac{1}{2}\int_0^t g^2 ds + \int_0^t g dB_s \text{who satisfies} Y_t = -\frac{1}{2} g^2 dt + g dB_t.$$

Using the Itô Formula for a function $u(x) = e^x$ gives us:

$$\begin{aligned} du(t, Y_t) &= \frac{du(t, Y_t)}{dt} dt + \frac{du(t, Y_t)}{dx} dY_t + \frac{1}{2}\frac{d^2 u(t.Y_t)}{dx^2} d(Y, Y)_t \\ &= e^Y \left(-\frac{1}{2} g^2 dt + g dB_t + \frac{1}{2} q^2 dt\right) \\ &= g(X_1) dB_t. \end{aligned}$$

I.5 Existence and Uniqueness of Stochastic Differential Equations

I.5.1 Notations and definitions

Let be $(\omega, \mathcal{F}, P)$ a probability space; $(B_t)_{t\ge 0}$ a brownian motion with value in R^d and x is a random variable with value in R^n independent of $(B_t)_{t\ge 0}$ We pose : $\mathcal{F}_t = \sigma(X, B_s, s \le t)$. $T \le 0$ and the functions are measurable and bounded:

$$b : R^n \times [0, T] \to R^n. \tag{I.1}$$

$$\sigma : R^n \times [0, T] \to M^{n\times m}. \tag{I.2}$$

So the problem is to solve the SDE:

$$\begin{cases} dX_t = b(t, X_l) dt + \sigma(t, X_t) dB_t \\ X(0) = x \end{cases} \tag{I.3}$$

which can be written in this integral form:

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s \text{ for } 0 \le t \le T \tag{I.4}$$

such as: The coefficient b is called the drift and σ is called the diffusion.

Definition 2. *Let be d and m positive integers, and Let b and σ two functions locally measurable defined on $R^+ \times R^d$ and values respectively in $M_{d\times n}$ and R^d, where $M_{d\times n}(R)$ designates the set of matrices $d \times m$*

with real coefficients. we notice: $\sigma = (\sigma_{ij})_{1\leq i\leq d.1\leq j\leq m}$ *and* $b = (b_i)_{1\leq i\leq d}$.

The solution of the equation:

$$\begin{cases} dX_t = b\left(t, X_l\right)dt + \sigma\left(t, X_t\right)dB_t \\ X(0) = x \end{cases}$$

such as:

$$X_t = x + \int_0^t b\left(s, X_s\right)ds + \int_0^t \sigma\left(s, X_s\right)dB_s$$

for $0 \leq t \leq Tor, fori \in 1, ..., d,$

$$X_t^i = x^i + \int_0^t b_i\left(s, X_s\right)ds + \sum_{j=1}^m \int_0^t \sigma_{ij}\left(s, X_s\right)dB_s^j, for 0 \leq t \leq T$$

The question is: What conditions should be applied on b *and* σ *to get the existence and uniqueness of a solution of (1.6).*

I.5.2 The Existence and Uniqueness Theorem

Theorem 4. *let* b *and* σ *be a borilien functions . We suppose that there exists a constant* L *such that : for every* $t \in [0;T]$ *, for evry* $x, \hat{x} \in R$*:*

$$|b(t,x) - b(t,\hat{x})| + |\sigma(t,x) - \sigma(t,\hat{x})| \leq L|x - \hat{x}| \tag{I.5}$$

$$|b(t,x)| + \|\sigma(t,x)\| \leq L(1 + |x|) \tag{I.6}$$

$$E\left(|X(0)|^2\right) < +\infty \tag{I.7}$$

Then there exists a unique solution X *of the SDE*

$$\begin{cases} dX_t = b\left(t, X_t, u_t\right)dt + \sigma\left(t, X_t\right)dB_t \\ X(0) = x \end{cases}$$

Proof 1. *Let us denote by* H^2 *Banach space made up of processes* X_t *progressively measurable, such as:*

$$E\left(\sup_{0\leq t\leq T} |X_t|^2\right) < +\infty$$

provided with the norm:

$$\|X\| = \left[E\left(\sup_{0\leq t\leq T} |X_t|^2\right) < +\infty\right]^{\frac{1}{2}}$$

We denote by H_c^2 the subspace of H^2 formed by continuous processes.

Existence, we build the solution by the approximation method of Picard, We pose:

$$X_t^0 = x$$

$$X_t^1 = x + \int_0^t \sigma(s,x)dB_s + \int_0^t b(s,x)ds$$
$$X_t^n = x + \int_0^t \sigma\left(s, X_s^{n-1}\right)dB_s + \int_0^t b\left(s, X_s^{n-1}\right)ds$$

as: Stochastic integrals are well defined because it is clear by induction that for each n, X_t^n is continuous and adapted, so the process $\sigma(s, X_s^{n-1})$

Let's fix a real number $T \geq 0$, Let's check first by recurrence on n that there is a constant C_n such as for every $t \in [0,T]$.

$$E\left[\left(X_t^n\right)^2\right] \leq C_n \tag{I.8}$$

This increase if $n = 0$. Then, if it is true when $n-1$, we use:

$$|\sigma(s,y)| \leq K' + K|y|. \quad \forall s \in [0,T] \cdot y \in R$$

for writing

$$\begin{aligned}
E\left[\left(X_t^n\right)^2\right] &\leq 3\left(|x|^2 + E\left[\left(\int_0^t \sigma\left(s, X_s^{n-1}\right)dB_s\right)^2\right] + E\left[\left(\int_0^t b\left(s, X_s^{n-1}\right)ds\right)^2\right]\right)\\
&\leq 3\left(|x|^2 + E\left[\left(\int_0^t \sigma\left(s, X_s^{n-1}\right)^2 ds\right)\right] + tE\left[\left(\int_0^t b\left(s, X_s^{n-1}\right)^2 ds\right)\right]\right)\\
&\leq 3\left(|x|^2 + 4(1+T)E\left[\int_0^l \left(K^{r2} + K^2\left(X_s^{n-1}\right)^2\right)ds\right]\right)\\
&\leq C_n \text{ with } C_n = 3\left(|x|^2 + 4T(1+T)\left(K^{r2} + K^2 C_{n-1}\right)\right)
\end{aligned}$$

To justify the calculation of second order of the stochastic integral.

we have to use the fact that $E\left[\left(\int_0^t \sigma\left(s, X_s^{n-1}\right)^2 ds\right)\right] \leq \infty$

which follows from the above increase for σ and the hypothesis of induction.

The increase (1.7), and the hypothesis on σ result in the local martingale $\int_0^t \sigma\left(s, X_s^{n-1}\right)dB_s$ is for each n a real martingale bounded in L^2 on the interval $[0,T]$. We use this remark to increase by induction:

$$E\left[\sup_{0\leq t\leq T}\left|X_t^{n-1} - X_t^n\right|^2\right]$$

We have

$$X_t^{n+1} - X_t^n = \int_0^t \left(\sigma\left(s, X_s^n\right) - \sigma\left(s, X_s^{n-1}\right)\right)dB_s + \int_0^t \left(b\left(s, X_s^n\right) - b\left(s, X_s^{n-1}\right)\right)ds$$

Using the Doob inequality, we get:

$$\begin{aligned}
& E\left[\sup_{0\leq s\leq t}\left|X_s^{n-1}-X_s^n\right|^2\right] \\
\leq & 2E\left[\sup_{0\leq s\leq t}\int_0^s \left(\sigma\left(u,X_u^n\right)-\sigma\left(u,X_u^{n-1}\right)\right)dB_u\right|^2+\sup_{0\leq s\leq t}\left|_0^s\left(b\left(u,X_u^n\right)-b\left(u,X_u^{n-1}\right)\right)du\ \right|\Bigg] \\
\leq & 2\left(4E\left[\left(\int_0^t \left(\sigma\left(u,X_u^n\right)-\sigma\left(u,X_u^{n-1}\right)\right)dB_u\right)^2\right]+E\left[\left(\int_0^t \left(b\left(u,X_u^n\right)-b\left(u,X_u^{n-1}\right)\right)du\right)^2\right]\right) \\
\leq & 2\left(4E\left[\int_0^t \left(\sigma\left(u,X_u^n\right)-\sigma\left(u,X_u^{n-1}\right)\right)^2du\right]+TE\left[\int_0^t \left(b\left(u,X_u^n\right)-b\left(u,X_u^{n-1}\right)\right)^2du\right]\right) \\
\leq & 2(4+T)K^2E\left[\int_0^t \left|X_t^{n-1}-X_t^n\right|^2du\right] \\
\leq & C_TE\left[\int_0^t \sup_{0\leq r\leq u}\left|X_r^n-X_r^{n-1}\right|^2du\right]
\end{aligned}$$

In note $C_T=2(4+T)K^2$. *and* $g_n(u)=E\left[\sup_{0\leq s\leq t}\left|X_s^{n-1}-X_s^n\right|^2\right]$, *so we see that:*

$$g_{n(t)}\leq C_T'\left(C_T\right)^n\frac{T^n}{n!}$$

From this last inequality we get:

$$\begin{aligned}
\textstyle\sum_{n\geq 0}\left\|\sup_{0\leq t\leq t}\left|X_t^{n+1}-X_t^n\right|\right\|_{L^1} &\leq \textstyle\sum_{n\geq 0}\|\sup_{0\leq t\leq t}\left|X_t^{n+1}-X_t^n\right|\|_{L^2} \\
&\leq \sqrt{C_T'}\textstyle\sum_{n\geq 0}\frac{(CT)^{\frac{n}{2}}}{\sqrt{n}}<\infty
\end{aligned}$$

