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List of symbols

We use the following notations throughout this thesis

Notation

- $\mathbb{R}$: set of real numbers.
- $\mathbb{R}^d$: the d-dimensional Euclidean space.
- $\mathbb{S}^d$: the space of second-order symmetric tensors on $\mathbb{R}^d$
- Ω: an open, bounded, and connected set in $\mathbb{R}^d$ with Lipschitz boundary.
- $\Gamma = \partial\Omega$: the boundary of Ω.
- $\Gamma_i (i = \overline{1,3}, a, b)$: a partition of the boundary Γ.
- $meas\Gamma_1$: the $d-1$ Lebesgue measure of the Γ_1.
- ν: the unit outward normal vector to Γ.
- u_ν: the normal component of the vector u.
- u_τ: the tangential part of the vector u.
- σ_ν: the normal component of the tensor σ.
- σ_τ: the tangential part of the tensor σ.
- $C^1(\overline{\Omega})$: the space of continuously differentiable real functions on $\overline{\Omega}$.
- $H = L^2(\Omega)^d$.
- $\mathscr{H} = L^2(\Omega)_s^{d\times d}$.
- $H_1 = H^1(\Omega)^d$.
- $\mathscr{H}_1 = \{\sigma \in \mathscr{H} \mid \operatorname{Div}\sigma \in H\}$.
- $H^{\frac{1}{2}}(\Gamma)$: the Sobolev space of order $\frac{1}{2}$ on Γ.

- $H^{-\frac{1}{2}}(\Gamma)$: the dual space of $H^{\frac{1}{2}}(\Gamma)$.
- H'_Γ dual of H_Γ.
- $\gamma : H_1 \to H_\Gamma$: the trace mapping for vector functions.

  If X is a real Hilbert space and $d \in \mathbb{N}^*$, we use the following notation:
- $X^d = \{x = (x_i) \mid x_i \in X, i = \overline{1,d}\}$.
- $X_s^{d\times d} = \{x = (x_{ij}) \mid x_{ij} = x_{ji} \in X, i, j = \overline{1,d}\}$.
- $(.,.)_X$: the scalar product of X.
- $\|.\|_X$: the norm on X.
- 2^X: the set of all parts of X.
- X': the dual space of X.
- $(.,.)_{X'\times X}$: the duality product between H' and H.

  If moreover $[0,T]$ is a time interval, $k \in \mathbb{N}$ and $1 \leq p < +\infty$, we denote by
- $C(0,T;X)$: the space of continuous functions from $[0,T]$ to X.
- $C^1(0,T;X)$: the space of continuously differentiable functions from $[0,T]$ in X.
- $L^p(0,T;X)$: the space of measurable functions of $(0,T)$ in X.
- $W^{k,p}(0,T;X)$: is the Sobolev space of parameters k and p of $(0,T)$ in X.

  For a function u, we denote by
- $\dot{u}$: the first derivative of u with respect to time.
- ∂u: the (classical) sub-differential of u.
- ∇u: the gradient of u.
- $\operatorname{Div} u$: the divergence of u.
- $\operatorname{dom}\psi = \{v \in X \mid \psi(v) < +\infty\}$: domain of ψ.
- $epi\psi = \{(v,\alpha) \in X \times \mathbb{R} \mid \psi(v) < \alpha\}$: epigraph of ψ.

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Introduction

Contact mechanics is a very extensive field which embraces several phenomena of contact between deformable solids or between a body and a rigid base. These contact phenomena appear abundantly in our daily life as in the mechanics of structures, and more particularly in the automotive sector, aeronautics (cracks in composites and fiber/matrix interfaces), energy production (assembly of structures, cracking in welded joints) and transmission systems. As a result, a considerable effort has been done in its modelling and numerical simulations. See for instance [28, 50, 55] and the references therein.

Piezoelectricity is the property possessed by certain bodies of becoming electrically polarized under the action of a mechanical constraint called the direct effect of piezoelectricity. The inverse piezoelectric effect is the ability of a body to deform when subjected to an electric field. More generally, the direct effect makes it possible to produce sensors (pressure sensor, etc.). While the reverse effect is used in the manufacture of actuators such as piezoelectrically controlled injectors in automobiles, piezoelectric dampers which play a fundamental role in attenuating unwanted vibrations and ultrasonic transducers which apply in the field of medical imaging in ultrasound for example. Many crystalline materials exhibit piezoelectric behavior. A few materials exhibit the phenomenon strongly enough to be used in applications that take advantage of their properties. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate and polyvinylidene fluoride (a polymer film). They form one of the categories of so-called intelligent materials. They are adaptive, scalable and combine both material and functionality. The mathematical study of piezoelectric phenomena is due to Voigt 1894 whose work is gathered in a work which serves as a reference [58] . Other works followed [32, 41, 43, 45].

Processes of adhesion are important in many industrial settings, adhesion is either direct (it takes place only for very smooth and extremely clean materials (mica or silicon for example), or mediated by an intermediate material. To model the process when bonding is not permanent, and debonding may take place, there is a need to add the adhesion process to the description of contact. Based on the ideas of M. Frémond [22, 23]; the idea is to introduce an internal variable of surface called adhesion field, which takes its values between zero and one and which describes the fractional density of active links on the contact surface. These models can be found in a large number of recent publications, see for example [17, 33, 35] where problems of contact with adhesion and without friction have been studied and [12] where numerical treatments applied to the interface have been developed. An application of the adhesive contact theory can be found in [48] where the importance of adhesion between implanted bone and tissue has been described.

Damage is a very important phenomenon in engineering because it directly affects the structure of machines. Because of the importance of this topic, an increasing number of engineering publications dealing with damage models have appeared. Models taking into consideration the inuence of the internal damage of the material on the contact process have been studied mathematically. Mathematical analysis of one-dimensional problems can be found in [26] . The first models of mechanical damage from thermodynamic considerations appeared in [21]. Recent general models in [24, 25, 27, 40] derive from the principle of virtual power. In all these works the damage of the material is described by a damage function a restricted to have values between zero and one. When $\alpha = 1$, there is no damage in the material, when $\alpha = 0$, the material is completely damaged, and when $0 < \alpha < 1$, there is partial damage and the system has reduced capacity.

The general theory of thermo-piezoelectricity was introduced by taking into account the finite speed of thermal turbulence propagation by Chandrasekhariah [18, 19]. The study of the effect of heat conduction on shift in the frequencies of a freely vibrating linear thermo-piezoelectric body with the help of perturbation methods was made by Yang and Batra [59] . Some problems in thermo-mechanics of contact with damage have been studied in [3, 8, 4, 40, 51].

This thesis represents a contribution to the analysis of some contact problems between a deformable body and a foundation, under the assumption of small deformations, we study quasi-static processes for electro-viscoelastic, electro-elastic-viscoplastic materials taking into consideration the influence of the internal damage of the material and a thermal effect. The boundary conditions are normal compliance or instantaneous normal response or Signorini with adhesion. The friction laws used are versions of Coulomb's law with or without adhesion. Electrical conditions are introduced in cases where the foundation is insulating or conductive. Our study of contact phenomena includes the following steps: mathematical modeling, variational analysis including results of existence and uniqueness of the solution.

This thesis consists of five chapters. which we briefly describe. In the first chapter, we begin by review some results concerning functional spaces, some elements of nonlinear analysis, variational equations and inequalities, Gronwall's lemma and some theorems which will be of great use for proofs. Then, we define the physical framework, the laws of behavior of the different materials, the boundary conditions. In the second chapter we consider a quasistatic problem with instantaneous normal response. We derive a variational formulation of the mechanical problem, for which we show that there is a unique weak solution using fixed point technique. Third chapter is devoted to the analysis of a thermo-electro-elastic-viscoplastic contact problem with damage, for which we derive a variational formulation and establish a result of existence and uniqueness of a weak solution. While in the fourth chapter, one is interested in the study of a problem of contact with a version of the law of friction of Coulomb. The contact is modelled by a normal compliance condition. We present a variational formulation of the problem and we prove the existence and uniqueness of the solution. Finally in the fifth chapter we study of electro elastic-viscoplastic contact problem with damage and adhesion. Proofs are based on a general result of evolution equations with maximal monotone operators, parabolic inequalities, differential equations and fixed point.

Chapter 1

Preliminaries and modelling

This chapter represents a brief reminder of the some results concerning functional spaces, elements of non linear analysis, standard results inequalities and equalities and Gronwall's lemma. Then, we recall of the mechanics of continuums where we will introduce the physical frameworks used in this thesis, the laws of behavior of the various materials, the boundary conditions of contact. In addition, we specify in this chapter contact processes with adhesion, electrical conditions at the contact surface and thermal conditions.

1.1 Preliminaries

1.1.1 Function spaces

We present in this paragraph preliminary material from functional analysis which will be used in subsequent chapters. Some of the results are stated without proofs, since they are standard and can be found in many references. We give some reminders about spaces of real-valued functions which help us to understand the properties of spaces appropriate to mechanics. We will define the spaces of continuous functions, continuously differentiable, p-integrable functions and Sobolev spaces. For more details on this subject we refer the reader, for example, to references [1, 11, 49].

Given an open set Ω of $\mathbb{R}^d$ and we note $\partial\Omega = \Gamma$. Let $x = (x_1, ..., x_d)$ be an element of $\mathbb{R}^d$ and let $\varpi = (\varpi_1, ..., \varpi_d) \in \mathbb{N}^d$ a multi-index such that $|\varpi| = \sum_{i=1}^{d} \varpi_i$, we pose

$$D^{\varpi} u(x) = \frac{\partial^{|\varpi|} u(x)}{\partial x_1^{\varpi_1} \dots \partial x_d^{\varpi_d}}.$$

Spaces of continuous and continuously differentiable functions

The space of uniformly continuous functions on Ω is denoted by $C(\bar{\Omega})$ and it is a Banach space for the norm given by

$$\|u\|_{C(\bar{\Omega})} = \sup\{|u(x)| : x \in \Omega\} = \max\{|u(x)| : x \in \bar{\Omega}\}.$$

For $m \geq 0$, the space $C^m(\bar{\Omega})$ defined by

$$C^m(\bar{\Omega}) = \left\{ u \in C(\bar{\Omega}) \mid D^{\varpi} u \in C(\bar{\Omega}) \text{ for all } \varpi \text{ such that } |\varpi| \leq m \right\}.$$

It is the space of continuous functions on $\bar{\Omega}$ whose derivatives of order at most m are also continuous on $\bar{\Omega}$. It is also a Banach space, it is endowed with the norm

$$\|u\|_{C^m(\bar{\Omega})} = \sum_{|\varpi|\leq m} \left|D^{\varpi} u\right|_{C(\bar{\Omega})}.$$

The space $C^\infty(\Omega)$ denotes the space of indefinitely differentiable functions

$$C^\infty(\bar{\Omega}) = \bigcap_{m=0}^{\infty} C^m(\bar{\Omega}).$$

Let u be a function in Ω, the support of u is defined by

$$\text{supp } u = \overline{\{u \in \Omega \mid u(x) \neq 0\}}.$$

If supp u is a proper subset of Ω, we say that u is a function with compact support in Ω. The space of indefinitely differentiable functions with compact support is given by

$$C_0^\infty(\Omega) = \left\{u \in C^\infty(\bar{\Omega}) \mid \operatorname{supp} u \subset \Omega\right\}.$$

Lebesgue spaces L^p

For $p \in [1, \infty[$, $L^p(\Omega)$ denotes the space of measurable functions in the Lebesgue sense defined on Ω with values in $\mathbb{R}$

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ Lebesgue measurable and } |u|^p \text{ Lebesgue integrable on } \Omega\}.$$

It is a Banach space, the norm of which is given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}.$$

If $p = \infty$ and $u : \Omega \to \mathbb{R}$ is measurable, then

$$\|u\|_{L^\infty(\Omega)} = \text{sup ess}(u) = \inf\{c : |u(x)| \leq c\}.$$

The $L^\infty(\Omega)$ space is also a Banach space.

The space $L^2(\Omega)$ endows with the dot product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx \quad \forall u, v \in L^2(\Omega),$$

is a Hilbert space. In addition, the Cauchy-Schwarz inequality corresponding to the Hölder inequality is verified, i.e.

$$\int_\Omega |u(x)v(x)|dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

Sobolev spaces

Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. We define the Sobolev spaces by

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \text{ such that } D^{\varpi} u \in L^p(\Omega) \text{ with } |\varpi| \leq k \right\}.$$

The norm over the space $W^{k,p}(\Omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\varpi| \leq k} \|D^{\varpi} u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty, \\ \max_{|\varpi| \leq k} \|D^{\varpi} u\|_{L^{\infty}(\Omega)} & \text{if } p = \infty. \end{cases}$$

For $p = 2$, we denote by $H^k(\Omega)$ the space $W^{k,2}(\Omega)$ and the previous norm comes from a scalar product

$$(u, v)_k = \int_{\Omega} \sum_{|\varpi| \leq k} D^{\varpi} u(x) D^{\varpi} v(x) dx, \quad u, v \in H^k(\Omega).$$

For further details on Sobolev spaces we refer the reader to eg [1, 11].

Theorem 1.1. *The Sobolev spaces $W^{k,p}(\Omega)$, for $k \in \mathbb{N}$ and $p \in [1, +\infty]$, endowed with the norm $\|.\|$, are Banach spaces. Moreover, the spaces $H^k(\Omega)$, for any integer k, are Hilbert spaces.*

Functional spaces in contact mechanics

To model a mechanical problem, we will need to introduce of specific function spaces. We give in this paragraph the spaces as well as some of their properties.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d (d = 2, 3)$, "." and $\|.\|$ represent respectively the scalar product and the euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$

$$\begin{aligned} & u.v = u_i v_i, \quad \|v\| = (v.v)^{\frac{1}{2}}, \quad \forall u, v \in \mathbb{R}^d, \\ & \sigma.\tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\| = (\tau.\tau)^{\frac{1}{2}}, \quad \forall \sigma, \tau \in \mathbb{S}^d. \end{aligned}$$

We introduce the following spaces

$$\begin{aligned} & H = \left\{ u = (u_i) \mid u_i \in L^2(\Omega) \right\}, \\ & \mathscr{H} = \left\{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\}, \\ & H_1 = \left\{ u = (u_i) \mid u_i \in H^1(\Omega) \right\}, \\ & \mathscr{H}_1 = \left\{ \sigma \in \mathscr{H} \mid \sigma_{ij,j} \in H \right\}. \end{aligned}$$

Here $\varepsilon : H_1 \to \mathscr{H}$ and $\mathrm{Div} : \mathscr{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(u) = \left(\varepsilon_{ij}(u) \right), \quad \varepsilon_{ij}(u) = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right), \quad \mathrm{Div}(\sigma) = \sigma_{ij,j}.$$

The sets H, H_1, $\mathscr{H}$ and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products

$$(u,v)_H = \int_\Omega u_i v_i dx \quad \forall u,v \in H, \quad (\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \sigma,\tau \in \mathscr{H},$$

$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathscr{H}}, \quad \forall u,v \in H_1,$$
$$(\sigma,\tau)_{\mathscr{H}_1} = (\sigma,\tau)_{\mathscr{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_H, \quad \sigma,\tau \in \mathscr{H}_1,$$

the associated norms are denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathscr{H}}$ and $\|.\|_{\mathscr{H}_1}$.

For any vector field $u \in H_1$ we use the notation u to denote the trace γu of u on Γ. We recall that the trace application $\gamma : H_1 \to L^2(\Gamma)^d$ is linear and continuous, but is not surjective. The image of H_1 by this application is denoted by $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ this subspace is continuously injected into $L^2(\Gamma)^d$.

Now, we define the closed subspaces of $H^1(\Omega)^d$

$$\mathscr{V} = \left\{ \zeta \in H^1(\Omega) \mid \zeta = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\}, \tag{1.1}$$

Let $\mathscr{V}_1$ denote the closed subspace of $L^2(\Omega)$ given by

$$\mathscr{V}_1 = \left\{ \zeta \in L^2(\Omega) \mid \varepsilon_{ij}(\zeta) \in L^2(\Omega) \right\} = H^1(\Omega), \tag{1.2}$$

and we denote by $\mathscr{V}'$ the dual space of $\mathscr{V}$, and $\mathscr{V}_1'$the dual space of $\mathscr{V}_1$. Let V denote the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{ v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_1 \right\}. \tag{1.3}$$

Also, Green's following formula is satisfied

$$\int_\Omega \sigma.\varepsilon(u) dx + \int_\Omega \operatorname{Div}\sigma.v = \int_\Gamma \sigma\nu.v da \quad \forall v \in H^1(\Omega)^d. \tag{1.4}$$

Korn's inequality

Let $meas\Gamma_1 > 0$. Then there exists a constant $C_k > 0$ dependent only from Ω and Γ_1 such that

$$\|\varepsilon(v)\|_{\mathscr{H}} \geq C_k \|v\|_{H_1}, \quad \forall v \in V. \tag{1.5}$$

A proof of this inequality can be found in ([44] p.79).

We consider in the space V, the inner product and the associated norm given by

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathscr{H}}, \; \|v\|_V = \|\varepsilon(v)\|_{\mathscr{H}}, \; u,v \in V. \tag{1.6}$$

By the Korn inequality, it comes that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V and therefore $(V,(.,.)_V)$ is a real Hilbert space. Moreover, using Sobolev's trace theorem, (1.5) and (1.6), there exists a constant C_0 depending only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq C_0 \|v\|_V, \quad \forall v \in V. \tag{1.7}$$

For a scalar function, which represents the adhesion field on the contact surface Γ_3, we define the set

$$\mathcal{Q} = \{\beta : \Gamma_3 \times [0,T] \to \mathbb{R} \quad 0 \leq \beta(t) \leq 1 \quad \text{on} \quad \Gamma_3\}. \tag{1.8}$$

Below, we define the Sobolev spaces associated with the electric unknowns (electric displacement field D and the electric potential.) Of the electro-mechanical problems which will be presented in next chapters of this thesis For the electric displacement field we use the Hilbert space

$$\mathcal{W} = \left\{D \in H \mid \operatorname{div} D \in L^2(\Omega)\right\}, \tag{1.9}$$

endowed with the inner products

$$(D,E)_{\mathcal{W}} = (D,E)_H + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)},$$

and the associated norm $\|.\|_{\mathcal{W}}$. The electric potential field is to be found in

$$W = \left\{\xi \in H^1(\Omega), \xi = 0 \text{ on } \Gamma_a\right\}. \tag{1.10}$$

Since $meas(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds:

$$\|\nabla\zeta\|_H \geq c_F \|\zeta\|_{H^1(\Omega)}, \quad \forall \zeta \in W, \tag{1.11}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a.

Here and below we write ξ for the trace of an element $\xi \in H^1(\Omega)$, on $\Gamma, E(\varphi) = -\nabla\varphi$, while ∇ is the gradient operator

$$\nabla\xi = (\xi_{,i}), \quad \forall \xi \in W.$$

On W we use the inner product

$$(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H, \tag{1.12}$$

and $\|.\|_W$ the associated norm. It follows from (1.11)-(1.12) that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W and therefore $(W, \|.\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exist a positive constant $\tilde{C}_0$ dependent only of Ω, Γ_a a and Γ_3, such that

$$\|\phi\|_{L^2(\Gamma_3)} \leq \tilde{C}_0 \|\phi\|_W, \quad \forall \phi \in W. \tag{1.13}$$

Moreover, when $D \in \mathcal{W}$ is a regular function, the following Green's type formula holds

$$(D, \nabla\zeta)_H + (\operatorname{div} D, \zeta)_{L^2(\Omega)} = \int_\Gamma D \cdot \nu \zeta da, \quad \forall \zeta \in H^1(\Omega). \tag{1.14}$$

Spaces of vector-valued functions

Spaces of vector-valued functions will be needed in the study of time-dependent variational problems. Let $(X, \|.\|_X)$ be a Banach space. In the contact problems studied in this thesis $[0,T]$ denotes the time interval where $T > 0$.

Spaces $C^m(0,T;X)$

we denote by $C([0,T];X)$ the spaces of continuous functions on $[0,T]$ with values in X, with the norm

$$\|u\|_{C(0,T;X)} = \max_{t\in[0,T]} \|u(t)\|_X.$$

The space $C(0,T;X)$ is a Banach space. For $m \geq 0$, we define the space

$$C^m(0,T;X) = \left\{u \in C(0,T;X) : u^{(j)} \in C(0,T;X), j = 1,2,\ldots m\right\}.$$

It is a Banach space with the norm

$$\|u\|_{C^m(0,T;X)} = \sum_{j=0}^{m} \max_{t\in[o,T]} \left\|u^{(j)}\right\|_X.$$

In particular, $C^1([0,T];X)$ denotes the space of continuously differentiable functions on $[0,T]$ with values in X. It is a Banach space with the norm

$$\|u\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|u(t)\|_X + \max_{t\in[0,T]} \|\dot{u}(t)\|_X.$$

Spaces $L^p(0,T;X)$

For $p \in [1,\infty)$, we denote by $L^p(0,T;X)$ the set of classes of measurable functions such that for all $t \in [0,T]$, the function defined by $\|u(t)\|_X$ is in $L^p([0,T])$. For $p = \infty$, $L^\infty(0;T;X)$ is the set of measurable functions u defined on $[0,T]$, such that there exists $C > 0$, $\|u(t)\|_X \leq C$, almost everywhere in $[0,T]$

Remark 1.2. (1) The space $L^p(0,T;X)$ are endowed with the norms

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T |u(t)|_X^p dt\right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty, \\ \inf\{c > 0 \mid \|u(t)\|_X < c \text{ a.e.t } \in (0,T)\} & \text{if } p = +\infty. \end{cases}$$

(2) For all $1 \leq p < \infty$, Banach spaces $C([0,T];X)$ is dense in $L^p(0,T;X)$. (3) The space $L^2(0,T;X)$ is a Hilbert space, if X is a Hilbert space.

Spaces $W^{k,p}(0,T;X)$

For $k \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$. We define the space

$$W^{k,p}(0,T;X) = \left\{u \in L^p(0,T;X) : \|u^{(m)}\|_{L^p(0,T;X)} < \infty, \forall m \leq k\right\}.$$

The norm is defined by

$$\|u\|_{W^{k,p}(0,T;X)} = \begin{cases} \left(\int_0^T \sum_{0\leq m\leq k} \|u^{(m)}(t)\|_X^p dt\right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty, \\ \max_{0\leq m\leq k} \ \operatorname{ess\,sup}_{t\in[0,T]} \|u^{(m)}(t)\|_X & \text{if } p = +\infty. \end{cases}$$

If X is a Hilbert space and $p = 2$, then the space

$$H^k(0,T;X) \equiv W^{k,2}(0,T;X),$$

is a Hilbert space with the inner product

$$(u,v)_{H^k(0,T;X)} = \sum_{0\le m\le k} \int_0^T \left(u^{(m)}(t), v^{(m)}(t)\right)_X dt.$$

1.1.2 Some elements of nonlinear analysis

In this paragraph we will recall some notions of nonlinear analysis. some results on monotonic operators, lower convex and semi-continuous functions, differentiability and sub differentiability. We will also mention some of the lemmas made in thesis [13]. For more details on the reminders appearing in this section, we refer for example to references [10, 53, 52].

Linear operators

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two normed spaces and let $L : X \to Y$ be an operator. an operator $L : X \to Y$ is linear if

$$L(\alpha u_1 + \beta u_2) = \alpha L(u_1) + \beta L(u_2) \quad \forall u_1, u_2 \in X, \quad \alpha, \beta \in \mathbb{R}.$$

A linear operator is continuous if and only if it is bounded, i.e. there exists a constant $M > 0$ such that

$$\|L(u)\|_Y \le M\|u\|_X \quad \forall u \in X.$$

Nonlinear operators

In the study of the mechanical problems approached in the thesis, the non-linearity of the operators used implies the use of properties that we described below.

We begin here with a brief reminder of the strongly monotonic and Lipschitz operators. For this, we consider a Hilbert space X endowed with the scalar product $(.,.)_X$ and the associated norm $\|.\|_X$. Let $A : X \to X$ a non-linear operator

Definition 1.3. Operator A is said:

(1) monotonous if

$$(Av - Aw, v - w)_X \ge 0 \quad \forall v, u \in X,$$

(2) strongly monotonic if there exists m> 0 such that

$$(Av - Aw, v - w)_X \ge m\|v - w\|_X^2 \quad \forall v, w \in X,$$

(3) of Lipschitz if there exists M> 0 such that

$$\|Av - Aw\|_X \le M\|v - w\|_X \quad \forall v, w \in X.$$

In the study of evolutionary nonlinear equations we will consider operators defined on a normed space with values in its dual. X' the dual space of X, denoting by $(.,.)_{X'\times X}$ for the duality product between X' and X, an extension of Definition 1.3 to this case is as follows.

Definition 1.4. Let $A : X \to X'$ an operator defined on X. Operator A is said
(1) monotonous if
$$(Au - Av, u - v)_{X' \times X} \geq 0 \quad \forall u, v \in X.$$
(2) hemicontinuous if
$\forall u, v \in X$ the application $t \to A(u + tv) : \mathbb{R} \to X'$ is continuous.

Banach fixed-point theorem

The Banach fixed-point theorem will be used several times in this work to prove the existence of solutions to variational problems in contact mechanics.

Theorem 1.5. *Let X be a Banach space, and K be a nonempty and closed set in a Banach space X, $A : K \to K$ an operator satisfies Definition 1.3 (3) with $0 < M < 1$. Then operator A admits a unique fixed point $u \in K$, i.e. $Au = u$ and we call A a contracting operator.*

Proof of Theorem 1.5 can be found in ([28], Ch. 1), for instance.

We also need a version of the Banach fixed point theorem which we recall in what follows. To this end, for an operator A, we define its powers inductively by the formula $A^m = A(A^{m-1})$ for $m \geq 2$.

Theorem 1.6. *Assume that K is a nonempty closed subset of a Banach space X and $A : K \longrightarrow K$. Assume also that $A^m : K \longrightarrow K$ is a contraction for some positive integer m. Then A has a unique fixed point.*

Proof. By Theorem 1.5, the mapping A^m has a unique fixed point $u \in K$. From
$$A^m(u) = u,$$
we obtain
$$A^m(A(u)) = A(A^m(u)) = Au.$$
Thus $Au \in K$ is also a fixed point of A^m. Since A^m has a unique fixed point, we must have
$$Au = u,$$
i.e., u is a fixed point of A. The uniqueness of a fixed point of A follows easily from that of A^m.

□

Lax-Milgram Theorem

Definition 1.7. A bilinear form $a : X \times X \to \mathbb{R}$ is continuous if there exists $M > 0$ such that
$$|a(u, v)| \leq M \|u\|_X \|v\|_X \quad \forall u, v \in X.$$

Definition 1.8. A bilinear form $a : X \times X \to \mathbb{R}$ is said to be coercive if there is a constant $m > 0$ such that
$$a(u, u) \geq m \|u\|_X^2 \quad \forall u \in X.$$

Theorem 1.9. *Let X be a Hilbert space, $a : X \times X \to \mathbb{R}$ a continuous coercive bilinear form. Let $l : X \to \mathbb{R}$ be a continuous linear form. Then, there exists a unique solution $u \in X$ which satisfies*
$$a(u, v) = l(v) \quad \forall v \in X.$$

Convex lower semicontinuous functions

Convex lower semicontinuous functions represent a decisive component in the study of variational inequalities. To introduce them, we consider a function ψ defined on a real vector space X and with value in $]-\infty,+\infty]$. Such a function is said to be proper if it is not identically equal to $+\infty$, i.e. if there exists $v_0 \in X$ such that $\psi(v_0) < +\infty$. The function ψ is said to be convex if

$$\psi(tv+(1-t)w) \leq t\psi(v)+(1-t)\psi(w) \quad \forall v,w \in X, t \in [0,1].$$

The function ψ is said to be strictly convex if this last inequality is strict for all $v,w \in X$ such that $v \neq w$. For any function $\psi : X \to]-\infty,+\infty]$, we define the domain and the epigraph of ψ respectively by:

$$\operatorname{dom}\psi = \{v \in X \mid \psi(v) < +\infty\}, \tag{1.15}$$

$$epi\psi = \{(v,\alpha) \in X \times \mathbb{R} \mid \psi(v) < \alpha\}. \tag{1.16}$$

It is clear that the following results can be established

1) ψ is proper if and only if $\operatorname{dom}\psi \neq \emptyset$.

2) The domain of ψ is a convex set of X if ψ is convex.

3) ψ is convex if and only if $epi\psi$ is a convex set of $X \times \mathbb{R}$.

We now assume the vector space X endowed with a topology (in general X will be a normalized vector space or even more particularly a Hilbert space). A function $\psi : X \to]-\infty,+\infty]$ is said to be lower semi-continuous (l.s.c) if

$$\liminf_n \psi(v_n) \geq \psi(v),$$

for all $v \in X$ and $v_n \to v$ in X. The lower semi-continuity property can be characterized by

Lemma 1.10. *Let $\psi : X \to]-\infty,+\infty]$, then the following properties are equivalent*

1) ψ is lower semi-continuous.

2) The epigraph of ψ is closed in $X \times \mathbb{R}$.

Since in a normalized vector space every convex set is simultaneously closed for the strong topology and the weak topology, the previous lemma leads to the following result.

Theorem 1.11. *Let X be a Hilbert space and $\psi : X \to]-\infty,+\infty]$ a convex and proper function. Then ψ is lower semi-continuous if and only if ψ is lower semi-continuous with respect to the weak topology of X.*

Let now be $K \subset X$. We call an indicator function of K, the function $\varphi_K : X \to]-\infty,+\infty]$ defined by

$$\varphi_K(v) = \begin{cases} 0 \text{ si } v \in K, \\ +\infty \text{ si } v \notin K. \end{cases} \tag{1.17}$$

Using this definition, we can easily prove the following result

Lemma 1.12. *K is a convex, closed and non-empty set of X if and only if the indicator function φ_K is convex, l.s.c and proper.*

Sub-differentiability

Let X be a real Hilbert space endowed with the inner product $(.,.)_X$ and the associated norm $\|.\|_X$. We now denote by 2^X the set of all the parts of X. A function $\psi : X \to]-\infty,+\infty]$ is said to be Gateaux-differentiable at the point $v \in X$ if there is an element $\nabla\psi(v) \in X$ such that:

$$\lim_{t\to 0} \frac{\psi(v+tw)-\psi(v)}{t} = (\nabla\psi(v),w)_X \quad \forall w \in X.$$

The element $\nabla\psi(v)$ is called the differential in the sense of Gateaux of ψ in v. The function ψ is said to be Gateaux-differentiable if it is Gateaux-differentiable at any point of X, in this case the operator $\nabla\psi : X \to X$ which associates each element $v \in X$ by the element $\nabla\psi(v)$ is called the gradient operator of ψ.

The convexity of the Gateaux-differentiable functions can be characterized as follows:

Lemma 1.13. *Let $\psi : X \to \mathbb{R}$ be a Gateaux-differentiable function. So ψ is a convex function if and only if*

$$\psi(w)-\psi(v) \geq (\nabla\psi(v),w-v)_X \quad \forall v,w \in X. \tag{1.18}$$

The inequality (1.18) suggests a generalization of the notion of gradient to convex functions. We say that the function $\psi : X \to]-\infty,+\infty]$ is subdifferentiable at a point $v \in X$ if there exists $f \in X$ such that

$$\psi(w)-\psi(v) \geq (f,w-v)_X \quad \forall w \in X. \tag{1.19}$$

The element f is then called a sub-gradient of ψ in v and the set of sub-gradients of ψ in v is called the sub-differential of ψ in v and is denoted $\partial\psi(v)$

$$\partial\psi(v) = \{f \in X \mid \psi(w)-\psi(v) \geq (f,w-v)_X \; \forall w \in X\}. \tag{1.20}$$

We denote by $\mathrm{dom}\,\partial\psi$ the set defined by

$$\mathrm{dom}(\partial\psi) = \{v \in X \mid \partial\psi(v) \neq \phi\}. \tag{1.21}$$

Using (1.21) and (1.20) and (1.15), it follows

$$\mathrm{dom}(\partial\psi) \subset \mathrm{dom}\,\psi. \tag{1.22}$$

The multivalued operator $v \to \partial\psi(v) : X \to 2^X$ is called the sub-differential of ψ. The function ψ is said to be subdifferential if it is subdifferential at any point of X, i.e. if $\mathrm{dom}(\partial\psi) = X$. Using the inequality (1.19), we have

Lemma 1.14. *Let $\psi : X \to]-\infty,+\infty]$ be a sub-differentiable function, Then ψ is convex, proper and lower semi-continuous.*

Lemma 1.15. *Let $\psi : X \to]-\infty,+\infty]$ be a convex and Gateaux-differentiable function. Then ψ is sub-differentiable and $\partial\psi(v) = \{\nabla\psi(v)\}$ for all $v \in X$.*

Remark 1.16. Let us explain the subdifferential of the indicator function of a set $K \subset \mathbb{R}^N$; using (1.22), (1.15) and (1.17), it results $\mathrm{dom}(\partial\varphi_K) \subset K$ and from (1.20) it comes

$$f \in \partial\varphi_K \Longleftrightarrow v \in K \text{ et } f.(w-v) \leq 0 \quad \forall w \in K. \tag{1.23}$$

Using this relation, it follows that if v is inside K, then $\partial\varphi_K(v) = \{0_{\mathbb{R}^N}\}$.

