



People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of El Oued
Faculty of Exact Sciences



Laboratory of operators Theory and PDE, Foundations and Applications

A Doctoral Thesis

Submitted in Fulfillment of the Requirements for the Degree of
'Doctorate Es-Sciences'

Field : Mathematics

Specialization : Applied Mathematics

Analysis of certain degenerate nonlinear reaction-diffusion systems

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Academic Year: 2025-2026 /1447-1448 AH

ملخص

في هذه الأطروحة، قمنا بدراسة ديناميكيات الانفجار لحلول الأنظمة شبه الخطية المكافئة من التفاعل والانتشار المتدهور، حيث ينشأ الانفجار في المشتق الزمني. في ظل ظروف مناسبة للشروط الابتدائية، نحدد زمن حدوث الانفجار. ولإكمال التحليل النظري، قمنا بتصميم وتطبيق مخططاً عددياً للفروق المحدودة في MATLAB لتقريب حل الجملة الخطية. تلتقط عمليات المحاكاة العددية تطور التدرجات المكانية الحادة والقمم البارزة بالقرب من منطقة الانفجار، مما يوفر رؤية قيمة حول تكوين وبنية نمط الانفجار. تُشكل هذه الدراسة الحسابية خطوة أولية نحو التحديد الدقيق لمجموعة الحلول المنفجرة وتوصيفها.

الكلمات المفتاحية: نظام التفاعل والانتشار غير الخطي، مشكلة الانحلال، وقت الانفجار، انفجار الحل، مشكلة شبه منفصلة، مخطط الفروق المحدودة، تقدير الخطأ، الاستقرار والتقارب.

Abstract

In this thesis, we investigate the blow-up dynamics of solutions to a semilinear degenerate parabolic reaction–diffusion system, in which the degeneracy arises in the time derivative. Under suitable conditions on the initial data, we establish the occurrence of finite-time blow-up. To complement the theoretical analysis, we design and implement a finite-difference numerical scheme in MATLAB to approximate the solution of the discretized system. The numerical simulations capture the development of sharp spatial gradients and pronounced peaks near the blow-up region, offering valuable insight into the formation and structure of the blow-up profile. This computational study constitutes a preliminary step toward the precise identification and characterization of the blow-up set.

Key words : Nonlinear reaction–diffusion system, Degenerate problem, Blow-up time, Blow-up of solution, Semi-discrete problem, Finite differences scheme, Error estimation, Stability and convergence.

Résumé

Dans cette thèse, nous étudions la dynamique d'explosion des solutions d'un système de réaction-diffusion parabolique semi-linéaire dégénéré, la dégénérescence portant sur la dérivée temporelle. Sous certaines conditions initiales, nous établissons l'existence d'une explosion en temps fini. Pour compléter l'analyse théorique, nous concevons et implémentons un schéma numérique aux différences finies sous MATLAB afin d'approximer la solution du système discrétisé. Les simulations numériques mettent en évidence le développement de gradients spatiaux marqués et de pics prononcés au voisinage de la zone d'explosion, offrant ainsi un éclairage précieux sur la formation et la structure du profil d'explosion. Cette étude numérique constitue une étape préliminaire vers l'identification et la caractérisation précises de l'ensemble des solutions explosives.

Mots clés : Système de réaction-diffusion non linéaire, Problème dégénéré, Temps d'explosion, Explosion de la solution, Problème semi-discret, Schéma aux différences finies, Estimation d'erreur, Stabilité et convergence.

Acknowledgments

First of all, I would like to thank God for giving me the strength and patience to complete this work.

I would like to express my sincere gratitude to my supervisor, Professor NISSE Lamine at the University of El-Oued, for his patience, guidance, and especially his wise advice, which helped me to accomplish this thesis.

I would also like to express my sincere gratitude to Dr DOUDI Nadjat, MCA at the University of El-Oued, for the continuous support of my thesis, as well as for her patience, motivation, enthusiasm, and knowledge.

My sincere thanks also go to the members of the jury, who have given me the great honor of evaluating this thesis.

In addition, I would like to thank all the teachers of the Department of Mathematics for their valuable support and help.

Finally, I thank all my relatives and friends whose support and encouragement have been of great help to me.

Dedication

I dedicate this humble work:

*To the soul of my late father, may God have mercy on him, who was my support in life.
Though he is absent from my sight, he remains ever present in my heart and in my prayers.*

*To my beloved mother, whose sincere prayers have always been my source of success, and
thanks to whom I have been able to rise again after every setback.*

*To my dear husband, my companion and support, who never ceased to stand by me with his
encouragement and guidance.*

*To my dear children, who are the reason for my success and the driving force behind my
continued pursuit of excellence.*

I dedicate as well:

*To all the members of my family who have surrounded me with encouragement and prayed
for my success at every stage.*

To all my friends, and everyone who supported and helped me, even with a kind word.

Sidhoum Bariza

Contents

List of figures	viii
List of tables	ix
Notations	xi
Introduction	1
1 Preliminary	8
1.1 Hölder spaces	8
1.2 Limited expansions and Taylor formulas	10
1.2.1 Taylor-Young formula	10
1.2.2 Taylor-Lagrange formula	11
1.2.3 Approximation of the first and second derivatives in 1D	11
1.3 Some useful inequalities	12
1.3.1 Hölder's inequality	12
1.3.2 Young's inequality	12
1.3.3 Poincaré inequality	13
1.3.4 Minkowski inequality	13
1.3.5 Jensen's inequality	13
1.4 Sub-solution and super-solution method for PDEs	14
1.4.1 Fundamental principle of the method	15
1.5 Reaction-diffusion equations	16
1.6 Degenerate problems	18
1.6.1 Examples of degenerate problems	18
1.7 Blow-up phenomenon in mathematical setting	19
1.7.1 Preliminary definitions on blow-up phenomena	20
1.7.2 Blow-up in Ordinary Differential Equations (ODEs)	21
1.7.3 Blow-up in Partial Differential Equations (PDEs)	21

1.8	Explosion phenomenon in some applied fields	22
1.8.1	Blow-up of solution in some chemical problems	23
1.8.2	Mathematical modeling of biological explosions in medicine	25
2	Theoretical and numerical studies blow-up of a degenerate nonlinear reaction-diffusion problem	28
2.1	Position of problem	28
2.2	The continuous problem	30
2.2.1	Local existence and uniqueness of the solution	30
2.2.2	Regularity of the solution	31
2.2.3	Blow-up of continuous solution	32
2.3	The semi-discrete problem	33
2.3.1	Properties of the matrix $\mathcal{A}_h$ and $\mathcal{B}_h$	34
2.3.2	The blow-up of solution	36
2.3.3	Properties of the solution to the semi-discrete problem	36
2.3.4	Convergence of the solution to the semi-discrete problem	37
2.4	The fully-discrete problem	43
2.4.1	The θ -scheme	43
2.4.2	Local truncation error estimation	45
2.4.3	Stability and convergence (for pre-blow-up phase)	47
3	Theoretical and numerical studies of the blow-up of a degenerate nonlinear reaction-diffusion problem with source terms	50
3.1	Position of problem	50
3.2	Study of the continuous problem	52
3.3	Study of the semi-discrete problem	53
3.3.1	Properties of the matrices $\mathcal{A}_h$ and $\mathcal{B}_h$	54
3.3.2	The blow-up of solution	55
3.3.3	Properties of the solution to the semi-discrete problem	56
3.3.4	Convergence of the solution to the semi-discrete problem	57
3.4	Finite difference schemes	61
3.4.1	Matrix form	62
3.4.2	Non-fixed time-step for stability	63
3.4.3	Local truncation error estimation	63
3.4.4	Stability and convergence of the θ -scheme (pre-blow-up phase)	64

4	Numerical results	68
4.1	Results for problem (2.1)	68
4.1.1	Point of Infinite Kinetics (PIK) or blow-up point	70
4.2	Results for problem (3.1)	73
4.2.1	Setup and initial conditions	73
4.2.2	Numerical blow-up times for different meshes	73
4.2.3	Effect of different initial conditions	74
4.2.4	Discussion	75
4.3	Algorithm used	76
4.4	Conclusion	76
	General conclusion	78
	Bibliography	79

List of Figures

4.1	Numerical solution $(\mathcal{U}, \mathcal{V})(x_i, t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = v_0(x) = \sin(\pi x)$.	69
4.2	Numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = v_0(x) = \sin(\pi x)$.	69
4.3	Numerical solution $(\mathcal{U}, \mathcal{V})(x_i, t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = x^3(1-x)^2$, $v_0(x) = (1-x^3)\sin(\frac{\pi}{2}x)$.	70
4.4	Numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $(\mathcal{U}_0)_i = (x_i^3(1-x_i)^2)$, $(\mathcal{V}_0)_i = (1-x_i^3)\sin(\frac{\pi}{2}x_i)$.	70
4.5	The PIK of numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ for $p = 2$, $\theta = 0.91$ and $u_0(x) = v_0(x) = \sin(\pi x)$.	71
4.6	The PIK of numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ for $p = 2$, $\theta = 0.91$ and $u_0 = x^3(1-x)^2$, $v_0 = (1-x^3)\sin(\frac{\pi}{2}x)$.	72
4.7	Numerical solution $(\mathcal{U}, \mathcal{V})(x_i, t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = x^3(1-x)^2$, $v_0(x) = (1-x^3)\sin(\frac{\pi}{2}x)$, and with source terms	75

List of Tables

4.1	Numerical blow-up times and the central processing unit times (CPUTs). . .	69
4.2	Numerical blow-up times, maximum norms, and CPU times for $u_0 = v_0 = \sin(\pi x)$, $p = 2$, $\theta = 0.91$, with source terms.	74
4.3	Numerical blow-up times, maximum norms, and CPU times for $u_0(x) = x^3(1 - x)^2$, $v_0(x) = (1 - x^3) \sin(\frac{\pi}{2}x)$, $p = 2$, $\theta = 0.91$, with source terms. . . .	74

Notations

Ω	: An open interval of $\mathbb{R}$.
$\partial\Omega$	: Boundary of Ω .
$\bar{\Omega} = \Omega \cup \partial\Omega$	: The adhesion of Ω .
$\frac{\partial u}{\partial t}, u_t$	: Partial derivative of u with respect to t .
$\frac{\partial u}{\partial x}, u_x$	: Partial derivative of u with respect to x .
$\frac{\partial^2 u}{\partial x^2}, u_{xx}$	: Second partial derivative of u with respect to x .
∇u	: $\left(\frac{\partial u}{\partial x_1}(x), \frac{\partial u}{\partial x_2}(x), \dots, \frac{\partial u}{\partial x_n}(x) \right) = \text{Grad } u, x \in \Omega \dots \mathbb{R}^N$.
$\nabla \cdot u$	: $\sum_{i=1}^{i=n} \frac{\partial u}{\partial x_i}(x) = \text{div } u, x \in \Omega \dots \mathbb{R}^N$.
Δu	: $\sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2}(x) =$ Laplacien operator of $u, x \in \Omega \dots \mathbb{R}^N$.
$L^2(]0, 1[\times]0, t_0[)$	: $\left\{ u :]0, 1[\times]0, t_0[\rightarrow \mathbb{R}; u \text{ is mesurable and } \int_0^{t_0} \int_0^1 u(x, t) ^2 < \infty \right\}$.
$C(\Omega)$	: Set of continuous functions on Ω .
$C^k(\Omega)$	: Space of k times continuously differentiable functions on Ω .
$C^{4,2}(]0, 1[\times]0, T[)$	: $\left\{ u :]0, 1[\times]0, T[\rightarrow \mathbb{R}; \partial_x^m u, \partial_t^k u \text{ are exist and continuous for all } 0 \leq m \leq 4, 0 \leq k \leq 2 \right\}$.
$C^\alpha(]0, 1[\times]0, T[)$	: $\left\{ u \in C(]0, 1[\times]0, T[); \exists L \geq 0, \forall x, y \in]0, 1[, \forall t, s \in]0, T[; u(x, t) - u(y, s) \leq L \left(x - y + t - s ^{\frac{1}{2}} \right)^\alpha \right\}, 0 < \alpha < 1$.
$C^{2+\alpha}(]0, 1[\times]0, T[)$	: $\{u; \partial_x u, \partial_x^2 u, \partial_t u \in C^\alpha(]0, 1[\times]0, T[)\}$.
x_i	: Discrete spatial variable.
$u_i = u(x_i, t)$	: Semi discrete approximation of u at point x_i .
$u_i^n = u(x_i, t_n)$	: Discrete approximation of u at point (x_i, t_n) .
T_b^u	: Blow-up time of u .
T_b^v	: Blow-up time of v .
Time CPU	: Central Processing Unit time.
PIK	: Point of Infinite Kinetics.

Introduction

In recent decades, considerable attention has been devoted to the study of finite-time blow-up phenomena in solutions to nonlinear and degenerate parabolic equations and systems. Blow-up refers to the situation where a solution becomes unbounded in finite time, typically due to strong nonlinearities or degeneracies in the diffusion terms or in temporal derivative. This behavior has been widely observed in models from physics, chemistry, and biology, including heat conduction in heterogeneous media, chemical reactions, and population dynamics.

The mathematical analysis of blow-up phenomena began with classical studies such as Kaplan [24], who investigated the growth of solutions of quasilinear parabolic equations and provided foundational results on blow-up rates. Later, Samarskii et al. [40] developed a comprehensive theory for blow-up problems in quasilinear parabolic equations, focusing on both analytical and numerical aspects. Galaktionov and Vázquez [21] provided a detailed classification of blow-up behaviors and discussed self-similar structures arising in nonlinear parabolic systems. The monograph by Quittner and Souplet [37] offers an extensive treatment of superlinear parabolic problems, addressing global existence, blow-up, and steady states.

The theoretical framework for degenerate equations was further developed by Coddington and Levinson [13], who established fundamental results in the theory of ordinary differential equations that form the backbone of blow-up analysis. Chipot et al. [12] investigated stationary solutions and blow-up for semilinear parabolic equations under nonlinear boundary conditions, linking boundary effects to finite-time singularities. Boni [3] examined blow-up and asymptotic behavior for nonlinear parabolic equations with nonlinear boundary conditions, demonstrating how boundary nonlinearities affect blow-up rates.

In the context of degenerate systems, Du [14] studied blow-up for degenerate reaction–diffusion equations with localized sources and provided sharp conditions for finite-time blow-up. Shi et al. [41] and Zhou et al. [47] extended these results to singular and nonlocal systems, revealing that blow-up strongly depends on the spatial degeneracy and nonlocal sources. Li et al. [33] considered global existence and blow-up in degenerate and singular

parabolic systems with localized sources, showing the crucial influence of nonlinearity on solution profiles.

In particular, Belyacine [2] studied certain classes of degenerate semilinear parabolic systems and proposed a spatial truncation technique to remove degeneracy by restricting the domain to $]\varepsilon_1, 1 - \varepsilon_2[$. Using sub- and supersolution methods, the existence and uniqueness of positive local solutions were established. Later, Nisse and Belyacine [35] analyzed the blow-up behavior of such systems and proved that under suitable initial data, blow-up occurs precisely at the degenerate boundary points.

Floater [15] considered the scalar degenerate parabolic problem

$$\begin{cases} x u_t(x, t) = u_{xx}(x, t) + u^p(x, t), & x \in]0, 1[, t > 0, \\ u(0, t) = u(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in [0, 1], \end{cases} \quad (1)$$

and proved that for $1 < p \leq 2$ and concave, sufficiently large initial data, u blows up in finite time at the degenerate boundary $x = 0$. M. Kouche [26] generalized this result to a coupled nonlinear system of degenerate semilinear parabolic equations

$$\begin{cases} x u_t(x, t) = u_{xx}(x, t) + v^p(x, t), & x \in]0, 1[, t > 0, \\ x v_t(x, t) = v_{xx}(x, t) + u^p(x, t), & x \in]0, 1[, t > 0, \\ u(x, t) = v(x, t) = 0, & x = 0, 1, t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in [0, 1]. \end{cases} \quad (2)$$

He showed that, under suitable sufficient conditions on the initial data, the solution (u, v) of the system blows up in finite time. In particular, for concave and sufficiently large initial data u_0, v_0 and $1 < p \leq 2$, the blow-up occurs at $x = 0$, which corresponds to the degeneracy point of the system.

The importance of such studies lies in their ability to describe singular behaviors that occur in physical processes governed by diffusion–reaction mechanisms. Du’s analysis [14] and the extensions by Li et al. [33] and Shi et al. [41] provide a strong theoretical foundation for the degenerate parabolic system investigated here.

From a numerical standpoint, numerous authors have explored finite difference and adaptive time-stepping methods for approximating blow-up times and profiles. Abia et al. [1] used explicit and linearly implicit Euler finite difference schemes to compute the numerical blow-up solution and estimate the blow-up times for the single equation

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) + u^p(x, t) + f(x, t), & x \in]0, 1[, 0 < t < T, \\ u(0, t) = u(1, t) = 0, & 0 < t < T, \\ u(x, 0) = u_0(x), & x \in (0, 1), \end{cases} \quad (3)$$

where $p = 2$ and $u_0(x) = 20 \sin(\pi x)$. The time step k_n was dynamically adjusted according to the formula

$$k_n = \begin{cases} \min\left(\frac{h^2}{2}, \frac{h^\beta}{\|\mathcal{U}_h^n\|_\infty}\right), & \text{for explicit scheme,} \\ \frac{h^\beta}{\|\mathcal{U}_h^n\|_\infty}, & \text{for implicit scheme,} \end{cases}$$

for $n \geq 0$, $\beta > 0$, where h denotes the spatial step and $\mathcal{U}_h^n$ represents the numerical solution vector at time t_n of the discrete problem.

Rasheed et al. [39] employed both explicit and linearly implicit Euler finite difference schemes to compute the numerical blow-up solution and estimate the blow-up times for problem (3), considering the cases $p = 3, 4, 5$ with initial condition $u_0(x) = 100(x - x^2)$. To enhance numerical convergence and obtain more accurate results, they used a specially designed time-step formula depending on p

$$k_n = \begin{cases} \min\left(\frac{h^2}{3}, \frac{h^\beta}{\|\mathcal{U}_h^n\|_\infty^p}\right), & \text{for explicit scheme,} \\ \frac{h^\beta}{\|\mathcal{U}_h^n\|_\infty^p}, & \text{for implicit scheme,} \end{cases}$$

for $n \geq 0$, $\beta > 0$. Computational simulations were performed to validate the theoretical blow-up results and illustrate the solution behavior near the blow-up points. Motivated by these works, we also incorporate source terms in our numerical experiments in order to examine their influence on the blow-up dynamics while preserving consistency with the underlying theoretical framework.

Recently, Khalil et al. [25] studied the finite difference approximation for a coupled system of two one-dimensional semilinear parabolic equations with blow-up and source terms

$$\begin{cases} u_t = u_{xx} + f(v, x, t) + f_1(x, t), & x \in]0, 1[, t > 0, \\ v_t = v_{xx} + g(u, x, t) + f_2(x, t), & x \in]0, 1[, t > 0, \\ u(x, t) = v(x, t) = 0, & x = 0, 1, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in [0, 1]. \end{cases} \quad (4)$$

They formulated the semi-discrete problem and demonstrated that, under suitable assumptions on the initial data and boundary conditions, the numerical blow-up solution and the blow-up time of the semi-discrete problem converge to their theoretical counterparts.

They also constructed fully discrete formulations using standard finite difference methods (explicit and implicit Euler schemes) with a variable time-stepping strategy and analyzed the consistency, stability, and convergence of these schemes.

Cho and Okamoto [11] extended these methods to axially symmetric nonlinear heat equations, whereas Cho and Lu [7] proposed new schemes for coupled parabolic systems exhibiting blow-up. Similarly, Rasheed et al. [38, 39] analyzed the numerical estimation of blow-up times for semilinear heat equations with gradient terms, confirming convergence to analytical blow-up times.

Polyanin and Shingareva [36] presented analytical and numerical approaches for nonlinear blow-up problems, including non-local transformations and exact solutions. In [5] Z. Chebana et al. are discussed the blow-up at finite-time of exact and numerical solutions for a class of semi-linear heat equation with nonlinear nonlocal conditions of second type. N. Yilmaz [46] proved that under suitable assumptions on the initial data, solutions with negative initial energy to a logarithmic Petrovsky equation with variable exponent experience finite-time blow-up.

Hamadneh et al. [22] extended the study of finite-time blow-up to fractional reaction–diffusion equations, combining analytical proofs with numerical simulations. Mo-mani et al. [34] investigated weak solutions and blow-up phenomena in nonlinear reaction–diffusion equations with integral conditions, supported by numerical experiments. These contributions highlight the strong interaction between theoretical and computational aspects of blow-up dynamics.