Thus, the series $\sup_t\left|X_l^{n+1}-X_l^n\right|$ *converges P-p.s and* X^n *converges uniformly on* $[0,T]$ *towards a process* X *continue. Furthermore* $X\in H_c^2$ *since convergence takes place in* H^2. *We check that* X *is a solution of the equation (1.6).*

Going to the limit in the recurrence equation for X^n, *we find that* X *is solution of (1.6).*

Uniqueness: suppose that X, $\hat{X}$ *solutions of the equation (1.6), for* $0\leq t\leq T$.

$$X(t)-\hat{X}(t)=\int_0^t b\left(s,X_s\right)-b\left(s,\hat{X}_s\right)ds+\int_0^t\sigma\left(s,X_s\right)-\sigma\left(s,\hat{X}_s\right)dB_s$$

since $(a+b)^2\leq 2a^2+2b^2$ *we can estimate*

$$E(|X(t)-\hat{X}(t)|)^2\leq 2E\left(\left|\int_0^t b\left(s,X_s\right)-b\left(s,\hat{X}_s\right)ds\right|^2\right)+2E\left(\left|\int_0^t\sigma\left(s,X_s\right)-\sigma\left(s,\hat{X}_s\right)dB_s\right|^2\right)$$

According to Cauchy - Schawrz's inequality, we have:

$$\begin{aligned}
E\left(\left|\int_0^t b\left(s,X_s\right)-b\left(s,\hat{X}_s\right)ds\right|^2\right) &\leq TE\left(\int_0^t\left|b\left(s,X_s\right)-b\left(s,\hat{X}_s\right)\right|^2ds\right) \\
&\leq L^2T\int_0^t E\left(\left|X_s-\hat{X}_s\right|^2\right)ds
\end{aligned}$$

In a semilary way:

Itô's isometry gives us:

$$E\left(\int_0^t \sigma\left(s, X_s\right)-\sigma\left(s, \hat{X}_s\right) dB_s\right|^2\Bigg) = E\left(\int_0^t \left|\sigma\left(s, X_s\right)-\sigma\left(s, \hat{X}_s\right)\right|^2 ds\right)$$
$$\leq L^2 \int_0^t E\left(\left|X_s-\hat{X}_s\right|^2\right) ds$$

For a constant c, we have:

$$E|X(t)-\hat{X}(t)|^2 \leq c\int_0^t E\left(\left|X_s-\hat{X}_s\right|^2\right) ds,\ for\ 0 \leq t \leq T$$

Let be $\phi(t)=E\left(|X(t)-\hat{X}(t)|^2\right)$*, then:*

$$\phi(t) \leq c\int_0^t \phi(s) ds \forall 0 \leq t \leq T$$

Using Granwall's lemma, with $c_0 = 0$ *involved* $\phi = 0$*.Therefore* $X(T) = \hat{X}$ *p.s , for* $0 \leq t \leq T$*, and* $X(r) = \hat{X}(r)$ *for all rational* $0 \leq r \leq T$*, for sets of zero probability.*

X *and* $\hat{X}$ *are almost certainly continuous trajectories;*

$$P\left(\max_{0\leq t\leq T}|X(t) = \hat{X}(t)| > 0\right) = 0$$

I.6 Linear Stochastic Differential Equations

In this part, we are going to study some properties concerning the linear SDEs, in particular the inverse of the fundamental solution of a linear stochastic differential equation.

I.6.1 Case where Brownian motion is one-dimensional

We consider the following linear SDE:

$$\begin{cases} dX_t = [A(t)X_t + b(t)]\, dt + [C(t)X_t + \sigma(t)]\, dB_t \\ X(0) = x \end{cases} \tag{I.9}$$

where B_t *is an one-dimensional brownian motion and:*

$$\begin{cases} A(.), C(.) \in L^\infty[0,T] \times R^n \times R^n \\ b(.), \sigma(.) \in L^2[0,T] \times R^n \end{cases}$$

According to the theorem (1.6.2) this equation have a unique strong solution $X(.)$ *represented by:*

$$X_t = \Phi_t x + \Phi_t\int_0^t \Phi_s^{-1}[b(s) - C(s)\sigma(s)]ds + \Phi_t\int_0^t \Phi_s^{-1}\sigma(s)dB_s; t \in [\theta, T] \tag{I.10}$$

where $\Phi(.)$ is the unique solution of the following equation:

$$\begin{cases} d\Phi_t = A(t)\Phi_t dt + C(t)\Phi_t dB_t \\ \Phi_t(0) = I \end{cases} \tag{I.11}$$

I.6.2 Case where Brownian motion is d-dimensional

we will now talk about the case where $B(.)$ is of dimension d. In this case, we can write the differential equation as follows:

$$\begin{cases} dX_t = [A(t)X_t + b(t)]\,dt + \sum_{j=1}^{d}\left[C^j(t)X_t + \sigma^j(t)\right]dB_t^j \\ X(0) = x \end{cases} \quad (1.13)$$

Let Φ be the solution of the equation:

$$\begin{cases} d\Phi_t = A(t)\Phi_t dt + \sum_{j=1}^{d} C^j(t)\Phi_t dB_t^j \\ \Phi\left(t_0, t_0\right) = I_d \end{cases} \quad (1.14)$$

Noting by $\Phi\left(T, t_0\right)$ the solution of (1.13) with values in $R^d \otimes R^d$ with the initial condition $\Phi\left(t_0, t_0\right) = I_d$ So that $\Phi(T, t)$ represents the solver of equation (1.13). On the other hand, the resolving $\Phi(T, s)$ check the following property:

$$\Phi(t, s)\Phi\left(s, t_0\right) = \Phi\left(t, t_0\right), \forall t \geq s \geq t_0$$

Semilarily, we can prove that Φ^{-1} exists, and it is the solution of:

$$\begin{cases} d\Psi_t = \Psi_t\left[-A(t) + \sum_{j=1}^{d} C^j(t)^2\right]dt - \sum_{j=1}^{d} \Psi_t C^j(t)^2 dB_t^j \\ \Psi\left(t_0, t_0\right) = I_d \end{cases} \quad (1.15)$$

This equation (of the reverse) was obtained by J.M. Bismut.

Theorem 5. *The solution $\Phi(T, s)$ of equation (1.14) is invertible and its inverse $\Psi(T, s)$ is given by the following equation:*

$$d\Psi_t = -\Psi_t A(t)dt + \sum_{j\leq d} \Psi_t C^j(t)^2 dt - \sum_{j\leq d} \Psi_t C^j(t)^2 dB_t.$$

Proof 2. *It suffices to demonstrate that:*

$$d\left(\Phi_t \Psi_t\right) = d\left(\Psi_t \Phi_t\right) = 0$$

According to Itô's formula, we have:

$$
\begin{aligned}
d\left(\Phi_t \Psi_t\right) &= d\Phi_t \Psi_t + \Phi_t d\Psi_t + d\left\langle \Phi_., \Psi_- \right\rangle_t \\
d\left(\Phi_t \Psi_t\right) &= \Psi_t \left[A(t)\Phi_t dt + \sum_{j\leq d} C^j(t)\Phi_t dB_t^j \right] \\
+\Phi_t & \left[-\Psi_t A(t)dt + \sum_{j\leq d} \Psi_t C^j(t)^2 dt - \sum_{j\leq d} \Psi_t C^j(t) dB_t^j \right] \\
& - \sum_{j\leq d} C^j(t)\Phi_t \Psi_t C^j(t)dt \\
&= A(t)\Phi_t \Psi_t dt + \sum_{j\leq d} C^j(t)\Phi_t \Psi_t dB_t^j - \Phi_t \Psi_t A(t)dt \\
& + \sum_{j\leq d} \Phi_t \Psi_t C^j(t)^2 dt - \sum_{j\leq d} \Phi_t \Psi_t C^j(t) dB_t^j \\
& - \sum_{j\leq d} C^j(t)^2 \Phi_t \Psi_t dt = 0
\end{aligned}
$$

Which shows the desired result. The strong solution X_t of (1.10) is represented by:

$$X_t = \Phi_t x + \Phi_t \int_0^t \Psi_s \left[b(s) - \sum_{j=1}^{d} C^j(s)\sigma^j(s) \right] ds + \sum_{j=1}^{d} \Phi_t \int_0^t \Psi_s \sigma^j(s) dB_s^j$$

Which shows the desired result.

The strong solution X_t of (1.10) is represented by:

$$X_t = \Phi_t x + \Phi_t \int_0^t \Psi_s \left[b(s) - \sum_{j=1}^{d} C^j(s)\sigma^j(s) \right] ds + \sum_{j=1}^{d} \Phi_t \int_0^t \Psi_s \sigma^j(s) dB_s^j$$

CHAPTER

II

STOCHASTIC CONTROL

II.1 Stochastic control

We set out to explain the basic structure of a problem of control. In general, a control problem occurs according to the characteristics following ticks:

II.1.1 State of the system

A dynamic system is characterized by its state at any time which can be discrete or continuous. While considering its continuous variation. The horizon (the interval variation of time) can be finite or infinite. The quantitative variables are represented by the state of the system and they are in finite number with real values.
At the time t, the state of the system will be noted $X_T(\omega)$ for $\omega \in \Omega$ a measurable space with a probability P.

II.2 Control

A control is a process $(u_t)_t$ suitable for filtration and takes its values in a space of the control A.