Differential inclusions

Let X be a Hilbert space and $A : D(A) \subset X \to 2^X$ a multivalued operator, where $D(A)$ is the domain of A defined by

$$D(A) = \{u \in X : Au \neq \varnothing\}. \tag{1.24}$$

The graph of A denoted by $\mathrm{Gr}(A)$ is defined by

$$\mathrm{Gr}(A) = \{(u,w) \in X \times X : w \in Au\}. \tag{1.25}$$

Definition 1.17. Operator $A : X \to X^2$ is said

(i) monotonous if

$$\forall (u_1,w_1) \in \mathrm{Gr}(A), \forall (u_2,w_2) \in Gr(A) : (w_1 - w_2, u_1 - u_2)_X \geq 0.$$

(ii) maximal monotonic if A is monotonic and there is no monotonic operator $B : X \to 2^X$ such that $\mathrm{Gr}(A)$ is a proper set of $\mathrm{Gr}(B)$, which is equivalent to the implication

$$[(w_1 - w_2, u_1 - u_2)_X \geq 0, \forall (u_1,w_1) \in Gr(A)] \Rightarrow (u_2,w_2) \in Gr(A).$$

Proposition 1.18. *The sub-differential $\partial \varphi_K : X \to 2^X$ of the indicator function of a closed convex set K of space X is a monotonic maximal operator.*

Proposition 1.19. *The sum of a monotonic maximal operator and a monotonic and Lipschitz operator is a monotonic maximal operator.*

Finally, we recall an abstract result relating to the existence and uniqueness of a solution of a certain differential inclusion class which will be used in the Fifth chapter of this thesis.

Theorem 1.20. *Let $A : D(A) \subset X \to 2^X$ be a multivalued operator such that $A + \omega I_X$ is a maximal monotonic operator for some real numbers ω. Then, for all $f \in W^{1,1}(0,T;X)$ and $u_0 \in D(A)$, there exists a unique function $u \in W^{1,\infty}(0,T;X)$, satisfying*

$$\dot{u}(t) + Au(t) \ni f(t) \ a.e.\ t \in (0,T), \tag{1.26}$$

$$u(0) = u_0. \tag{1.27}$$

For the proof of this theorem see [6], page 32.

1.1.3 Variational inequalities and evolution equations

The modelling of several classes of physical problems leads to mathematical problems such as elliptic and parabolic variational inequalities, as well as for ordinary differential equations. In this paragraph we give a result of existence and uniqueness for this type of problem.

Now we consider V and H two real Hilbert spaces such that the inclusion map of $(V, \|.\|_V)$ in $(H, \|.\|_H)$ is continuous and dense. Identifying the dual of H with itself, i.e. we can write the Gelfand triplet $V \subset H \subset V'$. The notations $\|.\|_V$, $\|.\|_{V'}$ and $(.,.)_{V' \times V}$ represent the norms on V, V' and the product of duality between V' and V, respectively.

Evolutionary variational inequalities

We recall the following standard result for parabolic variational inequalities (see [6], p. 124).

Theorem 1.21. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a(.,.) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\lambda > 0$ and γ,*

$$a(v,v) = \gamma \|v\|_H^2 \geq \lambda \|v\|_V^2, \quad \forall v \in H.$$

Then, for every $u_0 \in K$ and $S \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap L^2(0,T;V)$, such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$(\dot{u}(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \geq (S(t), v - u(t))_H, \quad \forall v \in K.$$

In order to obtain existence and uniqueness results in study other types of variational inequality which arise in Contact Mechanics. Given the operators $A : V \to V$ and $B : V \to V$, the functions $j : V \times V \to \mathbb{R}$ and $f : [0,T] \to V$. We consider the following problems

Problem I Find $u : [0,T] \to V$ such that

$$\begin{aligned}(A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + j(u(t), v)& \\ - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V, t \in [0,T],&\end{aligned} \tag{1.28}$$

$$u(0) = u_0. \tag{1.29}$$

Problem II Find a function $u : [0,T] \to V$ such that

$$\begin{aligned}(A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + j(\dot{u}(t), v)& \\ - j(\dot{u}(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V, t \in [0,T],&\end{aligned} \tag{1.30}$$

$$u(0) = u_0. \tag{1.31}$$

Finally, when $j : V \to \mathbb{R}$, we consider the problem

Problem III Find a function $u : [0,T] \to V$ such that

$$\begin{aligned}(A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + j(v)& \\ - j(\dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V, t \in [0,T],&\end{aligned} \tag{1.32}$$

$$u(0) = u_0. \tag{1.33}$$

We remark that one of the common features of the problems **I**, **II** and **III** above is that they involve the time derivative of the unknown function u. As a consequence, an initial condition is needed. For this reason we refer to such problems as evolutionary problems. To solve these inequalities, we assume that,

$$\begin{cases} (a) \text{ There exists } m_A > 0 \text{ such that} \\ (Au_1 - Au_2, u_1 - u_2)_V \geq m_A \|u_1 - u_2\|_V^2, \quad \forall u_1, u_2 \in V. \\ (b) \text{ There exists } L > 0 \text{ such that} \\ \|Au_1 - Au_2\|_V \leq L \|u_1 - u_2\|_V, \forall u_1, u_2 \in V. \end{cases} \tag{1.34}$$

The functional $j : V \times V \to \mathbb{R}$ satisfied

$$\begin{cases} \text{(a) For all } u \in V, j(u,.) \text{ is convex and l.s.c.} \\ \text{(b) There exists } \alpha > 0 \text{ such that} \\ j(u_1,v_2) - j(u_1,v_1) + j(u_2,v_1) - j(u_2,v_2) \\ \leq \alpha \|u_1 - u_2\|_V \|v_1 - v_2\|_V, \quad \forall u_1,u_2,v_1,v_2 \in V. \end{cases} \tag{1.35}$$

The operator $B : V \to V$ satisfies

$$\exists L_B > 0 \text{ such that } \|Bu - Bv\|_V \leq L_B \|u - v\|_V, \forall u,v \in V. \tag{1.36}$$

Finally, we assume that

$$f \in C([0,T],V). \tag{1.37}$$
$$u_0 \in V. \tag{1.38}$$

Theorem 1.22. *Let V be a Hilbert space and assume that (1.34)-(1.38). Then*

*1) There is a unique solution $u \in C^1([0,T];V)$ to problem **I**.*

*2) If u_1 and u_2 are two solutions of the problem **I** corresponding to the data $f_1, f_2 \in C([0,T];V)$, then there exists $c > 0$ such that*

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_H \leq c\left(\|f_1(t) - f_2(t)\|_H + \|u_1(t) - u_2(t)\|_H\right), \quad \forall t \in [0,T]. \tag{1.39}$$

Theorem 1.23. *Let V be a Hilbert space, assume that (1.34)-(1.38) hold and $m_A > \alpha$. Then there exists a unique solution $u \in C^1([0,T];V)$ to problem **II**.*

Corollary 1.24. *Let V be a Hilbert space and assume that (1.34), (1.36)-(1.38) hold and $j : V \to \mathbb{R}$ is convex and l.s.c,. Then there exists a unique solution u to problem **III**. Moreover, the solution satisfies $u \in C^1([0,T];V)$.*

The proof of Theorem 1.22 and 1.23 can be found in reference [56].

Ordinary differential equation

Theorem 1.25. *Let $V \subset H \subset V'$ be a Gelfand triple. Assume that $A : V \to V'$ is a hemicontinuous and monotone operator that satisfies*

$$(Av,v)_{V' \times V} \geq \lambda \|v\|_V^2 + \alpha \quad \forall v \in V, \tag{1.40}$$
$$\|Av\|_{V'} \leq C(\|v\|_V + 1) \quad \forall v \in V. \tag{1.41}$$

for some constants $\lambda > 0, C > 0$ and $\alpha \in \mathbb{R}$. Then, given $u_0 \in H$ and $f \in L^2(0,T;V')$, there exists a unique function u that satisfies

$$\begin{aligned} & u \in L^2(0,T;V) \cap C([0,T];H), \dot{u} \in L^2\left(0,T;V'\right), \\ & \dot{u}(t) + Au(t) = f(t) \quad a.e.\ t \in (0,T), \\ & u(0) = u_0. \end{aligned}$$

The previous abstract result can be found in [5, 6].

Now, we will recall a version of the Cauchy-Lipschitz theorem (see for example [57] , p.48).

Theorem 1.26. ***(Cauchy-Lipschitz)*** *Assume that* $(X, \|.\|_X)$ *is a real Banach space and* $T > 0$. *Let* $F(t,.) : X \to X$ *be an operator defined a.e.on* $(0,T)$ *satisfying the following conditions*

$$\begin{cases} \text{There exists } L_F > 0 \text{ such that} \\ \|F(t,x) - F(t,y)\|_X \leq L_F \|x-y\|_X, \qquad \forall x,y \in X, a.e.t \in (0,T), \end{cases}$$

and there exists $1 \leq p \leq +\infty$ *such that* $F(.,x) \in L^p(0,T;X), \forall x \in X$ *Then, for any* $x_0 \in X$, *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that*

$$\begin{cases} \dot{x}(t) = F(t,x(t)) \quad a.e.t \in (0,T), \\ x(0) = x_0. \end{cases}$$

1.1.4 Gronwall-type lemmas

We recall here the Gronwall type lemma which occurs in many contact problems, in particular to establish the uniqueness of the solution

Lemma 1.27. *Let* $m,n \in C(0,T;\mathbb{R})$ *such that* $m(t) \geq 0$ *and* $n(t) \geq 0$ *for all* $t \in [0,T]$, $a \geq 0$ *a constant and* $\psi \in C(0,T;\mathbb{R})$

(1) *if*

$$\psi(t) \leq a + \int_0^t m(s)ds + \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T],$$

so

$$\psi(t) \leq \left(a + \int_0^t m(s)ds\right) \exp\left(\int_0^t n(s)ds\right) \quad \forall t \in [0,T].$$

(2) *if*

$$\psi(t) \leq m(t) + a\int_0^t \phi(s)ds \quad \forall t \in [0,T],$$

so

$$\int_0^t \psi(s)ds \leq e^{\alpha t} \int_0^t m(s)ds \quad \forall t \in [0,T].$$

In the particular case $a = 0$, $n = 1$, part (1) of this lemma becomes

Corollary 1.28. *Let* $m \in C(0,T;\mathbb{R})$ *such that* $m(t) \geq 0$ *for all* $t \in [0,T]$. *If* $\psi \in C(0,T;\mathbb{R})$ *is a function such that*

$$\psi(t) \leq \int_0^t m(s)ds + \int_0^t \psi(s)ds \quad \forall t \in [0,T],$$

then, there exists $c > 0$ *such that*

$$\psi(t) \leq c\int_0^t m(s)ds \quad \forall t \in [0,T].$$

1.2 Modelling

In this part, we present the physical framework for the study of the boundary value problems presented in chapters 2-5. some notions of the mechanics of continuums, the equation of motion and equilibrium, the constitutive laws for the materials considered, as well as the boundary conditions of contact.

1.2.1 Physical setting

Consider a material body which occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2,3)$, with a regular boundary surface Γ, partitioned into three measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\Gamma_1 > 0$. Let ν be the unit outward normal vector on Γ . The body is embedded on Γ_1 in a fixed structure. On Γ_2 surface tractions of density f_2 act and on voluminal forces of density f_0 act. We assume that f_2 and f_0 vary very slowly with respect to time. Let $T > 0$ and let $[0,T]$ the time interval in question. In addition to the action of forces and tractions, the body is subjected to the action of electric charges of volume density q_0 and surface electric charges. To describe them, we consider a second partition of the boundary . into three measurable parts Γ_a, Γ_b and Γ_3 such that $meas(\Gamma_a) > 0$. We assume that the body is in frictional contact with or without adhesion with an insulating (or conductive) foundation on Γ_3, the electric potential vanishes on a and a surface electric charge of density q_2 is prescribed on Γ_b.

At each time instant Γ_3 is divided into two parts: one part where the body and the foundation are in contact, and the other part where they are separated. The boundary of the contact part is a free boundary, determined by the solution of the problem. For the sake of generality, we assume that in the reference configuration there exists a gap, denoted by $g = g(x)$, between Γ_3 and the foundation, which is measured along the outer normal ν.

Note that assumption $meas(\Gamma_1) > 0$ is necessary in quasistatic problems. Without this assumption, mathematically, the problem becomes noncoercive and many of the estimates and results below do not hold. This accurately reflects the physical situation, since when $\Gamma_1 = 0$ the body is not held in place, and it may move freely in space as a rigid body.

We denote by $u : \Omega \times [0,T] \to \mathbb{R}^d$ the field of displacements, $\sigma : \Omega \times \to \mathbb{S}^d$ the stress field,$\varphi : \Omega \times [0,T] \to \mathbb{R}$the electric potential field, $D : \Omega \times [0,T] \to \mathbb{R}^d$ the field of electric displacements, $\beta : \Gamma_3 \times, T] \to \mathbb{R}$ the bonding field, $\alpha : \Omega \times [0;T] \to \mathbb{R}$ the field of damage and a temperature field $\theta : \Omega \times [0,T] \to \mathbb{R}$.

We denote by u_ν and u_τ the normal and tangential component respectively of any vector field $u = u_\nu.\nu + u_\tau$ where $u_\nu = u.\nu$. Similarly, let and be the normal and tangential component respectively of the Cauchy stress tensor. He comes :

$$\sigma_\nu = (\sigma\nu)_\nu, \quad \sigma_\tau = (\sigma\nu)_\tau,$$

so

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu.$$

We note that σ_ν is a scalar, whereas σ_τ is a tangent vector to Γ. So we conclude

$$(\sigma\nu).u = \sigma_\nu u_\nu + \sigma_\tau \cdot u_\tau,$$

which will intervene throughout this thesis to establish the variational formulations of mechanical problems. The dots above a function represent the derivative with respect to time, i.e.

$$\dot{u} = \frac{du}{dt}, \quad \ddot{u} = \frac{d^2u}{dt^2},$$

where $\dot{u}$ denotes the velocity field and $\ddot{u}$ denotes the acceleration field. We denote by $\dot{u}_\nu$, $\dot{u}_\tau$ the normal and tangential components of the velocity vector $\dot{u}$ at the boundary such that

$$\dot{u}_\nu = \dot{u} \cdot \nu, \quad \dot{u}_\tau = \dot{u} - \dot{u}_\nu\nu.$$

The fundamental law of continuum mechanics expressing the equivalence between external forces and the tensor of accelerations for any system leads to the Cauchy equation of motion

$$\operatorname{Div}\sigma + f_0 = \rho\ddot{u} \quad \text{in} \quad \Omega\times(0,T), \tag{1.42}$$

where $\rho : \Omega \to \mathbb{R}_+$ designates the mass density. The evolutionary processes defined by (1.42) are called dynamic processes. In some situations, this equation can be further simplified. For example, in the case where $\dot{u} = 0$, it is a static process. In the case where the velocity field varies slowly with respect to time, i.e. the term $\rho\ddot{u}$ can be neglected, we are in the presence of a quasistatic process. In these two cases the equation (1.42) becomes

$$\operatorname{Div}\sigma + f_0 = 0 \quad \text{in} \quad \Omega\times(0,T). \tag{1.43}$$

In the case of a piezoelectric material, the constitutive law contains a new unknown, the electric field D, hence the need to introduce another equilibrium equation to manage it. This is the Maxwell-Gauss equation or charge conservation equation

$$\operatorname{div} D = q_0 \quad \text{in} \quad \Omega\times(0,T). \tag{1.44}$$

1.2.2 Constitutive laws

The description of our model is not yet complete, the previous equations alone are insufficient to describe the movement of the material body considered. it must be supplemented by other relations which characterize the behavior of each type of material and which one indicates under the general term the laws of behavior.

Elastic constitutive laws

A general elastic constitutive law is given by

$$\sigma = \mathscr{B}\varepsilon(u). \tag{1.45}$$

where $\mathscr{B}$ is the elasticity operator, assumed to be non linear. We allow $\mathscr{B}$ to depend on the location of the point; consequently, all that follows is valid for non homogeneous materials. We use the short-hand notation $\mathscr{B}\varepsilon(u)$ for $\mathscr{B}(x,\varepsilon(u))$. In particular, if $\mathscr{B}$ is a linear operator, (1.45) leads to the constitutive law of linearly elastic materials

$$\sigma_{ij} = b_{ijkl}\varepsilon_{kl}(u).$$

where σ_{ij} are the components of the stress tensor σ and b_{ijkl} are the components of the elasticity tensor $\mathscr{B}$. Clearly, The monotonic and Lipschitz assumptions can be replaced by these assumptions in this particular case, if all the components b_{ijkl} belong to $L^\infty(\Omega)$ and satisfy the usual properties of symmetry and ellipticity, i.e.

$$b_{ijkl} = b_{jikl} = b_{klij},$$

and there exists $m_{\mathscr{B}} > 0$ such that

$$b_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq m_{\mathscr{B}}\|\varepsilon\|^2 \quad \forall\varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d.$$

Viscoelastic constitutive laws

A general viscoelastic constitutive law with short memory is given by

$$\sigma = \mathscr{A}\varepsilon(\dot{u}) + \mathscr{B}\varepsilon(u). \tag{1.46}$$

We allow the viscosity operator $\mathscr{A}$ and the elasticity operator $\mathscr{B}$ to depend on the location of the point; consequently, all that follows is valid for non-homogeneous materials. We use the short-hand notation $\mathscr{A}\varepsilon(\dot{u})$ and $\mathscr{B}\varepsilon(u)$ for $\mathscr{A}(x,\varepsilon(\dot{u}))$ and $\mathscr{B}(x,\varepsilon(u))$, respectively. In particular, if $\mathscr{A}$ and $\mathscr{B}$ are linear operators, then (1.46) leads to the KelvinVoigt constitutive law

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{u}) + b_{ijkl}\varepsilon_{kl}(u),$$

where σ_{ij}, a_{ijkl} and b_{ijkl} denote the components of the stress tensor σ, the viscosity tensor $\mathscr{A}$ and the elasticity tensor $\mathscr{B}$,

A general viscoelastic constitutive law with long memory is given by

$$\sigma(t) = \mathscr{B}\varepsilon(u(t)) + \int_0^t \mathscr{M}(t-s,\varepsilon(u(s)))\mathrm{d}s. \tag{1.47}$$

We allow the elasticity operator $\mathscr{B}$ and the relaxation operator $\mathscr{M}$ to depend on the location of the point. Also, as shown in (1.47), the operator $\mathscr{M}$ is time-dependent.

Viscoplastic constitutive laws

The viscoplastic constitutive law can be written in the form

$$\dot{\sigma} = \mathscr{A}\varepsilon(\dot{u}) + \mathscr{G}(\sigma,\varepsilon), \tag{1.48}$$

where $\mathscr{A}$ represents the viscosity operator and $\mathscr{G}$ is the plasticity operator.

Elasto-viscoplastic constitutive laws

A general Elasto-viscoplastic constitutive law may be written as

$$\sigma(t) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) + \int_0^t \mathscr{G}(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s)),\varepsilon(u(s)))ds, \tag{1.49}$$

where the operators $\mathscr{A}$ and $\mathscr{B}$ are tensors of order four and not linear; their components a_{ijkl} and b_{ijkl} are called viscosity and elasticity coefficients respectively and $\mathscr{G}$ represents a non linear constitutive function which describes the viscoplastic behavior of the material.

This law modifies slightly the description of thermo-mechanical or electro-mechanical phenomena because here we must also take into consideration the temperature field, the electric displacement field $D = (Di)$ as well as the electric field $E(\varphi) = -\nabla\varphi = -(\varphi_{,i})$, where, recall, $\varphi_{,i} = \dfrac{\partial\varphi}{\partial x_i}$.

Viscoelastic constitutive law with long memory and damages

In this case the constitutive law is given by

$$\sigma(t) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) + \int_0^t \mathscr{M}((t-s), \varepsilon(u(t)), \alpha(s))ds, \tag{1.50}$$

$\mathscr{A}$ and $\mathscr{B}$ are functions of nonlinear viscosity and elasticity, respectively, $\mathscr{M}$ represents the tensor of viscoplasticity. The function α represents the damage whose evolution is described by the following differential inclusion

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \Psi(\varepsilon(u), \alpha). \tag{1.51}$$

The set of admissible damage functions K defined by

$$K = \left\{\xi \in H^1(\Omega) : 0 \leq \xi \leq 1 \quad \text{a.e. in } \Omega\right\}, \tag{1.52}$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of set K and Ψ is a given constitutive function which describes the sources of the damage in the system

Thermo-elasto-viscoplastic laws with damage

$$\begin{aligned} \sigma(t) &= \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) \\ &\quad + \int_0^t \mathscr{G}\left(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s)), \varepsilon\left(u(s)\right), \theta(s), \alpha(s)\right) ds, \end{aligned} \tag{1.53}$$

where $\mathscr{A}$ and $\mathscr{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathscr{G}$ is a nonlinear constitutive function which describes the viscoplastic behavior of the material. The evolution of the temperature field θ is governed by the heat equation, obtained from the conservation of energy and defined by the following parabolic equation

$$\dot{\theta} - k_0\Delta\theta = \Theta(\sigma - \mathscr{A}\varepsilon(\dot{\mathbf{u}})), \varepsilon\left(u\right), \theta, \alpha) + \rho, \tag{1.54}$$

where $k_0 > 0$, Θ is a non linear constitutive function that represents the heat generated by internal forces and ρ is the amount of heat in the volume during the time interval. α represents the damage field, the evolution of the damage field is described by

$$\dot{\alpha} - k_1\Delta\alpha + \partial\varphi_K(\alpha) \ni \Psi(\sigma - \mathscr{A}\varepsilon(\dot{u})), \varepsilon\left(u\right), \theta, \alpha), \tag{1.55}$$

where $k_1 > 0$, $\partial\varphi_K$ is the sub-differential of the indicator function on K defined previously.

Electro-elastic-viscoplastic constitutive law with damage

A constitutive law of an electro-elastic-viscoplastic material with damage is given by

$$\begin{aligned}
\sigma(t) &= \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) + \mathscr{E}^*E(\varphi(t)) \\
&\qquad + \int_0^t \mathscr{G}\left(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s) - \mathscr{E}^*E(\varphi(s))), \varepsilon\left(u(s)\right), \alpha(s)\right) ds, \\
D &= \mathscr{E}\varepsilon(u) + BE(\varphi).
\end{aligned} \tag{1.56}$$

where $\mathscr{A}$ and $\mathscr{B}$ are the viscosity and elasticity operators, respectively, and $\mathscr{G}$ is a nonlinear constitutive function describing the viscoplastic behavior of the material, where α is an internal variable describing the damage of the material caused by elastic deformations. $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathscr{E} = (e_{ijk})$ represent the third order piesoelectric tensor, and $\mathscr{E}^*$ denotes its transpose. B denotes the electric permittivity tensor.

Thermo-electroelastic-viscoplastic constitutive law

A constitutive law of a thermo-electro-elastic-viscoplastic material is given by

$$\begin{aligned}
\sigma &= \mathscr{A}\varepsilon\left(\dot{u}\right) + \mathscr{B}\varepsilon\left(u\right) - \mathscr{E}^*E(\varphi) \\
&\qquad + \int_0^t \mathscr{G}\left(\sigma(s) - \mathscr{A}\varepsilon\left(\dot{u}(s)\right) + \mathscr{E}^*E(\varphi)(s), \varepsilon\left(u(s)\right)\right) ds - C_e\theta, \\
D &= \mathscr{E}\varepsilon(u) + BE(\varphi), \\
\dot{\theta} &- \operatorname{div} K_c(\nabla\theta) = r(\dot{u}) + \mathbf{q},
\end{aligned} \tag{1.57}$$

where $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{G}$ are, respectively, nonlinear operators describing the purely viscous, the elastic and the viscoplastic properties of the material, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathscr{E} = (e_{ijk})$ represent the third order piesoelectric tensor, $\mathscr{E}^*$ is its transposition and B denotes the electric permittivity tensor, θ represent the temperature, $C_e = (c_{ij)}$ represents the thermal expansion tensor, K_c represent the thermal conductivity tensor, $\mathbf{q}$ represent the density of volume heat source and r is non linear function of velocity.

Thermo-electroelastic-viscoplastic constitutive law with damage

As described in the law above, but after adding the damage α, the constitutive law of a thermo-electro-elastic-viscoplastic material with damage is given by

$$\begin{aligned}
&\sigma = \mathscr{A}\varepsilon\left(\dot{u}\right) + \mathscr{B}\left(\varepsilon\left(u\right), \alpha\right) - \mathscr{E}^*E(\varphi) \\
&+ \int_0^t \mathscr{G}\left(\sigma(s) - \mathscr{A}\varepsilon\left(\dot{u}(s)\right) + \mathscr{E}^*E(\varphi)(s), \varepsilon\left(u(s)\right)\right) ds - C_e\theta, \\
&D = \mathscr{E}\varepsilon(u) + BE(\varphi), \\
&\dot{\theta} - \operatorname{div} K(\Delta\theta) = r(\dot{u}, \alpha) + \mathbf{q}, \\
&\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \Psi\left(\varepsilon(u), \alpha\right).
\end{aligned} \tag{1.58}$$

Constitutive law of electro-viscoelastic materials with long memory with damage and thermal effects

In this case the constitutive law is given by

$$
\begin{aligned}
\sigma(t) &= \mathscr{A}\varepsilon\left(\dot{u}(t)\right) + \mathscr{B}\left(\varepsilon\left(u(t)\right), \theta(t), \alpha(t)\right) \\
&\quad + \int_0^t \mathscr{M}\left(t-s, \varepsilon\left(u(s)\right), \theta(s), \alpha(s)\right) ds - (\mathscr{E})^* E\left(\varphi(t)\right), \\
D &= \mathscr{E}\varepsilon(u) + B\nabla(\varphi),
\end{aligned}
\tag{1.59}
$$

where $\mathscr{A}$ and $\mathscr{B}$ are the viscosity and elasticity operators, respectively, and $\mathscr{M}$ is the relaxation operator, where θ represents the absolute temperature. The temperature θ is described by the parabolic equation

$$
\dot{\theta} - \kappa_0 \Delta\theta = \Theta\left(\sigma, \varepsilon\left(u\right), \theta, \alpha\right) + \rho, \tag{1.60}
$$

where Θ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and ρ is a given volume heat source. α represents the damage field, the evolution of the damage field is described by (1.51)

1.2.3 Contact boundary conditions

Recall that the boundary Γ is divided into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that $mes(\Gamma_1) > 0$, we give in this paragraph the boundary conditions on each of the three parts.

The displacement boundary condition

The body is embedded in a fixed position on part Γ_1, the displacement field u is therefore zero

$$
u = \mathbf{0} \quad \text{on} \quad \Gamma_1 \times (0, T). \tag{1.61}
$$

The boundary condition of traction

A surface traction of density f_2 acts on Γ_2 and consequently the vector of Cauchy stresses satisfies

$$
\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2 \times (0, T). \tag{1.62}
$$

We now define the electrical boundary conditions for each of the two parts

Electrical boundary conditions

These conditions are determined from the two equations

$$
\varphi = 0 \quad \text{on} \quad \Gamma_a \times (0, T), \tag{1.63}
$$

$$
D.\nu = q_2 \quad \text{on} \quad \Gamma_b \times (0, T). \tag{1.64}
$$

Contact conditions with normal compliance

In this case, the foundation is assumed to be deformable and the contact zone is not known a priori. Normal stress satisfies the so-called normal compliance condition

$$-\sigma_\nu = p_\nu(u_\nu - g), \tag{1.65}$$

where u_ν is the normal displacement,,g represents the gap between the body and the foundation and p is a given positive function, called the normal compliance function. This condition indicates that the foundation exerts an action on the body depending on its penetration $u_\nu - g$.

Remark 1.29. (see [56]) The so-called bilateral contact describes the situation when contact between the body and the foundation is maintained during the process. It can be found in many machines and in moving parts and components of mechanical equipment. The corresponding contact condition is given by

$$u_\nu = 0 \text{ on } \Gamma_3. \tag{1.66}$$

In the engineering literature the bilateral contact is often described in terms of stress, with an equality of the form

$$-\sigma_\nu = F \text{ on } \Gamma_3, \tag{1.67}$$

where F is a given positive function. It results from (1.67) that σ_ν is negative and, therefore, the reaction of the foundation is towards the body.

Coulomb's law

It is one of the most widely used friction laws in mathematical literature. It is characterized by the intervention of normal stress in the friction bound and it can be stated as follows

$$\begin{cases} \|\sigma_\tau\| \leq \mu\,|\sigma_\nu|\,, \\ \|\sigma_\tau\| < \mu\,|\sigma_\nu| \Rightarrow \dot{u}_\tau = 0, \\ \|\sigma_\tau\| = \mu\,|\sigma_\nu| \Rightarrow \exists\lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda\dot{u}_\tau, \end{cases} \tag{1.68}$$

where μ is the coefficient of friction.

A quasi-static version of Coulomb's laws can be stated in the form

$$\begin{cases} \|\sigma_\tau\| \leq F_b, \\ -\sigma_\tau = F_b \dfrac{\dot{u}_\tau}{\|\dot{u}_\tau\|} \quad \text{ if } \dot{u}_\tau \neq 0, \end{cases} \tag{1.69}$$

Here, $\dot{u}_\tau$ is the relative tangential velocity or slip rate, and F_b represents the friction bound (this bound is not fixed and depends on the coefficient of friction and the solution of the problem). We can take for F_b the following choice

$$F_b = F_b(\sigma_\nu) = \mu|\sigma_\nu|,$$

where μ is the coefficient of friction.

Now, we replace the friction bound σ_ν of the law (1.68), by the normal compliance condition (1.65), so as to obtain the following conditions

$$\begin{cases} \|\sigma_\tau\| \le \mu p_\nu (u_\nu - g) \\ \|\sigma_\tau\| < \mu p_\nu (u_\nu - g) \Rightarrow \dot{u}_\tau = 0 \\ \|\sigma_\tau\| = \mu p_\nu (u_\nu - g) \Rightarrow \exists \lambda \ge 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau, \end{cases} \tag{1.70}$$

where p_ν is a positive function.

A quasistatic version of Coulomb's friction law used in several papers is given by

$$\begin{cases} -\sigma_\nu = p_\nu (u_\nu - g), \\ \|\sigma_\tau\| \le p_\tau (u_\nu - g), \\ \sigma_\tau = -p_\tau (u_\nu - g) \dfrac{\dot{u}_\tau}{\|\dot{u}_\tau\|} \text{ if } \quad \dot{u}_\tau \ne 0 \end{cases}, \tag{1.71}$$

where $p_e, e = \nu, \tau$ is a positive function. In (1.71), the tangential stress cannot exceed the friction bound $p_\nu(u_\nu - g)$. In addition, when the friction threshold is reached, the body begins to slide and the tangential stress tends to oppose the movement. This friction condition has been used in various papers, for example [47, 28].

Now we replace the frictional contact conditions with normal compliance (1.71) with the frictional contact conditions with a normal damping response, we get

$$\begin{cases} -\sigma_\nu(t) = p_\nu (\dot{u}_\nu(t)) \\ \|\sigma_\tau(t)\| \le p_\tau (\dot{u}_\nu(t)) \\ \sigma_\tau(t) = -p_\tau (\dot{u}_\nu(t)) \dfrac{\dot{u}_\tau(t)}{\|\dot{u}_\tau(t)\|} \text{ if } \dot{u}_\tau(t) \ne \mathbf{0} \end{cases} \quad \text{on} \quad \Gamma_3 \times (0, T). \tag{1.72}$$

Here, again, p_ν and p_τ represent given contact functions. This type of contact case has been considered by several authors (See for example [56])

Signorini type contact condition with adhesion

This condition models the contact with a rigid foundation, it is given by the following relations

$$\begin{aligned} & u_\nu \le 0, \sigma_\nu - \gamma_\nu R_\nu (u_\nu) \beta^2 \le 0, \\ & \left(\sigma_\nu - \gamma_\nu R_\nu (u_\nu) \beta^2\right) u_\nu = 0 \quad \text{ on } \Gamma_3 \times (0, T), \end{aligned} \tag{1.73}$$

where u_ν and σ_ν respectively represent the normal component of the displacement field and the stress field, β represent the adhesion field. The evolution of the adhesion field is described by a differential equation of the form

$$\begin{gathered} \dot{\beta} = -\left(\gamma_\nu \beta (R_\nu (u_\nu))^2 - \varepsilon_a\right)_+, \\ \beta(0) = \beta_0. \end{gathered} \tag{1.74}$$

γ_ν is the adhesion coefficient, R_ν is a truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if} \quad s < -L, \\ -s & \text{if} \quad -L \le s \le 0, \\ 0 & \text{if} \quad s > 0, \end{cases} \tag{1.75}$$

where $L > 0$ is the characteristic length of the link. If we choose in (1.75) L large enough, we can assume that $R_\nu(u_\nu) = -u_\nu$, hence the contact conditions (1.73) become

$$\begin{aligned} & u_\nu \le 0, \sigma_\nu + \gamma_\nu u_\nu \beta^2 \le 0, \\ & \left(\sigma_\nu + \gamma_\nu u_\nu \beta^2\right) u_\nu = 0 \quad \text{ on } \Gamma_3 \times (0,T). \end{aligned} \tag{1.76}$$

This conditions where used in [16], to model the unilateral adhesive contact. When the membership field is zero, the conditions (1.76) become

$$u_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\nu u_\nu = 0, \quad \text{sur } \Gamma_3 \times (0,T), \tag{1.77}$$

which are represent the classical contact conditions of Signorini, without adhesion.

Condition in the tangent plane

At points of Γ_3, the tangential stiffness generated by the glue is assumed to depend on adhesion and tangential displacement,

$$-\sigma_\tau = p_\tau(\beta) R_\tau(u_\tau),$$

where R_τ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v & \text{if } \|v\| \le L, \\ L\frac{v}{\|v\|} & \text{if } \|v\| > L. \end{cases}$$

Then, $p_\tau(\beta)$ acts as a spring constant, which increases with β; the traction is in the opposite direction to the displacement. The maximum modulus of the tangential tension is $p_\tau(1)L$. The tangential frictional pull is assumed to be much smaller than the adhesion and is therefore omitted. When it is not negligible, we must add the friction traction as was done in [12, 46, 48].

In this paragraph we will state the conditions of electrical contact, associated with the electro-mechanical problems introduced in [34, 2].

