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Ky-Fan norm and its applications in inequalities between spectral values

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Dedication

I dedicate my work to:

my mother and my father who were my first support and behind everything I am now, I love you both.

My sisters and brother are my life partners, I love you.

my grandmother, my deceased grandfather and grandmother, my aunt and all the family and loved ones.

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I thank **ALLAH** for helping me complete this thesis, and I pray to ALLAH to help me with more success and excellence to continue my studies.

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Abstract

Our main goal in this thesis, is to improve or find some important inequalities related to the usual operator norm and the numerical radius by using the $Ky-Fan$ norm defined on $\mathcal{K}(\mathcal{H})$ space of compact linear operators on the complex Hilbert space, we study the most important properties of this norm, trying to get satisfactory results.

Key-words : Complex Hilbert space, bounded linear operators, compact operators, spectrum, numerical range, usual operator norm, spectral radius, numerical radius.

ملخص

هدفنا الأساسي في هذه المذكرة، هو تحسين أو إيجاد بعض المتراجحات المهمة المتعلقة بالنظيم الاعتيادي للمؤثرات الخطية و نصف القطر الرقمي، وذلك باستعمال نظيم $Ky-Fan$ المعرف على $\mathcal{K}(\mathcal{H})$ فضاء المؤثرات الخطية المتراصة على فضاء هيلبرت المركب $\mathcal{H}$ درسنا أهم خواص هذا النظيم، محاولين الحصول على نتائج مرضية.

كلمات مفتاحية: فضاء هلبرتي مركب، مؤثر خطي محدود، مؤثر خطي متراص، طيف مؤثر خطي، الصورة الرقمية للمؤثر الخطي، النظيم الاعتيادي للمؤثرات الخطية، نصف القطر الطيفي، نصف القطر الرقمي.

Résumé

Notre objectif principal dans cette mémoire, est d'améliorer ou de trouver certaines inégalités importantes liées à la norme de l'opérateur usuel et au rayon numérique en utilisant la norme $Ky-Fan$. définie sur l'espace $\mathcal{K}(\mathcal{H})$ des opérateurs linéaires compacts sur l'espace de Hilbert complexe, nous étudier les propriétés les plus importantes de cette norme, en essayant d'obtenir des résultats satisfaisants.

Mots clés : Espace de Hilbert complexe, opérateurs linéaires bornés, opérateurs compacts, spectre, domaine numérique, norme usuelle des opérateurs, rayon spectral, rayon numérique, norme de $Ky-Fan$.

Contents

Notations

$\mathbb{N}$: The set of natural numbers.

$\mathbb{R}$: The set of real numbers.

$\mathbb{C}$: The set of complex numbers.

$\mathbb{F}$: $\mathbb{R}$ or $\mathbb{C}$.

$\mathcal{H}$: Complex Hilbert space.

$\langle ., . \rangle$: The inner product of $\mathcal{H}$.

$\overline{\mathcal{M}}$: The closure of $\mathcal{M}$.

$\mathcal{M}^{\perp}$:The orthogonal complement of $\mathcal{M}$.

$\oplus$:The sign of direct sum.

$\mathcal{B}(\mathcal{H})$: Banach algebra of all bounded linear operators on Hilbert space $\mathcal{H}$.

$\mathcal{I}(\mathcal{H})$: The set of invertible operators in $\mathcal{B}(\mathcal{H})$.

$\mathcal{K}(\mathcal{H})$: The set of compact operators in $\mathcal{B}(\mathcal{H})$.

T : A bounded linear operator defined on $\mathcal{H}$.

$\|T\|$: The norm of T.

T^{-1} : The inverse operator of T.

T^* : The adjoint operator of T.

$\Re(T)$: The real part of T.

$\Im(T)$: The imaginary part of T.

$|T|$: The absolute value of T.

$Ran(T)$: The range of T.

$Ker(T)$: The kernel of T.

$\sigma(T)$: The spectrum of T.

$\sigma_p(T)$: The point spectrum of T.

$\sigma_{ap}(T)$: The approximate point spectrum of T.

$\rho(T)$: The resolvent of T.

$r(T)$: Spectral radius of T.

$W(T)$: The numerical range of T.

$\omega(T)$: The numerical radius of T.

$s(T)$: The singularvalue of T.

$\|T\|_{(k)}$: The Ky-Fan norm of T.

Introduction

In mathematics, operator theory is one of the most important theorems in Functional Analysis. Operator theory is the study of linear operators on function spaces, and consideration may be given to nonlinear operators. The study, which depends heavily on the topology of function spaces. In the following, we will take a small overview about where this theory has come from and some of the people who have brought it to this point.

The spectral radius of an operator $T \in \mathcal{B}(\mathcal{H})$, is given by

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

The spectral radius, the usual norm on $T \in \mathcal{B}(\mathcal{H})$ and the numerical radius are very important values in the study of linear operators and its properties.

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with usual inner product $\langle ., . \rangle$. For $T \in \mathcal{B}(\mathcal{H})$. The numerical range of T, denoted by $W(T)$, was introduced by Toeplitz in [28] as

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$$

The Toeplitz-Hausdorff theorem has many proofs (see [28], [14] and [16]).
The numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$, is given by

$$\omega(T) = \sup\{|\langle Tx, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}$$

the numerical radius is an equivalent norm to the usual norm on $\mathcal{B}(\mathcal{H})$. Moreover, this concept was the subject of a wide literature carrying out many inequalities involving it. Some developments toward this subject have been done in [19], [18] by many researchers such as F. Kittaneh and S.S.Dragomir , and there are many recent results (see [27]).
The spectrum of a bounded linear operator of $T \in \mathcal{B}(\mathcal{H})$ defined as follows

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not invertible}\}$$

In this thesis, we will study the properties and some classifications of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$, as well as the spectrum and the spectral radius of an operator, numerical range and the numerical radius in detained. Lastly, we will define the $Ky - Fan$ norm on $\mathcal{K}(\mathcal{H})$, and we will study its properties and its applications, especially on the inequalities between the numerical radius and the usual norm on $\mathcal{B}(\mathcal{H})$.

This thesis is divided into this introduction four chapter, conclusion bibliography

In chapter 1, we give a selection of known properties in Hilbert spaces. Also we include some vital theorems such as Orthocomplements and Riesz's representation.

In chapter 2, the study of bounded linear operators. First, we define the adjoint of an operator, and we study some properties of the adjoint. Then, we offer some bounded linear operators classifications, and we go through their properties. Finally, we provide the definition of the square root and the absolute value of an operator, and we give some of their properties.

In chapter 3, we study the spectrum of an operator and its properties in detailed, and we provide the proof of every single property and theorem, then we consider the spectral radius with its known properties.
The numerical range and the numerical radius with their properties, also we provide a some kind of inequalities.

chapter 4 is the most important chapter in this thesis. First we define the compact operator and its properties, also we define the $Ky - Fan$ norm that is equivalent to the usual norm, and we study some of its properties. Then, we obtain inequalities involving the new norm, the usual norm and the numerical radius, and we obtain some new inequalities involving the usual norm and the numerical radius, also we get a refinement for some inequalities .

Chapter 1

Preliminaries

In this chapter, we present the most important definitions and theorems, around inner product space, Hilbert spaces and linear operator space, that we need throughout this thesis .

1.1 Inner product space and Hilbert space

Defnition 1.1 *Let X be a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). X is said to be a normed space if there exists a map $\|.\|$ that is defined from X to $\mathbb{R}_+$ satisfies the following:*

1) $\|x\| = 0$ *if and only if* $x = 0$,
2) $\|\lambda x\| = |\lambda|\|x\|$ *for any* $x \in X$ *and* $\lambda \in \mathbb{F}$,
3) $\|x + y\| \leq \|x\| + \|y\|$ *for any* $x, y \in X$.

And we denote $(X, \|.\|)$ is a normed space.

Defnition 1.2 *A sequence $\{x_n\} \in X$ is called Cauchy sequence if and only if:*

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N}, \quad n > N \text{ and } m > N \Longrightarrow |x_n - x_m| < \epsilon$$

.

Defnition 1.3 *Let $(X, \|.\|)$ be a normed space. Then $(X, \|.\|)$ is said to be a Banach space if every Cauchy sequence has a limit in X.*

Defnition 1.4 *Let X be a vector space over $\mathbb{F}$. An inner product (or a scalar product) on X is a map $\langle ., . \rangle : X \times X \longrightarrow \mathbb{F}$ such that the following three properties hold:*

1) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y + z \rangle$ *for all* $x, y, z \in X$ *and* $\lambda \in \mathbb{F}$ *(Linearity in the first argument),*
2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *for all* $x, y \in X$ *(Conjugate symmetry),*
3) $\langle x, x \rangle > 0$ *for all* $x \in X \backslash \{0\}$ *(Positive-definiteness).*

The pair $(X, \langle ., . \rangle)$ is called inner product space.

Proposition 1.1 *In the following properties, which result almost immediately from the definition of an inner product, x, y and z are arbitrary vectors, and λ arbitrary scalar.*

1) $\langle 0, x\rangle = 0$ *for all* $x \in X$,

2) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x+z\rangle$,

3) $\langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$.

Theorem 1.1 *Any inner product space* $(X, \langle ., .\rangle)$ *is a normed space where* $\|x\| = |\langle x, x\rangle|^{\frac{1}{2}}$ *and this norm is called the associated norm of the inner product.*

Theorem 1.2 *(Cauchy-Schwarz inequality)*
Let $(X, \langle ., .\rangle)$ *be an inner product space. Then we have*

$$|\langle x, y\rangle| \leq \|x\|\|y\| \quad \text{for all } x, y \in X$$

Moreover if $|\langle x, y\rangle| = \|x\|\|y\|$*, then* x *and* y *are linearly dependent.*

Corollary 1.1 *To have equality in the the Cauchy-Schwarz inequality for two vectors* x *and* y *(that is, to have* $|\langle x, y\rangle| = \|x\|\|y\|$*) it is necessary and sufficient that* x *and* y *are linearly dependent.*

Lemma 1.1 *Let* X *be an inner product space. Then* $\langle ., .\rangle : X \times X \longrightarrow \mathbb{F}$ *is continuous.*

Lemma 1.2 *Let* $(X, \langle ., .\rangle)$ *be an inner product space and suppose that* $(x_{n\in\mathbb{N}})$ *and* $(y_{n\in\mathbb{N}})$ *are convergent sequences in* X*, with* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$*. Then*

$$\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x, y\rangle$$

.

Theorem 1.3 *[6] (Buzano's inequality)*
Let $(X, \langle ., .\rangle)$ *be an inner product space and* $x, y, e \in X$ *with* $\|e\| = 1$*. Then we have*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + \langle x, y\rangle).$$

Proof. Let x, y, e be as in the theorem

$$\begin{aligned}
2|\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle| &= |\langle x, 2\overline{\langle e, y\rangle}e\rangle| - |\langle x, y\rangle| \\
&\leq |\langle x, 2\overline{\langle e, y\rangle}e\rangle - \langle x, y\rangle| \\
&= |\langle x, 2\overline{\langle e, y\rangle}e - y\rangle| \quad (use\, Cauchy - Schwarz\; inequality) \\
&\leq \|x\|\|2\overline{\langle e, y\rangle}e - y\|
\end{aligned}$$

So we have

$$2|\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle| \leq \|x\|\|2\overline{\langle e, y\rangle}e - y\| \tag{1.1}$$

Let us calculate $\|2\overline{\langle e, y\rangle}e - y\|$

$$\begin{aligned}
\|2\overline{\langle e, y\rangle}e - y\|^2 &= \langle 2\overline{\langle e, y\rangle}e - y, 2\overline{\langle e, y\rangle}e - y\rangle \\
&= 4|\langle e, y\rangle|^2\|e\|^2 + \|y\|^2 - 2\langle e, y\rangle\overline{\langle e, y\rangle} - 2\langle e, y\rangle\langle y, e\rangle \quad (using \|e\|^2 = 1\; and \langle y, e\rangle = \overline{\langle e, y\rangle}) \\
&= 4|\langle e, y\rangle|^2 + \|y\|^2 - 4|\langle e, y\rangle|^2 \\
&= \|y\|^2
\end{aligned}$$

So $\|2\overline{\langle e, y\rangle}e - y\| = \|y\|$.
Using (1.1)

$$2|\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle| \leq \|x\|\|y\|.$$

Hence

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|).$$

Remark 1.1 *1) Let $x, y, z \in X$. From Buzano's inequality we obtain*

$$|\langle x, y\rangle\langle y, z\rangle| \leq \frac{1}{2}(\|x\|\|z\| + |\langle x, z\rangle|)\|y\|^2. \tag{1.2}$$

2) Buzano's inequality is a generalization of Cauchy-Schwarz inequality, just put $z = x$ in (1.2) to obtain Cauchy-Schwarz inequality.

Proposition 1.2 *Let $(X, \langle .,.\rangle)$ be an inner product space. Then we have*

1) $\forall x, y \in X : \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2\|y\|^2)$ (Parallelogram law),

2) $\forall x, y \in X : \|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\langle x, y\rangle$ (Polarization identity),

3) $\forall x, y, z \in X : \|x - y\|\,\|z\| \ + \ \|y - z\|\,\|x\| \ \geq \ \|x - z\|\,\|y\|$ (Ptolemy's inequality),

4) if X is a real vector space, then $\forall x, y \in X : \langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$,

5) if X is a complex vector space, then

$$\forall x, y \in X : \langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) \ \text{(Polarization identity)}$$

. Where $\|x\| = |\langle x, x\rangle|^{\frac{1}{2}}$ and i is the imaginary complex numbers.

Defnition 1.5 *Let $\mathcal{H}$ be a vector space over $\mathbb{C}$. $\mathcal{H}$ is said to be a Hilbert space if it is an inner product space and $\mathcal{H}$ with associated norm is a Banach space, and we denote $(\mathcal{H}, \langle .,.\rangle)$ is a complex Hilbert space.*

Example 1.1 *1) A Hilbert space is $\mathbb{C}^n$ with the inner product:*

$$\forall x, y \in \mathbb{C}^n \quad \langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y_i}.$$

2) A Hilbert space is $\ell^2(\mathbb{C}) = \{x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{C};\ \sum_{n=1}^{\infty} |x_i|^2 < \infty\}$. with the inner product:

$$\forall x, y \in \ell^2(\mathbb{C}) \quad \langle x, y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i}\,.$$

3) A Hilbert space is $L^2(A, \mu) = \{f : A \longrightarrow \mathbb{C}; f$ is measurable and $\int_A |f(x)|^2 d\mu < \infty\}$ with the inner product:

$$f, g \in L^2(A, \mu) \quad \langle f, g\rangle = \int_A f(x)\overline{g(x)}d\mu.$$

From now on we consider $(\mathcal{H}, \langle .,.\rangle)$ a complex Hilbert space, and $\|.\|$ is the associated norm of the inner product.

1.2 Orthocomplements and Riesz's representation of Hilbert Space

Defnition 1.6 *Two vectors $x, y \in \mathcal{H}$ are said to be orthogonal if $\langle x, y\rangle = 0$, We also write in this case $x \perp y$.*

Defnition 1.7 *Let $\mathcal{M}$ be subset of $\mathcal{H}$. The orthogonal complement, written $\mathcal{M}^{\perp}$, of $\mathcal{M}$ is the set*

$$\mathcal{M}^{\perp} = \{x \in \mathcal{H} : \langle x, y\rangle = 0 \quad \forall y \in M\}.$$

Corollary 1.2 *If $x, y \in \mathcal{H}\backslash\{0\}$are orthogonal. Then they are linearly independent.*

Lemma 1.3 *(Pythagoras). If $x, y \in \mathcal{H}$ are orthogonal, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Defnition 1.8 *$\mathcal{M}$ is said to be a closed linear subspace of $\mathcal{H}$ if it is a linear subspace of $\mathcal{H}$, and $\overline{\mathcal{M}} = \mathcal{M}$.*

Theorem 1.4 *Let $\mathcal{M}$ be a subsets of $\mathcal{H}$ then*

1) *$\mathcal{M}^{\perp}$ is a closed linear subspace of $\mathcal{H}$,*

2) *$\mathcal{M} \subseteq (\mathcal{M}^{\perp})^{\perp}$,*

3) *If $\mathcal{M}$ is a linear set of $\mathcal{H}$, then $\mathcal{M} \cap \mathcal{M}^{\perp} = \emptyset$.*

Defnition 1.9 *Let $\mathcal{M}$ and $\mathcal{N}$ be two linear subspaces of $\mathcal{H}$. Then $\mathcal{H}$ is said to be the direct sum of $\mathcal{M}$ and $\mathcal{N}$, and we write $H = \mathcal{M} \oplus \mathcal{N}$, if*

1) *$\mathcal{H} = \mathcal{M} + \mathcal{N}$,*

2) *$\mathcal{M} \cap \mathcal{N} = \{0\}$.*

Lemma 1.4 *Let $\mathcal{M}$ and $\mathcal{N}$ be two linear subspaces of $\mathcal{H}$. Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ if and only if for every $x \in \mathcal{H}$ there exist unique vectors $y \in \mathcal{M}$ and $z \in \mathcal{N}$ such that $x = y + z$.*

Theorem 1.5 *(Orthogonal Decomposition)*
Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$. Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$is the direct sum of $\mathcal{M}$ and $\mathcal{M}^{\perp}$ i.e. for every $x \in \mathcal{H}$ there exist a unique $y \in \mathcal{M}$ and a unique $z \in \mathcal{M}^{\perp}$ such that $x = y + z$.

Proposition 1.3 *Let $\mathcal{M}$ and $\mathcal{N}$ be two subsets of $\mathcal{H}$ then*

1) *$\mathcal{H}^{\perp} = \{0\}$ and $\{0\}^{\perp} = \mathcal{H}$,*

2) *If $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{N}^{\perp} \subseteq \mathcal{M}^{\perp}$,*

3) *$(\mathcal{M}^{\perp})^{\perp} = \overline{\mathcal{M}}$,*

4) *$\mathcal{H} = \overline{\mathcal{M}} \oplus \mathcal{M}^{\perp}$.*

Defnition 1.10 *Let φ be a mapping from $\mathcal{H}$ to $\mathbb{C}$. Then φ is said to be a linear function if it satisfies:*

$$\forall x, y \in \mathcal{H}, \forall \lambda \in \mathbb{C} \quad \varphi(\lambda x + y) = \lambda\varphi(x) + \varphi(y)$$

Defnition 1.11 *Let φ be a linear function from $\mathcal{H}$ to $\mathbb{C}$. Then φ is said to be a bounded linear function if it satisfies:*

$$\exists c > 0, \forall x \in \mathcal{H} : \quad |\varphi(x)| \le c\|x\|.$$

Defnition 1.12 *The set of all bounded linear forms on $\mathcal{H}$ is called the topological dual of $\mathcal{H}$ and is denoted $\mathcal{H}'$, where the addition and the external product are defined as follows*

1) $\forall x \in \mathcal{H} \quad (\varphi + \psi)(x) = \varphi(x) + \psi(x),$

2) $\forall x \in \mathcal{H} \quad (\lambda\varphi)(x) = \lambda\varphi(x).$

Proposition 1.4 *Let $f \in \mathcal{H}'$ and is a normed space when we equip it with the following norm Let $f \in \mathcal{H}'$then*

$$\|f\|_{\mathcal{H}'} = \sup\{|f(x)| : \|x\| = 1\}.$$

Theorem 1.6 *Let $\mathcal{H}'$ the topological dual of $\mathcal{H}$, then $(\mathcal{H}', \|\|_{\mathcal{H}'})$ is a Banach space.*

Defnition 1.13 *For all $\varphi \in \mathcal{H}'$, the range of $x \in \mathcal{H}$ by φ is denoted $\langle \varphi, x\rangle$.Then*

$$\varphi \in \mathcal{H}' \qquad\qquad \varphi : \mathcal{H} \longrightarrow \mathbb{F},\ x \longrightarrow \langle \varphi, x\rangle = \varphi(x)$$

Is called duality product.

Theorem 1.7 *(Riesz's representation theorem)*
Let y be any arbitrary fixed in $\mathcal{H}$, we define $\varphi(x)$ by

$$\varphi(x) = \langle x, y\rangle \qquad\qquad \forall x \in \mathcal{H}$$

Then $\varphi \in \mathcal{H}'$ such that $\|\varphi\|_{\mathcal{H}'} = \|y\|$. Conversely, for any $\varphi \in \mathcal{H}'$ there exists a unique $y \in \mathcal{H}$ such that

$$\varphi(x) = \langle x, y\rangle \qquad\qquad \forall x \in \mathcal{H}$$

1.3 Matrix of a Linear Transformations and $\mathcal{B}(\mathcal{H})$ Space

Defnition 1.14 *A linear transformation is a map $T : V \longrightarrow W$ between vector spaces which preserves vector addition and scalar multiplication. It satisfies*

1) $T(v_1 + v_2) = Tv_1 + Tv_2 \quad \forall v_1, v_2 \in V,$

2) $T(\lambda v) = \lambda Tv \quad \forall v \in V$ *and* $\forall \lambda \in \mathbb{R}.$

Theorem 1.8 *Every linear transformation $T : V \longrightarrow W$ is uniquely determined by the values $T(v_1), T(v_2), \cdots, T(v_n)$ when $v_1, v_2, \cdots, v_n$ form a basis of V .*

Remark 1.2 *$v \in V$ can be expressed as a linear combination*

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some unique coefficients $\alpha_1, \alpha_2, \cdots, \alpha_n$ and this implies that

$$T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n).$$

Theorem 1.9 *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exiset a unique matrix A such that*

$$T(x) = Ax \quad \textit{for all} \;\; x \in \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose ith column is the vector $T(e_i)$, where e_i is the ith column of the identity matrix in $\mathbb{R}^n$

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$$

Example 1.2 *Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 17 & -20 \\ 12 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17x_1 & -20x_2 \\ 12x_1 & -14x_2 \end{bmatrix}$$

Using the standard basis e_1, e_2 of $\mathbb{R}^2$, we find that

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 17 \\ 12 \end{bmatrix} = 17 \cdot e_1 + 12 \cdot e_2$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -20 \\ -14 \end{bmatrix} = -20 \cdot e_1 - 14 \cdot e_2$$

The matrix of T with respect to the standard basis is

$$A = \begin{bmatrix} 17 & -20 \\ 12 & -14 \end{bmatrix}$$

Defnition 1.15 *A mapping $T : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be a linear operator on $\mathcal{H}$ if it satisfies the following properties*

1) $T(x + y) = Tx + Ty \;\; \forall x, y \in \mathcal{H}$,

2) $T(\lambda x) = \lambda Tx \;\; \forall x \in \mathcal{H}$ *and* $\forall \lambda \in \mathbb{C}$.

Defnition 1.16 *A linear operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be bounded linear operator if there exists $\exists\, c > 0$ such that for every $x \in \mathcal{H}$ we have that*

$$\|Tx\| \leq c\|x\|$$

We denoted the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Defnition 1.17 *The set of all bounded linear operators on $\mathcal{H}$ is an unitary algebra over $\mathbb{C}$ $\mathcal{B}(\mathcal{H})$, where the addition, the external product and the product are defined as follows*
Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$

1) $\forall x \in \mathcal{H}$, $(T + S)x = Tx + Sx$,

2) $\forall x \in \mathcal{H}$, $(\lambda T)x = \lambda Tx$,

3) $\forall x \in \mathcal{H}$, $(TS)x = T(Sx)$.

And the unitary element is I the identity($Ix = x, \forall x \in \mathcal{H}$), moreover $\mathcal{B}(\mathcal{H})$ is a Banach algebra when we equip it with the following norm
Let $T \in \mathcal{B}(\mathcal{H})$, then

$$\|T\| = \inf\{c > 0 : \|Tx\| \leq c\|x\| \quad for\ all\ x \in \mathcal{H}\}$$

Theorem 1.10 *Let $T \in \mathcal{B}\mathcal{H})$. Then*

1) $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$,
2) $\|Tx\| \leq \|T\|\|x\| \ \forall x \in \mathcal{H}$.

Lemma 1.5 *Let $T \in \mathcal{B}\mathcal{H})$. Then*

$$\forall x, y \in \mathcal{H} : \quad \|T(x) - T(y)\| \leq \|T\|\|x - y\|$$

Lemma 1.6 *Let $T \in \mathcal{B}\mathcal{H})$. Then*

1) $\|T\| = \sup\{\frac{\|Tx\|}{\|x\|} : x \in \mathcal{H} \backslash \{0\}\}$,
2) $\|T\| = \sup\{|\langle Tx, y\rangle| : \|x\| = \|y\| = 1\}$.

Theorem 1.11 *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\|TS\| \leq \|T\|\|S\|$$

Corollary 1.3 *Let $T \in \mathcal{B}(\mathcal{H})$. Then for any $n \geq 1$, $T^n \in \mathcal{B}(\mathcal{H})$ and $\|T^n\| \leq \|T\|^n$.*

Defnition 1.18 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1) *The kernel of T is the set*

$$Ker(T) = \{x \in \mathcal{H} : Tx = 0\},$$

2) *The range of T is the set*

$$Ran(T) = \{Tx : x \in \mathcal{H}\}.$$

$Ker(T)$ and $Ran(T)$ are linear subspaces of $\mathcal{H}$.

Proposition 1.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then then $Ker(T)$ is closed linear subspaces of $\mathcal{H}$.*

Remark 1.3 *This is not necessarily true for $Ran(T)$.*

Theorem 1.12 *Let$T \in \mathcal{B}(\mathcal{H})$. Then*

$$\forall x \in \mathcal{H} \qquad \langle Tx, x\rangle = 0 \quad \Longrightarrow \quad T = 0.$$

Proof. Let $x, y \in \mathcal{H}$ suppose that $\langle Tx, x\rangle = 0$ for any $x \in \mathcal{H}$, then
Since (use Generalized polarization identity).

$$\langle Tx, y\rangle = \frac{1}{4}\{\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle\}.$$

$$\Longrightarrow \langle Tx, y\rangle = 0 \ \forall x, y \in \mathcal{H}.$$

Setting $y = Tx$ we get

$$\|Tx\|^2 = \langle Tx, Tx\rangle = 0 \ \forall x \in \mathcal{H}$$

$$\Longrightarrow Tx = 0 \ \forall x \in \mathcal{H}.$$

There fore $T = 0$.