The importance of blow-up analysis has recently expanded to nonlocal and viscoelastic models. Choucha et al. [8] investigated systems of nonlocal singular viscoelastic equations with sources and distributed delay terms, proving global existence and blow-up results. In a related study, Choucha and Boulaaras [9] analyzed a viscoelastic plate equation with logarithmic nonlinearity and variable exponents, establishing global existence, general decay, and blow-up of solutions. Furthermore, Choucha et al. [10] examined a coupled nonlinear viscoelastic Kirchhoff system with fractional boundary conditions and variable exponents, presenting results on blow-up, growth, and decay rates. More recently, Laribi et al. [28] studied the exponential decay of laminated beams with nonlinear time-varying delays and microtemperature effects, showing how dynamic boundary terms influence the overall stability.

Fujita [20] analyzed the classical semilinear heat equation and established conditions under which solutions blow up in finite time, showing that sufficiently large initial data or certain nonlinear exponents lead to singularities. Levine [32] extended these results to more general parabolic equations, providing nonexistence and instability theorems that link the growth of solutions to properties of the nonlinear terms.

Overall, these developments demonstrate that blow-up behavior remains a central challenge in the analysis of nonlinear parabolic systems. The combination of degeneracy, nonlinearity, and nonlocal effects requires sophisticated analytical and numerical tools to capture the underlying dynamics accurately.

The aim of this thesis is to develop a numerical method to compute blow-up (amplification) solutions of a one-dimensional degenerate semilinear reaction–diffusion system. We consider the following initial–boundary value problem

$$\begin{cases} x u_t(x, t) = u_{xx}(x, t) + v^p(x, t), & x \in]0, 1[, \, t > 0, \\ (1 - x) v_t(x, t) = v_{xx}(x, t) + u^p(x, t), & x \in]0, 1[, \, t > 0, \\ u(x, t) = v(x, t) = 0, & x \in \{0, 1\}, \, t > 0, \\ u(x, 0) = u_0(x) \geq 0, \, v(x, 0) = v_0(x) \geq 0, & x \in [0, 1], \end{cases} \quad (5)$$

where $p > 1$, and the initial data u_0 and v_0 belong to an appropriate functional space. We say that the numerical solution u (resp. v) blows up in a finite time T if

$$\lim_{t \nearrow T} (\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty) = +\infty.$$

The degeneracy in (5) appears in the time derivatives: the coefficients of u_t and v_t vanish when $x \rightarrow 0$ and $x \rightarrow 1$, respectively, which alters the parabolic nature of the problem. The theoretical and numerical approaches mentioned above motivate the development of robust methods capable of detecting and reproducing blow-up profiles in such degenerate systems.

Belyacine [2] proposed a technique of truncating the domain near the boundary points, which removes the degeneracy and reduces system (5) to a sequence of non-degenerate problems that admit a positive local solution. Under suitable assumptions on the nonlinear part, and using the method of sub- and super-solutions, it was shown that the truncated system admits a unique local solution. Specifically, the domain $]0, 1[$ is replaced by $]\varepsilon_1, 1 - \varepsilon_2[$, with $\varepsilon_1, \varepsilon_2 > 0$, so that system (5) becomes non-degenerate on $]\varepsilon_1, 1 - \varepsilon_2[\times]0, T[$.

In [35], Niss and Belyacine analyzed the blow-up of system (5) and showed that, under suitable conditions on the initial data, the solution blows up in finite time. They also established sufficient conditions for blow-up of the semi-discrete system. The system (5) is degenerate at $x = 0$ and $x = 1$, since the coefficients of u_t and v_t vanish at the boundaries, which affects the parabolic character of the problem.

In the second work, we developed a numerical method for computing blow-up solutions of a one-dimensional degenerate semilinear parabolic system, including possible source terms.

We consider the following initial-boundary value problem

$$\begin{cases} x u_t(x, t) = u_{xx}(x, t) + v^p(x, t) + g_1(x, t), & x \in]0, 1[, t > 0, \\ (1 - x) v_t(x, t) = v_{xx}(x, t) + u^p(x, t) + g_2(x, t), & x \in]0, 1[, t > 0, \\ u(x, t) = v(x, t) = 0, & x = 0, 1, t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in [0, 1], \end{cases} \quad (6)$$

where $p > 1$, and the initial conditions u_0 and v_0 belong to a suitable functional space, e.g., a weighted Sobolev space [1, 11, 15]. The source terms g_1 and g_2 are assumed to be sufficiently smooth, i.e., $g_1, g_2 \in C^1(\overline{\Omega} \times [0, T])$, so that the theoretical analysis is valid. The numerical solution u (resp. v) is said to blow up in finite time T .

At the end of this introduction, we briefly present the contents of the chapters included in this thesis:

First chapter

This chapter serves as an introductory foundation for the mathematical tools used throughout the thesis. We begin by recalling essential notions from functional analysis, including basic concepts related to functional spaces and their fundamental properties. We then present limited developments and approximation techniques for certain derivatives, followed by a selection of useful inequalities that will be employed in later chapters. In addition, we provide a brief explanation of the subsolution and supersolution method, along with an outline of the maximum principle, both of which play a central role in the analysis of partial differential equations. Finally, we explored the importance of explosive solutions and their applications in various scientific and applied fields.

Second chapter:

Recalls the theoretical results on finite-time blow-up for the continuous and semi-discrete versions of problem (5). We present the results obtained by Niss and Belyassine in [2] regarding the conditions and assumptions that enable the solution to be blowed up on both sides of the domain for continuous and semi-discrete problems. In the end of this chapter, we introduce the numerical discretization based on the finite difference θ -scheme. Theoretical analysis indicates that, under appropriate regularity assumptions on the exact solution and suitable restrictions on the discretization parameters, the θ -scheme is consistent, stable, and convergent in the pre-blow-up regime.

Third chapter:

In this chapter, we developed and analyzed a fully discrete θ -scheme for a coupled non-

linear partial differential equation system with source terms (6). The proposed numerical method combines spatial discretization with an adaptive time-stepping strategy and provides an accurate approximation of the system dynamics, including finite-time blow-up phenomena.

A key aspect of this work is the explicit treatment of the nonlinear source terms (u^p, v^p) , which play a fundamental role in shaping the solution profile, governing the onset of blow-up, and influencing the overall numerical stability of the scheme. Theoretical investigations show that, under suitable assumptions on the regularity of the exact solution and appropriate constraints on the discretization parameters, the θ -scheme is consistent, stable, and convergent in the pre-blow-up phase.

Fourth chapter:

In this chapter, we present a numerical experiment. All numerical computations are implemented using MATLAB (R2020a). Our objective is to demonstrate that the results established in the second and third chapters can be observed through numerical simulations.

This work has resulted in the following publications:

B. Sidhoum, N. Doudi, L. Nisse, A. Alharbi and S. Boulaaras, *Theoretical and numerical studies of the blowup of a degenerate nonlinear reaction-diffusion problem with source terms*, APPLIED MATHEMATICS IN SCIENCE AND ENGINEERING, VOL. 34, NO. 1, 2611905 (2026). <https://doi.org/10.1080/27690911.2025.2611905>

B. Sidhoum, N. Doudi and L. Nisse, *blow-up behavior in degenerate nonlinear reaction-diffusion equations: theory and numerical simulation*, Palestine Journal of Mathematics, Vol 15(2)(2026) , 707–724.

Chapter 1

Preliminary

This chapter serves as an introductory foundation for the mathematical tools used throughout the thesis. We begin by recalling essential notions from functional analysis, including basic concepts related to functional spaces and their fundamental properties. We then present limited developments and approximation techniques for certain derivatives, followed by a selection of useful inequalities that will be employed in later chapters. In addition, we provide a brief explanation of the subsolution and supersolution method, both of which play a central role in the analysis of partial differential equations.

1.1 Hölder spaces

Let I be an interval in $\mathbb{R}$, X a Banach space, and E a convex open subset of X .

We recall some notions of regularity for functions defined on I or X .

Definition 1.

1. A function f is said to be globally Lipschitz continuous with respect to x if there exists a constant $L \geq 0$ such that

$$\forall x_1, x_2 \in E, \forall t \in I, \quad \|f(t, x_1) - f(t, x_2)\|_X \leq L\|x_1 - x_2\|_X.$$

2. A function f is said to be locally Lipschitz continuous with respect to x if for every $(x_0, t_0) \in I \times E$, there exist a neighborhood V of (x_0, t_0) and a constant $L(x_0, t_0) \geq 0$ such that

$$\forall (t, x_1) \in V, \forall (t, x_2) \in V, \quad \|f(t, x_1) - f(t, x_2)\|_X \leq L(x_0, t_0)\|x_1 - x_2\|_X.$$

Definition 2. A function $f : I \rightarrow X$ is said to be Hölder continuous of order α , $0 < \alpha < 1$, on I if there exists a constant $L \geq 0$ such that

$$\|f(t) - f(s)\|_X \leq L|t - s|^\alpha, \quad \text{for all } s, t \in I.$$

The function f is said to be locally Hölder on I if it is Hölder continuous in a neighborhood of every point $t \in I$.

We denote by $C^\alpha(I, X)$ the family of all Hölder continuous functions of order α on I .

Definition 3. Let $\Omega \subset \mathbb{R}^n$ be a domain, $0 < \alpha < 1$, and $m \in \mathbb{N}$. We define C^α and $C^{m+\alpha}$ by

$$C^\alpha(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ bounded, } \exists L \geq 0 \text{ such that } |u(x) - u(y)| \leq L \|x - y\|^\alpha, \forall x, y \in \Omega\},$$

$$C^{m+\alpha}(\Omega) = \{u \in C^\alpha(\Omega) \text{ such that } D^\beta u \in C^\alpha(\Omega), |\beta| \leq m\}.$$

The spaces $C^\alpha(\Omega)$ and $C^{m+\alpha}(\Omega)$ are endowed with the respective norms

$$\|u\|_{C^\alpha(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

$$\|u\|_{C^{m+\alpha}(\Omega)} = \sum_{|\beta| \leq m} \|D^\beta u\|_{C^\alpha(\Omega)},$$

and are Banach spaces.

Remark 1.

1. The spaces $C^{m+\alpha}(\Omega)$ are called Hölder spaces.
2. By definition, the Hölder space $C^{2+\alpha}(D)$ is

$$C^{2+\alpha}(D) = \left\{ u \in C^\alpha(D) \text{ such that } D_{x_i} u, D_{x_i x_j}^2 u, D_t u \in C^\alpha(D) \right\}.$$

Remark 2. The space $C^{2+\alpha}(D)$, endowed with the norm

$$\|u\|_{C^{2+\alpha}(D)} = \|u\|_{C^\alpha(D)} + \sum_{i=1}^n \|D_{x_i} u\|_{C^\alpha(D)} + \sum_{i,j=1}^n \|D_{x_i x_j}^2 u\|_{C^\alpha(D)} + \|D_t u\|_{C^\alpha(D)},$$

is a Banach space.

For more details regarding holder spaces and their characteristics, see [18], [27].

1.2 Limited expansions and Taylor formulas

In mathematical analysis, limited developments (or series expansions) are powerful tools that allow us to approximate complicated functions by simpler ones, usually polynomials. The central idea is that near a given point, a smooth function can be expressed as a polynomial plus a remainder term that becomes small when approaching this point. This provides both theoretical insights and practical applications in approximation, numerical analysis, and applied sciences [23], [44].

A key instrument in this context is the Taylor formula. Given a function that is sufficiently differentiable at a point x_0 , the Taylor formula of order n expresses the function near x_0 as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x),$$

where $R_n(x)$ is the remainder term (or error of approximation).

If $R_n(x) \rightarrow 0$ as $x \rightarrow x_0$, we obtain the limited development of order n of f at x_0 .

This framework provides a systematic way to approximate functions locally by polynomials, making it easier to study limits, compute approximations, and solve differential equations.

1.2.1 Taylor-Young formula

Theorem 1. (*Taylor-Young Formula of order n*) Let $x_0 \in \mathbb{R}$ and f be a function defined on a neighborhood of x_0 . If f is n times differentiable at x_0 , then

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n) \\ &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o((x - x_0)^n). \end{aligned}$$

Other formulation: by posing $h = x - x_0$,

$$\begin{aligned} f(x_0 + h) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}h^n + o(h^n) \\ &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}h^k + o(h^n). \end{aligned}$$

1.2.2 Taylor-Lagrange formula

If f is of class C^n on $[x_0, x]$ and $(n + 1)$ times differentiable on $[x_0, x]$ (or more simply if f is $(n + 1)$ times differentiable on $[x_0, x]$), then

$$\exists \xi \in]x_0, x[, \quad f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}. \quad (1.1)$$

1.2.3 Approximation of the first and second derivatives in 1D

The principle of the finite difference method (FD) consists in giving a certain number of points in the domain, which we will denote, $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We approximate the differential operator by a quotient in such a way as to deduce a system of equations in terms of discrete unknowns representing approximations of unknown u at the discretization points [23].

According to the Taylor expansion to order 3 in the neighborhood of point x_i and using points x_{i+1} and x_{i-1} , we obtain the second-order scheme (**center**) to approximate the first derivative of u at x_i

$$\frac{\partial u}{\partial x}(x_i) = \frac{u_{i+1} - u_{i-1}}{2h} + o(h^2). \quad (1.2)$$

The first derivative can also be approximated using first-order decentered schemes:

- **Forward formula:** According to the Taylor series expansion in the neighborhood of x_i , we have

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_{i+1}) - u(x_i)}{h} + o(h).$$

- **Backward formula:** If we use point x_{i-1} , we have

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_i) - u(x_{i-1})}{h} + o(h).$$

For the second derivative, performing a Taylor expansion in the neighborhood of x_i of order 4, we obtain the following centered scheme of order 2

$$\frac{\partial^2 u}{\partial^2 x}(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + o(h^2). \quad (1.3)$$

- **Forward formula:** Suppose the function u is C^3 continuous in the neighborhood of x .

According to the Taylor series expansion in the neighborhood of x_i , we have

$$\begin{aligned} u(x_{i+1}) &= u(x_i + h) = u(x_i) + h \frac{\partial u}{\partial x}(x_i) + \frac{h^2}{2} \frac{\partial^2 u}{\partial^2 x}(x_i) + o(h^3), \\ u(x_{i+2}) &= u(x_i + 2h) = u(x_i) + 2h \frac{\partial u}{\partial x}(x_i) + 2h^2 \frac{\partial^2 u}{\partial^2 x}(x_i) + o(h^3), \end{aligned}$$

this yields,

$$\frac{\partial^2 u}{\partial^2 x}(x_i) = \frac{u_{i+2} - 2u_{i+1} + u_i}{h^2} + o(h).$$

- **Backward formula:** If we use point x_{i-1} , we have

$$\begin{aligned} u(x_{i-1}) &= u(x_i - h) = u(x_i) - h \frac{\partial u}{\partial x}(x_i) + \frac{h^2}{2} \frac{\partial^2 u}{\partial^2 x}(x_i) + o(h^3), \\ u(x_{i-2}) &= u(x_i - 2h) = u(x_i) - 2h \frac{\partial u}{\partial x}(x_i) + 2h^2 \frac{\partial^2 u}{\partial^2 x}(x_i) + o(h^3), \end{aligned}$$

this yields,

$$\frac{\partial^2 u}{\partial^2 x}(x_i) = \frac{u_i - 2u_{i-1} + u_{i-2}}{h^2} + o(h).$$

1.3 Some useful inequalities

1.3.1 Hölder's inequality

Theorem 2. [4] Let $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ with $1 \leq p \leq +\infty$ and q the conjugate exponent ($\frac{1}{p} + \frac{1}{q} = 1$). Then $u, v \in L^1(\Omega)$ and

$$\int_{\Omega} |u(x)v(x)| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \quad (1.4)$$

Remark 3. In particular, if $p = 2$ and $q = 2$, we obtain the Cauchy-Schwarz inequality

$$\int_{\Omega} |u(x)v(x)| dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}. \quad (1.5)$$

1.3.2 Young's inequality

Proposition 1. [6] Let $1 < p < +\infty$. Then for all $a, b \geq 0$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $q \in]1, +\infty[$ is the conjugate exponent of p defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Moreover, equality holds if and only if $a^p = b^q$.

Remark 4. A simple case of Young's inequality with exponents $p = q = 2$ and $(\varepsilon > 0)$ is

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$

1.3.3 Poincaré inequality

Theorem 3. [4] Let Ω be a bounded domain. Then there exists a constant $c > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq c \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega). \quad (1.6)$$

1.3.4 Minkowski inequality

Let $p \in [1, +\infty[$ and two functions $f, g \in L^p(\Omega)$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

that is,

$$\left(\int |f + g|^p dx \right)^{\frac{1}{p}} \leq \left(\int |f|^p dx \right)^{\frac{1}{p}} + \left(\int |g|^p dx \right)^{\frac{1}{p}}. \quad (1.7)$$

Minkowski's inequality is also valid in the particular case of vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$), and even for series ($n = \infty$)

$$\left(\sum_{k=1}^n |x_k + y_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^n |y_k|^p \right)^{1/p}.$$

1.3.5 Jensen's inequality

Definition 4 (Convexity). A function f is convex on I if and only if for all $\lambda \in [0, 1]$ and all $x, y \in I$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Moreover, for any $z \in [x, y]$, we have

$$z = \lambda x + (1 - \lambda)y \quad \text{for some } \lambda \in [0, 1],$$

and since f is convex,

$$f(z) = f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Geometrically, this means that the point $(z, f(z))$ lies below the barycenter (i.e., the straight-line segment) joining the points $(x, f(x))$ and $(y, f(y))$. In other words,

$$f \text{ is convex on } I \iff \forall x, y \in I,$$

the graph of f between x and y lies below the chord connecting the points $A = (x, f(x))$ and $B = (y, f(y))$.

Different forms of Jensen's inequality: Jensen's inequality is a generalization of the convexity inequality to several numbers.

Theorem 4 (Integral form). *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f \in C([0, 1],]a, b[)$, and $\Phi :]a, b[\rightarrow \mathbb{R}$ a convex function. Then*

$$\Phi \left(\int_0^1 f(x) dx \right) \leq \int_0^1 \Phi(f(x)) dx.$$

Theorem 5 (Discrete Form). *Let f be a convex function. Let $(x_1, \dots, x_n)$ be a family of real numbers belonging to the domain of f .*

Let $(\lambda_1, \dots, \lambda_n)$ be a family of real numbers in the interval $[0, 1]$ such that

$$\sum_{i=1}^n \lambda_i = 1.$$

Then, we have

$$f \left(\sum_{i=1}^n \lambda_i x_i \right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

Many important elementary results in analysis follow from this, such as the arithmetic-geometric mean inequality: if $(x_i)_{0 \leq i < n}$ is a family of strictly positive real numbers, then

$$\frac{1}{n} \sum_{i=0}^{n-1} x_i \geq \left(\prod_{i=0}^{n-1} x_i \right)^{1/n}.$$

1.4 Sub-solution and super-solution method for PDEs

The *sub-solution and super-solution method* is a fundamental analytical technique for establishing the existence of solutions to nonlinear partial differential equations. The main idea is to construct two auxiliary functions: a *subsolution*, which lies below any possible solution, and a *supersolution*, which lies above any solution. These functions provide effective bounds within which an actual solution must lie [29, 30, 31].

Definition 5. Let $\mathcal{L}(u) = 0$ denote a (possibly nonlinear) partial differential equation posed in a domain Ω together with suitable boundary or initial conditions.

- A function $\underline{u}$ is called a subsolution if it satisfies

$$\mathcal{L}(\underline{u}) \leq 0 \quad \text{in } \Omega,$$

together with the boundary/initial conditions in the inequality sense (e.g., $\underline{u} \leq g$ on $\partial\Omega$).

- A function $\bar{u}$ is called a supersolution if

$$\mathcal{L}(\bar{u}) \geq 0 \quad \text{in } \Omega,$$

and $\bar{u} \geq g$ on $\partial\Omega$.

If the subsolution and supersolution satisfy the ordering

$$\underline{u} \leq \bar{u} \quad \text{in } \Omega,$$

they form what is known as an *ordered pair of sub- and supersolutions*.