1.2.4 Electrical conditions at the contact surface

On part Γ_3 of the surface. We assume that the foundation is electrically conductive and its potential is held at φ_0. The electrical condition on Γ_3 is given by

$$\mathrm{D}.\, \nu = \psi(u_\nu - g)\,\phi(\varphi - \varphi_0) \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{1.78}$$

where ψ and ϕ are given functions which will be described later. This condition represents a regularized condition which can be obtained from the following considerations.

When there is no contact at a point on the surface (i.e., $u_\nu \leq g$), the gap between the body and the base is assumed to be insulating (say it is filled with air) and the component normal of the electric displacement field vanishes so that there is no free electric charge on the surface. Thereby

$$u_\nu < g \Rightarrow \mathbf{D}.\nu = 0. \tag{1.79}$$

During the contact process, (i.e., when $u_\nu \geq g$) the normal component of the electric displacement field or the free electric charge is assumed to be proportional to the potential difference between the surface of the body and the foundation, with a positive constant k as a proportionality factor. Thereby,

$$u_\nu \geq g \Rightarrow \mathbf{D}.\nu = k\left(\varphi - \varphi_0\right). \tag{1.80}$$

Let's combine (1.79), (1.80) to get

$$\text{D. } \nu = k_{\chi_{(0,\infty)}}\left(u_\nu - g\right)\left(\varphi - \varphi_0\right) \quad \text{ on } \quad \Gamma_3 \times (0,T). \tag{1.81}$$

where $\chi_{[0,\infty)}$ is the characteristic function of the interval $[0,\infty)$, which is given by

$$\chi_{[0,\infty)}(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ 1 & \text{if} \quad r \geq 0. \end{cases} \tag{1.82}$$

The condition (1.81) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions can be considered as over-idealizations in several applications.

To make it more realistic, we regularize condition (1.81) and write it as (1.78) in which $k_{\chi[0;+\infty)}(u_\nu - g)$ is replaced with which is a regular function which will be described below, and φ_L is the truncation function

$$\phi_L(s) = \begin{cases} -L_\phi & \text{if} \quad s < -L_\phi, \\ s & \text{if} \quad -L_\phi \leq s \leq L_\phi, \\ L_\phi & \text{if} \quad s > L_\phi, \end{cases} \tag{1.83}$$

where $L\phi.$ is a very large positive constant. This way, the difference $\varphi - \varphi_0$ is replaced by $\psi(\varphi - \varphi_0)$. Note that this truncation poses no practical limitation on the applicability of the model since $L\phi.$ can be arbitrarily large and therefore in applications $\psi(\varphi - \varphi_0) = \varphi - \varphi_0$ The reasons for the regularization (1.78) of (1.81) are mathematical.

First, we need to avoid discontinuities in the electric charges when the contact is made and so we regularize the function $k\chi_{[0,\infty)}$ in (1.81) by a Lipschitz function ψ. A possible choice is the following example

$$\psi(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ k\delta r & \text{if} \quad 0 \leq r \leq \frac{1}{\delta}, \\ k & \text{if} \quad r > \delta, \end{cases} \tag{1.84}$$

where is a large enough parameter. This choice means that during the contact process, the electrical conductivity increases with the contact through the roughness of the surface, and stabilizes when the penetration $u_\nu - g$ reaches the value $\dfrac{1}{\delta}$.

Second, we need the $\psi(\varphi-\varphi_0)$ term to make the $\varphi-\varphi_0$ term bounded. Note that when $\psi=0$ in (1.78), we get

$$\text{D.}\ \nu = 0 \quad \text{on} \quad \Gamma_3 \times (0,T). \tag{1.85}$$

which decouples electrical and mechanical issues on the mating surface. The condition (1.85) models the case where the obstacle is a perfect insulator and was used in [7, 38, 54] . The condition (1.78) instead of (1.85), introduces a strong coupling between the mechanical and electrical boundary conditions . It will be used in chapter 4 of the dissertation.

1.2.5 Thermal conditions

The boundary condition associated with the temperature derived from the Fourier condition

$$-k_{ij}\frac{\partial\theta}{\partial\nu}\nu_j = k_e\left(\theta-\theta_R\right) - h_\tau\left(|\dot{u}_\tau|\right), \quad \text{on}\,\Gamma_3\times[0,T], \tag{1.86}$$

$$\theta = 0, \quad \text{in}\,\Gamma_1\cup\Gamma_2\times[0,T], \tag{1.87}$$

where θ_R is the foundation temperature, k_e represents the temperature exchange between the deformable body and the foundation, h_τ the tangential function. (1.87) is the temperature boundary condition on $\Gamma_1\cup\Gamma_2$. We also consider the conditions

$$k_0\frac{\partial\theta}{\partial\nu} + B\theta = 0, \quad \text{on}\,\Gamma\times[0,T], \tag{1.88}$$

$$\frac{\partial\alpha}{\partial\nu} = 0, \quad \text{on}\,\Gamma\times[0,T], \tag{1.89}$$

where (1.88) represents a Fourier boundary condition for temperature and (1.89) represents a homogeneous Neumann boundary condition for the damage domain on Γ where $\dfrac{\partial\alpha}{\partial\nu}$ represents the normal derivative of α.

Chapter 2

Thermo-electro elastic-viscoplastic contact problem

This chapter is devoted to the mathematical analysis of a problem of contact between a thermo-electro-elastic-viscoplastic body and an obstacle. The process is assumed to be quasi-static. The contact is modeled by a general normal damped response condition with friction law and heat exchange. We obtain a variational formulation of the models, then, we prove the existence and uniqueness of a weak solution. The proof is based on arguments parabolic inequalities, on differential equations and fixed point arguments. This chapter was the subject of the publication [30].

2.1 Formulation of the problem

We place ourselves in the physical framework that we presented in Chapter 1 of this thesis. The body is described by the thermo-electroelastic-viscoplastic constitutive law. The contact is modeled by a general normal damped response condition with friction law and heat exchange. Under these considerations, the considered mechanical problem is formulated as follows.

Problem $P1$

Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$, and an electric displacement field $D : \Omega \times (0,T) \to \mathbb{R}^d$ such that

$$\begin{aligned}
&\sigma = \mathscr{A}\varepsilon(\dot{u}) + \mathscr{B}\varepsilon(u) - \mathscr{E}^* E(\varphi) \\
&+ \int_0^t \mathscr{G}(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s)) + \mathscr{E}^* E(\varphi)(s), \varepsilon(u(s)))\,ds - C_e\theta \quad \text{in } \Omega \times (0,T), && (2.1)\\
&D = \mathscr{E}\varepsilon(u) + BE(\varphi) \quad \text{in } \Omega \times (0,T), && (2.2)\\
&\dot{\theta} - \operatorname{div}(K\nabla\theta) = r(\dot{u}) + \mathbf{q}, \quad \text{in } \Omega \times (0,T), && (2.3)\\
&\operatorname{Div}\sigma + f_0 = 0 \quad \text{in } \Omega \times (0,T), && (2.4)\\
&\operatorname{div} D - q_0 = 0 \quad \text{in } \Omega \times (0,T), && (2.5)\\
&u = 0 \quad \text{on } \Gamma_1 \times (0,T), && (2.6)\\
&\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), && (2.7)\\
&-\sigma_\nu = p_\nu(\dot{u}_\nu), \quad \text{on } \Gamma_3 \times (0,T) && (2.8)\\
&\begin{cases} \|\sigma_\tau\| \le p_\tau(\dot{u}_\nu) \\ \sigma_\tau = -p_\tau(\dot{u}_\nu)\dfrac{\dot{u}_\tau}{\|u_\tau\|} \text{ if } \quad \dot{u}_\tau \neq 0 \end{cases} \quad \text{on } \Gamma_3 \times (0,T), && (2.9)\\
&-k_{ij}\frac{\partial\theta}{\partial x_i}\nu_j = k_e(\theta - \theta_R) + h_\tau(|\dot{u}_\tau|) \quad \text{on } \Gamma_3 \times (0,T), && (2.10)\\
&\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), && (2.11)\\
&D.\nu = q_2 \quad \text{on } \Gamma_b \times (0,T), && (2.12)\\
&\theta = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), && (2.13)\\
&u(0) = u_0, \quad \theta(0) = \theta_0, \quad \text{in } \Omega. && (2.14)
\end{aligned}$$

We now describe the notations in (2.1)-(2.14) and provide some comments on the equalities and boundary conditions. First, the equations (2.1)-(2.3) represent the electro-elastic-viscoplastic constitutive law with thermal effects, The equations (2.4) and (2.5) are the equilibrium equations for the stress and electric displacement fields, which we have already seen in (1.43) and (1.44). Equations (2.6)-(2.7) are the displacement-traction conditions. (2.8)-(2.9) represents the contact conditions for part Γ_3 that were discussed in Chapter 1. Condition (2.10), represent, on Γ_3, a Fourier boundary condition for the temperature. (2.11) and (2.12) represent the electric boundary conditions. Equation (2.13) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$. Finally, The functions u_0 and θ_0 in (2.14) are the initial data.

2.2 Variational formulation

The viscosity operator $\mathscr{A}:\Omega\times\mathbb{S}^d\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}(a)\text{ There exists } L_{\mathscr{A}}>0 \quad \text{such that}\\ \|\mathscr{A}(x,\omega_1)-\mathscr{A}(x,\omega_2)\|\leq L_{\mathscr{A}}\|\omega_1-\omega_2\|,\\ \quad\text{for all } \omega_1,\omega_2\in\mathbb{S}^d,\ \text{a.e } x\in\Omega.\\ (b)\text{ There exists } m_{\mathscr{A}}>0 \quad \text{such that}\\ (\mathscr{A}(x,\omega_1)-\mathscr{A}(x,\omega_2)).(\omega_1-\omega_2)\geq m_{\mathscr{A}}\|\omega_1-\omega_2\|^2,\\ \quad\text{for all } \omega_1,\omega_2\in\mathbb{S}^d,\ \text{a.e } x\in\Omega.\\ (c)\text{ The mapping } x\mapsto\mathscr{A}(x,\omega) \text{ is Lebesgue measurable on } \Omega,\\ \quad\text{for any } \omega\in\mathbb{S}^d.\\ (d)\text{ The mapping } x\mapsto\mathscr{A}(x,0)\in\mathscr{H}.\end{cases}\tag{2.15}$$

The elasticity operator $\mathscr{B}:\Omega\times\mathbb{S}^d\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}\text{(a) There exists } L_{\mathscr{B}}>0 \quad \text{such that}\\ \|\mathscr{B}(x,\omega_1)-\mathscr{B}(x,\omega_2)\|\leq L_{\mathscr{B}}\|\omega_1-\omega_2\|,\\ \text{for all } \omega_1,\omega_2\in\mathbb{S}^d,\ \text{a.e. } x\in\Omega.\\ \text{(b) The mapping } x\mapsto\mathscr{B}(x,\omega) \text{ is Lebesgue measurable on } \Omega,\\ \qquad\qquad\text{for all } \omega\in\mathbb{S}^d.\\ \text{(c) The mapping } x\mapsto\mathscr{B}(x,0)\in\mathscr{H}.\end{cases}\tag{2.16}$$

We assume that the visco-plasticity operator $\mathscr{G}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\longrightarrow\mathbb{R}$ satisfies

$$\begin{cases}\text{(a) There exists a constant } L_{\mathscr{G}}>0 \quad \text{such that}\\ \|\mathscr{G}(x,\sigma_1,\varsigma_1)-\mathscr{G}(x,\sigma_2,\varsigma_2)\|\leq L_{\mathscr{G}}(\|\sigma_1-\sigma_2\|+\|\varsigma_1-\varsigma_2\|),\\ \text{for all } t\in(0,T),\sigma_1,\sigma_2,\varsigma_1,\varsigma_2\in\mathbb{S}^d,\ \text{a.e. } x\in\Omega.\\ \text{(b) The mapping } x\mapsto\mathscr{G}(x,\sigma,\varsigma) \text{ is Lebesgue measurable on } \Omega,\\ \text{for all } \sigma,\varsigma\in\mathbb{S}^d, t\in(0,T),\\ \text{(c) The mapping } x\mapsto\mathscr{G}(x,0,0)\in\mathscr{H}.\end{cases}\tag{2.17}$$

The thermal expansion operator $C_e:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases}\text{(a) There exists } L_{C_e}>0 \text{ such that}\\ \|C_e(x,\mu_1)-C_e(x,\mu_2)\|\leq L_{C_e}\|\mu_1-\mu_2\| \text{ for all } \mu_1,\mu_2\in\mathbb{R},\ \text{a.e. } x\in\Omega.\\ \text{(b) } C_e=\left(c_{ij}\right), c_{ij}=c_{ji}\in L^\infty(\Omega).\\ \text{(c) The mapping } x\mapsto C_e(x,\mu) \text{ is Lebesgue measurable on } \Omega,\\ \text{for any } \mu\in\mathbb{R}.\\ \text{(d) The mapping } x\mapsto C_e(x,0)\in\mathscr{H}.\end{cases}\tag{2.18}$$

The thermal conductivity operator $K:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases}\text{(a) There exists } L_K>0 \text{ such that}\\ \|K(x,\mu_1)-K(x,\mu_2)\|\le L_K\|\mu_1-\mu_2\|, \text{ for all } \mu_1,\mu_2\in\mathbb{R}, \text{ a.e. } x\in\Omega.\\ \text{(b) } k_{ij}=k_{ji}\in L^\infty(\Omega), k_{ij}\alpha_i\alpha_j\le c_k\alpha_i\alpha_j \text{ for some } c_k>0,\\ \text{for all } (\alpha_i)\in\mathbb{R}.\\ \text{(c) The mapping } x\mapsto k(x,0) \text{ belongs to } L^2(\Omega).\end{cases} \tag{2.19}$$

Electric permittivity operator $B=\left(b_{ij}\right):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies

$$\begin{cases}\text{(a) } B(\varepsilon,E)=\left(b_{ij}(\varepsilon)E_j\right) \text{ for all } E=(E_i)\in\mathbb{R}^d, \text{ a.e. } \varepsilon\in\Omega.\\ \text{(b) } b_{ij}=b_{ji}\in L^\infty(\Omega), 1\le i,j\le d.\\ \text{(c) There exists a constant } m_B>0 \text{ such that}\\ \qquad BE.E\ge m_B\|E\|^2, \text{ for all } E=(E_i)\in\mathbb{R}^d, \text{ a.e. in } \Omega.\end{cases} \tag{2.20}$$

The piezoelectric operator $\mathscr{E}:\Omega\times\mathbb{S}^d\to\mathbb{R}^d$ satisfies

$$\begin{cases}(a) \quad \mathscr{E}=\left(f_{ijk}\right), f_{ijk}\in L^\infty(\Omega), 1\le i,j,k\le d.\\ (b) \quad \mathscr{E}(\mathbf{x})\sigma\cdot\tau=\sigma\cdot\mathscr{E}^*\tau, \text{ for all } \sigma\in\mathbb{S}^d, \text{ and all } \tau\in\mathbb{R}^d.\end{cases} \tag{2.21}$$

The tangential function $p_e:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}_+$, $e=\nu,\tau$ satisfies

$$\begin{cases}\text{(a) There exists } L_e>0 \text{ such that}\\ \quad \|p_e(x,\mu_1)-p_e(x,\mu_2)\|\le L_e\|\mu_1-\mu_2\|\\ \text{for all } \mu_1,\mu_2\in\mathbb{R}, \text{ a.e. } x\in\Gamma_3\\ \text{(d) For any } \mu\in\mathbb{R}, x\mapsto p_e(x,\mu) \text{ is Lebesgue measurable on } \Gamma_3\\ \text{(c) The mapping } x\mapsto p_e(x,0) \text{ belongs to } L^2(\Gamma_3).\end{cases} \tag{2.22}$$

We assume that the boundary and initial data θ_R, k_e, u_0 and θ_0 the volume of forces f_0 and f_2 and the charges densities q_0, q_2, the heat source density $\mathbf{q}$, satisfy

$$\theta_R\in C(0,T;L^2(\Gamma_3)),\ k_e\in L^\infty(\Omega,\mathbb{R}_+), \tag{2.23}$$

$$u_0\in V, \quad \theta_0\in\mathscr{X}, \tag{2.24}$$

$$f_0\in C\left(0,T;L^2(\Omega)^d\right),\ f_2\in C\left(0,T;L^2\left(\Gamma_2\right)^d\right), \tag{2.25}$$

$$q_0\in C\left(0,T;L^2(\Omega)\right),\ q_2\in C\left(0,T;L^2\left(\Gamma_b\right)\right), \tag{2.26}$$

$$\mathbf{q}\in C\left(0,T;L^2\left(\Omega\right)\right). \tag{2.27}$$

The function $r:V\to L^2(\Omega)$ satisfies that there exists a constant $L_r>0$ such that

$$\|r(u_1)-r(u_2)\|_{L^2(\Omega)}\le L_r\|u_1-u_2\|_V, \forall u_1,u_2\in V. \tag{2.28}$$

We We note that the tangential function h_τ satisfies (2.22). Next. We define six mappings $j:V\times V\to\mathbb{R}$, $f:[0,T]\to V$, $q:[0,T]\to W$, respectively, by

$$j(u,v)=\int_{\Gamma_3}p_\nu(u_\nu)v_\nu da+\int_{\Gamma_3}p_\tau(u_\nu)\|v_\tau\|da, \tag{2.29}$$

$$(f(t),v)_V=\int_\Omega f_0(t)\cdot v dx+\int_{\Gamma_2}f_2(t)\cdot v da, \tag{2.30}$$

$$(q(t),\zeta)_W=\int_\Omega q_0(t)\zeta dx-\int_{\Gamma_b}q_2(t)\zeta da, \tag{2.31}$$

for all $u,v \in V$, $\zeta \in W$ and $t \in [0,T]$. Note that

$$f \in C(0,T;V), \quad q \in C(0,T;W). \tag{2.32}$$

Suppose problem (2.1)-(2.14) has a regular solution (u,σ,θ) and let $v \in V$, using Green's formula (1.4), it comes that

$$\begin{aligned}\int_{\Omega} \sigma \cdot (\varepsilon(v)-\varepsilon(\dot{u}))dx = \int_{\Omega} f_0 \cdot (v-\dot{u})dx + \int_{\Gamma_2} f_2 \cdot (v-\dot{u})da \\ + \int_{\Gamma_3} \sigma_\nu (v_\nu - \dot{u}_\nu)\, da + \int_{\Gamma_3} \sigma_\tau \cdot (v_\tau - \dot{u}_\tau)\, da.\end{aligned} \tag{2.33}$$

Then we use the friction law (2.9) to see that

$$\sigma_\tau \cdot (v_\tau - \dot{u}_\tau) \geq p_\tau(\dot{u}_\nu)\,|\dot{u}_\tau| - p_\tau(\dot{u}_\nu)\,|v_\tau| \qquad \text{a.e. on } \Gamma_3. \tag{2.34}$$

Moreover, on the points of Γ_3 where $\dot{u}_\tau \neq 0$ we have

$$\begin{aligned}\sigma_\tau \cdot (v_\tau - \dot{u}_\tau) &= -p_\tau(\dot{u}_\nu)\,\frac{\dot{u}_\tau}{|\dot{u}_\tau|}(v_\tau - \dot{u}_\tau) \\ &\geq p_\tau(\dot{u}_\nu)\,|\dot{u}_\tau| - p_\tau(\dot{u}_\nu)\,|v_\tau| \qquad \text{a.e. on } \Gamma_3.\end{aligned} \tag{2.35}$$

Because $\dot{u}_\tau.\dot{u}_\tau = \|\dot{u}_\tau\|^2$ and $\dot{u}_\tau.v_\tau \leq \|\dot{u}_\tau\|\|v_\tau\|$; on the points of Γ_3 where $\dot{u}_\tau = 0$ we have

$$\begin{aligned}\sigma_\tau \cdot (v_\tau - \dot{u}_\tau) &= \sigma_\tau \cdot v_\tau \geq -|\sigma_\tau|\,|v_\tau| \\ &\geq -p_\tau(\dot{u}_\nu)\,|v_\tau| = p_\tau(\dot{u}_\nu)\,|\dot{u}_\tau| - p_\tau(\dot{u}_\nu)\,|v_\tau|,\end{aligned}$$

because $\|\sigma_\tau\| \leq p_\tau(\dot{u}_\nu)$ and $\|\dot{u}_\tau\| = 0$. By integrating (2.34) over Γ_3 we obtain

$$\int_{\Gamma_3} \sigma_\tau \cdot (v_\tau - \dot{u}_\tau)\, da \geq \int_{\Gamma_3} p_\tau(\dot{u}_\nu)\,(|\dot{u}_\tau| - |v_\tau|)\, da. \tag{2.36}$$

We now combine (2.36) and (2.33), then we use the condition (2.8) to find

$$\begin{aligned}&\int_{\Omega} \sigma \cdot (\varepsilon(v)-\varepsilon(\dot{u}))dx + \int_{\Gamma_3} p_\nu(\dot{u}_\nu)\,(v_\nu - \dot{u}_\nu)\, da \\ &+ \int_{\Gamma_3} p_\tau(\dot{u}_\nu)\,(|v_\tau| - |\dot{u}_\tau|)\, da \geq \int_{\Omega} f_0 \cdot (v-\dot{u})dx + \int_{\Gamma_2} f_2 \cdot (v-\dot{u})da.\end{aligned}$$

We use the notations (2.29) and (2.30) to deduce that

$$(\sigma(t), \varepsilon(v)-\varepsilon(\dot{u}(t)))_{\mathscr{H}} + j(\dot{u}(t),v) - j(\dot{u}(t),\dot{u}(t)) \geq (\mathbf{f}(t), v-\dot{u}(t))_V.$$

Now for the temperature we have for all

$$\begin{aligned}\int_{\Omega} \dot{\theta}\tau dx - \int_{\Omega} \operatorname{div}(K\nabla\theta)\tau dx = \int_{\Omega} r(\dot{u})\tau dx + \int_{\Omega} q\tau dx \\ \int_{\Omega} \dot{\theta}\tau dx - \int_{\Omega} \left(K_{ij}\theta_{,i}\right)_{,j}\tau dx = \int_{\Omega} r(\dot{u})\tau dx + \int_{\Omega} q\tau dx,\end{aligned}$$

according to Green's formula

$$\int_\Omega \dot{\theta}\tau dx - \int_\Gamma \left(k_{ij}\theta_{,i}\right)\nu_j\cdot\tau ds + \int_\Omega \left(k_{ij}\theta_{,j}\right)\tau_{,j}dx = \int_\Omega r(\dot{u})\tau dx + \int_\Omega q\tau dx.$$

According to the condition (2.10)

$$\int_\Omega \dot{\theta}\tau dx + \int_\Gamma (k_e(\theta-\theta_R) + h_\tau(|\dot{u}_\tau|))\tau ds + \int_\Omega (K\nabla\theta)\nabla\tau dx$$
$$= \int_\Omega r(\dot{u})\tau dx + \int_\Omega q\tau dx,$$

$$\int_\Omega \dot{\theta}\tau dx + \int_{\Gamma_3}(k_e\theta)\tau ds + \int_\Omega (K\nabla\theta)\nabla\tau dx = \int_\Omega r(\dot{u})\tau dx$$
$$-\int_{\Gamma_3} h_\tau(|\dot{u}_\tau|)\tau ds + \int_\Omega q\tau dx + \int_{\Gamma_3} k_e(\theta_R)\tau ds.$$

We pose $Q:[0,T]\to\mathscr{V}'$, $K:\mathscr{V}\to\mathscr{V}'$, and $R:V\to\mathscr{V}'$

$$(Q(t),\eta)_{\mathscr{V}'\times\mathscr{V}} = \int_{\Gamma_3} k_e\theta_R(t)\eta da + \int_\Omega q(t)\eta dx. \tag{2.37}$$

$$(K\tau,\eta)_{\mathscr{V}'\times\mathscr{V}} = \sum_{i,j=1}^d \int_\Omega k_{ij}\frac{\partial\tau}{\partial x_j}\frac{\partial\eta}{\partial x_i}dx + \int_{\Gamma_3} k_e\tau\eta da. \tag{2.38}$$

$$(Rv,\eta)_{\mathscr{V}'\times\mathscr{V}} = \int_\Omega r(\dot{v})\eta dx - \int_{\Gamma_3} h_\tau(|v_\tau|)\eta da. \tag{2.39}$$

Therefore, we obtain

$$\dot{\theta}(t) + K\theta(t) = R\dot{\mathbf{u}}(t) + Q(t), \quad \text{in } \mathscr{V}'.$$

On the other hand, using Green's formula (1.14) for the electrical unknowns of the problem as well as the conditions (2.5), (2.11), and (2.12) we obtain

$$(D(t),\nabla\psi)_{L^2(\Omega)^d} + \int_\Omega q_0(t)\psi dx = \int_{\Gamma_b}\psi q_2 da \quad \forall\psi\in W.$$

keeping in mind (2.31) we obtain

$$(D(t),\nabla\psi)_{L^2(\Omega)^d} + (q(t),\psi)_W = 0 \tag{2.40}$$

Combine this last equality with (2.2) to obtain

$$(B\nabla\varphi(t),\nabla\psi)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u(t)),\nabla\psi)_{L^2(\Omega)^d} = (q(t),\psi)_W$$
$$\forall\psi\in W, \quad \forall t\in[0,T].$$

According to the above, we deduce the following variational formulation of the mechanical problem (2.1)-(2.14)

Problem PV

Find a displacement field $u : (0,T) \to V$, a stress field $\sigma : (0,T) \to \mathscr{H}$, an electric potential $\varphi : (0,T) \to W$, and a temperature $\theta : (0,T) \to \mathscr{X}$ such that

$$\begin{aligned}\sigma(t) &= \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) + \mathscr{E}^*\nabla\varphi(t) \\ &+ \int_0^t \mathscr{G}(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s)) - \mathscr{E}^*\nabla\varphi(s), \varepsilon(u(s)))\,ds - C_e\theta(t),\end{aligned} \tag{2.41}$$

$$(\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}} + j(\dot{u}(t), v) - j(\dot{u}(t), \dot{u}(t)) \geq (\mathbf{f}(t), v - \dot{u}(t))_V, \tag{2.42}$$

$$(B\nabla\varphi(t), \nabla\phi)_H - (\mathscr{E}\varepsilon(u(t)), \nabla\phi)_H = (q(t), \phi)_W, \quad \forall\phi \in W, t \in (0,T) \tag{2.43}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t), \quad \text{in } \mathscr{V}', \tag{2.44}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0. \tag{2.45}$$

To study problem PV we use the smallness assumption

$$L_\nu + L_\tau < \frac{m_{\mathscr{A}}}{C_0^2}, \tag{2.46}$$

where L_ν, L_τ, $m_{\mathscr{A}}$ and C_0 are given in (2.22), (2.15) and (1.7), respectively Our main existence and uniqueness result for Problem PV is in the following section.

2.3 Result of existence and uniqueness

Theorem 2.1. *Assume that (2.15)-(2.28) hold, and the smallness assumption (2.46). Then there exists a unique solution $(u, \sigma, \varphi, \theta, D)$ to problem PV. Moreover, the solution has the regularity*

$$u \in C^1(0,T;V), \tag{2.47}$$

$$\varphi \in C(0,T;W), \tag{2.48}$$

$$\sigma \in C(0,T;\mathscr{H}), \tag{2.49}$$

$$\theta \in C\left(0,T;L^2(\Omega)\right) \cap L^2(0,T;\mathscr{X}) \cap W^{1,2}\left(0,T;\mathscr{X}'\right), \tag{2.50}$$

$$D \in C(0,T;\mathscr{W}). \tag{2.51}$$

The functions u, σ, φ, θ, and D which satisfy (2.41)-(2.45) are called a weak solution of the contact problem P. We conclude that under the above assumptions, the mechanical problem (2.1) - (2.14) has a unique weak solution satisfying (2.47)-(2.51). The proof of Theorem 2.1, is carried out is several steps and is based on arguments of evolutionary quasivariational inequalities, differential equations and fixed points. We denote by C a constant whose value may change from line to line when no confusing can arise. Let $\eta \in C(0,T;\mathscr{H})$, we consider the following variational problem.

Problem $\mathscr{P}^1_\eta$

Find a displacement field $u_\eta : [0,T] \to V$ such that for all $t \in [0,T]$

$$\begin{aligned}
&(\mathscr{A}\varepsilon(\dot{u}_\eta(t)), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)))_{\mathscr{H}} + (\mathscr{B}\varepsilon(u_\eta(t)), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)))_{\mathscr{H}} \\
&+ (\eta(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)))_{\mathscr{H}} + j(\dot{u}_\eta(t), v) - j(\dot{u}_\eta(t), \dot{u}_\eta(t)) \geq (f(t), v - \dot{u}_\eta(t))_V, \quad (2.52) \\
&\forall v \in V, \text{ a.e. } t \in (0,T), \\
&u_\eta(0) = u_0. \quad (2.53)
\end{aligned}$$

We have the following result for $\mathscr{P}^1_\eta$

Lemma 2.2. *There exists a unique solution $u_\eta \in C^1(0,T;V)$ to the problem (2.52) and (2.53).*

Proof. Introducing now the operators $A : V \to V$ and $B : V \to V$ defined as

$$(Au, v)_V = (\mathscr{A}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \quad \forall u, v \in V, \tag{2.54}$$

$$(Bu, v)_V = (\mathscr{B}\varepsilon(u)\varepsilon(v))_{\mathscr{H}}, \quad \forall u, v \in V. \tag{2.55}$$

So,Ineq. (2.52) can be rewritten

$$\begin{aligned}
(A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + j(\dot{u}_\eta(t), v) \\
- j(\dot{u}_\eta(t), \dot{u}_\eta(t)) \geq (\mathbf{f}_\eta(t), v - \dot{u}(t))_V,
\end{aligned} \tag{2.56}$$

where

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \eta(t), \quad \text{a.e.t } \in [0,T].$$

Using (2.54)-(2.55) and (2.15)-(2.16) it follows that A and B are Lipschitz continuous operators. Using again (2.54) and (2.15) we deduce that A is a strongly monotone operator on V

$$\begin{aligned}
(Au_1 - Au_2, u_1 - u_2)_V &= (\mathscr{A}(\varepsilon(u_1)) - \mathscr{A}(\varepsilon(u_2)), \varepsilon(u_1) - \varepsilon(u_2))_{\mathscr{H}} \\
&\geq m_{\mathscr{A}} \|\varepsilon(u_1) - \varepsilon(u_2)\|^2_{\mathscr{H}} \geq C\|u_1 - u_2\|^2_V
\end{aligned}$$

We use (2.29), (2.22) and (1.7) to see that the functional j defined in (2.29) satisfies condition (1.35)(a). With (2.22) and (1.7) again we get

$$j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \leq \tilde{c}_0^2 (L_\nu + L_\tau)\|u_1 - u_2\|_V \|v_1 - v_2\|_V \tag{2.57}$$

For all u_1, u_2, v_1 and $v_2 \in V$, which shows that the functional j verifies the condition (1.35)(b) Finally, note that $\mathbf{f}_\eta \in C([0,T];V)$, $u_0 \in V$ and smallness assumption (2.46) and we use classical arguments of functional analysis concerning evolutionary quasivariational inequality [56] that there exists a unique solution $u_\eta \in C^1(0,T;V)$ to the problem $\mathscr{P}^1_\eta$

□

In the next step we use the solution u_η, obtained in Lemma 2.2, to construct the following variational problem for the electrical potential.

Problem $\mathscr{P}_\eta^2$

Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$(B\nabla\varphi_\eta(t),\nabla\zeta)_H - (\mathscr{E}\varepsilon(u_\eta(t)),\nabla\zeta)_H = (q(t),\zeta)_W, \text{ for all } \zeta \in W, t \in (0,T). \tag{2.58}$$

We have the following result for problem $\mathscr{P}_\eta^2$

Lemma 2.3. *Problem (2.58) has unique solution φ_η which satisfies the regularity (2.48). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (2.58) corresponding to $\eta_1,\eta_2 \in C([0,T];\mathscr{H})$, then there exists $C>0$ such that*

$$\left\|\varphi_{\eta_1}(t)-\varphi_{\eta_2}(t)\right\|_W \le C\left\|u_{\eta_1}(t)-u_{\eta_2}(t)\right\|_V, \quad \forall t\in[0,T]. \tag{2.59}$$

Proof. we consider the form $S : W\times W \to \mathbb{R}$

$$S(\varphi,\phi) = (B\nabla\varphi,\nabla\phi)_H, \quad \forall\varphi,\phi\in W, \tag{2.60}$$

we use (1.11), (2.20), (2.60) and defined $(\varphi,\psi)_W$ to show that the form S is bilinear continuous, symmetric and coercive on W, moreover using (2.31) and the Riesz representation theorem we may define an element $\xi_\eta : [0,T] \to W$ such that

$$(\xi_\eta(t),\phi)_W = (q(t),\phi)_W + (\mathscr{E}\varepsilon(u_\eta(t)),\nabla\phi)_H, \quad \forall\phi\in W, t\in(0,T),$$

we apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t)\in W$ such that

$$S(\varphi_\eta(t),\phi) = (\xi_\eta(t),\phi)_W, \quad \forall\phi\in W. \tag{2.61}$$

It follows from (2.61) that φ_η is a solution of the equation (2.58). Let $\varphi_{\eta_i}=\varphi_i$, and $u_{\eta_i}=u_i$ for $i=1,2$. We use (2.58) to obtain

$$\|\varphi_1(t)-\varphi_2(t)\|_W \le C\|u_1(t)-u_2(t)\|_V, \quad \forall t\in[0,T].$$

Now since $u_\eta \in C^1(0,T;V)$, so implies that $\varphi_\eta \in C(0,T;W)$. This completes the proof. □

In the third step, we use the displacement field u_η obtained in Lemma 2.2 to consider the following variational problem.