Remark 1.4 *Theorem 1.12 is not correct in real Hilbert spaces.*
Considering the following real Hilbert space $(\mathbb{R}^2, \langle ., . \rangle)$ *and let* $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *note that* $\langle Tx, Tx \rangle = 0$ *for all* $x \in \mathbb{R}^2$ *but* $T \neq 0$.

Corollary 1.4 *Let* $T, S \in \mathcal{B}(\mathcal{H})$. *If* $\langle Tx, x \rangle = \langle Sx, x \rangle$ *for all* $x \in \mathcal{H}$, *then* $T = S$.

Defnition 1.19 *Let* $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ *be a sequence of operators and* $T \in \mathcal{B}(\mathcal{H})$. *Then*

1) $(T_n)_{n\in\mathbb{N}}$ *is said to be uniformly convergent to* T *if*

$$\lim_{n\longrightarrow\infty} \|T_n - T\| = 0$$

And we write $T_n \xrightarrow{u} T$,

2) $(T_n)_{n\in\mathbb{N}}$ *is said to be strongly convergent to* T *if*

$$\forall x \in \mathcal{H} \lim_{n\longrightarrow\infty} \|T_n x - Tx\| = 0$$

And we write $T_n \xrightarrow{s} T$,

3) $(T_n)_{n\in\mathbb{N}}$ *is said to be weakly convergent to* T *if*

$$\forall x, y \in \mathcal{H} \quad \lim_{n\longrightarrow\infty} \langle x, T_n y - Ty \rangle = 0$$

And we write $T_n \xrightarrow{w} T$.

Proposition 1.6 *Let* $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ *be a sequence of operators and* $T \in \mathcal{B}(\mathcal{H})$. *Then*

$$T_n \xrightarrow{u} T \Rightarrow T_n \xrightarrow{s} T \Rightarrow T_n \xrightarrow{w} T.$$

Chapter 2

Definitions and basic properties of Bounded Linear Operators

This chapter contains some definitions and basic properties of linear operators on Hilbert spaces, and some other propeities and results.

2.1 The inverse and the adjoint of a linear operator

Defnition 2.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is said to be invertible if there exists S operator such that*

$$TS = ST = I$$

I is the identity operator.
We write $S = T^{-1}$ is the inverse of T.

Theorem 2.1 *Let $T \in \mathcal{B}(\mathcal{H})$. If T is bijective, then $T^{-1} \in \mathcal{B}(\mathcal{H})$, we denotes to the set of all invertible operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{I}(\mathcal{H})$.*

Theorem 2.2 *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\| < 1$. Then $I - T$ is invertible and the inverse given by*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T_n \quad \text{and} \quad \|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|}.$$

Corollary 2.1 *Let $T \in \mathcal{B}(\mathcal{H})$. If T is bijective, then T is invertible.*

Lemma 2.1 *Let $\mathcal{I}(\mathcal{H})$ be the set of all invertible operators in $\mathcal{B}(\mathcal{H})$. Then $\mathcal{I}(\mathcal{H})$ is open.*

Proof. Let $T \in \mathcal{I}(\mathcal{H})$ and let $\epsilon = \|T^{-1}\|^{-1}$, let's prove that $B(T, \epsilon) \subset \mathcal{I}(\mathcal{H})$ where

$$B(T, \epsilon) = \{S \in \mathcal{B}(\mathcal{H}) : \|T - S\| < \epsilon\}$$

Let $S \in B(T, \epsilon) \iff \|T - S\| < \epsilon$,then

$$\begin{aligned} \|(T - S)T^{-1}\| &\leq \|T - S\|\|T^{-1}\| \\ &< \epsilon\|T^{-1}\| \\ &= \|T^{-1}\|^{-1}\|T^{-1}\| = 1 \end{aligned}$$

Then $\|(T - S)T^{-1}\| < 1 \Longrightarrow I - (T - S)T^{-1}$is invertible (using Theorem 2.2).
But $I - (T - S)T^{-1} = I - I + ST^{-1} = ST^{-1}$, which means that ST^{-1} is invertible.
We have $S = ST^{-1}T$, since ST^{-1} and T are invertible $\Longrightarrow S$ is invertible $\Longrightarrow S \in \mathcal{I}(\mathcal{H})$.
Therefore $B(T, \epsilon) \subset \mathcal{I}(\mathcal{H})$, then $\mathcal{I}(\mathcal{H})$ is open.

Lemma 2.2 *Let $T, S \in I(\mathcal{H})$, and $n \in \mathbb{N}$. Then*

1) The inverse of T^{-1} is T,

2) The inverse of ST is $T^{-1}S^{-1}$.

Corollary 2.2 *Let $T \in I(\mathcal{H})$, and $n \in \mathbb{N}$. Then the inverse of T^n is T^{-n}.*

Theorem 2.3 *$F : \mathcal{I}(\mathcal{H}) \longrightarrow \mathcal{I}(\mathcal{H})$ defined by $F(T) = T^{-1}$ is continuous.*

Proof. Let $(T_n) \subset \mathcal{I}(\mathcal{H})$ and $T \in \mathcal{I}(\mathcal{H})$ such that $\lim\limits_{n\longrightarrow\infty} \|T_n - T\| = 0$. Since $T_n \xrightarrow[n\longrightarrow\infty]{} T$, then: $\exists n_0 \in N\ \forall n \geq n_0$:

$$\begin{aligned}
\|T_n - T\| < \frac{1}{2\|T^{-1}\|} &\Longrightarrow \forall n \geq n_0\ \|I - T^{-1}T_n\| = \|T^{-1}(T - T_n)\| \leq \|T^{-1}\|\|T_n - T\| < \frac{1}{2}. \\
&\Longrightarrow \forall n \geq n_0\ \|I - T^{-1}T_n\| < \frac{1}{2} \\
&\Longrightarrow \forall n \geq n_0\quad T^{-1}T_n \text{ is invertible}
\end{aligned}$$

And $\|(T^{-1}T_n)^{-1}\| \leq \dfrac{1}{1 - \|I - T^{-1}T_n\|} \leq 2$ (by Theorem 2.2), then $\forall n \geq n_0 \|(T^{-1}T_n)^{-1}\| < 2$. We have

$$\begin{aligned}
\forall n \geq n_0\quad \|T_n^{-1}\| &= \|T_n^{-1}TT^{-1}\| \\
&\leq \|T_n^{-1}T\|\|T^{-1}\| \\
&= \|(T^{-1}T_n)^{-1}\|\|T^{-1}\| < 2\|T^{-1}\|
\end{aligned}$$

Then

$$\forall n \geq n_0\quad \|T_n^{-1}\| < 2\|T^{-1}\| \Longrightarrow \sup_{n\in\mathbb{N}} \|T_n^{-1}\| \leq \max\{\|T_1^{-1}\|, \|T_2^{-1}\|, \cdots, \|T_{n_0}^{-1}\|, 2\|T^{-1}\|\} < \infty.$$

Then

$$\sup_{n\in\mathbb{N}} \|T_n^{-1}\| < \infty \Longleftrightarrow \exists\, c > 0\ \ \forall n \in \mathbb{N} :\quad \|T_n^{-1}\| \leq c.$$

Since $\|T_n^{-1} - T^{-1}\| = \|T_n^{-1}(T_n - T)T^{-1}\| \leq \|T_n^{-1}\|\|T_n - T\|\|T^{-1}\| \leq c\|T_n - T\|\|T^{-1}\| \xrightarrow[n\longrightarrow\infty]{} 0$. Then

$$\lim_{n\longrightarrow\infty} \|T_n^{-1} - T^{-1}\| = 0 \Longleftrightarrow F(T_n) \xrightarrow[n\longrightarrow\infty]{} F(T).$$

Therefore F is continuous.

Lemma 2.3 *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then*

$$\forall x \in \mathcal{H} :\quad \|Tx\| \geq \|T^{-1}\|^{-1}\|x\|.$$

Proof.$\forall y \in \mathcal{H} \|T^{-1}y\| \leq \|T^{-1}\|\|y\|$, let $x \in \mathcal{H}$ setting $y = Tx$, and since T is invertible $\|T^{-1}\| \neq 0$, we get :

$$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\| \Longrightarrow \|Tx\| \geq \|T^{-1}\|^{-1}\|x\| \text{ for all } x \in \mathcal{H}.$$

Lemma 2.4 *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator. Assume that there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in \mathcal{H}$, then $Ran(T)$ is closed.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \subset Ran(T)$ and $y \in \mathcal{H}$ such that $y_n \xrightarrow[n\longrightarrow\infty]{} y$, since $(y_n)_{n\in\mathbb{N}} \subset Ran(T)$, there exists $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $\forall n \in \mathbb{N}$ $y_n = Tx_n$, based on the assumption, we have:
$\forall n, m \in \mathbb{N} : \|y_n - y_m\| = \|Tx_n - Tx_m\| \geq \alpha\|x_n - x_m\|$, since $(y_n)_{n\in\mathbb{N}}$ is convergent, then $\lim\limits_{n,m\longrightarrow\infty} \|y_n - y_m\| = 0 \Longrightarrow \lim\limits_{n,m\longrightarrow\infty} \|x_n - x_m\| = 0 \Longleftrightarrow (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
Since $(H, \|\|)$ is a Banach space, then there exists $x \in \mathcal{H}$ such that $x_n \xrightarrow[n\longrightarrow\infty]{} x$.
We know that $\forall n \in \mathbb{N} y_n = Tx_n$, using the boundedness of T, we obtain :

$$y = \lim_{n\longrightarrow\infty} y_n = \lim_{n\longrightarrow\infty} Tx_n = T(\lim_{n\longrightarrow\infty} x_n) = Tx \Longrightarrow y = Tx, \text{then} y \in Ran(T).$$

Hence $\overline{Ran(T)} = Ran(T) \Longleftrightarrow Ran(T)$ is closed.

Theorem 2.4 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$T \textit{is invertible} \Longleftrightarrow \begin{cases} \exists\alpha > 0, \quad \|Tx\| \geq \|x\| \ \textit{for all}\, x \in \mathcal{H} \\ \overline{Ran(T)} = \mathcal{H} \end{cases}$$

Proof. Since T is invertible, then $Ran(T) = \mathcal{H}$ which means that $Ran(T)$ is dense in $\mathcal{H}$.

And from Lemma 2.3 we have that:

$$\forall x \in \mathcal{H} : \|Tx\| \geq \|T^{-1}\|^{-1}\|x\|.$$

Since $Ran(T)$ is dense in $\mathcal{H} \Longleftrightarrow \overline{Ran(T)} = \mathcal{H}$, and since there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in \mathcal{H}$, and using Lemma 2.4 then $Ran(T)$ is closed, then $Ran(T) = \mathcal{H}$.
On the other hand, let

$$x \in Ker(T) \Longleftrightarrow Tx = 0 \Longrightarrow 0 = \|Tx\| \geq \alpha\|x\| \Longrightarrow \|x\| = 0 \Longleftrightarrow x = 0.$$

Then $Ker(T) \subset \{0\} \Longrightarrow Ker(T) = \{0\}$ Hence T is bijective, using Corollary 2.1 we conclude that T is invertible.

Theorem 2.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a unique $T^* \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \forall x, y \in \mathcal{H}$$

And $\|T\| = \|T^*\|$.

Proof. Fix $y \in \mathcal{H}$ Define $f_y(x) = \langle Tx, y\rangle, \ x \in \mathcal{H}$.
Since T is bounded linear operator and using Cauchy-Schwarz inequality, we can show that f_y is a continuous linear functional on $\mathcal{H}$, that is, f_y is linear and

$$|f_y(x)| \leq \|T\|\|x\|\|y\|, \quad \forall x \in \mathcal{H}.$$

Then by Riesz representation theorem, there exists a unique $z \in \mathcal{H}$ such that $f_y(x) = \langle x, z\rangle$.
Therefore for each $y \in \mathcal{H}$, there exists a unique $z \in \mathcal{H}$ such that $f_y(x) = \langle x, z\rangle = \langle Tx, y\rangle$.
Now, we define $T^*y = z$, for each $y \in \mathcal{H}$. Thus

$$\langle x, T^*y\rangle = \langle Tx, y\rangle \ \ \forall x, y \in \mathcal{H}. \tag{2.1}$$

We prove that $T^* \in \mathcal{B}(\mathcal{H})$.
Consider $y, z \in \mathcal{H}$ and $\lambda \in \mathbb{F}$, we get

$$\begin{aligned} \langle x, T^*(\lambda y + z)\rangle &= \langle Tx, \lambda y + z\rangle \\ &= \overline{\lambda}\langle Tx, y\rangle + \langle Tx, z\rangle \\ &= \langle x, \lambda T^* y\rangle + \langle x, T^* z\rangle \\ &= \langle x, \lambda T^* y + T^* z\rangle, \ \forall x \in \mathcal{H}. \end{aligned}$$

Hence $T^*(\lambda y + z) = \lambda T^* y + T^* z, \ \ \forall y, z \in \mathcal{H}, \forall \lambda \in \mathbb{F}$.
That T^* is leniar. Since the equation (2.1) is satisfied for all elements in $\mathcal{H}$, choose $x = T^* y$ and using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|T^* y\|^2 &\leq \|T\|\|T^* y\|\|y\|, \ \ \forall y \in \mathcal{H}. \\ \Rightarrow \|T^* y\| &\leq \|T\|\|y\|, \forall y \in \mathcal{H}. \end{aligned}$$

Therefore, T^* is bounded and $\|T^*\| \leq \|T\|$. By a similar argument using (2.1), we can show that $\|T\| \leq \|T^*\|$.
Hence T^* is bounded and $\|T\| = \|T^*\|$
Then $T^* \in \mathcal{B}(\mathcal{H})$. We just still need to prove the uniqueness of T^*.
Assume that there exists T_1^* and T_2^* such that

$$\langle Tx, y\rangle = \langle x, T_1^* y\rangle = \langle x, T_2^* y\rangle \ \ \forall x, y \in \mathcal{H}$$

$$\langle x, (T_1^* - T_2^*)y\rangle = 0 \ \ \forall x, y \in \mathcal{H}$$

Hence $T_1^* y - T_2^* y = 0 \ \ \forall y \in \mathcal{H}$, that $T_1^* = T_2^*$.

Defnition 2.2 *The operator T^* is called the adjoint of T.*

Remark 2.1 *If we have an operator S such that $\langle x, Sy\rangle = \langle Tx, y\rangle, \forall x, y \in \mathcal{H}$, then by uniqueness $T^* = S$.*

Example 2.1 *1) Let $I \in \mathcal{B}(\mathcal{H})$ be the identity operator on $\mathcal{H}$. Then we have $I^* = I$,*

2) $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \longmapsto Ax$. Then $\langle Ax, y\rangle = (Ax)^T y = (x^T A^T)y = x^T(A^T y) = \langle x, A^T y\rangle$. Therefore, the adjoint A^* is the transpose A^T of A.

Lemma 2.5 *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\lambda, \mu \in \mathbb{F}$. Then*

1) $(\lambda T + \mu S)^* = \overline{\lambda} T^* + \overline{\mu} S^*$,

2) $(TS)^* = S^* T^*$.

Proof. 1)For all $x, y \in \mathcal{H}$,that

$$\begin{aligned} \langle x, (\lambda T + \mu S)^* y\rangle &= \langle(\lambda T + \mu S)x, y\rangle \\ &= \lambda\langle Tx, y\rangle + \mu\langle Sx, y\rangle \\ &= \langle x, \overline{\lambda} T^* y\rangle + \langle x, \overline{\mu} S^* y\rangle \\ &= \langle x, (\overline{\lambda} T^* + \overline{\mu} S^*)y\rangle \end{aligned}$$

Henc $(\lambda T + \mu S)^* = \overline{\lambda} T^* + \overline{\mu} S^*$.
2) For all $x, y \in \mathcal{H}$,that

$$\begin{aligned} \langle x, (TS)^* y\rangle &= \langle(TS)x, y\rangle \\ &= \langle Sx, T^* y\rangle \\ &= \langle x, S^* T^* y\rangle \end{aligned}$$

Henc $(TS)^* = S^* T^*$.

Theorem 2.6 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1) $(T^*)^* = T$,

2) $\|T^*T\| = \|TT^*\| = \|T\|^2$,

3) *The function $f : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ by $f(T) = T^*$ is continuous.*

Proof. 1) We have $\langle x, (T^*)^*y\rangle = \langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle \;\; \forall x, y \in \mathcal{H}$.
Henc $(A^*)^*y = Ay \;\; \forall y \in \mathcal{H}$, then $(A^*)^* = A$.
2) Recall that the norm of composition operators $\|SH\| \leq \|S\|\|H\|$. Hence, we get

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2 \tag{2.2}$$

For the other inequality, consider

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \\ &\leq \|x\|\|T^*Tx\| \\ &\leq \|TT^{*\|\|\|x\|^2} \end{aligned}$$

$$\|T\|^2 = \sup_{\|x\|\leq 1} \|Tx\|^2 \leq \sup_{\|x\|\leq 1} \|T^*T\|\|x\|^2 \leq \|T^*T\|$$

For the other inequality, consider

$$\|T\|^2 \leq \|T^*T\| \tag{2.3}$$

From (2.2) and (2.3) we obtain $\|T^*T\| = \|T\|^2$.
3) We have $\|f(T) - f(S)\| = \|T^* - S^*\| = \|(T - S)^*\| = \|T - S\|$.
That, for all $\epsilon > 0$ choose $\delta = \epsilon$ we have: $\forall \epsilon > 0, \; \exists \delta = \epsilon, \; \|T - S\| < \epsilon \Rightarrow \|T^* - S^*\| < \epsilon$.

Lemma 2.6 *Let $T \in \mathcal{H}$. That*

1) $Ker(T) = Ran(T^*)^{\perp}$,

2) $Ker(T^*) = Ran(T)^{\perp}$,

3) $Ker(T)^{\perp} = \overline{Ran(T^*)}$,

4) $Ker(T^*) = \{0\}$ *if and only if $Ran(T)$ dense in $\mathcal{H}$.*

Proof. 1) Suppose $x \in Ker(T)$, then $Tx = 0$. Thus $\langle T^*y, x\rangle = \langle y, Tx\rangle = 0$, for all $y \in \mathcal{H}$.
Therefore, $x \in Ran(T^*)^{\perp}$,that $Ker(T) \subset Ran(T^*)^{\perp}$.
Suppose $x \in Ran(T^*)^{\perp}$, that $T^*Tx \in Ran(T^*)$.
We have

$$\begin{aligned} \langle x, T^*Tx\rangle = 0 &\Rightarrow \langle Tx, Tx\rangle = 0 = \|Tx\|^2 = 0 \\ &\Rightarrow Tx = 0 \; and \; x \in Ker(T) \\ &\Rightarrow Ran(T^*)^{\perp} \subset Ker(T). \end{aligned}$$

Therefore $Ker(T) = Ran(T^*)^{\perp}$.
2) We have $Ker(T^*) = (Ran((T^*)^*))^{\perp}$, and since $(T^*)^* = T$, hence $Ker(T^*) = (Ran(T))^{\perp}$.
3) We have

$$Ker(T) = Ran(T^*)^{\perp} \Rightarrow (Ker(T))^{\perp} = (Ran(T^*)^{\perp})^{\perp} = \overline{Ran(T^*)}$$

4) We have $Ker(T^*) = 0 \Longleftrightarrow (Ran(T))^{\perp} = 0 \Longleftrightarrow \overline{Ran(T)} = \mathcal{H}$. Then $Ker(T^*) = \{0\}$ if and only if $Ran(T)$ dense in $\mathcal{H}$.

Corollary 2.3 *Let $T \in \mathcal{H}$. That $Ker(T^*)^\perp = \overline{Ran(T)}$.*

Lemma 2.7 *Let $T \in \mathcal{H}$ is invertible if and only if T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.*

Proof. Suppose T is invertible, that T^{-1} exist and we have $TT^{-1} = T^{-1}T = I$, so

$$(TT^{-1})^* = (T^{-1}T)^* = I^* = I$$

And $(T^{-1})^*T^* = T^*(T^{-1})^* = I$.
Hence T^* is invertible, and$(T^*)^{-1} = (T^{-1})^*$. Reciprocally if T^* is invertible.

Corollary 2.4 *Let $T \in \mathcal{B}(\mathcal{H})$*

$$T \textit{ is invertible.} \Longleftrightarrow \begin{cases} \exists \alpha > 0, \quad \|Tx\| \geq \alpha\|x\| \quad \textit{for all} \quad x \in \mathcal{H}. \\ Ker(T^*) = \{0\} \end{cases}$$

Proof. By Lemma2.6 and Theorem 2.4.

2.2 Classes of Operators on Hilbert Spaces

Defnition 2.3 *Let $T \in \mathcal{B}(\mathcal{H})$ is said to be*

1. *Self-adjoint operator if : $T^* = T$,*
2. *Normal operator if : $TT^* = T^*T$,*
3. *Positive operator if : $\forall x \in \mathcal{H} : \langle Tx, x\rangle \geq 0$, and we denote $T \geq 0$,*
4. *Unitary operator if : $TT^* = T^*T = I$,*
5. *Isometry operator if : $T^*T = I$,*
6. *Projection operator if : $T^2 = T$,*
7. *Orthogonal projection operator if : $T^2 = T = T^*$,*
8. *Quasinormal operator if : $T(T^*T) = (T^*T)T$,*
9. *Hyponormal operator if : $T^*T \geq TT^*$.*

Lemma 2.8 *Let $\mathcal{S}(\mathcal{H})$ be the set of all self-adjoint operators in $\mathcal{B}(\mathcal{H})$. Then*

1) For all $\lambda, \mu \in \mathbb{R}$ and $T, S \in \mathcal{S}(\mathcal{H})$, then $\lambda T + \mu S \in \mathcal{S}(\mathcal{H})$,

2) $\mathcal{S}(\mathcal{H})$ is a closed subset of $\mathcal{B}(\mathcal{H})$.

Proof.1) Let $\lambda, \mu \in \mathbb{R}$ and $T, S \in \mathcal{S}(\mathcal{H})$, then

$$(\lambda T + \mu S)^* = \lambda T^* + \mu S^* = \lambda T + \mu S.$$

2) Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathcal{H})$ convergent to $T \in \mathcal{B}(\mathcal{H})$.
The function $T \longmapsto T^*$ is continuous, then $T_n^* \longrightarrow T^*$.
But $\forall n \in \mathbb{N}, \;\; T_n = T_n^* \Rightarrow Tn \longrightarrow T^*$, from the uniqueness of the limit $T = T^*$.
Then $T \in \mathcal{S}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ is closed.

Proposition 2.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1) $TT^*, T^*T \in \mathcal{S}(\mathcal{H})$,

2) There exist self-adjoint operators R *and* S *such that* $T = R + iS$ *where* $R = \dfrac{1}{2}(T + T^*)$ *and* $S = \dfrac{1}{2i}(T - T^*)$.

Proof. 1) Let $T \in \mathcal{B}(\mathcal{H})$, then

$$(TT^*)^* = (T^*)^*T^* = TT^*$$

Hence $TT^* \in \mathcal{S}(\mathcal{H})$, the same for T^*T.
2) We have $R = \dfrac{1}{2}(T + T^*)$ and $S = \dfrac{1}{2i}(T - T^*)$, then

$$T = R + iS = \frac{1}{2}(T + T^*) + i\frac{1}{2i}(T - T^*) = T.$$

And $R^* = (\dfrac{1}{2}(T + T^*))^* = \dfrac{1}{2}(T^* + (T^*)^*) = R$, $S = (\dfrac{1}{2i}(T - T^*))^* = -\dfrac{1}{2i}(T^* - (T^*)^*) = S$.
Hence $R, S \in \mathcal{S}(\mathcal{H})$.

Theorem 2.7 *If* $T \in \mathcal{B}(\mathcal{H})$*is self-adjoint, then* $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

Proof. Let $m = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. Note that $|\langle Tx, x\rangle| \leq m$ for each $x \in \mathcal{H}$ with $\|x\| = 1$.
If $\|x\| = 1$, then by the $Cauchy - Schwartz$ inequality,

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| = \|Tx\|.$$

Hence $m \leq \|T\|$.
To prove the other way, let $x, y \in \mathcal{H}$. Then $\langle T(x \pm y), x \pm y\rangle = \langle Tx, x\rangle + 2\Re\langle Tx, y\rangle + \langle Ty, y\rangle$.
Therefore

$$\begin{aligned}
4\Re\langle Tx, y\rangle &= \langle T(x + y), x + y\rangle - \langle T(x - y), x - y\rangle \\
&\leq |\langle T(x + y), x + y\rangle| + |\langle T(x - y), x - y\rangle| \\
&\leq m\left(\|x + y\|^2 + \|x - y\|^2\right) \\
&= 2m(\|x\|^2 + \|y\|^2) \text{ (by the parallelogram law).}
\end{aligned}$$

Now $\langle Tx, y\rangle = |\langle Tx, y\rangle| \exp(i\theta)$ for some real θ. Substituting $x\exp(i\theta)$ in the above equation, we get

$$\begin{aligned}
4\Re\langle Tx\exp(-\imath\theta), y\rangle &\leq 2m(\|x\|^2 + \|y\|^2)4|\langle Tx, y\rangle| \\
&\leq 2m(\|x\|^2 + \|y\|^2).
\end{aligned}$$

Substituting $y = \dfrac{\|x\|}{\|Tx\|}\, Tx$ in place of y in the above equation, we get $\|Tx\| \leq m\|x\|$.
Hence $\|T\| = m$.

Theorem 2.8 *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then*

1) Let $\lambda = \inf_{\|x\|=1}\langle Tx, x\rangle$*. If there exists an* $x_1 \in \mathcal{H}$ *such that* $\|x_1\| = 1$ *and* $\lambda = \langle Tx_1, x_1\rangle$*, then* λ *is an eigenvalue of* T *with corresponding eigenvector* x_1,

2) Let $\mu = \sup_{\|x\|=1}\langle Tx, x\rangle$*. If there exists a* $x_2 \in \mathcal{H}$*,* $\|x_2\| = 1$ *such that* $\mu = \langle Tx_2, x_2\rangle$*, then* μ *is eigenvalue value of* T *with corresponding eigenvector* x_2.