1.4.1 Fundamental principle of the method

The method relies on two key principles:

1. **Existence via monotone iteration.** Starting from the subsolution, one constructs an increasing sequence of approximate solutions, while starting from the supersolution one constructs a decreasing sequence. Under appropriate conditions (such as monotonicity of the nonlinearity and the validity of a maximum principle), both sequences converge to a function u satisfying

$$\underline{u} \leq u \leq \bar{u},$$

and

$$\mathcal{L}(u) = 0.$$

2. **Use of maximum principles.** Maximum principles for elliptic and parabolic equations ensure that the iterative sequences remain trapped between the sub- and supersolutions. This prevents divergence or blow-up and enforces the convergence of the iterative scheme.

Example 1. *Consider the semilinear elliptic problem*

$$-\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

A function $\underline{u}$ is a subsolution if

$$-\Delta \underline{u} \leq f(\underline{u}),$$

while $\bar{u}$ is a supersolution if

$$-\Delta \bar{u} \geq f(\bar{u}),$$

with $0 \leq \underline{u} \leq \bar{u}$ in Ω . Under these conditions, monotone iteration or fixed-point theorems yield a solution u such that

$$\underline{u} \leq u \leq \bar{u}.$$

1.5 Reaction–diffusion equations

Reaction–diffusion systems model the evolution of physical, chemical, or biological quantities distributed in space and governed by the combined effects of diffusion and local reactions. Diffusion acts as a smoothing mechanism that transports matter from regions of high concentration to regions of low concentration, thereby reducing spatial gradients. In contrast, reaction mechanisms locally modify the concentrations through production, consumption, or interaction between species, and are typically described by nonlinear laws.

We aim to describe the evolution of the concentrations of one or more interacting substances subject to two fundamental processes:

- **Diffusion:** spatial spreading of substances, leading to homogenization of concentrations.
- **Local reactions:** transformation and interaction of species through chemical, biological, or physical mechanisms.

Such processes arise in numerous scientific fields, including chemistry, biology, physics, and geology. The interplay between diffusion and nonlinear reactions can generate complex dynamical behaviors such as traveling waves, oscillations, and spatial pattern formation. In particular, diffusion-driven instabilities may lead to the emergence of non-homogeneous structures known as Turing patterns.

Reaction–diffusion equations typically take the general form

$$\partial_t U(t, x) = D\Delta U(t, x) + R(U, t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \quad (1.8)$$

Here $U(t, x) \in \mathbb{R}^e$ denotes the vector of concentrations of e interacting species, $D \in \mathcal{M}_e(\mathbb{R})$ is a diagonal diffusion matrix containing the diffusion coefficients, and $R(U, t, x) \in \mathbb{R}^e$ represents the vector of local reaction terms.

Solutions of reaction–diffusion equations may exhibit a wide range of qualitative behaviors, including convergence to steady states, oscillatory dynamics, traveling waves, and spatial pattern formation. Under certain conditions on the nonlinearities and initial data, solutions may also experience finite-time blow-up, characterized by the unbounded growth of concentrations in finite time. The rigorous analysis of existence, uniqueness, stability, and blow-up phenomena constitutes a central topic in the mathematical theory of reaction–diffusion systems.

Some of the most fundamental reaction–diffusion systems are of the form

$$\begin{cases} \partial_t u(t, x) = D_u \Delta u(t, x) + f(u, v), \\ \partial_t v(t, x) = D_v \Delta v(t, x) + g(u, v). \end{cases} \quad (1.9)$$

There exists a large class of models that differ by their diffusion coefficients and nonlinear interaction terms. Representative examples include:

Name of the system	Diffusion	Nonlinearities
Fisher/KPP	$D_u = 1, D_v = 0$	$f(u, v) = u(1 - u - v),$ $g(u, v) = u$
FitzHugh–Nagumo	$D_u = \varepsilon, D_v = 0$	$f(u, v) = u(1 - u)(u - \alpha) - v,$ $g(u, v) = \gamma u - \beta v$
Gray–Scott	$D_u \neq 0, D_v \neq 0$	$f(u, v) = -uv^2 + F(1 - u),$ $g(u, v) = uv^2 - (F + k)v$
Brusselator	$D_u \neq 0, D_v \neq 0$	$f(u, v) = a - (b + 1)u + u^2v,$ $g(u, v) = bu - u^2v$

Each of these systems admits stationary solutions that may lose stability under suitable parameter conditions, leading to the formation of non-homogeneous spatial structures.

From a computational perspective, the nonlinear and potentially singular behavior of solutions requires robust numerical discretization techniques. Finite difference, finite element, and finite volume methods are commonly employed to approximate solutions, often combined with implicit or semi-implicit time-stepping schemes to ensure stability. The convergence and stability analysis of such numerical methods, particularly in the presence of strong nonlinearities or blow-up phenomena, represents an essential component of the mathematical study of reaction–diffusion systems.

1.6 Degenerate problems

A partial differential equation is said to be *degenerate* when the differential operator loses its uniform ellipticity or uniform parabolicity in a subset of the domain. More precisely, consider a PDE of the form

$$u_t - \nabla \cdot (a(x, u) \nabla u) = f(x, u) \quad \text{in } \Omega \times (0, T).$$

The problem is called *degenerate* if the diffusion coefficient $a(x, u)$ satisfies

$$a(x, u) \geq 0 \quad \text{but} \quad a(x, u) = 0 \text{ at some points of } \Omega,$$

so that the operator

$$-\nabla \cdot (a(x, u) \nabla u),$$

fails to remain uniformly elliptic or uniformly parabolic. In this case, the equation may change type or lose regularity, leading to weaker diffusion properties and requiring special analytical and numerical techniques.

1.6.1 Examples of degenerate problems

In this section we present several classical examples of degenerate partial differential equations, illustrating different types of degeneracy.

1. Porous medium equation (PME).

$$u_t = \Delta(u^m), \quad m > 1.$$

This equation is *degenerate parabolic* because

$$\Delta(u^m) = \nabla \cdot (mu^{m-1} \nabla u),$$

and the diffusion coefficient mu^{m-1} vanishes whenever $u = 0$.

2. Fast diffusion equation (FDE).

$$u_t = \Delta(u^m), \quad 0 < m < 1.$$

The diffusion becomes singular or degenerate at points where $u = 0$, causing a loss of uniform parabolicity.

3. The p -Laplacian equation.

$$-\Delta_p u := -\nabla \cdot (|\nabla u|^{p-1} \nabla u) = f(x), \quad p > 1.$$

This operator is *degenerate elliptic* since

$$|\nabla u|^{p-1} = 0 \quad \text{whenever } \nabla u = 0.$$

4. Degenerate quasilinear parabolic equation.

$$u_t = \nabla \cdot (a(u) \nabla u), \quad a(u) \geq 0,$$

with $a(u_0) = 0$ for some u_0 . The equation becomes degenerate in regions where the diffusion coefficient $a(u)$ vanishes.

5. Heat equation with vanishing diffusion.

$$u_t = a(x) \Delta u,$$

where $a(x) \geq 0$ but vanishes on part of Ω . The problem is degenerate parabolic on the subset where $a(x) = 0$.

6. Stefan problem (Phase-Change). The governing equation

$$u_t - \Delta u = 0,$$

holds only in the liquid region. In the solid region, the diffusion effectively disappears, producing a degenerate structure and a moving free boundary.

7. Black–Scholes equation with degenerate volatility.

$$u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + r x u_x - r u = 0,$$

where $\sigma(0) = 0$. The diffusion term vanishes at $x = 0$, yielding a degenerate parabolic equation.

1.7 Blow-up phenomenon in mathematical setting

The blow-up phenomenon is one of the fundamental behaviors that arise in the study of the properties of solutions to various mathematical models, especially in the context of

nonlinear equations. It describes the loss of boundlessness of a solution in finite time or as a given variable approaches a critical value, where the solution tends to infinity according to a specific mechanism determined by the nature of the system under consideration.

For problems exhibiting blow-up behavior, the solution is generally defined only locally in time, in the sense that its existence is guaranteed on a maximal interval $[0, T]$, where T denotes the maximal existence time at which the solution becomes unbounded. In contrast, the existence of a global solution, defined for all $t \geq 0$, implies that no blow-up occurs, since the solution remains bounded on the entire time interval.

1.7.1 Preliminary definitions on blow-up phenomena

Definition 6 (Maximal existence time). *Let u be a solution of a differential equation defined for $t \geq 0$, the number $T > 0$ is called the maximal existence time if u exists and remains regular on $[0, T]$, and cannot be extended to any larger interval $[0, T + \varepsilon]$ for any $\varepsilon > 0$.*

A solution is called global if its maximal existence time satisfies $T = +\infty$. Otherwise, if $T < +\infty$, the solution exists only on a finite time interval and is referred to as a local solution.

The maximal existence time is a critical time beyond which the solution may blow-up.

Definition 7 (Blow-up). *Let u be a solution of an evolution equation defined on $[0, T]$. We say that the solution blow-up in finite time if the maximal existence time satisfies*

$$T < +\infty$$

, and the solution becomes unbounded as $t \rightarrow T$.

More precisely, blow-up occurs if there exists a norm (typically L^2, H^1 , etc.) such that

$$\lim_{t \rightarrow T} \|u(t)\| = +\infty,$$

in this case, T is called the "blow-up time", and the solution cannot be extended beyond this time as a regular solution.

Definition 8 (Blow-up point blow-up set). *Let $u(x, t)$ be a solution of an evolution equation defined on $\mathbb{R} \times [0, T]$, where $T < +\infty$ is the maximal existence time.*

A point $x_0 \in \mathbb{R}$ is called a "blow-up point" if the solution becomes unbounded in any neighborhood of x_0 as $t \rightarrow T$.

Meaning, there exists a sequence (x_m, t_m) such that

$$t_m \rightarrow T, \quad x_m \rightarrow x_0 \quad \text{and} \quad u(x_m, t_m) \rightarrow +\infty.$$

The set of all blow-up points is often defined as

$$B = \{x_0 \in \mathbb{R}; \exists(x_m, t_m) \rightarrow (x_0, T) \text{ such that } |u(x_m, t_m)| \rightarrow +\infty\}.$$

1.7.2 Blow-up in Ordinary Differential Equations (ODEs)

Perhaps the most natural example in which the blow-up phenomenon appears is in ordinary differential equations

$$\begin{cases} \frac{du}{dt} = u^2, & t > 0, \\ u(0) = a > 0. \end{cases}$$

The unique solution is defined only on a finite interval $[0, T)$, where $T = 1/a$. It is explicitly given by

$$u(t) = \frac{1}{T - t},$$

and satisfies $\lim_{t \rightarrow T^-} u(t) = +\infty$.

In general, the variable t is interpreted as time, and one says that the solution blows up at the finite time $t = T$. From this example, blow-up can be described as the phenomenon for which the solution is not globally defined because it tends to infinity in finite time.

A first extension of this example consists in considering the ordinary differential equation of the form $\frac{du}{dt} = u^p$, $p > 1$, or more generally

$$\begin{cases} \frac{du}{dt} = f(u), & t > 0, \\ u(0) = a > 0, \end{cases} \quad (1.10)$$

where f is a continuous function such that $f(u) > 0$ for $u \geq 0$. In this case, the blow-up condition is

$$F(z) = \int_a^z \frac{ds}{f(s)} < C \quad \text{for all } z > a, \quad (1.11)$$

where F is an antiderivative of f and C is a positive constant. Concretely, the blow-up time is $T = F(\infty)$.

1.7.3 Blow-up in Partial Differential Equations (PDEs)

Here, the study of blow-up is somewhat more difficult and mathematically more interesting since the problem under consideration involves several variables. The typical example

is the partial differential equation in which the solution $u(x, t)$ depends on a spatial variable $x \in \mathbb{R}^n$, $n \geq 1$, and time t .

A special class of this type of evolution equation is what are called reaction-diffusion equations, which emerged in the 20th century to model processes in physics and biology. They are generally of the form

$$\frac{\partial u}{\partial t} = \operatorname{div} A(u, \nabla u, x, t) + B(u, \nabla u, x, t), \quad (1.12)$$

where $\operatorname{div} A$ is a second-order linear operator.

The prototype is the semilinear heat equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u). \quad (1.13)$$

To get a clear idea, one considers $u \geq 0$ as temperature. The term $\operatorname{div} A$ in (1.12) represents diffusion, and B may model reaction, absorption, as well as convection. These equations are mostly studied on a spatial domain $\Omega \subseteq \mathbb{R}^n$ and are supplemented with the initial condition $u(x, 0) = u_0(x)$ as well as the boundary condition if Ω is not the whole of $\mathbb{R}^n$.

If we have techniques to guarantee the positivity of the solution as long as it exists and is regular, then we obtain the maximal existence time $0 < t < T$, $T = T_{\max}$, whether finite or not. The case where T may be finite occurs when the solution u is bounded for all $0 < t < T$ but

$$\lim_{t \rightarrow T} \|u(\cdot, t)\|_{\infty} = \infty,$$

where $\|u(\cdot, t)\|_{\infty} = \sup_{x \in \mathbb{R}} |u(x, t)|$. Then we say that u blows up in finite time T , which is the blow-up time.

One may observe that there is another situation in which the reaction-diffusion equation has no global (regular) solution, and this occurs when, in (1.13) $f(u) = -u^{-p}$, $p > 0$. In this case, it is conceivable that there exists a finite time T such that $u(x, t) > 0$ for all $0 < t < T$ but

$$\lim_{t \rightarrow T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{x \in \mathbb{R}} u(x, t)$. Then the reaction term blows up, and the regular solution ceases to exist. This phenomenon is another form of blow-up, called "singular extinction".

1.8 Explosion phenomenon in some applied fields

Blow-up phenomena represent one of the most striking behaviors that arise in mathematical models across a wide range of scientific disciplines. A blow-up occurs when the

solution of a system becomes unbounded in finite time, signaling the emergence of extreme or singular events. Such behavior is not merely a mathematical curiosity but reflects critical transitions in real-world systems. In economics, blow-up may correspond to uncontrollable inflation or financial crises; in medicine, it can model rapid cellular proliferation or the explosive spread of infectious diseases. Similarly, environmental models exhibit blow-up in scenarios of population explosion or unstable ecological interactions, while in chemistry, finite-time blow-up appears in autocatalytic reactions and thermal runaway processes. Understanding these phenomena provides essential insight into the mechanisms driving abrupt changes, helps identify thresholds beyond which systems become unstable, and supports the development of predictive and preventive strategies in diverse scientific fields.

In the study of differential equations, it is not necessarily the case that the solution follows a stable path; rather, the solution may vanish under certain conditions. Instead, for each system there exists a certain threshold beyond which the solution is considered to blow up. At this point, the solution values reach unrealistic levels or become unacceptable from an applied perspective. This is an indication that the system has lost its stability or gone out of control. It may also suggest that the differential model has reached the limits of its validity and is no longer capable of representing reality in an acceptable way.

1.8.1 Blow-up of solution in some chemical problems

Solution blow-up refers to the phenomenon where the value of a variable, concentration of reactants, or temperature increases to infinity within a finite time. In exothermic reactions, the increase in temperature accelerates the reaction rate, thereby producing more heat that may lead to explosion. This is particularly evident in autocatalytic reactions where the product accelerates its own formation, causing the concentration to increase at an unlimited rate. This type of behavior is extremely dangerous in chemical and petrochemical industries, as it can cause significant material and human damage [16].

Example 2 (Thermal runaway in hydrogen peroxide decomposition with metals).

***Scientific background:** In the chemical industry, numerous incidents have been recorded resulting from the rapid decomposition of hydrogen peroxide (H_2O_2) in the presence of certain metallic catalysts. This leads to a spontaneous thermal acceleration (thermal runaway).*

To describe the phenomenon of thermal runaway mathematically, we rely on an energy balance equation within the reactor. This equation takes into account both the heat generated by the chemical reaction inside the reactor and the heat exchanged with the external environment.

By using the Arrhenius law to model the reaction rate, we obtain a nonlinear equation capable of explaining the rapid increase in temperature, which may lead to thermal runaway.

Mathematical model: Energy balance equation

$$\frac{dT}{dt} = \frac{\Delta H}{\rho c_p} \cdot k_0 e^{-E/RT} \cdot C - h(T - T_{env}),$$

where:

ΔH : enthalpy of reaction.

ρ : density.

c_p : specific heat capacity.

k_0 : pre-exponential factor.

C : concentration.

h : heat transfer coefficient.

T : temperature.

R : universal gas constant.

E : activation energy.

T_{env} : environment temperature.

$k(T) = k_0 e^{-E/RT}$: arrhenius reaction equation.

Frank-Kamenetskii Equation: After performing certain suitable transformations and a change of variables, the energy balance equation can be rewritten in the form of the Frank-Kamenetskii equation as follows:

$$u_t - \Delta u = \lambda e^u.$$

Here, u represents the temperature.

The term e^u describes the increase in the reaction rate with the rise in temperature.

When the heat generated by the chemical reaction exceeds the heat lost through diffusion, a thermal runaway occurs, which corresponds mathematically to the blow-up of the solution.

This phenomenon has been experimentally studied by Frikha and others [19], particularly in the case of hydrogen peroxide decomposition in the presence of chromium compounds.

The study showed that the interaction between heat generation and reaction kinetics, governed by Arrhenius-type laws, may lead to thermal runaway when the generated heat exceeds the system's ability to dissipate it.

1.8.2 Mathematical modeling of biological explosions in medicine

The concept of finite-time explosion has gained significant importance in the field of medicine. This concept indicates that a variable in a biological system, such as cell count, virus concentration, degree of inflammation, or severity of infection, may reach extremely high values within a finite time according to specific mathematical models. Although the concept of infinity is not actually realized in biological reality, it describes tremendous acceleration and represents biological collapse or disease catastrophe.

In modern medicine, the explosion of solutions represents an analytical framework for understanding certain rapidly escalating phenomena, such as:

- Viral replication during acute stages of infection.
- Increase in cytokines in cases of immune storms.
- Aggressive cancerous growth.

The importance of this concept lies in the fact that it allows doctors and researchers to identify critical tipping points that precede an explosion; this contributes to the development of early intervention strategies to prevent the patient's clinical condition from deteriorating.

Example 3 (Accelerated cancer growth). *In the aggressive stages of some tumors, growth exceeding exponential growth is observed due to cancer cells secreting factors that stimulate their own division.*

Mathematical model of modified gompertz model:

$$\frac{dN}{dt} = r \cdot N^p \ln(N), \quad p > 1,$$

when $N > 1$, then

$$\frac{dN}{dt} > r \cdot N^p.$$

Therefore, an explosion occurs in a shorter time than standard exponential growth.

The modified growth model allows for the use of multiple nonlinear limits, which under aggressive conditions leads to a mathematical explosion in tumor volume, reflecting clinical cases of rapid spread.

Real-life example: *Several tumors exhibit non-linear and very rapid growth, for example: sarcomas and lymphomas that can double in size in less than 72 hours (Tumor Volume Doubling Time - TVDT).*

Example 4 (Diabetes and hypertension). *[17]. The interaction between diabetes and hypertension has been widely recognized in biomedical literature as a bidirectional nonlinear feedback system, often referred to as a cardiometabolic coupling.*

In this context, an increase in blood glucose levels contributes to vascular dysfunction and elevated blood pressure, while hypertension, in turn, affects insulin sensitivity and glucose regulation.

Where coupling appears medically as follows?

High blood sugar levels lead to:

- Blood vessel damage.*
- Increased insulin resistance.*
- Sodium retention.*

These factors lead to high blood pressure.

High blood pressure also leads to:

- Poor blood flow to the pancreas.*
- Insulin secretion disorder.*

These factors lead to high blood sugar.

In this example, the phenomenon of blow-up of solutions explains the rapid deterioration of physiological conditions, where vital variables exceed their normal bounds within a finite time. Clinically, this corresponds to conditions such as hyperthermia crisis, acute diabetic complications, and rapid organ failure, reflecting a breakdown of the system from a stable state to a state of collapse within a short time interval.

This mutual influence motivates mathematical representations in the form of nonlinear coupled dynamical systems (ODE complex model)

$$\begin{cases} \frac{dS(t)}{dt} = a - b S(t) + \alpha P(t), \\ \frac{dP(t)}{dt} = c - d P(t) + \beta S(t), \end{cases}$$

where

$S(t)$: blood sugar level.

$P(t)$: blood pressure.

a, c : natural production.

b, d : body regulation.

α, β : mutual influence force.

$p, q > 1$: non-linearity.

The condition $p, q > 1$ reflects the role of a critical threshold separating stable regimes from regimes of uncontrolled growth.

A blow-up of solution in this problem will occur if one of the following conditions is met:

α, β are large.

$p, q > 1$.

b, d are weak.