Problem $\mathscr{P}_\eta^3$

Find the temperature field $\theta_\eta : (0,T) \to L^2(\Omega)$

$$\dot\theta_\eta(t) + K\theta_\eta(t) = R\dot u_\eta(t) + Q(t), \quad \text{in } \mathscr{V}', \quad \text{a.e.t } \in [0,T], \tag{2.62}$$
$$\theta_\eta(0) = \theta_0. \tag{2.63}$$

Lemma 2.4. *There exists a unique solution θ_η to the auxiliary problem $\mathscr{P}_\eta^3$ satisfying (2.50).*

Proof. The result follows from classical first order evolution equation given in Refs. [6, 55]. Here the Gelfand triple is given by

$$\mathscr{V} \subset L^2(\Omega) = \left(L^2(\Omega)\right)' \subset \mathscr{V}'.$$

The operator K is linear and coercive. By Korn's inequality, we have

$$(K\tau,\tau)_{\mathscr{V}'\times\mathscr{V}} \geq C\|\tau\|^2_{\mathscr{V}}.$$

Given (2.38) and for all $\tau,\eta \in \mathscr{V}$ we have

$$(K\tau,\eta)_{\mathscr{V}'\times\mathscr{V}} \leq \sum_{i,j=1}^{d} \|k_{i,j}\|_{L^\infty(\Gamma_3)} \|\tau_{,i}\|_{L^2(\Omega)} \|\eta_{,i}\|_{L^2(\Omega)} + \|k_e\|_{L^\infty(\Gamma_3)} \|\eta\|_{L^2(\Omega)}$$

that imply

$$(K\tau,\eta)_{\mathscr{V}'\times\mathscr{V}} \leq C\|\tau\|_{\mathscr{V}}\|\eta\|_{\mathscr{V}}. \tag{2.64}$$

From the expression of the operator R, $\dot{u}_\eta \in L^2(0,T;V) \Rightarrow R\dot{u}_\eta \in L^2(0,T;\mathscr{V}')$, as $Q \in L^2(0,T;\mathscr{V}')$ then

$$R\dot{u}_\eta + Q \in L^2\left(0,T;\mathscr{V}'\right) \tag{2.65}$$

Using linear and coercion operators K and using (2.64) and the regularity (2.65),$\theta_0 \in L^2(\Omega)$ imply that all the conductions of Theorem 1.25 hold, so we conclude that there exists a single function θ_η and who satisfies

$$\theta_\eta \in L^2(0,T;\mathscr{V}) \cap C\left(0,T;L^2(\Omega)\right), \quad \dot{\theta}_\eta \in L^2\left(0,T;\mathscr{V}'\right), \tag{2.66}$$

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t) \quad \text{a.e. } t \in (0,T), \tag{2.67}$$

$$\theta_\eta(0) = \theta_0. \tag{2.68}$$

□

In the fourth step, we use u_η, φ_η and θ_η obtained in Lemmas 2.2, 2.3 and 2.4, respectively to construct the following Cauchy problem for the stress field.

Problem $\mathscr{P}^4_\eta$

Find the stress field $\sigma_\eta : [0,T] \to \mathscr{H}$ which is a solution of the problem

$$\sigma_\eta(t) = \mathscr{B}(\varepsilon(u_\eta(t))) + \int_0^t \mathscr{G}(\sigma_\eta(s),\varepsilon(u_\eta(s)))\,ds - C_e\theta_\eta(t), \text{ a.e. } t \in (0,T). \tag{2.69}$$

Lemma 2.5. *$\mathscr{P}^4_\eta$ has a unique solutions $\sigma_\eta \in C(0,T;\mathscr{H})$. Moreover, if σ_{η_i}, u_{η_i} and θ_{η_i} represent the solutions of Problems $\mathscr{P}^4_\eta$, $\mathscr{P}^1_\eta$ and $\mathscr{P}^3_\eta$ respectively, for $\eta_i \in C(0,T;\mathscr{H})$, $i = 1,2$, then there exists $C > 0$ such that*

$$\begin{aligned}\left\|\sigma_{\eta_1}(t) - \sigma_{\eta_2}(t)\right\|^2_{\mathscr{H}} \leq & C\left(\left\|u_{\eta_1}(t) - u_{\eta_2}(t)\right\|^2_V\right. \\ & \left. + \left\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\right\|^2_{L^2(\Omega)} + \int_0^t \left\|u_{\eta_1}(s) - u_{\eta_2}(s)\right\|^2_V\right).\end{aligned} \tag{2.70}$$

Proof. Let $\Sigma_\eta : C(0,T;\mathscr{H}) \to C(0,T;\mathscr{H})$ be the operator given by

$$\Sigma_\eta \sigma(t) = \mathscr{B}(\varepsilon(u_\eta(t))) + \int_0^t \mathscr{G}(\sigma_\eta(s), \varepsilon(u_\eta(s)))\,ds - C_e\theta_\eta(t), \tag{2.71}$$

Let $\sigma_i \in C(0,T;\mathscr{H})$, $i=1,2$ and $t_1 \in (0,T)$. Using hypothesis (2.17) and Holder's inequality, we find

$$\left\|\Sigma_\eta\sigma_1(t_1) - \Sigma_\eta\sigma_2(t_1)\right\|_{\mathscr{H}}^2 \le L_{\mathscr{G}}^2 T \int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2\,ds.$$

Integration on the time interval $(0,t_2) \subset (0,T)$, it follows that

$$\int_0^{t_2} \left\|\Sigma_\eta\sigma_1(t_1) - \Sigma_\eta\sigma_2(t_1)\right\|_{\mathscr{H}}^2 dt_1 \le L_{\mathscr{G}}^2 T \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2\,ds dt_1.$$

Therefore

$$\left\|\Sigma_\eta\sigma_1(t_2) - \Sigma_\eta\sigma_2(t_2)\right\|_{\mathscr{H}}^2 \le L_{\mathscr{G}}^4 T^2 \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2\,ds dt_1.$$

For $t_1, t_2, ..., t_p \in (0,T)$, we generalize the procedure above by recurrence on p. We obtain the inequality

$$\begin{aligned}&\left\|\Sigma_\eta\sigma_1(t_p) - \Sigma_\eta\sigma_2(t_p)\right\|_{\mathscr{H}}^2 \\ &\qquad \le L_{\mathscr{G}}^{2p} T^p \int_0^{tp} \cdots \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2\,ds dt_1 \ldots dt_{p-1}.\end{aligned}$$

Which implies

$$\left\|\Sigma_\eta\sigma_1(t_p) - \Sigma_\eta\sigma_2(t_p)\right\|_{\mathscr{H}}^2 \le \frac{L_{\mathscr{G}}^{2p} T^{p+1}}{p!} \int_0^{T} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2\,ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\left\|\Sigma_\eta\sigma_1 - \Sigma_\eta\sigma_2\right\|_{C(0,T;\mathscr{H})}^2 \le \frac{L_{\mathscr{G}}^{2p} T^{p+2}}{p!} \|\sigma_1 - \sigma_2\|_{C(0,T;\mathscr{H})}^2.$$

It follows from this inequality that for large p enough, the operator Σ_η^p is a contraction on the Banach space $C(0,T;\mathscr{H})$, and therefore there exists a unique element $\sigma \in C(0,T;\mathscr{H})$ such that $\Sigma_\eta\sigma = \sigma$. Moreover, σ is the unique solution of Problem $\mathscr{P}_\eta^4$, and using (2.69), the regularity of u_η, θ_η, and the properties of the operators $\mathscr{B}$, $\mathscr{G}$ and C_e it follows that $\sigma \in C(0,T;\mathscr{H})$. Consider now $\eta_1, \eta_2 \in C(0,T;\mathscr{H})$ and for $i=1,2$, denote $u_{\eta_i} = u_i$, $\theta_{\eta_i} = \theta_i$, and $\sigma_{\eta_i} = \sigma_i$. We have

$$\sigma_i(t) = \mathscr{B}(\varepsilon(u_i(t))) + \int_0^t \mathscr{G}(\sigma_i(s), \varepsilon(u_i(s)))\,ds - C_e\theta_i(t), \qquad \text{a.e. } t \in (0,T). \tag{2.72}$$

and using the properties (2.16), (2.17) and (2.18) of $\mathscr{B}$, $\mathscr{G}$ and C_e we find

$$\begin{aligned}\|\sigma_1(t) - \sigma_2(t)\|_{\mathscr{H}}^2 \le C\Big(&\|u_1(t) - u_2(t)\|_V^2 + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \\ &+ \int_0^t \|u_1(s) - u_2(s)\|_V^2\,ds + \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2\,ds\Big), \quad \forall t \in [0,T].\end{aligned} \tag{2.73}$$

We use Gronwall argument in the previous inequality to deduce (2.70), which concludes the proof of Lemma 2.5.

□

Finally, as a consequence of these results and using the properties of the operators $\mathscr{G}$, $\mathscr{E}$ and C_e, for $t \in (0,T)$, we consider the element $\Lambda\eta(t) \in \mathscr{H}$ defined by

$$\begin{aligned}(\Lambda\eta(t),v)_{\mathscr{H}\times V} = (\mathscr{E}^*\nabla\varphi_\eta(t),\varepsilon(v))_{\mathscr{H}} + (C_e\theta_\eta(t),\varepsilon(v))_{\mathscr{H}}\\ + \left(\int_0^t \mathscr{G}\left(\sigma_\eta,\varepsilon\left(u_\eta(s)\right)\right)ds,\varepsilon(v)\right)_{\mathscr{H}}, \forall v \in V,\end{aligned} \tag{2.74}$$

Here, for every $\eta \in C(0,T;\mathscr{H})$. u_η, φ_η, θ_η and σ_η represent the displacement field, the electric potential field, the temperature field and the stress field, obtained in Lemmas 2.2, 2.3, 2.4 and 2.5 respectively. We have the following result.

Lemma 2.6. *The mapping Λ has a fixed point $\eta^* \in C(0,T;\mathscr{H})$, such that $\Lambda\eta^* = \eta^*$.*

Proof. Let $t \in (0,T)$ and $\eta_1,\eta_2 \in C(0,T;\mathscr{H})$. We use the notation that $u_{\eta_i} = u_i$, $\dot{u}_{\eta_i} = \dot{u}_i$, $\theta_{\eta_i} = \theta_i$, $\varphi_{\eta_i} = \varphi_i$, and $\sigma_{\eta_i} = \sigma_i$ for $i = 1,2$. Let us start by using (1.7), (2.17), (2.18) and (2.21), we have

$$\begin{aligned}\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|^2_{\mathscr{H}} \leq& \|\mathscr{E}^*\nabla\varphi_1(t) - \mathscr{E}^*\nabla\varphi_2(t)\|^2_{\mathscr{H}}\\ &+ \|C_e\theta_1(t)) - C_e\theta_2(t)\|^2_{\mathscr{H}} + \int_0^t \|\mathscr{G}\left(\sigma_1(s),\varepsilon(u_1(s))\right) - \mathscr{G}\left(\sigma_2(s),\varepsilon(u_2(s))\right)\|^2_{\mathscr{H}}ds\\ \leq& C\left(\|\varphi_1(t) - \varphi_2(t)\|^2_W + \|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)}\right.\\ &\left.+ \int_0^t \|\sigma_1(s) - \sigma_2(s)\|^2_{\mathscr{H}}ds + \int_0^t \|u_1(s) - u_2(s)\|^2_V ds\right).\end{aligned} \tag{2.75}$$

We use estimates (2.70), (2.59) to obtain

$$\begin{aligned}\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|^2_{\mathscr{H}} \leq C\left(\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} + \|u_1(s) - u_2(s)\|^2_V\right.\\ \left.+ \int_0^t \|u_1(s) - u_2(s)\|^2_V ds\right).\end{aligned} \tag{2.76}$$

Using inequality (2.52) for $\eta = \eta_1$, we find

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{u}_1(t)),\varepsilon(v) - \varepsilon(\dot{u}_1(t)))_{\mathscr{H}} + (\mathscr{B}\varepsilon(u_1(t)),\varepsilon(v) - \varepsilon(\dot{u}_1(t)))_{\mathscr{H}}\\ &+ (\eta_1(t),\varepsilon(v) - \varepsilon(\dot{u}_1(t)))_{\mathscr{H}} + j(\dot{u}_1(t),v) - j(\dot{u}_1(t),\dot{u}_1(t))\\ &\qquad \geq (f(t),v - \dot{u}_1(t))_V, \forall v \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{2.77}$$

for $\eta = \eta_2$, we find

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{u}_2(t)),\varepsilon(v) - \varepsilon(\dot{u}_2(t)))_{\mathscr{H}} + (\mathscr{B}\varepsilon(u_2(t)),\varepsilon(v) - \varepsilon(\dot{u}_2(t)))_{\mathscr{H}}\\ &+ (\eta_2(t),\varepsilon(v) - \varepsilon(\dot{u}_2(t)))_{\mathscr{H}} + j(\dot{u}_2(t),v) - j(\dot{u}_2(t),\dot{u}_2(t))\\ &\qquad \geq (f(t),v - \dot{u}_2(t))_V, \forall v \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{2.78}$$

we take $v=\dot{u}_2(t)$ in (2.77) and $v=\dot{u}_1(t)$ in (2.78), add the two inequalities to obtain

$$\begin{aligned}(\mathscr{A}\varepsilon(\dot{u}_1(t))-\mathscr{A}\varepsilon(\dot{u}_2(t)),\varepsilon(\dot{u}_1(t))-\varepsilon(\dot{u}_2(t)))_{\mathscr{H}}\\ \leq(\mathscr{B}\varepsilon(u_1(t))-\mathscr{B}\varepsilon(u_2(t)),\varepsilon(\dot{u}_2(t))-\varepsilon(\dot{u}_1(t)))_{\mathscr{H}}\\ +(\eta_1(t)-\eta_2(t),\varepsilon(\dot{u}_2(t))-\varepsilon(\dot{u}_1(t)))+j(\dot{u}_1(t),\dot{u}_2(t))\\ -j(\dot{u}_1(t),\dot{u}_1(t))+j(\dot{u}_2(t),\dot{u}_1(t))-j(\dot{u}_2(t),\dot{u}_2(t)),\end{aligned}$$

then we use assumptions (2.15), (2.16) and (2.22) to find

$$\begin{aligned}m_{\mathscr{A}}\|\dot{u}_1-\dot{u}_2\|_V^2\leq&L_{\mathscr{B}}\|u_1-u_2\|_V\|\dot{u}_1-\dot{u}_2\|_V+\|\eta_1-\eta_2\|_{\mathscr{H}}\|\dot{u}_1-\dot{u}_2\|_V\\ &+c_0^2(L_\nu+L_\tau)\|\dot{u}_1-\dot{u}_2\|_V^2\end{aligned}\tag{2.79}$$

It follows that

$$\|\dot{u}_1-\dot{u}_2\|_V\leq C(\|u_1-u_2\|_V+\|\eta_1-\eta_2\|_{\mathscr{H}}).\tag{2.80}$$

Since $u_i(t)=\int_0^t\dot{u}_i(s)ds+u_0,\forall t\in[0,T]$, we have

$$\|u_1(t)-u_2(t)\|_V\leq\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|_V\,ds.\tag{2.81}$$

Using (2.80), (2.81) and the Gronwall's inequality, we find

$$\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|_V\,ds\leq\int_0^t\|\eta(s)-\eta(s)\|_{\mathscr{H}}ds,\tag{2.82}$$

Then, we find

$$\|u_1(t)-u_2(t)\|_V\leq C\int_0^t\|\eta_1-\eta_2\|_{\mathscr{H}}\,ds,\quad t\in[0,T].\tag{2.83}$$

Let $\theta_{\eta_i}=\theta_i$, and $u_{\eta_i}=u_i$ for $i=1,2$. Let $t\in\mathbb{R}^+$ be fixed. Then, we have

$$\begin{aligned}&\left(\dot{\theta}_1(t)-\dot{\theta}_2(t),\theta_1(t)-\theta_2(t)\right)_{\mathscr{V}'\times\mathscr{V}}+(K\theta_1(t)-K\theta_2(t),\theta_1(t)-\theta_2(t))_{\mathscr{V}'\times\mathscr{V}}\\ &\quad=(R\dot{\mathbf{u}}_1(t)-R\dot{\mathbf{u}}_2(t),\theta_1(t)-\theta_2(t))_{\mathscr{V}'\times\mathscr{V}}.\end{aligned}$$

We integrate the above equality over $(0,t)$ and we use the strong monotonicity of K and the Lipschitz continuity of $R:V\to\mathscr{V}'$ to deduce that

$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2ds\leq C\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|_V^2ds,$$

It follows now from (2.82), that

$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2\leq C\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}}^2\,ds,\quad\forall t\in[0,T].$$

Form the previous inequality and estimates (2.83) and (2.76) it follows now that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{\mathscr{H}}^2 \leq C\int_0^T \|\eta_1(s) - \eta_2(s)\|_{\mathscr{H}}^2\, ds. \tag{2.84}$$

Reiterating this inequality m times we obtain

$$\|\Lambda^m\eta_1 - \Lambda^m\eta_2\|_{C(0,T;\mathscr{H})}^2 \leq \frac{C^m T^m}{m!}\|\eta_1 - \eta_2\|_{C(0,T;\mathscr{H})}^2.$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathscr{H})$, and so Λ has a unique fixed point.

□

We are now ready to prove Theorem 2.1.

Existence

Let $\eta^* \in C(0,T;\mathscr{H})$ be the fixed point of Λ and

$$u = u_{\eta^*}, \quad \theta = \theta_{\eta^*}, \quad \varphi_{\eta^*} = \varphi, \tag{2.85}$$

$$\sigma = \mathscr{A}\varepsilon(\dot{u}) + \mathscr{E}^*\nabla\varphi(t) + \sigma_{\eta^*}, \tag{2.86}$$

$$D = \mathscr{E}\varepsilon(u) + B\nabla(\varphi). \tag{2.87}$$

We prove that $(u,\sigma,\theta,\varphi,D)$ satisfies (2.41)-(2.45) and (2.47)-(2.51). Indeed, we write (2.69) for $\eta^* = \eta$ and use (2.85)-(2.86) to obtain that (2.41) is satisfied. Now we consider (2.52) for $\eta^* = \eta$ and use (2.85) to find

$$\begin{gathered}(\mathscr{A}\varepsilon(\dot{u}(t)),\varepsilon(v-\dot{u}(t)))_{\mathscr{H}} + (\mathscr{B}\varepsilon(u(t)),\varepsilon(v)-\varepsilon(\dot{u}(t)))_{\mathscr{H}}\\ +(\eta^*(t),v-\dot{u}(t))_{\mathscr{H}} + j(\dot{u}(t),v) - j(\dot{u}(t),\dot{u}(t)) \geq (f(t),v-\dot{u}(t))_V\\ \forall v\in V, t\in[0,T].\end{gathered} \tag{2.88}$$

The equalities $\Lambda\eta^* = \eta^*$ combined with (2.74), (2.85) and (2.86) show that for all $v \in V$,

$$\begin{aligned}(\eta^*(t),v)_{\mathscr{H}\times V} &= (\mathscr{E}^*\nabla\varphi(t),\varepsilon(v))_{\mathscr{H}} - (C_e\theta(t),\varepsilon(v))_{\mathscr{H}},\\ &+\left(\int_0^t \mathscr{G}(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s)) - \mathscr{E}^*\nabla\varphi(t),\varepsilon(u(s)))ds,\varepsilon(v)\right)_{\mathscr{H}},\end{aligned} \tag{2.89}$$

We substitute (2.89) in (2.88)) and use (2.41) to see that (2.42) is satisfied. We write now (2.58) for $\eta = \eta^*$ and use (2.85) to find (2.43). From (2.62) and (2.85) we see that (2.44) is satisfied. Next, (2.45), The regularities (2.47), (2.48), (2.49) and (2.50) follow from Lemmas 2.2, 2.3, 2.4 and 2.5. Let now $t_1,t_2 \in [0,T]$, from (1.11), (2.20), (2.21) and (2.87), we conclude that there exists a positive constant $C > 0$ verifying

$$\|D(t_1) - D(t_2)\|_H \leq C(\|\varphi(t_1) - \varphi(t_2)\|_W + \|u(t_1) - u(t_2)\|_V).$$

The regularity of u and φ given by (2.47) and (2.48) implies

$$D \in C(0,T;H). \tag{2.90}$$

We choose $\phi \in D(\Omega)^d$ in (2.43) and using (2.31) we find

$$\operatorname{div} D(t) = q_0(t), \quad \forall t \in [0,T], \tag{2.91}$$

Property $D \in C(0,T;\mathscr{W})$ follows from (2.26),(2.90) and (2.91) which concludes the existence part the Theorem 2.1.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ. and the unique solvability of the Problems $\mathscr{P}_\eta^1$, $\mathscr{P}_\eta^2$, $\mathscr{P}_\eta^3$ and $\mathscr{P}_\eta^4$ which completes the proof.

Chapter 3

Thermo-electro-elastic-viscoplastic contact problem with damage

This chapter is devoted to the mathematical study of a quasi-static contact problem between a thermo electro elastic-viscoplastic body with damage, and an obstacle, the so-called foundation. we assume that the normal stress is prescribed on the contact surface and we use the quasistatic version of Coulombs law of dry friction. After having established the variational formulation and having posed the necessary hypotheses, we establish a result of existence and uniqueness of the solution. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

3.1 Formulation of the problem

We place ourselves in the physical framework that we presented in Chapter 1 of this thesis. The classical formulation of the mechanical problem may be stated as follows

Problem $P2$

Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$, an electric displacement field $D : \Omega \times (0,T) \to \mathbb{R}^d$, and a damage field $\alpha : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned}
&\sigma = \mathscr{A}\varepsilon(\dot{u}) + \mathscr{B}(\varepsilon(u),\alpha) - \mathscr{E}^* E(\varphi) \\
&+ \int_0^t \mathscr{G}(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s)) + \mathscr{E}^* E(\varphi)(s), \varepsilon(u(s)))\,ds - C_e\theta \qquad \text{in } \Omega \times (0,T), && (3.1)\\
&D = \mathscr{E}\varepsilon(u) + BE(\varphi) \qquad \text{in } \Omega \times (0,T), && (3.2)\\
&\dot{\theta} - \operatorname{div} K(\Delta\theta) = r(\dot{u},\alpha) + \mathbf{q}, \quad \text{in } \Omega \times (0,T), && (3.3)\\
&\dot{\alpha} - k\Delta\alpha + \partial\varphi_{\mathscr{Y}}(\alpha) \ni S(\varepsilon(u),\alpha), \quad \text{in } \Omega \times (0,T), && (3.4)\\
&\operatorname{Div}\sigma + f_0 = 0 \qquad \text{in } \Omega \times (0,T), && (3.5)\\
&\operatorname{div} D - q_0 = 0 \qquad \text{in } \Omega \times (0,T), && (3.6)\\
&u = 0 \qquad \text{on } \Gamma_1 \times (0,T), && (3.7)\\
&\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0,T), && (3.8)\\
&-\sigma_\nu = F \quad \text{on } \Gamma_3 \times (0,T) && (3.9)\\
&\begin{cases} \|\sigma_\tau\| \le \mu\,|\sigma_\nu| \\ \sigma_\tau = -\mu\,|\sigma_\nu| \dfrac{\dot{u}_\tau}{\|\dot{u}_\tau\|} \quad \text{if} \quad \dot{u}_\tau \neq 0 \end{cases} \quad \text{on } \Gamma_3 \times (0,T), && (3.10)\\
&-k_{ij}\frac{\partial\theta}{\partial x_i}\nu_j = k_e(\theta - \theta_R) + h_\tau(|\dot{u}_\tau|) \quad \text{on } \Gamma_3 \times (0,T), && (3.11)\\
&\frac{\partial\alpha}{\partial\nu} = 0 \qquad \text{on } \Gamma \times (0,T), && (3.12)\\
&\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), && (3.13)\\
&D.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), && (3.14)\\
&\theta = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), && (3.15)\\
&u(0) = u_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0, \qquad \text{in } \Omega. && (3.16)
\end{aligned}$$

First, equations (3.1)-(3.4) represent the electro-elastic-viscoplastic constitutive law with damage and thermal effects. Equations (3.5) and (3.6) represent the equilibrium equations for the stress and electric displacement fields. Equations (3.7)-(3.8) are the displacement-traction conditions. Frictional contact conditions of the form (3.9) and (3.10) describe the contact on the surface Γ_3, where F is a given positive function. The above relations assert that the tangential stress is bounded by the normal stress multiplied by the value of the friction coefficient μ, (3.11), (3.12) represent, respectively on Γ, a Fourier boundary condition for the temperature and an homogeneous Neumann boundary condition for the damage field on Γ. (3.13) and (3.14) represent the electric boundary conditions. Equation (3.15) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$. Finally, The functions u_0, θ_0 and α_0 in (3.16) are the initial data.

3.2 Variational formulation

Before establishing a variational formulation to the problem $P2$, we need some assumptions. We assume that the operator $\mathscr{A}$, the visco-plasticity operator $\mathscr{G}$, The thermal expansion operator C_e, The thermal conductivity operator K, Electric permittivity operator B, The piezoelectric operator $\mathscr{E}$, and the tangential function h_τ verify the hypotheses (2.15),(2.17), (2.18), (2.19), (2.20), (2.21) and (2.22) of the previous chapter. The elasticity operator $\mathscr{B}:\Omega\times\mathbb{S}^d\times\mathbb{R}\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}\text{(a) There exists } L_{\mathscr{B}}>0 \quad \text{such that}\\ \|\mathscr{B}(x,\omega_1,\alpha_1)-\mathscr{B}(x,\omega_2,\alpha_2)\|\leq L_{\mathscr{B}}(\|\omega_1-\omega_2\|+\|\alpha_1-\alpha_2\|),\\ \text{for all } \omega_1,\omega_2\in\mathbb{S}^d,\alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e. } x\in\Omega.\\ \text{(b) The mapping } x\mapsto\mathscr{B}(x,\omega,\alpha) \text{ is Lebesgue measurable on } \Omega,\\ \qquad\qquad \text{for all } \omega\in\mathbb{S}^d,\alpha\in\mathbb{R}.\\ \text{(c) The mapping } x\mapsto\mathscr{B}(x,0,0)\in\mathscr{H}.\end{cases} \tag{3.17}$$

The function $S:\Omega\times\mathbb{S}^d\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\begin{cases}\text{(a) There exists a constant } L_S>0 \quad \text{such that}\\ \|S(x,\omega_1,\alpha_1)-S(x,\omega_2,\alpha_2)\|\leq L_S(\|\omega_1-\omega_2\|+\|\alpha_1-\alpha_2\|),\\ \text{for all } \omega_1,\omega_2\in\mathbb{S}^d,\ \text{for all } \alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e. } x\in\Omega.\\ \text{(b) The mapping } x\mapsto S(x,\omega,\alpha) \text{ is Lebesgue measurable on } \Omega,\\ \qquad\qquad \text{for all } \omega\in\mathbb{S}^d,\ \text{for all } \alpha\in\mathbb{R}.\\ \text{(c) The mapping } x\mapsto S(x,0,0)\in L^2(\Omega).\end{cases} \tag{3.18}$$

We assume that the friction coefficient μ, the normal stress F, the boundary and initial data θ_R, k_e, α_0, u_0 and θ_0 the volume of forces f_0 and f_2 and the charges densities q_0, q_2 the heat source density $\mathbf{q}$ the microcrack diffusion coefficient k_0 satisfy

$$\begin{aligned}&\mu\in L^\infty(\Gamma_3),\ \mu\geq 0 \text{ a.e. on } \Gamma_3,\\ &F\in L^2(\Gamma_3),\ F\geq 0 \text{ a.e. on } \in\Gamma_3,\end{aligned} \tag{3.19}$$

$$\theta_R\in C(0,T;L^2(\Gamma_3)),\ k_e\in L^\infty(\Omega,\mathbb{R}_+), \tag{3.20}$$

$$u_0\in V,\quad \alpha_0\in K,\quad \theta_0\in\mathscr{V}, \tag{3.21}$$

$$f_0\in C\left(0,T;L^2(\Omega)^d\right),\ f_2\in C\left(0,T;L^2(\Gamma_2)^d\right), \tag{3.22}$$

$$q_0\in C\left(0,T;L^2(\Omega)\right),\ q_2\in C\left(0,T;L^2(\Gamma_b)\right), \tag{3.23}$$

$$k_0>0,\quad \mathbf{q}\in C\left(0,T;L^2(\Omega)\right). \tag{3.24}$$

The function $r:V\times\mathbb{R}\to L^2(\Omega)$ satisfies that there exists a constant $L_r>0$ such that

$$\begin{aligned}\|r(u_1,\xi_1)-r(u_2,\xi_2)\|_{L^2(\Omega)}\leq L_r(\|u_1-u_2\|_V+\|\xi_1-\xi_2)\|)\\ \forall u_1,u_2\in V,\quad \xi_1,\xi_2\in\mathbb{R}.\end{aligned} \tag{3.25}$$

We introduce the following bilinear form $a:H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$, by

$$a(\zeta,\xi)=k_0\int_\Omega\nabla\zeta\cdot\nabla\xi dx. \tag{3.26}$$

Next. We define six mappings $j : V \to \mathbb{R}$, $f : [0,T] \to V$, $q : [0,T] \to W$, $Q : [0,T] \to \mathscr{V}'$, $K : \mathscr{V} \to \mathscr{V}'$, and $R : V \to \mathscr{V}'$ respectively, by

$$j(v) = \int_{\Gamma_3} \mu F \|v_\tau\| da, \quad \forall v \in V, \tag{3.27}$$

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v da, \tag{3.28}$$

$$(q(t), \zeta)_W = \int_\Omega q_0(t) \zeta dx - \int_{\Gamma_b} q_2(t) \zeta da, \tag{3.29}$$

$$(Q(t), \eta)_{\mathscr{V}' \times \mathscr{V}} = \int_{\Gamma_3} k_e \theta_R(t) \eta da + \int_\Omega \mathbf{q}(t) \eta dx. \tag{3.30}$$

$$(K\tau, \eta)_{\mathscr{V}' \times \mathscr{V}} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \eta da. \tag{3.31}$$

$$(Rv, \eta)_{\mathscr{V}' \times \mathscr{V}} = \int_\Omega r(v) \eta dx + \int_{\Gamma_3} h_\tau(|v_\tau|) \eta da. \tag{3.32}$$

for all $u, v \in V$, $\zeta \in W$, $\eta, \tau \in \mathscr{V}$ and $t \in [0,T]$. Note that

$$f \in C(0,T;V), \quad q \in C(0,T;W). \tag{3.33}$$

Similar to before. Using standard arguments based on Green's formula (1.4) and (1.14) , we obtain the following variational formulation (3.1)-(3.16).

Problem *PV*

Find a displacement field $u : (0,T) \to V$, a stress field $\sigma : (0,T) \to \mathscr{H}$, an electric potential $\varphi : (0,T) \to W$, a damage field $\alpha : (0,T) \to H^1(\Omega)$, and a temperature $\theta : (0,T) \to \mathscr{V}$ such that

$$\begin{aligned} &\sigma(t) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}(\varepsilon(u(t)), \alpha(t)) + \mathscr{E}^* \nabla \varphi(t) \\ &+ \int_0^t \mathscr{G}(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s)) - \mathscr{E}^* \nabla \varphi(s), \varepsilon(u(s))) ds - C_e \theta(t), \end{aligned} \tag{3.34}$$

$$D = \mathscr{E}\varepsilon(u) - B\nabla(\varphi), \tag{3.35}$$

$$(\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))_{\mathscr{H}} + j(v) - j(\dot{u}(t)) \geq (\mathbf{f}(t), v - \dot{u}(t))_V, \tag{3.36}$$

$$(B\nabla\varphi(t), \nabla\phi)_H - (\mathscr{E}\varepsilon(u(t)), \nabla\phi)_H = (q(t), \phi)_W, \quad \forall \phi \in W, t \in (0,T) \tag{3.37}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t), \quad \text{in } \mathscr{V}', \tag{3.38}$$

$$\begin{aligned} &\alpha(t) \in K, (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\geq (S(\varepsilon(u(t)), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)}, \forall \xi \in K, t \in (0,T), \end{aligned} \tag{3.39}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0. \tag{3.40}$$

Our main existence and uniqueness result for Problem *PV* is in the following section.

3.3 Result of existence and uniqueness

Theorem 3.1. *Assume that (3.17)-(3.25) hold, Then there exists a unique solution* $(u,\sigma,\varphi,\theta,\alpha,D)$ *to problem* PV. *Moreover, the solution has the regularity*

$$u \in C^1(0,T;V), \tag{3.41}$$
$$\varphi \in C(0,T;W), \tag{3.42}$$
$$\sigma \in C(0,T;\mathscr{H}), \tag{3.43}$$
$$\theta \in C\left(0,T;L^2(\Omega)\right) \cap L^2(0,T;\mathscr{V}) \cap W^{1,2}\left(0,T;\mathscr{V}'\right), \tag{3.44}$$
$$\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{3.45}$$
$$D \in C(0,T;\mathscr{W}). \tag{3.46}$$

The functions u, σ, φ, θ, α, and D which satisfy (3.34)-(3.40) are called a weak solution of the contact problem P. We conclude that, under the assumptions (3.17)-(3.25), the mechanical problem (3.1)-(3.16) has a unique weak solution satisfying (3.41)-(3.46). The proof of Theorem 3.1, is carried out is several steps and is based on arguments of evolutionary variational inequalities, parabolic inequalities, differential equations and fixed points. We denote by C a constant whose value may change from line to line when no confusing can arise. Let $\eta \in C(0,T;\mathscr{H})$ and $\lambda \in C(0,T;L^2(\Omega))$ we consider the following variational problem.