Proos.1) Let $\alpha \in \mathbb{C}$ and $v \in \mathcal{H}$, by defintion of λ, we have

$$\langle T(x_1 + \alpha v), x_1 + \alpha v\rangle \geq \lambda\langle x_1 + \alpha v, x_1 + \alpha v\rangle.$$

Hence
$\langle Tx_1, x_1\rangle + \overline{\alpha}\langle Tx_1, v\rangle + \langle Tv, x_1\rangle + |\alpha|^2\langle Tv, v\rangle \geq \lambda\langle x_1, x_1\rangle + \lambda\overline{\alpha}\langle v, x_1\rangle + \lambda\langle v, x_1\rangle + \lambda|\alpha|^2\langle v, v\rangle$.
That is

$$\langle Tx_1, x_1\rangle + 2\alpha\Re\langle Tv, x_1\rangle + |\alpha|^2\langle Tv, v\rangle \geq \lambda\langle x_1, x_1\rangle + \lambda|\alpha|^2\langle v, v\rangle + 2\Re\alpha\lambda\langle x_1, v\rangle.$$

Substituting $\lambda = \langle Tx_1, x_1\rangle$ and taking $\alpha = r\langle v, (T - \lambda I)x_1\rangle$, $r \in \mathbb{R}$, we can conclude that $\langle v, (T - \lambda I)x_1\rangle = 0$ for each $v \in \mathcal{H}$. Hence $Tx_1 = \lambda x_1$.
The proof of the other statement follows by taking $-T$ in place of T.

Theorem 2.9 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$T \textit{is self-adjoint} \Longleftrightarrow \langle Tx, x\rangle \in \mathbb{R} \ \textit{ for all } x \in \mathcal{H}.$$

Proof. Assume that T is self-adjoint, then $\forall x \in \mathcal{H}, \langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle$.
Then $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$, therefore $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.
Conversely, suppose that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$, then
$\langle Tx, x\rangle = \langle x, Tx\rangle = \langle T^*x, x\rangle \Longrightarrow \langle (T - T^*)x, x\rangle = 0$ for all $x \in \mathcal{H}$
We get $T - T^* = 0 \Longrightarrow T = T^*$, hence T is self-adjoint.

Lemma 2.9 *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator, for all $\lambda \in \mathbb{C}$, then $T - \lambda I$ is normal operator.*

Proof. We have $TT^* = T^*T$, then

$$\begin{aligned}
(T - \lambda I)(T - \lambda I)^* &= (T - \lambda I)(T^* - \overline{\lambda}I) \\
&= TT^* - \overline{\lambda}T - \lambda T^* + \lambda\overline{\lambda}I \\
&= T^*T - \lambda T^* - \overline{\lambda}T + \lambda\overline{\lambda}I \\
&= T^*(T - \lambda I) - \overline{\lambda}I(T - \lambda I) \\
&= (T^* - \overline{\lambda}I)(T - \lambda I) \\
&= (T - \lambda I)^*(T - \lambda I)
\end{aligned}$$

Hence $T - \lambda I$ is normal operator.

Theorem 2.10 *(Fuglede's theorem) Let $T \in \mathcal{B}(\mathcal{H})$ be normal operator and $S \in \mathcal{B}(\mathcal{H})$. Then*

$$TS = ST \Longrightarrow T^*S = ST^*.$$

Lemma 2.10 *Let $T \in \mathcal{B}(\mathcal{H})$ be normal operator, Then*

1) $\|Tx\| = \|T^*x\|$ *for all* $x \in \mathcal{H}$,

2) *If there exists* $\alpha > 0$, $\|Tx\| \geq \alpha\|x\|$ *for all* $x \in \mathcal{H}$. *Then* $Ker(T^*) = 0$.

Proof.

$$\begin{aligned}
1) T \textit{ is normal} \Longleftrightarrow T^*T - TT^* = 0 &\Longleftrightarrow \forall x \in \mathcal{H} : \langle (T^*T - TT^*)x, x\rangle = 0 \\
&\Longleftrightarrow \forall x \in \mathcal{H} : \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle \\
&\Longleftrightarrow \forall x \in \mathcal{H} : \langle Tx, Tx\rangle = \langle T^*x, T^*x\rangle \\
&\Longleftrightarrow \forall x \in \mathcal{H} : \|Tx\|^2 = \|T^*x\|^2 \\
&\Longleftrightarrow \forall x \in \mathcal{H} : \|Tx\| = \|T^*x\|.
\end{aligned}$$

2) T is normal $\Longleftrightarrow \forall x \in \mathcal{H} : \|Tx\| = \|T^*x\|$, let $x \in Ker(T^*) \Longleftrightarrow T^*x = 0$
Then $0 = \|T^*x\| = \|Tx\| \geq \alpha\|x\| \Longrightarrow \|x\| = 0 \Longleftrightarrow x = 0 \Longrightarrow Ker(T^*) = 0$.

Corollary 2.5 *Let $T \in \mathcal{H}$ is a normal operator. Then the following are equivalent*

1) T is invertible,

2) There exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in \mathcal{H}$.

Theorem 2.11 *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator and $n \in \mathbb{N}$, then $\|T^n\| = \|T\|^n$.*

Proof. For all $n \in \mathbb{N}$: $\|T^n\| \leq \|T\|^n$ is always held.
So we need to prove that $\|T\|^n \leq \|T^n\|$ for all $n \in \mathbb{N}$, in case T is normal.
By induction. First, for $n = 2$, let $x \in \mathcal{H}$ then:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\|.$$

On the other hand, we have:

$$\|T^2x\|^2 = \langle T^2x, T^2x\rangle = \langle T^*TTx, Tx\rangle = \langle TT^*Tx, Tx\rangle = \langle T^*Tx, T^*Tx\rangle = \|T^*Tx\|^2$$

Then $\|T^2x\| = \|T^*Tx\|$ for all $x \in \mathcal{H}$, therefore $\|Tx\|^2 \leq \|T^2x\|\|x\| \leq \|T^2\|\|x\|^2$ for all $x \in \mathcal{H}$
Thus $\|T\|^2 \leq \|T^2\|$.
Second, suppose that $\|T\|^n \leq \|T^n\|$ is true, and let's prove that $\|T\|^{n+1} \leq \|T^{n+1}\|$ is also true.
Let $x \in \mathcal{H}$:

$$\|T^nx\|^2 = \langle T^nx, T^nx\rangle = \langle T^*T^nx, T^{n-1}x\rangle \leq \|T^*T^nx\|\|T^{n-1}x\|$$

But

$$\|T^*T^nx\|^2 = \langle T^*T^nx, T^*T^nx\rangle = \langle TT^*T^nx, T^nx\rangle = \langle T^*T^{n+1}x, T^nx\rangle = \langle T^{n+1}x, T^{n+1}x\rangle = \|T^{n+1}x\|^2.$$

Then

$$\forall x \in \mathcal{H} : \|T^*T^nx\| = \|T^{n+1}x\|.$$

Thus

$$\forall x \in \mathcal{H} : \|T^nx\|^2 \leq \|T^{n+1}x\|\|T^{n-1}x\|.$$

Therefore

$$\|T^n\|^2 \leq \|T^{n+1}\|\|T^{n-1}\| \leq \|T^{n+1}\|\|T\|^{n-1}.$$

So that

$$\|T^n\|^2 \leq \|T^{n+1}\|\|T\|^{n-1}.$$

But we have that $\|T\|^{2n} \leq \|T^n\|^2$, then $\|T\|^{2n} \leq \|T^{n+1}\|\|T\|^{n-1}$, thus $\|T\|^{n+1} \leq \|T^{n+1}\|$.
Hence $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$.

Theorem 2.12 *Let $T \in \mathcal{B}(\mathcal{H})$ be a strictly positive operator with closed range $Ran(T)$, then T is invertible.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ such that $T > 0 \Longleftrightarrow \forall x \in \mathcal{H}\backslash 0, \ \ \langle Tx, x\rangle > 0$.
Assume that $\{0\} \subsetneq Ker(T) \Longrightarrow \exists x_0 \in \mathcal{H} \ \{0\} : x_0 \in Ker(T) \Longrightarrow Tx_0 = 0 \Longrightarrow \langle Tx_0, x_0\rangle = 0$.
But this contradicts $T > 0$, then $Ker(T) = \{0\}$.
On the other hand, since $T > 0 \Longrightarrow T$ is self-adjoint$\Longrightarrow Ker(T) = Ker(T^*) = \{0\}$.
We know that $Ker(T^*) = (Ran(T))^\perp = \{0\} \Longrightarrow \overline{Ran(T)} = H$.
Since $Ran(T)$ is closed $\Longrightarrow Ran(T) = H$.
Hence T is bijective $\Longrightarrow T$ is invertible.

Corollary 2.6 *If $\mathcal{H}$ is finite-dimensional and if $T \in \mathcal{B}(\mathcal{H})$ is strictly positive, then T is invertible.*

Proposition 2.2 *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator and $n \in \mathbb{N}$. Then*

1) T^n is positive,

2) If T is invertible, then T^{-1} is also positive.

Proof. 1) Since T is positive $\Longrightarrow$ T is self-adjoint, if n is even, then there exists $k \in \mathbb{N}$ such that $n = 2k$.
Let $x \in \mathcal{H}$

$$\langle T^n x, x\rangle = \langle T^{2k}x, x\rangle = \langle T^k x, (T^k)^* x\rangle = \langle T^k x, (T^*)^k x\rangle = \langle T^k x, T^k x\rangle = \|T^k x\|^2 \geq 0.$$

In this case T^n is positive.
If n is odd, then there exists $k \geq 0$ such that $n = 2k+1$. Let $x \in \mathcal{H}$

$$\langle T^n x, x\rangle = \langle T^{2k+1}x, x\rangle = \langle T^{k+1}x, T^k x\rangle = \langle TT^k x, T^k x\rangle \geq 0 \quad (\textit{becaus } T \textit{ is positive}).$$

Overall Tn is positive for all $n \in \mathbb{N}$.
2) Assume that T is invertible, and we have $\forall x \in \mathcal{H} : \langle Tx, x\rangle \geq 0$, then

$$\forall x \in \mathcal{H} : \langle TT^{-1}x, T^{-1}x\rangle \geq 0$$

Thus $\forall x \in \mathcal{H}\langle x, T^{-1}x\rangle \geq 0 \Longrightarrow \forall x \in \mathcal{H} : \langle T^{-1}x, x\rangle \geq 0$, therefore T^{-1} is positive.

Theorem 2.13 *(Generalized Schwarz inequality) Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then*

$$\forall x, y \in \mathcal{H} : |\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle.$$

Proof. Let $t \in \mathbb{R}$ and $x, y \in \mathcal{H}$, since T is positive we have :

$$0 \leq \langle T(x+ty), x+ty\rangle = t^2\langle Ty, y\rangle + t(\langle Tx, y\rangle + \langle Ty, x\rangle) + \langle Tx, x\rangle$$

But $\quad \langle Ty, x\rangle = \langle y, T^*x\rangle = \langle y, Tx\rangle = \langle Tx, y\rangle \Longrightarrow \langle Tx, y\rangle + \langle Ty, x\rangle = 2R\langle Tx, y\rangle.$

Then $\quad \forall x, y \in \mathcal{H} \quad \forall t \in \mathbb{R} : t^2\langle Ty, y\rangle + 2tR\langle Tx, y\rangle + \langle Tx, x\rangle \geq 0.$

Since $\quad t^2\langle Ty, y\rangle + 2tR\langle Tx, y\rangle + \langle Tx, x\rangle \leq t^2\langle Ty, y\rangle + 2t|\langle Tx, y\rangle| + \langle Tx, x\rangle.$

Thus $\quad \forall x, y \in \mathcal{H}, \forall t \in \mathbb{R} : t^2\langle Ty, y\rangle + 2t|\langle Tx, y\rangle| + \langle Tx, x\rangle \geq 0.$

Therefore $\forall x, y \in \mathcal{H} : \Delta = 4|\langle Tx, y\rangle|^2 - 4\langle Tx, x\rangle\langle Ty, y\rangle \leq 0.$

Hence $\forall x, y \in \mathcal{H}|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle.$

Lemma 2.11 *Let $\mathcal{U}(\mathcal{H})$ be the set of all unitary operator in $\mathcal{B}(\mathcal{H})$, Then $\mathcal{U}(\mathcal{H})$ is a closed subset of $\mathcal{B}(\mathcal{H})$.*

Proof. Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{U}(\mathcal{H})$ convergent to $T \in \mathcal{B}(\mathcal{H})$.
The function $T \longmapsto T^*$ is continuous, then $T_n^* \longrightarrow T^*$. and sinec the multiplication map

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \ni (X, Y) \longmapsto XY \in \mathcal{B}(\mathcal{H})$$

is continuous, have $T^*T = \lim\limits_{n\longrightarrow\infty} T_n^*T_n$ and $TT^* \lim\limits_{n\longrightarrow\infty} T_nT_n^*$, so we immediately get $T^*T = TT^*$.
Then $T \in \mathcal{U}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ is closed.

Corollary 2.7 *Let $T \in \mathcal{U}(\mathcal{H})$, then $T^* = T^{-1} \in \mathcal{U}(\mathcal{H})$.*

Theorem 2.14 *Let $T \in \mathcal{B}(\mathcal{H})$ is unitary if and only if $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in \mathcal{H}$.*

Proof. $T \in \mathcal{U}(\mathcal{H})$, then

$$\forall x \in \mathcal{H}\ \|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = \langle x, TT^*x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2.$$

And

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = \langle x, x\rangle = \|x\|^2$$

Hence $\|Tx\| = \|T^*x\| = \|x\|$.

Proposition 2.3 *Let $T, S \in \mathcal{B}(\mathcal{H})$, Then*

1) $T, S \in \mathcal{U}(\mathcal{H})$, then $TS \in \mathcal{U}(\mathcal{H})$,

2) $T \in \mathcal{U}(\mathcal{H})$ if and only if T is an isometry with $Rar(T) = \mathcal{H}$.

Proof. 1) $T, S \in \mathcal{U}(\mathcal{H})$, then

$$TS(TS)^* = T(SS^*)T^* = TT^* = I = S^*S = S^*(T^*T)S = (TS)^*TS$$

Hence $TS \in \mathcal{U}(\mathcal{H})$.
2) If $T \in \mathcal{U}(\mathcal{H}) \Longrightarrow T$ is an invertible isometry, and since T is invertible, then $Ran(T) = \mathcal{H}$.
Now suppose that T is an isometry with $Ran(T) = \mathcal{H}$, since T is an isometry $\Longrightarrow Ker(T) = 0$.
And we have that $R(T) = \mathcal{H} \Longrightarrow T$ is bijective $\Longrightarrow T$ is invertible.
We have that $T^*T = I \Longrightarrow T^*TT^{-1} = T^{-1} \Longrightarrow T^{-1} = T^*$.
Then $T^*T = T^{-1}T = TT^{-1} = TT^* = I \Longrightarrow T^*T = TT^* = I$.
Therefore T is unitary.

Proposition 2.4 *Let $T \in \mathcal{B}(\mathcal{H})$, Then*

$$T \textit{ is isometry} \Longleftrightarrow \langle Tx, Ty\rangle = \langle x, y\rangle\ \forall x, y \in \mathcal{H}.$$

Proof.

$$\begin{aligned}
\forall x, y \in \mathcal{H}\ \ \langle Tx, Ty\rangle = \langle x, y\rangle &\Longleftrightarrow \forall x, y \in \mathcal{H} : \langle T^*Tx, y\rangle - \langle x, y\rangle = 0\\
&\Longleftrightarrow \forall x, y \in \mathcal{H} : \langle (T^*T - I)x, y\rangle = 0\\
&\Longleftrightarrow T^*T = I\\
&\Longleftrightarrow T \textit{ is isometry.}
\end{aligned}$$

Theorem 2.15 *$T \in \mathcal{B}(\mathcal{H})$ is hyponormal if and only if $\|Tx\| \geq \|T^*x\|$ for all $x \in \mathcal{H}$.*

Proof.

$$\begin{aligned}
T \textit{ is hyponormal} \Longleftrightarrow T^*T - TT^* \geq 0 &\Longleftrightarrow \forall x \in \mathcal{H} : \langle (T^*T - TT^*)x, x\rangle \geq 0\\
&\Longleftrightarrow \forall x \in \mathcal{H} : \langle T^*Tx, x\rangle \geq \langle TT^*x, x\rangle\\
&\Longleftrightarrow \forall x \in \mathcal{H} : \langle Tx, Tx\rangle \geq \langle T^*x, T^*x\rangle\\
&\Longleftrightarrow \forall x \in \mathcal{H} : \|Tx\|^2 \geq \|T^*x\|^2\\
&\Longleftrightarrow \forall x \in \mathcal{H} : \|Tx\| \geq \|T^*x\|
\end{aligned}$$

Proposition 2.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*
T is positive $\Longrightarrow$ T is self-adjoint $\Longrightarrow$ $T$is normal $\Longrightarrow$ $T$is quasinormal $\Longrightarrow$ $T$is hyponormal.

Proof. T is positive $\Longrightarrow$ T is self-adjoint $\Longrightarrow$ $T$is normal (clear).
Let T be a quasinormal operator $\Longleftrightarrow$ $(T^*T)T = T(T^*T)$, then $(T^*T)T = (TT^*)T$.
Thus $\forall x \in \mathcal{H} : (T^*T - TT^*)Tx = 0 \Longrightarrow \forall x \in Ran(T) : (T^*T - TT^*)x = 0$.
Let's prove that $\forall x \in \overline{Ran(T)} : (T^*T - TT^*)x = 0$, let $x \in \overline{Ran(T)} \Longleftrightarrow \exists (x_n)_{n\in\mathbb{N}} \subset Ran(T)$ such that $x_n \xrightarrow[n\longrightarrow\infty]{} x$
$\forall n \in \mathbb{N} : (T^*T - TT^*)x_n = 0$, since T and T^* are bounded, we obtain :
$\lim\limits_{n\longrightarrow\infty} (T^*T - TT^*)x_n = 0 \Longrightarrow (T^*T - TT^*) \lim\limits_{n\longrightarrow\infty} x_n = 0 \Longrightarrow (T^*T - TT^*)x = 0$.
Then $\forall x \in \overline{Ran(T)} : (T^*T - TT^*)x = 0$, therefore

$$(T^*T)_{|\overline{Ran(T)}} = (TT^*)_{|\overline{Ran(T)}} \tag{2.4}$$

From theorem of orthogonal decomposition, we have $\mathcal{H} = \overline{Ran(T)} \oplus (Ran(T))^{\perp}$, also we know that $(Ran(T))^{\perp} = Ker(T^*)$.
Then $\mathcal{H} = Ran(T) \oplus Ker(T^*)$, let $x \in Ker(T^*) \Longleftrightarrow T^*x = 0$. Then:
$\forall x \in Ker(T^*) : \langle (T^*T - TT^*)x, x\rangle = \langle T^*Tx - TT^*x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$, then

$$(T^*T - TT^*)_{|Ker(T^*)} \geq 0 \tag{2.5}$$

So $\forall x \in \mathcal{H}$, there exist $x_1 \in Ran(T)$ and $x_2 \in Ker(T^*)$ such that $x = x_1 + x_2$. Using (2.4), we get:

$$\begin{aligned}
\langle (T^*T - TT^*)x, x\rangle &= \langle (T^*T - TT^*)(x_1 + x_2), x_1 + x_2\rangle \\
&= \langle (T^*T - TT^*)x_1, x_1\rangle + \langle (T^*T - TT^*)x_2, x_2\rangle \\
&+ \langle (T^*T - TT^*)x_1, x_2\rangle + \langle (T^*T - TT^*)x2, x1\rangle \\
&= \langle (T^*T - TT^*)x_2, x_2\rangle + \langle x_1, (T^*T - TT^*)x_2\rangle + \langle T^*Tx_2, x_1\rangle \\
&= \langle (T^*T - TT^*)x_2, x_2\rangle + \langle x_1, T^*Tx_2\rangle + \langle T^*Tx_2, x_1\rangle
\end{aligned}$$

If $x_1 \in Ran(T)$, then $\exists y \in \mathcal{H}$ such that $x_1 = Ty$, using the fact that $(T^*T)T = T(T^*T)$ and $x_2 \in Ker(T^*)$, we get:

$$\langle x_1, T^*Tx_2\rangle = \langle Ty, T^*Tx_2\rangle = \langle T^*TTy, x_2\rangle = \langle TT^*Ty, x_2\rangle = \langle T^*Ty, T^*x_2\rangle = 0.$$

And since $\langle T^*Tx_2, x_1\rangle = \langle x_1, T^*Tx_2\rangle$, then $\langle x_1, T^*Tx_2\rangle = \langle T^*Tx_2, x_1\rangle = 0$ for all $x_1 \in Ran(T)$.
Let $x_1 \in Ran(T)$, then there exists $(y_n) \subset Ran(T)$ such that $\lim\limits_{n\longrightarrow\infty} y_n = x_1$, then:

$$\begin{aligned}
\forall n \in \mathbb{N} : \langle y_n, T^*Tx_2\rangle = \langle T^*Tx_2, y_n\rangle = 0 &\Longrightarrow \lim_{n\longrightarrow\infty} \langle y_n, T^*Tx_2\rangle = \lim_{n\longrightarrow\infty} \langle T^*Tx_2, y_n\rangle = 0 \\
&\Longrightarrow \langle \lim_{n\longrightarrow\infty} y_n, T^*Tx_2\rangle = \langle T^*Tx_2, \lim_{n\longrightarrow\infty} y_n\rangle = 0 \\
&\Longrightarrow \langle x_1, T^*Tx_2\rangle = \langle T^*Tx_2, x_1\rangle = 0 \; for \; all x_1 \in Ran(T).
\end{aligned}$$

Then $\langle (T^*T - TT^*)x, x\rangle = \langle (T^*T - TT^*)x_2, x_2\rangle \geq 0$ for all $x \in \mathcal{H}$ $(by(2.5))$.

2.3 Square root and the absolute value

Defnition 2.4 *Let $T \in \mathcal{B}(\mathcal{H})$. A square root of T is a positive operator $S \in \mathcal{B}(\mathcal{H})$ such that $T = S^2$.*

Defnition 2.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then the commutant of T is denoted by $\{T\}$ such that*

$$\{T\} = \{S \in \mathcal{B}(\mathcal{H}) : TS = ST\}$$

Corollary 2.8 *Let $T \in \mathcal{B}(\mathcal{H})$. then*

1) *$\{T\} \subset \{T^n\}$ for all $n \in \mathbb{N}$,*

2) *If $\lambda \in \mathbb{C}\ \{0\}$, then $\{\lambda T\} = \{T\}$,*

3) *$\{T\} \subset \{p(T)\}$ where $p(T)$ is a polynomial of T.*

Defnition 2.6 *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ be a sequence of self-adjoint operators, then $(T_n)_{n\in\mathbb{N}}$is said to be bounded monotone increasing, if there exists $T \in \mathcal{B}(\mathcal{H})$ such that $T_1 \le T_2 \le \cdots \le T_n \le \cdots \le T$.*

Theorem 2.16 *If $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ a sequence of positive operators is bounded monotone increasing, then there exists a positive operator $T \in \mathcal{B}(\mathcal{H})$ such that $T_n \xrightarrow{s} T$.*

Corollary 2.9 *If $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ is bounded monotone increasing, then there exists $T \in \mathcal{B}(\mathcal{H})$ such that $T_n \xrightarrow{s} T$.*

Theorem 2.17 *Let $T \in \mathcal{B}(\mathcal{H})$ be positive. Then T possesses a unique positive square root denoted by $T^{\frac{1}{2}}$. Moreover, if $S \in \mathcal{B}(\mathcal{H})$ is such that $TS = ST$, then $T^{\frac{1}{2}}S = ST^{\frac{1}{2}}$.*

Proof. First of all, if $T = 0 \Longrightarrow S = 0$ verifies $S^2 = T$, $S = 0$ is the unique positive square root of $T = 0$, because if $\exists S' \in \mathcal{B}(\mathcal{H})$ a square root of T such that
$S' \neq 0 \Longrightarrow \|S'x\|^2 = \langle S'x, S'x\rangle = \langle (S')^2x, x\rangle = 0, \forall x \in \mathcal{H}$.
Then $S' = 0 \Longrightarrow T^{\frac{1}{2}} = 0$ is the unique positive square root and $\{T\} = \{0\} = \{T^{\frac{1}{2}}\}$.
Assume that $T \neq 0$, we know that $T \le \|T\|I$, then $\dfrac{1}{\|T\|}T \le I$, let $T' = \dfrac{1}{\|T\|}T$, thus $T' \le I$.
Let S_n be defined as follows :

$$S_0 = 0 \quad \textit{and} \quad \forall n \in \mathbb{N} \quad S_{n+1} = S_n + \frac{1}{2}(T' - S_n^2)$$

Note that S_n is written as a polynomial of T' , and its coefficients are real, for example $S_1 = \dfrac{1}{2}T'$ and $S_2 = T' - \dfrac{1}{8}(T')^2$ and so on, and it can be easily proved by induction. By Lemma 2.8 S_n is self-adjoint for all $n \in \mathbb{N}$, moreover $\{T'\} \subset \{S_n\}$ for all $n \in \mathbb{N}$ by Corollary 2.8. We have that :

$$S_0 \le S_1 \le \cdots \le S_n \le \cdots \le I \tag{2.6}$$

Since $\forall n \ge 0 :\ I - S_{n+1} = I - S_n - \dfrac{1}{2}T' + \dfrac{1}{2}S_n^2 = \dfrac{1}{2}(I - T') + \dfrac{1}{2}(S_n^2 + I - 2S_n)$.
Then $\forall n \ge 0 :\ I - S_{n+1} = \dfrac{1}{2}(I - T') + \dfrac{1}{2}(S_n - I)^2$.
Since $T' \le I$, and $(S_n - I)^2 \ge 0$ for all $n \ge 0$, therefore $\forall n \ge 0 :\ I - S_{n+1} \ge 0$
Since $S_0 = 0$, we obtain that $\forall n \ge 0 :\ S_n \le I$.
Now we just need to prove that, $\forall n \ge 0 : S_n \le S_{n+1}$

By induction. First, for $n = 0$, we have $S_0 = 0$ and $S_1 = \frac{1}{2}T' \Longrightarrow S_0 \leq S_1$
Second, assume that $S_n \leq S_{n+1}$ is true, and let's prove that $S_{n+1} \leq S_{n+2}$ is held.