In this coupled system, the blow-up of solutions refers to the occurrence of a finite time T^ such that at least one of the state variables becomes unbounded. More precisely,*

$$\lim_{t \rightarrow T^*} (S(t) + P(t)) = +\infty.$$

This behavior reflects the effect of strong nonlinear coupling between $S(t)$ and $P(t)$, which may lead to an uncontrolled growth of the solution in finite time.

Chapter 2

Theoretical and numerical studies blow-up of a degenerate nonlinear reaction-diffusion problem

In this chapter, we present the theoretical results related to problem (5), concerning the existence and uniqueness of the solution for the continuous problem. We also address the blow-up of solutions for both the continuous and the semi-discrete problems.

Furthermore, we show that the solution of the semi-discrete problem converges to that of the continuous problem. Similarly, the blow-up time of the semi-discrete problem is proven to converge to the corresponding blow-up time of the continuous problem.

At the end of this chapter, we introduce the numerical discretization based on the finite difference θ -scheme. Theoretical analysis indicates that, under appropriate regularity assumptions on the exact solution and suitable restrictions on the discretization parameters, the θ -scheme is consistent, stable, and convergent in the pre-blow-up regime.

2.1 Position of problem

We consider the following time-degenerate coupled nonlinear reaction-diffusion system

$$\left\{ \begin{array}{ll} x u_t(x, t) = u_{xx}(x, t) + v^p(x, t), & x \in]0, 1[, \ t > 0, \\ (1 - x) v_t(x, t) = v_{xx}(x, t) + u^p(x, t), & x \in]0, 1[, \ t > 0, \\ u(x, t) = v(x, t) = 0, & x \in \{0, 1\}, \ t > 0, \\ u(x, 0) = u_0(x) \geq 0, \ v(x, 0) = v_0(x) \geq 0, & x \in [0, 1], \end{array} \right. \quad (2.1)$$

where $1 < p \leq 2$, and the initial data $u_0, v_0 \in C^{2+\alpha}([0, 1])$, $u_0 = v_0 = 0$ on $\{0, 1\}$.

The system is supplemented with homogeneous Dirichlet boundary conditions and non-negative initial data.

Analytical description: The model is of "time-degenerate coupled nonlinear reaction-diffusion system", where, the diffusion is driven by the second-order spatial operators, and nonlinear reaction occurs through the interaction terms.

The coupling is realized through "nonlinear reaction terms of power-type", namely u^p and v^p , which describe mutual nonlinear feedback between the two components u and v , meaning that each component influences the evolution of the other in a bidirectional manner through nonlinear interaction.

The system is "time-degenerate" due to the coefficients x and $1 - x$ multiplying the time derivatives, which vanish at $x = 0$ and $x = 1$, leading to degeneracy effects in the temporal evolution, which leads to a loss of the local temporal dynamics and causes the equation to change its nature from an evolutionary behavior in the interior to a quasi-static behavior at the boundary, it means that time no longer directly influences the evolution of the solution at the boundary points.

The diffusion mechanism is classical with constant coefficients.

The non-negative initial data are essential in the qualitative analysis of solution, particularly in the study of "finite-time blow-up phenomena".

Physical and chemical interpretation: The present model can be interpreted as a one-dimensional reaction-diffusion system describing the evolution of two interacting chemical species A and B with spatial diffusion and nonlinear mutual interaction, where u is the concentration of A and v the concentration of B .

The diffusion terms u_{xx} and v_{xx} represent the classical transport mechanism driven by gradients of the concentrations u and v , while the nonlinear terms u^p and v^p describe strong nonlinear reaction effects between the two species.

From a physical and chemical point of view, the system models an autocatalytic interaction in which each component enhances the production of the other through power-law nonlinearities.

This behavior similar, in a qualitative sense, to Arrhenius-type kinetics, where the reaction rate increases rapidly with the state of the system, potentially leading to runaway effects.

In the classical Arrhenius model, the source of acceleration in the reaction rate is primarily the temperature, where an increase in temperature leads to a rapid enhancement of the reaction rate. In contrast, in the present model, the acceleration mechanism is driven by the concentrations u and v , indeed, increasing values of u and v amplify the reaction

rate through the power-type nonlinearities u^p and v^p , resulting in a strong positive feedback effect.

Moreover, the presence of the time - degenerate coefficients x and $1 - x$ introduces a spatially heterogeneous medium in which the effective temporal evolution is weakened near the boundary points $x = 0$ and $x = 1$, reflecting a non-uniform response of the system across the spatial domain.

Overall, the model describes a balance between diffusion, which tends to smooth spatial variations, and strong nonlinear reaction mechanisms. That may lead to rapid growth and finite - time blow-up phenomena under suitable conditions.

2.2 The continuous problem

In this section, we present results obtained by Belyacine in [2] on the blow-up in finite time of solution of continuous problem (2.1).

Assuming that there exists a classical positive solution (u, v) of (2.1) on $[0, 1] \times [0, T[$, that is, $u, v \in C([0, 1] \times [0, T]) \cap C^{2,1}([0, 1] \times]0, T[)$.

The system (2.1) is said to be degenerate at $x = 0$ and $x = 1$ in the sense that the coefficient of u tends to 0 as $x \rightarrow 0$, and the coefficient of v tends to 0 as $x \rightarrow 1$.

2.2.1 Local existence and uniqueness of the solution

Belyassine proposed in [2] a technique of truncating the domain in the vicinity of the boundary points, which allows the degeneracy to be removed and brings this system to a sequence of non-degenerate problems that admit a positive local solution. Under certain assumptions on the nonlinear part, and using the method of sub and over-solution, they show that the new system admits a unique local solution. The idea is to truncate the domain $]0, 1[$ by considering the interval $]\varepsilon, 1 - \varepsilon[$, such that system (2.1) becomes non-degenerate on $]\varepsilon, 1 - \varepsilon[\times]0, T[$.

She obtained the following truncated system

$$\begin{cases} x u_t(x, t) = u_{xx}(x, t) + v^p(x, t), & \text{in }]\varepsilon, 1 - \varepsilon[\times]0, T[, \\ (1 - x) v_t(x, t) = v_{xx}(x, t) + u^p(x, t), & \text{in }]\varepsilon, 1 - \varepsilon[\times]0, T[, \\ u(x, t) = v(x, t) = 0, & x \in \{\varepsilon, 1 - \varepsilon\}, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in]\varepsilon, 1 - \varepsilon[. \end{cases} \quad (2.2)$$

By applying the local existence theorems of solution for non-degenerate parabolic system, Belyassine [2] obtained the following results:

Lemma 1. *There exists $t_0 > 0$ such that the system (2.2) admits a solution $(u_\varepsilon, v_\varepsilon)$ on $]\varepsilon_1, 1 - \varepsilon_2[\times]0, t_0[$, where $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$.*

The following lemma provides an important property of the sequences (u_ε) and (v_ε) .

Lemma 2. *Let $\delta_1 < \delta'_1$ and $\delta_2 < \delta'_2$, and suppose that $(u_{\delta_1\delta_2}, v_{\delta_1\delta_2})$ and $(u_{\delta'_1\delta'_2}, v_{\delta'_1\delta'_2})$ are solutions of the system (2.2) on $]0, t_0[$. Then*

$$u_{\delta'_1\delta'_2} < u_{\delta_1\delta_2},$$

$$v_{\delta'_1\delta'_2} < v_{\delta_1\delta_2},$$

for all $x \in]\delta'_1, 1 - \delta'_2[$ and $t \in]0, t_0[$.

Theorem 6. *There exists $t_0 > 0$, independent of $\varepsilon_1, \varepsilon_2$, and a function $h \in C^{2,1}([0, 1] \times [0, t_0])$ such that the problem (2.2) admits a solution $(u_{\varepsilon_1\varepsilon_2}, v_{\varepsilon_1\varepsilon_2})$ satisfying*

$$\begin{aligned} u_{\varepsilon_1\varepsilon_2} &\leq h, \\ v_{\varepsilon_1\varepsilon_2} &\leq h, \end{aligned} \quad \text{on }]\varepsilon_1, 1 - \varepsilon_2[\times]0, t_0[.$$

Corollary 1. *The sequences (u_ε) and (v_ε) converge pointwise to functions u, v on $]0, 1[\times]0, t_0[$ as $\varepsilon \rightarrow 0$. Moreover, $u, v \in L^p([0, 1] \times [0, t_0])$.*

2.2.2 Regularity of the solution

This section aims to prove that the functions u and v , obtained in the previous corollary, are classical solutions to problem (2.2), by relying on a set of appropriate estimates.

Lemma 3. *Let $Q =]a, b[\times]0, t_2[$ be a rectangular domain such that $0 < a < b < 1$ and $0 < t_2 < t$. There exists a constant $C' > 0$, independent of ε , such that*

$$\|u_\varepsilon\|_{C^\alpha(Q)} \leq C',$$

$$\|v_\varepsilon\|_{C^\alpha(Q)} \leq C'.$$

Lemma 4. *Let $Q =]a, b[\times]0, t_2[$ be a rectangular domain such that $0 < a < b < 1$ and $0 < t_2 < t_0$. There exists a constant $C'' > 0$, independent of ε , such that*

$$\|u_\varepsilon\|_{C^{2+\alpha}(Q)} \leq C'',$$

$$\|v_\varepsilon\|_{C^{2+\alpha}(Q)} \leq C''.$$

Theorem 7. *If $u_0, v_0 \in C^{2+\alpha}(0, 1)$ for $\alpha \in]0, 1[$, then (u, v) is a classical solution of system (2.2) in $]0, 1[\times]0, t_0[$.*

Proposition 2. *The solution of system (2.2) is unique and it is positive on $]0, 1[\times]0, t_0[$.*

2.2.3 Blow-up of continuous solution

Definition 9. *The solution (u, v) of the continuous problem (2.1) is said to blow-up in finite time, if there exists a finite time T_b such that $\|u(\cdot, t)\|_\infty < \infty$ and $\|v(\cdot, t)\|_\infty < \infty$ for $t \in [0, T_b[$, but*

$$\lim_{t \rightarrow T_b} \|u(\cdot, t)\|_\infty + \lim_{t \rightarrow T_b} \|v(\cdot, t)\|_\infty = \infty,$$

where $\|u(\cdot, t)\|_\infty = \sup_{0 \leq x \leq 1} |u(x, t)|$ and $\|v(\cdot, t)\|_\infty = \sup_{0 \leq x \leq 1} |v(x, t)|$.

The time T_b is called the blow-up time of the solution (u, v) .

Definition 10. *A point $x \in]0, 1[$ is an blow-up point of (2.1) if there exists a sequence (x_m, t_m) such that*

$$t_m \rightarrow T, x_m \rightarrow x \text{ and } u(x_m, t_m) \rightarrow \infty, v(x_m, t_m) \rightarrow \infty \text{ if } m \rightarrow \infty.$$

For finding a suffisantly conditions for each the solution of system (2.1) is blow-up at finite time $T > 0$, they apply the eigenfunction technique (Fourier coefficient method) originally introduced by S. Kaplan [24], to our particular system (2.1), by considering the following corresponding spectral problem (of Sturm-Liouville)

$$\begin{cases} \frac{d^2 \phi(x)}{dx^2} = -\lambda x(1-x)\phi(x), & x \in]0, 1[, \\ \phi(0) = \phi(1) = 0, \\ \phi(x) > 0, & \text{in }]0, 1[. \end{cases} \quad (2.3)$$

There is a unique positive real eigenvalue $\lambda > 0$, associated with an eigenfunction $\phi \in C([0, 1] \cap C^2(]0, 1[))$, which is a solution of problem (2.3) on $]0, 1[$, see [13].

Let $(u, v)(x, t)$ be a solution to (2.1) on $[0, 1] \times [0, T[$, we set

$$\begin{cases} \mathcal{U}(t) = \int_0^1 x \phi(x) u(x, t) dx, & t \in [0, T[, \\ \mathcal{V}(t) = \int_0^1 (1-x) \phi(x) v(x, t) dx, & t \in [0, T[, \end{cases}$$

and

$$\mathcal{W}(t) = \mathcal{U}(t) + \mathcal{V}(t).$$

Lemma 5. [35]. For $p > 1$, there exists $C > 0$ such that

$$\frac{d\mathcal{W}(t)}{dt} \geq -\lambda\mathcal{W}(t) + \frac{C}{2^{p-1}}\mathcal{W}^p(t), \forall t \in]0, T[.$$

As an immediate consequence we have a criterion on the initial data of the continuous problem (2.1), which is sufficient for the blow-up of the solution in a finite time. This is given in the next theorem.

Theorem 8. [35]. If $p > 1$, C a constant defined in Lemma 5, and the initial data u_0 and v_0 satisfy

$$\int_0^1 x\phi(x)u_0(x)dx + \int_0^1 (1-x)\phi(x)v_0(x)dx > 2\left(\frac{\lambda}{C}\right)^{\frac{1}{p-1}},$$

then the solution (u, v) of the system (2.1) blows up in a finite time T .

2.3 The semi-discrete problem

In this section, we present the results obtained by Nisse and Belyassine in [35], they study an approximation of the system (2.1) by the online method (OLM) to discretize the system (2.1) in space. They choose a constant step $h = 1/(N + 1)$ where N is a positive integer, and $x_i = ih$, $i = 0, \dots, N + 1$ in $]0, 1[$.

They use the finite difference method to approximate the derivatives. This allows us to obtain a second-order scheme called centered to approximate the second derivative of u and v with respect to x .

They obtain the following system of ordinary differential equations

$$\begin{cases} \frac{\partial u_i}{\partial t}(t) = \frac{1}{x_i h^2} (u_{i+1} - 2u_i + u_{i-1}) + \frac{1}{x_i} v_i^p; & i = 1, \dots, N; t > 0, \\ \frac{\partial v_i}{\partial t}(t) = \frac{1}{(1-x_i) h^2} (v_{i+1} - 2v_i + v_{i-1}) + \frac{1}{(1-x_i)} u_i^p; & i = 1, \dots, N; t > 0, \\ u_0(t) = u_{N+1}(t) = v_0(t) = v_{N+1}(t) = 0; & t > 0, \\ u_i(0) = u_0(x_i) \geq 0, v_i(0) = v_0(x_i) \geq 0, \end{cases} \quad (2.4)$$

such that $u_i = u(x_i, t)$, $v_i = v(x_i, t)$ for $1 \leq i \leq N$.

We see that the scheme (2.4) can be written in the following matrix form

$$\begin{cases} \frac{\partial \mathcal{U}_h}{\partial t}(t) = A_h \mathcal{U}_h + \mathcal{F}_1(\mathcal{V}_h); & t > 0, \\ \frac{\partial \mathcal{V}_h}{\partial t}(t) = B_h \mathcal{V}_h + \mathcal{F}_2(\mathcal{U}_h); & t > 0, \end{cases} \quad (2.5)$$

where

$$\mathcal{A}_h = \frac{1}{h^3} \begin{bmatrix} -\frac{2}{1} & 1 & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & -\frac{2}{2} & \frac{1}{2} & \ddots & & \vdots \\ 0 & \frac{1}{3} & -\frac{2}{3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{N-1} \\ 0 & \cdots & \cdots & 0 & \frac{1}{N} & -\frac{2}{N} \end{bmatrix},$$

$$\mathcal{B}_h = \frac{1}{h^3} \begin{bmatrix} -\frac{2}{N} & \frac{1}{N} & 0 & \cdots & \cdots & 0 \\ \frac{1}{N-1} & -\frac{2}{N-1} & \frac{1}{N-1} & \ddots & & \vdots \\ 0 & \frac{1}{N-2} & -\frac{2}{N-2} & \frac{1}{N-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{1} & -\frac{2}{1} \end{bmatrix},$$

$$\mathcal{U}_h = (u(x_1, t), u(x_2, t), \dots, u(x_N, t))^T, \quad \mathcal{V}_h = (v(x_1, t), v(x_2, t), \dots, v(x_N, t))^T,$$

and

$$\mathcal{F}_1 = \left(\frac{v_1^p}{h}, \frac{v_2^p}{2h}, \dots, \frac{v_N^p}{Nh} \right)^T, \quad \mathcal{F}_2 = \left(\frac{u_1^p}{Nh}, \frac{u_2^p}{(N-1)h}, \dots, \frac{u_N^p}{h} \right)^T.$$

2.3.1 Properties of the matrix $\mathcal{A}_h$ and $\mathcal{B}_h$

As in [35], we decompose matrix $\mathcal{A}_h$ as follows

$$\mathcal{A}_h = \mathcal{A}_h^s + \mathcal{T}_h^1,$$

where $\mathcal{A}_h^s$ is a symmetric tridiagonal matrix and $\mathcal{T}_h^1$ is a positive upper triangular matrix defined as follows

$$\mathcal{A}_h^s = \frac{1}{h^3} \begin{bmatrix} -2 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & -\frac{2}{2} & \frac{1}{3} & \ddots & & \vdots \\ 0 & \frac{1}{3} & -\frac{2}{3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{N} \\ 0 & \cdots & \cdots & 0 & \frac{1}{N} & -\frac{2}{N} \end{bmatrix}; \mathcal{T}_h^1 = \frac{1}{h^3} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \frac{1}{6} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \frac{1}{N(N-1)} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

Similarly, decompose $\mathcal{B}_h$ as follows

$$\mathcal{B}_h = \mathcal{B}_h^s + \mathcal{T}_h^2,$$

where $\mathcal{B}_h^s$ is a symmetric tridiagonal matrix and $\mathcal{T}_h^2$ is a positive lower triangular matrix defined as follows

$$\mathcal{B}_h^s = \frac{1}{h^3} \begin{bmatrix} -\frac{2}{N} & \frac{1}{N} & 0 & \cdots & \cdots & 0 \\ \frac{1}{N} & -\frac{2}{N-1} & \frac{1}{N-1} & \ddots & & \vdots \\ 0 & \frac{1}{N-1} & -\frac{2}{N-2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & -\frac{2}{1} \end{bmatrix}; \mathcal{T}_h^2 = \frac{1}{h^3} \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{N(N-1)} & 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \frac{1}{2} & 0 \end{bmatrix}$$

Lemma 6. *The matrix $\mathcal{A}_h^s$ has N distinct negative real eigenvalues*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N,$$

where λ_1 is the largest eigenvalue.

Lemma 7. *The eigenvector components associated with the eigenvalue λ_1 are positive.*

Remark 5. *We note $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ the eigenvector associated with λ_1 having strictly positive components such that $\sum_{i=1}^N \phi_i = 1$.*

Remark 6. *The vector*

$$\omega = (\phi_N, \phi_{N-1}, \dots, \phi_1)^T = (\phi_{N-i+1})_{1 \leq i \leq N}$$

is an eigenvector of the matrix $\mathcal{B}_h^s$ associated with the eigenvalue λ_1 .

2.3.2 The blow-up of solution

To show the blow-up finite time of the system solution (2.5), we use the Kaplan method. We pose

$$\begin{cases} \varphi_h(t) = \langle \mathcal{U}_h(t), \phi \rangle; & t \in]0, T[, \\ \psi_h(t) = \langle \mathcal{V}_h(t), \omega \rangle; & t \in]0, T[, \end{cases} \quad (2.6)$$

where T is the maximum existence time of the solution $(\mathcal{U}_h, \mathcal{V}_h)$.