Problem $\mathscr{P}^1_\eta$

Find a displacement field $u_\eta : [0,T] \to V$ such that for all $t \in [0,T]$

$$\begin{aligned}
&\left(\mathscr{A}\varepsilon\left(\dot{u}_\eta(t)\right), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t))\right)_{\mathscr{H}} + \left(\mathscr{B}\left(\varepsilon\left(u_\eta(t)\right), \alpha_\lambda(t)\right), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t))\right)_{\mathscr{H}} \\
&+ (\eta(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)))_{\mathscr{H}} + j(v) - j(\dot{u}_\eta(t)) \geq (f(t), v - \dot{u}_\eta(t))_V, \qquad (3.47)\\
&\forall v \in V, \text{ a.e. } t \in (0,T),
\end{aligned}$$

$$u_\eta(0) = u_0. \tag{3.48}$$

We have the following result for $\mathscr{P}^1_\eta$

Lemma 3.2. *(1) There exists a unique solution* $u_\eta \in C^1(0,T;V)$ *to the problem (3.47) and (3.48).*

(2) If u_1 *and* u_2 *are two solutions of (3.47) and (3.48) corresponding to the data* η_1, $\eta_2 \in C([0,T];\mathscr{H})$, *then there exists* $C > 0$ *such that*

$$\|u_1(t) - u_2(t)\|_V \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathscr{H}}\, ds, \quad t \in [0,T]. \tag{3.49}$$

Proof. Let us introduce operators $A : V \to V$ and $B : V \times H^1(\Omega) \to V$

$$(Au, v)_V = (\mathscr{A}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \quad \forall u, v \in V, \tag{3.50}$$
$$(B(u,\alpha), v)_V = (\mathscr{B}(\varepsilon(u),\alpha)\varepsilon(v))_{\mathscr{H}}, \quad \forall u, v \in V, \quad \alpha \in H^1(\Omega). \tag{3.51}$$

Therefore, (3.47) can be rewritten as follows

$$\begin{aligned}(A\dot{u}(t), v-\dot{u}(t))_V + (B(u(t),\alpha(t)), v-\dot{u}(t))_V + j(v)\\ - j(u(t)) \geq (\mathbf{f}_\eta(t), v-\dot{u}(t))_V\,,\end{aligned} \tag{3.52}$$

where

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \eta(t), \quad \text{a.e.t } \in [0,T].$$

We use assumption (2.15) to see that A is a strongly monotone Lipschitz continuous operator. Also, it follows from (3.17) that B is a Lipschitz continuous operator and we use (1.7) to see that the functional j defined in (3.27) satisfies

$$j(v) \leq c_0 \|\mu\|_{L^\infty(\Gamma_3)} \|F\|_{L^2(\Gamma_3)} \|v\|_V, \quad \forall v \in V,$$

so the seminorm j is continuous and, therefore, it is a convex lower semicontinuous function on V. Finally, note that $\mathbf{f}_\eta \in C([0,T];V)$ and $u_0 \in V$ and we use classical arguments of functional analysis concerning evolutionary variational inequalities [10, 56] we can easily prove the existence and uniqueness of u_η satisfying (3.41). Using inequality (3.47) for $\eta = \eta_1$, $u_{\eta_1} = u_1$, $\dot{u}_{\eta_1} = \dot{u}_1$, we find

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{u}_1(t)), \varepsilon(v) - \varepsilon(\dot{u}_1(t)))_{\mathscr{H}} + (\mathscr{B}(\varepsilon(u_1(t)), \alpha_\lambda(t)), \varepsilon(v) - \varepsilon(\dot{u}_1(t)))_{\mathscr{H}}\\ &+ (\eta_1(t), \varepsilon(v) - \varepsilon(\dot{u}_1(t)))_{\mathscr{H}} + j(v) - j(\dot{u}_1(t)) \geq (f(t), v - \dot{u}_1(t))_V\,,\\ &\forall v \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{3.53}$$

for $\eta = \eta_2$, $u_{\eta_2} = u_2$, $\dot{u}_{\eta_2} = \dot{u}_2$, we find

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{u}_2(t)), \varepsilon(v) - \varepsilon(\dot{u}_2(t)))_{\mathscr{H}} + (\mathscr{B}(\varepsilon(u_2(t)), \alpha_\lambda(t)), \varepsilon(v) - \varepsilon(\dot{u}_2(t)))_{\mathscr{H}}\\ &+ (\eta_2(t), \varepsilon(v) - \varepsilon(\dot{u}_2(t)))_{\mathscr{H}} + j(v) - j(\dot{u}_2(t)) \geq (f(t), v - \dot{u}_2(t))_V\,,\\ &\forall v \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{3.54}$$

we take $v = \dot{u}_2(t)$ in (3.53) and $v = \dot{u}_1(t)$ in (3.54), add the two inequalities to obtain

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{u}_1(t)) - \mathscr{A}\varepsilon(\dot{u}_2(t)), \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t)))_{\mathscr{H}}\\ &\qquad \leq (\mathscr{B}(\varepsilon(u_1(t)), \alpha_\lambda(t)) - \mathscr{B}(\varepsilon(u_2(t)), \alpha_\lambda(t)), \varepsilon(\dot{u}_2(t)) - \varepsilon(\dot{u}_1(t)))_{\mathscr{H}}\\ &\qquad + (\eta_1(t) - \eta_2(t), \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t))),\end{aligned}$$

then we use assumptions (2.15) and (3.17) to derive

$$\|\dot{u}_1 - \dot{u}_2\|_V \leq C(\|u_1 - u_2\|_V + \|\eta_1 - \eta_2\|_{\mathscr{H}})\,. \tag{3.55}$$

Since $u_i(t) = \int_0^t \dot{u}_i(s)ds + u_0, \forall t \in [0,T]$, we have

$$\|u_1(t) - u_2(t)\|_V \leq \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V\, ds. \tag{3.56}$$

Using (3.55), (3.56) and the Gronwall's inequality, we find

$$\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V\, ds \leq \int_0^t \|\eta(s) - \eta(s)\|_{\mathscr{H}} ds, \tag{3.57}$$

which concludes the proof of Lemma 3.2.

□

In the next step we use the solution u_η, obtained in Lemma 3.2, to construct the following variational problem for the electrical potential.

Problem $\mathscr{P}_\eta^2$

Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$(B\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathscr{E}\varepsilon(u_\eta(t)), \nabla\zeta)_H = (q(t), \zeta)_W, \text{ for all } \zeta \in W, t \in (0,T). \tag{3.58}$$

We have the following result for problem $\mathscr{P}_\eta^2$

Lemma 3.3. *Problem (3.58) has unique solution φ_η which satisfies the regularity (3.42). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (3.58) corresponding to $\eta_1, \eta_2 \in C([0,T]; \mathscr{H})$, then there exists $C > 0$ such that*

$$\left\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\right\|_W \leq C \left\|u_{\eta_1}(t) - u_{\eta_2}(t)\right\|_V, \quad \forall t \in [0,T]. \tag{3.59}$$

Proof. See the proof of Lemma 2.3 (Chapter 2)

□

In the third step, we use the displacement field u_η obtained in Lemma 3.2 to consider the following variational problem.

Problem $\mathscr{P}_\eta^3$

Find the temperature field $\theta_\eta : (0,T) \to L^2(\Omega)$

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t), \quad \text{in } \mathscr{V}', \quad \text{a.e.t } \in [0,T], \tag{3.60}$$

$$\theta_\eta(0) = \theta_0. \tag{3.61}$$

Lemma 3.4. *There exists a unique solution θ_η to the auxiliary problem $\mathscr{P}_\eta^3$ satisfying (3.44). Moreover $\exists C > 0$ such that $\forall \eta_1, \eta_2 \in C(0,T; \mathscr{H})$.*

$$\left\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\right\|_{L^2(\Omega)}^2 \leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathscr{H}}^2 ds, \quad \forall t \in [0,T]. \tag{3.62}$$

Proof. The result follows from classical first order evolution equation given in Refs. [6, 55]. (See the proof of Lemma 2.4 (Chapter 2)). Here the Gelfand triple is given by

$$\mathscr{V} \subset L^2(\Omega) = \left(L^2(\Omega)\right)' \subset \mathscr{V}'.$$

The operator K is linear and coercive. By Korn's inequality, we have

$$(K\tau, \tau)_{\mathscr{V}' \times \mathscr{V}} \geq C\|\tau\|_{\mathscr{V}}^2.$$

Let $\theta_{\eta_i} = \theta_i$, and $u_{\eta_i} = u_i$ for $i = 1,2$. Let $t \in \mathbb{R}^+$ be fixed. Then, we have

$$\begin{aligned}&\left(\dot{\theta}_1(t) - \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{\mathscr{V}' \times \mathscr{V}} + (K\theta_1(t) - K\theta_2(t), \theta_1(t) - \theta_2(t))_{\mathscr{V}' \times \mathscr{V}}\\ &\quad = (R\dot{u}_1(t) - R\dot{u}_2(t), \theta_1(t) - \theta_2(t))_{\mathscr{V}' \times \mathscr{V}}.\end{aligned}$$

We integrate the above equality over $(0,t)$ and we use the strong monotonicity of K and the Lipschitz continuity of $R : V \to \mathscr{V}'$ to deduce that

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}ds \leq C\int_0^t \|\dot u_1(s)-\dot u_2(s)\|^2_V ds,$$

It follows now from (3.57), that

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)} \leq C\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_{\mathscr{H}} ds, \quad \forall t\in[0,T].$$

□

In the fourth step we let $\lambda \in C(0,T;L^2(\Omega))$

Problem $\mathscr{P}_\lambda$

Find the damage field $\alpha_\lambda : (0,T)\to L^2(\Omega)$ such that $\alpha_\lambda(t)\in\mathscr{Y}$ and

$$\begin{aligned}&(\dot\alpha_\lambda(t),\xi-\alpha_\lambda)_{L^2(\Omega)}+a(\alpha_\lambda(t),\xi-\alpha_\lambda(t))\\ &\qquad\geq(\lambda(t),\xi-\alpha_\lambda(t))_{L^2(\Omega)} \quad \forall\xi\in\mathscr{Y},\ \text{a.e. } t\in(0,T),\end{aligned} \tag{3.63}$$

$$\alpha_\lambda(0)=\alpha_0. \tag{3.64}$$

For the study of problem $\mathscr{P}_\lambda$, we have the following result.

Lemma 3.5. *There exists a unique solution α_λ to the auxiliary problem $\mathscr{P}_\lambda$ satisfying (3.45).*

Proof. The inclusion mapping of $\left(H^1(\Omega),\|\cdot\|_{H^1(\Omega)}\right)$ into $\left(L^2(\Omega),\|\cdot\|_{L^2(\Omega)}\right)$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega)\subset L^2(\Omega)\subset\left(H^1(\Omega)\right)'.$$

We use the notation $(.,.)_{(H^1(\Omega))'\times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $(H^1(\Omega))$. We have

$$(\alpha,\zeta)_{(H^1(\Omega))'\times H^1(\Omega)}=(\alpha,\zeta)_{L^2(\Omega)}, \quad \forall\alpha\in L^2(\Omega),\zeta\in H^1(\Omega),$$

and we note that K is a closed convex set in $(H^1(\Omega))$, using the definition (3.26) of the bilinear form a, for all $\psi,\zeta\in H^1(\Omega)$, we have $a(\psi,\zeta)=a(\zeta,\psi)$ and

$$|a(\psi,\zeta)|\leq k\|\nabla\psi\|_H\|\nabla\zeta\|_H\leq c\|\psi\|_{H^1(\Omega)}\|\zeta\|_{H^1(\Omega)},$$

Therefore, a is continuous and symmetric. Thus, for all $\psi\in H^1(\Omega)$, we have

$$a(\psi,\psi)=k\|\nabla\psi\|^2_H,$$

so

$$a(\psi,\psi)+(k+1)\|\psi\|^2_{L^2(\Omega)}\geq k\left(\|\nabla\psi\|^2_H+\|\psi\|^2_{L^2(\Omega)}\right),$$

which implies

$$a(\psi,\psi)+c_0\|\psi\|^2_{L^2(\Omega)} \geq c_1\|\psi\|^2_{H^1(\Omega)} \text{ with } c_0=k+1 \text{ and } c_1=k.$$

Finally, we use that condition $\alpha_0 \in K$, and we use a standard result for parabolic variational inequalities (see [6], p. 124), we find that there exists a unique function $\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)$, such that $\alpha(0)=\alpha_0$, $\alpha(t)\in K$ for all $t\in[0,T]$ and for almost all $t\in(0,T)$

$$\begin{aligned}&(\dot{\alpha}_\lambda(t),\zeta-\alpha_\lambda)_{(H^1(\Omega))'\times H^1(\Omega)}+a(\alpha_\lambda(t),\zeta-\alpha_\lambda(t))\geq(\lambda(t),\zeta-\alpha_\alpha(t))_{L^2(\Omega)},\\&\forall\zeta\in K,\end{aligned}$$

□

In the fifth step, we use u_η, φ_η, θ_η and α_λ obtained in Lemmas 3.2, 3.3, 3.4 and 3.5, respectively to construct the following Cauchy problem for the stress field.

Problem $\mathscr{P}_{\eta,\lambda}$

Find the stress field $\sigma_{\eta,\lambda}:[0,T]\to\mathscr{H}$ which is a solution of the problem

$$\begin{gathered}\sigma_{\eta,\lambda}(t)=\mathscr{B}(\varepsilon(u_\eta(t),\alpha_\lambda(t)))+\int_0^t\mathscr{G}\left(\sigma_{\eta,\lambda}(s),\varepsilon(u_\eta(s))\right)ds-C_e\theta_\eta(t),\\ \text{a.e. } t\in(0,T).\end{gathered}\tag{3.65}$$

Lemma 3.6. *$\mathscr{P}_{\eta,\lambda}$ has a unique solutions $\sigma_{\eta,\lambda}\in C(0,T;\mathscr{H})$. Moreover, if $\sigma_{\eta_i,\lambda_i}$, u_{η_i}, θ_{η_i} and α_{λ_i} represent the solutions of Problems $\mathscr{P}_{\eta,\lambda}$, $\mathscr{P}^1_\eta$, $\mathscr{P}^3_\eta$ and, $\mathscr{P}_\lambda$ respectively, for $(\eta_i,\lambda_i)\in C(0,T;\mathscr{H}\times L^2(\Omega))$, $i=1,2$, then there exists $C>0$ such that*

$$\begin{aligned}\left\|\sigma_{\eta_1,\lambda_1}(t)-\sigma_{\eta_2,\lambda_2}(t)\right\|^2_{\mathscr{H}}\leq&C\Big(\left\|u_{\eta_1}(t)-u_{\eta_2}(t)\right\|^2_V+\left\|\alpha_{\lambda_1}(t)-\alpha_{\lambda_2}(t)\right\|^2_{L^2(\Omega)}\\&+\left\|\theta_{\eta_1}(t)-\theta_{\eta_2}(t)\right\|^2_{L^2(\Omega)}+\int_0^t\left\|u_{\eta_1}(s)-u_{\eta_2}(s)\right\|^2_V\Big).\end{aligned}\tag{3.66}$$

Proof. Let $\Sigma_{\eta,\lambda}:C(0,T;\mathscr{H})\to C(0,T;\mathscr{H})$ be the operator given by

$$\Sigma_{\eta,\lambda}\sigma(t)=\mathscr{B}(\varepsilon(u_\eta(t),\alpha_\lambda(t)))+\int_0^t\mathscr{G}\left(\sigma_{\eta,\lambda}(s),\varepsilon(u_\eta(s))\right)ds-C_e\theta_\eta(t),\tag{3.67}$$

Let $\sigma_i\in W^{1,\infty}(0,T;\mathscr{H})$, $i=1,2$ and $t_1\in(0,T)$. Using hypothesis (2.17) and Holder's inequality, we find

$$\left\|\Sigma_{\eta,\lambda}\sigma_1(t_1)-\Sigma_{\eta,\lambda}\sigma_2(t_1)\right\|^2_{\mathscr{H}}\leq L^2_{\mathscr{G}}T\int_0^{t_1}\|\sigma_1(s)-\sigma_2(s)\|^2_{\mathscr{H}}ds.$$

Integration on the time interval $(0,t_2)\subset(0,T)$, it follows that

$$\int_0^{t_2}\left\|\Sigma_{\eta,\lambda}\sigma_1(t_1)-\Sigma_{\eta,\lambda}\sigma_2(t_1)\right\|^2_{\mathscr{H}}dt_1\leq L^2_{\mathscr{G}}T\int_0^{t_2}\int_0^{t_1}\|\sigma_1(s)-\sigma_2(s)\|^2_{\mathscr{H}}dsdt_1.$$

Therefore

$$\left\|\Sigma_{\eta,\lambda}\sigma_1(t_2)-\Sigma_{\eta,\lambda}\sigma_2(t_2)\right\|_{\mathscr{H}}^2 \le L_{\mathscr{G}}^4 T^2 \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2\, ds dt_1.$$

For $t_1,t_2,\dots,t_n \in (0,T)$, we generalize the procedure above by recurrence on n. We obtain the inequality

$$\begin{aligned}&\left\|\Sigma_{\eta,\lambda}\sigma_1(t_n)-\Sigma_{\eta,\lambda}\sigma_2(t_n)\right\|_{\mathscr{H}}^2\\ &\qquad\le L_{\mathscr{G}}^{2n}T^n \int_0^{tn}\cdots\int_0^{t_2}\int_0^{t_1} \|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2\, ds dt_1\dots dt_{n-1}.\end{aligned}$$

Which implies

$$\left\|\Sigma_{\eta,\lambda}\sigma_1(t_n)-\Sigma_{\eta,\lambda}\sigma_2(t_n)\right\|_{\mathscr{H}}^2 \le \frac{L_{\mathscr{G}}^{2n}T^{n+1}}{n!}\int_0^T \|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2\, ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\left\|\Sigma_{\eta,\lambda}\sigma_1-\Sigma_{\eta,\lambda}\sigma_2\right\|_{C(0,T;\mathscr{H})}^2 \le \frac{L_{\mathscr{G}}^{2n}T^{n+2}}{n!}\|\sigma_1-\sigma_2\|_{C(0,T;\mathscr{H})}^2.$$

It follows from this inequality that for large n enough, the operator $\Sigma_{\eta,\lambda}^n$ is a contraction on the Banach space $C(0,T;\mathscr{H})$, and therefore there exists a unique element $\sigma \in C(0,T;\mathscr{H})$ such that $\Sigma_{\eta,\lambda}\sigma=\sigma$. Moreover, σ is the unique solution of Problem $\mathscr{P}_{\eta,\lambda}$. Consider now $(\eta_1,\lambda_1),(\eta_2,\lambda_2)\in C(0,T;\mathscr{H}\times L^2(\Omega))$ and for $i=1,2$, denote $u_{\eta_i}=u_i$, $\theta_{\eta_i}=\theta_i$, $\alpha_{\lambda_i}=\alpha_i$ and $\sigma_{\eta_i,\lambda_i}=\sigma_i$. We have

$$\begin{aligned}\sigma_i(t) = \mathscr{B}(\varepsilon(u_i(t),\alpha_i(t))) + \int_0^t \mathscr{G}(\sigma_i(s),\varepsilon(u_i(s)))\,ds - C_e\theta_i(t),\\ \text{a.e. } t\in(0,T).\end{aligned} \tag{3.68}$$

and using the properties (3.17), (2.17) and (2.18) of $\mathscr{B}$, $\mathscr{G}$ and C_e we find

$$\begin{aligned}&\|\sigma_1(t)-\sigma_2(t)\|_{\mathscr{H}}^2\\ &\le C\Big(\|u_1(t)-u_2(t)\|_V^2 + \|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Omega)}^2 + \|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2\\ &\quad + \int_0^t \|u_1(s)-u_2(s)\|_V^2\,ds + \int_0^t \|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2\,ds\Big), \quad \forall t\in[0,T].\end{aligned} \tag{3.69}$$

We use Gronwall argument in the previous inequality to deduce (3.66), which concludes the proof of Lemma 3.6.

□

Finally, as a consequence of these results and using the properties of the operators $\mathscr{G}$, $\mathscr{E}$, C_e and the function S, for $t\in(0,T)$, we consider the element

$$\Lambda(\eta,\lambda)(t) = \Big(\Lambda^1(\eta,\lambda)(t),\Lambda^2(\eta,\lambda)(t)\Big) \in \mathscr{H}\times L^2(\Omega), \tag{3.70}$$

defined by

$$
\begin{aligned}
\left(\Lambda^1(\eta,\lambda)(t),v\right)_{\mathscr{H}\times V} &= \left(\mathscr{E}^*\nabla\varphi_\eta(t),\varepsilon(v)\right)_{\mathscr{H}} + \left(C_e\theta_\eta(t),\varepsilon(v)\right)_{\mathscr{H}} \\
&+\left(\int_0^t \mathscr{G}\left(\sigma_{\eta,\lambda},\varepsilon\left(u_\eta(s)\right)\right)ds,\varepsilon(v)\right)_{\mathscr{H}}, \forall v\in V,
\end{aligned} \tag{3.71}
$$

$$
\Lambda^2(\eta,\lambda)(t) = S\left(\varepsilon\left(u_\eta(t)\right),\alpha_\lambda(t)\right). \tag{3.72}
$$

Here, for every $(\eta,\lambda)\in C(0,T;\mathscr{H}\times L^2(\Omega))$. u_η, φ_η, θ_η, α_λ and $\sigma_{\eta,\lambda}$ represent the displacement field, the electric potential field, the temperature field, the damage field and the stress field, obtained in Lemmas 3.2, 3.3, 3.4, 3.5 and 3.6 respectively. We have the following result.

Lemma 3.7. *The mapping Λ has a fixed point $(\eta^*,\lambda^*)\in C(0,T;\mathscr{H}\times L^2(\Omega))$, such that $\Lambda(\eta^*,\lambda^*)=(\eta^*,\lambda^*)$.*

Proof. Let $t\in(0,T)$ and $(\eta_1,\lambda_1),(\eta_2,\lambda_2)\in C\left(0,T;\mathscr{H}\times L^2(\Omega)\right)$. We use the notation that $u_{\eta_i}=u_i$, $\dot{u}_{\eta_i}=\dot{u}_i$, $\theta_{\eta_i}=\theta_i$, $\varphi_{\eta_i}=\varphi_i$, $\alpha_{\lambda_i}=\alpha_i$ and $\sigma_{\eta_i,\lambda_i}=\sigma_i$ for $i=1,2$. Let us start by using (1.7), (2.17), (2.18) and (2.21), we have

$$
\begin{aligned}
&\|\Lambda^1(\eta_1,\lambda_1)(t)-\Lambda^1(\eta_2,\lambda_2)(t)\|^2_{\mathscr{H}} \le \|\mathscr{E}^*\nabla\varphi_1(t)-\mathscr{E}^*\nabla\varphi_2(t)\|^2_{\mathscr{H}} \\
&\quad+\|C_e\theta_1(t))-C_e\theta_2(t)\|^2_{\mathscr{H}}+\int_0^t\|\mathscr{G}\left(\sigma_1(s),\varepsilon(u_1(s))\right)-\mathscr{G}\left(\sigma_2(s),\varepsilon(u_2(s))\right)\|^2_{\mathscr{H}}ds \\
&\le C\left(\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}\right. \\
&\quad\left.+\int_0^t\|\sigma_1(s)-\sigma_2(s)\|^2_{\mathscr{H}}ds+\int_0^t\|u_1(s)-u_2(s)\|^2_V ds\right).
\end{aligned} \tag{3.73}
$$

We use estimates (3.59), (3.66) to obtain

$$
\begin{aligned}
&\|\Lambda^1(\eta_1,\lambda_1)(t)-\Lambda^1(\eta_2,\lambda_2)(t)\|^2_{\mathscr{H}} \le C\left(\|\alpha_1(t))-\alpha_2(t)\|^2_{L^2(\Omega)}\right. \\
&\quad\left.+\|\theta_1(t))-\theta_2(t)\|^2_{L^2(\Omega)}+\|u_1(s)-u_2(s)\|^2_V+\int_0^t\|u_1(s)-u_2(s)\|^2_V ds\right).
\end{aligned} \tag{3.74}
$$

By similar arguments, from (3.72), (3.18) we obtain

$$
\begin{aligned}
&\left\|\Lambda^2(\eta_1,\lambda_1)(t)-\Lambda^2(\eta_2,\lambda_2)(t)\right\|^2_{\mathscr{H}} \le C\left(\|u_1(t)-u_2(t)\|^2_V\right. \\
&\quad\left.+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\right), \qquad \text{a.e. } t\in(0,T).
\end{aligned} \tag{3.75}
$$

It follows now from (3.75), (3.74) and (3.70) that

$$
\begin{aligned}
&\|\Lambda(\eta_1,\lambda_1)(t)-\Lambda(\eta_2,\lambda_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)} \le C\left(\|\alpha_1(t))-\alpha_2(t)\|^2_{L^2(\Omega)}\right. \\
&\quad\left.+\|\theta_1(t))-\theta_2(t)\|^2_{L^2(\Omega)}+\|u_1(s)-u_2(s)\|^2_V+\int_0^t\|u_1(s)-u_2(s)\|^2_V ds\right).
\end{aligned} \tag{3.76}
$$

Form (3.63), deduced that

$$
\begin{aligned}
&(\dot{\alpha}_1-\dot{\alpha}_2,\alpha_1-\alpha_2)_{L^2(\Omega)}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2) \\
&\qquad\le(\lambda_1-\lambda_2,\alpha_1-\alpha_2)_{L^2(\Omega)}, \text{ a.e. } t\in(0,T).
\end{aligned}
$$

integrate inequality with respect to time, using the initial conditions $\alpha_1(0)=\alpha_2(0)=\alpha_0$, and inequality $a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\ge 0$, we find

$$\tfrac{1}{2}\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)} \leq C\int_0^t \left(\lambda_1(s)-\lambda_2(s),\alpha_1(s)-\alpha_2(s)\right)_{L^2(\Omega)} ds,$$

which implies

$$\begin{aligned}&\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)}\\ &\qquad\leq C\left(\int_0^t \left\|\lambda_1(s)-\lambda_2(s)\right\|^2_{L^2(\Omega)} ds+\int_0^t \left\|\alpha_1(s)-\alpha_2(s)\right\|^2_{L^2(\Omega)} ds\right).\end{aligned}$$

This inequality combined with the Gronwall inequality leads to

$$\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)} \leq C\int_0^t \left\|\lambda_1(s)-\lambda_2(s)\right\|^2_{L^2(\Omega)} ds\,, \forall t\in[0,T]. \tag{3.77}$$

Form the previous inequality and estimates (3.77), (3.76), (3.62) and (3.49) it follows now that

$$\begin{aligned}&\left\|\Lambda(\eta_1,\lambda_1)(t)-\Lambda(\eta_2,\lambda_2)(t)\right\|^2_{\mathscr{H}\times L^2(\Omega)}\\ &\qquad\leq C\int_0^T \left\|(\eta_1,\lambda_1)(s)-(\eta_2,\lambda_2)(s)\right\|^2_{\mathscr{H}\times L^2(\Omega)} ds.\end{aligned} \tag{3.78}$$

Reiterating this inequality m times we obtain

$$\begin{aligned}&\left\|\Lambda^m(\eta_1,\lambda_1)-\Lambda^m(\eta_2,\lambda_2)\right\|^2_{C(0,T;\mathscr{H}\times L^2(\Omega))}\\ &\qquad\leq \frac{C^mT^m}{m!}\left\|(\eta_1,\lambda_1)-(\eta_2,\lambda_2)\right\|^2_{C(0,T;\mathscr{H}\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathscr{H}\times L^2(\Omega))$, and so Λ has a unique fixed point.

□

Now we have every thing that is required to prove Theorem 3.1.

Existence

Let $(\eta^*,\lambda^*)\in C(0,T;\mathscr{H}\times L^2(\Omega))$ be the fixed point of Λ and

$$u=u_{\eta^*},\quad \theta=\theta_{\eta^*},\quad \varphi_{\eta^*}=\varphi,\quad \alpha=\alpha_{\lambda^*} \tag{3.79}$$

$$\sigma=\mathscr{A}\varepsilon(\dot u)+\mathscr{E}^*\nabla\varphi(t)+\sigma_{\eta^*\lambda^*}, \tag{3.80}$$

$$D=\mathscr{E}\varepsilon(u)+B\nabla(\varphi). \tag{3.81}$$

We prove that $(u,\sigma,\theta,\varphi,\alpha,D)$ satisfies (3.34)-(3.40) and (3.41)-(3.46). Indeed, we write (3.65) for $\eta^*=\eta,\quad \lambda^*=\lambda$ and use (3.79)-(3.80) to obtain that (3.34) is satisfied. Now we consider (3.47) for $\eta^*=\eta,\quad \lambda^*=\lambda$ and use (3.79) to find

$$\begin{gathered}(\mathscr{A}\varepsilon(\dot u(t)),\varepsilon(v-\dot u(t)))_{\mathscr{H}}+(\mathscr{B}(\varepsilon(u(t)),\alpha(t)),\varepsilon(v)-\varepsilon(\dot u(t)))_{\mathscr{H}}\\ +(\eta^*(t),v-\dot u(t))_{\mathscr{H}}+j(v)-j(\dot u(t))\geq(f(t),v-\dot u(t))_V\\ \forall v\in V,t\in[0,T].\end{gathered} \tag{3.82}$$

The equalities $\Lambda^1(\eta^*,\lambda^*) = \eta^*$ and $\Lambda^2(\eta^*,\lambda^*) = \lambda^*$. combined with (3.71)-(3.72), (3.79) and (3.80) show that for all $v \in V$,

$$\begin{aligned}(\eta^*(t),v)_{\mathscr{H}\times V} &= (\mathscr{E}^*\nabla\varphi(t),\varepsilon(v))_{\mathscr{H}} - (C_e\theta(t),\varepsilon(v))_{\mathscr{H}},\\ &+\left(\int_0^t \mathscr{G}(\sigma(s)-\mathscr{A}\varepsilon(\dot u(s))-\mathscr{E}^*\nabla\varphi(t),\varepsilon(u(s)))ds,\varepsilon(v)\right)_{\mathscr{H}},\end{aligned} \tag{3.83}$$

$$\lambda^*(t) = S(\varepsilon(u(t)),\alpha(t)). \tag{3.84}$$

We substitute (3.83) in (3.82)) and use (3.34) to see that (3.36) is satisfied. We write now (3.58) for $\eta = \eta^*$ and use (3.79) to find (3.37). From (3.60) and (3.79) we see that (3.38) is satisfied. We write (3.63) for $\lambda = \lambda^*$ and use (3.79) and (3.84) to find that (3.39) is satisfied Next, (3.40), The regularities (3.41), (3.42), (3.44) and (3.45) follow from Lemmas 3.2, 3.3, 3.4 and 3.5. The regularity $\sigma \in C(0,T;\mathscr{H})$ follows from Lemmas 3.6. Let now $t_1,t_2 \in [0,T]$, from (1.11), (2.20), (2.21) and (3.81), we conclude that there exists a positive constant $C > 0$ verifying

$$\|D(t_1)-D(t_2)\|_H \le C(\|\varphi(t_1)-\varphi(t_2)\|_W + \|u(t_1)-u(t_2)\|_V).$$

The regularity of u and φ given by (3.41) and (3.42) implies

$$D \in C(0,T;H). \tag{3.85}$$

We choose $\phi \in D(\Omega)^d$ in (3.37) and using (3.29) we find

$$\operatorname{div} D(t) = q_0(t), \quad \forall t \in [0,T], \tag{3.86}$$

Property (3.46) follows from (3.23),(3.85) and (3.86) which concludes the existence part the Theorem 3.1.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ. and the unique solvability of the Problems $\mathscr{P}^1_\eta$, $\mathscr{P}^2_\eta$, $\mathscr{P}^3_\eta$, $\mathscr{P}_\lambda$ and $\mathscr{P}_{\eta,\lambda}$ which completes the proof.

Chapter 4

Thermo-electro-viscoelastic contact problem with damage

In this chapter we consider a mathematical model which describes the quasistatic process of a contact with friction and damage between a body having a nonlinear thermo-electro-viscoelastic constitutive law and an obstacle. the contact is described with the normal compliance condition and a version of Coulombs law of friction. we derive a variational formulation for the mechanical problem. Then, we provide the existence of a weak solution to the problem. The proof is based on arguments of time-dependent variational inequalities, parabolic inequalities, on differential equations and fixed point arguments. The content of this chapter was the subject of the publication [31]

4.1 Formulation of the problem

We place ourselves in the physical framework that we presented in Chapter 1 of this thesis. The body is described by Constitutive law of electro-viscoelastic materials with long memory with damage and thermal effects. The contact is described with the normal compliance condition and a version of Coulomb's law of friction. Under these considerations, the considered mechanical problem be stated as follows.