$$\begin{aligned} S_{n+2} - S_{n+1} &= S_{n+1} + \frac{1}{2}T' - \frac{1}{2}S_{n+1}^2 - S_n - \frac{1}{2}T' + \frac{1}{2}S_n^2 \\ &= \frac{1}{2}S_n^2 - S_n + S_{n+1} - \frac{1}{2}S_{n+1}^2 \\ &= \frac{1}{2}(I + S_n^2 - 2S_n) - \frac{1}{2}(I + S_{n+1}^2 - 2S_{n+1}) \\ &= \frac{1}{2}(I - S_n)^2 \frac{1}{2}(I - S_{n+1})^2 \end{aligned}$$

We have $I - S_n - (I - S_{n+1}) = S_{n+1} - S_n \geq 0$, then $I - S_{n+1} \leq I - S_n$.
Therefore $(I - S_{n+1})^2 \leq (I - S_n)^2$ (by Corollary 2.8), so that $\frac{1}{2}(I - S_n)^2 - \frac{1}{2}(I - S_{n+1})^2 \geq 0$.
Hence $S_{n+2} - S_{n+1} \geq 0$, which means that $S_{n+2} \geq S_{n+1}$.
Then (2.6) is held. By applying Theorem 2.16 on $(S_n)_{n\in\mathbb{N}}$, $\exists\, S \in \mathcal{B}(\mathcal{H})$, a positive operator such that $S_n \xrightarrow{s} S$.
Since $S_{n+1} = S_n + \frac{1}{2}(T' - S_n^2)$, then $S = S + \frac{1}{2}(T' - S^2)$, thus $S^2 = T'$.
Now let's prove that $\{T'\} \subset \{S\}$, let $R \in \{T'\}$ and $x \in \mathcal{H}$. Using the fact that R is continuous and $\{T'\} \subset \{S_n\}$, we obtain :

$$RSx = R(\lim_{n\longrightarrow\infty} S_n x) = \lim_{n\longrightarrow\infty} RS_n x = \lim_{n\longrightarrow\infty} S_n Rx = SRx, \; \textit{hence } RS = SR$$

Then $\{T'\} \subset \{S\}$.
Now let's prove the uniqueness of S that satisfies $S^2 = T'$, assume that there exists a positive operator $S' \in \mathcal{B}(\mathcal{H})$ such that $(S')^2 = T'$. Then:

$$T'S' = (S')^2 S' = S'T' \Longrightarrow S' \in \{T'\} \Longrightarrow S' \in \{S\} \;\; (\textit{since } \{T\} \subset \{S\})$$

Then $(S - S')(S + S') = S^2 - (S')^2 - SS' + S'S = 0$ (since $S^2 = (S')^2 = T'$ and $S'S = SS'$)
Thus $(S - S')x = 0$ for all $x \in R(S + S')$. Let's prove that $(S - S')x = 0$ for all $x \in \overline{R(S + S')}$.
Let $x \in \overline{R(S + S')}$, then there exists $(x_n)_{n\in\mathbb{N}} \subset R(S + S')$ such that $\lim_{n\longrightarrow\infty} x_n = x$, then:

$$\forall n \in \mathbb{N} : (S - S')x_n = 0 \Longrightarrow \lim_{n\longrightarrow\infty} (S - S')x_n = 0 \Longrightarrow (S - S') \lim_{n\longrightarrow\infty} x_n = 0 \Longrightarrow (S - S')x = 0.$$

Then $(S - S')x = 0$ for all $x \in \overline{R(S + S')}$. By orthogonal decomposition we have $\mathcal{H} = \overline{R(S + S')} \oplus R(S + S')^{\perp}$.
And $R(S+S')^{\perp} = N((S+S')^*) = N(S+S')$ (since $S+S' \geq 0$), then $\mathcal{H} = \overline{R(S + S')} \oplus N(S+S')$.
Let $x \in N(S + S') \Longleftrightarrow Sx + S'x = 0$
Since $\langle Sx, x\rangle, \langle S'x, x\rangle \geq 0$, and $\langle Sx + S'x, x\rangle = \langle Sx, x\rangle + \langle S'x, x\rangle = 0$.
Then $\langle Sx, x\rangle = \langle S'x, x\rangle = 0$, and since $S \geq 0$, then $\exists R \in \mathcal{B}(\mathcal{H})$ a positive operator such that $R^2 = S$. Then:

$$\langle Sx, x\rangle = \langle R^2 x, x\rangle = \langle Rx, Rx\rangle = \|Rx\|^2 = 0 \Longrightarrow Rx = 0 \Longrightarrow Sx = RRx = 0 \Longrightarrow Sx = 0.$$

Since $Sx + S'x = 0 \Longrightarrow S'x = 0$ as well, then $Sx = S'x = 0$ for all $x \in N(S + S')$.
Therefore $(S - S')x = Sx - S'x = 0$ for all $x \in N(S + S')$.
Thus $(S - S')x = 0$ for all $x \in \mathcal{H}$, then $S = S'$. Hence S is unique.
We get that there is a unique positive operator $S^2 = T'$ and $\{T'\} \subset \{S\}$, and since $T' = \frac{1}{\|T\|}T$.
Then $T = \|T\|S^2 = (\|T\|^{\frac{1}{2}}S)^2$ and $\{T\} = \{T'\} \subset \{S\} = \{(\|T\|S)^{\frac{1}{2}}\}$. (Corollary 2.8).

The uniqueness of S gives the uniqueness of $\|T\|^{\frac{1}{2}}S$. then $T^{\frac{1}{2}} = \|T\|^{\frac{1}{2}}S$ is the unique positive square root of T, and $\{T\} \subset \{T^{\frac{1}{2}}\}$.
Then for all $T \geq 0$, there exists a unique positive square root $\sqrt{T}$, and $\{T\} \subset \{T^{\frac{1}{2}}\}$, and $T^{\frac{1}{2}}$ is a limit of a polynomial of T as we have seen in the proof.

Corollary 2.10 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two positive operator such that $TS = ST$. Then*

$$(TS)^{\frac{1}{2}} = T^{\frac{1}{2}}S^{\frac{1}{2}}.$$

Proof. Since T is positive, it admits a unique positive square root, which we denote by $T^{\frac{1}{2}}$. Since S commutes with T, it commutes with $T^{\frac{1}{2}}$ as well.
Let $x \in \mathcal{H}$. We may write (remembering that positive operators are necessarily self-adjoint)

$$\langle TSx, x\rangle = \langle T^{\frac{1}{2}}T^{\frac{1}{2}}Sx, x\rangle = \langle T^{\frac{1}{2}}Sx, T^{\frac{1}{2}}x\rangle = \langle ST^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle \geq 0$$

as S is positive. Therefore, $TS \geq 0$. Since T and S are positive, both $T^{\frac{1}{2}}$ and $S^{\frac{1}{2}}$ exist and are welldefined. Since T and S also commute, TS is positive and it makes sense then to define $(TS)^{\frac{1}{2}}$. If we come to show that

$$(T^{\frac{1}{2}}S^{\frac{1}{2}})^2 = TS$$

then by the uniqueness of the square root, the desired result follows. Now since T and S commute, so do their square roots and we have

$$(T^{\frac{1}{2}}S^{\frac{1}{2}})^2 = T^{\frac{1}{2}}S^{\frac{1}{2}}T^{\frac{1}{2}}S^{\frac{1}{2}} = T^{\frac{1}{2}}T^{\frac{1}{2}}S^{\frac{1}{2}}S^{\frac{1}{2}} = TS \Longrightarrow (T^{\frac{1}{2}}S^{\frac{1}{2}})^2 = TS$$

The proof is complete.

Proposition 2.6 *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then*

1) $\|T^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$,

2) $Ker(T^{\frac{1}{2}}) = Ker(T)$,

3) $\overline{Ran(T^{\frac{1}{2}})} = \overline{Ran(T)}$.

Proof.1) Let $x \in \mathcal{H}$, then

$$\|T^{\frac{1}{2}}x\|^2 = \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle = \langle Tx, x\rangle \Longrightarrow \sup_{\|x\|=1} \|T^{\frac{1}{2}}x\|^2 = \sup_{\|x\|=1} \langle Tx, x\rangle$$

Since $T \geq 0$, then T is self-adjoint, by Theorem 2.7 $\|T\| = \sup_{\|x\|=1}\langle Tx, x\rangle$.

Then $\|T^{\frac{1}{2}}\|^2 = \|T\|$, hence $\|T^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$.

2) $Ker(T) \subset Ker(T^{\frac{1}{2}})$, let $x \in Ker(T) \Longleftrightarrow Tx = 0$, and we have $\|T^{\frac{1}{2}}x\|^2 = \langle Tx, x\rangle = 0$.

Then

$$\|T^{\frac{1}{2}}x\| = 0 \Longleftrightarrow T^{\frac{1}{2}}x = 0 \Longleftrightarrow x \in Ker(T^{\frac{1}{2}})$$

Therefore $Ker(T) \subset Ker(T^{\frac{1}{2}})$.

$Ker(T^{\frac{1}{2}}) \subset Ker(T)$, let $x \in Ker(T^{\frac{1}{2}}) \Longleftrightarrow T^{\frac{1}{2}}x = 0 \Longrightarrow T^{\frac{1}{2}}T^{\frac{1}{2}}x = 0 \Longrightarrow Tx = 0$.

Then $x \in Ker(T)$, thus $Ker(T^{\frac{1}{2}}) \subset Ker(T)$. Hence $Ker(T^{\frac{1}{2}}) = Ker(T)$.

3) Since $Ker(T^{\frac{1}{2}}) = Ker(T)$, and we have $Ker(T^{\frac{1}{2}}) = Ran(T^{\frac{1}{2}})^{\perp}$ and $Ker(T) = Ran(T)^{\perp}$ (T and $T^{\frac{1}{2}}$ are self-adjoint)
Then $Ran(T^{\frac{1}{2}})^{\perp} = Ran(T)^{\perp} \Longrightarrow (Ran(T^{\frac{1}{2}})^{\perp})^{\perp} = (R(T)^{\perp})^{\perp} \Longrightarrow \overline{Ran(T^{\frac{1}{2}})} = \overline{Ran(T)}$.

Corollary 2.11 *Let T, $S \in \mathcal{B}(\mathcal{H})$ be two positive operators, then*

$$TS = ST \Longleftrightarrow T^{\frac{1}{2}}S^{\frac{1}{2}} = S^{\frac{1}{2}}T^{\frac{1}{2}}$$

Proof. ($\Longrightarrow$) Suppose that $TS = ST$. Since $TS = ST$ and $\{T\}' \subset \{T^{\frac{1}{2}}\}'$, then $T^{\frac{1}{2}}S = ST^{\frac{1}{2}}$. And since $T^{\frac{1}{2}}S = ST^{\frac{1}{2}}$ and $\{S\}' \subset \{S^{\frac{1}{2}}\}'$, hence $T^{\frac{1}{2}}S^{\frac{1}{2}} = S^{\frac{1}{2}}T^{\frac{1}{2}}$.

($\Longleftarrow$) Assume that $T^{\frac{1}{2}}S^{\frac{1}{2}} = S^{\frac{1}{2}}T^{\frac{1}{2}}$, then:

$$TS = T^{\frac{1}{2}}T^{\frac{1}{2}}S^{\frac{1}{2}}S^{\frac{1}{2}} = T^{\frac{1}{2}}S^{\frac{1}{2}}T^{\frac{1}{2}}S^{\frac{1}{2}} = S^{\frac{1}{2}}T^{\frac{1}{2}}S^{\frac{1}{2}}T^{\frac{1}{2}} = S^{\frac{1}{2}}S^{\frac{1}{2}}T^{\frac{1}{2}}T^{\frac{1}{2}} = ST$$

Thus $TS = ST$.

Proposition 2.7 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two positive operators such that $TS = ST$. Then*

1) *If $S \leq T \Longrightarrow S^{\frac{1}{2}} \leq T^{\frac{1}{2}}$,*

2) *$(T+S)^{\frac{1}{2}} \leq T^{\frac{1}{2}} + S^{\frac{1}{2}}$.*

Proof. 1) If $S \leq T$, then $T - S \geq 0$, let

$x \in \mathcal{H} : \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})(T^{\frac{1}{2}} + S^{\frac{1}{2}})x, (T^{\frac{1}{2}} + S^{\frac{1}{2}})x\rangle = \langle (T-S)x, (T^{\frac{1}{2}} + S^{\frac{1}{2}})x\rangle = \langle (T^{\frac{1}{2}} + S^{\frac{1}{2}})(T-S)x, x\rangle$.

Since $T^{\frac{1}{2}}, S^{\frac{1}{2}} \geq 0$, then $T^{\frac{1}{2}} + S^{\frac{1}{2}} \geq 0$, and we have that $T - S \geq 0$, and since

$$(T^{\frac{1}{2}} + S^{\frac{1}{2}})(T - S) = (T - S)(T^{\frac{1}{2}} + S^{\frac{1}{2}})$$

Then, we get $(T^{\frac{1}{2}} + S^{\frac{1}{2}})(T - S) \geq 0$, therefore

$$\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})(T^{\frac{1}{2}} + S^{\frac{1}{2}})x, (T^{\frac{1}{2}} + S^{\frac{1}{2}})x\rangle \geq 0 \; for \; all \; x \in \mathcal{H}$$

Thus $\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})y, y\rangle \geq 0$ for all $y \in Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})$.

Let $y \in Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})$, $\exists (y_n)_{n\in\mathbb{N}} \subset Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})$ such that $\lim_{n\longrightarrow\infty} yn = y$, then:

$\forall n \in \mathbb{N} : \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})y_n, y_n\rangle \geq 0 \Longrightarrow \lim_{n\longrightarrow\infty} \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})y_n, y_n\rangle \geq 0 \Longrightarrow \langle \lim_{n\longrightarrow\infty} (T^{\frac{1}{2}} - S^{\frac{1}{2}})y_n, \lim_{n\longrightarrow\infty} y_n\rangle \geq 0$

Therefore $\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})y, y\rangle \geq 0$ for all $y \in \overline{Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})}$, thus

$$T^{\frac{1}{2}} - S^{\frac{1}{2}}{}_{|\overline{Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})}} \geq 0 \tag{2.7}$$

Using orthogonal decomposition theorem, we get $\mathcal{H} = \overline{Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})} \oplus (Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}}))^{\perp}$.
But $(Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}}))^{\perp} = Ker((T^{\frac{1}{2}} + S^{\frac{1}{2}})^*) = Ker(T^{\frac{1}{2}} + S^{\frac{1}{2}})$, then
$\mathcal{H} = \overline{Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})} \oplus Ker(T^{\frac{1}{2}} + S^{\frac{1}{2}})$.
Let $x \in Ker(T^{\frac{1}{2}} + S^{\frac{1}{2}}) \Longleftrightarrow T^{\frac{1}{2}}x + S^{\frac{1}{2}}x = 0 \Longleftrightarrow S^{\frac{1}{2}}x = -T^{\frac{1}{2}}x$, then:

$$\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x, x\rangle = \langle 2T^{\frac{1}{2}}x, x\rangle = 2\langle T^{\frac{1}{2}}x, x\rangle \geq 0$$

And $\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x, x\rangle = \langle -2S^{\frac{1}{2}}x, x\rangle = -2\langle S^{\frac{1}{2}}x, x\rangle \leq 0$

Thus $\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x, x\rangle = 0$ for all $x \in Ker(T^{\frac{1}{2}} + S^{\frac{1}{2}})$.
Let $x \in \mathcal{H}$, there exist $x_1 \in \overline{Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})}$ and $x_2 \in Ker(T^{\frac{1}{2}} + S^{\frac{1}{2}})$ such that $x = x_1 + x_2$, then

$$\begin{aligned}
\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x, x\rangle &= \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})(x_1 + x_2), x_1 + x_2\rangle \\
&= \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_1\rangle + \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_2, x_2\rangle \\
&+ \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_2\rangle + \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_2, x_1\rangle \\
&= \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_1\rangle + \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_2\rangle + \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_2, x_1\rangle
\end{aligned}$$

If $x_1 \in Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})$, there exists $y \in \mathcal{H}$ such that $x_1 = (T^{\frac{1}{2}} + S^{\frac{1}{2}})y$, recall that $x_2 \in Ker(T^{\frac{1}{2}} + S^{\frac{1}{2}})$, and using the fact that $(T^{\frac{1}{2}} + S^{\frac{1}{2}})(T^{\frac{1}{2}} - S^{\frac{1}{2}}) = (T^{\frac{1}{2}} - S^{\frac{1}{2}})(T^{\frac{1}{2}} + S^{\frac{1}{2}})$, we obtain :

$$\begin{aligned}
\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_2\rangle &= \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})(T^{\frac{1}{2}} + S^{\frac{1}{2}})y, x_2\rangle \\
&= \langle (T^{\frac{1}{2}} + S^{\frac{1}{2}})(T^{\frac{1}{2}} - S^{\frac{1}{2}})y, x_2\rangle \\
&= \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})y, (T^{\frac{1}{2}} + S^{\frac{1}{2}})x_2\rangle = 0
\end{aligned}$$

And since $\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_2, x_1\rangle = \overline{\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_2\rangle}$

$$\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_2\rangle = \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_2, x_1\rangle = 0 \quad \forall x_1 \in Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})$$

And we can easily extend it to $\overline{Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})}$, and we get:

$$\langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_2\rangle = \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_2, x_1\rangle = 0 \quad \forall x_1 \in \overline{Ran(T^{\frac{1}{2}} + S^{\frac{1}{2}})}$$

Then $\qquad \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x, x\rangle = \langle (T^{\frac{1}{2}} - S^{\frac{1}{2}})x_1, x_1\rangle \geq 0$ (using 2.7)
Thus $T^{\frac{1}{2}} - S^{\frac{1}{2}} \geq 0$, hence $S^{\frac{1}{2}}T^{\frac{1}{2}}$.
2) We have that $T^{\frac{1}{2}}S^{\frac{1}{2}} = S^{\frac{1}{2}}T^{\frac{1}{2}}$, then:

$$T + S - (T^{\frac{1}{2}} + S^{\frac{1}{2}})^2 = T + S - (T + S + 2T^{\frac{1}{2}}S^{\frac{1}{2}}) = -2T^{\frac{1}{2}}S^{\frac{1}{2}}$$

Since $T^{\frac{1}{2}}S^{\frac{1}{2}} \geq 0$, then $-2T^{\frac{1}{2}}S^{\frac{1}{2}} \leq 0$, which means that $T + S - (T^{\frac{1}{2}} + S^{\frac{1}{2}})^2 \leq 0$.
Thus $T + S \leq (T^{\frac{1}{2}} + S^{\frac{1}{2}})^2$, since $(T + S)(T^{\frac{1}{2}} + S^{\frac{1}{2}})^2 = (T^{\frac{1}{2}} + S^{\frac{1}{2}})^2(T + S)$, and using 2) we get:

$$(T + S)^{\frac{1}{2}} \leq ((T^{\frac{1}{2}} + S^{\frac{1}{2}})^2)^{\frac{1}{2}}$$

Since $T^{\frac{1}{2}} + S^{\frac{1}{2}} \geq 0$, then $(T + S)^{\frac{1}{2}} \leq T^{\frac{1}{2}} + S^{\frac{1}{2}}$.

Corollary 2.12 *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator, then*

$$T \leq I \Longleftrightarrow T^2 \leq T \Longleftrightarrow \|T\| \leq 1$$

Theorem 2.18 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two positive operators. Then*

$$\|T + S\| \leq \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|T^{\frac{1}{2}}S^{\frac{1}{2}}\|^2}\right). \tag{2.8}$$

Corollary 2.13 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two positive operators. Then*

$$\|T + S\| \leq \max\{\|T\|, \|S\|\} + \|\sqrt{T}\sqrt{S}\| \tag{2.9}$$

Proof. By (2.8), we have that :

$$\begin{aligned}
\|T + S\| &\leq \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\|T^{\frac{1}{2}}S^{\frac{1}{2}}\|^2}\right) \\
&\leq \frac{1}{2}\left(\|T\| + \|S\| + |\|T\| - \|S\|| + 2\|T^{\frac{1}{2}}S^{\frac{1}{2}}\|\right)
\end{aligned}$$

Since $\max\{\|T\|, \|S\|\} = \dfrac{1}{2}(\|T\| + \|S\| + |\|T\| - \|S\||)$, then:

$$\|T + S\| \leq \frac{1}{2}\left(2\max\{\|T\|, \|S\|\} + 2\|T^{\frac{1}{2}}S^{\frac{1}{2}}\|\right)$$

Hence

$$\|T + S\| \leq \max\{\|T\|, \|S\|\} + \|T^{\frac{1}{2}}S^{\frac{1}{2}}\|.$$

Defnition 2.7 *Let $T \in \mathcal{B}(\mathcal{H})$, the absolute value of T is the unique positive square root of the positive operator T^*T, and we denote it by $|T|$, that is $|T| = (T^*T)^{\frac{1}{2}}$.*

Proposition 2.8 *$T \in \mathcal{B}(\mathcal{H})$. Then*

1) $\||T|\| = \|T\|$,

2) $|T| = T$ if and only if $T \geq 0$,

3) $|T| = |T^*|$ if and only if T is normal.

Proof.1) $\||T|\| = \|(T^*T)^{\frac{1}{2}}\| = \|T^*T\|^{\frac{1}{2}} = (\|T\|^2)^{\frac{1}{2}} = \|T\|$. (we used $\|T^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$ and $\|T^*T\| = \|T\|^2$).
2) First assume that $|T| = T$, since $(T^*T)^{\frac{1}{2}} = T$ and $(T^*T)^{\frac{1}{2}} \geq 0$, then $T \geq 0$. Now suppose that $T \geq 0$, then:

$$T^* = T \Longrightarrow T^*T = T^2 \Longrightarrow (T^*T)^{\frac{1}{2}} = T \Longleftrightarrow |T| = T.$$

3) Assume that $|T| = |T^*|$, then:

$$(T^*T)^{\frac{1}{2}} = (TT^*)^{\frac{1}{2}} \Longrightarrow ((T^*T)^{\frac{1}{2}})^2 = TT^{*2} \Longrightarrow T^*T = TT^* \Longleftrightarrow T\text{is normal.}$$

Suppose that T is normal, then $T^*T = TT^* \Longrightarrow (T^*T)^{\frac{1}{2}} = (TT^*)^{\frac{1}{2}} \Longleftrightarrow |T| = |T^*|$.

Proposition 2.9 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be such that $TS = ST$. If T is normal, then $|T||S| = |S||T|$.*

Proof. Since $TS = ST$ and T is normal, we have $T^*S = ST^*$. We then clearly have from the previous two relations:

$$TS = ST \Longrightarrow T^*TS = T^*ST = ST^*T$$

Hence

$$|T|S = S|T|.$$

Since $|T|$ is self-adjoint, the previous equality gives (by taking adjoints) $|T|S^* = S^*|T|$. Hence

$$S^*|T|S = |T|S^*S \Longrightarrow S^*S|T| = |T|S^*S \Longrightarrow |S||T| = |T||S|,$$

as required.

Theorem 2.19 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be self-adjoint such that TS is normal. Then*

$$|TS| = |T||S|.$$

Since T and S are self-adjoint, we may write

$$S(TS) = STS = (TS)^*S.$$

Since TS and $(TS)^*$ are normal, the Fuglede-Putnam theorem gives

$$S(TS)^* = (TS)^*S\text{or merely}S^2T = TS^2.$$

Consequently, $S^2T^2 = TS^2T = T^2S^2$. On the other hand, we easily see that

$$|TS|^2 = (TS)^*TS = TS(TS)^* = TS^2T = T^2S^2.$$

and so

$$|TS| = (T^2S^2)^{\frac{1}{2}} = (T^2)^{\frac{1}{2}}(S^2)^{\frac{1}{2}} = |T||S|,$$

as required.

Theorem 2.20 *Let $T, S \in \mathcal{B}(\mathcal{H})$ be such that $TS = ST$. If T is normal, then*

$$|TS| = |T||S|.$$

Proof. Since $TS = ST$ and T is normal, we get $T^*S = ST^*$ or $TS^* = S^*T$. Hence

$$|TS|^2 = (TS)^*TS = S^*T^*TS = T^*S^*TS = T^*TS^*S.$$

We have, $T^*TS^*S = S^*ST^*T$. Consequently,

$$|TS| = (T^*TS^*S)^{\frac{1}{2}} = (T^*T)^{\frac{1}{2}}(S^*S)^{\frac{1}{2}} = |T||S|.$$

as required.

Corollary 2.14 *Let $T \in \mathcal{B}(\mathcal{H})$ be normal and invertible. Then*

$$|T^{-1}| = |T|^{-1}.$$

Proof. It is clear that

$$I = |TT^{-1}| = |T||T^{-1}|.$$

So, the self-adjoint $|T|$ is right invertible and so it is invertible and:

$$|T|^{-1} = |T^{-1}|.$$

Chapter 3

Spectral Values: Definitions and Properties

We give in this chapter the main concepts of the spectrum, numerical range, spectral radius and numerical radius of operators and give the basic results concerning them that will be used later.