II.2.1 Admissible control

An admissible control is a process u_t where $t \in [0, T]$ measurable $\mathcal{F}_t-$adapted with values in a borelian A of $\mathbf{R}^n$. Noting by U the set of all admissible controls.

$$U = \{u : (0, T] \times \Omega \to A \ . \ \textit{such that } u \textit{ is measurable and } \mathcal{F}_t - \ \textit{adapted} \} \, . \tag{II.1}$$

II.2.2 Optimal Control

We say that a control u^* is optimal if :

$$J(u^*) = \inf\{J(u); \forall u \in U\} \tag{II.2}$$

II.2.3 Almost optimal control

Let $\epsilon > 0$, the control u^ϵ is almost optimal, if for every control $u \in \mathcal{U}$ we have:

$$J(u^\epsilon) \leq J(u) + \epsilon$$

II.2.4 Cost or performance criterion

The goal of optimal control is to minimize (or to maximize) a functional. $J(u) = E\left[\int_t^T \ell(t, X_\iota, u_t)\, dt + g(X_T)\right]$ *on the set of all admissible controls.*

The Function l is the integral cost function, g is the final or terminal cost. We then defines the function:

$$V(t, x) = \inf_{u \in U} J(t, x, u) \tag{II.3}$$

Usually ; the cost is given by:

$$J(u) = E(g(X_T)) \tag{II.4}$$

where g is a function: $R^d \to R^d$ of class C^1, $|g_x(X)| \leq C(1 + |x|)$ and derivative bounded. That is to say:

$$|g_x(X)| \leq M, \textit{where } g_x \textit{is the gradient of } g \textit{ on } x.$$

If, starting from a state x at instant t, we define for any control process u that the cost is given by:

$$J(u) = E[g(X_T)]$$

The value function associated with this stochastic control problem is given by:

for $(t,x) \in [0,T] \times \mathbf{R}^n$ and $\forall u \in U$:

$$V(t,x) = \inf_{u \in U} J(t,x,u) \tag{II.5}$$

When we seek to maximize a gain, instead of minimizing a cost, then we will write:

$$V(t,x) = \sup_{u \in U} J(t,x,u) = -\inf_{u \in U} J(-(t,x,u)) \tag{II.6}$$

An admissible control $u^* \in U$ is optimal if:

$$V(t,x) = J(t,x,u^*) \tag{II.7}$$

II.3 Stochastic control resolution methods

In this paragraph, we propose to recall the two famous approaches in the solving stochastic control problems:

The principles of dynamic programming of Bellman which has an infinitesimal version of the Hamilton Jacobi Bellman equation and of the Pontryagin maximum which will be at the center of our interest in this work which consists of seeking the conditions optimality requirements satisfied by optimal control u^*.

II.3.1 The principle of dynamic programming

The principle of dynamic programming was initiated in the 1950s by R.Bellman which is a fundamental principle for stochastic control theory. In the context of the diffusion process and even more generally for Markov process controls.

The basic idea of this principle is to consider a family of control problems to different initial states and establish relationships between the associated value functions.

The dynamic programming equation leads to a second order strongly nonlinear parabolic partial differential equation, called Hamilton-Jacobi-Bellman.

II.3.2 Hamilton-Jacobi-Bellman equation(HJB)

The equation of HJB is the infinitesimal version of the principle of programming dynamic. Assuming that the value function V is C^2.

The formal derivation of the equation of HJB is given by:

$$\frac{\partial V}{\partial t}(t,x)+\inf_{u\in U}\left[\mathcal{L}_u V(t,x)+f(t,x,u)\right]=0,\forall(t,x)\in[0,T]\times R^n, \tag{II.8}$$

where $\mathcal{L}_u$ is defined by:

$$\mathcal{L}_u V=b(x,u)D_x(V)+\frac{1}{2}\operatorname{tra}\left[\sigma(x,u)\bar{\sigma}(x,u)D_x^2(V)\right] \tag{II.9}$$

However; when trying to maximize a gain then from (II.8):

$$-\frac{\partial V}{\partial t}(t,x)-\sup_{u\in U}\left[\mathcal{L}_u V(t,x)+f(t,x,u)\right]=0 \tag{II.10}$$

We often write this as:

$$-\frac{\partial v}{\partial t}(t,x)-H\left(t,x,D_xV(t,x),D_x^2(V)\right)=0,\forall(t,x)\in[0,T]\times R^n, \tag{II.11}$$

where:

$$H(t,x,p,M)=\sup_{u\in U}\left[b(x,u)p_t+\frac{1}{2}\operatorname{tra}[\sigma\dot{\sigma}(t,x,u)M+f(t,x,u)]\right] \tag{II.12}$$

This equation (II.11) is called the dynamic programming equation or HJB equation.

The function H is called the Hamiltonian.

II.3.3 The Maximum Principle of Pontryagin

The principle of Pontryagin's maximum has been used in control theory optimal. It provides the necessary optimality conditions to minimize a cost function $J(u)$ while using the Lagrange approach in calculating variations.Derivative functional $J(u)$ with respect to a certain disturbance parameter must be positive. $\frac{dJ(u_\theta)}{d\theta}|_{\theta=0}\geq 0$.

II.3.4 deterministic control

The principle of the maximum in the deterministic case of control ($\sigma=0$), was formulated by Soviet mathematician Lev Semyonovich Pontryagin [13] *in* 1950.

Recent results for the study of optimal control in the deterministic case have been treated by Felming [7], [8]*, where the author presents fundamental results in the control theory.*

The general problem of optimal control is considered as a differential system governed by the following equation:

$$\begin{cases} dX_t = b\left(t, X_t, u_t\right) dt, t \in [0, T] \\ X(0) = x \end{cases} \tag{II.13}$$

For any control $u \in \mathcal{U}$ with $\mathcal{U}$ is the set of admissible controls on $[0, T]$, then all the controls are well defined on $[0, T]$.

we define the cost of the associated trajectory by:

$$J(u) = \int_0^T \ell\left(s, X_s, u_s\right) ds + g\left(X_T\right) \tag{II.14}$$

The aim is to minimize the function $J(u)$ on a set $\mathcal{U}$ off all admissible controls. So a control u^* is optimal if:

$$J\left(u^*\right) = \min\{J(u), u \in \mathcal{U}\} \tag{II.15}$$

Under the following assumptions:

$$b, \ell : [0, T] \times R^d \times A \rightarrow \mathrm{R}^d$$

$$|b(t, x, u) - b(t, y, u)| \leq K|x - y| \tag{II.16}$$

$$|b(t, x, u)| + |\ell(t, x, u)| \leq c(1 + |x|) \tag{II.17}$$

as $b(t, x, .)$, $\ell(t, x, .)$ of $\mathbf{A} \rightarrow \mathbf{R}^d$ are continuous in u and uniformly in (t, x). b, ℓ are of the class C^1 on x.

The Hamiltonian of the system is defined as:

$$H\left(t, X_t, u_t, p_t\right) \hat{=} p_t b\left(t, X_t, u_t\right) - \ell\left(t, X_t, u_t\right) \tag{II.18}$$

We therefore have the following statement:

Theorem 6. *Principle of Pontryagin [13]*

Let (X^*, u^*) the optimal solution of (2.4) and (2.5), then there is a processes $p(t)$ and $\mathcal{F}-$adapted, solution of the following equation:

$$\begin{cases} dp_t = -H_x\left(t, x_t, u_t, p_t\right) dt \\ \qquad p(T) = -g_x\left(X_T\right) \end{cases}$$

such as:

$$H\left(t, X_t^*, u_t^*, p_t\right) = \max_{u \in U} H\left(t, X_t^*, u, p_t\right).P - p.s$$

Example 2. *We consider the following equation:* $\frac{dx_t^u}{dt} = a_t x_t + b_t u_t$ *and the cost function is given by:* $J(u) = \int_0^T \left(m_s x_s^2 + n_s u_s^2\right) ds + dx_T^2$ *with* $m_s > 0$, $n_s > 0$ *and* $d > 0$.

The Hamiltonian of the system is:

$$H\left(t, X_t, u_t, P_t\right) = P_t\left(a_t x_t + b_t u_t\right) - m_t x_t^2 - n_t u_t^2$$

where:

$$dP_t = -P_t a_t + 2m_t x_t$$

$$P(T) = -2dx_t$$

Hamiltonian maximization is:

$$H_u\left(t, X_t, u_t\right) = 0 \Leftrightarrow P_t b_t - 2n_t u_t = 0$$

$$\Rightarrow u_t = \frac{b_t P_t}{2n_t}$$

To identify u_t*, we have to solve the equation:*

$$\begin{pmatrix} x_t \\ P_t' \end{pmatrix} = \begin{pmatrix} a_t x_t + \frac{b_t^2}{2n_t} P_t \\ -a_t P_t + 2m_l x_t \end{pmatrix} = \begin{pmatrix} a_t & \frac{b_t^2}{2n_t} \\ 2m_t & -a_t \end{pmatrix} \begin{pmatrix} x_t \\ P_t \end{pmatrix}$$
$$\begin{pmatrix} x_t' \\ P_t' \end{pmatrix} = A_t \begin{pmatrix} x_t \\ P_t \end{pmatrix}$$
$$\begin{pmatrix} x_t \\ P_t \end{pmatrix} = \exp\left(-\int_0^t A_s ds\right) \begin{pmatrix} x_0 \\ P_0 \end{pmatrix}$$

It remains to calculate x_0 *and* P_0*:*

$$\begin{pmatrix} x_t \\ P_t \end{pmatrix} = \exp\left(-\int_0^t A_s ds\right) \begin{pmatrix} x_0 \\ P_0 \end{pmatrix}$$

then :

$$\begin{pmatrix} x_0 \\ \\ P_0 \end{pmatrix} = \exp\left(\int_0^T A_s ds\right)\begin{pmatrix} x_T^* \\ \\ -2dx_T^* \end{pmatrix}$$
$$= x_T^* \exp\left(\int_0^T A_s ds\right)\begin{pmatrix} 1 \\ \\ -2d \end{pmatrix}$$

See Bahlali [1]

CHAPTER

III

MAXIMUM PRINCIPLE IN THE REGULAR CASE

III.1 Some properties of controlled SDE

The optimality necessary conditions for optimal control problems are made by Pontryagin, Boltyanskii and Gankrelidze. These conditions are known as name of Maximum Principle of Pontryagin.