Let us also pose

$$\begin{cases} \Phi_h(t) = \varphi_h(t) + \psi_h(t), \\ \Phi_h(0) = \varphi_h(0) + \psi_h(0). \end{cases}$$

Lemma 8. [35] *Let $p > 1$ and $\mu = -\lambda_1$. If the components of $(\mathcal{U}_h, \mathcal{V}_h)(0)$ are non-negative, then there exists a constant $c > 0$ such that*

$$\frac{d\Phi_h(t)}{dt} \geq -\mu\Phi_h(t) + \frac{c}{2^{p-1}}\Phi_h^p(t), t \in]0, T[,$$

where

$$c = \min_{1 \leq i \leq N} \left(\frac{\phi_i}{ih\phi_{N-i+1}} \right) = \min_{1 \leq i \leq N} \left(\frac{\phi_{N-i+1}}{(N-i+1)h\phi_i} \right).$$

Theorem 9. [35] *Let $p > 1, \mu = -\lambda_1$ and c the constant given in Lemma 8. If $\Phi_h(0) > 2c^{\frac{1}{1-p}}\mu^{\frac{1}{p-1}}$, then the solution blows up in finite time*

$$T_h \leq \frac{1}{\mu(p-1)} \log \left(\frac{2^{c/(p-1)}}{2^{c/(p-1)} - \mu\Phi_h(0)^{1-p}} \right).$$

2.3.3 Properties of the solution to the semi-discrete problem

According to (2.4), we define

$$\mathcal{W}_h(t) = (\omega_1(t), \dots, \omega_N(t), \omega_{N+1}(t), \dots, \omega_{2N}(t))^T,$$

where, for $t \geq 0$

$$\omega_i = \begin{cases} u_i, & \text{si } 1 \leq i \leq N, \\ v_{i-N}, & \text{si } N+1 \leq i \leq 2N, \end{cases}$$

are solutions of the nonlinear ordinary differential equations

$$\frac{d\omega_i(t)}{dt} - \delta_h^2(\omega_i(t)) = F_i(\omega_i(t)), \quad 1 \leq i \leq 2N, \quad (2.7)$$

where

$$\delta_h^2(\omega_i(t)) = \begin{cases} \frac{1}{ih^3}(u_{i+1} - 2u_i + u_{i-1}), & 1 \leq i \leq N, \\ \frac{1}{(2N+1-i)h^3}(v_{i+1-N} - 2v_{i-N} + v_{i-1-N}), & N+1 \leq i \leq 2N, \end{cases} \quad (2.8)$$

and

$$F_i(\omega_i(t)) = \begin{cases} \frac{\omega_{i+N}^p}{ih}, & 1 \leq i \leq N, \\ \frac{\omega_{i-N}^p}{(2N+1-i)h}, & N+1 \leq i \leq 2N, \end{cases} \quad (2.9)$$

with

$$\omega_i(0) = u_0(x_i) \text{ si } 1 \leq i \leq N \text{ and } \omega_i(0) = v_0(x_{i-N}) \text{ si } N+1 \leq i \leq 2N.$$

We set $\omega_0(t) = \omega_{2N+1}(t) = 0$, by convention, and we note that the existence and uniqueness of the solutions of ordinary differential equations are established if the nonlinear part is a locally Lipschitzian continuous function.

Proposition 3. [2]. *Let $\mathcal{W}_h(t) \in C^1([0, T[, \mathbb{R}^{2N})$ be solution of (2.11) on $]0, T[$, such that*

$$\omega_i(0) \geq 0, \quad 1 \leq i \leq 2N,$$

then

$$\omega_i(t) \geq 0, \quad 1 \leq i \leq 2N, \forall t \in]0, T[.$$

2.3.4 Convergence of the solution to the semi-discrete problem

In this paragraph we show that for each fixed interval $[0, T]$ wherem the continuous solution (u, v) is defined, the solution $(\mathcal{U}_h, \mathcal{V}_h)$ of the semi-discrete problem (2.4) converges to (u_h, v_h) when $h \rightarrow 0$.

Let

$$\begin{aligned} \tau_h &= \mathcal{U}_h(t) - u_h(t), \\ \varrho_h &= \mathcal{V}_h(t) - v_h(t), \end{aligned}$$

where

$$u_h(t) = (u(x_1, t), \dots, u(x_N, t))^T \text{ and } v_h(t) = (v(x_1, t), \dots, v(x_N, t))^T.$$

Lemma 9. *Let $z(x, t)$ and $y(x, t)$ be two positive functions, $C_1'', C_2'' > 0$. Then, for all i and t*

$$\begin{aligned} -C_1''(y(x_i, t) - |\varrho_h(t)|) &\leq C_1''|\varrho_h(t) - y(x_i, t)|, \\ -C_2''(z(x_i, t) - |\tau_h(t)|) &\leq C_2''|\tau_h(t) - z(x_i, t)|. \end{aligned}$$

Theorem 10. *Suppose that problem (2.1) has a solution $(u, v) \in (C^{4,1}([0, 1] \times [0, T]))^2$ and the initial condition in (2.4) satisfies*

$$|\mathcal{W}_h(0) - \omega_h(0)|_h = O(1) \text{ when } h \rightarrow 0, \quad (2.10)$$

where $\omega_h(t) = (u(x_1, t), \dots, u(x_N, t), v(x_1, t), \dots, v(x_N, t))^T$. Then for h sufficiently small, the semi-discrete problem (2.4) admits a unique solution $\mathcal{W}_h \in C^1([0; T]; \mathbb{R}^{2N})$ such as

$$\max_{0 \leq t \leq T} |\mathcal{W}_h(t) - \omega_h(t)|_\infty = O(|\mathcal{W}_h(0) - \omega_h(0)|_\infty + h^2) \text{ when } h \rightarrow 0. \quad (2.11)$$

Proof. Let $\alpha, \beta > 0$ such that

$$\begin{aligned} \|u(\cdot, t)\|_\infty &< \alpha, \\ \|v(\cdot, t)\|_\infty &< \beta, \end{aligned} \quad \text{for } t \in [0, T]. \quad (2.12)$$

The semi-discrete problem (2.4) admits for each h , a unique solution $(\mathcal{U}_h(t), \mathcal{V}_h(t)) \in C^1([0, T_b^h[, \mathbb{R}^{2N})$. Let $t(h)$ be the largest value of $t > 0$ such that $t < \min\{T, T_b^h\}$

$$\begin{aligned} |\mathcal{U}_h(t) - u_h(t)|_\infty &< 1, \\ |\mathcal{V}_h(t) - v_h(t)|_\infty &< 1, \end{aligned} \quad \text{for } t \in [0, t(h)]. \quad (2.13)$$

The existence of $t(h) > 0$ for h sufficiently small is justified by the relation (2.10).

By the triangle inequality, we obtain

$$\begin{aligned} |\mathcal{U}_h(t)|_\infty &\leq \|u(\cdot, t)\|_\infty + |\mathcal{U}_h(t) - u_h(t)|_\infty, \\ |\mathcal{V}_h(t)|_\infty &\leq \|v(\cdot, t)\|_\infty + |\mathcal{V}_h(t) - v_h(t)|_\infty, \end{aligned} \quad \text{for } t \in [0, t(h)].$$

Which implies that

$$\begin{aligned} |\mathcal{U}_h(t)|_\infty &\leq 1 + \alpha, \\ |\mathcal{V}_h(t)|_\infty &\leq 1 + \beta, \end{aligned} \quad \text{for } t \in [0, t(h)]. \quad (2.14)$$

Using the Taylor development, we have for $t \in]0, t(h)[$,

$$\begin{cases} ih \frac{d\tau_i(t)}{dt} - \delta^2 \tau_i(t) = p\eta^{p-1} \varrho_i(t) + O(h^2), \\ (N+1-i)h \frac{d\varrho_i(t)}{dt} - \delta^2 \varrho_i(t) = p\xi^{p-1} \tau_i(t) + O(h^2), \end{cases} \quad (2.15)$$

where

$$\begin{cases} \delta^2 \tau_i(t) = \frac{\tau_{i-1}(t) - 2\tau_i(t) + \tau_{i+1}(t)}{h^2}, \\ \delta^2 \varrho_i(t) = \frac{\varrho_{i-1}(t) - 2\varrho_i(t) + \varrho_{i+1}(t)}{h^2}, \end{cases} \quad \text{for all } 1 \leq i \leq N,$$

and

$$\begin{aligned} \xi_h(t) &= \lambda_t \mathcal{U}_h(t) + (1 - \lambda_t) u_h(t), \\ \eta_h(t) &= \lambda_t \mathcal{V}_h(t) + (1 - \lambda_t) v_h(t), \end{aligned} \quad \text{for } \lambda_t \in]0, 1[.$$

By (2.12) and (2.14), there exist positive constants $\mathcal{C}^1, \mathcal{C}^2$ and $\mathcal{K}$ such that

$$\begin{cases} ih \frac{d\tau_i(t)}{dt} - \delta^2 \tau_i(t) \leq \mathcal{C}_1 |\varrho_i(t)| + \mathcal{K}h^2, \\ (N+1-i)h \frac{d\varrho_i(t)}{dt} - \delta^2 \varrho_i(t) \leq \mathcal{C}_2 |\tau_i(t)| + \mathcal{K}h^2. \end{cases} \quad (2.16)$$

We consider the two comparison functions

$$\begin{aligned} z(x, t) &= \exp((L+1)t + \mathcal{C}x^2) (|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2), \\ y(x, t) &= \exp((L+1)t + \mathcal{C}x^2) (|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2), \end{aligned}$$

where $L, \mathcal{C}, \mathcal{Q}$ are constants.

Substituting these two functions into problem (2.1), we obtain

$$\begin{cases} xz_t(x, t) - z_{xx}(x, t) = (x(L+1) - 2\mathcal{C} - 4\mathcal{C}^2x^2) z(x, t), \\ (1-x)y_t(x, t) - y_{xx}(x, t) = ((1-x)(L+1) - 2\mathcal{C} - 4\mathcal{C}^2x^2) y(x, t), \\ z(x, 0) = \exp(\mathcal{C}x^2) (|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2), \\ y(x, 0) = \exp(\mathcal{C}x^2) (|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2). \end{cases} \quad (2.17)$$

By posing

$$z_i(t) = z(x_i, t) \text{ and } y_i(t) = y(x_i, t),$$

knowing that

$$\begin{aligned} \left(\frac{\partial^2 z}{\partial x^2} \right)_i &= \frac{z_{i-1}(t) - 2z_i(t) + z_{i+1}(t)}{h^2} + O(h^2), \\ \left(\frac{\partial^2 y}{\partial x^2} \right)_i &= \frac{y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)}{h^2} + O(h^2). \end{aligned}$$

According to (2.17), we obtain

$$\begin{cases} x_i \frac{dz_i(t)}{dt} - \delta^2 z_i(t) = (x_i(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_i^2) z_i(t) + O(h^2), \\ (1-x_i) \frac{dy_i(t)}{dt} - \delta^2 y_i(t) = ((1-x_i)(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_i^2) y_i(t), \\ z_i(0) = \exp(\mathcal{C}x_i^2) (|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2), \\ y_i(0) = \exp(\mathcal{C}x_i^2) (|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2), \end{cases} \quad (2.18)$$

as

$$\begin{aligned} z(x, t) &= \left(\frac{|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2}{|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2} \right) y(x, t), \\ y(x, t) &= \left(\frac{|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2}{|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2} \right) z(x, t), \end{aligned}$$

we can choose $L, \mathcal{C}, \mathcal{Q}$ large enough so that

$$\begin{cases} x_i \frac{dz_i(t)}{dt} - \delta^2 z_i(t) \geq (x_i(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_i^2) y_i(t) + \mathcal{K}h^2, \\ (1-x_i) \frac{dy_i(t)}{dt} - \delta^2 y_i(t) \geq ((1-x_i)(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_i^2) z_i(t) + \mathcal{K}h^2, \\ z_i(0) > \tau_i(0), y_i(0) > \varrho_i(0). \end{cases} \quad (2.19)$$

So

$$\begin{cases} ih \frac{dz_i(t)}{dt} - \delta^2 z_i(t) \geq \mathcal{C}'_1 y_i(t) + \mathcal{K}h^2, \\ (N+1-i)h \frac{dy_i(t)}{dt} - \delta^2 y_i(t) \geq \mathcal{C}'_2 z_i(t) + \mathcal{K}h^2, \\ z_i(0) > \tau_i(0), y_i(0) > \varrho_i(0), \end{cases} \quad (2.20)$$

where $\mathcal{C}'_1 \geq \mathcal{C}_1, \mathcal{C}'_2 \geq \mathcal{C}_2$.

Assume that

$$\begin{aligned} e_h(t) &= z_h(t) - \tau_h(t), \\ s_h(t) &= y_h(t) - \varrho_h(t). \end{aligned}$$

From (2.16) and (2.20), we obtain

$$\begin{cases} ih \frac{de_i(t)}{dt} - \delta^2 e_i(t) - \mathcal{C}'_1 y_i(t) + \mathcal{C}_1 |\varrho_i(t)| \geq 0, \\ (N+1-i)h \frac{ds_i(t)}{dt} - \delta^2 s_i(t) - \mathcal{C}'_2 z_i(t) + \mathcal{C}_2 |\tau_i(t)| \geq 0, \\ e_i(0) > 0; s_i(0) > 0. \end{cases} \quad (2.21)$$

Taking into account Lemma 9, we obtain

$$\begin{cases} ih \frac{de_i(t)}{dt} - \delta^2 e_i(t) + \mathcal{C}''_1 |\varrho_i(t) - y_i(t)| \geq 0, \\ (N+1-i)h \frac{ds_i(t)}{dt} - \delta^2 s_i(t) + \mathcal{C}''_2 |\tau_i(t) - z_i(t)| \geq 0, \\ e_i(0) > 0; s_i(0) > 0, \end{cases}$$

where $\mathcal{C}''_1 = \min\{\mathcal{C}_1, \mathcal{C}'_1\}$ and $\mathcal{C}''_2 = \min\{\mathcal{C}_2, \mathcal{C}'_2\}$. So

$$\begin{cases} ih \frac{de_i(t)}{dt} - \delta^2 e_i(t) + \mathcal{C}''_1 |s_i(t)| \geq 0, \\ (N+1-i)h \frac{ds_i(t)}{dt} - \delta^2 s_i(t) + \mathcal{C}''_2 |e_i(t)| \geq 0, \\ e_i(0) > 0; s_i(0) > 0, \end{cases} \quad (2.22)$$

for $1 \leq m \leq M$. Equivalently

$$\begin{cases} \frac{de_i(t)}{dt} - \delta_h^2 e_i(t) + \frac{\mathcal{C}''_1}{ih} |s_i(t)| \geq 0, \\ \frac{ds_i(t)}{dt} - \delta_h^2 s_i(t) + \frac{\mathcal{C}''_2}{(N+1-i)h} |e_i(t)| \geq 0, \\ e_i(0) > 0; s_i(0) > 0, \end{cases}$$

where δ_h^2 is the finite difference operator, already defined in the framework of Proposition 3.

Now, we pose

$$\mathcal{S}_h(t) = (e_1(t), \dots, e_N(t), s_1(t), \dots, s_N(t)),$$

and

$$\mathcal{H}(|\mathcal{S}_i(t)|) = \begin{cases} \frac{\mathcal{C}''_1}{ih} |s_i(t)|, 1 \leq i \leq N, \\ \frac{\mathcal{C}''_2}{(N+1-i)h} |e_i(t)|, 1 \leq i \leq N. \end{cases}$$

Then, we can define the following differential inequality

$$\begin{cases} \frac{d\mathcal{S}_i(t)}{dt} - \delta_h^2 \mathcal{S}_i(t) + \mathcal{H}(|\mathcal{S}_i(t)|) \geq 0, \\ \mathcal{S}_i(0) > 0, \end{cases} \quad (2.23)$$

from Proposition 3, we deduce that

$$\begin{aligned} z_i(t) &\geq \tau_i(t), \\ y_i(t) &\geq \varrho_i(t), \end{aligned} \quad \text{for } t \in]0, t(h)[, 1 \leq i \leq N. \quad (2.24)$$

In the same way we can also show that

$$\begin{aligned} z_i(t) &> -\tau_i(t), \\ y_i(t) &> -\varrho_i(t), \end{aligned} \quad \text{for } t \in]0, t(h)[, 1 \leq i \leq N, \quad (2.25)$$

where

$$\begin{aligned} -\tau_h &= -\mathcal{U}_h(t) + u_h(t), \\ -\varrho_h &= -\mathcal{V}_h(t) + v_h(t), \end{aligned}$$

which implies that

$$\begin{aligned} z_i(t) &> |\tau_i(t)|, \\ y_i(t) &> |\varrho_i(t)|, \end{aligned} \quad \text{for } t \in]0, t(h)[, 1 \leq i \leq N.$$

We deduce that

$$\begin{aligned} |\mathcal{U}_h(t) - u_h(t)|_\infty &\leq \exp((L+1)t + \mathcal{C}) (|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2), \\ |\mathcal{V}_h(t) - v_h(t)|_\infty &\leq \exp((L+1)t + \mathcal{C}) (|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2), \end{aligned}$$

for $t \in]0; t(h)[$. So, we have

$$|\mathcal{W}_h(t) - \omega_h(t)|_\infty \leq \exp((L+1)t + \mathcal{C}) (|\mathcal{W}_h(0) - \omega_h(0)|_\infty + \mathcal{Q}h^2), t \in]0; t(h)[.$$

Let us show that $t(h) = T$. Let's suppose that $T > t(h)$. From (2.13), we have

$$\begin{aligned} 1 &= |\mathcal{W}_h(t(h)) - \omega_h(t(h))|_\infty \\ &\leq \exp((L+1)T + \mathcal{C}) (|\mathcal{W}_h(0) - \omega_h(0)|_\infty + \mathcal{Q}h^2), t \in]0; t(h)[. \end{aligned}$$

As

$$\lim_{h \rightarrow 0} \exp((L+1)T + \mathcal{C}) (|\mathcal{W}_h(0) - \omega_h(0)|_\infty + \mathcal{Q}h^2) = 0,$$

we deduce that $1 \leq 0$, which is impossible. Therefore $t(h) = T$. $\square$

Theorem 11. [2] *Under hypotheses of Theorem 10, if T_b and T_h are respectively the blow-up finite time of continuous problem (2.1) and semi-discret problem (2.4) and if there exists a continuous and non-decreasing function*

$$F : [0, \infty[\rightarrow \mathbb{R}$$

where

$$|\mathcal{W}_h|_\infty \leq F\left(\frac{1}{T_h - t}\right), \quad t < T_h,$$

then

$$\lim_{h \rightarrow 0} T_h = T_b.$$

2.4 The fully–discrete problem

2.4.1 The θ -scheme

In this section, we derive the θ -scheme fully-discrete finite difference formulas for the problem (2.1), by approximating the time derivative in problem (2.4), using a linear combination between forward and backward finite difference. We consider that $\mathcal{U}_i^n$ and $\mathcal{V}_i^n$, are the approximate values of $u(x_i, t_n)$ and $v(x_i, t_n)$, respectively, where $t_{n+1} = t_n + k_n$, N be a positive integer, and consider the grid: $x_i = ih, 0 \leq i \leq N + 1$, $h = \frac{1}{N+1}$, $x_{i+1} = x_i + h$, and h is the space-step, and k_n is the non fixed time-step.

However, in order to derive θ -scheme formula formally, we approximate the time-derivative in problem (2.4), at the mesh points (x_i, t_n) using the central finite difference formula, and $\delta_x^2 \mathcal{U}(t)$ and $\delta_x^2 \mathcal{V}(t)$ are approximated as follows

$$\delta_x^2 \mathcal{U}_i(t_n) = \theta \delta_x^2 \mathcal{U}_i(t_{n+1}) + (1 - \theta) \delta_x^2 \mathcal{U}_i(t_n),$$

$$\delta_x^2 \mathcal{V}_i(t_n) = \theta \delta_x^2 \mathcal{V}_i(t_{n+1}) + (1 - \theta) \delta_x^2 \mathcal{V}_i(t_n).$$

$\theta = 0$: The scheme is *explicit* and conditionally stable.

$\theta = 1$: The scheme is *implicit* and unconditionally stable.

$0 < \theta < 1$: The scheme is *semi-implicit*, offering a balance between stability and accuracy.

The nonlinear terms are taken at the mesh points (x_i, t_n) , that yields the linear θ -scheme finite difference formula

$$\begin{aligned} x_i \frac{\mathcal{U}_i^{n+1} - \mathcal{U}_i^n}{k_n} &= \theta \delta_x^2 \mathcal{U}_i^{n+1} + (1 - \theta) \delta_x^2 \mathcal{U}_i^n + (\mathcal{V}_i^n)^p, \\ (1 - x_i) \frac{\mathcal{V}_i^{n+1} - \mathcal{V}_i^n}{k_n} &= \theta \delta_x^2 \mathcal{V}_i^{n+1} + (1 - \theta) \delta_x^2 \mathcal{V}_i^n + (\mathcal{U}_i^n)^p, \end{aligned} \quad (2.26)$$

for $i = 1, \dots, N$, where

$$\delta_x^2 \mathcal{U}_i^n = \frac{\mathcal{U}_{i+1}^n - 2\mathcal{U}_i^n + \mathcal{U}_{i-1}^n}{h^2} \quad \text{and} \quad \delta_x^2 \mathcal{V}_i^n = \frac{\mathcal{V}_{i+1}^n - 2\mathcal{V}_i^n + \mathcal{V}_{i-1}^n}{h^2}.$$

With boundary conditions discretization: $\mathcal{U}_0^n = \mathcal{U}_{N+1}^n = \mathcal{V}_0^n = \mathcal{V}_{N+1}^n = 0$ and discrete initial conditions $\mathcal{U}_i^0 = u_0(x_i)$, $\mathcal{V}_i^0 = v_0(x_i)$; $i = 0, \dots, N + 1$.