3.1 Spectrum and resolvant of linear operator

Defnition 3.1 *Let $T \in \mathcal{B}(\mathcal{H})$. The resolvent set of T is denoted by $\rho(T)$, and it is defined as follows*

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \textit{is invertible}\}.$$

And the spectrum set of T is given by

$$\sigma(T) = \mathbb{C} \backslash \rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \textit{is not invertible}\}.$$

Defnition 3.2 *A complex number $\lambda \in \mathbb{C}$ is called an eigenvalue of $T \in \mathcal{B}(\mathcal{H})$ if there exists $x \in \mathcal{H} \backslash \{0\}$ such that $Tx = \lambda x$.*

Corollary 3.1 *Let λ is an eigenvalue of $T \in \mathcal{B}(\mathcal{H})$ then, $\lambda \in \sigma(T)$.*

Proof.

$$\begin{aligned} \text{Since } \lambda \text{ is an eigenvalue of } T &\Longrightarrow \exists x \in \mathcal{H} \backslash \{0\} : Tx = \lambda x \\ &\Longrightarrow (T - \lambda I)x = 0, \ \ 0 \neq x \in \mathcal{H} \\ &\Longrightarrow x \in Ker(T - \lambda I) \\ &\Longrightarrow Ker(T - \lambda I) \neq \{0\} \\ &\Longrightarrow (T - \lambda I) \text{is not invertible} \\ &\Longrightarrow \lambda \in \sigma(T) \end{aligned}$$

Defnition 3.3 *Let $T \in \mathcal{B}(\mathcal{H})$, the resolvent function of T is the map R defined by*

$$\begin{aligned} R : \rho(T) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ \lambda &\longmapsto (T - \lambda I)^{-1} \end{aligned}$$

Lemma 3.1 *(The resolvent equation)*
Let $T \in \mathcal{B}(\mathcal{H})$, and let R be the resolvent function of T. Then

$$R(T; \lambda) - R(T; \lambda_0) = (\lambda - \lambda_0) R(T; \lambda) R(T; \lambda_0), \quad \forall \lambda, \lambda_0 \in \sigma(T).$$

Proof. Let $T \in \mathcal{B}(\mathcal{H})$.

$$\begin{aligned} R(T;\lambda) - R(T;\lambda_0) &= (T-\lambda I)^{-1} - (T-\lambda_0 I)^{-1} \\ &= (T-\lambda I)^{-1}((T-\lambda_0 I) - (T-\lambda I))(T-\lambda_0 I)^{-1} \\ &= (\lambda-\lambda_0)R(T;\lambda)R(T;\lambda_0). \end{aligned}$$

Lemma 3.2 *If $T, S \in \mathcal{B}(\mathcal{H})$ and $TS = ST$, then for any $\lambda \in \rho(T)$ and $\mu \in \rho(S)$ the operators $T, S, R(T;\lambda)$ and $R(S;\mu)$ all commute with each other.*

Proof. It is clear that $TR(T;\lambda) = R(T;\lambda)T$,$TR(S;\mu) = R(S;\mu)T$,$SR(T;\lambda) = R(T;\lambda)S$ and $SR(S;\mu) = R(S;\mu)S$.
If $\lambda = \mu$, then $R(T;\lambda)R(S;\mu) = R(S;\mu)R(T;\lambda)$. Assume that $\lambda \neq \mu$, from (Lemma 1.1) we have :

$$R(T;\lambda)R(S;\mu) = \frac{1}{\lambda-\mu}(R(T;\lambda) - R(S;\mu)) = \frac{1}{\mu-\lambda}(R(S;\mu) - R(T;\lambda)) = R(S;\mu)R(T;\lambda)$$

Remark 3.1 *If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$. This is because*

$$R(T;\lambda) = (T-\lambda)^{-1} = \lambda^{-1}\left(\frac{T}{\lambda} - I\right)^{-1} = \lambda\sum_{n=0}^{\infty}\left(\frac{T}{\lambda}\right)^n$$

This series converges uniformly and absolutely because $\|\dfrac{T}{\lambda}\| \leq 1$

Theorem 3.1 *Let $T \in \mathcal{B}(\mathcal{H})$. then $\sigma(T) \neq \emptyset$.*

Proof. Suppose that $\sigma(T) = \emptyset$, then $\rho(T) = \mathbb{C}$. Fix $h, k \in \mathcal{H}$ and define $f_{h,k} : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$f_{h,k}(z) = \langle h, (T-\lambda I)^{-1}k\rangle.$$

One shows that this function is analytic on all of $\mathbb{C}$ and uses the Neumann series to show that as $|z| \longrightarrow \infty$ we have $|f_{h,k}(z)| \longrightarrow 0$. This in turn implies that $f_{h,k}$ is bounded. A result in complex analysis says that any bounded entire function is constant. Since it tends to 0 it must be the 0 function. Setting $z = 0$ we see that $\langle h, T^{-1}k\rangle = 0$ for every pair of vectors. Choosing $h = T^{-1}k$ yields that $\|T^{-1}k\| = 0$ so that $T^{-1} = 0$, a contradiction.
Hence, $\sigma(T)$ must be non-empty.

Theorem 3.2 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1) $\sigma(T)$ is a closed set,

2) If $|\lambda| > \|T\|$, then $\lambda \notin \sigma(T)$.

Proof. 1) Define the map $F : \mathbb{C} \longrightarrow \mathcal{B}(\mathcal{H})$ by $F(\lambda) = T - \lambda I$. We have $\|F(\lambda) - F(\mu)\| = |\lambda - \mu|$, so that F is continuous.
We have $\sigma(T) = F^{-1}(\mathcal{B}(\mathcal{H})\backslash G)$ where G denotes the set of invertible operators. Since G is open (Lemma 2.1), we deduce that $T \in \mathcal{B}(\mathcal{H})\backslash G$ and $\sigma(T)$ are closed.
2) Suppose that $|\lambda| > \|T\|$. Then $\|\lambda^{-1}T\| < 1$, and it follows that $I - \lambda^{-1}T$ is invertible (Theorem 2.2). Hence, $T - \lambda I = -\lambda(I - \lambda^{-1}T)$ is also invertible. This show that $\lambda \notin \sigma(T)$.

Remark 3.2 *1) The spectrum of T is closed and bonded, then $\sigma(T)$ is compact,*

2) It is clear that $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$.

Lemma 3.3 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\sigma(T^*) = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma(T)\}.$$

Proof. Let $T \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} \lambda \in \rho(T) &\iff T - \lambda I \text{ is invertible} \\ &\iff (T - \lambda I)^* = T^* - \overline{\lambda} I \text{ is invertible} \\ &\iff \overline{\lambda} \in \rho(T^*) \end{aligned}$$

Theorem 3.3 *Let $T \in \mathcal{B}(\mathcal{H})$*

1) for all P is a polynomial then $\sigma(P(T)) = P(\sigma(T))$,

2) If T is invertible then $\sigma(T^{-1}) = \{\lambda^{-1} \in \mathbb{C} : \lambda \in \sigma(T)\}$.

Proof. 1) Let $\lambda \in \sigma(T)$ and consider $Q(x) = P(x) - P(\lambda)$ then λ is a root of $Q(x)$ and so there is a polynomial $B \in \mathbb{C}[x]$ such that $Q(x) = (x - \lambda)B(x)$. Thus

$$P(T) - P(\lambda) = (T - \lambda I)B(T) = B(T)(T - \lambda I)$$

Since $(T - \lambda I)$ is not invertible for $\lambda \in \sigma(T)$, $P(T) - P(\lambda)$is not invertible, $so P(\lambda) \in \sigma(P(T))$
Conversely, let $\lambda \in \sigma(P(T))$ then, by factoring we obtain that

$$P(x) - \lambda = \alpha(x - \mu_1) \cdots (x - \mu_n)$$

and

$$P(T) - \lambda = \alpha(T - \mu_1 I) \cdots (T - \mu_n I)$$

Since $\lambda \in \sigma(P(T))$, $P(T) - \lambda$ is not invertible and so there is some $i \in \{1, \cdots, n\}$ such that $T - \mu i$ is not invertible.
This $\mu i \in \sigma(T)$ but $P(\mu i) = \lambda$ by the first part of this discussion
2) Since T is invertible, then $0 \notin \sigma(T)$, therefore λ^{-1} is defined when $\lambda \in \sigma(T)$.
Let $\lambda \in \mathbb{C} \backslash \{0\}$ $T^{-1} - \lambda^{-1} I = -\lambda^{-1} T^{-1}(T - \lambda I)$, and $\lambda^{-1} T^{-1}$ is invertible for all $\lambda \in \mathbb{C} \backslash \{0\}$. Then

$$\begin{aligned} \lambda \in \sigma(T) &\iff T - \lambda I \text{ is not invertible.} \\ &\iff -\lambda^{-1} T^{-1}(T - \lambda I) \text{ is not invertible.} \\ &\iff T^{-1} - \lambda^{-1} I \text{ is not invertible.} \\ &\iff \lambda^{-1} \in \sigma(T^{-1}). \end{aligned}$$

Hence $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

Corollary 3.2 *Let $U \in \mathcal{B}(\mathcal{H})$ is unitary operator,then*

$$\sigma(U) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

Proof. Since $\|U\| = \|U^*\| = 1$ and $U^{-1} = U^*$, then:

$$\sigma(U) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

$$\sigma(U^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

Then

$$\begin{aligned}\sigma(U) = \{\lambda \in \mathbb{C} : \lambda^{-1} \in \sigma(U^{-1}) = \sigma(U^*)\} \Longrightarrow \sigma(U) &= \{\lambda \in \mathbb{C} : |\lambda^{-1}| \leq 1\} \\ &= \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}\end{aligned}$$

Hence $\sigma(U) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$

Theorem 3.4 *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.*

Proof. Let $\lambda \in \rho(TS)\backslash\{0\} \Longleftrightarrow TS - \lambda I$ is invertible.
Set $R = (TS - \lambda I)^{-1}$, then:

$$(TS - \lambda I)R = R(TS - \lambda I) = I \Longrightarrow TSR - \lambda R = RTS - \lambda R = I$$

Therefore $\quad TSR = RTS = I + \lambda R.$

$$\begin{aligned}(ST - \lambda I)(SRT - I) &= STSRT - ST - \lambda SRT + \lambda I \\ &= S(I + \lambda R)T - ST - \lambda SRT + \lambda I \\ &= ST + \lambda SRT - ST - \lambda SRT + \lambda I \\ &= \lambda I\end{aligned}$$

Since $\lambda \neq 0$, then : $\quad (ST - \lambda I)\frac{1}{\lambda}(SRT - I) = I.$

$$\begin{aligned}(SRT - I)(ST - \lambda I) &= SRTST - \lambda SRT - ST + \lambda I \\ &= S(I + \lambda R)T - \lambda SRT - ST + \lambda I \\ &= ST + \lambda SRT - \lambda SRT - ST + \lambda I \\ &= \lambda I\end{aligned}$$

Since $\lambda \neq 0$, then : $\quad \frac{1}{\lambda}(SRT - I)(ST - \lambda I) = I$. Therefore $ST - \lambda I$ is invertible, then $\lambda \in \rho(ST)\backslash\{0\}$.
Hence $\rho(TS)\backslash\{0\} = \rho(ST)\backslash\{0\} \Longleftrightarrow \sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.

Theorem 3.5 *Let $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\sigma(T) \subset \mathbb{R}$.*

Proof. Suppose $\lambda = a + ib \in \mathbb{C}$ with $b \neq 0$, then for any $x \in \mathcal{H}$

$$\begin{aligned}\|(T - \lambda I)x\|^2 &= \langle (T - \lambda I)x, (T - \lambda I)x \rangle \\ &= \|(T - aI)x\|^2 + b^2\|x\|^2 + \langle Tx, (-ib)x \rangle + \langle (-ib)x, T(x) \rangle \\ &= \|(T - aI)x\|^2 + b^2\|x\|^2 \\ &\geq b^2\|x\|^2\end{aligned}$$

where the third equality holds because $\langle T(x), x \rangle = \langle x, T(x) \rangle$ which implies that

$$\langle Tx, (-ib)x \rangle + \langle (-ib)x, T(x) \rangle = ib\langle T(x), x \rangle - ib\langle x, T(x) \rangle = 0$$

Then $\|(T - \lambda I)x\| \geq b\|x\|$
Hence, $(T - \lambda I)$ is invertible, and $\lambda \notin \sigma(T)$.
Therefore $\forall \lambda \in \mathbb{C}$ such that $b \neq 0$ we have $\lambda \in \rho(T) \Longleftrightarrow \lambda \notin \sigma(T)$.
Thus if$\lambda \in \rho(T)$, then $b = 0$, which means that $\sigma(T) \subset \mathbb{R}$.

Corollary 3.3 *Let $T \in \mathcal{B}(\mathcal{H})$ be positive. then $\sigma(T) \subset \mathbb{R}_+$*

Proof. Since $T$is positive $\Longrightarrow T$ is self-adjoint $\Longrightarrow \sigma(T) \subset \mathbb{R}$.
Suppose $\lambda \in \mathbb{R}$ with $\lambda < 0$, then for any $x \in \mathcal{H}$

$$-\lambda\|x\|^2 = -\lambda\langle x, x\rangle \leq \langle Tx, x\rangle - \lambda\langle x, x\rangle = \langle (T-\lambda)x, x\rangle \leq \|(T-\lambda)x\|\|x\|$$

Thus $-\lambda\|x\| \leq \|(T-\lambda)x\|$. $T-\lambda I$ is invertible, hence $\lambda \in \rho(T)$.
Therefore for all $\lambda < 0$, then $\lambda \in \rho(T) \Longleftrightarrow \lambda \notin \sigma(T)$, thus if $\lambda \in \sigma(T)$.
Then $\lambda \geq 0$, hence $\sigma(T) \subset \mathbb{R}_+$.

Defnition 3.4 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

- *The point spectrum of T is the set:*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : Ker(T-\lambda I) \neq \{0\}\}.$$

- *The approximate point spectrum of T is the set:*

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists (x_n) \subset \mathcal{H} : \|x_n\| = 1 \ \ and \lim_{n\longrightarrow\infty}(T-\lambda I)x_n = 0\}.$$

Remark 3.3 *We let $\sigma_p(T)$ denote the set of eigenvalues of T.*

Lemma 3.4 *If $\mathcal{H}$ is a finite-dimensional Hilbert space, then $\sigma(T) = \sigma_p(T) \ \ \forall T \in \mathcal{B}(\mathcal{H})$.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ is a finite-dimensional, then $dim(\mathcal{H}) = dim(Ker(T)) + dim(Ran(T))$
If $\lambda \in \sigma(T) \Longrightarrow T-\lambda I$ is not invertible

$$\begin{aligned}
\bullet If \ \ T-\lambda I \ \ \text{is not injective} \ &\Longleftrightarrow \ Ker(T-\lambda I) \neq 0 \\
&\Longleftrightarrow \ \exists \ 0 \neq x \in Ker(T-\lambda I) \\
&\Longleftrightarrow \ Tx = \lambda x \\
&\Longleftrightarrow \ \lambda \in \sigma_p(T)
\end{aligned}$$

$$\begin{aligned}
\bullet If \ T-\lambda I \ \text{is not surjective} \ &\Longleftrightarrow \ dim(Ran(T-\lambda I)) \neq \ dim(\mathcal{H}) \\
&\Longleftrightarrow \ dim(Ker(T-\lambda I)) \neq 0 \\
&\Longleftrightarrow \ \exists \ 0 \neq x \in Ker(T-\lambda I) \\
&\Longleftrightarrow \ Tx = \lambda x \\
&\Longleftrightarrow \ \lambda \in \sigma_p(T)
\end{aligned}$$

Hence $\sigma(T) = \sigma_p(T)$.

Proposition 3.1 *Let $T \in \mathcal{B}(\mathcal{H})$, Then*

$$\sigma_p(T) \subset \sigma_{ap}(T)$$

Proof. Let $\lambda \in \sigma_p(T)$, so there is a vector $x \in \mathcal{H}$ such that $Tx = \lambda x$ and $x \neq 0$, it sufficies taking $x_n = \dfrac{x}{\|x\|}$ for all $n \in \mathbb{N}$, then $\lambda \in \sigma_{ap}(T)$.

3.2 Spectral radius

Defnition 3.5 *Let $T \in \mathcal{B}(\mathcal{H})$. The spectral radius of T is denoted by $r(T)$, and it is defined as follows*

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

$r(T)$ defined when $\sigma(T) \neq 0$.

Example 3.1 *Let $U \in \mathcal{B}(\mathcal{H})$ is unitary. $r(U) = 1$.*

Remark 3.4 *It is clear $r(T) \leq \|T\|$.*

Lemma 3.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1) $r(T^*) = r(T)$,

2) $r(T^n) = r(T)^n \ \ \forall n \in \mathbb{N}$,

3) If $T \geq 0$, then $r(T^{\frac{1}{2}}) = r(T)^{\frac{1}{2}}$.

Proof. 1) Since $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$(Lemma 3.3), then:

$$r(T^*) = \sup\{|\lambda| : \lambda \in \sigma(T^*)\} = \sup\{|\overline{\lambda}| : \lambda \in \sigma(T)\} = \sup\{|\lambda| : \lambda \in \sigma(T)\} = r(T)$$

Hence $r(T^*) = r(T)$.
2) We have $\sigma(T^n) = (\sigma(T))^n$, then:

$$r(T^n) = \sup\{|\lambda| : \lambda \in \sigma(T^n)\} = \sup\{|\lambda|^n : \lambda \in \sigma(T)\} = (\sup\{|\lambda| : \lambda \in \sigma(T)\})^n = r(T)^n$$

Therefore $r(T^n) = r(T)^n$
3) We have $\sigma(T^{\frac{1}{2}}) = \sigma(T)^{\frac{1}{2}}$, then

$$r(T^{\frac{1}{2}}) = \sup\{\lambda : \lambda \in \sigma(T^{\frac{1}{2}})\} = \sup\{\sqrt{\lambda} : \lambda \in \sigma(T)\} = \sqrt{\sup\{\lambda : \lambda \in \sigma(T)\}} = r(T)^{\frac{1}{2}}$$

As desired.

Proposition 3.2 *Let $T, S \in \mathcal{B}(\mathcal{H})$, then*

1) $r(\alpha T) = |\alpha| r(T)$,

2) $r(TS) = r(ST)$.

Proof. 1) We have $\sigma(\alpha T) = \alpha\sigma(T) = \{\alpha\lambda : \lambda \in \sigma(T)\}$, then:

$$r(\alpha T) = \sup\{\lambda : \lambda \in \sigma(\alpha T)\} = \sup\{|\alpha\lambda| : \lambda \in \sigma(T)\} = |\alpha| \sup\{|\lambda| : \lambda \in \sigma(T)\} = |\alpha| r(T)$$

Therefore $r(\alpha T) = |\alpha| r(T)$.
2) We have $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$, then:

$$\sup\{\lambda : \lambda \in \sigma(TS) \cup \{0\}\} = \sup\{\lambda : \lambda \in \sigma(ST) \cup \{0\}\}$$

But

$$\sup\{\lambda : \lambda \in \sigma(TS) \cup \{0\}\} = \sup\{\lambda : \lambda \in \sigma(TS)\}$$
$$\sup\{\lambda : \lambda \in \sigma(ST) \cup \{0\}\} = \sup\{\lambda : \lambda \in \sigma(ST)\}$$

Then

$$\sup\{\lambda : \lambda \in \sigma(TS)\} = \sup\{\lambda : \lambda \in \sigma(ST)\}$$

Hence

$$r(TS) = r(ST)$$

Theorem 3.6 *(Gelfand's formula) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$r(T) = \lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}}$$

Proof. 1) Let $a_n = \log \|T^n\| \ \ \forall n \in \mathbb{N}$,We want to show that $\dfrac{a_n}{n}$ conveges.
Since $\forall n, m \in \mathbb{N} : \|T^{n+m}\| \leq \|T^n\| \|T^m\|$
We have

$$a_{pm} \leq pa_m \ \text{ and } \ a_{n+m} \leq a_n + a_m$$

.
2) We write $n = pm + q$ with $0 \leq q < m$ and $p \in \mathbb{N}$. Then :

$$\frac{a_n}{n} \leq \frac{pa_m}{n} + \frac{a_q}{n}$$

Since $\dfrac{pm}{n} \xrightarrow[n \longrightarrow \infty]{} 1$, and $\dfrac{q}{n} \xrightarrow[n \longrightarrow \infty]{} 0$, then $\dfrac{p}{n} \xrightarrow[n \longrightarrow \infty]{} \dfrac{1}{m}$ It follows that :

$$\limsup_{n \longrightarrow \infty} \frac{a_n}{n} \leq \frac{a_m}{m}$$

Then we take $m \longrightarrow \infty$ to obtain

$$\limsup_{n \longrightarrow \infty} \frac{a_n}{n} \leq \liminf_{m \longrightarrow \infty} \frac{a_m}{m}$$

which implies that $\dfrac{a_n}{n}$ is converges.
3) Let us denote $\lim\limits_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}}$ by ρ.
We shall show $r(T) = \rho$.

- First, we expand $(\lambda - T)^{-1}$ formally as a Neumann series

$$\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda} \right)^n$$

  From the definition of ρ, we see that this Neumann series converges if $\dfrac{\rho}{|\lambda|} < 1$ and diverges if $\dfrac{\rho}{|\lambda|} > 1$.

- For if $\dfrac{\rho}{|\lambda|} < 1$ it means that we can find an N such that for any $n \geq N$

$$\frac{\|T^n\|^{\frac{1}{n}}}{|\lambda|} < \eta < 1 \ \text{ or } \ \frac{\|T^n\|}{|\lambda|^n} < \eta^n$$

  with $\eta < 1$. Thus, the Neumann series converges. Similar argument for the divergence proof.

- Thus, we get $f\{\lambda : |\lambda| > \rho\} \subset \rho(T)$ from the convergence argument and $f\{\lambda : |\lambda| < \rho\} \subset \sigma(T)$ from the divergence argument. These two imply $\rho = r(T)$.

Corollary 3.4 *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then $r(T) = \|T\|$.*

Proof. we have $\forall n \in \mathbb{N} : \|T^n\| = \|T\|^n \Longrightarrow \lim_{n\longrightarrow\infty} \|T^n\|^{\frac{1}{n}} = \|T\|$(Theorem 2.11)
Since

$$r(T) = \lim_{n\longrightarrow\infty} \|T^n\|^{\frac{1}{n}} \Longrightarrow r(T) = \|T\|$$

Remark 3.5 *Notice that $r(T) = 0$ does not imply $T = 0$. For instant, the matrix*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has zero spectral radius but it is not zero. A bounded operator is called nilponent if $r(T) = 0$.

Theorem 3.7 *Let $T \in \mathcal{B}(\mathcal{H})$, such that T commutes with S i.e. $TS = ST$. Then :*

$$r(TS) \leq r(T)r(S)$$

Proof. $r(TS) = \lim_{n\longrightarrow\infty} \|TS^n\|^{\frac{1}{n}}$, since $TS = ST$, then $(TS)^n = T^nS^n$, therefore:

$$r(TS) = \lim_{n\longrightarrow\infty} \|T^nS^n\|^{\frac{1}{n}} \leq \lim_{n\longrightarrow\infty} \|T^n\|^{\frac{1}{n}}\|S^n\|^{\frac{1}{n}} \leq \lim_{n\longrightarrow\infty} \|T^n\|^{\frac{1}{n}} \lim_{n\longrightarrow\infty} \|S^n\|^{\frac{1}{n}} = r(T)r(S)$$

Hence $r(TS) = r(T)r(S)$.

3.3 Numerical range of a linear operator

Defnition 3.6 *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of T is denoted by $W(T)$, and it is defined as follows*

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

Defnition 3.7 *Let $A \in \mathcal{M}_n(\mathbb{C})$. The numerical range of A is denoted by $W(A)$, and it is defined as follows*

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$$

And x^* its conjugate transpose.

Example 3.2 *In $\mathbb{C}^2$ let T be the operator defined by the matrix*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If $x = (x_1, x_2)$, $\|x\| = |x_1| + |x_2| = 1$, we have $Tx = (x_2, 0)$ and $\langle Tx, x\rangle = x_2\bar{x}_1$.
Notice that $|\langle Tx, x\rangle| = |x_2||x_1| \leq \frac{1}{2}(|x_1|^2 + |x_2|^2) \leq \frac{1}{2}$, and thus

$$W(t) \subset \left\{z : |z| \leq \frac{1}{2}\right\}.$$

Letting $z = re^{i\theta}$, $0 \leq r \leq \frac{1}{2}$, if we choose $x = (\cos\alpha, e^{i\theta}\sin\alpha)$, where $\sin 2\alpha = 2r \leq 1$, and $0 \leq \alpha \leq \frac{\pi}{4}$, we see that

$$\langle Tx, x\rangle = e^{i\theta}\sin\alpha\cos\alpha = re^{i\theta}.$$

Then $W(T) = \{z : |z| \leq \frac{1}{2}\}$.

Proposition 3.3 *Let $T, S \in \mathcal{B}(\mathcal{H})$. $\alpha, \beta \in \mathbb{C}$ and $U \in \mathcal{B}(\mathcal{H})$ be unitary. Then*

1) $W(T) = \{\alpha\}$ if and only if $T = \alpha I$,

2) $W(\alpha T + \beta I) = \alpha W(T) + \beta = \alpha z + \beta : z \in W(T)$,

3) $W(T^*) = \{\overline{z} : z \in W(T)\}$,

4) $W(\Re T) = \Re(W(T))$, and $W(\Im T) = \Im(W(T))$,

5) $W(T + S) \subset W(T) + W(S)$,

6) $W(U^*TU) = W(T)$.

Proof. 1) If $T = \alpha I$, then $W(T) = \{\langle \alpha x, x\rangle : \|x\| = 1\} = \{\alpha\}$.

$$\begin{aligned}
\text{Assume that } W(T) = \{\alpha\} &\implies \forall x \in \mathcal{H} where \|x\| = 1 : \langle Tx, x\rangle = \alpha. \\
&\implies \forall x \in \mathcal{H} \langle Tx, x\rangle = \alpha\|x\|^2 = \alpha\langle x, x\rangle \\
&\implies \forall x \in \mathcal{H} \langle (T - \alpha I)x, x\rangle = 0. \\
&\implies T = \alpha I \text{ as desired.}
\end{aligned}$$

2) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle (\alpha T + \beta I)x, x\rangle = \alpha\langle Tx, x\rangle + \beta\langle x, x\rangle = \alpha\langle Tx, x\rangle + \beta \implies W(\alpha T + \beta I) = \alpha W(T) + \beta.$$

3) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \implies W(T^*) = \{\overline{z} : z \in W(T)\}.$$

4) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, we know that $\Re(T) = \dfrac{1}{2}(T + T^*)$ and $\Im(T) = \dfrac{1}{2i}(T - T^*)$, then :

$$\langle \Re(T)x, x\rangle = \langle \frac{1}{2}(T + T^*)x, x\rangle = \frac{1}{2}\langle Tx, x\rangle + \frac{1}{2}\langle T^*x, x\rangle = \frac{1}{2}(\langle Tx, x\rangle + \langle Tx, x\rangle) = \Re(\langle Tx, x\rangle).$$

Thus $W(\Re(T)) = \Re(W(T))$, and processing the same way, we obtain $W(\Im(T)) = \Im(W(T))$.
5) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then:

$$\langle (T + S)x, x\rangle = \langle Tx, x\rangle + \langle Sx, x\rangle \in W(T) + W(S) \implies W(T + S) \subset W(T) + W(S).$$

6) Let $x \in \mathcal{H}$ such that $\|x\| = 1$, and since U is unitary, we have $\|Ux\| = \|x\| = 1$, then :

$$\langle U^*TUx, x\rangle = \langle TUx, Ux\rangle \in W(T) \implies W(U^*TU) \subset W(T).$$

Since U is surjective, then for all $\|x\| = 1$, $\exists y \in \mathcal{H}$ such that $x = Uy$, we have $\|y\| = \|Uy\| = \|x\| = 1$, then:

$$\langle Tx, x\rangle = \langle TUy, Uy\rangle = \langle U^*TUy, y\rangle \in W(U^*TU) \implies W(T) \subset W(U^*TU).$$

Hence $W(U^*TU) = W(T)$.