This chapter is devoted to the study of the principle of the stochastic maximum in the case where the coefficients are differentiable.

The optimal control problem consists in minimizing a cost function given by:

$$J(u) = E[g(X_T)] \tag{III.1}$$

where: X_T is a solution in T of the next SDE:

$$\begin{cases} dX_t = b\,(t, X_t, u_t)\,dt + \sigma\,(t, X_t)\,dB_t \\ X(0) = x \end{cases} \tag{III.2}$$

To establish the necessary conditions for optimality, we suppose that the cost function $J(u)$ is differentiable and admits a minimum noted u^* which:

$$J\,(u^*) = \min\{J(u), u \in U\}. \tag{III.3}$$

Then we proceed by a perturbation of u^* on $[t_0, t_0 + \theta]$, where θ is pretty small, we therefore obtain a control u_θ admissible and $\mathcal{F}_t-$adapted.

The final result of the maximum principle is deduced from the derivative by in relation to θ of the parturbated cost function $J(u_\theta)$ on point $\theta = 0$, for that, we assumes that the coefficients are differentiable. All the necessary conditions satisfied by the control u^* are called necessary conditions of optimality which are also known as the maximum principle of Pontryagin.

III.1.1 Definitions and Notations

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}\,, P\right)$ a filtered probability space with the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ which satisfies the usual conditions. $B = \{B_t : 0 \leq t \leq T\}$ a brownian motion on $\mathbf{R}^d$, defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}\,, P\right)$ with b and σ two measurable functions which:

$$b : [0, T] \times R^d \times A \to R^d$$

$$\sigma : [0, T] \times R^d \to R^d \otimes R^d$$

Our objective is to minimize a given cost function in the form:

$$J(u) = E[g(X_T)] \tag{III.4}$$

where X_T is the solution of the controlled SDE $(III.2)$ taken at terminal time T and the function g.

In order to properly define our problem, we give necessary hypotheses such as than:

The functions g, b and σ verify the liner growth on x, uniformly in $(t, u) \in [0, T] \times A$ as: The functions g, b and σ are derivable in x and with continuous and bounded derivatives. there is a constance C, such as for every $(t, u) \in [0, T] \times A$; we have for $x \in \mathbf{R}^d$:

$$|b(t,x,u)-b(t,y,u)|+|\sigma(t,x)-\sigma(t,x)|\leq C(|x-y|) \tag{III.5}$$

$$|b(t,x,u)|+|\sigma(t,x)|\leq C(1+|x|) \tag{III.6}$$

$$g(x)\leq C(1+|x|) \tag{III.7}$$

The functions g, b and σ are derivable in x and with continuous and bounded derivatives.

for every $(t,x)\in[0,T]\times\mathbf{R}^d$, the function $b(t,x,.):A\to\mathbf{R}^d$ is continuous.

The hypotheses $(III.5)$ and $(III.6)$ ensuring the existence and the strong uniqueness of the solution of the SDE $(III.2)$ for every control $u\in U$.

The case where the cost function contains an integral term:

$$J(u)=E\left[\int_0^T \ell\left(t,X_t,u_t\right)dt+g\left(X_T\right)\right] \tag{III.8}$$

Where: $\ell:[0,T]\times\mathbf{R}^d\times A\to\mathbf{R}^d$

It is treated in the same way as the previous case because we will not lose generality if we put $\ell=0$.

It suffices to add a one-dimensional equation to the SDE $(III.2)$ as following:

$$\begin{cases} dX_{d+1}(t)=\ell\left(t,X_t,u_t\right)dt \\ X_{d+1}(0)=0 \end{cases} \tag{III.9}$$

Noting by $X_t=(X_t,X_{d+1}(t))$ the solution of the equation:

$$\begin{cases} d\bar{X}_t=\bar{b}\left(t,\bar{X}_t,u_t\right)dt+\bar{\sigma}\left(t,\bar{X}_t\right)dB_t \\ \bar{X}(0)=(x,0) \end{cases} \tag{III.10}$$

We obtain:

$$J(u)=E\left[X_{d+1}(T)+g\left(X_T\right)\right]=E\left[\bar{g}\left(\bar{X}_T\right)\right] \tag{III.11}$$

The following theorem confirms the existence and uniqueness of the solution of the SDE $(III.2)$, it is due to Ito. See Kushner [12]

Theorem 7. *Let b and σ two continuous functions verify the hypotheses $(III.5)$ and $(III.6)$. The equation $(III.2)$ admits a unique strong solution with defined continuous trajectories on $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},P\right)$, for $p\geq 1$:*

$$E\left[\left(\sup_{0\leq s\leq T}|X_s|^p\right)\right]<+\infty$$

III.1.2 Cost function

We have:

$$\begin{aligned} |J(u)| &= |E\left[g\left(X_T\right)\right]| \\ &\leq E\left[\left|g\left(X_T\right)\right|\right] \end{aligned}$$

And according to the hypothesis we obtain:

$$\begin{aligned} |J(u)| &\leq c\left[1 + E\left(|X_T|\right)\right] \\ &\leq c\left[1 + E\left(\sup_{t\leq T}|X_l|\right)\right] \end{aligned}$$

we put $p = 1$ in the inequality of the theorem $(III.11)$, we obtain:

$$E\left(\sup_{t\leq T}|X_T|\right) < +\infty$$

then:

$$|J(u)| < +\infty, \text{ for } u \in \mathcal{U}$$

the cost function is well defined and the problem too.

Now suppose that the cost function $J(u)$ admits a minimum that we note u^* then:

$$J(u^*) \leq J(u) \tag{III.12}$$

We call X^* the solution of the equation of $(III.2)$ corresponding to the control u^* optimal trajectory.

III.1.3 perturbation and estimation of the solution

To obtain the necessary optimality conditions, we compare u^* to controls which are sufficiently close to it.

We define the next perturbation:

$$u_\theta = \begin{cases} u_t^* \text{ if } t \in [0, t_0] \\ v \text{ if } t \in [t_0, t_0 + \theta] \\ u_t^* \text{ if } t \in [t_0 + \theta, T] \end{cases}$$

with $v \in A$, $t_0 \in [0, T]$, θ very small.

u_θ is a measurable $\mathcal{F}-$adapted process in A, so u_θ is a process admissible.

We call u_θ the strong perturbation of u^*.

By definition u_θ differs from u^* on the interval $[t_0, t_0+\theta]$, and if $\theta = 0$ we have $u_\theta = u^*$.

When the control takes place on both the derivative and diffusion terms, we therefore uses another kind of perturbation which is the following(See Bensoussan[2, 3]):

$$u_\theta(t) = u^*(t) + \theta v(t) \tag{III.13}$$

We will now see how this perturbation of u^* transfers to the solution of the equation $(III.2)$ corresponding to the control u_θ and also to the cost function $J(u_\theta)$.

To give estimates of the solutions of the SDE $(III.2)$, we define the following distance:

$$d\left(u_\theta, \hat{u}\right) = P \otimes dt\left\{(w,t) \in \Omega \times [0,T] u_\theta(w,t) \neq u^*(w,t)\right\} \tag{III.14}$$

Lemma 1. *Let u and v be two controls. Let's put X_t^u and X_t^v the solution of the equation (III.2), So there is a constant K such as:*

$$E\left(\sup_{0\le t\le T}\left|X_t^u - X_t^v\right|\right)^2 \le K[d(u,v)]^{\frac{1}{2}}$$

Remark 1. *1/ In the case where u and v differ only over an interval of length θ, then we have a better estimate of the solutions.*

In fact:

$$\begin{aligned}
E\left[\sup_{t\le T}\left|X_t^u - X_t^v\right|^2\right] &\le k_1 E\left(\int_{t_0}^{t_0+\theta}\left|b\left(t, X_t^v, u_t\right) - b\left(t, X_t^v, v_t\right)\right| dt\right)^2 \\
&\le k_1\theta^2 E\left(\sup_{1\le T}\left|b\left(t, X_t^v, u_i\right) - b\left(t, X_t^y \cdot v_t\right)\right|^2 dt\right) \\
&\le M\theta^2 \\
&= Md(u,v)^2
\end{aligned}$$

2/ if b is bounded, we have:

$$E\left[\sup_{1\le T}\mid X_1^u - X_t^{y/2}\right] \le Md(u,v)$$

Estimation of the solution

We will now propose another estimate which will relate to the variation of trajectories X_t^θ and X_t^* in relation to the distance $d(u_\theta, u^*)$. we therefore obtain the following estimate:

Lemma 2. *Let X_t^θ and X_t^* be the solutions of the equation (3.2) corresponding to $\hat{u}$ and u_θ. So we have* *the following estimate:*

$$E\left[\sup_{t\le T}\left|X_t^\theta - X_t^*\right|^2\right] \le M\theta^2$$

and we get

$$\lim_{\theta\to 0} E\left[\sup_{t\le T}\left|X_t^\theta - X_t^*\right|^2\right] = 0$$

Proof 3. *By definition X_t^θ and X_t^* are solutions of the equation (III.2) compared at controls u^θ and u^* respectively, they are given by the following equations:*

$$X_i^* = x + \int_0^t b\left(s, X_s^*, \hat{u}_s\right) ds + \int_0^t \sigma\left(s, X_s^*\right) dB_s$$

$$X_i^\theta = x + \int_0^t b\left(s, X_s^\theta, \hat{u}_s\right) ds + \int_0^t \sigma\left(s, X_s^\theta\right) dB_s$$

And like the strong perturbation given by the following definition:

$$u_\theta = \begin{cases} u_t^* \ \ si\ t \in [0, t_0] \\ v \ \ si\ t \in [t_0, t_0+\theta] \\ u_t^* \ \ si\ t \in [t_0+\theta, T] \end{cases}$$

We deduce

$$\begin{cases} X_t^\theta = X_t^* \qquad si\ t \le t_0 \\ dX_t^\theta = b\left(t, X_t^\theta, u_t^*\right) dt + \sigma\left(t, X_t^\theta\right) dB_t \ \ si\ t_0+\theta \le t \le T \end{cases}$$

So

$$\begin{aligned}
\left|X_t^\theta - X_t^*\right|^2 &= \left|x + \int_0^t b\left(s, X_s^\theta, u_s^\theta\right) ds + \int_0^t \sigma\left(s, X_s^\theta\right) dB_s - x - \int_0^t b\left(s, X_s^*, u_s^*\right) ds - \int_0^t \sigma\left(s, X_s^*\right) dB_s\right|^2 \\
&=| \int_0^t \left[b\left(s, X_s^\theta, u_s^\theta\right) - b\left(s, X_s^*, u_s^*\right)\right] ds + \int_0^t \left[\sigma\left(s, X_s^\theta\right) - \sigma\left(s, X_s^*\right)\right] dB_s\Big]^2 \\
&\le 2\left|\int_0^t \left[b\left(s, X_s^\theta, u_s^\theta\right) - b\left(s, X_s^*, u_s^*\right)\right] ds\right|^2 + 2 \,|\int_0^t \left[\sigma\left(s, X_s^\theta\right) - \sigma\left(s, X_s^*\right)\right] dB_s\Big]^2 \\
&\le c_1 \int_0^t \left|\left[b\left(s, X_s^\theta, u_s^\theta\right) - b\left(s, X_s^*, u_s^*\right)\right]\right|^2 ds + c \,|\int_0^t \left[\sigma\left(s, X_s^\theta\right) - \sigma\left(s, X_s^*\right)\right] dB_s^2
\end{aligned}$$

We add and subtract the term $b(s, X_s^*, u_s^\theta)$

$$\begin{aligned}
\left|X_t^\theta - X_t^*\right|^2 \le& c_1 \int_0^t \left|\left[b\left(s, X_s^\theta, u_s^\theta\right) - b\left(s, X_s^*, u_s^*\right) + b\left(s, X_s^*, u_s^\theta\right) - b\left(s, X_s^*, u_s^\theta\right)\right]\right|^2 ds \\
&+ c\left|\int_0^t \left[\sigma\left(s, X_s^\theta\right) - \sigma\left(s, X_s^*\right)\right] dB_s\right|^2 \\
\le& c_2 \int_0^t \left|b\left(s, X_s^\theta, u_s^\theta\right) - b\left(s, X_s^*, u_s^\theta\right)\right|^2 ds + c_2 \int_0^t \left|b\left(s, X_s^*, u_s^\theta\right) - b\left(s, X_s^*, u_s^*\right)\right|^2 ds \\
&+ c\left|\int_0^t \left[\sigma\left(s, X_s^\theta\right) - \sigma\left(s, X_s^*\right)\right] dB_s\right|^2,
\end{aligned}$$

The coefficient $b(.,X_s^*,u_s^\theta)$ *is Lipscizien in* X*, so we have:*

$$\left|b\left(s,X_s^\theta,u_s^\theta\right)-b\left(s,X_s^*,u_s^\theta\right)\right|\leq k_1\left|X_s^\theta-X_s^*\right|$$

Therefore :

$$\begin{aligned}\left|X_t^\theta-X_t^*\right|^2\leq & c_2\int_0^t k_1^2\left|X_s^\theta-X_s^*\right|^2ds+c_2\int_0^t\left|b\left(s,X_s^*,u_s^\theta\right)-b\left(s,X_s^*,u_s^*\right)\right|^2ds\\ &+c\left|\int_0^t\left[\sigma\left(s,X_s^\theta\right)-\sigma\left(s,X_s^*\right)\right]dB_s\right|^2\\ E\left[\sup_{t\leq T}\left|X_t^\theta-X_t^*\right|^2\right]\leq & c_2k_1^2\int_0^t E\left(\left|X_s^\theta-X_s^*\right|^2\right)ds+c_2\int_0^t E\left(\left|b\left(s,X_s^*,u_s^\theta\right)-b\left(s,X_s^*,u_s^*\right)\right|^2\right)ds\\ &+cE\left[\sup_{i\leq T}\left|\int_0^t\left[\sigma\left(s,X_s^\theta\right)-\sigma\left(s,X_s^*\right)\right]dB_s\right|^2\right]\end{aligned}$$

By the inequality of Burkholder-Davis-Gandy:

$$\left|\sigma\left(s,X_s^\theta\right)-\sigma\left(s,X_s^*\right)\right|\leq k_2\left|X_s^\theta-X_s^*\right|$$

so

$$\begin{aligned}E\left[\mathrm{Sup}_{t\leq T}\left|X_t^\theta-X_i^*\right|^2\leq\left(c_2k_1^2+c_3k_2^2\right)\int_0^t E\left(\mathrm{Sup}_{t\leq T}\left|X_s^\theta-X_s^*\right|^2\right)ds\right.\\ +c_2\int_0^t E\left(\left|b\left(s,X_s^*,u_s^\theta\right)-b\left(s,X_s^*,u_s^*\right)\right|^2\right)ds\end{aligned}$$

By definition, u_θ *does not differ from optimal control* u^* *on* $[t_0,t_0+\theta]$ *Let's pose* $K=(c_2k_1^2+c_3k_2^2)$*:*

$$\begin{aligned}E\left[\sup_{t\leq T}\left|X_t^\theta-X_t^*\right|^2\right]&\leq K\int_0^t E\left(\left|X_s^\theta-X_s^*\right|^2\right)ds+c_2E\left(\sup_{t\leq T}\left|\int_{t_0}^{t_0+\theta}\left(b\left(s,X_s^*,v\right)-b\left(s,X_s^*,u_s^*\right)\right)ds\right|^2\right)\\ &\leq K\int_0^t E\left(\left|X_s^\theta-X_s^*\right|^2\right)ds+c_2\theta^2E\left[\sup_{t\leq T}\mid b\left(t,X_t^*,v\right)-b\left(t,X_t^*,u_t^*\right)\right]\Big]^2\end{aligned}$$

The coefficient $b\left(t,X_t,u_t\right)$*is linearly growing, then*

$$\begin{aligned}E\left[\sup_{t\leq T}\left|X_t^\theta-X_t^*\right|^2\right]&\leq K\int_0^t E\left(\left|X_s^\theta-X_s^*\right|^2\right)ds+c_3\theta^2E\left[\left(1+\sup_{t\leq T}\left|X_s^*\right|\right)^2\right]\\ &\leq K\int_0^t E\left(\left|X_s^\theta-X_s^*\right|^2\right)ds+c_\downarrow\theta^2\end{aligned}$$

By Gronwal's lemma we get:

$$\begin{aligned}E\left[\sup_{t\leq T}\left|X_t^\theta-X_t^*\right|^2\right]&\leq c_4\theta^2e^{\int_0^t kds}\\ &\leq M\theta^2\end{aligned}$$

where$E\left[\sup_{t\leq T}\left|X_t^\theta-X_t^*\right|^2\right]\underset{\theta\to 0}{\to}0$. *The perturbation of* u^* *results in the solution of equation (3.2) and also on the associated cost function.*

We will see how this perturbation of u^* *results in the variation of the cost.*

Lemma 3. *Let $J(u)$ be a cost function defined by $J(u) = E\left(g\left(X_T\right)\right)$.*

So

$$\left|J\left(u^{\theta}\right) - J\left(u^{*}\right)\right| \leq K\theta$$

Proof 4. *Since $g \in \mathcal{C}^1$ and $|g_x(X)| \leq M$ so g is a Lipschitzian function:*

$$\begin{aligned}
\left|J\left(u^{\theta}\right) - J\left(u^{*}\right)\right| &= \left|E\left(g\left(X_T^{\theta}\right)\right) - E\left(g\left(X_T^{*}\right)\right)\right| \\
&= \left|E\left[g\left(X_T^{\theta}\right) - g\left(X_T^{*}\right)\right]\right| \\
&\leq E\left|g\left(X_T^{\theta}\right) - \left(X_T^{*}\right)\right| \\
&\leq E\left[K\left|X_T^{\theta} - X_T^{*}\right|\right] \\
&\leq KE\left[\left|X_T^{\theta} - X_T^{*}\right|\right] \\
&\leq K\theta.
\end{aligned}$$

The application $\theta \to J\left(u_{\theta}\right)$ is continuous on $\theta = 0$.