Thus the θ -scheme formula takes the form

$$\begin{cases} \mathcal{A}\mathcal{U}^{n+1} = (\mathcal{A}\mathcal{A})\mathcal{U}^n + k_n F_{v,p}^n & n = 0, \dots, M - 1, \\ \mathcal{B}\mathcal{V}^{n+1} = (\mathcal{B}\mathcal{B})\mathcal{V}^n + k_n F_{u,p}^n & n = 0, \dots, M - 1, \end{cases} \quad (2.27)$$

where

$$\mathcal{U}^n = (\mathcal{U}_1^n, \dots, \mathcal{U}_N^n)^T, \quad \mathcal{V}^n = (\mathcal{V}_1^n, \dots, \mathcal{V}_N^n)^T,$$

with $\mathcal{U}^0 = (u_0(x_1), u_0(x_2), \dots, u_0(x_N))^T$ and $\mathcal{V}^0 = (v_0(x_1), v_0(x_2), \dots, v_0(x_N))^T$, where

$$\mathcal{A} = \begin{bmatrix} (x_1 + 2\theta\gamma_1) & -\theta\gamma_1 & 0 & \cdots & 0 \\ -\theta\gamma_2 & (x_2 + 2\theta\gamma_2) & -\theta\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\theta\gamma_{N-1} \\ 0 & \cdots & 0 & -\theta\gamma_N & (x_N + 2\theta\gamma_N) \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} (1 - x_1) + 2\theta\gamma_1 & -\theta\gamma_1 & 0 & \cdots & 0 \\ -\theta\gamma_2 & (1 - x_2) + 2\theta\gamma_2 & -\theta\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\theta\gamma_{N-1} \\ 0 & \cdots & 0 & -\theta\gamma_N & (1 - x_N) + 2\theta\gamma_N \end{bmatrix}$$

$$\mathcal{AA} = \begin{bmatrix} (x_1 - 2(1 - \theta)\gamma_1) & (1 - \theta)\gamma_1 & 0 & \cdots & 0 \\ (1 - \theta)\gamma_2 & (x_2 - 2(1 - \theta)\gamma_2) & (1 - \theta)\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (1 - \theta)\gamma_{N-1} \\ 0 & \cdots & 0 & (1 - \theta)\gamma_N & (x_N - 2(1 - \theta)\gamma_N) \end{bmatrix}$$

$$\mathcal{BB} = \begin{bmatrix} (1 - x_1) - 2(1 - \theta)\gamma_1 & (1 - \theta)\gamma_1 & 0 & \cdots & 0 \\ (1 - \theta)\gamma_2 & (1 - x_2) - 2(1 - \theta)\gamma_2 & (1 - \theta)\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (1 - \theta)\gamma_{N-1} \\ 0 & \cdots & 0 & (1 - \theta)\gamma_N & (1 - x_N) - 2(1 - \theta)\gamma_N \end{bmatrix}$$

where $\gamma_n = \frac{k_n}{h^2}$, $F_{v,p}^n = ((\mathcal{V}_1^n)^p, \dots, (\mathcal{V}_N^n)^p)^T$ and $F_{u,p}^n = ((\mathcal{U}_1^n)^p, \dots, (\mathcal{U}_N^n)^p)^T$.

In addition, in order to guarantee that the stability (convergence) condition of θ -scheme is satisfied, for $p = 2$ the following non-fixed time-step formula is considered

$$k_n = \min \left(\frac{h^2}{2}, \frac{h^\alpha}{\|\mathcal{U}_h^n\|_\infty^p}, \frac{h^\alpha}{\|\mathcal{V}_h^n\|_\infty^p} \right), \quad \alpha \geq 1.$$

2.4.2 Local truncation error estimation

Assume that, the exact solution $(u, v)(x, t)$ is smooth ($u, v \in C^{4,2}([0, 1] \times [0, T])$) and remains bounded for all $t < T_b$ (no blow up). So, we have the following Local Truncation error estimation.

Theorem 12. (Local truncation error of θ -scheme) Let $((\tau^u)_i^n, (\tau^v)_i^n)$ be the local truncation error of the θ -scheme formulas (2.27) at the mesh point (x_i, t_n) . Then, there exist $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \geq 0$, such that

$$|(\tau^u)_i^n| \leq \mathcal{C}_1 k + \mathcal{C}_2 h^2, \quad |(\tau^v)_i^n| \leq \mathcal{C}_3 k + \mathcal{C}_4 h^2, \quad (2.28)$$

where $k = \max_n |k_n|$.

Proof. Substitute the exact solution $(u, v)(x_i, t_n)$ into the numerical scheme, it result the local truncation error

$$(\tau^u)_i^n = x_i \frac{u_i^{n+1} - u_i^n}{k_n} - \theta \delta_x^2 u_i^{n+1} - (1 - \theta) \delta_x^2 u_i^n - (v_i^n)^p,$$

$$(\tau^v)_i^n = (1 - x_i) \frac{v_i^{n+1} - v_i^n}{k_n} - \theta \delta_x^2 v_i^{n+1} - (1 - \theta) \delta_x^2 v_i^n - (u_i^n)^p.$$

We assume the continuous solution $(u, v)(x, t)$ is smooth for $t < T_b$. Applying Taylor expansion to show that the exact solution verifies the scheme with a good order.

We first notice that

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{k_n} = \frac{\partial u}{\partial t}(x_i, t_n) + \frac{k_n}{2} \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + O(k_n^2).$$

Then

$$\frac{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n))}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_i, t_n) + O(h^2).$$

Similarly

$$\begin{aligned} \frac{u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1}))}{h^2} &= \frac{\partial^2 u}{\partial x^2}(x_i, t_{n+1}) + O(h^2) \\ &= x_i \frac{\partial u}{\partial t}(x_i, t_{n+1}) - (v_i^{n+1})^p + O(h^2) \\ &= x_i \frac{\partial u}{\partial t}(x_i, t_n) + x_i k_n \frac{\partial^2 u}{\partial t^2}(x_i, t_n) \\ &\quad - (v_i^n)^p + O(k_n + h^2). \end{aligned}$$

Replacing all these terms in the scheme, we obtain the consistency error in (x_i, t_n)

$$\begin{aligned} (\tau^u)_i^n &= x_i \frac{\partial u}{\partial t}(x_i, t_n) + x_i \frac{k_n}{2} \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + x_i O(k_n^2) \\ &\quad - \theta \left[x_i \frac{\partial u}{\partial t}(x_j, t_n) + x_i k_n \frac{\partial^2 u}{\partial t^2}(x_i, t_n) - (v_i^n)^p + O(k_n + h^2) \right] \\ &\quad - (1 - \theta) \left[\frac{\partial^2 u}{\partial x^2}(x_i, t_n) - O(h^2) \right] - (v_i^n)^p \\ &= (1 - \theta) \left[x \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - v^p \right] (x_i, t_n) \\ &\quad + k_n \left(\frac{1}{2} - \theta \right) \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + O(k_n + h^2) \\ &= k_n \left(\frac{1}{2} - \theta \right) \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + O(k_n + h^2). \end{aligned}$$

In the same way we can prove that

$$(\tau^v)_i^n = k_n \left(\frac{1}{2} - \theta \right) \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + O(k_n + h^2).$$

This concludes the proof of the theorem. □

2.4.3 Stability and convergence (for pre-blow-up phase)

Assume a perturbation $e_i^n = u_i^n - \mathcal{U}_i^n$. Using discrete energy method or maximum norm estimates, we can show

$$\|e^{n+1}\| \leq (1 + C\Delta t)\|e^n\| + O(\Delta t^2 + h^2).$$

If $\theta \geq 1/2$, the scheme is unconditionally stable in the linear case and reasonably stable in nonlinear case up to T_b .

In order to investigate the stability for the θ -scheme formulas, we need to recall the following definition.

Definition 11. (*The stability condition*). Set $(e_u)_i^n = u_i^n - \mathcal{U}_i^n$, $(e_v)_i^n = v_i^n - \mathcal{V}_i^n$, $i = 1, \dots, N$, where (u_i^n, v_i^n) and $(\mathcal{U}_i^n, \mathcal{V}_i^n)$ are the exact and numerical solutions of the coupled system (2.1) of two PDEs, respectively, at the mesh point (x_i, t_n) . For any arbitrary initial rounding error $((e_u)_i^0, (e_v)_i^0)$, $i = 1, \dots, N$, we say that the numerical solution (the difference approximation) is stable, if there exists a positive number β independent on the space-step (h) and time-step (k), such that

$$\|\mathcal{E}_u^n\| \leq \beta \max \{ \|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\| \}, \|\mathcal{E}_v^n\| \leq \beta \max \{ \|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\| \},$$

where $\|\mathcal{E}_u^n\| = \max_{1 \leq i \leq N} |(e_u)_i^n|$ and $\|\mathcal{E}_v^n\| = \max_{1 \leq i \leq N} |(e_v)_i^n|$, $n = 1, 2, \dots$

Since the blow-up occurred near $x = 0$ and $x = 1$, we will prove convergence and stability on a interval $]\varepsilon_1, 1 - \varepsilon_2[$ such that $\varepsilon_1, \varepsilon_2 > 0$.

Theorem 13. The θ -scheme (2.26) is stable, if $1 \geq 2(1 - \theta)\frac{k}{h^2}$, where $k = \max_{1 \leq i \leq N} k_n$.

Proof. We begin with the first equation of the discrete coupled system (2.26). We substitute the exact and approximate solutions (u_i^n, v_i^n) and $(\mathcal{U}_i^n, \mathcal{V}_i^n)$, we obtain

$$\begin{aligned} x_i \frac{(e_u)_i^{n+1} - (e_u)_i^n}{k_n} = & \theta \frac{(e_u)_{i+1}^{n+1} - 2(e_u)_i^{n+1} + (e_u)_{i-1}^{n+1}}{h^2} \\ & + (1 - \theta) \frac{(e_u)_{i+1}^n - 2(e_u)_i^n + (e_u)_{i-1}^n}{h^2} + (v_i^n)^p - (\mathcal{V}_i^n)^p, \end{aligned} \quad (2.29)$$

which implies

$$\begin{aligned} & (x_i + 2\theta\gamma_n)(e_u)_i^{n+1} - \theta\gamma_n(e_u)_{i+1}^{n+1} - \theta\gamma_n(e_u)_{i-1}^{n+1} \\ = & (x_i - 2(1 - \theta)\gamma_n)(e_u)_i^n + (1 - \theta)\gamma_n(e_u)_{i+1}^n + (1 - \theta)\gamma_n(e_u)_{i-1}^n \\ & + k_n((v_i^n)^p - (\mathcal{V}_i^n)^p), \end{aligned} \quad (2.30)$$

then, we have

$$\begin{aligned}
 (x_i + 2\theta\gamma_n)(e_u)_i^{n+1} &= \theta\gamma_n(e_u)_{i+1}^{n+1} + \theta\gamma_n(e_u)_{i-1}^{n+1} \\
 &\quad + (x_i - 2(1-\theta)\gamma_n)(e_u)_i^n \\
 &\quad + (1-\theta)\gamma_n(e_u)_{i+1}^n + (1-\theta)\gamma_n(e_u)_{i-1}^n \\
 &\quad + k_n((v_i^n)^p - (\mathcal{V}_i^n)^p),
 \end{aligned} \tag{2.31}$$

which gives

$$\begin{aligned}
 (x_i + 2\theta\gamma_n) |(e_u)_i^{n+1}| &\leq \theta\gamma_n |(e_u)_{i+1}^{n+1}| + \theta\gamma_n |(e_u)_{i-1}^{n+1}| \\
 &\quad + (x_i - 2(1-\theta)\gamma_n) |(e_u)_i^n| \\
 &\quad + (1-\theta)\gamma_n |(e_u)_{i+1}^n| + (1-\theta)\gamma_n |(e_u)_{i-1}^n| \\
 &\quad + k_n |(v_i^n)^p - (\mathcal{V}_i^n)^p|,
 \end{aligned} \tag{2.32}$$

Assume that $|(e_u)_{j_0}^{n+1}| = \max_{1 \leq j \leq N} |(e_u)_j^{n+1}|$ and $|(e_u)_{i_0}^n| = \max_{1 \leq i \leq N} |(e_u)_i^n|$, so we obtain

$$\begin{aligned}
 (x_i + 2\theta\gamma_n) |(e_u)_{j_0}^{n+1}| &\leq \theta\gamma_n |(e_u)_{j_0}^{n+1}| + \theta\gamma_n |(e_u)_{j_0}^{n+1}| \\
 &\quad + (x_i - 2(1-\theta)\gamma_n) |(e_u)_{i_0}^n| \\
 &\quad + (1-\theta)\gamma_n |(e_u)_{i_0}^n| + (1-\theta)\gamma_n |(e_u)_{i_0}^n| \\
 &\quad + k_n |(v_i^n)^p - (\mathcal{V}_i^n)^p|.
 \end{aligned} \tag{2.33}$$

So, by using CFL condition $(2(1-\theta)\gamma_n \leq 1)$ and lipshitz condition, we obtain

$$\begin{aligned}
 x_i |(e_u)_{j_0}^{n+1}| &\leq x_i |(e_u)_{i_0}^n| + |(v_i^n)^p - (\mathcal{V}_i^n)^p| \\
 &\leq x_i |(e_u)_{i_0}^n| + kl_1 |v_i^n - \mathcal{V}_i^n|,
 \end{aligned} \tag{2.34}$$

then,

$$\|\mathcal{E}_u^{n+1}\| \leq \|\mathcal{E}_u^n\| + \frac{kl_1}{x_i} \|\mathcal{E}_v^n\| \leq \|\mathcal{E}_u^n\| + \frac{kl_1}{\varepsilon_1} \|\mathcal{E}_v^n\|, \tag{2.35}$$

finally,

$$\begin{aligned}
 \|\mathcal{E}_u^{n+1}\| &\leq \left(1 + \frac{kl_1}{\varepsilon_1}\right) \max(\|\mathcal{E}_u^n\|, \|\mathcal{E}_v^n\|) \leq \dots \leq \left(1 + \frac{kl_1}{\varepsilon_1}\right)^{n+1} \max(\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|) \\
 &\leq \left(1 + \frac{Tl_1}{M\varepsilon_1}\right)^M \max(\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|). \\
 &\leq C \max(\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|).
 \end{aligned} \tag{2.36}$$

$$(\lim_{M \rightarrow \infty} \left(1 + \frac{Tl_1}{M\varepsilon_1}\right)^M = \exp(\frac{Tl_1}{\varepsilon_1})).$$

In the same way for the second equation of (2.26), we obtain

$$\begin{aligned} \|\mathcal{E}_v^{n+1}\| &\leq \left(1 + \frac{kl_2}{\varepsilon_1}\right) \max(\|\mathcal{E}_u^n\|, \|\mathcal{E}_v^n\|) \leq \dots \leq \left(1 + \frac{kl_2}{\varepsilon_1}\right)^{n+1} \max(\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|). \\ &\leq C \max(\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|). \end{aligned} \quad (2.37)$$

where l_1 and l_2 are the constants of lipshitz for the functions u^p and v^p respectively. $\square$

Theorem 14. *The θ -scheme formula (2.26) is convergent, if $1 \geq 2(1 - \theta)\gamma_n$, where $k = \max_{1 \leq i \leq N} k_n$.*

Proof. The same steps can be followed for the stability proof, taking into account what needs to be changed. $\square$

Remark 7. *In the last chapter, we present the algorithm used, together with the most significant numerical results concerning the blow-up of solutions at the points discussed in this chapter, along with additional numerical findings.*

Chapter 3

Theoretical and numerical studies of the blow-up of a degenerate nonlinear reaction–diffusion problem with source terms

In this chapter, we focus on the phenomenon of blow-up of solutions for semilinear and degenerate (time-derivative) parabolic equation systems with additional source terms. We show that, under suitable conditions on the initial data and on the growth of the source term, the solution exhibits finite-time blow-up.

This study constitutes a first step toward determining the set of blow-up points in the presence of external forcing.

3.1 Position of problem

Theoretical studies on blow-up for nonlinear parabolic equations with different boundary conditions have been extensively investigated. For instance, Boni [3] studied blow-up and asymptotic behavior for nonlinear parabolic equations with nonlinear boundary conditions. Chipot et al. [12] analyzed stationary solutions, blow-up, and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions. Du [14] investigated blow-up for a degenerate reaction–diffusion system with nonlinear localized sources. Floater [15] studied blow-up at the boundary for degenerate semilinear parabolic equations. Galaktionov and Vázquez [21] provided a comprehensive study on blow-up problems in nonlinear parabolic equations. Hamadneh et al. [22] analyzed finite-time blow-up problems for nonlinear fractional reaction-diffusion equations. Li et al. [33] studied global

existence and blow-up for degenerate and singular parabolic systems with localized sources. Momani et al. [34] examined weak solutions, blow-up, and numerical studies for nonlinear reaction-diffusion problems with integral conditions. Niss and Belyacine [35] studied blow-up for semilinear degenerate parabolic systems. Quittner and Souplet [37] provided a general theory of superlinear parabolic problems, including blow-up and steady states. Samarskii et al. [40] investigated blow-up in quasilinear parabolic equations. Shi et al. [41] analyzed blow-up for degenerate and singular parabolic systems with nonlocal sources and absorptions. Zhou et al. [47] studied blow-up for degenerate and singular parabolic systems with nonlocal sources.

Moreover, recent developments in functional analysis, fractional calculus, and wavelet techniques have been applied to study related problems in parabolic and reaction-diffusion equations. For example, Abia et al. [1] and Cho et al. [11] investigated Herz spaces in the context of degenerate parabolic systems. Cho and Lu [7] applied fractional-wavelet analysis to study blow-up solutions. These methods, including fractional calculus of the Lerch zeta function, provide useful tools for analyzing degeneracy, source terms, and blow-up behavior.

Since the theoretical estimation of the blow-up time is generally not feasible in most situations, many researchers have focused on computing the blow-up time numerically. For example, Abia et al. [1] studied blow-up phenomena for semidiscretizations of reaction-diffusion equations. Cho and Lu [7] analyzed numerical solutions for a parabolic system with blow-up. Cho et al. [11] proposed finite difference schemes for an axisymmetric nonlinear heat equation exhibiting blow-up. Khalil et al. [25] developed numerical finite difference approximations for a coupled parabolic system with blow-up. Polyanin and Shingareva [36] discussed non-local transformations and numerical methods for nonlinear blow-up problems. Rasheed et al. [38, 39] investigated numerical blow-up solutions for semilinear parabolic and heat equations. These studies also analyzed the influence of source terms on numerical blow-up.

The purpose of this work is to develop a numerical method for computing blow-up solutions of a one-dimensional degenerate semilinear parabolic system, including possible source terms. We consider the following initial-boundary value problem

$$\begin{cases} x u_t(x, t) = u_{xx}(x, t) + v^p(x, t) + g_1(x, t), & x \in]0, 1[, t > 0, \\ (1 - x) v_t(x, t) = v_{xx}(x, t) + u^p(x, t) + g_2(x, t), & x \in]0, 1[, t > 0, \\ u(x, t) = v(x, t) = 0, & x = \{0, 1\}, t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in [0, 1], \end{cases} \quad (3.1)$$

where $1 < p \leq 2$, and the initial conditions u_0 and v_0 belong to a suitable functional space, e.g., a weighted Sobolev space [1, 11, 15]. The source terms g_1 and g_2 are assumed to be

sufficiently smooth, i.e., $g_1, g_2 \in C^1(\bar{\Omega} \times [0, T])$, so that the theoretical analysis is valid. The numerical solution u (resp. v) is said to blow up in finite time T .

3.2 Study of the continuous problem

In this section, we present results obtained by Niss and Belyacine [35] on the blow-up of solutions of the continuous problem (3.1) in finite time. We assume the existence of a classical positive solution (u, v) of (3.1) on $[0, 1] \times [0, T[$, i.e., $u, v \in C([0, 1] \times [0, T]) \cap C^{2,1}([0, 1[\times]0, T])$.

The eigenfunction technique (Fourier coefficient method) originally introduced by Kaplan [24] can be applied to system (3.1) by considering the corresponding spectral problem

$$\begin{cases} \phi''(x) + \lambda x(1-x)\phi(x) = 0, & x \in]0, 1[, \\ \phi(0) = \phi(1) = 0, \\ \phi(x) > 0, & x \in]0, 1[. \end{cases} \quad (3.2)$$

There exists a unique positive real eigenvalue $\lambda > 0$ associated with an eigenfunction $\phi \in C([0, 1]) \cap C^2(]0, 1[)$, which is a solution of problem (3.2) on $(0, 1)$, see [13].