Problem $P3$

Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$, an electric displacement field $D : \Omega \times (0,T) \to \mathbb{R}^d$, and a damage field $\alpha : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\sigma(t) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}(\varepsilon(u(t)), \theta(t), \alpha(t)) + \int_0^t \mathscr{M}(t-s, \varepsilon(u(s)), \theta(s), \alpha(s))\,ds - (\mathscr{E})^* E(\varphi(t)) \quad \text{in } \Omega \times (0,T), \tag{4.1}$$

$$D = \mathscr{E}\varepsilon(u) + B\nabla(\varphi) \quad \text{in } \Omega \times (0,T), \tag{4.2}$$

$$\dot{\theta} - \kappa_0 \Delta\theta = \Theta(\sigma, \varepsilon(u), \theta, \alpha) + \rho \quad \text{in } \Omega \times (0,T), \tag{4.3}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni S(\varepsilon(u), \alpha), \quad \text{in } \Omega \times (0,T), \tag{4.4}$$

$$\operatorname{Div}\sigma + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{4.5}$$

$$\operatorname{div} D - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{4.6}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{4.7}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{4.8}$$

$$-\sigma_\nu = p_\nu(u_\nu - g) \quad \text{on } \Gamma_3 \times (0,T) \tag{4.9}$$

$$\begin{cases} \|\sigma_\tau\| \le \mu p_\nu(u_\nu - g) \\ \|\sigma_\tau\| < \mu p_\nu(u_\nu - g) \Rightarrow \dot{u}_\tau = 0 \\ \|\sigma_\tau\| = \mu p_\nu(u_\nu - g) \Rightarrow \text{there exists } \lambda \ge 0 \\ \qquad\qquad \text{such that } \sigma_\tau = -\lambda \dot{u}_\tau, \end{cases} \quad \text{on } \Gamma_3 \times (0,T) \tag{4.10}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{4.11}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{4.12}$$

$$D.\nu = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{4.13}$$

$$D \cdot \nu = \psi(u_\nu - g)\phi_l(\varphi - \varphi_0) \quad \text{on } \Gamma_3 \times (0,T), \tag{4.14}$$

$$k_0 \frac{\partial\theta}{\partial\nu} + B\theta = 0 \quad \text{on } \Gamma \times (0,T), \tag{4.15}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0, \quad \text{in } \Omega. \tag{4.16}$$

First, equations (4.1) and (4.2) represent the thermo-electro-viscoelastic constitutive law with long term-memory and damage. Equation (4.3) represents the energy conservation where Θ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and ρ is a given volume heat source The evolution of the damage field is governed by the inclusion of parabolic type given by the relation (4.4). Equations (4.5) and (4.6) represent the equilibrium equations for the stress and electric displacement fields. Equations (4.7)-(4.8) are the displacement-traction conditions. Frictional contact conditions of the form (4.9) and (4.10) have been used in various papers [28, 47]. and which describe the contact on the surface Γ_3 , described by the normal compliance function p. The relation (4.11) describes a homogeneous Neumann boundary condition. (4.12) and (4.13) represent the electric boundary conditions. Equality (4.14) represents the electrical condition on the potential contact surface, (4.15) represent a Fourier boundary condition for the temperature. Finally, The functions u_0, θ_0 and α_0 in (4.16) are the initial data.

4.2 Variational formulation

In the study of the problem P, we consider the following assumptions. The elasticity operator $\mathscr{B}:\Omega\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathscr{B}}>0 \quad \text{such that} \\ \|\mathscr{B}(x,\varepsilon_1,\theta_1,\alpha_1)-\mathscr{B}(x,\varepsilon_2,\theta_2,\alpha_2)\|\leq L_{\mathscr{B}}(\|\varepsilon_1-\varepsilon_2\|+\|\theta_1-\theta_2\| \\ +\|\alpha_1-\alpha_2\|),\ \text{for all } \varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\ \text{a.e. } x\in\Omega. \\ \text{(b) The mapping } x\mapsto\mathscr{B}(x,\varepsilon,\theta,\alpha) \text{ is Lebesgue measurable on } \Omega, \\ \qquad\qquad \text{for all } \varepsilon\in\mathbb{S}^d,\theta,\alpha\in\mathbb{R}. \\ \text{(c) The mapping } x\mapsto\mathscr{B}(x,0,0,0)\in\mathscr{H}. \end{cases} \tag{4.17}$$

The relaxation function $\mathscr{M}:\Omega\times(0,T)\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_{\mathscr{M}}>0 \quad \text{such that} \\ \|\mathscr{M}(x,t,\varepsilon_1,\theta_1,\alpha_1)-\mathscr{M}(x,t,\varepsilon_2,\theta_2,\alpha_2)\|\leq L_{\mathscr{M}}(\|\varepsilon_1-\varepsilon_2\|+\|\theta_1-\theta_2\| \\ +\|\alpha_1-\alpha_2\|),\ \text{for all } t\in(0,T),\varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\theta_1,\theta_2,\alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e. } x\in\Omega. \\ \text{(b) The mapping } x\mapsto\mathscr{M}(x,t,\varepsilon,\theta,\alpha) \text{ is Lebesgue measurable on } \Omega, \\ \text{for all } \varepsilon,\in\mathbb{S}^d,t\in(0,T),\ \text{for all } \theta,\alpha\in\mathbb{R}. \\ \text{(c) The mapping } x\mapsto\mathscr{M}(x,t,\varepsilon,\theta,\alpha) \text{ is continuous in } \Omega, \\ \text{for all } \varepsilon,\in\mathbb{S}^d,t\in(0,T),\ \text{for all } \theta,\alpha\in\mathbb{R}. \\ \text{(d) The mapping } x\mapsto\mathscr{M}(x,t,0,0,0)\in\mathscr{H}. \end{cases} \tag{4.18}$$

The function $\Theta:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\begin{cases} (a)\text{There exists a constant } L_\Theta>0 \quad \text{such that} \\ \|\Theta(x,\sigma_1,\varepsilon_1,\theta_1,\alpha_1)-\Theta(x,\sigma_2,\varepsilon_2,\theta_2,\alpha_2)\| \\ \qquad\qquad \leq L_\Theta(\|\sigma_1-\sigma_2\|+\|\varepsilon_1-\varepsilon_2\|+\|\theta_1-\theta_2\|+\|\alpha_1-\alpha_2\|), \\ \forall\sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\quad \forall\theta_1,\theta_2\in\mathbb{R},\quad \forall\alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e. } x\in\Omega. \\ (b)\text{ The mapping } x\mapsto\Theta(x,\sigma,\varepsilon,\theta,\alpha) \text{ is Lebesgue measurable on } \Omega, \\ \forall\sigma,\varepsilon\in\mathbb{S}^d,\forall\theta,\alpha\in\mathbb{R}. \\ (c)\text{ The mapping } x\mapsto\Theta(x,0,0,0,0)\in L^2(\Omega). \end{cases} \tag{4.19}$$

The surface electrical conductivity function $\psi:\Gamma_3\times\mathbb{R}\to\mathbb{R}^+$ satisfies

$$\begin{cases} \text{(a) There exists } L_\psi>0 \text{ such that} \\ \|\psi(.,u_1)-\psi(.,u_2)\|\leq L_\psi\|u_1-u_2\|,\ \text{for all } u_1,u_2\in\mathbb{R}. \\ \text{(b) There exists } M_\psi>0 \text{ such that } \|\psi(x,u)\|\leq M_\psi, \\ \qquad\qquad \text{for all } u\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3. \\ \text{(c)} x\mapsto\psi(x,) \text{ is measurable on } \Gamma_3,\ \text{for all } u\in\mathbb{R}, \\ \text{(d)} x\mapsto\psi(x,u)=0, \quad \text{for all } u\leq 0. \end{cases} \tag{4.20}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists } L_\nu > 0 \text{ such that} \\ \quad \|p_\nu(x,u_1) - p_\nu(x,u_2)\| \le L_\nu \|u_1 - u_2\| \\ \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(d) For any } u \in \mathbb{R}, x \mapsto p_\nu(x,u) \text{ is measurable on } \Gamma_3 \\ \text{(e) } x \mapsto p_\nu(x,u) = 0, \text{ for all } u \le 0. \end{array}\right. \tag{4.21}$$

Similarly, we assume that the operator $\mathscr{A}$ satisfies the assumption (2.15), In addition, we suppose that the function S, the electric permittivity operator B, the piezoelectric operator $\mathscr{E}$ satisfy the assumptions (3.18), (2.20) and (2.21) respectively. Next, We assume that the gap function g the initial potential φ_0 the friction coefficient μ, the volume heat source ρ, the initial data α_0, u_0 and θ_0 the volume of forces f_0 and f_2 and the charges densities q_0, q_2, the energy coefficient k_0, the microcrack diffusion coefficient k_1 satisfy

$$g \in L^2(\Gamma_3),\ g \ge 0 \text{ a.e. on } \Gamma_3,\ \varphi_0 \in L^2(\Gamma_3), \tag{4.22}$$

$$\mu \in L^\infty(\Gamma_3),\ \mu \ge 0 \text{ a.e. on } \Gamma_3, \tag{4.23}$$

$$u_0 \in V, \quad \alpha_0 \in K, \quad \theta_0 \in H^1(\Omega), \tag{4.24}$$

$$f_0 \in C\left(0,T;L^2(\Omega)^d\right),\ f_2 \in C\left(0,T;L^2(\Gamma_2)^d\right), \tag{4.25}$$

$$q_0 \in C\left(0,T;L^2(\Omega)\right),\ q_2 \in C\left(0,T;L^2(\Gamma_b)\right), \tag{4.26}$$

$$B > 0, \quad k_i > 0, \quad i = 0,1, \quad \rho \in C\left(0,T;L^2(\Omega)\right). \tag{4.27}$$

We introduce the following continuous functional $a_i : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $i = 0,1$ by

$$a_0(\zeta,\xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi dx + B \int_\Gamma \zeta\xi d\gamma, \tag{4.28}$$

$$a_1(\zeta,\xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi dx. \tag{4.29}$$

Next. We define four mappings $j : V \times V \to \mathbb{R},\ h : V \times W \to W$, $f : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$j(u,v) = \int_{\Gamma_1} p_\nu(u_\nu - g) v_\nu da + \int_{\Gamma_3} \mu p_\nu(u_\nu - g) \|v_\tau\| da, \tag{4.30}$$

$$(h(u,\varphi),\zeta)_W = \int_{\Gamma_3} \psi(u_\nu - g) \phi_l(\varphi - \varphi_0) \zeta da, \tag{4.31}$$

$$(f(t),v)_V = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v da, \tag{4.32}$$

$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta dx - \int_{\Gamma_b} q_2(t)\zeta da, \tag{4.33}$$

for all $u, v \in V$, $\varphi, \zeta \in W$ and $t \in [0,T]$. Note that

$$f \in C(0,T;V), \quad q \in C(0,T;W). \tag{4.34}$$

By Using Green's formulas (1.4) and (1.14), we obtain the following variational formulation of the mechanical problem (4.1)-(4.16).

Problem *PV*

Find a displacement field $u:(0,T)\to V$, a stress field $\sigma:(0,T)\to\mathscr{H}$, an electric potential $\varphi:(0,T)\to W$, a damage field $\alpha:(0,T)\to H^1(\Omega)$, and a temperature $\theta:(0,T)\to H^1(\Omega)$ such that

$$\begin{aligned}\sigma(t)=\mathscr{A}\varepsilon(\dot u(t))+\mathscr{B}(\varepsilon(u(t)),\theta(t),\alpha(t))+\\ \int_0^t\mathscr{M}(t-s,\varepsilon(u(s)),\theta(s),\alpha(s))\,ds-(\mathscr{E})^*E(\varphi(t))\end{aligned}\tag{4.35}$$

$$(\sigma(t),\varepsilon(v)-\varepsilon(\dot u(t))_{\mathscr{H}}+j(u(t),v)-j(u(t),\dot u(t))\geq(\mathbf{f}(t),v-\dot u(t))_V\tag{4.36}$$

$$(B\nabla\varphi(t),\nabla\zeta)_H-(\mathscr{E}\varepsilon(u(t)),\nabla\zeta)_H+(h(u(t),\varphi(t)),\zeta)_W=(q(t),\zeta)_W,\tag{4.37}$$

$$\begin{aligned}&(\dot\theta(t),v)_{L^2(\Omega)}+a_0(\theta(t),v)=(\psi(\sigma(t),\varepsilon(u(t)),\theta(t)),v)_{L^2(\Omega)}\\ &\quad+(\rho(t),v)_{L^2(\Omega)},\quad\forall v\in H^1(\Omega),\ \text{a.e. } t\in(0,T),\end{aligned}\tag{4.38}$$

$$\begin{aligned}&\alpha(t)\in K,(\dot\alpha(t),\xi-\alpha(t))_{L^2(\Omega)}+a(\alpha(t),\xi-\alpha(t))\\ &\geq(S(\varepsilon(u(t)),\alpha(t)),\xi-\alpha(t))_{L^2(\Omega)},\forall\xi\in K,t\in(0,T),\end{aligned}\tag{4.39}$$

$$u(0)=u_0,\quad\theta(0)=\theta_0,\quad\alpha(0)=\alpha_0.\tag{4.40}$$

Our main existence and uniqueness result for Problem *PV* is in the following section.

4.3 Result of existence and uniqueness

Theorem 4.1. *Assume that (4.17)-(4.27) and (2.15), (3.18), (2.20) and (2.21) hold, Then there exists a unique solution $(u,\sigma,\varphi,\theta,\alpha,D)$ to problem PV. Moreover, the solution has the regularity*

$$u\in C^1(0,T;V),\tag{4.41}$$
$$\varphi\in C(0,T;W),\tag{4.42}$$
$$\sigma\in C(0,T;\mathscr{H}),\tag{4.43}$$
$$\theta\in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right),\tag{4.44}$$
$$\alpha\in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right),\tag{4.45}$$
$$D\in C(0,T;\mathscr{W}).\tag{4.46}$$

The functions u, σ, φ, θ, α, and D which satisfy (4.35)-(4.40) are called a weak solution of the contact problem P. We conclude that under the above assumptions, the mechanical problem (4.1)-(4.16) has a unique weak solution satisfying (4.41)-(4.46). It follows from (4.37) that $\operatorname{div}D-q_0=0$ for all $t\in(0,T)$, and therefore the regularity (4.42) of φ, combined with (2.20),(2.21), and (4.26) implies (4.46). The proof of theorem 4.1, is carried out is several steps and is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed points. We denote by C a constant whose value may change from line to line when no confusing can arise. Let $\eta\in C(0,T;\mathscr{H})$ and $g\in C(0,T;V)$ we consider the following variational problem.

Problem $\mathscr{P}_\eta^1$

Find a displacement field $u_\eta : [0,T] \to V$ and a stress field $\sigma_\eta : [0,T] \to \mathscr{H}$ such that for all $t \in [0,T]$

$$\sigma_\eta = \mathscr{A}\left(\varepsilon\left(\dot{u}_\eta\right)\right) + \eta, \tag{4.47}$$

$$\left(\sigma_\eta(t), \varepsilon\left(w - \dot{u}_\eta\right)\right)_{\mathscr{H}} + j(u_\eta, w) - j\left(u_\eta, \dot{u}_\eta\right) \geq \left(\mathbf{f}(t), w - \dot{u}_\eta\right)_V, \quad \forall w \in V, \tag{4.48}$$

$$u_\eta(0) = u_0. \tag{4.49}$$

We consider the following variational inequality

Problem $\mathscr{P}_\eta^2$

Find $v_{\eta g} : \Omega \times (0,T) \to V$ such that

$$\begin{aligned} \left(\mathscr{A}\varepsilon\left(v_{\eta g}\right), \varepsilon\left(w - v_{\eta g}\right)\right)_{\mathscr{H}} + j(g,w) - j\left(g, v_{\eta g}\right) \\ \geq \left(\mathbf{f}, w - v_{\eta g}\right)_V - \left(\eta, \varepsilon\left(w - v_{\eta g}\right)\right)_{\mathscr{H}}, \quad \forall w \in V. \end{aligned} \tag{4.50}$$

Lemma 4.2. *For all $g \in C(0,T;V)$, $\eta \in C(0,T;V)$, $\mathscr{P}_\eta^2$ has a unique solution with the regularity $v_{\eta g} \in C(0,T;V)$.*

Proof. Using Riesz Representation Theorem, we may define an element $\mathbf{F} \in C(0,T;V)$ by

$$(\mathbf{F}(t), v)_V = (\mathbf{f}(t), v)_V - (\eta(t), \varepsilon(v))_{\mathscr{H}}, \ \forall v \in V.$$

We define the operator $A : V \to V$ such that

$$(Au, v) = (\mathscr{A}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \forall u, v \in V. \tag{4.51}$$

It follows from (4.51) and (2.15)(a) that for all $u_1, u_2 \in V$ and $v \in V$ we have

$$\begin{aligned} \|(Au_1 - Au_2, v)_V\| &= \|(\mathscr{A}\left(\varepsilon\left(u_1\right)\right) - \mathscr{A}\left(\varepsilon\left(u_2\right)\right), \varepsilon(v))\|_{\mathscr{H}}, \\ &\leq \|\mathscr{A}\left(\varepsilon\left(u_1\right)\right) - \mathscr{A}\left(\varepsilon\left(u_2\right)\right)\|_{\mathscr{H}} \|\varepsilon(v)\|_{\mathscr{H}}, \\ &\leq L_{\mathscr{A}} \|\varepsilon\left(u_1\right) - \varepsilon\left(u_2\right)\|_{\mathscr{H}} \|\varepsilon(v)\|_{\mathscr{H}}, \\ &\leq L_{\mathscr{A}} \|u_1 - u_2\|_V \|v\|_V, \end{aligned}$$

which shows that $A : V \to V$ is Lipschitz continuous. Now, by (4.51) and (2.15)(b) we find

$$\begin{aligned} (Au_1 - Au_2, u_1 - u_2)_V &= \left(\mathscr{A}\left(\varepsilon\left(u_1\right)\right) - \mathscr{A}\left(\varepsilon\left(u_2\right)\right), \varepsilon\left(u_1\right) - \varepsilon\left(u_2\right)\right)_{\mathscr{H}} \\ &\geq m_{\mathscr{A}} \|\varepsilon\left(u_1\right) - \varepsilon\left(u_2\right)\|_{\mathscr{H}}^2 \geq C \|u_1 - u_2\|_V^2. \end{aligned}$$

And according to Korn's inequality, it comes

$$(Au_1 - Au_2, u_1 - u_2)_V \geq C \|u_1 - u_2\|_V^2,$$

i.e., that $A : V \to V$ is a strongly monotone operator on V. And we can easily check that $v \mapsto j(g(t), v)$ is convex lower semicontinuous and proper. It follows from classical results for elliptic variational inequalities (see [10]) that there exists a unique $v_{\eta g} \in V$, which is a solution of (4.50). To establish its regularity by showing that $v_{\eta g} \in C(0,T;V)$. We let $t_1, t_2 \in [0,T]$ and denote $\eta_i = \eta(t_i)$, $g_i = g(t_i)$, $f_i = f(t_i)$ and $v_i = v_{\eta g}(t_i)$. Using the relation (4.50) we find that

$$\begin{aligned}
&(\mathscr{A}\varepsilon(v_1) - \mathscr{A}\varepsilon(v_2), \varepsilon(v_1 - v_2))_{\mathscr{H}} \leq (f_1 - f_2, v_1 - v_2)_V \\
&+ (\eta_1 - \eta_2, \varepsilon(v_1 - v_2))_{\mathscr{H}} + j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2).
\end{aligned} \tag{4.52}$$

Moreover, we have

$$(\mathscr{A}\varepsilon(v_1) - \mathscr{A}\varepsilon(v_2), \varepsilon(v_1 - v_2))_{\mathscr{H}} \geq m_{\mathscr{A}} \|v_1 - v_2\|_V^2. \tag{4.53}$$

From the definition of the functional j given by (4.30), we have

$$\begin{aligned}
&j(g_1, v_1) - j(g_1, v_2) + j(g_2, v_1) - j(g_2, v_2) \\
&\quad \leq c_0^2 L_V \|\mu\|_{L^\infty(\Gamma_3)} \|g_1 - g_2\|_V \|v_1 - v_2\|_V.
\end{aligned} \tag{4.54}$$

Using these bounds in (4.52), we obtain

$$\|v_1 - v_2\|_V \leq C(\|f_1 - f_2\|_V + \|\eta_1 - \eta_2\|_{\mathscr{H}} + \|g_1 - g_2\|_V), \tag{4.55}$$

then the conclusion that $v_{\eta g} \in C([0,T];V)$ follows from the continuity of f, σ and g in their respective spaces V and $\mathscr{H}$.

□

With the help of Lemma 4.2, we are in a position to show the following existence and uniqueness result for Problem $\mathscr{P}_\eta^1$.

Lemma 4.3. *There exists a unique solution to Problem $\mathscr{P}_\eta^1$ such that $u_\eta \in C^1(0,T;V)$ and $\sigma_\eta \in C(0,T;\mathscr{H})$.*

Proof. We consider an operator $\Lambda_\eta : C(0,T;V) \to C(0,T;V)$ defined by

$$\Lambda_\eta g = g_\eta,\ g \in C(0,T;V), \tag{4.56}$$

where

$$g_\eta(t) = u_0 + \int_0^t v_{\eta g}(s)ds,\ t \in (0,T). \tag{4.57}$$

and $v_{\eta g}$ is the solution of (4.50). We will show that this operator has a unique fixed point $g_\eta \in C([0,T];V)$. To this end, let $g_1, g_2 \in C([0,T];V)$ and denote by $v_i = v_{\eta g_i}$, $i = 1,2$, the corresponding solutions of (4.50). Using (4.56) and (4.57) we have

$$\left\|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)\right\|_V \leq \int_0^t \|v_1(s) - v_2(s)\|_V ds, \quad \forall t \in [0,T]. \tag{4.58}$$

Using estimates similar to those in the proof of Lemma 4.2 (see (4.52)-(4.55)) we see that

$$\|v_1(s) - v_2(s)\|_V \leq C\|g_1(s) - g_2(s)\|_V, \quad s \in [0,T]. \tag{4.59}$$

Taking into account (4.58) we obtain

$$\left\|\Lambda_\eta g_1(t)-\Lambda_\eta g_2(t)\right\|_V \leq C\int_0^t \|g_1(s)-g_2(s)\|_V\,ds, \quad \forall t\in[0,T]. \tag{4.60}$$

Let is introduce the following notations

$$\begin{cases} I_1=\displaystyle\int_0^t \|g_1(s)-g_2(s)\|_V\,ds,\\ \vdots\\ I_k=\displaystyle\int_0^t\int_0^{s_{k-1}}\cdots\int_0^{s_1}\|g_1(r)-g_2(r)\|_V\,, \end{cases}$$

and by induction, by denoting by Λ_η^m the m power of the operator Λ_η, we obtain

$$\begin{aligned}&\left\|\Lambda_\eta^m g_1(t)-\Lambda_\eta^m g_2(t)\right\|_V\\ &\qquad\leq C^m\left(\sum_{k=1}^m C_m^k I^{m-k}\|g_1(t)-g_2(t)\|_V\right),\end{aligned}$$

for all $t\in(0,T)$ and $m\in\mathbb{N}$,

$$\begin{aligned} I^{m-k}\|g_1-g_2\|_V &= \int_{(m-k)\text{ fois}}\cdot\int\|g_1-g_2\|_V\\ &\leq \int_0^s\int\cdots\int_{(m-k)\text{ fois}}\|g_1-g_2\|_{C(0,T;V)}\\ &\leq \frac{t^{m-k}}{k!}\|g_1-g_2\|_{C(0,T;V)}\\ &\leq \frac{T^{m-k}}{k!}\|g_1-g_2\|_{C(0,T;V)}\,,\end{aligned}$$

$$\begin{aligned}&\|\Lambda_\eta^m g_1(t)-\Lambda_\eta^m g_2(t)\|_{C(0,T;V)}\\ &\qquad\leq C^m\left(\sum_{k=1}^m C_m^k\frac{T^{m-k}}{k!}\|g_1(t)-g_2(t)\|_{C(0,T;V)}\right)\\ &\qquad\leq \frac{(CT)^m}{m!}\|g_1(t)-g_2(t)\|_{C(0,T;V)}\,,\end{aligned} \tag{4.61}$$

which implies that for a sufficiently large m the operator Λ_η^m is a contraction on $C(0,T;V)$, Thus Λ_η has a unique fixed point $g_\eta^*\in C([0,T];V)$. Next, let $v_\eta\in C([0,T];V)$ and $\sigma_\eta\in C([0,T];\mathscr{H})$ be given by

$$v_\eta=v_{\eta g_\eta^*},\ \sigma_\eta=\sigma_{\eta g_\eta^*}=\mathscr{A}\varepsilon\left(v_{\eta g_\eta^*}\right)+\eta. \tag{4.62}$$

Moreover, using (4.57) and (4.62), we let $u_\eta:[0,T]\to V$ be the function

$$u_\eta(t)=u_0+\int_0^t v_\eta(s)ds, \quad \forall t\in[0,T]. \tag{4.63}$$

Clearly, (4.47) and (4.49) are satisfied. Moreover, by (4.62), (4.63) and (4.56), (4.57), it follows that $u_\eta = g^*_\eta$ and $\dot{u}_\eta = v_\eta$. Therefore, if $g = g^*_\eta$ in (4.50), then we obtain (4.48). To prove the regularity of σ_η, we choose $w = v_{\eta g}(t) \pm \varphi$ in (4.48) where $\varphi \in C_0^\infty(\Omega)^d$, $t \in (0,t)$ we find

$$(\sigma_{\eta g}, \varepsilon(\varphi))_{\mathscr{H}} = (\mathbf{F}, \varphi)_V, \quad \forall \varphi \in C_0^\infty(\Omega)^d, \text{ on } [0,T]. \tag{4.64}$$

Using (4.32) we deduce

$$\operatorname{Div} \sigma_\eta + \mathbf{f}_0 = 0, \quad \text{ on } [0,T], \tag{4.65}$$

and then, assumption (4.25) and equation (4.65) imply that $\sigma_\eta \in C([0,T];\mathscr{H})$. This establishes the existence part in Lemma 4.3. The uniqueness is a consequence of the uniqueness of the fixed point of the operator Λ_η defined by (4.56), (4.57) and the unique solvability of the Problem $\mathscr{P}^2_\eta$ and relations (2.15) and (4.47).

□

In the second step we use the displacement field u_η obtained in Lemma 4.3, to construct the following variational problem for the an electrical potential.

Problem $\mathscr{P}^3_\eta$

Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$\begin{aligned}(B\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathscr{E}\varepsilon(u_\eta(t)), \nabla\zeta)_H + (h(u_\eta(t), \varphi_\eta(t)), \zeta)_W \\ = (q(t), \zeta)_W, \text{ for all } \zeta \in W, t \in (0,T).\end{aligned} \tag{4.66}$$

We have the following result for problem $\mathscr{P}^3_\eta$

Lemma 4.4. *Problem (4.66) has unique solution φ_η which satisfies the regularity (4.42). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (4.66) corresponding to $\eta_1, \eta_2 \in C([0,T];\mathscr{H})$, then there exists $C > 0$ such that*

$$\left\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\right\|_W \le C \left\|u_{\eta_1}(t) - u_{\eta_2}(t)\right\|_V \quad \forall t \in [0,T]. \tag{4.67}$$

Proof. Let $t \in [0;T]$. We use the Riesz Representation Theorem to define the operator $A_\eta(t) : W \to W$ by

$$(A_\eta(t)\varphi, \xi)_W = (B\nabla\varphi_\eta(t), \nabla\xi)_W - (\mathscr{E}\varepsilon(u_\eta(t)), \nabla\xi)_W + (h(u_\eta(t)), \varphi_\eta(t)), \xi)_W. \tag{4.68}$$

For all$\varphi, \xi \in W$. Let $\varphi_1, \varphi_2 \in W$, we use the assumptions (2.20) and (4.31) to find

$$\begin{aligned}(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \ge m_B \|\varphi_1 - \varphi_2\|^2_W \\ + \int_{\Gamma_3} \psi(u_{\eta\nu}(t) - g)(\phi(\varphi_1 - \varphi_0) - \phi(\varphi_2 - \varphi_0))(\varphi_1 - \varphi_2)\, da,\end{aligned}$$

and, by the condition (4.20)(a) combined with the monotonicity of the function , defined by (1.83), we obtain

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \ge m_B \|\varphi_1 - \varphi_2\|^2_W, \tag{4.69}$$

then $A_\eta(t)$ is a strongly monotone operator on W. On the other hand, using again (2.20)-(2.21), (4.20) and (4.31) we have

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \xi)_W \le c_B \|\varphi_1 - \varphi_2\|_W \|\xi\|_W + \int_{\Gamma_3} M_\psi |\varphi_1 - \varphi_2| |\xi| da, \quad \forall \xi \in W, \tag{4.70}$$

where c_B represents a positive constant which depends on the permittivity operator B. It follows from (4.70) and (1.13) that

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \xi)_W \le \left(c_B + M_\psi \tilde{c}_0^2\right) \|\varphi_1 - \varphi_2\|_W \|\xi\|_W,$$

which implies

$$\|A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2\|_W \le \left(c_B + M_\psi \tilde{c}_0^2\right) \|\varphi_1 - \varphi_2\|_W,$$

which shows that $A_\eta : W \to W$ is Lipschitz continuous. Since A_η is a strongly monotone Lipschitz continuous operator on W, we deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$A_\eta(t)\varphi_\eta(t) = q(t). \tag{4.71}$$

We now combine (4.68) and (4.71) to deduce that $\varphi_\eta(t) \in W$ solution of the variational equation (4.66) We now prove that $\varphi_\eta(t) \in W^{1,p}(0,T;W)$. For this end, consider $t_1, t_2 \in [0,T]$ and, for simplicity, we write $\varphi(t_i) = \varphi_i$, $u_{\eta\nu}(t_i) = u_i$, $q(t_i) = q_i$, for $i = 1,2$ Using (4.66),(2.20), (2.21) and (4.31) we find

$$\begin{aligned} m_{\mathbf{B}} \|\varphi_1 - \varphi_2\|_W^2 &\le c_\varepsilon \|u_1 - u_2\|_V \|\varphi_1 - \varphi_2\|_W + \|q_1 - q_2\|_W \|\varphi_1 - \varphi_2\|_W \\ &+ \int_{\Gamma_3} |\psi(u_1 - g)\phi(\varphi_1 - \varphi_0) - \psi(u_2 - g)\phi(\varphi_2 - \varphi_0)| |\varphi_1 - \varphi_2| da. \end{aligned} \tag{4.72}$$

We use the bounds $|\psi(u_i - g)| \le M_\psi$, $\phi_L(\varphi_i - \varphi_0)| \le L$, the Lipschitz continuity of functions ψ and ϕ_L, and the inequality (1.13) to obtain

$$\begin{aligned} &\int_{\Gamma_3} |\psi(u_1 - g)\phi_L(\varphi_1 - \varphi_0) - \psi(u_2 - g)\phi_L(\varphi_2 - \varphi_0)| |\varphi_1 - \varphi_2| da \\ &\qquad \le M_\psi \int_{\Gamma_3} |\varphi_1 - \varphi_2|^2 da + L_\psi L \int_{\Gamma_3} |u_1 - u_2| |\varphi_1 - \varphi_2| da \\ &\qquad \le M_\psi \tilde{c}_0^2 \|\varphi_1 - \varphi_2\|_W^2 + L_\psi L c_0 \tilde{c}_0 \|u_1 - u_2\|_V \|\varphi_1 - \varphi_2\|_W da. \end{aligned}$$

Inserting the last inequality in (4.72) yield

$$m_{\mathbf{B}} \|\varphi_1 - \varphi_2\|_W^2 \le \left(c_\varepsilon + L_\psi L_\psi c_0 \tilde{c}_0\right) \|u_1 - u_2\|_V + \|q_1 - q_2\|_W + M_\psi \tilde{c}_0^2 \|\varphi_1 - \varphi_2\|_W, \tag{4.73}$$

It now follows from the inequality (4.73) that

$$\|\varphi_1 - \varphi_2\|_W \le c(\|u_1 - u_2\|_V + \|q_1 - q_2\|_W), \tag{4.74}$$

where c is a positive constant. Note also that the hypotheses (4.26) combined with the definition (4.32) imply that $q \in W^{1,p}(0,T;W)$. So, since $u \in C^1([0,T];V)$. The inequality (4.74) implies that $\varphi_\eta \in W^{1,p}(0,T;W)$. Let now be $\eta_1, \eta_2 \in C([0,T];H)$ and, for simplicity, write $\varphi_{\eta_i} = \varphi_i$; $u_{\eta_i} = u_i$ for $i = 1,2$. We use (4.66) and with arguments similar to those used in the proof of (4.73) to obtain that

$$m_{\mathbf{B}} \|\varphi_1(t) - \varphi_2(t)\|_W^2 \leq \left(c_\varepsilon + L_\psi L_\psi c_0 \tilde{c}_0\right) \|u_1(t) - u_2(t)\|_V + M_\psi \tilde{c}_0^2 \|\varphi_1(t) - \varphi_2(t)\|_W, \tag{4.75}$$

This inequality leads to (4.67).

□

For $\chi \in C(0,T;L^2(\Omega))$, we consider the following variational problem.

Problem $\mathscr{P}_\chi$

Find the temperature field $\theta_\chi : (0,T) \to L^2(\Omega)$

$$\begin{aligned} &\left(\dot{\theta}_\chi(t), \mathbf{v}\right)_{L^2(\Omega)} + a_0\left(\theta_\chi(t), \mathbf{v}\right) = (\chi(t) + \rho(t), \mathbf{v})_{L^2(\Omega)}, \\ &\qquad\qquad\qquad\qquad \forall \mathbf{v} \in L^2(\Omega), \text{ a.e. } t \in (0,T), \end{aligned} \tag{4.76}$$

$$\theta_\chi(0) = \theta_0, \text{ in } \Omega. \tag{4.77}$$

Lemma 4.5. *There exists a unique solution θ_λ to the auxiliary problem $\mathscr{P}_\lambda$ satisfying (4.44).*

Proof. The inclusion of the trace of $(\mathscr{V}_1; \|.\|_{\mathscr{V}_1})$ in $(L^2(\Omega), \|.\|_{L^2(\Omega)})$ is continuous and dense, we can write the Gelfand Triplet

$$\mathscr{V}_1 \subset L^2(\Omega) \subset \mathscr{V}_1'.$$

We use the notation $(.,.)_{\mathscr{V}_1' \times \mathscr{V}_1}$ to denote the duality product between $\mathscr{V}_1'$ and $\mathscr{V}_1$, we have

$$(\zeta, \xi)_{\mathscr{V}_1' \times \mathscr{V}_1} = (\zeta, \xi)_{L^2(\Omega)}.$$

We consider the linear operator $A_0 : \mathscr{V}_1 \to \mathscr{V}_1'$ defined by

$$(A_0 \zeta, \xi)_{\mathscr{V}_1' \times \mathscr{V}_1} = \mathfrak{a}_0(\zeta, \xi), \quad \forall \zeta, \xi \in \mathscr{V}_1.$$

we use this last equality and the de definition (4.28), we can write for all $\zeta, \xi \in \mathscr{V}_1$

$$\left|(A_0 \zeta, \xi)_{\mathscr{V}_1' \times \mathscr{V}_1}\right| \leq k_0 \int_\Omega |\nabla \zeta . \nabla \xi| dx + k_2 \int_\Gamma |\zeta \xi| da,$$

by using Hölder's inequality, we get

$$\left|(A_0 \zeta, \xi)_{\mathscr{V}_1' \times \mathscr{V}_1}\right| \leq k_0 \|\nabla \zeta\|_{L^2(\Omega)^d} \|\nabla \xi\|_{L^2(\Omega)^d} + k_2 \|\zeta\|_{L^2(\Gamma)} \|\xi\|_{L^2(\Gamma)}. \tag{4.78}$$

Keeping in mind Sobolev trace Theorem, the inequality (4.78) becomes

$$\|A_0 \zeta\|_{\mathscr{V}_1'} \leq C \|\zeta\|_{\mathscr{V}_1},$$

which shows that $A_0 : \mathscr{V}_1 \to \mathscr{V}_1'$ is continuous and therefore hemicontinuous. Easy to make sure that

$$(A_0 \zeta, \zeta)_{\mathscr{V}_1' \times \mathscr{V}_1} \geq 0,$$

i.e, that $A_0 : \mathscr{V}_1 \to \mathscr{V}_1'$ is a monotone operator.