Corollary 3.5 *Let $T \in \mathcal{B}(\mathcal{H})$, then*

1) $W(T) \subseteq \mathbb{R}$ if and only if T is self-adjoint, *2) If $\mathcal{H}$ is a finite dimensional space, then $W(T)$ is compact,*

3) For all $z \in W(T)$, then $z \leq \|T\|$.

Proof. 1) Let $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, sinec $\forall x \in \mathcal{H}\ \ \langle Tx, x\rangle \in \mathbb{R}$, then $W(T) \subseteq \mathbb{R}$.

2) Since $\mathcal{H}$ is a finite dimensional space, then the unit sphere $S_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\| = 1\}$ is compact. Define $f : S_{\mathcal{H}} \longrightarrow \mathbb{C}$ by $f(x) = \langle Tx, x\rangle$, then f is continuous, since: $\forall x, y \in S_{\mathcal{H}}$:

$$\begin{aligned} |f(x) - f(y)| = |\langle Tx, x\rangle - \langle Ty, y\rangle + \langle Ty, x\rangle - \langle Ty, x\rangle| &= |\langle Tx - Ty, x\rangle + \langle Ty, x - y\rangle| \\ &\leq |\langle Tx - Ty, x\rangle| + |\langle Ty, x - y\rangle| \\ &\leq \|T\|\|x - y\|\|x\| + \|T\|\|x - y\|\|y\| \\ &\leq 2\|T\|\|x - y\| \end{aligned}$$

Therefore f is continuous, and since $W(T) = f(S_{\mathcal{H}})$, then $W(T)$ is compact.
3) Let $z \in W(T)$ and $\|x\| = 1$, then

$$z = \langle Tx, x\rangle \leq |\langle Tx, x\rangle| \leq \|T\|\|x\|^2 = \|T\|$$

As desired.

Theorem 3.8 *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma_{ap}(T) \subset \overline{W(T)}$.*

Proof. Let us take $\lambda \in \sigma_{ap}(T)$ and a sequence $\{x_n\}$ of unit vectors and $\lim\limits_{n \longrightarrow \infty}(T - \lambda I)x_n = 0$ that satisfies

$$\|(T - \lambda I)x_n\| \longrightarrow 0$$

The Cauchy-Schwarz inequality then gives us that

$$\|\langle (T - \lambda I)x_n, x_n\rangle\| \leq \|\langle (T - \lambda I)x_n\rangle\|\|x_n\| = \|(T - \lambda I)x_n\| \longrightarrow 0.$$

From the inequality above we can conclude that $\langle Tx_n, x_n\rangle \longrightarrow \lambda$, so $\lambda \in \overline{W(T)}$.

Proposition 3.4 *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma_p(T) \subset W(T)$.*

Proof. Let $\lambda \in \sigma_p(T)$, then there exists $x \in \mathcal{H}$ where $\|x\| = 1$ such that $Tx = \lambda x$, then:

$$\lambda = \lambda\|x\|^2 = \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle \Longrightarrow \lambda = \langle Tx, x\rangle \Longrightarrow \lambda \in W(T)$$

Therefore $\sigma_p(T) \subset W(T)$.

Theorem 3.9 *(Toeplitz-Hausdorff Theorem). Let $T \in \mathcal{B}(\mathcal{H})$. Then $W(T)$ is convex.*

Proof. Let $\xi = \langle Tx, x\rangle$ and $\eta = \langle Ty, y\rangle$ for any $\|x\| = \|y\| = 1$. We want to prove that the segment joining ξ and η is in $W(T)$. If $\xi = \eta$, then the segment is ξ and $\xi \in W(T)$.
Suppose that $\xi \neq \eta$, choose complex numbers a, b such that $a\xi + b = 1$ and $a\eta + b = 0$. Indeed $a = \dfrac{1}{\xi - \eta}$ and $b = \dfrac{-\eta}{\xi - \eta}$ are the desired numbers. we have $W(aT + bI) = aW(T) + b$. Consequently $\{0, 1\} \in W(aT + bI)$ (since $a\xi + b = 1$ and $a\eta + b = 0$ where $\xi, \eta \in W(T)$). Then we have the following:

$$\forall t \in [0, 1] : t\xi + (1 - t)\eta \in W(T) \Longleftrightarrow t \in W(aT + bI)$$

- Let $t\xi + (1-t)\eta \in W(T)$, then $\exists z \in \mathcal{H}$ where $\|z\| = 1$ such that $\langle Tz, z\rangle = t\xi + (1-t)\eta$, therefore:

$$a\langle Tz, z\rangle + b = ta\xi + (1-t)a\eta + b = t(a\xi + b) + (1-t)(a\eta + b) = t$$

Then $t = a\langle Tz, z\rangle + b$. Hence $t \in W(aT + bI)$.

- Let $t \in W(aT + bI)$, then $\exists z \in \mathcal{H}$ where $\|z\| = 1$ such that $t = a\langle Tz, z\rangle + b$, therefore:

$$\begin{aligned} t = \frac{1}{\xi - \eta}\langle Tz, z\rangle - \frac{\eta}{\xi - \eta} \quad &\Longrightarrow \quad \langle Tz, z\rangle = t(\xi - \eta) + \eta = t\xi + (1-t)\eta \\ &\Longrightarrow \quad t\xi + (1-t)\eta = \langle Tz, z\rangle \end{aligned}$$

Hence $t\xi + (1-t)\eta \in W(T)$.
Then to prove that $t\xi + (1-t)\eta \in W(T)$ for all $t \in [0,1]$, it is enough to prove that $[0,1] \subset W(aT + bI)$.
Set $S = aT + bI$, then:

$$\langle Sx, x\rangle = a\langle Tx, x\rangle + b = a\xi + b = 1$$

Therefore $\langle Sx, x\rangle = 1$, and the same $\langle Sy, y\rangle = 0$. Define $g : \mathbb{R} \longrightarrow \mathbb{C}$, by:

$$g(\theta) = \langle Sy, x\rangle e^{-i\theta} + \langle Sx, y\rangle e^{i\theta}$$

It is obvious that g is continuous.
Let $\theta \in \mathbb{R}$, and using $e^{-i(\theta+\pi)} = -e^{-i\theta}$ and $e^{i(\theta+\pi)} = -e^{i\theta}$, we get :

$$g(\theta + \pi) = \langle Sy, x\rangle e^{-i(\theta+\pi)} + \langle Sx, y\rangle e^{i(\theta+\pi)} = -\langle Sy, x\rangle e^{-i\theta} + \langle Sx, y\rangle e^{i\theta} = -g(\theta)$$

Then $g(\theta + \pi) = -g(\theta)$ for all $\theta \in \mathbb{R}$, which means that $g(\pi) = -g(0)$, then $\Im g(\pi) = -\Im g(0)$.
Since $\Im g$ is continuous, and the sign of $\Im g(0)$ and $\Im g(\pi)$ are different (because $\Im g(\pi) = -\Im g(0)$), by intermediate value theorem, there exists $\theta_0 \in [0, \pi]$ such that $\Im g(\theta_0) = 0$.
Put $\hat{x} = e^{i\theta_0}x$. $\hat{x}$ and y are linearly independent, if not, $\exists \alpha \in \mathbb{C}$ such that $y = \alpha\hat{x}$ where $|\alpha| = 1$ (since $\|y\| = 1$).
We have $0 = \langle Sy, y\rangle = |\alpha|^2\langle S\hat{x}, \hat{x}\rangle = |\alpha|^2\langle Sx, x\rangle = 1$, contradiction. Then, $\hat{x}$ and y are linearly independent. Therefore $\|(1-t)y + t\hat{x}\| \neq 0$ for all $t \in [0,1]$. Put

$$\phi(t) = \frac{(1-t)y + t\hat{x}}{\|(1-t)y + t\hat{x}\|} \quad \text{for all } t \in [0,1]$$

Clearly $\|\phi(t)\| = 1$ for all $t \in [0,1]$. Furthermore $\langle S\phi(0), \phi(0)\rangle = 0$ and $\langle S\phi(1), \phi(1)\rangle = 1$.
Define $f(t) = \langle S\phi(t), \phi(t)\rangle$ for all $t \in [0,1]$. Set $\beta = \|(1-t)y + t\hat{x}\|$, using the fact that $g(e^{i\theta_0}) \in \mathbb{R}$, we get:

$$\begin{aligned} f(t) = \langle S\phi(t), \phi(t)\rangle &= \frac{(1-t)^2}{\beta^2}\langle Sy, y\rangle + \frac{t^2}{\beta^2}\langle S\hat{x}, \hat{x}\rangle + \frac{t(1-t)}{\beta^2}(\langle Sy, \hat{x}\rangle + \langle S\hat{x}, y\rangle) \\ &= \frac{t^2}{\beta^2} + \frac{t(1-t)}{\beta^2}(e^{-i\theta_0}\langle Sy, x\rangle + e^{i\theta_0}\langle Sx, y\rangle) \\ &= \frac{t^2}{\beta^2} + \frac{t(1-t)}{\beta^2}g(e^{i\theta_0}) \in \mathbb{R} \ \text{ for all } t \in [0,1] \end{aligned}$$

Thus f is a real-valued function. Moreover f is clearly continuous and $f([0,1]) \subset W(S)$. (since $\|\phi(t)\| = 1$)
Since $f(0) = 0$, $f(1) = 1$ and f is continuous, then $[0,1] \subset f([0,1])$ (by intermediate value theorem).
And since $f([0,1]) \subset W(S)$, thus $[0,1] \subset W(S)$.
Hence $t\xi + (1-t)\eta \in W(T)$ for all $t \in [0,1]$, therefore $W(T)$ is convex.

3.4 Numerical radius inequalities

Defnition 3.8 *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical radius of T is denoted by $\omega(T)$, and it is defined as follows*

$$\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$$

Corollary 3.6 *The numerical radius ω defines a norm on $\mathcal{B}(\mathcal{H})$.*

Proof • If $T = 0$, it is clear that $\omega(T) = 0$.
Now assume that $\omega(T) = 0$, then:

$$\begin{aligned} \forall \lambda \in W(T) : \lambda = 0 &\iff \forall x \in \mathcal{H}, \|x\| = 1 : \langle Tx, x\rangle = 0 \\ &\implies \forall x \in \mathcal{H}\langle Tx, x\rangle = 0 \\ &\implies T = 0. \end{aligned}$$

Hence $T = 0 \iff \omega(T) = 0$.
• Let $T \in \mathcal{B}(\mathcal{H})$ and $\mu \in \mathbb{C}$. We have $W(\mu T) = \mu W(T)$, we get:
$\omega(\mu T) = \sup\{|\lambda| : \lambda \in W(\mu T)\} = \sup\{|\mu\lambda| : \lambda \in W(T)\} = |\mu| \sup\{|\lambda| : \lambda \in W(T)\}$
Thus $\omega(\alpha T) = |\alpha|\omega(T)$.
• Let $T, S \in \mathcal{B}(\mathcal{H})$. We have $W(T + S) \subset W(T) + W(S)$, we obtain:

$$\begin{aligned} \omega(T + S) = \sup\{|\lambda| : \lambda \in W(T + S)\} &\leq \sup\{|\lambda| : \lambda \in W(T) + W(S)\} \\ &= \sup\{|\mu_1 + \mu_2| : \mu_1 \in W(T), \mu_2 \in W(S)\} \\ &\leq \sup\{|\mu_1| : \mu_1 \in W(T)\} + \sup\{|\mu_2| : \mu_2 \in W(S)\} \\ &= \omega(T) + \omega(S). \end{aligned}$$

Then $\omega(T + S) \leq \omega(T) + \omega(S)$. Therefore ω is a norm on $\mathcal{B}(\mathcal{H})$.

Theorem 3.10 *(Equivalent norm) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega(T) \leq \|T\| \leq 2\,\omega(T) \tag{3.1}$$

Proof. Let $T \in \mathcal{B}(\mathcal{H})$. Then:

- Let $x \in \mathcal{B}(\mathcal{H})$ such that $\|x\| = 1$, then:

  $$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$$

  Thus $\sup\{|\langle Tx, x\rangle| : \|x\| = 1\} \leq \|T\|$, therefore $\omega(T) \leq \|T\|$.

- Let $x, y \in \mathcal{H}$, then:

  $$\begin{aligned} |\langle T\frac{x}{\|x\|}, \frac{x}{\|x\|}\rangle| \leq \omega(T) &\implies \frac{1}{\|x\|^2}|\langle Tx, x\rangle| \leq \omega(T) \\ &\implies |\langle Tx, x\rangle| \leq \omega(T)\|x\|^2 \end{aligned}$$

  We have that

  $$\langle Tx, y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle).$$

  Therefore

  $$\begin{aligned} |\langle Tx, y\rangle| &\leq \frac{1}{4}\omega(T)(\|x + y\|^2 + \|x - y\|^2 + \|x + iy\|^2 + \|x - iy\|^2) \\ &= \omega(T)(\|x\|^2 + \|y\|^2) \end{aligned}$$

Thus $|\langle Tx, y\rangle| \leq \omega(T)(\|x\|^2 + \|y\|^2)$ for all $x, y \in \mathcal{H}$, then:

$$\|T\| \leq \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| \leq \sup_{\|x\|=\|y\|=1} \omega(T)(\|x\|^2 + \|y\|^2) = 2\omega(T).$$

Hence $\|T\| \leq 2\omega(T)$.

Theorem 3.11 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1) If $w(T) = \|T\|$, then $r(T) = \|T\|$,

2) If $\lambda \in W(T)$ and $|\lambda| = \|T\|$, then $\lambda \in \sigma_p(T)$.

Proof. 1) Since $\omega(T) = \|T\|$, then $\exists (x_n) \subset \mathcal{H}$ where $\|x_n\| = 1$ such that $\lim\limits_{n\longrightarrow\infty} |\langle Tx_n, x_n\rangle| = \|T\|$. Set $\lambda = \lim\limits_{n\longrightarrow\infty} \langle Tx_n, x_n\rangle$, so $|\lambda| = \|T\|$, then:

$$|\langle Tx_n, x_n\rangle| \leq \|Tx_n\|\|x_n\| = \|Tx_n\| \leq \|T\|$$

Thus $\lim\limits_{n\longrightarrow\infty} \|Tx_n\| = \|T\|$, therefore:
$\|(T-\lambda I)x_n\|^2 = \langle (T-\lambda I)x_n, (T-\lambda I)x_n\rangle = \|Tx_n\|^2 + |\lambda|^2 - \lambda\langle Tx_n, x_n\rangle - \lambda\langle x_n, Tx_n\rangle$

$$\begin{aligned} &\Longrightarrow \quad \|(T-\lambda I)x_n\|^2 \xrightarrow[n\longrightarrow\infty]{} 2\|T\|^2 - 2|\lambda|^2 = 0 \\ &\Longrightarrow \quad \lim_{n\longrightarrow\infty} \|(T-\lambda I)x_n\| = 0 \end{aligned}$$

Therefore $\lambda \in \sigma_{ap}(T)$, then $\lambda \in \sigma(T)$, and since $r(T) \leq \|T\|$ and $|\lambda| = \|T\|$, hence $r(T) = \|T\|$ as required.
2) Since $\lambda \in W(T)$, then $\exists x \in \mathcal{H}$ where $\|x\| = 1$ and $\lambda = \langle Tx, x\rangle$. Since $\|T\| = |\langle Tx, x\rangle|$, then:

$$|\lambda| = |\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\| = |\langle Tx, x\rangle| \Longrightarrow |\langle Tx, x\rangle| = \|Tx\|\|x\|$$

Then Tx and x are linearly dependent i.e. $\exists\ \eta \in \mathbb{C}$ such that $Tx = \eta x$, then:

$$\lambda = \langle Tx, x\rangle = \langle \eta x, x\rangle = \eta \Longrightarrow \lambda = \eta \Longrightarrow Tx = \lambda x$$

Hence $\lambda \in \sigma_p(T)$ as desired.

Theorem 3.12 *Let $T \in \mathcal{B}(\mathcal{H})$, If $Ran(T) \perp Ran(T^*)$, then $w(T) = \frac{1}{2}\|T\|$.*

Proof. We have that $\mathcal{H} = Ker(T) \oplus (Ker(T))^\perp$, and since $(Ker(T))^\perp = \overline{Ran(T^*)}$, then $\mathcal{H} = Ker(T) \oplus \overline{Ran(T^*)}$. Since $Ran(T) \perp Ran(T^*)$, we can easily prove that $Ran(T) \perp \overline{R(T^*)}$.
Let $x \in \mathcal{H}$ where $\|x\| = 1$, then $\exists x_1 \in Ker(T)$ and $\exists x_2 \in \overline{Ran(T^*)}$ such that $x = x_1 + x_2$ and $\|x_1\|^2 + \|x_2\|^2 = 1$.
Using the fact that $x_1 \in Ker(T)$ and $Ran(T) \perp \overline{R(T^*)}$, we get:

$$\begin{aligned} \langle Tx, x\rangle = \langle T(x_1 + x_2), x_1 + x_2\rangle &= \langle Tx_2, x_1\rangle \\ &\leq \|T\|\|x_1\|\|x_2\| \\ &\leq \frac{1}{2}\|T\|(\|x_1\|^2 + \|x_2\|^2) \\ &= \frac{1}{2}\|T\| \end{aligned}$$

Thus $\omega(T) \leq \|T\|$, and we always have $\frac{1}{2}\|T\| \leq \omega(T)$ by (3.1).
Hence $\omega(T) = \frac{1}{2}\|T\|$.

Lemma 3.6 *[24] Let $T \in \mathcal{B}(\mathcal{H})$, $x \in \mathcal{H}$ such that $\|x\| = 1$ and $n \in \mathbb{N}$. Then*

$$1 - \langle T^n x, x\rangle = \frac{1}{n}\sum_{j=1}^{n} \|x_j\|^2 (1 - u_j \langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle).$$

Where $u_j = e^{2\pi j i}$ and $x_j = (\prod_{k=1,k\neq j}^{n}(1 - u_j T))x$ for all $j = 1, 2, \cdots, n$.

Proof. We have these two well-known polynomial identities:

$$1 - z^n = \prod_{k=1}^{n}(1 - u_k z) \quad \text{and} \quad 1 = \frac{1}{n}\sum_{j=1}^{n}\prod_{k=1,k\neq j}^{n}(1 - u_k z).$$

Since these identities remain valid when we replace z by T, then we have :

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} \|x_j\|^2 (1 - u_j \langle T\frac{x}{\|x_j\|}, \frac{x}{\|x_j\|}\rangle) &= \frac{1}{n}\sum_{j=1}^{n}\langle (I - u_j T)x_j, x_j\rangle \\
&= \frac{1}{n}\sum_{j=1}^{n}\langle (I - T^n)x, x_j\rangle \\
&= \langle (I - T^n)x, \frac{1}{n}\sum_{j=1}^{n}\prod_{k=1,k\neq j}^{n}(1 - u_k T)x_j\rangle \\
&= \langle (I - T^n)x, x\rangle \\
&= 1 - \langle T^n x, x\rangle
\end{aligned}$$

Theorem 3.13 *[24] (Power inequality) Let $T \in \mathcal{B}(\mathcal{H})$, and $n \in \mathbb{N}$. Then*

$$\omega(T^n) \leq (\omega(T))^n$$

Proof. By the homogeneity of ω, it suffices to assume that $\omega(T) \leq 1$ and prove $\omega(T^n) \leq 1$. If we do, then:

$$\omega(\frac{1}{\omega(T)}T) = \frac{\omega(T)}{\omega(T)} = 1 \Longrightarrow \omega((\frac{1}{\omega(T)}T)^n) = \omega(\frac{1}{(\omega(T))^n}T^n) = \frac{1}{(\omega(T))^n}\omega(T^n) \leq 1$$

Then $\omega(T^n) \leq (\omega(T))^n$.
Assume that $\omega(T) \leq 1$. Let $x \in \mathcal{H}$ where $\|x\| = 1$ and $z \in \mathbb{C}$ such that $|z| \leq 1$, then :

$$\Re(1 - z\langle Tx, x\rangle) = 1 - \Re(z\langle Tx, x\rangle) \geq 1 - |z\langle Tx, x\rangle| \geq 1 - |z|\omega(T) \geq 1 - |z| \geq 0$$

Then $\Re(1 - z\langle Tx, x\rangle) \geq 0$. If $\Re(1 - z^n\langle T^n x, x\rangle) \geq 0$, then $\omega(T^n) \leq 1$, since:

$$\Re(1 - z^n\langle T^n x, x\rangle) \geq 0 \Longrightarrow \Re(z^n\langle T^n x, x\rangle) \leq 1$$

Choose $z^n = \dfrac{\overline{\langle T^n x, x\rangle}}{|\langle T^n x, x\rangle|}$, note that $|z| = 1$, then:

$$\Re(\frac{\overline{\langle T^n x, x\rangle}}{|\langle T^n x, x\rangle)|}\langle T^n x, x\rangle) \leq 1 \Longrightarrow |\langle T^n x, x\rangle| \leq 1 \Longrightarrow \omega(T^n) \leq 1$$

Therefore, to prove $\omega(T^n) \leq 1$, it is enough to prove $\Re(1 - z^n\langle T^n x, x\rangle) \geq 0\ \forall |z| \leq 1$.
By Lemma 3.6, we have $1 - z^n\langle T^n x, x\rangle = 1 - \langle (zT)^n x, x\rangle = \dfrac{1}{n}\sum_{j=1}^n \|x_j\|^2 (1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle)$.
Then:

$$\begin{aligned}
\Re(1 - \langle (zT)^n x, x\rangle) &= \Re(\frac{1}{n}\sum_{j=1}^n \|x_j\|^2 (1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle))\\
&= \frac{1}{n}\sum_{j=1}^n \|x_j\|^2 \Re(1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle)
\end{aligned}$$

Since $|zu_j| = |e^{\frac{2\pi j i}{n}} z| = |z| \leq 1$ for all $j \in \{1, \cdots, n\}$, and we have that $\Re(1 - z\langle Tx, x\rangle) \geq 0$. Then $\Re(1 - zu_j\langle T\frac{x_j}{\|x_j\|}, \frac{x_j}{\|x_j\|}\rangle \geq 0$ for all $j \in \{1, \cdots, n\}$, therefore $\Re(1 - \langle (zT)^n x, x\rangle) \geq 0$. Hence $\omega(T^n) \leq 1$.

Theorem 3.14 *[11] Let $T \in \mathcal{B}(\mathcal{H})$, Then*

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{3.2}$$

Proof. Let $\|x\| = 1$, using Theorem 2.15, we get:

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \sqrt{\langle |T|x, x\rangle}\sqrt{\langle |T^*|x, x\rangle}\\
&\leq \frac{1}{2}(\langle |T|x, x\rangle + \langle |T^*|x, x\rangle)\\
&= \frac{1}{2}\langle (|T| + |T^*|)x, x\rangle\\
&\leq \frac{1}{2}\||T| + |T^*|\|
\end{aligned}$$

Theorem 3.15 *[18] Let $T \in \mathcal{B}(\mathcal{H})$, Then*

$$\omega(T) \leq \frac{1}{2}(\|T\| + \sqrt{\|T^2\|}) \tag{3.3}$$

Proof. By (3.2), we have:

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\|$$

And by (2.8), we have:

$$\||T| + |T^*|\| \leq \max\{\||T|\|, \||T^*|\|\} + \|\sqrt{|T|}\sqrt{|T^*|}\|$$

Since $\||T|\| = \||T^*|\| = \|T\|$, then $\max\{\||T|\|, \||T^*|\|\} = \|T\|$.
We have $\|\sqrt{|T|}\sqrt{|T^*|}\| \leq \sqrt{\||T||T^*|\|}$, and $\||T||T^*|\| = \|T^2\|$. Then $\|\sqrt{|T|}\sqrt{|T^*|}\| \leq \sqrt{\|T^2\|}$, therefore:

$$\||T| + |T^*|\| \leq \|T\| + \sqrt{\|T^2\|}$$

Hence

$$\omega(T) \leq \frac{1}{2}(\|T\| + \sqrt{\|T^2\|})$$

Theorem 3.16 *[9] Let $T \in \mathcal{B}(\mathcal{H})$, Then*

$$\omega(T)^2 \leq \frac{1}{2}\left(\omega(T^2) + \|T\|^2\right).$$

Proof. Let $x \in \mathcal{H}$ such that $\|x\| = 1$. Using Theorem 1.3, we obtain:

$$\begin{aligned} |\langle Tx, x\rangle|^2 = |\langle Tx, x\rangle\langle x, T^*x\rangle| &\leq \frac{1}{2}(\|Tx\|\|T^*x\| + \langle Tx, T^*x\rangle) \\ &= \frac{1}{2}(\|Tx\|\|T^*x\| + \langle T^2x, x\rangle) \\ &\leq \frac{1}{2}\left(\omega(T^2) + \|T\|^2\right). \end{aligned}$$

As desired.

Chapter 4

Ky-Fan norm: Definitions and basic properties

In this chapter we give some general properties about compact operators, $Ky-Fan$ norm and its properties.