Let $(u, v)(x, t)$ be a solution to (3.1) on $[0, 1] \times [0, T[$. We set

$$\mathcal{U}(t) = \int_0^1 x \phi(x) u(x, t) dx, \quad \mathcal{V}(t) = \int_0^1 (1-x) \phi(x) v(x, t) dx, \quad t \in [0, T),$$

$$\mathcal{W}(t) = \mathcal{U}(t) + \mathcal{V}(t).$$

Lemma 10. [35] *Let $p > 1$ and assume that the source terms $g_1(x, t), g_2(x, t) \in C^1(\bar{\Omega} \times [0, T])$ are nonnegative. Then there exists a constant $C > 0$ such that*

$$\frac{d\mathcal{W}(t)}{dt} \geq -\lambda \mathcal{W}(t) + \frac{C}{2^{p-1}} \mathcal{W}^p(t) + \int_0^1 \phi(x) g_1(x, t) dx + \int_0^1 \phi(x) g_2(x, t) dx, \quad \forall t \in]0, T[.$$

Definition 12. *The solution (u, v) of the continuous problem (3.1) is said to blow up in finite time if there exists a finite time $T_b > 0$ such that*

$$\|u(\cdot, t)\|_\infty < \infty \quad \text{and} \quad \|v(\cdot, t)\|_\infty < \infty, \quad \forall t \in [0, T_b[,$$

but

$$\lim_{t \rightarrow T_b} \|u(\cdot, t)\|_\infty + \lim_{t \rightarrow T_b} \|v(\cdot, t)\|_\infty = \infty,$$

where

$$\|u(\cdot, t)\|_\infty = \sup_{0 \leq x \leq 1} |u(x, t)|, \quad \|v(\cdot, t)\|_\infty = \sup_{0 \leq x \leq 1} |v(x, t)|.$$

The time T_b is called the blow-up time of the solution (u, v) .

Definition 13. A point $x \in]0, 1[$ is called a blow-up point of the problem (3.1) if there exists a sequence (x_m, t_m) such that

$$t_m \rightarrow T_b, \quad x_m \rightarrow x, \quad \text{and} \quad u(x_m, t_m) \rightarrow \infty, \quad v(x_m, t_m) \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

An immediate consequence is a sufficient criterion on the initial data for blow-up in finite time.

Theorem 15. [35] Let $p > 1$, C be the constant defined in Lemma 10, and assume the source terms $g_1, g_2 \geq 0$. If the initial data u_0, v_0 satisfy

$$\int_0^1 x \phi(x) u_0(x) dx + \int_0^1 (1-x) \phi(x) v_0(x) dx > 2 \left(\frac{\lambda}{C} \right)^{\frac{1}{p-1}},$$

then the solution (u, v) of system (3.1) blows up in finite time T_b .

3.3 Study of the semi-discrete problem

We study an approximation of system (3.1) using the online method (OLM) to discretize in space. Let $h = 1/(N+1)$ be the spatial step with N a positive integer and $x_i = ih$, $i = 0, \dots, N+1$, in $[0, 1]$.

We use the finite difference method to approximate the derivatives, obtaining a second-order centered scheme for the second derivative of u and v . The resulting system of ordinary differential equations with source terms reads:

$$\begin{cases} \frac{du_i}{dt}(t) = \frac{1}{x_i h^2} (u_{i+1} - 2u_i + u_{i-1}) + \frac{1}{x_i} v_i^p + \frac{1}{x_i} g_1(x_i, t), & i = 1, \dots, N, \quad t > 0, \\ \frac{dv_i}{dt}(t) = \frac{1}{(1-x_i) h^2} (v_{i+1} - 2v_i + v_{i-1}) + \frac{1}{1-x_i} u_i^p + \frac{1}{1-x_i} g_2(x_i, t), & i = 1, \dots, N, \quad t > 0, \\ u_0(t) = u_{N+1}(t) = 0, \quad v_0(t) = v_{N+1}(t) = 0, & t > 0, \\ u_i(0) = u_0(x_i) \geq 0, \quad v_i(0) = v_0(x_i) \geq 0, & i = 1, \dots, N. \end{cases} \quad (3.3)$$

Let $\mathcal{U}_h(t) = (u_1, \dots, u_N)^T$ and $\mathcal{V}_h(t) = (v_1, \dots, v_N)^T$. Then system (3.3) can be written in matrix form as

$$\begin{cases} \frac{d\mathcal{U}_h}{dt}(t) = A_h \mathcal{U}_h + \mathcal{F}_1(\mathcal{V}_h) + \mathcal{G}_1(t), & t > 0, \\ \frac{d\mathcal{V}_h}{dt}(t) = B_h \mathcal{V}_h + \mathcal{F}_2(\mathcal{U}_h) + \mathcal{G}_2(t), & t > 0, \end{cases} \quad (3.4)$$

where

$$\mathcal{F}_1 = \left(\frac{v_1^p}{h}, \frac{v_2^p}{2h}, \dots, \frac{v_N^p}{Nh} \right)^T, \quad \mathcal{F}_2 = \left(\frac{u_1^p}{Nh}, \frac{u_2^p}{(N-1)h}, \dots, \frac{u_N^p}{h} \right)^T,$$

$$\mathcal{G}_1(t) = \left(\frac{g_1(x_1, t)}{x_1}, \dots, \frac{g_1(x_N, t)}{x_N} \right)^T, \quad \mathcal{G}_2(t) = \left(\frac{g_2(x_1, t)}{1-x_1}, \dots, \frac{g_2(x_N, t)}{1-x_N} \right)^T.$$

The matrices A_h and B_h represent the discretized second derivatives with degeneracy factors:

$$A_h = \frac{1}{h^3} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1/2 & -1 & 1/2 & \ddots & \vdots \\ 0 & 1/3 & -2/3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1/(N-1) \\ 0 & \cdots & 0 & 1/N & -2/N \end{bmatrix},$$

$$B_h = \frac{1}{h^3} \begin{bmatrix} -2/N & 1/N & 0 & \cdots & 0 \\ 1/(N-1) & -2/(N-1) & 1/(N-1) & \ddots & \vdots \\ 0 & 1/(N-2) & -2/(N-2) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1/2 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

3.3.1 Properties of the matrices $\mathcal{A}_h$ and $\mathcal{B}_h$

As in [35], we decompose the matrix $\mathcal{A}_h$ as

$$\mathcal{A}_h = \mathcal{A}_h^s + \mathcal{T}_h^1,$$

where $\mathcal{A}_h^s$ is a symmetric tridiagonal matrix and $\mathcal{T}_h^1$ is a positive upper triangular matrix

$$\mathcal{A}_h^s = \frac{1}{h^3} \begin{bmatrix} -2 & 1/2 & 0 & \cdots & 0 \\ 1/2 & -1 & 1/3 & \ddots & \vdots \\ 0 & 1/3 & -2/3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1/N \\ 0 & \cdots & 0 & 1/N & -2/N \end{bmatrix}, \quad \mathcal{T}_h^1 = \frac{1}{h^3} \begin{bmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 0 & 0 & 1/6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1/(N(N-1)) \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, the matrix $\mathcal{B}_h$ is decomposed as

$$\mathcal{B}_h = \mathcal{B}_h^s + \mathcal{T}_h^2,$$

where $\mathcal{B}_h^s$ is symmetric tridiagonal and $\mathcal{T}_h^2$ is a positive lower triangular matrix

$$\mathcal{B}_h^s = \frac{1}{h^3} \begin{bmatrix} -2/N & 1/N & 0 & \cdots & 0 \\ 1/N & -2/(N-1) & 1/(N-1) & \ddots & \vdots \\ 0 & 1/(N-1) & -2/(N-2) & \ddots & 1/2 \\ 0 & \cdots & 0 & 1/2 & -2 \end{bmatrix}, \quad \mathcal{T}_h^2 = \frac{1}{h^3} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1/(N(N-1)) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1/2 & 0 \end{bmatrix}.$$

Lemma 11. [35] *The matrix $\mathcal{A}_h^s$ has N distinct negative real eigenvalues*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N,$$

where λ_1 is the largest eigenvalue.

Lemma 12. [35] *The components of the eigenvector associated with λ_1 are strictly positive.*

Remark 8. Let $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ denote the eigenvector associated with λ_1 , normalized such that $\sum_{i=1}^N \phi_i = 1$.

Remark 9. The vector $\omega = (\phi_N, \phi_{N-1}, \dots, \phi_1)^T$ is an eigenvector of the matrix $\mathcal{B}_h^s$ associated with the eigenvalue λ_1 .

3.3.2 The blow-up of solution

To show finite-time blow-up of the semidiscrete system solution (3.4), we use the Kaplan method [35]. Define

$$\begin{cases} \varphi_h(t) = \langle \mathcal{U}_h(t), \phi \rangle, & t \in (0, T), \\ \psi_h(t) = \langle \mathcal{V}_h(t), \omega \rangle, & t \in (0, T), \end{cases} \quad (3.5)$$

where T is the maximal existence time of the solution $(\mathcal{U}_h, \mathcal{V}_h)$.

Let

$$\Phi_h(t) = \varphi_h(t) + \psi_h(t), \quad \Phi_h(0) = \varphi_h(0) + \psi_h(0).$$

Theorem 16. [35] *Let $p > 1$, $\mu = -\lambda_1$, and*

$$c = \min_{1 \leq i \leq N} \left(\frac{\phi_i}{ih\phi_{N-i+1}} \right) = \min_{1 \leq i \leq N} \left(\frac{\phi_{N-i+1}}{(N-i+1)h\phi_i} \right).$$

Assume the semidiscrete system includes nonnegative source terms $\mathcal{G}_1(t) \geq 0$ and $\mathcal{G}_2(t) \geq 0$.

If

$$\Phi_h(0) > 2c^{\frac{1}{1-p}} \mu^{\frac{1}{p-1}},$$

then the solution blows up in finite time T , and a rigorous lower bound for T can be obtained analogously to the source-free case.

3.3.3 Properties of the solution to the semi-discrete problem

Define

$$\mathcal{W}_h(t) = (\omega_1(t), \dots, \omega_N(t), \omega_{N+1}(t), \dots, \omega_{2N}(t))^T,$$

where

$$\omega_j(t) = \begin{cases} u_i(t), & 1 \leq i \leq N, \\ v_{i-N}(t), & N+1 \leq i \leq 2N, \end{cases}$$

and $u_i(t), v_i(t)$ satisfy the nonlinear ODE system

$$\frac{d\omega_i(t)}{dt} - \delta_h^2(\omega_i(t)) = F_i(\omega(t)), \quad 1 \leq i \leq 2N, \quad (3.6)$$

with

$$\delta_h^2(\omega_i(t)) = \begin{cases} \frac{1}{ih^3}(u_{i+1} - 2u_i + u_{i-1}), & 1 \leq i \leq N, \\ \frac{1}{(2N+1-i)h^3}(v_{i+1-N} - 2v_{i-N} + v_{i-1-N}), & N+1 \leq i \leq 2N, \end{cases}$$

and

$$F_i(\omega(t)) = \begin{cases} \frac{\omega_{i+N}^p}{ih} + g_1(x_i, t), & 1 \leq i \leq N, \\ \frac{\omega_{i-N}^p}{(2N+1-i)h} + g_2(x_{i-N}, t), & N+1 \leq i \leq 2N, \end{cases}$$

where $g_1(x_i, t), g_2(x_{i-N}, t)$ are nonnegative discrete source terms.

The initial and boundary conditions are

$$\omega_i(0) = \begin{cases} u_0(x_i), & 1 \leq i \leq N, \\ v_0(x_{i-N}), & N+1 \leq i \leq 2N, \end{cases} \quad \omega_0(t) = \omega_{2N+1}(t) = 0.$$

Proposition 4. [2, 1] Let $\mathcal{W}_h(t) \in C^1([0, T[, \mathbb{R}^{2N})$ be a solution of (3.6) with nonnegative

initial data

$$\omega_i(0) \geq 0, \quad 1 \leq i \leq 2N.$$

Then, including the effect of nonnegative source terms,

$$\omega_i(t) \geq 0, \quad 1 \leq i \leq 2N, \quad \forall t \in [0, T[.$$

3.3.4 Convergence of the solution to the semi-discrete problem

In this subsection, we show that for each fixed interval $[0, T]$, where the continuous solution (u, v) is defined, the solution $(\mathcal{U}_h, \mathcal{V}_h)$ of the semi-discrete problem (3.3), including smooth source terms, converges to (u_h, v_h) as $h \rightarrow 0$.

Define the discrete errors

$$\begin{aligned} \tau_h(t) &= \mathcal{U}_h(t) - u_h(t), \\ \varrho_h(t) &= \mathcal{V}_h(t) - v_h(t), \end{aligned}$$

where

$$u_h(t) = (u(x_1, t), \dots, u(x_N, t))^T, \quad v_h(t) = (v(x_1, t), \dots, v(x_N, t))^T.$$

Lemma 13. *Let $z(x, t)$ and $y(x, t)$ be two positive functions, and $\mathcal{C}_1'', \mathcal{C}_2'' > 0$. Then, for all $1 \leq i \leq N$ and $t \in [0, T[$*

$$\begin{aligned} -\mathcal{C}_1'' (y(x_i, t) - |\varrho_h(t)|) &\leq \mathcal{C}_1'' |\varrho_h(t) - y(x_i, t)|, \\ -\mathcal{C}_2'' (z(x_i, t) - |\tau_h(t)|) &\leq \mathcal{C}_2'' |\tau_h(t) - z(x_i, t)|. \end{aligned}$$

Theorem 17. *Suppose that problem (3.1), with sufficiently smooth source terms $g_1(x, t)$ and $g_2(x, t) \in C([0, 1] \times [0, T])$, has a solution $(u, v) \in (C^{4,1}([0, 1] \times [0, T]))^2$, and the initial condition in (3.3) satisfies*

$$|\mathcal{W}_h(0) - \omega_h(0)|_\infty = O(1) \quad \text{as } h \rightarrow 0, \tag{3.7}$$

where

$$\omega_h(t) = (u(x_1, t), \dots, u(x_N, t), v(x_1, t), \dots, v(x_N, t))^T.$$

Then, for h sufficiently small, the semi-discrete problem (3.3), including source terms, admits a unique solution $\mathcal{W}_h \in C^1([0, T]; \mathbb{R}^{2N})$ such that

$$\max_{0 \leq t \leq T} |\mathcal{W}_h(t) - \omega_h(t)|_\infty = O\left(|\mathcal{W}_h(0) - \omega_h(0)|_\infty + h^2\right) \quad \text{as } h \rightarrow 0. \tag{3.8}$$

Remark 10. *The convergence rate $O(h^2)$ is preserved even with the inclusion of sufficiently smooth source terms g_1 and g_2 , as they only contribute to the local truncation error in the finite difference approximation.*

Proof of Theorem 10 – Part 1. Let $\alpha, \beta > 0$ be such that

$$\|u(\cdot, t)\|_\infty < \alpha, \quad \|v(\cdot, t)\|_\infty < \beta, \quad t \in [0, T]. \quad (3.9)$$

For each $h > 0$, the semi-discrete problem (3.3), including smooth source terms $g_1(x_i, t)$ and $g_2(x_i, t)$, admits a unique solution

$$(\mathcal{U}_h(t), \mathcal{V}_h(t)) \in C^1([0, T_b^h], \mathbb{R}^{2N}),$$

where T_b^h is the maximal existence time.

Define $t(h)$ as the largest time satisfying $t < \min\{T, T_b^h\}$ such that

$$|\mathcal{U}_h(t) - u_h(t)|_\infty < 1, \quad |\mathcal{V}_h(t) - v_h(t)|_\infty < 1, \quad t \in [0, t(h)]. \quad (3.10)$$

The existence of such $t(h) > 0$ for small h follows from (3.7).

By the triangle inequality,

$$|\mathcal{U}_h(t)|_\infty \leq \|u(\cdot, t)\|_\infty + |\mathcal{U}_h(t) - u_h(t)|_\infty, \quad |\mathcal{V}_h(t)|_\infty \leq \|v(\cdot, t)\|_\infty + |\mathcal{V}_h(t) - v_h(t)|_\infty. \quad (3.11)$$

Hence,

$$|\mathcal{U}_h(t)|_\infty \leq 1 + \alpha, \quad |\mathcal{V}_h(t)|_\infty \leq 1 + \beta, \quad t \in [0, t(h)]. \quad (3.12)$$

Using Taylor expansion and the mean-value theorem, the discrete error equations for $1 \leq i \leq N$ read

$$\begin{cases} ih \frac{d\tau_i}{dt}(t) - \delta^2 \tau_i(t) = p\eta^{p-1} \varrho_i(t) + g_1(x_i, t) + O(h^2), \\ (N+1-i)h \frac{d\varrho_i}{dt}(t) - \delta^2 \varrho_i(t) = p\xi^{p-1} \tau_i(t) + g_2(x_i, t) + O(h^2), \end{cases} \quad (3.13)$$

where δ^2 denotes the standard second-order centered difference approximation, and

$$\xi_h(t) = \lambda_t \mathcal{U}_h(t) + (1 - \lambda_t) u_h(t), \quad \eta_h(t) = \lambda_t \mathcal{V}_h(t) + (1 - \lambda_t) v_h(t), \quad \lambda_t \in (0, 1).$$

From (3.9) and (3.12), there exist constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{K} > 0$ such that

$$\begin{cases} ih \frac{d\tau_i}{dt}(t) - \delta^2 \tau_i(t) \leq \mathcal{C}_1 |\varrho_i(t)| + |g_1(x_i, t)| + \mathcal{K}h^2, \\ (N+1-i)h \frac{d\varrho_i}{dt}(t) - \delta^2 \varrho_i(t) \leq \mathcal{C}_2 |\tau_i(t)| + |g_2(x_i, t)| + \mathcal{K}h^2. \end{cases} \quad (3.14)$$

Here, the source terms contribute only to the local truncation error and do not affect the $O(h^2)$ convergence rate, provided they are bounded and sufficiently smooth.

Define comparison functions

$$\begin{aligned} z(x, t) &= \exp((L+1)t + \mathcal{C}x^2) (|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2), \\ y(x, t) &= \exp((L+1)t + \mathcal{C}x^2) (|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2), \end{aligned}$$

with constants $L, \mathcal{C}, \mathcal{Q} > 0$ chosen sufficiently large.

Discretizing in space at $x_i = ih$, we obtain

$$\begin{cases} x_i \frac{dz_i}{dt} - \delta^2 z_i = (x_i(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_i^2) z_i + O(h^2), \\ (1-x_i) \frac{dy_i}{dt} - \delta^2 y_i = ((1-x_i)(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_i^2) y_i + O(h^2), \\ z_i(0) = \exp(\mathcal{C}x_i^2) (|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2), \\ y_i(0) = \exp(\mathcal{C}x_i^2) (|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2). \end{cases} \quad (3.15)$$

By construction, z_i and y_i dominate the discrete errors τ_i and ϱ_i , and the discrete comparison principle yields

$$|\tau_i(t)| \leq z_i(t), \quad |\varrho_i(t)| \leq y_i(t), \quad t \in [0, t(h)], \quad 1 \leq i \leq N.$$

Finally, this gives the estimate

$$\max_{1 \leq i \leq N} (|\tau_i(t)|, |\varrho_i(t)|) = O(|\mathcal{W}_h(0) - \omega_h(0)|_\infty + h^2), \quad t \in [0, t(h)].$$

Using Lemma 13, we can replace $|\varrho_i|$ and $|\tau_i|$ in (3.14) by the differences between the approximations and the super-solutions, yielding

$$\begin{cases} ih \frac{de_i}{dt} - \delta^2 e_i + \mathcal{C}_1'' |s_i| \geq 0, \\ (N+1-i)h \frac{ds_i}{dt} - \delta^2 s_i + \mathcal{C}_2'' |e_i| \geq 0, \\ e_i(0) > 0, \quad s_i(0) > 0, \end{cases}$$

with $\mathcal{C}_1'' = \min\{\mathcal{C}_1, \mathcal{C}_1'\}$ and $\mathcal{C}_2'' = \min\{\mathcal{C}_2, \mathcal{C}_2'\}$.