On the other hand, by means of (4.78), as $k_2 > 0$ and for all $\zeta \in \mathscr{V}_1$ we obtain

$$(A_0\zeta,\zeta)_{\mathscr{V}_1'\times\mathscr{V}_1} \geq k_0\|\nabla\zeta\|^2_{L^2(\Omega)^d},$$

and from where

$$(A_0\zeta,\zeta)_{\mathscr{V}_1'\times\mathscr{V}_1} \geq k_0\|\zeta\|^2_{\mathscr{V}_1} - k_0\|\zeta\|^2_{L^2(\Omega)}.$$

Then, A satisfies condition (1.40) of Theorem 1.25 with $w = k_0$ and $\lambda = -k_0\|\xi\|^2_{L^2(\Omega)}$, according to the above we have also A_0 satisfies the condition (1.41). It now follows from the regularity (4.24), (4.27) and $\chi \in L^2(0,T;\mathscr{V}_1')$ that $q_\chi = \chi + q \in L^2(0,T;L^2(\Omega))$ and $\theta_0 \in L^2(\Omega)$. Finally, we notice that all the conditions of Theorem 1.25 hold, so we conclude that there exists a unique function θ_χ which satisfies

$$\theta_\chi \in L^2(0,T;V)\cap C\left([0,T];L^2(\Omega)\right), \quad \dot{\theta}_\chi \in L^2\left(0,T;\mathscr{V}_1'\right), \tag{4.79}$$

$$\dot{\theta}_\chi(t) + A_0\theta_\chi(t) = q_\chi(t) \quad \text{a.e.t } \in (0,T), \tag{4.80}$$

$$\theta_\chi(0) = \theta_0. \tag{4.81}$$

□

In the third step we let $\lambda \in L^2(0,T;L^2(\Omega))$

Problem $\mathscr{P}_\lambda$

Find the damage field $\alpha_\lambda : (0,T) \to L^2(\Omega)$ such that $\alpha_\lambda(t) \in K$ and

$$\begin{aligned}(\dot{\alpha}_\lambda(t),\xi-\alpha_\lambda)_{L^2(\Omega)} &+ a_1(\alpha_\lambda(t),\xi-\alpha_\lambda(t)) \\ &\geq (\lambda(t),\xi-\alpha_\lambda(t))_{L^2(\Omega)} \quad \forall\xi\in K,\ \text{a.e. } t\in(0,T),\end{aligned} \tag{4.82}$$

$$\alpha_\lambda(0) = \alpha_0. \tag{4.83}$$

For the study of problem $\mathscr{P}_\lambda$, we have the following result.

Lemma 4.6. *There exists a unique solution α_λ to the auxiliary problem $\mathscr{P}_\lambda$ satisfying (4.45).*

The above lemma follows from a standard result for parabolic variational inequalities, see the proof of Lemma 3.5 (Chapter3). Finally, as a consequence of these results and using the properties of the operator $\mathscr{G}$ the operator $\mathscr{E}$, the functions Θ and S for $t \in (0,T)$, we consider the element

$$\Lambda(\eta,\chi,\lambda)(t) = \left(\Lambda^1(\eta,\chi,\lambda)(t),\Lambda^2(\eta,\chi,\lambda)(t),\Lambda^3(\eta,\chi,\lambda)(t)\right) \in \mathscr{H}\times L^2(\Omega)\times L^2(\Omega), \tag{4.84}$$

defined by

$$\begin{aligned}(\Lambda^1(\eta,\lambda,\chi)(t),v)_{\mathscr{H}\times V} &= (\mathscr{B}(\varepsilon(u(t),\theta_\chi(t),\alpha_\lambda(t)),\varepsilon(v))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\eta(t),\varepsilon(v))_{\mathscr{H}} \\ &+ \left(\int_0^t \mathscr{M}\left(t-s,\varepsilon\left(u_\eta(s)\right),\theta_\chi(s),\alpha_\lambda(s)\right)ds,\varepsilon(v)\right)_{\mathscr{H}}, \forall v\in V,\end{aligned} \tag{4.85}$$

$$\Lambda^2(\eta,\chi,\lambda)(t) = \Theta\left(\sigma_\eta,\varepsilon\left(u_\eta(t)\right),\theta_\chi(t),\alpha_\lambda(t)\right). \tag{4.86}$$

$$\Lambda^3(\eta,\chi,\lambda)(t) = S\left(\varepsilon\left(u_\eta(t)\right),\alpha_\lambda(t)\right). \tag{4.87}$$

Here, for every $(\eta,\chi,\lambda)\in C(0,T;\mathscr{H}\times L^2(\Omega)\times L^2(\Omega))$. u_η, φ_η, θ_χ, α_λ and σ_η represent the displacement field, the electric potential field, the temperature field, the damage field and the stress field , obtained in Lemmas 4.2, 4.3, 4.4,4.5 and 4.6 respectively. We have the following result.

Lemma 4.7. *The mapping Λ has a fixed point $(\eta,\chi,\lambda)\in C(0,T;\mathscr{H}\times L^2(\Omega)\times L^2(\Omega))$, such that $\Lambda(\eta^*,\chi^*,\lambda^*)=(\eta^*,\chi^*,\lambda^*)$.*

Proof. Let $t\in(0,T)$ and $(\eta_1,\chi_1,\lambda_1),(\eta_2,\chi_2,\lambda_2)\in C\left(0,T;\mathscr{H}\times L^2(\Omega)\times L^2(\Omega)\right)$. We use the notation that $u_{\eta_i}=u_i$, $\dot{u}_{\eta_i}=v_{\eta_i}=v_i$, $\theta_{\chi_i}=\theta_i$, $\varphi_{\eta_i}=\varphi_i$, $\alpha_\eta=\alpha_i$ and $\sigma_{\eta_i,\theta_i}=\sigma_i$ for $i=1,2$. Let us start by using (4.17), (4.18) and (2.21), we have

$$\begin{aligned}
&\|\Lambda^1(\eta_1,\chi_1,\lambda_1)(t)-\Lambda^1(\eta_2,\chi_2,\lambda_2)(t)\|^2_{\mathscr{H}}\\
&\quad\leq\|\mathscr{B}(\varepsilon(u_1(t)),\theta_1(t),\alpha_1(t))-\mathscr{B}(\varepsilon(u_2(t)),\theta_2(t),\alpha_2(t))\|^2_{\mathscr{H}}\\
&\quad+\|\mathscr{E}^*\nabla\varphi_1(t)-\mathscr{E}^*\nabla\varphi_2(t)\|^2_{\mathscr{H}}\\
&\quad+\int_0^t\|\mathscr{M}(t-s,\varepsilon(u_1(s)),\theta_1(s),\alpha_1(s))-\mathscr{M}(t-s,\varepsilon(u_2(s)),\theta_2(s),\alpha_2(s))\|^2_{\mathscr{H}}\,ds,\\
&\quad\leq C\Big(\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|u_1(t)-u_2(t)\|^2_V+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}\\
&\quad+\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}+\int_0^t\|u_1(s)-u_2(s)\|^2_V\,ds\\
&\quad+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds\Big).
\end{aligned}\tag{4.88}$$

By similar arguments, from (4.86), (4.19) we obtain

$$\begin{aligned}
&\left\|\Lambda^2(\eta_1,\chi_1,\lambda_1)(t)-\Lambda^2(\eta_2,\chi_2,\lambda_2)(t)\right\|^2_{\mathscr{H}}\\
&\leq C\Big(\|\sigma_1(t)-\sigma_2(t)\|^2_V+\|u_1(t)-u_2(t)\|^2_V\\
&\quad+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big),\qquad\text{a.e. } t\in(0,T).
\end{aligned}\tag{4.89}$$

Similarly, using (4.87), (3.18) implies

$$\begin{aligned}
&\left\|\Lambda^3(\eta_1,\chi_1,\lambda_1)(t)-\Lambda^3(\eta_2,\chi_2,\lambda_2)(t)\right\|^2_{\mathscr{H}}\\
&\qquad\qquad\leq C\Big(\|u_1(t)-u_2(t)\|^2_V+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big).
\end{aligned}\tag{4.90}$$

It follows now from (4.88), (4.89) and (4.90) that

$$\begin{aligned}
&\|\Lambda(\eta_1,\chi_1,\lambda_1)(t)-\Lambda(\eta_2,\chi_2,\lambda_2)(t)\|^2_{\mathscr{H}}\\
&\qquad\leq C\Big(\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\sigma_1(t)-\sigma_2(t)\|^2_{\mathscr{H}}\\
&\qquad+\|u_1(t)-u_2(t)\|^2_V+\int_0^t\|u_1(s)-u_2(s)\|^2_V\,ds\\
&\qquad+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds\\
&\qquad+\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds\Big).
\end{aligned}\tag{4.91}$$

Taking into account that

$$\sigma_i(t) = \mathscr{A}\left(\varepsilon\left(\dot{u}_i(t)\right)\right) + \eta_i(t), \quad \forall t \in [0,T], \tag{4.92}$$

it follows that

$$\begin{aligned}
&\left(\mathscr{A}\left(\varepsilon\left(v_1(s)\right)\right) - \mathscr{A}\left(\varepsilon\left(v_2(s)\right)\right), \varepsilon\left(v_1(s) - v_2(s)\right)\right)_{\mathscr{H}} \\
&\quad \leq j\left(v_1(s), v_2(s)\right) + j\left(v_2(s), v_1(s)\right) - j\left(v_1(s), v_1(s)\right) - j\left(v_2(s), v_2(s)\right) \\
&\qquad - \left(\eta_1(s) - \eta_2(s), \varepsilon(v_1(s) - v_2(s))\right)_{\mathscr{H}}.
\end{aligned}$$

So, by using (2.15), (1.7) and (4.30), we deduce that

$$\begin{aligned}
m_{\mathscr{A}} \left\|v_1(s) - v_2(s)\right\|_V^2 \leq & c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \left\|v_1(s) - v_2(s)\right\|_V^2 \\
& + \left\|\eta_1(s) - \eta_2(s)\right\|_{\mathscr{H}} \left\|v_1(s) - v_2(s)\right\|_V^2,
\end{aligned}$$

which, implies

$$\left\|v_1(s) - v_2(s)\right\|_V \leq C \left\|\eta_1(s) - \eta_2(s)\right\|_{\mathscr{H}}. \tag{4.93}$$

Moreover, from (4.63), we obtain

$$\left\|u_1(t) - u_2(t)\right\|_V \leq \int_0^t \left\|v_1(s) - v_2(s)\right\|_V ds. \tag{4.94}$$

So, using the inequality above, we find

$$\begin{aligned}
\int_0^t \left\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\right\|_V ds &\leq C \int_0^t \int_0^s \left\|\eta_1(r) - \eta_2(r)\right\|_{\mathscr{H}} dr ds \\
&\leq \int_0^T \left\|\eta_1(s) - \eta_2(s)\right\|_{\mathscr{H}} ds.
\end{aligned} \tag{4.95}$$

From (4.76) we deduce that

$$\left(\dot{\theta}_1 - \dot{\theta}_2, \theta_1 - \theta_2\right)_{L^2(\Omega)} + a_0\left(\theta_1 - \theta_2, \theta_1 - \theta_2\right) + \left(\chi_1 - \chi_2, \theta_1 - \theta_2\right)_{L^2(\Omega)} = 0.$$

We integrate this equality with respect to time, using the initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$ and inequality $a_0\left(\theta_1 - \theta_2, \theta_1 - \theta_2\right) \geq 0$ to find

$$\frac{1}{2}\left\|\theta_1(t) - \theta_2(t)\right\|_{L^2(\Omega)}^2 \leq \int_0^t \left(\chi_1(s) - \chi_2(s), \theta_1(s) - \theta_2(s)\right)_{L^2(\Omega)} ds,$$

which implies that

$$\left\|\theta_1(t) - \theta_2(t)\right\|_{L^2(\Omega)}^2 \leq \int_0^t \left\|\chi_1(s) - \chi_2(s)\right\|_{L^2(\Omega)}^2 ds + \int_0^t \left\|\theta_1(s) - \theta_2(s)\right\|_{L^2(\Omega)}^2 ds.$$

This inequality combined with Gronwall's inequality leads to

$$\left\|\theta_1(t) - \theta_2(t)\right\|_{L^2(\Omega)}^2 \leq C \int_0^t \left\|\chi_1(s) - \chi_2(s)\right\|_{L^2(\Omega)}^2 ds. \tag{4.96}$$

Form (4.82), deduced that

$$\begin{aligned}
&\left(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2\right)_{L^2(\Omega)} + a_1\left(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2\right) \\
&\qquad\qquad \leq \left(\lambda_1 - \lambda_2, \alpha_1 - \alpha_2\right)_{L^2(\Omega)}, \text{ a.e. } t \in (0,T).
\end{aligned}$$

integrate inequality with respect to time, using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, and inequality $a_1\left(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2\right) \geq 0$, we find

$$\tfrac{1}{2}\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)} \le C\int_0^t \left(\lambda_1(s)-\lambda_2(s),\alpha_1(s)-\alpha_2(s)\right)_{L^2(\Omega)} ds,$$

which implies

$$\begin{aligned}&\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)}\\ &\qquad \le C\left(\int_0^t \left\|\lambda_1(s)-\lambda_2(s)\right\|^2_{L^2(\Omega)} ds + \int_0^t \left\|\alpha_1(s)-\alpha_2(s)\right\|^2_{L^2(\Omega)} ds\right).\end{aligned}$$

This inequality combined with the Gronwall inequality leads to

$$\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)} \le C\int_0^t \left\|\lambda_1(s)-\lambda_2(s)\right\|^2_{L^2(\Omega)} ds\,, \forall t\in[0,T]. \tag{4.97}$$

Form the previous inequality and estimates (4.97), (4.96), (4.95) and (4.91) it follows now that

$$\begin{aligned}&\left\|\Lambda(\eta_1,\chi_1,\lambda_1)(t)-\Lambda(\eta_2,\chi_2,\lambda_2)(t)\right\|^2_{\mathscr{H}\times L^2(\Omega)\times L^2(\Omega)}\\ &\qquad \le C\int_0^T \left\|(\eta_1,\chi_1,\lambda_1)(s)-(\eta_2,\chi_2,\lambda_2)(s)\right\|^2_{\mathscr{H}\times L^2(\Omega)\times L^2(\Omega)} ds.\end{aligned} \tag{4.98}$$

Reiterating this inequality m times we obtain

$$\begin{aligned}&\left\|\Lambda^m(\eta_1,\chi_1,\lambda_1)-\Lambda^m(\eta_2,\chi_2,\lambda_2)\right\|^2_{C(0,T;\mathscr{H}\times L^2(\Omega)\times L^2(\Omega))}\\ &\qquad \le \frac{C^mT^m}{m!}\left\|(\eta_1,\chi_1,\lambda_1)-(\eta_2,\chi_2,\lambda_2)\right\|^2_{C(0,T;\mathscr{H}\times L^2(\Omega)\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathscr{H}\times L^2(\Omega)\times L^2(\Omega))$, and so Λ has a unique fixed point.

□

We have now all that is needed to complete the proof of Theorem 4.1.

Existence

Let $(\eta^*,\chi^*,\lambda^*)\in C(0,T;\mathscr{H}\times L^2(\Omega)\times L^2(\Omega))$ be the fixed point of Λ defined by (4.84)-(4.87) and let (u_η,σ_η) be the solution of problem $\mathscr{P}^1_\eta$. Let φ_η be the solution of problem $\mathscr{P}^3_\eta$ for $\eta=\eta^*$, let θ_{χ^*} be the solution of problem $\mathscr{P}_\chi$ for $\chi=\chi^*$ and let α_{λ^*} be the solution of problem $\mathscr{P}_\lambda$ for $\lambda=\lambda^*$. The equalities $\Lambda^1(\eta^*,\chi^*,\lambda^*)=\eta^*$, $\Lambda^2(\eta^*,\chi^*,\lambda^*)=\chi^*$ and $\Lambda^3(\eta^*,\chi^*,\lambda^*)=\lambda^*$ combined with (4.85)-(4.87) show that (4.35)-(4.39) are satisfied. Next (4.40) and the regularity (4.41)-(4.45) follow from Lemmas 4.3, 4.4, 4.5 and 4.6. Which concludes the existence part of the theorem.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ. and the unique solvability of the Problems $\mathscr{P}^1_\eta$, $\mathscr{P}^3_\eta$, $\mathscr{P}_\chi$ and $\mathscr{P}_\lambda$ which completes the proof.

Chapter 5

Electro elastic-viscoplastic contact problem with damage and adhesion

In this chapter we study a mathematical problem describing the adhesive contact between an electro elastic-viscopalastic material with damage and a foundation. The contact is modelled by the Signorini condition. The evolution of the damage is described by an inclusion of parabolic type. The adhesion process is modelled by a bonding field on the contact surface. The process is quasistatic. We derive a weak formulation of the system, then we prove the existence of a unique weak solution to the problem. The proof is based on a general result on evolution equations with maximal monotone operators, parabolic inequalities, differential equations and fixed point.

5.1 Formulation of the problem

We place ourselves in the physical framework that we presented in Chapter 1 of this thesis. The body is described by Electro-elastic-viscoplastic constitutive law with damage and adhision. The adhesion process is modelled by a bonding field on the contact surface. The contact is modelled by the Signorini condition. Under these considerations, the considered mechanical problem is formulated as follows.

Problem $P4$

Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, an electric displacement field $D : \Omega \times (0,T) \to \mathbb{R}^d$, a damage field $\alpha : \Omega \times (0,T) \to \mathbb{R}$, and a bonding field $\beta : \Gamma_3 \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned}
&\sigma(t) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) + \mathscr{E}^* E(\varphi(t)) \\
&+ \int_0^t \mathscr{G}\left(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s) - \mathscr{E}^* E(\varphi(s))), \varepsilon\left(u(s)\right), \alpha(s)\right) ds \quad \text{in } \Omega \times (0,T),
\end{aligned} \tag{5.1}$$

$$D = \mathscr{E}\varepsilon(u) + BE(\varphi) \qquad \text{in } \Omega \times (0,T), \tag{5.2}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\psi_K(\alpha) \ni S\left(\sigma - \mathscr{A}\varepsilon(\dot{u}) - \mathscr{E}^*\nabla(\varphi), \varepsilon(u), \alpha\right), \quad \text{in } \Omega \times (0,T), \tag{5.3}$$

$$\operatorname{Div}\sigma + f_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{5.4}$$

$$\operatorname{div} D - q_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{5.5}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{5.6}$$

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{5.7}$$

$$\begin{cases} u_\nu \leq 0 \\ \sigma_\nu - \gamma_\nu \beta^2 R_\nu\left(u_\nu\right) \leq 0, \\ \left(\sigma_\nu - \gamma_\nu \beta^2 R_\nu\left(u_\nu\right)\right) u_\nu = 0 \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{5.8}$$

$$-\sigma_\tau = p_\tau(\beta) R_\tau\left(u_\tau\right) \qquad \text{on } \Gamma_3 \times (0,T), \tag{5.9}$$

$$\dot{\beta}(t) = -\left(\gamma_\nu \beta(t) R_\nu\left(u_\nu(t)\right)^2 - \varepsilon_a\right)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{5.10}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{5.11}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{5.12}$$

$$D.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{5.13}$$

$$u(0) = u_0, \quad \alpha(0) = \alpha_0, \qquad \text{in } \Omega. \tag{5.14}$$

$$\beta(0) = \beta_0, \qquad \text{on } \Gamma_3. \tag{5.15}$$

First, equations (5.1) and (5.2) represent the electro elastic-viscoplastic constitutive law with damage, The evolution of the damage field is governed by the inclusion of parabolic type given by the relation (5.3). Equations (5.4) and (5.5) represent the equilibrium equations for the stress and electric displacement fields. Equations (5.6)-(5.7) are the displacement-traction conditions. The condition (5.8) represents the boundary conditions of contact with adhesion of the Signorini type that we have already introduced in (1.73). The relation (5.9) means that the resistance to tangential movement is generated by the glue and the tangential frictional pull is negligible. Thus, it depends only on the intensity of adhesion and the tangential displacement, but as long as it does not exceed the length of the link L recalled in the first part of this thesis Equation (5.10) represents the ordinary differential equation which describes the evolution of the bonding field, The relation (5.11) describes a homogeneous Neumann boundary condition. (5.12) and (5.13) represent the electric boundary conditions. Finally, The functions u_0, α_0 and β_0 in (5.14) and (5.15) are the initial data.

5.2 Variational formulation

In the study of the problem $P4$, we consider the following assumptions. The viscosity operator $\mathscr{A}:\Omega\times\mathbb{S}^d\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}\text{(a) } \mathscr{A}(x,\xi)=\left(a_{ijkl}(x)\xi_{kl}\right) \text{ for all } \xi\in\mathbb{S}^d, \text{ a.e. } x\in\Omega.\\ \text{(b) } a_{ijkl}=a_{klij}=a_{jikl}\in L^\infty(\Omega),\\ \text{(c) There exists } m_{\mathscr{A}}>0 \text{ such that}\\ \quad a_{ijkl}\xi_{ij}\xi_{kl}\ge m_{\mathscr{A}}\|\xi\|^2, \text{ for all } \xi=\left(\xi_{ij}\right)\in\mathbb{S}^d, \text{ a.e. } x\in\Omega.\end{cases}\tag{5.16}$$

The elasticity operator $\mathscr{B}:\Omega\times\mathbb{S}^d\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}\text{(a) There exists } L_{\mathscr{B}}>0 \quad \text{such that}\\ \|\mathscr{B}(x,\xi_1)-\mathscr{B}(x,\xi_2)\|\le L_{\mathscr{B}}\|\xi_1-\xi_2\|, \text{ for all } \xi_1,\xi_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega.\\ \text{(b) The mapping } x\mapsto\mathscr{B}(x,\xi) \text{ is Lebesgue measurable on } \Omega,\\ \text{for any } \xi\in\mathbb{S}^d.\\ \text{(c) The mapping } x\mapsto\mathscr{B}(x,0)\in\mathscr{H}.\end{cases}\tag{5.17}$$

The visco-plasticity operator $\mathscr{G}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\begin{cases}\text{(a) There exists a constant } L_{\mathscr{G}}>0 \quad \text{such that}\\ \|\mathscr{G}(x,\sigma_1,\xi_1,\beta_1)-\mathscr{G}(x,\sigma_2,\xi_2,\beta_2)\|\\ \qquad\qquad \le L_{\mathscr{G}}\left(\|\sigma_1-\sigma_2\|+\|\xi_1-\xi_2\|+\|\beta_1-\beta_2\|\right),\\ \text{for all } \xi_1,\xi_2,\sigma_1,\sigma_2\in\mathbb{S}^d, \text{ for all } \beta_1,\beta_2\in\mathbb{R}, \text{ a.e. } x\in\Omega.\\ \text{(b) The mapping } x\mapsto\mathscr{G}(x,\sigma,\xi,\beta) \text{ is Lebesgue measurable on } \Omega,\\ \qquad\qquad \text{for all } \sigma,\xi\in\mathbb{S}^d, \text{ for all } \beta\in\mathbb{R}.\\ \text{(c) The mapping } x\mapsto\mathscr{G}(x,0,0,0)\in\mathscr{H}.\end{cases}\tag{5.18}$$

The function $S:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\begin{cases}\text{(a) There exists a constant } L_S>0 \quad \text{such that}\\ \|S(x,\sigma_1,\xi_1,\beta_1)-S(x,\sigma_2,\xi_2,\beta_2)\|\\ \qquad\qquad \le L_S\left(\|\sigma_1-\sigma_2\|+\|\xi_1-\xi_2\|+\|\beta_1-\beta_2\|\right),\\ \text{for all } \xi_1,\xi_2,\sigma_1,\sigma_2\in\mathbb{S}^d, \text{ for all } \beta_1,\beta_2\in\mathbb{R}, \text{ a.e. } x\in\Omega.\\ \text{(b) The mapping } x\mapsto S(x,\sigma,\xi,\beta) \text{ is Lebesgue measurable on } \Omega,\\ \qquad\qquad \text{for all } \sigma,\xi\in\mathbb{S}^d, \text{ for all } \beta\in\mathbb{R}.\\ \text{(c) The mapping } x\mapsto S(x,0,0,0)\in L^2(\Omega).\end{cases}\tag{5.19}$$

We assume that the tangential contact function $p_\tau:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}_+$ satisfies

$$\begin{cases}\text{(a) There exists } L_\tau>0 \text{ such that}\\ \quad \|p_\tau(x,\alpha_1)-p_\tau(x,\alpha_2)\|\le L_\tau\|\alpha_1-\alpha_2\|,\\ \text{for all } \alpha_1,\alpha_2\in\mathbb{R}, \text{ a.e. } x\in\Gamma_3.\\ \text{(b) There exists } M_\tau>0 \text{ such that}\\ \quad \|p_\tau(x,\alpha)\|\le M_\tau, \text{ for all } \alpha\in\mathbb{R}, \text{ a.e. } x\in\Gamma_3.\\ \text{(c) For any } \alpha\in\mathbb{R},\ x\mapsto p_\tau(x,\alpha) \text{ is measurable on } \Gamma_3.\\ \text{(d) The mapping } x\mapsto p_\tau(x,0) \text{ belongs to } L^2(\Gamma_3).\end{cases}\tag{5.20}$$

Similarly, we assume that the electric permittivity operator B, the piezoelectric operator $\mathscr{E}$ satisfy the assumptions (2.20) and (2.21) respectively. Next, we assume that the adhesion coefficients and the limit bound satisfies

$$\gamma_\nu \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0 \text{ a.e. on } \Gamma_3. \tag{5.21}$$

The initial bonding field satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \text{ a.e. on } \Gamma_3. \tag{5.22}$$

and the initial damage field satisfies

$$\alpha_0 \in K. \tag{5.23}$$

Since $meas(\Gamma_1) > 0$ and the viscosity operator satisfies condition (5.16), it follows that the space V given by (1.3) is a Hilbert space endowed with the scalar product

$$(u,v)_V = (\mathscr{A}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}. \tag{5.24}$$

For Signorini problem, we use the convex subset of admissible displacements fields given by

$$U_{ad} = \{u \in V \mid u_\nu \leq 0 \text{ on } \Gamma_3\}, \tag{5.25}$$

and we consider the regularity assumption

$$u_0 \in U_{ad}. \tag{5.26}$$

Finally, the given potential satisfies

$$\varphi_0 \in L^2(\Gamma_3). \tag{5.27}$$

The forces, traction, volume and surface free charge densities satisfy

$$f_0 \in W^{1,1}\left(0,T;L^2(\Omega)^d\right), \quad f_2 \in W^{1,1}\left(0,T;L^2(\Gamma_2)^d\right), \tag{5.28}$$

$$q_0 \in C\left(0,T;L^2(\Omega)\right), \quad q_2 \in C\left(0,T;L^2(\Gamma_b)\right). \tag{5.29}$$

$$q_2(t) = 0 \text{ on } \Gamma_3, \quad \forall t \in (0,T). \tag{5.30}$$

Note that we need to impose assumption (5.30) for physical reasons. Indeed the foundation is assumed to be insulator. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi,\zeta) = k\int_\Omega \nabla\xi.\nabla\zeta dx, \tag{5.31}$$

and the microcrack diffusion coefficient verifies

$$k > 0. \tag{5.32}$$

Next. We define three mappings $j : L^2(\Gamma_3) \times V \times V \to \mathbb{R},$, $f : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$j(\beta,u,v) = \int_{\Gamma_3} p_\tau(\beta) R_\tau(u_\tau)\cdot v_\tau da - \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu da, \tag{5.33}$$

$$(f(t),v)_V = \int_\Omega f_0(t)\cdot v dx + \int_{\Gamma_2} f_2(t)\cdot v da, \tag{5.34}$$

$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta dx - \int_{\Gamma_b} q_2(t)\zeta da, \tag{5.35}$$

for all $u,v \in V$, $\varphi,\zeta \in W$ and $t \in (0,T)$. Note that

$$f \in W^{1,1}(0,T;V), \quad q \in C(0,T;W). \tag{5.36}$$

Suppose that $(u,\sigma,\varphi,\beta,\alpha)$ are regular functions satisfying (5.1)-(5.16). On the one hand, according to a standard procedure based on Green's formula (1.4), (1.14) combined with the conditions (5.4), (5.6), (5.7) and the definition of f, we have

$$(\sigma(t),\varepsilon(v)-\varepsilon(u(t)))_{\mathscr{H}} = (f(t),v-u(t))_V + \int_{\Gamma 3} \sigma\nu\cdot(v-u(t))da, \forall v \in V,$$

and taking into account the decomposition of the Cauchy tensor, we obtain

$$(\sigma(t),\varepsilon(v)-\varepsilon(u(t)))_{\mathscr{H}} = (f(t),v-u(t))_V + \int_{\Gamma 3} \sigma_\nu(t)(v_\nu - u_\nu(t))da$$
$$+ \int_{\Gamma 3} \sigma_\tau(t)\cdot(v_\tau - u_\tau(t))da \quad \forall v \in V.$$

Now, using (5.8) and (5.9) we find

$$u(t) \in U_{ad}, \quad (\sigma(t),\varepsilon(v)-\varepsilon(u(t)))_{\mathscr{H}} \geq (f(t),v-u(t))_V$$
$$+ \int_{\Gamma 3} \gamma_\nu \beta^2 R_\nu(u_\nu)(v_\nu - u_\nu(t))da - \int_{\Gamma 3} p_\tau(\beta) R_\tau(u_\tau)\cdot(v_\tau - u_\tau(t))da$$
$$\forall v \in U_{ad}.$$

We combine this last inequality with (5.33) to obtain

$$u(t) \in U_{ad}, \quad (\sigma(t),\varepsilon(v)-\varepsilon(u(t)))_{\mathscr{H}} + j(\beta(t),u(t),v-u(t))$$
$$\geq (f(t),v-u(t))_V \quad \forall v \in U_{ad}.$$

For the damage and electric field similarly in deducing the variational formulation of the previous problems. In accordance with the above, we derive the following variational formulation for the mechanical problem (5.1)-(5.15)

Problem *PV*

Find a displacement field $u : (0,T) \to V$, a stress field $\sigma : (0,T) \to \mathscr{H}$, an electric potential $\varphi : (0,T) \to W$, a damage field $\alpha : (0,T) \to H^1(\Omega)$, and a bonding field $\beta : (0,T) \to L^2(\Gamma_3)$, such that

$$\begin{aligned}\sigma(t) =& \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}(\varepsilon u(t)) + \mathscr{E}^*\nabla\varphi(t)\\ &+ \int_0^t \mathscr{G}\left(\sigma(s) - \mathscr{A}\varepsilon(\dot{u}(s) - \mathscr{E}^*\nabla\varphi(t)), \varepsilon\left(u(s), \alpha(s)\right)\right) ds,\end{aligned} \tag{5.37}$$

$$\begin{aligned}&u(t) \in U_{ad}, (\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_{\mathscr{H}} + j(\beta(t), u(t), v - u(t))\\ &\qquad \geq (f(t), v - u(t))_V, \quad \forall v \in U_{ad}, \quad \text{a.e. } t \in (0,T),\end{aligned} \tag{5.38}$$

$$(B\nabla\varphi(t), \nabla\phi)_H - (\mathscr{E}\varepsilon(u(t)), \nabla\phi)_H = (q(t), \phi)_W \quad \forall\phi \in W, t \in (0,T), \tag{5.39}$$

$$\begin{aligned}&\alpha(t) \in K, (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t))\\ &\qquad \geq \left(S\left(\sigma(t) - \mathscr{A}\varepsilon(\dot{u}(t)) - \mathscr{E}^*\nabla\varphi(t), \varepsilon(u(t)), \alpha(t)\right), \xi - \alpha(t)\right)_{L^2(\Omega)},\\ &\qquad \forall\xi \in K, t \in (0,T),\end{aligned} \tag{5.40}$$

$$\dot{\beta}(t) = -\left(\gamma_\nu\beta(t)R_\nu\left(u_\nu(t)\right)^2 - \varepsilon_a\right)_+ . \text{ a.e. } t \in (0,T), \tag{5.41}$$

$$u(0) = u_0, \quad \beta(0) = \beta_0. \tag{5.42}$$

Our main result, concerning the well-posedness of the problem PV , is stated next and proved in next section. We also take into account this note, which we need later.

Remark 5.1. We note that, in Problem P and in Problem PV, we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, (5.41)-(5.42) guarantees that $\beta(x,t) \leq \beta_0(x)$ and, therefore, assumption (5.22) shows that $\beta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\beta(x,t_0) = 0$ at time t_0, then it follows from (5.41)-(5.42) that $\dot{\beta}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\beta(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \beta(x,t) \leq 1$ for all $t \in [0,T]$, a.e. $x \in \Gamma_3$.