4.1 Compact operators on Hilbert space

Defnition 4.1 *Let $\mathcal{H}$ be a Hilbert space, we call an orthonormal system $\{x_n : n \in \mathbb{N}^*\}$ if*

$$\langle x_i, x_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Defnition 4.2 *Let $x_1, x_2, \cdots, x_n \in \mathcal{H}$. The span of $x_1, x_2, \cdots, x_n$, written span$\{x_1, x_2, \cdots, x_n\}$ is the set of all linear combinations of $x_1, x_2, \cdots, x_n$, so*

$$span\{x_1, x_2, \cdots, x_n\} = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}\}$$

Defnition 4.3 *A set $\{x_n : n \in \mathbb{N}^*\}$ is a basis of $\mathcal{H}$ if*

1) $\{x_n\}$ is orthonormal system,

2) For all $x \in \mathcal{H}$.

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n.$$

every $x \in \mathcal{H}$ can be expressed uniquely in the form

$$x = \sum_{n=1}^{\infty} \alpha_n x_n$$

Defnition 4.4 *The space is called about $\mathcal{H}$ that it is separable if and only if it has a countable orthonormal basis.*

Defnition 4.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is called a finite rank if $Ran(T)$ is finite dimensional, we denoted by $r(T) = dim(Ran(T))$.*

Defnition 4.6 *A linear operator is compact if, for any bounded sequence $\{x_n\} \in \mathcal{H}$, the sequence $\{Tx_n\} \in \mathcal{H}$ contains a convergent subsequence.*

The set of compact linear operator will be denoted by $\mathcal{K}(\mathcal{H})$.

Theorem 4.1 *Let $T \in \mathcal{K}(\mathcal{H})$. Then T is bounded. Thus $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.*

Proof. Suppose that T is not bounded. Then for each integer $n \geq 1$ there exists a unit vector x_n such that $\|Tx_n\| \geq n$. Since the sequence $\{x_n\}$ is bounded, by the compactness of T there exists a subsequence $\|Tx_{n_{(r)}}\|$ which converges.
This contradicts $\|Tx_{n_{(r)}}\| \geq n(r)$. (ie. convergence implies boundedness)

Theorem 4.2 *Let $T, S \in \mathcal{K}(\mathcal{H})$, and $\alpha, \beta \in \mathbb{C}$, then $\alpha T + \beta S$ is compact.*
Thus $\mathcal{K}(\mathcal{H})$ is a linear subspace of $\mathcal{B}(\mathcal{H})$.

Proof. Let $\{x_n\}$ be a bounded sequence in X. Since S is compact, there is a subsequence $\{x_{n(r)}\}$ such that $\{Tx_{n(r)}\}$ converges. Then, since $\{x_{n(r)}\}$ is bounded and T is compact, there is a subsequence $\{x_{n(r(s))}\}$ of the sequence $\{x_{n(r)}\}$ such that $\{Sx_{n(r(s))}\}$ converges.Since the sum of convergent sequences converges, it follows that the sequence $\{Tx_{n(r(s))} + Sx_{n(r(s))}\}$ converges. Thus $\alpha T + \beta S$ is compact.

Lemma 4.1 *Let S be a compact operator on a separable Hilbert space $\mathcal{H}$ and suppose that $\{T_n\} \subseteq \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ are such that for each $x \in \mathcal{H}$, the sequence $T_n x \longrightarrow Tx$. Then $T_n S \longrightarrow TS$ in the norm of $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose that $\|T_n S - TS\| \nrightarrow 0$. Then there exists a $\delta > 0$ and a subsequence $\{T_{n(r)}S\}$ such that

$$\|T_{n(r)}S - TS\| > \delta.$$

Choose unit vectors $\{x_{n(s)}\}$ of $\mathcal{H}$ such that

$$\|(T_{n(r)}S - TS)(x_{n(s)})\| > \delta.$$

Since S is compact, we get a subsequence $\{x_{n(r)}\}$ of $\{x_{n(s)}\}$ such that $Sx_{n(r)}$ is convergent. Assume that $Sx_{n(r)} \longrightarrow y$. Then

$$\delta < \|(T_{n(r)}S - TS)x_{n(r)}\| \leq \|(T_{n(s)} - T)(Sx_{n(r)} - y)\| + \|(T_{n(r)} - T)y\|. \tag{4.1}$$

Since $Sx_{n(r)} \longrightarrow y$, there exists n such that for $n(r) > n$,

$$\|Sx_{n(r)} - y\| < \frac{\delta}{8C}.$$

Also as $T_{n(r)}y \longrightarrow Ty$ for ach $y \in \mathcal{H}$, there exists m such that $n(r) > m$ implies

$$\|(T - T_{n(r)})y\| < \frac{\delta}{4}.$$

Since $\{T_n\} \subseteq \mathcal{B}(\mathcal{H})$ is bounded, then $\|T_n\| \leq C$ and $\|Tx\| = \lim \|T_n x\| \leq C$.
Hence $\|T - T_{n(r)}\| \leq 2C$. Now from 4.1

$$\delta < \|(T_{n(r)}S - TS)x_{n(r)}\| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}.$$

a contradiction.

Theorem 4.3 *Every compact operator on a separable Hilbert space $\mathcal{H}$ is a norm limit of a sequence of finite rank operators. In other words , the set of finite rank operators is dense in the space of compact operators.*

Proof. Let $\{x_n : n \in \mathbb{N}^*\}$ be an orthonormal basis for $\mathcal{H}$ and $\mathcal{H}_n := span\{x_k\}_{k=1}^n$. Then the orthogonal projections $P_n : \mathcal{H} \longrightarrow \mathcal{H}$ defined by

$$P_n y = \sum_{i=1}^{n} \langle y, x_i \rangle x_i.$$

has the property that $P_n y \longrightarrow y$ for each $x \in \mathcal{H}$.
Now, if K is compact, then by Lemma 4.1, it follows that $P_n K \longrightarrow K$ in the operator norm of $\mathcal{B}(\mathcal{H})$.
Here $Ran(P_n K) \subseteq Ran(P_n) = \mathcal{H}_n$ is finite dimensional.

Example 4.1 1) *If $dim(\mathcal{H}) < \infty$, the identity operator I is a compact operator,*

2) *Let $\mathcal{H} = \ell_2$ and $\{e_n\}$ be the standard orthonormal basis of $\mathcal{H}$.
Define $D : \mathcal{H} \longrightarrow \mathcal{H}$ by*

$$D(x_1, x_2, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{2}, \cdots)$$

Then D is bounded. Next, we show that $D \in \mathcal{K}(\mathcal{H})$. Define $D_n : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$D_n x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i.$$

Then D_n is finite rank bounded operator and $D_n \longrightarrow D$ as $n \longrightarrow 1$. Hence by Theorem 4.3, D is compact,

3) *Let $k(\cdot, \cdot) \in L^2[a, b]$. Define $K : L^2[a, b] \longrightarrow L^2[a, b]$ by*

$$(Kf)(s) = \int_a^b k(s,t) f(t) dt, \text{ for all } f \in L^2[a, b].$$

It can be verified that $K \in \mathcal{B}(L^2[a, b])$. Let $\{x_n : n \in \mathbb{N}\}$ be an orthonormal basis for $L^2[a, b]$. Then $\varphi_{m,n}(s, t) = x_{n(s)} x_{m(t)}$ for all $s, t \in [a, b]$ and for all $m, n \in \mathbb{N}$ forms an orthonormal basis for $L^2([a, b] \times [a, b])$. Hence

$$k(s,t) = \sum_{n,m=1}^{\infty} \langle k(s,t), \varphi_{m,n}(s,t) \rangle \varphi_{m,n}(s,t)$$

Let

$$k_N(s,t) = \sum_{n,m=1}^{N} \langle k(s,t), \varphi_{m,n}(s,t) \rangle \varphi_{m,n}(s,t)$$

Now define $k_N : L^2[a, b] \longrightarrow L^2[a, b]$ by

$$(K_N f)(s) = \int_a^b k_N(s,t) f(t) dt, \text{ for all } f \in L^2[a, b].$$

*Note that K_N is a finite rank operator and $K_N \longrightarrow K$ as $N \longrightarrow \infty$.
Hence by Theorem 4.3, $K \in K(L^2[a, b])$.*

Theorem 4.4 *$T, S \in \mathcal{B}(\mathcal{H})$ and at least on of the operators T, S is compact, then $TS \in \mathcal{B}(\mathcal{H})$ is compact.*

Proof. Let $\{x_n\}$ be a bounded sequence in X.

- If S is compact then there is a subsequence $\{x_{n(r)}\}$ such that $\{Sx_{n(r)}\}$ converges. Since T is bounded (and so is continuous), the sequence $\{TSx_{n(r)}\}$ converges. Thus TS is compact,
- If S is bounded but not compact the the sequence $\{Sx_n\}$ is bounded. Then since T must be compact, there is a subsequence $\{Sx_{n(r)}\}$ such that $\{TSx_{n(r)}\}$ converges, and again TS is compact.

Theorem 4.5 *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1) T is compact $\Longleftrightarrow T^*T$ or TT^* is compact,

2) T is compact $\Longleftrightarrow T^*$ is compact.

Proof.1) If T is compact, then T^*T is compact by Theorem 4.4. To prove the converse, assume that T^*T is compact. If $\{x_n\} \subseteq \mathcal{H}$ is a bounded sequence with bound $M > 0$, then T^*Tx_n has a convergent subsequence namely $T^*Tx_{n(r)}$,say $T^*Tx_n \longrightarrow y$. For notational convenience we denote the subsequence $\{x_{n(k)}\}$ by $\{x_n\}$.
For $n > m$, we have

$$\begin{aligned}\|Tx_n - Tx_m\|^2 &= \langle T(x_n - x_m), T(x_n - x_m)\rangle \\ &= \langle T^*T(x_n - x_m), (x_n x_m)\rangle \\ &\leq \|T^*T(x_n - x_m)\|\|x_n - x_m\| \\ &\leq 2M\|T^*T(x_n - x_m)\|.\end{aligned}$$

Hence $\{Tx_n\}$ is Cauchy and hence convergent.
Similarly, $TT^* \in \mathcal{K}(\mathcal{H})$.
2) Let $\{x_n\} \subseteq \mathcal{H}$ be a bounded sequence. As TT^* is compact, $\{TT^*x_n\}$ has a convergent subsequence, say $TT^*x_{n(r)}$ converging to x. Now for $r > s$.

$$\begin{aligned}\|T^*x_{n(r)} - T^*x_{n(s)}\|^2 &= \langle TT^*(x_{n(r)} - x_{n(s)}), x_{n(r)} - x_{n(s)}\rangle \\ &\leq \|TT^*(x_{n(r)} - x_{n(s)})\|\|x_{n(r)} - x_{n(s)}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

That is $\{T^*x_{n(k)}\}$ is Cauchy, hence convergent. Thus T^* is compact.
By the above argument, T^* is compact implies that $T^{**} = T$ is compact.

Proposition 4.1 *Let $T \in \mathcal{B}(\mathcal{H})$, we have*

1) If T has finite rank then T is compact,

2) If either $dim(\mathcal{H})$ is finite then T is compact.

Proof. 1) Since T has finite rank, the space $Ran(T)$ is a finite-dimensional.
Furthermore, for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ is bounded in $Ran(T)$, so by the Bolzano-Weierstrass theorem this sequence must contain a convergent subsequence.
Hence T is compact.

2) If $dim(\mathcal{H})$ is finite then $r(T) \leq dim(\mathcal{H})$, so $r(T)$ is finite. Thus, the result follows from the previous part of this proof.

Defnition 4.7 *A complex number $\lambda \in \mathbb{C}$ is called an eigenvalue of $T \in \mathcal{B}(\mathcal{H})$ if there exists a vector $0 \neq x \in \mathcal{H}$ such that $Tx = \lambda x$. The vector x is called an eigenvector for T corresponding to the eigenvalue λ.*

Example 4.2 *Let $\mathcal{H} = \ell^2$. Define $T : \mathcal{H} \longrightarrow \mathcal{H}$ by*

$$T(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots). \quad for\ all\ (x_1, x_2, x_3, \cdots) \in \mathcal{H}$$

Let$\{e_n : n \in \mathbb{N}\}$ denote the standard orthonormal basis for $\mathcal{H}$. Then $Te_n = \frac{1}{n}e_n$. Hence $\{\frac{1}{n}\}$ is a set of eigenvalues of T with corresponding eigenvectors e_n.

Proposition 4.2 *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, then eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. Let λ and μ be distinct eigenvalues of T and x and y be the corresponding eigenvectors. Then $Tx = \lambda x$ and $Ty = \mu y$. Now

$$\mu\langle x, y\rangle = \langle x, \mu y\rangle = \langle x, Ty\rangle = \langle Tx, y\rangle = \lambda\langle x, y\rangle.$$

Since $\lambda, \mu \in \mathbb{R}$ and distict, it follows that $\langle x, y\rangle = 0$.

Theorem 4.6 *[8] Let $T \in \mathcal{B}(\mathcal{H})$ is compact and self-adjoint, then at least one of the numbers $\|T\|$ or $-\|T\|$ is an eigenvalue.*

Proof. By Theorem 2.8, there exists a sequence $\{x_n\} \subseteq \mathcal{H}$ with $\|x_n\| = 1$ for each n such that and $\lambda \in \mathbb{R}$ such that $\langle Tx_n, x_n\rangle \longrightarrow \lambda$, where $\lambda = +\|T\|$ or $\lambda = -\|T\|$.
Now

$$\begin{aligned} \|Tx_n - \lambda x_n\|^2 &= \|Tx_n\|^2 + \lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \\ &\leq 2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \longrightarrow 0\ as\ n \longrightarrow \infty. \end{aligned}$$

Since T is compact, there exists a sub sequence $\{Tx_{n(r)}\}$ of $\{Tx_n\}$ such that $Tx_{n(r)} \longrightarrow y$. So $Tx_{n(r)} - \lambda x_{n(r)} \longrightarrow 0$ as $n \longrightarrow \infty$. That is $x_n \longrightarrow \dfrac{1}{\lambda}y$. Hence $y = \lim Tx_{n(r)} = \dfrac{1}{\lambda}Ty$.
Hence λ is an eigenvalue.

Corollary 4.1 *If $T \in \mathcal{K}(\mathcal{H})$ be self-adjoint, then $\max_{\|x\|=1} |\langle Tx, x\rangle| = \|T\|$.*

Theorem 4.7 *Let $T \in \mathcal{K}(\mathcal{H})$, and $\lambda \neq 0$, then*

$$dim(Ker(T - \lambda I)) < \infty.$$

Proof. *Assume by contradiction that $dim(Ker(T - \lambda I)) = \infty$. Then we can find a sequence orthonormal $\{e_n\}_{n\in\mathbb{N}} \subset Ker(T - \lambda I)$, then $\{e_n\} \in Ker(T - \lambda I)$, we have $Te_n = \lambda e_n$.*
For $n \neq m$:

$$\|\lambda e_n - \lambda e_m\|^2 = 2\lambda^2.$$

so the sequence $\{\lambda e_n\}$ cannot have a convergent subsequence. This gives a contradiction and prove that $dim(Ker(T - \lambda I)) < \infty$.

Theorem 4.8 *Let $T \in \mathcal{K}(\mathcal{H})$, then*

1) *If $dim(\mathcal{H}) = \infty$, then $0 \in \sigma(T)$,*

2) *$0 \neq \lambda \in \sigma(T)$ if and only if it is an eigenvalue of T,*

3) *$\sigma(T)\backslash\{0\}$ is finite or is given by a sequence of eigenvalues tending to 0.*

Proof. 1) Assume that 0 belongs to the resolvent set of T. Then I is the composition of the compact operator T with the bounded operator T^{-1}, so I is a compact operator. This implies that $dim(\mathcal{H}) < \infty$.

2)Let $\lambda \in \mathbb{R}^*$ (or $\mathbb{C}^*$). Then we have $T - \lambda I = \lambda(\lambda^{-1}T - I)$. Since $\lambda^{-1}T$ is compact, Theorem 4.7 shows that $(T - \lambda)$ is bijective (with bounded inverse) if and only if it is injective, so λ is in the resolvent set of T if and only if it is not an eigenvalue.
Moreover, if λ is an eigenvalue of T we have

$$dim(Ker(T - \lambda I)) = dim(Ker(\lambda^{-1}T - I)) < \infty.$$

3) Since K is a bounded operator, the set of eigenvalues of K is bounded in $\mathbb{R}(\mathbb{C})$.
Assume that $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of distinct non-zero eigenvalues of T tending to some λ. We prove that $\lambda = 0$. For $n \in \mathbb{N}$ we consider $\varphi_n \in \mathcal{H}\backslash\{0\}$ such tha $T\varphi_n = \lambda_n\varphi_n$.
Then for $n \in \mathbb{N}$ we set $\mathcal{H}_n = span\{\varphi_0, \varphi_1, \cdots, \varphi_{n-1}\}$ and we consider $u_n \in \mathcal{H}_n$ such that $\|u_n\| = 1$ and $u_n \in \mathcal{H}_{n-1}^{\perp}$ if $n \geq 1$. Then for $j \in \mathbb{N}$ and $k > j$ we have

$$\left\| \frac{Tu_k}{\lambda_k} - \frac{Tu_j}{\lambda_j} \right\| = \left\| \frac{Tu_k - \lambda_k u_k}{\lambda_k} - \frac{Tu_j - \lambda_j u_j}{\lambda_j} + u_k - u_j \right\| \geq 1$$

Since $Tu_k - \lambda_k u_k$, $Tu_j - \lambda_j u_j$, $u_j \in \mathcal{H}_{k-1}$. If $\lambda \neq 0$ we obtain a contradiction with the compactness of T.

Remark 4.1 *If T is compact operator and $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then the restriction operator $T|_{\mathcal{M}}$ is also compact.*

Theorem 4.9 *[31] (The Spectral theorem). Suppose $T \in \mathcal{K}(\mathcal{H})$ be self-adjoint. Then there exists a system of orthonormal vectors $x_1, x_2, x_3, \cdots$ of eigenvectors of T and corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq 0$.*

$$Ty = \sum_{k=1}^{\infty} \lambda_k \langle y, x_k \rangle x_k \qquad \textit{for all } y \in \mathcal{H}.$$

If $\{\lambda_n\}$ is infinite, then $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$. The series on the right hand side converges in the operator norm of $\mathcal{B}(\mathcal{H})$.

Proof. We prove the theorem step by step.

Step 1 : Construction of eigenvectors

We use Theorem 4.6 repeatedly for constructing eigenvalues and eigenvectors.
Let $\mathcal{H}_1 = \mathcal{H}$ and $T_1 = T$. Then by Theorem 4.6, there exists an eigenvalue λ_1 of T_1 and an eigenvector x_1 such that $\|x_1\| = 1$ and $|\lambda_1| = \|T_1\|$.
Now $span\{x_1\}$ is a closed subspace of $\mathcal{H}_1$, hence by the projection theorem,

$$\mathcal{H}_1 = span\{x_1\} \oplus span\{x_1\}^{\perp}.$$

Let $\mathcal{H}_2 = span\{x_1\}^{\perp}$. Clearly $\mathcal{H}_2$ is a closed subspace of $\mathcal{H}_1$ and $T(\mathcal{H}_2) \subseteq \mathcal{H}_2$.
Now let $T_2 = T_1|_{\mathcal{H}_2}$. Then T_2 is a compact and self-adjoint operator in $\mathcal{B}(\mathcal{H}_2)$.
If $T_2 = 0$, then there is nothing to prove. Assume that $T_2 \neq 0$.
Then by Theorem 4.6, there exists an eigenvalue λ_2 of T_2 with $|\lambda_2| = \|T_2\|$ and a corresponding eigenvector x_2 with $\|x_2\| = 1$. Since T_2 is a restriction of T_1, $|\lambda_2| = \|T_2\| \leq \|T_1\| = |\lambda_1|$.

By the construction x_1 and x_2 are orthonormal.
Now let $\mathcal{H}_3 = span\{x_1, x_2\}^{\perp}$. Clearly $\mathcal{H}_3 \subseteq \mathcal{H}_2$. It is easy to show that $T(\mathcal{H}_3) \subseteq \mathcal{H}_3$.
The operator $T_3 = T|_{\mathcal{H}_3}$ is compact and self-adjoint.
Hence by Theorem 4.6, there exists an eigenvalue λ_3 of T_3 and a corresponding eigenvector x_3 with $\|x_3\| = 1$. Here $|\lambda_3| = \|T_3\|$.
Hence $|\lambda_3| \leq |\lambda_2| \leq |\lambda_1|$.

Proceeding in this way, either after some stage n, $T_n = 0$ or there exists a sequence λ_n of eigenvalues of T and corresponding vector x_n with $\|x_n\| = 1$ and $|\lambda_n| = \|T_n\|$. Also $|\lambda_{n+1}| \leq |\lambda_n|$ for each n.

Step 2 :If $\{\lambda_n\}$ is infinite, then $\lambda_n \longrightarrow 0$
If $\lambda_n \not\twoheadrightarrow 0$, there exists an $\epsilon > 0$ such that $|\lambda_n| \geq \epsilon$ for infinitely many n. If $n \neq m$,

$$\|Tx_n - Tx_m\|^2 = \|\lambda_n x_n - \lambda_m x_m\|^2 = \lambda_n^2 + \lambda_m^2 > \epsilon^2.$$

This shows that $\{Tx_n\}$ has no convergent subsequence, a contradiction to the compactness of T.
Hence $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Step 3 :Representation of T
Here we consider two cases

Case 1 :$T_n = 0$ for some n
Let $y_n = y - \sum_{k=1}^{n} \langle y, x_k \rangle x_k$. Then $y_n \perp x_i$ for $1 \leq i \leq n$. Hence

$$0 = T_n y_n = Ty - \sum_{k=1}^{n} \lambda_k \langle y, x_k \rangle x_k.$$

That is $Ty = \sum_{k=1}^{n} \lambda_k \langle y, x_k \rangle x_k$.

Case 2 :$T_n \neq 0$ for infinitely many n
For $y \in \mathcal{H}$, by Case 1, we have

$$\begin{aligned} \|Ty - \sum_{k=1}^{n} \lambda_k \langle y, x_k \rangle x_k\| = \|T_n y_n\| &\leq \|T_n\| \|y_n\| \\ &= |\lambda_n| \|x_n\| \leq |\lambda_n| \|x\| \longrightarrow 0 \end{aligned}$$

Hence $Ty = \sum_{k=1}^{\infty} \lambda_k \langle y, x_k \rangle x_k$.

Example 4.3 *Define $D : \ell_2 \longrightarrow \ell_2$ by*

$$D(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots) \quad (x_1, x_2, \cdots) \in \ell_2$$

Then $D \in \mathcal{K}(\mathcal{H})$ and self-adjoint. Note that $De_n = \frac{1}{n} e_n$. Hence $\dfrac{1}{n}$ is an eigenvalue with an eigenvector e_n. Also $\dfrac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$. We can also represent D as

$$D(x) = \sum_{n=1}^{\infty} \frac{1}{n} \langle x, e_n \rangle e_n, \text{ enien for all } x = (x_n) \in \ell_2.$$

Defnition 4.8 *An orthonormal system* $\{x_1, x_2, \cdots\}$ *of eigenvectors of* $T \in \mathcal{B}(\mathcal{H})$ *with corresponding non zero eigenvalues* $\lambda_1, \lambda_2, \lambda_3, \cdots$ *is called a basic system of eigenvalues and eigenvectors of* T *if*

$$Tx = \sum_k \lambda_k \langle x, x_k \rangle x_k$$

The Spectral Theorem guarantees the existence of a basic system of eigenvalues and eigenvectors for a compact self-adjoint operator.

Defnition 4.9 *If* $T \in \mathcal{K}(\mathcal{H})$ *and* λ *is an eigenvalue of* $|T| = \sqrt{T^*T}$, *we call* λ *a singularvalue of* T, *and we define*

$$s(T) = \lambda(|T|).$$

4.2 Ky-Fan norm and its properties

Defnition 4.10 *[13] Let* $T \in \mathcal{K}(\mathcal{H})$, $\{e_n\}_{n=1}^{\infty}$ *be orthonormal basis for* $\mathcal{H}$, *The trace of* T, *denoted by* Tr *is given by*

$$Tr(T) = \sum_{k\geq 1} \langle Te_k, e_k \rangle$$

where this sum equals ∞ *if it does not converge.*

Proposition 4.3 *Let* $T, S \in \mathcal{B}(\mathcal{H})$ *and* $\alpha \in \mathbb{C}$, *Then*

1) $Tr(T + S) = Tr(T) + Tr(S)$,

2) $Tr(\alpha T) = \alpha Tr(t)$

Proof.1) Let $T, S \in \mathcal{B}(\mathcal{H})$, then:

$$\begin{aligned} Tr(T+S) &= \sum_{k\geq 1} \langle (T+S)e_k, e_k \rangle \\ &= \sum_{k\geq 1} \langle Te_k, e_k \rangle + \langle Se_k, e_k \rangle \\ &= Tr(T) + Tr(S). \end{aligned}$$

2) Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$, then:

$$\begin{aligned} Tr(\alpha T) &= \sum_{k\geq 1} \langle \alpha Te_k, e_k \rangle \\ &= \sum_{k\geq 1} \alpha \langle Te_k, e_k \rangle \\ &= \alpha Tr(T). \end{aligned}$$

Defnition 4.11 *[22] Let* $T \in \mathcal{K}(\mathcal{H})$. *For* $p \in [1, \infty)$, *we define the Schatten* $p-norm$ *of as*

$$\|T\|_p = (Tr(|T|^p))^{\frac{1}{p}}$$

- *If* $p = 1$ *is called the trace norm (also known as the nuclea rnorm or the* $Ky\ Fan$ *norm),*
- *If* $p = 2$ *is called the* $Hilbert - Schmidt$ *norm,*
- *If* $p = \infty$ *is called the operator norm* $(\|.\|_\infty)$.