If the cost function $J\left(u_{\theta}\right)$ is differentiable, the necessary optimality conditions will be based on the differentiation $J\left(u_{\theta}\right)$.

The interest of the perturbation of optimal control u^* is to introduce a control u_{θ} on which we can derive the function $J\left(u_{\theta}\right)$.

We can note that the results obtained in the principle of the maximum are based largely on the differentiability of $J\left(u_{\theta}\right)$ in $\theta = 0$.

The study of the derivation of$J\left(u_{\theta}\right)$ in $\theta = 0$ is given by the following lemma:

Lemma 4. *If the application $\theta \to J\left(u_{\theta}\right)$is differentiable at the point $\theta = 0$ so:*

$$\left.\frac{dJ\left(u_{\theta}\right)}{d\theta}\right|_{\theta=0} \geq 0$$

Proof 5. *The application $\theta \to J\left(u_{\theta}\right)$is continuous to the point $\theta = 0$, and if this application is differentiable at the point $\theta = 0$, so we develop $J\left(u_{\theta}\right)$ on point u^* :*

$$J\left(u_{\theta}\right) = J\left(u^{*}\right) + \theta\left.\frac{dJ\left(u_{\theta}\right)}{d\theta}\right|_{\theta=0} + \varepsilon(\theta),$$

such as $\varepsilon(\theta) \underset{\theta\to 0}{\to} 0$ *So we have:*

$$J(u_\theta) - J(u^*) = \theta \left.\frac{dJ(u_\theta)}{d\theta}\right|_{\theta=0} + \varepsilon(\theta)$$

and like u^* *is optimal then*$J(u^*) \leq J(u_\theta)$ *for* $\theta \in [0,T]$,*that is to say* $J(u_\theta) - J(u^*) \geq 0 \forall\theta$, *which implies*

$$\frac{dJ(u_\theta)}{d\theta}\underset{\theta=0}{|} \geq 0$$

III.2 Linearization of the solution

We want to approach$\left(X_t^\theta - X_t^*\right)$ to another linear process solution of the equation $(III.2)$, noted by Z_t.Let X_t^θ is the trajectory corresponding to u_θ and X_t^* is the trajectory corresponding to u^*.

According to the definition of X_t^θ and X_t^*,

we have

$$\begin{aligned} X_t^\theta &= x + \int_0^t b\left(s, X_s^\theta, u_s^\theta\right) ds + \int_0^t \sigma\left(s, X_s^\theta\right) dB_s, \\ X_t^* &= x + \int_0^t b\left(s, X_s^*, u_s^*\right) ds + \int_0^t \sigma\left(s, X_s^*\right) dB_s \end{aligned}$$

Let Z_t solution of the following linear differential equation: $\forall t \in [t_0, T]$

$$\begin{cases} dZ_t = b_x\left(t, X_t^*, u_t^*\right) Z_t dt + \sum_{j\leq d} \sigma_x^j\left(t, X_t^*\right) Z_t dB_t^j \\ Z(t_0) = b\left(t_0, X_{t_0}^*, v\right) - b\left(t_0, X_{t_0}^*, u_t^*\right) \end{cases}$$

We have the following approximation:

Lemma 5. *Let* X_T^**and* X_T^θ *be the solutions of equation (III.2) corresponding to u * and* u_θ *(resp) in*T.

So we have

$$\lim_{\theta\to 0} E\left(\left|\frac{X_T^\theta - X_T^*}{\theta} - Z(T)\right|^2\right) = 0$$

$\left(\frac{1}{\theta}\left[X_T^\theta - X_T^*\right]\right.$ *converge in root mean square to* $\left.Z(T)\right)$.

Proof 6. *According to the definitions of* X_T^*, X_T^θ *and* Z_t*we have:*

$$dX_t^\theta = b\left(t, X_t^\theta, u_t^\theta\right) dt + \sigma\left(t, X_t^\theta\right) dB_t$$

$$dX_t^* = b\left(t, X_t^*, u_t^*\right) dt + \sigma\left(t, X_t^*\right) dB_t$$

$$dZ_t = b_x\left(t, X_t^\theta, u_t^\theta\right) Z_t dt + \sigma_x\left(t, X_t^\theta\right) Z_t dB_t$$

So we can define a process$\tilde{X}_t^\theta$*:*

$$\tilde{X}_t^\theta = \frac{1}{\theta}\left[X_t^\theta - X_t^*\right] - Z_t$$

Where $t \in [t_0, t_0+\theta]$ *: According to Itô's formula, we have for* $t \in [t_0, t_0+\theta]$ *:*

$$d\tilde{X}_t^\theta = \frac{1}{\theta}\left[dX_t^\theta - dX_t^* - \theta dZ_t\right]$$

$$= \frac{1}{\theta}\left[b\left(t, X_t^\theta, v\right) - b\left(t, X_t^*, u_t^*\right) - \theta b_x\left(t, X_t^\theta, u_t^\theta\right) Z_t\right] dt$$

$$+ \left[\sigma\left(t, X_t^\theta\right) - \sigma\left(t, X_t^*\right) - \sigma_x\left(t, X_t^\theta\right) Z_t\right] dB_t$$

Replacing X_t^θ *with* $X_t^\theta = X_t^* + \theta\left(\tilde{X}_t^\theta + Z_t\right)$. *We obtain*

$$d\tilde{X}_t^\theta = \frac{1}{\theta}\left[b\left(t, X_t^* + \theta\left(\tilde{X}_t^\theta + Z_t\right), v\right) - b\left(t, X_t^*, u_t^*\right) - \theta b_x\left(t, X_t^\theta, u_t^\theta\right) Z_t\right] dt$$
$$+ \left[\sigma\left(t, X_t^* + \theta\left(\tilde{X}_t^\theta + Z_t\right)\right) - \sigma\left(t, X_t^*\right) - \sigma_x\left(t, X_t^\theta\right) Z_t\right] dB_t$$

For $t = t_0$ *we have* $\tilde{X}_t^\theta = X_t^*$*,which implies:*

$$\tilde{X}_{t_0}^\theta = -Z_{t_0}$$

$$= -\left(b\left(t_0, X_{t_0}^*, v\right) - b\left(t_0, X_{t_0}^*, u_{t_0}^*\right)\right)$$

We'll have :

$$X_{t_0+\theta}^\sim = \frac{1}{\theta}\left[\int_{t_0}^{t_0+\theta}\left[b\left(s, X_s^* + \theta\left(\tilde{X}_s^\theta + Z_s\right), v\right) - b\left(t, X_s^*, u_s^*\right)\right] ds\right]$$
$$+_\theta\left[\int_{t_0}^{t_0+\theta}\left[\sigma\left(s, X_t^* + \theta\left(\tilde{X}_t^\theta + Z_t\right)\right) - \sigma\left(s, X_s^*\right)\right] dB_s\right] - \int_{t_0}^{t_0} b_x\left(t, X_s^*, u_s^*\right) Z_s ds$$
$$- \int_{t_0}^{t_0+\theta} \sigma_x\left(t, X_s^*\right) Z_s dB_s$$

By adding and subtracting the following terms:

$$\frac{1}{\theta}\left[\int_{t_0}^{t_0+\theta} b\left(t, X_s^*, v\right) ds\right], \frac{1}{\theta}\left[\int_{t_0}^{t_0+\theta} b\left(t, X_{t_0}^*, v\right) ds\right] \text{ and } \frac{1}{\theta}\left[\int_{t_0}^{t_0+\theta} b\left(t, X_{t_0}^*, u_{t_0}^*\right) ds\right]$$

We obtain:

$$X^{\theta}_{t_0+\theta} = \frac{1}{\theta}\left[\begin{array}{l} t_0+\theta \\ \\ t_0 \end{array}\right.$$

$$+\frac{1}{\theta}\left[\int_{t_0}^{t_0+\theta}\left[b\left(s,X_s^*,v\right)-b\left(s,X_{t_0}^*,v\right)\right]ds\right]$$

$$-\frac{1}{\theta}\left[\int_{L_0}^{t_0+\theta}\left[b\left(s,X_s^*,u_s^*\right)-b\left(s,X_{t_0}^*,u_{t_0}^*\right)\right]ds\right]$$

$$+\frac{1}{\theta}[\int_{t_0}^{t_s}+\theta\left[\sigma\left(s,X_s^*+\theta\left(X_t\right)ds\right]\right.$$

$$-\int_{t_0}^{t_0+\theta}b_x\left(s,X_s^*,u_s^*\right)Z_s ds-\int_{t_0}^{-\theta}\sigma\left(s,X_s^*\right)dB_s$$

We can deduce

$$\left|X^{\theta}t_0+\theta\right|^2 \leq c\left|\frac{1}{\theta}\int_{t_0}^{t_0+\theta}\left[b\left(s,X_s^*+\theta\left(X_s+Z_s\right),v\right)-b\left(s,X_s^*,v\right)\right]ds\right|^2$$

$$+c\left|\frac{1}{\theta}\int_{t_0}^{t_0+\theta}\left[b\left(s,X_s^*,v\right)-b\left(s,X_{t_0}^*,v\right)\right]ds\right|^2$$

$$+c\left|\frac{1}{\theta}\int_{t_0}^{t_0+\theta}\left[b\left(s,X_s^*,u_s^*\right)-b\left(s,X_{t_0}^*,u_{t_0}^*\right)\right]ds\right|^2$$

$$+c\mid\frac{1}{\theta}\int_{t_0}^{t_0}+\theta\left[\sigma\left(s,X_s^*+\theta\left(X_t+Z_t\right)\right)-\sigma\left(s,X_s^*\right)\right]dB_s|^2$$

$$+c\left|\int_{t_0}^{t_0+\theta}b_x\left(s,X_s^*,u_s^*\right)Z_s ds\right|^2+c\left|\int_{t_0}^{t_0+\theta}\sigma_x\left(s,X_s^*\right)dB_s\right|^2$$