5.3 Result of existence and uniqueness

Theorem 5.2. *Assume that (5.16)-(5.30) hold. Then there exists a unique solution* $(u, \sigma, \varphi, \beta, \alpha, D)$ *to problem PV. Moreover, the solution has the regularity*

$$u \in W^{1,\infty}(0,T;V), \tag{5.43}$$

$$\varphi \in C(0,T;W), \tag{5.44}$$

$$\sigma \in W^{1,\infty}(0,T;\mathscr{H}), \quad Div\sigma \in W^{1,\infty}(0,T;H), \tag{5.45}$$

$$\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{5.46}$$

$$\beta \in W^{1,\infty}\left(0,T;L^2\left(\Gamma_3\right)\right) \cap \mathscr{Q}. \tag{5.47}$$

$$D \in C(0,T;\mathscr{W}). \tag{5.48}$$

The functions u, σ, φ, α, β, and D which satisfy (5.37)-(5.42) are called a weak solution of the contact problem P. We conclude that, under the assumptions (5.16)-(5.30), the mechanical problem (5.1)-(5.15) has a unique weak solution satisfying (5.43)-(5.48). The proof of Theorem 5.2, is carried out is several steps and is based on a general result on evolution equations with maximal monotone operators, parabolic inequalities, differential equations and fixed point. We denote by C a constant whose value may change from line to line when no confusing can arise. Let $\eta \in L^\infty(0,T;\mathscr{H})$ be given, in the first step, we consider the following variational problem.

Problem $\mathscr{P}^1_\eta$

Find a displacement field $u_\eta : (0,T) \to V$, such that

$$\begin{aligned} &(\mathscr{A}\varepsilon(\dot{u}_\eta(t)),\varepsilon(v-u_\eta(t)))_{\mathscr{H}} + (\mathscr{B}\varepsilon(u_\eta(t)),\varepsilon(v-u_\eta(t)))_{\mathscr{H}} + (\eta(t),v-u_\eta(t))_{\mathscr{H}} \\ &\qquad \geq (f(t),v-u_\eta(t))_V\,, \forall v\in V, \text{ a.e. } t\in(0,T), \end{aligned} \tag{5.49}$$

$$u_\eta(0) = u_0. \tag{5.50}$$

We have the following result for $\mathscr{P}^1_\eta$

Lemma 5.3. *There exists a unique solution to Problem $\mathscr{P}^1_\eta$ with the regularity (5.43).*

Proof. We use the Riesz representation Theorem to define

$$(Bu,v)_V = (\mathscr{B}\varepsilon(u),\varepsilon(v))_{\mathscr{H}}, \quad \forall u,v\in V. \tag{5.51}$$

Now, from (5.24), (5.16), (5.17) and (5.51), we obtain that

$$\|Bu_1 - Bu_2\|_V \leq \frac{L_{\mathscr{B}}}{m_{\mathscr{A}}}\|u_1-u_2\|_V\,, \quad \forall u_1,u_2\in V, \tag{5.52}$$

that is, B is a Lipschitz continuous operator. Moreover, the operator

$$B + \frac{L_{\mathscr{B}}}{m_{\mathscr{A}}} I_V : V \to V,$$

is a monotone Lipschitz continuous operator on V. Let $\Psi_{U_{ad}} : V \to]-\infty;+\infty]$ denote the indicator function of the set U_{ad} and let $\partial\Psi_{U_{ad}}$ be the subdifferential of $\Psi_{U_{ad}}$, as U_{ad} is a nonempty, convex, closed part of V, it follows that $\partial\Psi_{U_{ad}}$ is a maximal monotone operator on V and $D(\partial\Psi_{U_{ad}}) = U_{ad}$. Moreover, the sum

$$\partial\Psi_{U_{ad}} + B + \frac{L_{\mathscr{B}}}{m_{\mathscr{A}}} I_V : U_{ad} \subset V \to 2^V,$$

is a maximal monotone operator. Let the function $f_\eta(t)\in V$ given by

$$(f_\eta(t),v)_V = (f(t),v)_V - (\eta(t),v)_{\mathscr{H}}, \quad \forall v\in U_{ad},$$

keeping in mind $\eta\in L^\infty(0,T;\mathscr{H})$, it follow that $f_\eta\in W^{1,\infty}(0,T;V)$. We take $A = \partial\Psi_{U_{ad}} + B : D(A) = U_{ad}\subset V \to 2^V$. Thus, conditions (5.26) allow us to apply classical result on evolution equations with maximal monotone operators which may be found in ([6], p. 32), that there exists a unique element $u_\eta\in W^{1,\infty}(0,T;V)$ such that

$$\begin{aligned} &\dot{u}_\eta(t) + Au_\eta(t) \ni f_\eta(t), \\ &u_\eta(0) = u_0. \end{aligned}$$

Therefore

$$\dot{u}_\eta(t) + \partial\Psi_{U_{ad}}(u_\eta(t)) + Bu_\eta(t) \ni f_\eta(t) \text{ a.e.t } \in (0,T), \tag{5.53}$$

$$u_\eta(0) = u_0. \tag{5.54}$$

For all $u, g \in V$, we have

$$g \in \partial\Psi_{Uad}(u) \Leftrightarrow u \in U_{ad}, (g, v-u)_V \leq 0, \quad \forall v \in U_{ad},$$

so after inclusion (5.53) we have $u_\eta(t) \in U_{ad}$ and the following variational inequality

$$\begin{aligned}(\dot{u}_\eta(t), v-u_\eta(t))_V + (Bu_\eta(t), v-u_\eta(t))_V \\ \geq (f_\eta(t), v-u_\eta(t))_V, \forall v \in U_{ad} \text{ a.e. } t \in (0,T).\end{aligned} \tag{5.55}$$

Then by using (5.24), (5.51), we deduce that there exists a unique solution of problem $\mathscr{P}_\eta^1$ and it satisfies (5.43).

□

In the second step we use the displacement field obtained in Lemma 5.3, to construct the following variational problem for the an electrical potential.

Problem $\mathscr{P}_\eta^2$

Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$(B\nabla\varphi_\eta(t), \nabla\phi)_H - (\mathscr{E}\varepsilon(u_\eta(t)), \nabla\phi)_H = (q(t), \phi)_W, \quad \forall \phi \in W, t \in (0,T). \tag{5.56}$$

We have the following result for problem $\mathscr{P}_\eta^2$

Lemma 5.4. *Problem (5.56) has unique solution φ_η which satisfies the regularity (5.44).*

Proof. See the proof of Lemma 2.3 (Chapter 2)

□

In the third step we let $\lambda \in L^2(0,T; L^2(\Omega))$

Problem $\mathscr{P}_\lambda$

Find the damage field $\alpha_\lambda : (0,T) \to H^1(\Omega)$ such that $\alpha_\lambda(t) \in K$ and

$$\begin{aligned}(\dot{\alpha}_\lambda(t), \xi - \alpha_\lambda)_{L^2(\Omega)} + a(\alpha_\lambda(t), \xi - \alpha_\lambda(t)) \\ \geq (\theta(t), \xi - \alpha_\lambda(t))_{L^2(\Omega)}, \quad \forall \xi \in K, \text{ a.e. } t \in (0,T),\end{aligned} \tag{5.57}$$

$$\alpha_\lambda(0) = \alpha_0. \tag{5.58}$$

We have the following result.

Lemma 5.5. *There exists a unique solution α_λ to the auxiliary problem $\mathscr{P}_\lambda$ satisfying (5.46).*

Proof. See the proof of Lemma 3.5 (Chapter 3)

□

Problem $\mathscr{P}_\beta$

Find a bonding field $\beta : (0,T) \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_\eta(t) = -\left(\gamma_\nu \beta_\eta(t) R_\nu\left(u_{\eta\nu}(t)\right)^2 - \varepsilon_a\right)_+ \text{ a.e. } t \in (0,T), \tag{5.59}$$

$$\beta_\eta(0) = \beta_0. \tag{5.60}$$

We obtain the following result.

Lemma 5.6. *There exists a unique solution β_η to Problem $\mathscr{P}_\beta$ and it satisfies $\beta_\eta \in W^{1,\infty}((0,T;L^2(\Gamma_3)) \cap \mathscr{Q}$.*

Proof. For the sake of simplicity we suppress the dependence of various functions on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -\left(\beta\left(\gamma_\nu\left(R_\nu\left(u_{\eta\nu}(t)\right)\right)^2 - \varepsilon_a\right)_+\right),$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. We show that F_η is Lipschitz continuous with respect to the second variable, uniformly in time; and, for all $\beta \in L^2(\Gamma_3)$, mapping $t \mapsto F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of the classical Cauchy-Lipschitz theorem given in ([57], p. 60) we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to Problem $\mathscr{P}_\beta$. Also, the arguments used in Remark 5.1 show that $0 \leq \beta_\eta(t) \leq 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathscr{Q}$ (1.8), we find that $\beta_\eta \in \mathscr{Q}$, which concludes the proof of Lemma 5.6.

□

Let us consider now the auxiliary problem

Problem $\mathscr{P}_{\eta,\lambda}$

Find the stress field $\sigma_{\eta,\lambda} : (0,T) \to \mathscr{H}$ which is a solution of the problem

$$\sigma_{\eta,\lambda}(t) = \mathscr{B}\left(\varepsilon\left(u_\eta(t)\right)\right) + \int_0^t \mathscr{G}\left(\sigma_{\eta,\lambda}(s), \varepsilon\left(u_\eta(s)\right), \alpha_\lambda(s)\right) ds, \qquad \text{a.e. } t \in (0,T). \tag{5.61}$$

Lemma 5.7. *$\mathscr{P}_{\eta,\lambda}$ has a unique solutions $\sigma_{\eta,\lambda} \in W^{1,\infty}(0,T;\mathscr{H})$. Moreover, if $\sigma_{\eta_i,\lambda_i}$, u_{η_i}, α_{η_i} represent the solutions of Problems $\mathscr{P}_{\eta,\lambda}$, $\mathscr{P}^1_\eta$ and, $\mathscr{P}_\lambda$ respectively, for $(\eta_i,\lambda_i) \in W^{1,\infty}(0,T;\mathscr{H} \times L^2(\Omega))$, $i = 1,2$ then there exists $C > 0$ such that*

$$\begin{aligned} \left\|\sigma_{\eta_1,\lambda_1}(t) - \sigma_{\eta_2,\lambda_2}(t)\right\|^2_{\mathscr{H}} \leq & C\left(\left\|u_{\eta_1}(t) - u_{\eta_2}(t)\right\|^2_V + \int_0^t \left\|u_{\eta_1}(s) - u_{\eta_2}(s)\right\|^2_V \right. \\ & \left. + \int_0^t \left\|\alpha_{\lambda_1}(s) - \alpha_{\lambda_2}(s)\right\|^2_{L^2(\Omega)} ds\right). \end{aligned} \tag{5.62}$$

Proof. Let $\Pi_{\eta,\lambda} : W^{1,\infty}(0,T;\mathscr{H}) \to W^{1,\infty}(0,T;\mathscr{H})$ be the operator given by

$$\Pi_{\eta,\lambda}\sigma(t) = \mathscr{B}\varepsilon\left(u_\eta(t)\right) + \int_0^t \mathscr{G}\left(\sigma(s), \varepsilon\left(u_\eta(s)\right), \alpha_\lambda(s)\right) ds. \tag{5.63}$$

Let $\sigma_i \in W^{1,\infty}(0,T;\mathscr{H})$, $i=1,2$ and $t_1 \in (0,T)$. Using hypothesis (5.18) and Holder's inequality, we find

$$\left\|\Pi_{\eta,\lambda}\sigma_1(t_1) - \Pi_{\eta,\lambda}\sigma_2(t_1)\right\|_{\mathscr{H}}^2 \leq L_{\mathscr{G}}^2 T \int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds.$$

Integration on the time interval $(0,t_2) \subset (0,T)$, it follows that

$$\int_0^{t_2} \left\|\Pi_{\eta,\lambda}\sigma_1(t_1) - \Pi_{\eta,\lambda}\sigma_2(t_1)\right\|_{\mathscr{H}}^2 dt_1 \leq L_{\mathscr{G}}^2 T \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds dt_1.$$

Therefore

$$\left\|\Pi_{\eta,\lambda}\sigma_1(t_2) - \Pi_{\eta,\lambda}\sigma_2(t_2)\right\|_{\mathscr{H}}^2 \leq L_{\mathscr{G}}^4 T^2 \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds dt_1.$$

For $t_1, t_2, ..., t_n \in (0,T)$, we generalize the procedure above by recurrence on n. We obtain the inequality

$$\begin{aligned}&\left\|\Pi_{\eta,\lambda}\sigma_1(t_n) - \Pi_{\eta,\lambda}\sigma_2(t_n)\right\|_{\mathscr{H}}^2 \\ &\qquad \leq L_{\mathscr{G}}^{2n} T^n \int_0^{tn} \cdots \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds dt_1 \dots dt_{n-1}.\end{aligned}$$

Which implies

$$\left\|\Pi_{\eta,\lambda}\sigma_1(t_n) - \Pi_{\eta,\lambda}\sigma_2(t_n)\right\|_{\mathscr{H}}^2 \leq \frac{L_{\mathscr{G}}^{2n} T^{n+1}}{n!} \int_0^{T} \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\left\|\Pi_{\eta,\lambda}\sigma_1 - \Pi_{\eta,\lambda}\sigma_2\right\|_{W^{1,\infty}(0,T;\mathscr{H})}^2 \leq \frac{L_{\mathscr{G}}^{2n} T^{n+2}}{n!} \|\sigma_1 - \sigma_2\|_{W^{1,\infty}(0,T;\mathscr{H})}^2.$$

It follows from this inequality that for large n enough, the operator $\Pi_{\eta,\lambda}^n$ is a contraction on the Banach space $W^{1,\infty}(0,T;\mathscr{H})$, and therefore there exists a unique element $\sigma \in W^{1,\infty}(0,T;\mathscr{H})$ such that $\Pi_{\eta,\lambda}\sigma = \sigma$. Moreover, σ is the unique solution of Problem $\mathscr{P}_{\eta,\lambda}$. Consider now $(\eta_1,\lambda_1),(\eta_2,\lambda_2) \in W^{1,\infty}(0,T;\mathscr{H} \times L^2(\Omega))$ and for $i=1,2$ denote $u_{\eta_i} = u_i$, $\alpha_{\lambda_i} = \alpha_i$, and $\sigma_{\eta_i,\lambda_i} = \sigma_i$. We have

$$\sigma_i(t) = \mathscr{B}\varepsilon(u_i(t)) + \int_0^t \mathscr{G}(\sigma_i(s), \varepsilon(u_i(s)), \alpha_i(s))\, ds, \quad \text{a.e. } t \in (0,T). \tag{5.64}$$

□

and using the properties (5.17) and (5.18) of $\mathscr{B}$ and $\mathscr{G}$ we find

$$\begin{aligned}&\|\sigma_1(t) - \sigma_2(t)\|_{\mathscr{H}}^2 \\ &\leq C\Big(\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds \\ &\quad + \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2 ds\Big), \quad \forall t \in (0,T).\end{aligned} \tag{5.65}$$

We use Gronwall argument in the previous inequality to deduce (5.62), which concludes the proof of Lemma 4.5. Finally, as a consequence of these results and using the properties of the operator $\mathscr{G}$ the operator $\mathscr{E}$, the function S for $t \in (0,T)$, we consider the element

$$\Lambda(\eta,\lambda)(t) = \left(\Lambda^1(\eta,\lambda)(t), \Lambda^2(\eta,\lambda)(t)\right) \in \mathscr{H} \times L^2(\Omega), \tag{5.66}$$

defined by

$$\begin{aligned}\left(\Lambda^1(\eta,\lambda)(t), v\right)_{\mathscr{H}\times V} &= \left(\mathscr{E}^*\nabla\varphi_\eta(t), \varepsilon(v)\right)_{\mathscr{H}} + j(\alpha_\eta(t), u_\eta(t), v) \\ &+ \left(\int_0^t \mathscr{G}\left(\sigma_{\eta,\lambda}(s), \varepsilon\left(u_\eta(s)\right), \alpha_\lambda(s)\right) ds, \varepsilon(v)\right)_{\mathscr{H}}, \forall v \in V,\end{aligned} \tag{5.67}$$

$$\Lambda^2(\eta,\lambda)(t) = S\left(\sigma_{\eta,\lambda}, \varepsilon\left(u_\eta(t)\right), \alpha_\lambda(t)\right). \tag{5.68}$$

Here, for every $(\eta,\lambda) \in W^{1,\infty}(0,T;\mathscr{H}\times L^2(\Omega))$. u_η, φ_η, α_λ, β_η and $\sigma_{\eta,\lambda}$ represent the displacement field, the electric potential field, the damage, a bonding field, field and the stress field, obtained in Lemmas 5.3, 5.4, 5.5, 5.6 and 5.7 respectively. We have the following result.

Lemma 5.8. *The mapping Λ has a fixed point $(\eta^*,\lambda^*) \in W^{1,\infty}(0,T;\mathscr{H}\times L^2(\Omega))$, such that $\Lambda(\eta^*,\lambda^*) = (\eta^*,\lambda^*)$.*

Proof. Let $t \in (0,T)$ and $(\eta_1,\lambda_1),(\eta_2,\lambda_2) \in W^{1,\infty}\left(0,T;\mathscr{H}\times L^2(\Omega)\right)$. We use the notation that $u_{\eta_i} = u_i$, $\alpha_{\lambda_i} = \alpha_i$, $\varphi_{\eta_i} = \varphi_i$, $\beta_\eta = \beta_i$ and $\sigma_{\eta_i,\lambda_i} = \sigma_i$ for $i = 1,2$. Let us start by using (1.7), (1.13) hypotheses (5.18), (2.21), (5.20), (5.21) and the definition of R_ν, R_τ and Remark 5.1 we have

$$\begin{aligned}&\left\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_2)(t)\right\|_{\mathscr{H}}^2 \\ &\le \|\mathscr{E}^*\nabla\varphi_1(t) - \mathscr{E}^*\nabla\varphi_2(t)\|_{\mathscr{H}}^2 \\ &\quad + \int_0^t \|\mathscr{G}\left(\sigma_1(s), \varepsilon\left(u_1(s)\right), \alpha_1(s)\right) - \mathscr{G}\left(\sigma_2(s), \varepsilon\left(u_2(s)\right), \alpha_2(s)\right)\|_{\mathscr{H}}^2 ds \\ &\quad + C\left(\left\|\beta_1^2(t)R_\nu\left(u_{1\eta\nu}(t)\right) - \beta_2^2(t)R_\nu\left(u_{2\eta\nu}(t)\right)\right\|_{L^2(\Gamma_3)}^2\right) \\ &\quad + C\left(\left\|p_\tau(\beta_1(t))R_\tau\left(u_{1\eta\tau}(t)\right) - p_\tau(\beta_2(t))R_\tau\left(u_{2\eta}\tau(t)\right)\right\|_{L^2(\Gamma_3)}^2\right),\end{aligned} \tag{5.69}$$

so we obtain

$$\begin{aligned}&\left\|\Lambda^1(\eta_1,\lambda_1)(t) - \Lambda^1(\eta_2,\lambda_2)(t)\right\|_{\mathscr{H}}^2 \\ &\le C\Big(\|\varphi_1(t) - \varphi_2(t)\|_W^2 + \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds \\ &+ \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds \\ &+ \|u_1(t) - u_2(t)\|_V^2 + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Omega)}^2\Big).\end{aligned} \tag{5.70}$$

We use estimate (5.62) to obtain

$$\begin{aligned}&\left\|\Lambda^1(\eta_1,\lambda_1)(t) - \Lambda^1(\eta_2,\lambda_2)(t)\right\|_{\mathscr{H}}^2 \\ &\le C\Big(\|\varphi_1(t) - \varphi_2(t)\|_W^2 + \|u_1(s) - u_2(s)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds \\ &\quad + \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}^2 + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2 ds\Big).\end{aligned} \tag{5.71}$$

By similar arguments, from (5.68), (5.62) and (5.19) we obtain

$$
\begin{aligned}
&\left\|\Lambda^2(\eta_1,\lambda_1)(t)-\Lambda^2(\eta_2,\lambda_2)(t)\right\|^2_{\mathscr{H}}\\
&\leq C\Big(\|u_1(t)-u_2(t)\|_V^2+\int_0^t\|u_1(s)-u_2(s)\|_V^2\\
&\quad+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds\Big), \qquad \text{a.e. } t\in(0,T).
\end{aligned}
\tag{5.72}
$$

Therefore,

$$
\begin{aligned}
&\|\Lambda(\eta_1,\lambda_1)(t)-\Lambda(\eta_2,\lambda_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)}\leq C\Big(\|\varphi_1(t)-\varphi_2(t)\|_W^2\\
&+\|u_1(t)-u_2(t)\|_V^2+\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\\
&+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\Big) \qquad \text{a.e. } t\in(0,T).
\end{aligned}
\tag{5.73}
$$

For the electric potential, we use (5.56) for $\eta=\eta_1$ to obtain

$$
(B\nabla\varphi_1,\nabla\phi)_H=(\mathscr{E}\varepsilon(u_1),\nabla\phi)_H+(q(t),\phi)_W \quad \forall\phi\in W,
\tag{5.74}
$$

and for $\eta=\eta_2$, we have

$$
(B\nabla\varphi_2,\nabla\phi)_H=(\mathscr{E}\varepsilon(u_2),\nabla\phi)_H+(q(t),\phi)_W \quad \forall\phi\in W,
\tag{5.75}
$$

we set $\phi=\varphi_1-\varphi_2$ and subtract (5.74) and (5.75) let's find

$$
(B\nabla(\varphi_1-\varphi_2),\nabla(\varphi_1-\varphi_2))_H=(\mathscr{E}\varepsilon(u_1-u_2),\nabla(\varphi_1-\varphi_2))_H,
$$

from (2.20) (2.21) we have

$$
\|\varphi_1(t)-\varphi_2(t)\|_W\leq C\|u_1(t)-u_2(t)\|_V.
\tag{5.76}
$$

Next, we use (5.55) to find that

$$
\begin{aligned}
&(\dot u_1(t)-\dot u_2(t),u_1(t)-u_2(t))_V\leq(\eta_2(t)-\eta_1(t),u_1(t)-u_2(t))_{\mathscr{H}}\\
&+(Bu_2(t)-Bu_1(t),u_1(t)-u_2(t))_V,
\end{aligned}
$$

using Cauchy-Schwarz inequality and (5.52) we obtain

$$
\begin{aligned}
&(\dot u_1(t)-\dot u_2(t),u_1(t)-u_2(t))_V\leq\|\eta_1(t)-\eta_2(t)\|_{\mathscr{H}}\|u_1(t)-u_2(t)\|_V\\
&\quad+\frac{L_{\mathscr{B}}}{m_{\mathscr{A}}}\|u_1(t)-u_2(t)\|_V^2.
\end{aligned}
$$

We integrate this inequality with respect to time and use the initial conditions $u_1(0)=u_2(0)=u_0$ to find that

$$
\begin{aligned}
&\frac{1}{2}\|u_1(t)-u_2(t)\|_V^2\leq\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}}\|u_1(s)-u_2(s)\|_V\,ds\\
&\quad+\frac{L_{\mathscr{B}}}{m_{\mathscr{A}}}\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds.
\end{aligned}
$$

Applying the inequality

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \quad a, b \in \mathbb{R},$$

we find that

$$\frac{1}{2}\|u_1(t) - u_2(t)\|_V^2 \leq \frac{1}{2}\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathscr{H}}^2 ds + \frac{1}{2}\int_0^t \|u_1(s) - u_2(s)\|_V^2 ds$$
$$+ \frac{L_{\mathscr{B}}}{m_{\mathscr{A}}}\int_0^t \|u_1(s) - u_2(s)\|_V^2 ds$$

where we obtain

$$\|u_1(t) - u_2(t)\|_V^2 \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathscr{H}}^2 ds + C\int_0^t \|u_1(s) - u_2(s)\|_V^2 ds,$$

and, after a Gronwall argument, we obtain

$$\|u_1(t) - u_2(t)\|_V^2 \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathscr{H}}^2 ds. \tag{5.77}$$

On the other hand, from the Cauchy problem (5.59)-(5.60) we can write

$$\beta_i(t) = \beta_0 - \int_0^t \left(\gamma_\nu \beta_i(s) R_\nu\left(u_{i\nu}(s)\right)^2 - \varepsilon_a\right)_+ ds.$$

Using now the definition of R_ν, the inequality $|R_\nu(u_\nu)| \leq L$, and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we obtain

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \leq C\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds$$
$$+ C\int_0^t \|u_{1\nu}(s) - u_{2\nu}(s)\|_{L^2(\Gamma_3)} ds.$$

We apply Gronwalls inequality to deduce that

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \leq C\int_0^t \|u_{1\nu}(s) - u_{2\nu}(s)\|_{L^2(\Gamma_3)} ds,$$

and, using (1.7) we obtain

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \leq C\int_0^t \|u_1(s) - u_2(s)\|_V ds. \tag{5.78}$$

Form (5.57), we deduce that

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \leq (\lambda_1 - \lambda_2, \alpha_1 - \alpha_2)_{L^2(\Omega)}, \quad \forall t \in (0, T). \tag{5.79}$$

Integrating the previous inequality with respect to time, using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$ and inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \geq 0$ to find

$$\frac{1}{2}\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 \leq \int_0^t (\lambda_1(s) - \lambda_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)} ds. \tag{5.80}$$

This inequality, combined with Gronwalls inequality, leads to

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)} \leq C\int_0^t \|\lambda_1(s)-\lambda_2(s)\|^2_{L^2(\Omega)}\,ds, \quad \forall t\in(0,T), \tag{5.81}$$

Form the previous inequality and estimates (5.73), (5.76), (5.77), (5.78) and (5.81) it follows now that

$$\begin{aligned}&\|\Lambda(\eta_1,\lambda_1)(t)-\Lambda(\eta_2,\lambda_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)}\\ &\qquad \leq C\int_0^T \|(\eta_1,\lambda_1)(s)-(\eta_2,\lambda_2)(s)\|^2_{\mathscr{H}\times L^2(\Omega)}\,ds.\end{aligned} \tag{5.82}$$

Reiterating this inequality m times we obtain

$$\begin{aligned}&\|\Lambda^m(\eta_1,\theta_1)-\Lambda^m(\eta_2,\theta_2)\|^2_{W^{1,\infty}(0,T;\mathscr{H}\times L^2(\Omega))}\\ &\qquad \leq \frac{C^mT^m}{m!}\|(\eta_1,\lambda_1)-(\eta_2,\lambda_2)\|^2_{W^{1,\infty}(0,T;\mathscr{H}\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $W^{1,\infty}(0,T;\mathscr{H}\times L^2(\Omega))$, and so Λ has a unique fixed point.

□

Now, we have all the ingredients needed to prove Theorem 5.2.

Existence

Let $(\eta^*,\lambda^*)\in W^{1,\infty}(0,T;\mathscr{H}\times L^2(\Omega))$ be the fixed point of Λ and

$$u=u_{\eta^*},\quad \varphi_{\eta^*}=\varphi,\quad \sigma=\mathscr{A}\varepsilon(\dot u)+\mathscr{E}^*\nabla\varphi(t)+\sigma_{\eta^*\lambda^*}, \tag{5.83}$$

$$\alpha=\alpha_{\lambda^*},\quad \beta=\beta_{\eta^*}. \tag{5.84}$$

$$D=\mathscr{E}\varepsilon(u)+B\nabla(\varphi). \tag{5.85}$$

We prove that $(u,\sigma,\varphi,\beta,\alpha,D)$ satisfies (5.37)-(5.42) and (5.43)-(5.48). Indeed, we write (5.61) for $\eta^*=\eta,\quad \theta^*=\theta$ and use (5.83)-(5.84) to obtain that (5.37) is satisfied. Now we consider (5.49) for $\eta^*=\eta$ and use the first equality in (5.83) to find

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot u_\eta(t)),\varepsilon(v-u_\eta(t)))_{\mathscr{H}}+(\mathscr{B}\varepsilon(u_\eta(t)),\varepsilon(v-u_\eta(t)))_{\mathscr{H}}+(\eta^*(t),\varepsilon(v-u)_\eta(t))_{\mathscr{H}}\\ &\qquad \geq (f(t),v-u_\eta(t))_V,\forall v\in V,\ \text{a.e. } t\in(0,T),\end{aligned} \tag{5.86}$$

The equalities $\Lambda^1(\eta^*,\lambda^*)=\eta^*$, and $\Lambda^2(\eta^*,\lambda^*)=\lambda^*$, combined with (5.67)-(5.68), (5.83) and (5.84) show that for all $v\in V$,

$$\begin{aligned}(\eta^*(t),v)_{\mathscr{H}\times V}&=(\mathscr{E}^*\nabla\varphi(t),\varepsilon(v))_{\mathscr{H}}+j(\beta(t),u(t),v)\\ &\quad+\left(\int_0^t \mathscr{G}(\sigma(s)-\mathscr{A}\varepsilon(\dot u(s))-\mathscr{E}^*\nabla\varphi(t),\varepsilon(u(s)),\alpha(s))ds,\varepsilon(v)\right)_{\mathscr{H}},\end{aligned} \tag{5.87}$$

$$\lambda^*(t)=S(\sigma(s)-\mathscr{A}\varepsilon(\dot u(s)-\mathscr{E}^*\nabla\varphi(t),\varepsilon(u(t)),\alpha(t)). \tag{5.88}$$

We substitute (5.87) in (5.86)) and use (5.37) to see that (5.38) is satisfied. We write now (5.57) for $\lambda = \lambda^*$ and use (5.84) and (5.88) to find (5.40). We write (5.56), (5.59) for $\eta = \eta^*$ and use (5.83)-(5.84) to find that (5.39), (5.41) are satisfied. Next, (5.42), The regularities (5.43), (5.44), (5.46) and (5.47) follow from Lemmas 5.3, 5.4, 5.6, 5.7. and the regularity (5.45) follows from lemma 5.5. Let now $t_1, t_2 \in [0,T]$, from (2.20), (2.21), (1.11) and (5.85), we conclude that there exists a positive constant $C > 0$ verifying

$$\|D(t_1) - D(t_2)\|_H \leq C\left(\|\varphi(t_1) - \varphi(t_2)\|_W + \|u(t_1) - u(t_2)\|_V\right).$$

The regularity of u and φ given by (5.43) and (5.44) implies

$$D \in C(0,T;H). \tag{5.89}$$

We choose $\phi \in D(\Omega)^d$ in (3.39) and using (5.35) we find

$$\operatorname{div} D_*(t) = q_0(t), \quad \forall t \in [0,T], \tag{5.90}$$

By (5.29) and (5.89) we obtain

$$D \in C(0,T;\mathscr{W}).$$

Which concludes the existence part of the theorem.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ.

Conclusion

This thesis introduce a contribution to the study of some boundary problems in contact mechanics. For this purpose, different constitutive laws have been considered such as: Electro-viscoelastic, Electro-Elastic-viscoplastic constitutive laws with internal damage of the material and a thermal effect. We have studied contact problems in a quasistatic process with boundary conditions, for which we couple a thermal effect and damage and adhesion or an electrical effect. For each of these problems, we give the variational formulation, then the existence and the uniqueness of the weak solution.

In the future, it would be interesting to extend the results of the present thesis by the existence and the uniqueness of these problems by the method of variational control as well as the numerical study.

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Abstract
The objective of this work is the study of some problems in Contact Mechanics for Electro-viscoelastic, electro-elastic-viscoplastic constitutive laws with internal damage of the material and a thermal effect with or without adhesion. We obtain variational formulations followed by the results of existence and uniqueness of weak solutions. The proof is based on arguments of evolutionary variational inequalities, parabolic inequalities, differential equations, and fixed point theorem.
Key words and phrases: Electro-viscoelasticity, quasistatic, elastic-viscoplastic, friction contact, damage, adhesion, compliance normal, normal damped response, Signorini's condition, temperature, differential equations, fixed point.
AMS Subject Classification:
74C10, 49J40, 74M10, 74M15, 47H10.

Résumé
L'objectif de ce travail est l'étude de quelques problèmes de Mécanique du Contact pour les lois de comportement électro-viscoélastiques, électro- élasto-viscoplastiques avec endommagement interne du matériau et effet thermique avec ou sans adhérence. On obtient des formulations variationnelles suivies des résultats d'existence et d'unicité des solutions faibles. La preuve est basée sur des arguments d'inégalités variationnelles évolutives, d'inégalités paraboliques, d'équations différentielles et du théorème du point fixe.
Mots clés: Electro-viscoélasticité, quasistatique, élasto-viscoplastique, frottement de contact, endommagement, adhérence, compliance normale, réponse normale amortie, condition de Signorini, température, équations différentielles, point fixe.
AMS :
74C10, 49J40, 74M10, 74M15, 47H10.

ملخص
الهدف من هذا العمل هو دراسة بعض المشاكل في ميكانيكا التلامس للقوانين التأسيسية اللزجة الكهربية ، والمرونة الكهربية اللزجة مع الضرر الداخلي للمادة والتأثير الحراري أثناء الالتصاق أو بدونه. نحصل على صيغ متغيرة متبوعة بنتائج وجود الحلول الضعيفة وتفردها. يعتمد الدليل على حجج عدم المساواة التطورية المتغيرة، وعدم المساواة المكافئ، والمعادلات التفاضلية، ونظرية النقطة الثابتة.
كلمات البحث : اللزوجة الكهربية ، اللزوجة المرنة ، شبه الساكنة ، اللزوجة المرنة ، تلامس الاحتكاك ، التلف ، الالتصاق ، الامتثال الطبيعي ، الاستجابة الطبيعية المخمدة ، حالة سنيوريني ، درجة الحرارة ، المعادلات التفاضلية ، النقطة الثابتة.
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