Remark 4.2 *Since* $|T| = (T^*T)^{\frac{1}{2}}$, *Then the Schatten* $p-norm$ *can be pet as*

$$\|T\|_p = (Tr((T^*T)^{\frac{p}{2}}))^{\frac{1}{p}}$$

Defnition 4.12 *Let* $T \in \mathcal{B}(\mathcal{H})$. *define the trace norm as*

$$\|T\|_1 = Tr(|T|)$$

Proof.• If $T = 0$, it is obvious that $\|T\|_1 = 0$.
Conversely, assume that $\|T\|_1 = 0$, then:

$$\begin{aligned}
\|T\|_1 = Tr(|T|) = 0 \implies & \sum_{k\geq 1}\langle |T|e_k, e_k\rangle = 0 \\
\implies & \sum_{k\geq 1}\langle (T^*T)^{\frac{1}{2}}e_k, e_k\rangle = 0 \\
\implies & \langle (T^*T)^{\frac{1}{2}}e_k, e_k\rangle = 0 \;\; for\ all\ k \geq 1 \quad ((T^*T)^{\frac{1}{2}} \text{ be positive}) \\
\implies & (T^*T)^{\frac{1}{2}} = 0 \quad (\text{ Use Generalized polarization identity}) \\
\implies & T^*T = 0 \\
\implies & \langle T^*Tx, x\rangle = 0 \quad \text{for all } x \in \mathcal{H} \\
\implies & \langle Tx, Tx\rangle = 0 \quad \text{for all } x \in \mathcal{H} \\
\implies & Tx = 0 \quad \text{for all } x \in \mathcal{H} \\
\implies & T = 0
\end{aligned}$$

Therefore $T = 0 \Longleftrightarrow \|T\|_1 = 0$.
• Let $\alpha \in$ and $T \in \mathcal{B}(\mathcal{H})$, then:

$$\begin{aligned}
\|\alpha T\|_1 = Tr(|\alpha T|) \implies & \sum_{k\geq 1}\langle |\alpha T|e_k, e_k\rangle \\
\implies & \sum_{k\geq 1}\langle (\alpha^2 T^*T)^{\frac{1}{2}}e_k, e_k\rangle \\
\implies & \sum_{k\geq 1}|\alpha|\langle Te_k, Te_k\rangle = |\alpha|\|T\|_1
\end{aligned}$$

Hence $\|\alpha T\|_1 = |\alpha|\|T\|_1$.

• Let $T, S \in \mathcal{B}(\mathcal{H})$, then:

$$\begin{aligned}
\|T + S\|_1 = Tr(|T + S|) \;\; \leq & \;\; Tr(|T| + |S|) \\
\leq & \;\; Tr(|T|) + Tr(|S|) \\
= & \;\; \|T\|_1 + \|S\|_1
\end{aligned}$$

Example 4.4 *Define* $T \in \mathcal{B}(\mathbb{F}^4)$ *by* $T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$.
A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

Thus

$$\sqrt{T^*T}(z_1, z_2, z_3, z_4) = (3z_1, 2z_2, 0, 3z_4).$$

Hence $\|T\|_1 = 3 + 3 + 2 + 0 = 8$.

Theorem 4.10 *[12] (Courant min-max theorem). Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $T \in \mathcal{K}(\mathcal{H})$ be a positive operator. If $k \in \mathbb{N}$ then*

$$\max_{dim\ S=k}\ \min_{x\in\ S,\|x\|=1} \langle Tx, x\rangle = \lambda_k(T) = s_k(T). \tag{4.2}$$

and

$$\min_{dim\ S=k-1}\ \max_{x\in\ S^{\perp},\|x\|=1} \langle Tx, x\rangle = \lambda_k(T) = s_k(T). \tag{4.3}$$

Proof. $|T|$ is compact and positive, so according to Theorem 4.9 there is an orthonormal set $\{e_n : n \in \mathbb{N}\}$ such that

$$Tx = \sum_{n=1}^{\infty} s_n(T)\langle x, e_n\rangle e_n.$$

For $k \in \mathbb{N}$, let $S_k = \bigvee_{n=k}^{\infty}\{e_n\}$. $S_k^{\perp} = \bigvee_{n=1}^{k-1}\{e_n\}$, so S_k has codimension $k-1$. (The codimension of a closed subspace of a Hilbert space is the dimension of its orthogonal complement.) If S is a k dimensional subspace of $\mathcal{H}$, then there is some $x \in S_k \cap S$ with $\|x\| = 1$. This is because if V is a closed subspace with codimension $k-1$ of a *Hilbert space* and W is a k dimensional subspace of the *Hilbert space*, then their intersection is a subspace of nonzero dimension. As $x \in S_k$, there are $\alpha_n \in \mathbb{C}$, $n \geq k$, with

$$x = \sum_{n=k}^{\infty} \alpha_n e_n \qquad\qquad \|x\|^2 = \sum_{n=k}^{\infty} |\alpha_n|^2.$$

As the sequence $s_n(T)$ is nonincreasing,

$$\begin{aligned}
\langle Tx, x\rangle &= \left\langle \sum_{n=1}^{\infty} s_n(T)\langle x, e_n\rangle e_n, x\right\rangle \\
&= \left\langle \sum_{n=1}^{\infty} s_n(T)\left\langle \sum_{m=k}^{\infty} \alpha_m e_m, e_n\right\rangle e_n, \sum_{m=k}^{\infty} \alpha_m e_m\right\rangle \\
&= \left\langle \sum_{n=1}^{\infty} s_n(T)\sum_{m=k}^{\infty} \alpha_m \delta_{m,n} e_n, \sum_{m=k}^{\infty} \alpha_m e_m\right\rangle \\
&= \left\langle \sum_{n=1}^{\infty} s_n(T)\alpha_n \chi_{\geq k(n)} e_n, \sum_{m=k}^{\infty} \alpha_m e_m\right\rangle \\
&= \left\langle \sum_{n=k}^{\infty} s_n(T)\alpha_n e_n, \sum_{m=k}^{\infty} \alpha_m e_m\right\rangle \\
&= \sum_{n=k}^{\infty} s_n(T)|\alpha_n|^2 \\
&\leq s_k(T)\sum_{n=k}^{\infty} |\alpha_n|^2 \\
&= s_k(T).
\end{aligned}$$

where we write

$$\chi_{\geq k(n)} = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$$

This shows that if $dimS = k$ then

$$\inf_{x\in S,\|x\|=1}\langle Tx, x\rangle \leq s_k(T).$$

Let $M = \inf_{x\in S,\|x\|=1}\langle Tx, x\rangle \leq s_k(T)$, and let $x_n \in S$, $\|x_n\| = 1$, with $\langle Tx_n, x_n\rangle \longrightarrow M$.
As S is a finite dimensional Hilbert space, the unit sphere in it is compact, so there a a subsequence $x_{a(n)}$ that converges to some $z \in S$, $\|z\| = 1$. We have

$$\begin{aligned}
|\langle Tz, z\rangle - \langle Tx_n, x_n\rangle| &\leq |\langle Tz, z\rangle - \langle Tx_n, z\rangle| + |\langle Tx_n, z\rangle - \langle Tx_n, x_n\rangle| \\
&= |\langle T(z - x_n), z\rangle| + |\langle Tx_n, z - x_n\rangle| \\
&\leq \|T\|\|z - x_n\|\|z\| + \|T\|\|x_n\|\|z - x_n\| \\
&= 2\|T\|\|z - x_n\|.
\end{aligned}$$

As $x_{a(n)} \longrightarrow z$, we get $\langle Tx_{a(n)}, x_{a(n)}\rangle \longrightarrow \langle Tz, z\rangle$. As $\langle Tx_n, x_n\rangle \longrightarrow M$, we get

$$\langle Tz, z\rangle = M = \inf_{x\in S,\|x\|=1}\langle Tx, x\rangle.$$

As $z \in S$ and $\|z\| = 1$, we have in fact

$$\min_{x\in S,\|x\|=1}\langle Tx, x\rangle = \inf_{x\in S,\|x\|=1}\langle Tx, x\rangle \leq s_k(T)$$

This is true for any k dimensional subspace of $\mathcal{H}$, so

$$\sup_{dimS=k}\min_{x\in S,\|x\|=1}\langle Tx, x\rangle \leq s_k(T).$$

If $S = S_{k+1}^{\perp}$ then $e_k \in S$, $\|e_k\| = 1$, and

$$\langle Te_k, e_k\rangle = \langle s_k(T)e_k, e_k\rangle = s_k(T).$$

so in fact

$$\max_{dimS=k}\min_{x\in S,\|x\|=1}\langle Tx, x\rangle = s_k(T).$$

which is the first of the two formulas that we want to prove.
For $k \geq 1$, let $S_k = \bigvee_{n=1}^{k}\{e_n\}$. If S is a $k-1$ dimensional subspace of $\mathcal{H}$, then $S^{\perp}$ is a closed subspace with codimension $k-1$, so the intersection of S_k and $S^{\perp}$ has nonzero dimension, and so there is some $x \in S_k \cap S^{\perp}$ with $\|x\| = 1$.

As $x \in S_k$ there are $\alpha_1, \cdots, \alpha_k$ with $x = \sum_{n=1}^{k} \alpha_n e_n$, giving

$$\begin{aligned}
\langle Tx, x \rangle &= \left\langle \sum_{n=1}^{\infty} s_n(T) \langle x, e_n \rangle e_n, x \right\rangle \\
&= \left\langle \sum_{n=1}^{\infty} s_n(T) \left\langle \sum_{m=1}^{k} \alpha_m e_m, e_n \right\rangle e_n, \sum_{m=1}^{k} \alpha_m e_m \right\rangle \\
&= \left\langle \sum_{n=1}^{\infty} s_n(T) \sum_{m=1}^{k} \alpha_m \delta_{m,n} e_n, \sum_{m=1}^{k} \alpha_m e_m \right\rangle \\
&= \left\langle \sum_{n=1}^{\infty} s_n(T) \alpha_n \chi_{\leq k(n)} e_n, \sum_{m=1}^{k} \alpha_m e_m \right\rangle \\
&= \left\langle \sum_{n=1}^{k} s_n(T) \alpha_n e_n, \sum_{m=1}^{k} \alpha_m e_m \right\rangle \\
&= \sum_{n=1}^{k} s_n(T) |\alpha_n|^2 \\
&\geq s_k(T) \sum_{n=1}^{k} |\alpha_n|^2 \\
&= s_k(T).
\end{aligned}$$

This shows that

$$\sup_{x \in S^{\perp}, \|x\|=1} \langle Tx, x \rangle \geq s_k(T).$$

Define $M = \sup_{x \in S^{\perp}, \|x\|=1} \langle Tx, x \rangle$. Because M is a supremum, there is a sequence x_n on the unit sphere in S_{k-1} such that $\langle Tx_n, x_n \rangle \longrightarrow M$. The unit sphere in S_{k-1} is compact because S_{k-1} is finite dimensional, so this sequence has a convergent subsequence $x_{a(n)} \longrightarrow z$. As

$$|\langle Tz, z \rangle - \langle Tx_n, x_n \rangle| \leq 2\|T\| \|z - x_n\|.$$

and $x_{a(n)} \longrightarrow z$, we get

$$\langle Tz, z \rangle = M = \sup_{x \in S^{\perp}, \|x\|=1} \langle Tx, x \rangle.$$

whence

$$\max_{x \in S^{\perp}, \|x\|=1} \langle Tx, x \rangle = \sup_{x \in S^{\perp}, \|x\|=1} \langle Tx, x \rangle \geq s_k(T).$$

As this is true for any $k-1$ dimensional subspace S,

$$\inf_{dim S = k-1} \max_{x \in S^{\perp}, \|x\|=1} \langle Tx, x \rangle \geq s_k(T).$$

But for $S = S_{k-1}$ we have $e_k \in S^{\perp}$, $\|e_k\| = 1$, and

$$\langle Te_k, e_k \rangle = \langle s_k(T) e_k, e_k \rangle = s_k(T).$$

which implies that

$$\min_{dim S = k-1} \max_{x \in S^{\perp}, \|x\|=1} \langle Tx, x \rangle = s_k(T).$$

Corollary 4.2 *T is positive operator $\Longrightarrow$ T is self-adjoint operator ($T^* = T$).*

$$s_k(T) = \lambda_k((T^*T)^{\frac{1}{2}}) = \lambda_k((T^2)^{\frac{1}{2}}) = \lambda_k(T).$$

Theorem 4.11 *[2] Let $T \in \mathcal{K}(\mathcal{H})$, then*

$$s_k(T) = \max_{dim\ S=k} \min_{x\in\ S, \|x\|=1} \|Tx\|.$$

Proof. By Theorem 4.11 we have

$$\begin{aligned} s_k^2(T) = \lambda_k(T^*T) &= \max_{dim\ S=k} \min_{x\in\ S, \|x\|=1} \langle T^*Tx, x\rangle \\ &= \max_{dim\ S=k} \min_{x\in\ S, \|x\|=1} \langle Tx, Tx\rangle \\ &= \max_{dim\ S=k} \min_{x\in\ S, \|x\|=1} \|Tx\|^2 \end{aligned}$$

Hence

$$s_k(T) = \max_{dim\ S=k} \min_{x\in\ S, \|x\|=1} \|Tx\|.$$

Remark 4.3 *Let $T \in \mathcal{B}(\mathcal{H})$. Then the singular values of T are the nonnegative square root of the eigenvalues of T^*T i.e.*

$$s_i = \sqrt{\lambda_i} \qquad i = 0, 1, 2, \cdots$$

Theorem 4.12 *[25] Let $T, S \in \mathcal{K}(\mathcal{H})$, then*

$$s_k(T + S) \leq s_k(T) + s_k(S).$$

Proof. By Theorem 4.11 we have

$$\begin{aligned} s_k(T+S) &= \min_{dim\ S=k-1} \max_{x\in\ S^\perp, \|x\|=1} \|(T+S)x\| \\ &\leq \min_{dim\ S=k-1} \max_{x\in\ S^\perp, \|x\|=1} (\|Tx\| + \|Sx\|) \\ &= \min_{dim\ S=k-1} \max_{x\in\ S^\perp, \|x\|=1} \|Tx\| + \min_{dim\ S=k-1} \max_{x\in\ S^\perp, \|x\|=1} \|Sx\| \\ &= s_k(T) + s_k(S) \end{aligned}$$

Theorem 4.13 *[10] Let $T \in \mathcal{K}(\mathcal{H})$ and $s_1(T) \geq s_2(T) \geq \cdots \geq 0$. the singular values of T. The Ky Fan $k-norm$ defined as*

$$\|T\|_{(k)} = \sum_{j=1}^{k} s_j(T) \qquad k = 1, 2, \cdots$$

Proof.

- The singular values of $T = 0$ are all zero, so $\|T\|_{(k)} = 0$.
  Furthermore, as singular values are always non-negative, there does not exist a operator T with negative singular values, so for $\|T\|_{(k)} = 0$, T can only be 0.
  Hence $T = 0 \iff \|T\|_{(k)} = 0$.

$$\begin{aligned} \|T\|_{(k)} = 0 &\implies \sum_{j=1}^{(k)} s_j(T) = 0 \\ &\implies s_j = 0 \quad for\ all\ 1 \leq j \leq k\ (s_j\ non-negative) \\ &\implies T = 0. \end{aligned}$$

- Let $\alpha \in$ and $T \in \mathcal{K}(\mathcal{H})$, then:
  The singular values of αT are eigenvalues of $\sqrt{\alpha^2 T^* T}$, so

$$\begin{aligned}
\sqrt{\alpha^2 T^* T} = |\alpha|\sqrt{T^* T} &\implies s_i(\alpha T) = |\alpha| s_i(T) \\
&\implies \|\alpha T\|_{(k)} = \sum_{j=1}^{k} |\alpha| s_j(T) \\
&\implies \|\alpha T\|_{(k)} = |\alpha| \sum_{j=1}^{k} s_j(T)
\end{aligned}$$

- Let $T, S \in \mathcal{K}(\mathcal{H})$, from Theorem 4.12 we have

$$\begin{aligned}
\|T + S\|_{(k)} &= \sum_{j=1}^{k} s_j(T + S) \\
&\leq \sum_{j=1}^{k} s_j(T) + \sum_{j=1}^{k} s_j(S) \\
&= \|T\|_{(k)} + \|S\|_{(k)}
\end{aligned}$$

Lemma 4.2 *[17] Let $T \in \mathcal{K}(\mathcal{H})$, then*

$$s_j(T) = s_j(T^*) \qquad j = 1, 2, \cdots$$

Proof. By Theorem 4.11, then

$$\begin{aligned}
s_k(T) &= \min_{dim\ S=k-1} \max_{x\in\ S^{\perp}, \|x\|=1} \|Tx\| \\
&= \min_{dim\ S=k-1} \max_{x\in\ S^{\perp}, \|x\|=1} \|T^* x\| \\
&= s_k(T^*).
\end{aligned}$$

As desired.

Corollary 4.3 *Let $T \in \mathcal{K}(\mathcal{H})$, then*

$$\|T\|_{(k)} = \|T^*\|_{(k)}.$$

Theorem 4.14 *[17] Let $T \in \mathcal{K}(\mathcal{H})$, and $S \in \mathcal{B}(\mathcal{H})$, then*

$$\|ST\|_{(k)} \leq \|S\| \|T\|_{(k)}.$$

Proof. By Theorem 4.11, then

$$s_k(T) = \min_{dim\ S=k-1} \max_{x\in\ S^{\perp}, \|x\|=1} \|Tx\|.$$

Hence

$$\begin{aligned}
s_k(ST) = \min_{dim\ S=k-1} \max_{x\in\ S^{\perp}, \|x\|=1} \|STx\| &\leq \min_{dim\ S=k-1} \max_{x\in\ S^{\perp}, \|x\|=1} (\|S\| \|Tx\|) \\
&= \|S\| \min_{dim\ S=k-1} \max_{x\in\ S^{\perp}, \|x\|=1} \|Tx\| \\
&= \|S\| \|T\|_{(k)}.
\end{aligned}$$

Corollary 4.4 *Let $T \in \mathcal{K}(\mathcal{H})$, and $S \in \mathcal{B}(\mathcal{H})$, then*

$$\|TS\|_{(k)} \leq \|S\|\|T\|_{(k)}.$$

Proof. Since $\|T\|_{(k)} = \|T^*\|_{(k)}$, then

$$\|TS\|_{(k)} = \|(TS)^*\|_{(k)} = \|S^*T^*\|_{(k)} \leq \|S^*\|\|T^*\|_{(k)} \leq \|S\|\|T\|_{(k)}.$$

As desired.

4.3 Inequalities

Lemma 4.3 *Let $T \in \mathcal{K}(\mathcal{H})$, then $\|T\| = \|T\|_1 = s_1(T)$.*

Recall that $s_1(T) \geq s_2(T) \geq \cdots$.

Proof. Let $T \in \mathcal{K}(\mathcal{H})$. By Corollary 4.1, then

$$\max_{\|x\|=1} |\langle T^*Tx, x\rangle| = \lambda_1(T^*T) = \|T\|^2.$$

Hence

$$s_1(T) = \|T\|.$$

Recall that $\lambda_1(T) \geq \lambda_2(T) \geq \cdots$.

Theorem 4.15 *Let $T \in \mathcal{K}(\mathcal{H})$,then*

$$\frac{1}{k}\|T\|_{(k)} \leq \|T\| \leq \|T\|_{(k)} \qquad k = 1, 2, \cdots .$$

Proof. • Since $\|T\| = s_1(T)$, then

$$\begin{aligned} \|T\| = s_1(T) &\leq s_1(T) + \sum_{j=2}^{k} s_j(T) \qquad k = 2, 3, \cdots \\ &= \sum_{i=1}^{k'=1} s_i(T) + \sum_{j=2}^{k} s_j(T) \qquad k = 2, 3, \cdots \\ &= \sum_{j=1}^{k} s_j(T) = \|T\|_{(k)} \qquad k = 1, 2, \cdots . \end{aligned}$$

• Let $T \in \mathcal{K}(\mathcal{H})$,then

$$\begin{aligned} \lambda(T) \in \sigma_p(T) &\implies \lambda(T) \leq \|T\| \\ &\implies (\lambda(T^*T))^{\frac{1}{2}} \leq (\|T\|^2)^{\frac{1}{2}} \\ &\implies \lambda(|T|) \leq \|T\| \\ &\implies s(T) \leq \|T\| \end{aligned}$$

For all $s_j(T)$, $j = 1, 2, \cdots, k$ and $k = 1, 2, \cdots$ we have

$$\begin{aligned} s_j(T) \leq \|T\| &\implies \sum_{j=1}^{k} s_j(T) \leq k\|T\| \\ &\implies \frac{1}{k}\|T\|_{(k)} \leq \|T\|. \end{aligned}$$

As desired.

Corollary 4.5 *Let $T \in \mathcal{K}(\mathcal{H})$,then*

$$\frac{1}{2k}\|T\|_{(k)} \leq \omega(T) \leq \|T\|_{(k)}.$$

Proof. By Theorem 3.10 and Theorem 4.15, we get inequality.

Remark 4.4 *$\|T\|_{(k)}$, $\|T\|$ and ω, are equivalent norms.*

Theorem 4.16 *Let $T \in \mathcal{K}(\mathcal{H})$, then*

$$(k-1)s_k(T) + \omega(T) \leq \|T\|_k \qquad k = 1, 2, \cdots .$$

Proof. Let $T \in \mathcal{K}(\mathcal{H})$, and by Theorem 3.10 we have

$$\omega(T) \leq \|T\|.$$

Thus

$$\omega(T) \leq \|T\| = s_1(T) \implies \omega(T) + \sum_{j=2}^{k} s_j(T) \leq \sum_{j=1}^{k} s_j(T) \qquad k = 2, 3, \cdots .$$

On the other hand, we have:

$$\begin{aligned} s_k(T) \leq s_j \qquad j = 1, 2, \cdots, k &\implies s_k(T) \leq s_j \qquad j = 2, \cdots, k \\ &\implies (k-1)s_k(T) \leq \sum_{j=2}^{k} s_j \qquad k = 2, 3, \cdots \end{aligned}$$

Hence

$$\begin{aligned} (k-1)s_k(T) + \omega(T) &\leq \omega(T) + \sum_{j=2}^{k} s_j \qquad k = 2, 3, \cdots \\ &\leq \|T\| + \sum_{j=2}^{k} s_j \qquad k = 2, 3, \cdots \\ &= \|T\|_k \qquad k = 1, 2, \cdots \end{aligned}$$

Therefore

$$(k-1)s_k(T) + \omega(T) \leq \|T\|_k \qquad k = 1, 2, \cdots .$$

Theorem 4.17 *Let $T \in \mathcal{K}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, then*

$$\omega(TS) \leq \frac{1}{2}\|S\| \left(\|T\| + \|T\|_{(k)}\right)$$

Proof. By Theorem 3.10 and Corollary 4.5 we have

$$\omega(T) \leq \frac{1}{2}(\|T\| + \|T\|_{(k)}).$$

Thus

$$\begin{aligned} \omega(TS) &\leq \frac{1}{2}(\|TS\| + \|TS\|_{(k)}) \\ &\leq \frac{1}{2}\|S\|(\|T\| + \|T\|_{(k)}). \end{aligned}$$

As desired.

Theorem 4.18 *Let $T \in \mathcal{K}(\mathcal{H})$,then*

$$\omega(T^n) \leq \frac{1}{2}\|T^{n-1}\|(\|T\| + \|T\|_{(k)}).$$

Proof. By Theorem 3.10 and Corollary 4.5 we have

$$\omega(T) \leq \frac{1}{2}(\|T\| + \|T\|_{(k)}).$$

Thus

$$\begin{aligned}
\omega(T^n) &\leq \frac{1}{2}(\|T^n\| + \|T^n\|_{(k)}) \\
&\leq \frac{1}{2}(\|T^{n-1}\|\|T\| + \|T^{n-1}\|\|T\|_{(k)}) \quad (using\ Corollary\ 4.4) \\
&\leq \frac{1}{2}\|T^{n-1}\|(\|T\| + \|T\|_{(k)}).
\end{aligned}$$

As desired.

Theorem 4.19 *Let $T \in \mathcal{K}(\mathcal{H})$,then*

$$\|T^n\|_{(k)} \leq \|T^{n-1}\|\|T\|_{(k)} \leq \|T\|_{(k)}^n.$$

Proof. By Corollary 4.4, then

$$\|T^n\|_{(k)} \leq \|T^{n-1}\|\|T\|_{(k)} \leq \|T\|_{(k)}^{n-1}\|T\|_{(k)} = \|T\|_{(k)}^n.$$

Corollary 4.6 *Let $T \in \mathcal{K}(\mathcal{H})$ is a normal operator, then*

$$\|T^n\|_{(k)} = \|T\|_{(k)}^n \iff \|T\|_{(k)} = s_1(T).$$

Proof. By Theorem 4.19, we have

$$\begin{aligned}
\|T^n\|_{(k)} = \|T\|_{(k)}^n &\iff \|T\|_{(k)}^n = \|T^{n-1}\|\|T\|_{(k)} \\
&\iff \|T\|_{(k)}^{n-1} = \|T^{n-1}\| = \|T\|^{n-1} \\
&\iff \|T\|_{(k)}^{n-1} = s_1(T)^{n-1} \\
&\iff \|T\|_{(k)} = s_1(T).
\end{aligned}$$

Theorem 4.20 *Let $T \in \mathcal{K}(\mathcal{H})$,then*

$$\omega(T)^2 \leq \frac{1}{4}\|T\|\left(3\|T\| + \|T\|_{(k)}\right).$$

Proof. Using Theorem 3.16, we obtain:

$$\begin{aligned}
\omega(T)^2 &\leq \frac{1}{2}\left(\omega(T^2) + \|T\|^2\right) \\
&\leq \frac{1}{2}\left(\frac{1}{2}\|T\|\left(\|T\| + \|T\|_{(k)}\right) + \|T\|^2\right) \\
&= \frac{3}{4}\|T\|^2 + \frac{1}{4}\|T\|\|T\|_{(k)} \\
&= \frac{1}{4}\|T\|\left(3\|T\| + \|T\|_{(k)}\right).
\end{aligned}$$

Therefore

$$\omega(T)^2 \leq \frac{1}{4}\|T\|\left(3\|T\| + \|T\|_{(k)}\right).$$

Conclusion

It this thesis, we give some preliminaries on Hilbert spaces and important theorems and properties. Then, we study bounded linear operators in detailed, and we provide some vital linear operator classes, also we define the square root and the absolute value of an operator. After that we present the following notions, spectral radius, numerical range and the numerical radius of an operator. And we provide some recent inequalities involving the usual operator norm and the numerical radius. Our work in this thesis, we define the $Ky - Fan$ norm on $\mathcal{K}(\mathcal{H})$ that is equivalent to the usual operator norm, and we study some of its properties. Also we see its applications on the inequalities between the usual operator norm and the numerical radius. We obtain some inequalities, and a refinement for others, as we see in the last chapter.

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