Since the coefficient b is Lipschitzian in x, $\left|b_x\left(t,X_t,u_t\right)\right| \leq k$ *and from the Cauchy-Schwartz inequality, we get:*

$$\left|X^{\theta}{}_{t_0+\theta}\right|^* \leq c^1_{\theta}k^2\int_{t_0}^{t_0+\theta}\left|X_s^{\theta}-X_s^*\right|ds$$

$$+c\frac{1}{\theta}k^2\int_{t_0}^{t_0+\theta}\left|X_s^{\theta}-X_{t_0}^*\right|ds$$

$$+c\left|\frac{1}{\theta}\int_{t_0}^{t_0+\theta}\left[b\left(s,X_s^*,u_s^*\right)-b\left(s,X_{t_0}^*,u_{t_0}^*\right)\right]ds\right|^2$$

$$+c\left|\frac{1}{\theta}\int_{t_0}^{t_0+\theta}\left[\sigma\left(s,X_s^*+\theta\left(\bar{X}_t+Z_t\right)\right)-\sigma\left(s,X_s^*\right)\right]dB_s\right|^2$$

$$+ck^2\int_{t_0}^{t_0+\theta}\left|Z_s\right|^2 ds+c\left|\int_{t_0}^{t_0+\theta}\sigma_x\left(s,X_s^*\right)dB_s\right|^2$$

By applying the Burkholder Davis Gandy inequality:

$$\left|\frac{1}{\theta}\int_{t_0}^{t_0+\theta}\left[\sigma\left(s,X_s^*+\theta\left(\begin{array}{c}-\theta \\ \\ X_t\end{array}+Z_t\right)\right)-\sigma\left(s,X_s^*\right)\right]dB_s\right|^2 \quad et \quad \left|\int_{t_0}^{t_0+\theta}\sigma_x\left(s,X_s^*\right)dB_s\right|^2$$

We get the following result:

$$E\left(\sup_{t\in[t_0,t_0+\theta]}\left|X^\theta t_0+\theta\right|^2\right)^2 \le c\frac{1}{\theta}k^2E\left(\sup_{t\in[t_0,t_0+\theta]}\left|X_s^\theta - X_s^*\right|^2\right)$$
$$+\,c\frac{1}{\theta}k^2E\left(\sup_{t\in[t_0,t_0+\theta]}\left|X_s^\theta - X_{t_0}^*\right|^2\right)$$
$$+\,c\frac{1}{\theta}E\left(\int_{t_0}^{t_0+\theta}\left|b\left(s,X_s^*,u_s^*\right)-b\left(s,X_{t_0}^*,u_{t_0}^*\right)\right|^2ds\right)$$
$$+\,c\frac{1}{\theta}E\left(\left|\int_{t_0}^{t_0+\theta}\left[\sigma\left(s,X_s^*+\theta\left(\bar{X}_t+Z_t\right)\right)-\sigma\left(s,X_s^*\right)\right]dB_s\right|^2\right)$$
$$+\,ck^2\int_{t_0}^{t_0+\theta}\left|Z_s\right|^2ds + cE\left(\sup_{t\in[t_0,t_0+\theta]}\left|\int_{t_0}^{t_0+\theta}\sigma_x\left(s,X_s^*\right)dB_s\right|^2\right)$$

So we will have:

$$E\left(\sup_{t\in[t_0,t_0+\theta]}\left|X^\theta{}_{t_0+\theta}\right|^2\right) \le ck^2(1+m)E\left(\sup_{t\in[t_0,t_0+\theta]}\left|X_s^\theta - X_s^*\right|^2\right)$$
$$+\,ck^2E\left(\sup_{t\in[t_0,t_0+\theta]}\left|X_t^\theta - X_{t_0}^*\right|^2\right)$$
$$+c\frac{1}{\theta}E\left(\int_{t_0}^{t_0+\theta}\left|b\left(s,X_s^*,u_s^*\right)-b\left(s,X_{t_0}^*,u_{t_0}^*\right)\right|^2ds\right)$$
$$+\,ck^2(1+m)E\left(\int_{t_0}^{t_0+\theta}\left|Z_s\right|^2ds\right)$$

When $\theta\to 0$ then the terms on the right converge to 0*. Where* t $\in[t_0+\theta,T]$ *:*

Since we have the following perturbation:

$$u_{\theta=}\begin{cases} u_t^* \;\; si\ t\in[0,t_0] \\ v \;\; si\ t\in[t_0,t_0+\theta] \\ u_t^* \;\; si\ t\in[t_0+\theta,T] \end{cases}$$

So we get:

$$dX_t^\theta = \frac{1}{\theta}\left[dX_t^\theta - dX_t^* - \theta dZ_t\right]$$
$$=\frac{1}{\theta}\left\{\left[b\left(t,X_t^\theta,u_t^*\right)-b\left(t,X_t^*,u_t^*\right)-\theta b_x\left(t,X_t^*,u_t^*\right)Z_t\right]dt\right.$$
$$\left.+\left[\sigma\left(t,X_t^\theta\right)-\sigma\left(s,X_t^*\right)-\theta\sigma_x\left(t,X_s^*\right)Z_t\right]dB_t\right\}$$
$$X_t^\theta = \frac{1}{\theta}\left[X_t^\theta - X_t^*\right]-Z_t \;\; Leads\ to\ X_t^\theta = X_t^*+\theta\left(X_t^{-\theta}+Z_t\right)$$

$$dX_t^\theta = \tfrac{1}{\theta}\left\{\left[b\left(t,X_t^*+\theta\left(X_t+Z_t\right),u_t^*\right)-b\left(t,X_t^*,u_t^*\right)-\theta b_x\left(t,X_t^*,u_t^*\right)Z_t\right]dt\right.$$
$$\left.+\left[\sigma\left(t,X_t^*+\theta\left(X_t+Z_t\right)\right)-\sigma\left(t,X_t^*\right)-\theta\sigma_x\left(t,X_s^*\right)Z_t\right]dB_t\right\}$$

Although the coefficients b and σ are of class C^1 in variable x, which allows us to do a first-order expansion of the form:

$$dX_t^\theta = \int_0^t b_x\left(t, X_t^* + \mu\theta\left(X_t + Z_t\right), u_t^*\right) X_t dt$$
$$+ \int_0^t \sigma_x\left(t, X_t^* + \mu\theta\left(X_t + Z_t\right)\right) X_t dB_t$$
$$+ \int_0^t \left[b_x\left(t, X_t^* + \mu\theta\left(X_t + Z_t\right), u_t^*\right) - b_x\left(t, X_t^*, u_t^*\right)\right] Z_t dt$$
$$+ \int_0^t \left[\sigma_x\left(t, X_t^* + \mu\theta\left(-\theta_t + Z_t\right)\right) - \sigma_x\left(t, X_t^*\right)\right] Z_t dB_t$$

Since b_x and σ_x are continuous and bounded, we can apply the Burkholder Davis Gandy inequality:

$$E\left(\left|X_t^\theta\right|^2\right) = E\left(\left|X_{t_0+\theta}^\theta\right|^2\right) + ME\left(\int_{t_0+\theta}^T \left|dX_t^{-\theta}\right|^2 ds\right)$$
$$+E\left[\int_{t_0+\theta}^T \left|Z_s\right|^2 ds \left|\int_0^1 (d\mu)\left[b_x\left(s, X_s^* + \mu\left(X_s^\theta - X_s^*\right), u_s^*\right) - b_x\left(s, X_s^*, u_s^*\right)\right] ds\right|^2\right]$$
$$+E\left[\int_{t_0+\theta}^T \left|Z_s\right|^2 ds \left|\int_0^1 (d\mu)\left[\sigma_x\left(s, X_s^* + \mu\left(X_s^\theta - X_s^*\right)\right) - \sigma_x\left(s, X_s^*\right)\right] ds\right|^2\right]$$

Finally, by applying Gronwall's lemma and the fact that $E\left(\sup_{0\le t\le T}\left|X_t\right|^p < +\infty\right)$ we get the result.

We will now give the first expression of the derivative of the cost function J. As the function g is differentiable in x, the following theorem gives the value of the derivative $J\left(u_\theta\right)$ at the point $\theta = 0$:

Theorem 8. *The application $\theta \to J\left(u_\theta\right)$ is differentiable at the point $\theta = 0$. Moreover, we have:*

$$\left.\frac{dJ\left(u^\theta\right)}{d\theta}\right|_{\theta=0} = E\left\langle g_x\left(X_t^*\right), Z(T)\right\rangle$$

, where g_x is the gradient from g to x.

Proof 7. *since the function g is of class C^1 then for almost any w there exists $\lambda(w) \in]0, T[$ such that:*

$$g\left(X_\theta(t)\right) - g\left(X^*(t)\right) = \left\langle g_x\left(X_t^* + \lambda\left(X_t^\theta - X_t^*\right)\right), \left(X_t^\theta - X_t^*\right)\right\rangle$$

Therefore:

$$\frac{1}{\theta}E\left[g\left(X_\theta(t)\right) - g\left(X^*(t)\right)\right] = \frac{1}{\theta}E\left[\left\langle g_x\left(X_t^* + \lambda\left(X_t^\theta - X_t^*\right)\right), \left(X_t^\theta - X_t^*\right)\right\rangle\right]$$
$$= E\left[\left\langle g_x\left(X_t^* + \lambda\left(X_t^\theta - X_t^*\right)\right), \frac{1}{\theta}\left(X_t^\theta - X_t^*\right)\right\rangle\right]$$

The fact that g_x is bounded: $\frac{X_T^\theta - X_T^*}{\theta} \overset{L^2(\Omega,F,P)}{\rightarrow} Z(T)$ *We obtain:*

$$\lim_{\theta\to 0}\frac{J(u_\theta)-J(u^*)}{\theta} = \lim_{\theta\to 0}\frac{1}{\theta}E\left[g\left(X_\theta(T)\right)-g\left(X^*(T)\right)\right]$$

$$= \lim_{\theta\to 0}E\left[\left\langle g_x\left(X_t^*+\lambda\left(X_T^\theta-X_T^*\right)\right), \frac{1}{\theta}\left(X_T^\theta-X_T^*\right)\right\rangle\right]$$

$$= E\left[\left\langle g_x\left(X_t^*\right), Z(T)\right\rangle\right]$$

Now we can announce the main result of this chapter which is the principle of the maximum in the regular case.

III.3 Maximum principle

Let $\phi(T,t)$ the resolvent of the equation $(III.2)$ with values in $R^d\otimes R^d$ solution of:

$$\begin{cases} d\Phi\left(t,t_0\right) = b_x\left(t,X_t^\star,u_i^\star\right)\Phi\left(t,t_0\right)+\sum_{j\leq d}\sigma_x^j\left(t.X_t^\star\right)\Phi\left(t,t_0\right)dB_t^j \\ \Phi\left(t_0,t_0\right) = I_d \end{cases} \tag{III.15}$$

By the uniqueness of the trajectory we have:

$$Z_t = \Phi\left(t,t_0\right)Z_{t_0} = \Phi\left(t,t_0\right)\left(b\left(t_0,X_{t_0}^*,v\right)-b\left(t_0,X_{t_0}^*,u_{t_0}^*\right)\right) \tag{III.16}$$

Uniqueness also gives us the following multiplicative property:

$$\Phi\left(t,t_0\right) = \Phi\left(t,t_1\right)\circ\Phi\left(t_1,t_0\right) \text{ for } t_0<t_1<t$$

according to (Theorem 1.6.2); $\phi(t,t_0)$ is invertible and its inverse $\Psi(t,t_0)$ the reverse equation :

$$\begin{cases} d\Psi_t = -\Psi_t b_x\left(t,X_i^\star,u_i^\star\right)dt+\sum_{j\leq d}\Psi_t\sigma_x^j\left(t,X_t^\star\right)dt-\sum_{j\leq d}\Psi_t\sigma_x^j\left(t,X_t^\star\right)dB_t^j \\ \Psi\left(t_0,t_0\right) = I_d \end{cases} \tag{III.17}$$

Theorem 9 (The stochastic maximum principle)**.** *Let u^* be an optimal control, X^* the solution of the equation $(III.2)$ corresponding at u^*.*

So there is a process $p(t)$ $\mathcal{F}_t-$adapted such as:

$$E\left\langle p(t),b\left(t,X_t^*,v\right)\right\rangle \leq E\left\langle p(t),b\left(t,X_t^*,u_t^*\right)\right\rangle \forall v\in A$$

where

$$p(t) = -E\left[\Phi^*(T,t)g_x\left(X_T^*\right)/\mathcal{F}_t\right]$$

We denote by $\Phi^*(T,t)$ the transpose of $\Phi(T,t)$.

Proof 8. *By the Theorem* (8) *we have:*

$$\frac{dJ\left(u_\theta\right)}{d\theta}=E\left\langle g_x\left(X_T\right), Z(T)\right\rangle$$

or

$$Z(T)=\Phi(T,t)Z(t)$$

$$=\Phi(T,t)\left[b\left(t, X_t^*, v\right)-b\left(t, X_t^*, u_t^*\right)\right]$$

and since the control u^* *is optimal we have:*

$$\frac{dJ\left(u_\theta\right)}{d\theta}\geq 0$$

so

$$E\left\langle g_x\left(X_T\right), Z(T)\right\rangle \geq 0$$

$$E\left\langle g_x\left(X_T\right), \Phi(T,t)\left[b\left(t, X_t^*, v\right)-b\left(t, X_t^*, u_t^*\right)\right]\right\rangle \geq 0$$

$$E\left\langle \Phi^t(T,t) g_x\left(X_T\right), b\left(t, X_t^*, v\right)\right\rangle - E\left\langle \Phi^t(T,t) g_x\left(X_T\right), b\left(t, X_t^*, u_t^*\right)\right\rangle \geq 0$$

Which give

$$E\left\langle \Phi^t(T,t) g_x\left(X_T\right), b\left(t, X_t^*, v\right)\right\rangle \geq E\left\langle \Phi^t(T,t) g_x\left(X_T\right), b\left(t, X_t^*, u_t^*\right)\right\rangle .$$

If we set

$$\hat{p}(t)=\Phi^t(T,t) g_x\left(X_T^*\right)$$

Noting that the process $\hat{p}(t)$ *is not* $\mathcal{F}_t$ *-adapted, we set*

$$p(t)=-E\left[\hat{p}(t)/\mathcal{F}_t\right]$$

so $p(t)$ *is a* $\mathcal{F}_t$ *-adapted process, which results in:*

$$E\left\langle E\left[\Phi^t(T,t) g_x\left(X_T\right)/F_t\right], b\left(t, X_t^*, v\right)\right\rangle \geq E\left\langle E\left[\Phi^t(T,t) g_x\left(X_T\right)\right], b\left(t, X_t^*, u_t^*\right)\right\rangle$$

We therefore deduce that $E\left\langle -E\left[\Phi^t(T,t) g_x\left(X_T\right)/\mathcal{F}_t\right], b\left(t, X_t^*, v\right)\right\rangle \geq E\left\langle -E\left[\Phi^t(T,t) g_x\left(X_T\right)\right], b\left(t, X_t^*, u_t^*\right)/\mathcal{F}_t\right\rangle$,

finally we obtain the following variational inequality:

$$E\left\langle p(t), b\left(t, X_t^*, v\right)\right\rangle \leq E\left\langle p(t), b\left(t, X_t^*, u_t^*\right)\right\rangle, \forall v \in A$$

Which completes the demonstration.

Corollary 1. *Let* u^* *be an optimal control, then there exists* $p(t)$ *a process* $\mathcal{F}_t-$ *adapted such as:*

$$\left\langle p(t), b\left(t, X_t^*, v\right)\right\rangle \leq \left\langle p(t), b\left(t, X_t^*, u_t^*\right)\right\rangle, \quad P.p.s,\ dt-p.p \forall v \in A$$

Proof 9. *We notice that if we use another perturbation such as: Let* $t_0 \in [0, T]$ *and if* $g \in \mathcal{F}_{t_0}$:

$$u_\theta = \begin{cases} v \ si \ (\omega, t) \in g \times [t_0, t_0 + \theta] \\ u_t^* \ sinon \end{cases}$$

We get the result of the previous theorem without expectation because $p(t)$ *and* $b(t, X_t, v)$ *are* $mathcal F_t$ *-adapted.*

Theorem 10. *Let* (X^*, u_t^*) *be the optimal solution. Then there exists a process* $p(t)$ *which is* $\mathcal{F}_t$ *-adapte such that:*

$$p(t) = -E\left[\Phi^t(T,t) g_x\left(X_T^*\right) / \mathcal{F}_t\right]$$

And

$$H\left(t, X_t^*, v, p_t\right) \leq H\left(t, X_t^*, u_t^*, p_t\right) \forall v \in A dt - p \cdot p., P - p.s$$

where H *is the Hamiltonian associated with our stochastic control problem of which it is defined by:*

$$H\left(t, X_t^*, u_t, p_t\right) = \left\langle p_t, b\left(t, X_t, u_t\right)\right\rangle$$

Everything we have done previously concerns the case of a function given by:

$$J(u) = E\left[g\left(X_T\right)\right]$$

But if the cost function is given in the following form:

$$J(u) = E\left[\int_0^T \ell\left(t, X_t, u_t\right) dt + g\left(X_T\right)\right]$$

In this case, the Hamiltonian is equal to:

$$H\left(t, X_t^*, u_t, p_t\right) = \left\langle p_t, b\left(t, X_t, u_t\right)\right\rangle - \ell\left(t, X_t, u_t\right)$$

This is how the theorem of the maximum principle becomes:

Theorem 11. *Let* (X^*, u_t^*) *be the optimal solution of our control problem.*

Then there exists a process $p(t)$ *which is* $\mathcal{F}_t -$ *adapted such that:*

1) $dp_t = -H\left(t, X_t^*, u_t, p_t\right) dt + q_t dB_t, p(T) = -g_x\left(X_T^*\right)$

2) $H\left(t, X_t^*, u_t^*, p_t\right) = \max_{v \in A} H\left(t, X_t^*, v, p_t\right) P - p.s, dt - p.p.$

See J.Yong [16], *Zhoo* [17, 18].

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