Dividing by ih and $(N+1-i)h$, we obtain

$$\begin{cases} \frac{de_i}{dt} - \delta_h^2 e_i + \frac{\mathcal{C}_1''}{ih} |s_i| \geq 0, \\ \frac{ds_i}{dt} - \delta_h^2 s_i + \frac{\mathcal{C}_2''}{(N+1-i)h} |e_i| \geq 0, \\ e_i(0) > 0, \quad s_i(0) > 0, \end{cases}$$

where δ_h^2 is the finite-difference operator defined in Proposition 4.

By the discrete comparison principle, it follows that

$$e_i(t) \geq 0, \quad s_i(t) \geq 0, \quad \forall t \in [0, T], \quad 1 \leq i \leq N,$$

so that the error between the semi-discrete solution and the exact solution is bounded by the super-solutions z_i and y_i .

Define

$$\mathcal{S}_h(t) = (e_1(t), \dots, e_N(t), s_1(t), \dots, s_N(t)),$$

and

$$\mathcal{H}(|\mathcal{S}_i(t)|) = \begin{cases} \frac{\mathcal{C}_1''}{ih} |s_i(t)|, & 1 \leq i \leq N, \\ \frac{\mathcal{C}_2''}{(N+1-i)h} |e_i(t)|, & 1 \leq i \leq N. \end{cases}$$

Then the discrete differential inequality reads

$$\begin{cases} \frac{d\mathcal{S}_i(t)}{dt} - \delta_h^2 \mathcal{S}_i(t) + \mathcal{H}(|\mathcal{S}_i(t)|) \geq 0, \\ \mathcal{S}_i(0) > 0, \end{cases} \quad (3.16)$$

which ensures, via Proposition 3, that

$$z_i(t) \geq \tau_i(t), \quad y_i(t) \geq \varrho_i(t), \quad \forall t \in]0, t(h)[, \quad 1 \leq i \leq N, \quad (3.17)$$

and similarly,

$$z_i(t) > |\tau_i(t)|, \quad y_i(t) > |\varrho_i(t)|, \quad \forall t \in]0, t(h)[, \quad 1 \leq i \leq N. \quad (3.18)$$

Consequently, the discrete error satisfies

$$\begin{cases} |\mathcal{U}_h(t) - u_h(t)|_\infty \leq \exp((L+1)t + \mathcal{C})(|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2), \\ |\mathcal{V}_h(t) - v_h(t)|_\infty \leq \exp((L+1)t + \mathcal{C})(|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2), \end{cases}$$

for $t \in]0, t(h)[$, and hence

$$|\mathcal{W}_h(t) - \omega_h(t)|_\infty \leq \exp((L+1)t + \mathcal{C})(|\mathcal{W}_h(0) - \omega_h(0)|_\infty + \mathcal{Q}h^2).$$

Finally, to show that $t(h) = T$, assume by contradiction that $t(h) < T$. Then, by definition of $t(h)$,

$$1 = |\mathcal{W}_h(t(h)) - \omega_h(t(h))|_\infty \leq \exp((L+1)T + \mathcal{C})(|\mathcal{W}_h(0) - \omega_h(0)|_\infty + \mathcal{Q}h^2),$$

but

$$\lim_{h \rightarrow 0} \exp((L+1)T + \mathcal{C})(|\mathcal{W}_h(0) - \omega_h(0)|_\infty + \mathcal{Q}h^2) = 0,$$

a contradiction. Hence, $t(h) = T$, and the proof is complete. $\square$

3.4 Finite difference schemes

In this section, we derive the fully-discrete θ -scheme for problem (3.1) by approximating the time derivative using a linear combination of forward and backward finite differences. Let $\mathcal{U}_i^n$ and $\mathcal{V}_i^n$ denote the approximate values of $u(x_i, t_n)$ and $v(x_i, t_n)$ on the grid

$$x_i = ih, \quad i = 0, 1, \dots, N+1, \quad h = \frac{1}{N+1}, \quad t_{n+1} = t_n + k_n,$$

where h is the spatial step and k_n is the (possibly nonuniform) time step.

The time derivative is approximated using a θ -weighted difference, while the spatial second derivatives are discretized as

$$\delta_x^2 \mathcal{U}_i(t_n) = \theta \delta_x^2 \mathcal{U}_i(t_{n+1}) + (1 - \theta) \delta_x^2 \mathcal{U}_i(t_n), \quad \delta_x^2 \mathcal{V}_i(t_n) = \theta \delta_x^2 \mathcal{V}_i(t_{n+1}) + (1 - \theta) \delta_x^2 \mathcal{V}_i(t_n),$$

with

$$\delta_x^2 \mathcal{U}_i^n = \frac{\mathcal{U}_{i+1}^n - 2\mathcal{U}_i^n + \mathcal{U}_{i-1}^n}{h^2}, \quad \delta_x^2 \mathcal{V}_i^n = \frac{\mathcal{V}_{i+1}^n - 2\mathcal{V}_i^n + \mathcal{V}_{i-1}^n}{h^2}.$$

The nonlinear source terms

$$f_1(x_i, t_n) = (\mathcal{V}_i^n)^p, \quad f_2(x_i, t_n) = (\mathcal{U}_i^n)^p$$

are evaluated explicitly at the previous time step t_n , resulting in the fully-discrete θ -scheme:

$$\begin{aligned} x_i \frac{\mathcal{U}_i^{n+1} - \mathcal{U}_i^n}{k_n} &= \theta \delta_x^2 \mathcal{U}_i^{n+1} + (1 - \theta) \delta_x^2 \mathcal{U}_i^n + f_1(x_i, t_n) + g_1(x_i, t_n), \\ (1 - x_i) \frac{\mathcal{V}_i^{n+1} - \mathcal{V}_i^n}{k_n} &= \theta \delta_x^2 \mathcal{V}_i^{n+1} + (1 - \theta) \delta_x^2 \mathcal{V}_i^n + f_2(x_i, t_n) + g_2(x_i, t_n), \end{aligned} \quad i = 1, \dots, N. \quad (3.19)$$

The boundary conditions are discretized as

$$\mathcal{U}_0^n = \mathcal{U}_{N+1}^n = \mathcal{V}_0^n = \mathcal{V}_{N+1}^n = 0,$$

and the initial conditions as

$$\mathcal{U}_i^0 = u_0(x_i), \quad \mathcal{V}_i^0 = v_0(x_i), \quad i = 0, \dots, N + 1.$$

3.4.1 Matrix form

Defining $\gamma_n = \frac{k_n}{h^2}$, the scheme (3.19) can be written in matrix form as

$$\begin{cases} \mathcal{A} \mathcal{U}^{n+1} = \mathcal{A} \mathcal{U}^n + k_n F_{v,p}^n, \\ \mathcal{B} \mathcal{V}^{n+1} = \mathcal{B} \mathcal{V}^n + k_n F_{u,p}^n, \end{cases} \quad (3.20)$$

where

$$\mathcal{U}^n = (\mathcal{U}_1^n, \dots, \mathcal{U}_N^n)^T, \quad \mathcal{V}^n = (\mathcal{V}_1^n, \dots, \mathcal{V}_N^n)^T,$$

and the source term vectors

$$F_{v,p}^n = ((f_1 + g_1)(x_1, t_n), \dots, (f_1 + g_1)(x_N, t_n))^T, \quad F_{u,p}^n = ((f_2 + g_2)(x_1, t_n), \dots, (f_2 + g_2)(x_N, t_n))^T.$$

The matrices are tridiagonal and defined by

$$\mathcal{A} = \begin{bmatrix} x_1 + 2\theta\gamma_1 & -\theta\gamma_1 & 0 & \cdots & 0 \\ -\theta\gamma_2 & x_2 + 2\theta\gamma_2 & -\theta\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\theta\gamma_{N-1} \\ 0 & \cdots & 0 & -\theta\gamma_N & x_N + 2\theta\gamma_N \end{bmatrix}$$

$$\begin{aligned}
 \mathcal{AA} &= \begin{bmatrix} x_1 - 2(1-\theta)\gamma_1 & (1-\theta)\gamma_1 & 0 & \cdots & 0 \\ (1-\theta)\gamma_2 & x_2 - 2(1-\theta)\gamma_2 & (1-\theta)\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (1-\theta)\gamma_{N-1} \\ 0 & \cdots & 0 & (1-\theta)\gamma_N & x_N - 2(1-\theta)\gamma_N \end{bmatrix} \\
 \mathcal{B} &= \begin{bmatrix} 1 - x_1 + 2\theta\gamma_1 & -\theta\gamma_1 & 0 & \cdots & 0 \\ -\theta\gamma_2 & 1 - x_2 + 2\theta\gamma_2 & -\theta\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\theta\gamma_{N-1} \\ 0 & \cdots & 0 & -\theta\gamma_N & 1 - x_N + 2\theta\gamma_N \end{bmatrix} \\
 \mathcal{BB} &= \begin{bmatrix} 1 - x_1 - 2(1-\theta)\gamma_1 & (1-\theta)\gamma_1 & 0 & \cdots & 0 \\ (1-\theta)\gamma_2 & 1 - x_2 - 2(1-\theta)\gamma_2 & (1-\theta)\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (1-\theta)\gamma_{N-1} \\ 0 & \cdots & 0 & (1-\theta)\gamma_N & 1 - x_N - 2(1-\theta)\gamma_N \end{bmatrix}
 \end{aligned}$$

This formulation explicitly incorporates the source terms while preserving the θ -scheme structure, making it suitable for stability and convergence analysis.

3.4.2 Non-fixed time-step for stability

In order to guarantee that the stability (convergence) condition of the explicit scheme is satisfied, for $p = 2$ we consider the following non-uniform time step:

$$k_n = \min\left(\frac{h^2}{2}, \frac{h^\alpha}{\|\mathcal{U}_h^n\|_\infty^p}, \frac{h^\alpha}{\|\mathcal{V}_h^n\|_\infty^p}\right), \quad \alpha \geq 1,$$

which allows us to control the growth of the nonlinear source terms and preserves the stability of the scheme near blow-up.

3.4.3 Local truncation error estimation

Assume that the exact solution $(u, v)(x, t)$ is sufficiently smooth, namely $u, v \in C^{4,2}$, and remains bounded for all $t < T_b$ (i.e., before blow-up). Then the local truncation error of the θ -scheme can be estimated as follows.

Theorem 18 (Local truncation error of the θ -scheme). *Let $((\tau^u)_i^n, (\tau^v)_i^n)$ denote the lo-*

cal truncation errors of the θ -scheme at the grid point (x_i, t_n) . Then there exist positive constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ such that

$$|(\tau^u)_i^n| \leq \mathcal{C}_1 k + \mathcal{C}_2 h^2, \quad |(\tau^v)_i^n| \leq \mathcal{C}_3 k + \mathcal{C}_4 h^2, \quad k = \max_n k_n. \quad (3.21)$$

Proof. Substituting the exact solution $(u, v)(x_i, t_n)$ into the numerical scheme (3.19), the local truncation errors satisfy

$$\begin{aligned} (\tau^u)_i^n &= x_i \frac{u_i^{n+1} - u_i^n}{k_n} - \theta \delta_x^2 u_i^{n+1} - (1 - \theta) \delta_x^2 u_i^n - (f_1 + g_1)(x_i, t_n), \\ (\tau^v)_i^n &= (1 - x_i) \frac{v_i^{n+1} - v_i^n}{k_n} - \theta \delta_x^2 v_i^{n+1} - (1 - \theta) \delta_x^2 v_i^n - (f_2 + g_2)(x_i, t_n). \end{aligned}$$

Since u and v are smooth, Taylor expansions give

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{k_n} = \partial_t u(x_i, t_n) + \frac{k_n}{2} \partial_{tt} u(x_i, t_n) + O(k_n^2),$$

and

$$\delta_x^2 u(x_i, t_n) = \partial_{xx} u(x_i, t_n) + O(h^2), \quad \delta_x^2 u(x_i, t_{n+1}) = \partial_{xx} u(x_i, t_{n+1}) + O(h^2),$$

with analogous expansions for v .

Collecting the terms yields

$$\begin{aligned} (\tau^u)_i^n &= k_n \left(\frac{1}{2} - \theta \right) \partial_{tt} u(x_i, t_n) + O(k_n + h^2), \\ (\tau^v)_i^n &= k_n \left(\frac{1}{2} - \theta \right) \partial_{tt} v(x_i, t_n) + O(k_n + h^2). \end{aligned}$$

Hence, the local truncation errors are of order $O(k_n + h^2)$, which proves the stated estimates. □

3.4.4 Stability and convergence of the θ -scheme (pre-blow-up phase)

Let $e_i^n = u_i^n - \mathcal{U}_i^n$ denote the perturbation of the numerical solution. Using discrete energy or maximum-norm estimates, we have

$$\|e^{n+1}\| \leq (1 + C\Delta t) \|e^n\| + O(\Delta t^2 + h^2).$$

For $\theta \geq 1/2$, the scheme is unconditionally stable in the linear case and maintains reasonable stability for the nonlinear problem with source terms up to T^* . The non-fixed time-step formula further controls growth from the source terms.

To investigate the stability of the θ -scheme (3.19), we recall the following definition.

Definition 14 (Stability condition). *Set*

$$(e_u)_i^n = u_i^n - \mathcal{U}_i^n, \quad (e_v)_i^n = v_i^n - \mathcal{V}_i^n, \quad i = 1, \dots, N,$$

where (u_i^n, v_i^n) and $(\mathcal{U}_i^n, \mathcal{V}_i^n)$ are the exact and numerical solutions of the coupled system (3.1) at (x_i, t_n) .

For any initial rounding errors $(e_u)_i^0, (e_v)_i^0$, the numerical solution is said to be stable if there exists a positive constant β , independent of the space-step h and time-step k , such that

$$\|\mathcal{E}_u^n\| \leq \beta \max\{\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|\}, \quad \|\mathcal{E}_v^n\| \leq \beta \max\{\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|\}, \quad n = 1, 2, \dots,$$

where

$$\|\mathcal{E}_u^n\| = \max_{1 \leq i \leq N} |(e_u)_i^n|, \quad \|\mathcal{E}_v^n\| = \max_{1 \leq i \leq N} |(e_v)_i^n|.$$

Since blow-up occurs near $x = 0$ and $x = 1$, we prove convergence and stability in the interval $]\varepsilon_1, 1 - \varepsilon_2[$, with $\varepsilon_1, \varepsilon_2 > 0$.

Theorem 19 (Stability of the θ -scheme). *The θ -scheme (3.19) is stable if*

$$1 \geq 2(1 - \theta) \frac{k}{h^2},$$

where $k = \max_{1 \leq i \leq N} k_n$.

Proof. Starting from the first equation of (3.19) and subtracting the numerical solution from the exact one, we get

$$\begin{aligned} x_i \frac{(e_u)_i^{n+1} - (e_u)_i^n}{k_n} &= \theta \frac{(e_u)_{i+1}^{n+1} - 2(e_u)_i^{n+1} + (e_u)_{i-1}^{n+1}}{h^2} \\ &\quad + (1 - \theta) \frac{(e_u)_{i+1}^n - 2(e_u)_i^n + (e_u)_{i-1}^n}{h^2} \\ &\quad + (v_i^n)^p - (\mathcal{V}_i^n)^p. \end{aligned}$$

Rewriting, we have

$$\begin{aligned} (x_i + 2\theta\gamma_n)(e_u)_i^{n+1} &= \theta\gamma_n(e_u)_{i+1}^{n+1} + \theta\gamma_n(e_u)_{i-1}^{n+1} \\ &\quad + (x_i - 2(1 - \theta)\gamma_n)(e_u)_i^n + (1 - \theta)\gamma_n[(e_u)_{i+1}^n + (e_u)_{i-1}^n] \\ &\quad + k_n|(v_i^n)^p - (\mathcal{V}_i^n)^p|. \end{aligned}$$

Taking the maximum over i and using the Lipschitz property of v^p , we get

$$\|(e_u)^{n+1}\| \leq \|(e_u)^n\| + \frac{k_n l_1}{\varepsilon_1} \|(e_v)^n\|,$$

where l_1 is the Lipschitz constant of v^p and $\varepsilon_1 > 0$ is the lower bound of x_i in the interval $]\varepsilon_1, 1 - \varepsilon_2[$.

Iterating in n yields

$$\|(e_u)^{n+1}\| \leq \left(1 + \frac{k_n l_1}{\varepsilon_1}\right)^{n+1} \max\{\|(e_u)^0\|, \|(e_v)^0\|\}.$$

Similarly, for the second equation:

$$\|(e_v)^{n+1}\| \leq \left(1 + \frac{k_n l_2}{\varepsilon_1}\right)^{n+1} \max\{\|(e_u)^0\|, \|(e_v)^0\|\},$$

where l_2 is the Lipschitz constant of u^p .

Hence, the scheme is stable in the maximum norm for the pre-blow-up phase with source terms. $\square$

Theorem 20 (Convergence of the θ -scheme). *The θ -scheme formula (3.19) is convergent in the maximum norm if*

$$1 \geq 2(1 - \theta)\gamma_n, \quad \text{where } \gamma_n = \frac{k_n}{h^2} \text{ and } k = \max_{1 \leq i \leq N} k_n.$$

Proof. Let the exact solution at the mesh points be denoted by $u_i^n = u(x_i, t_n)$ and $v_i^n = v(x_i, t_n)$, and the numerical solution by $\mathcal{U}_i^n$ and $\mathcal{V}_i^n$. We define the global errors as

$$(e_u)_i^n = u_i^n - \mathcal{U}_i^n, \quad (e_v)_i^n = v_i^n - \mathcal{V}_i^n, \quad i = 1, \dots, N.$$

Substituting the exact solution into the θ -scheme (3.19) yields the local truncation errors $(\tau^u)_i^n$ and $(\tau^v)_i^n$. Consequently, the global errors satisfy the following perturbed discrete system

$$\begin{aligned} x_i \frac{(e_u)_i^{n+1} - (e_u)_i^n}{k_n} &= \theta \frac{(e_u)_{i+1}^{n+1} - 2(e_u)_i^{n+1} + (e_u)_{i-1}^{n+1}}{h^2} \\ &\quad + (1 - \theta) \frac{(e_u)_{i+1}^n - 2(e_u)_i^n + (e_u)_{i-1}^n}{h^2} \\ &\quad + (v_i^n)^p - (\mathcal{V}_i^n)^p - (\tau^u)_i^n, \end{aligned}$$

$$\begin{aligned}
 (1 - x_i) \frac{(e_v)_i^{n+1} - (e_v)_i^n}{k_n} &= \theta \frac{(e_v)_{i+1}^{n+1} - 2(e_v)_i^{n+1} + (e_v)_{i-1}^{n+1}}{h^2} \\
 &\quad + (1 - \theta) \frac{(e_v)_{i+1}^n - 2(e_v)_i^n + (e_v)_{i-1}^n}{h^2} \\
 &\quad + (u_i^n)^p - (\mathcal{U}_i^n)^p - (\tau^v)_i^n.
 \end{aligned}$$

Since the analysis is restricted to the pre-blow-up interval $]\varepsilon_1, 1 - \varepsilon_2[$ with $\varepsilon_1, \varepsilon_2 > 0$, the nonlinear source terms are locally Lipschitz continuous. Hence, there exist constants $l_1, l_2 > 0$ such that

$$|(v_i^n)^p - (\mathcal{V}_i^n)^p| \leq l_1 |(e_v)_i^n|, \quad |(u_i^n)^p - (\mathcal{U}_i^n)^p| \leq l_2 |(e_u)_i^n|.$$

Applying the maximum norm and proceeding as in the stability analysis, we obtain

$$\begin{aligned}
 \|(e_u)^{n+1}\| &\leq \|(e_u)^n\| + \frac{k_n l_1}{\varepsilon_1} \|(e_v)^n\| + k_n \|(\tau^u)^n\|, \\
 \|(e_v)^{n+1}\| &\leq \|(e_v)^n\| + \frac{k_n l_2}{\varepsilon_1} \|(e_u)^n\| + k_n \|(\tau^v)^n\|.
 \end{aligned}$$

Let

$$E^n = \max\{\|(e_u)^n\|, \|(e_v)^n\|\}.$$

Then the above inequalities yield

$$E^{n+1} \leq \left(1 + \frac{k_n \max(l_1, l_2)}{\varepsilon_1}\right) E^n + k_n \max\{\|(\tau^u)^n\|, \|(\tau^v)^n\|\}.$$

Using the local truncation error estimates $\|(\tau^u)^n\| = O(k_n + h^2)$ and $\|(\tau^v)^n\| = O(k_n + h^2)$, and iterating the above inequality for $nk_n \leq T$, we conclude that

$$E^n \leq \exp\left(\frac{T \max(l_1, l_2)}{\varepsilon_1}\right) O(k + h^2).$$

Therefore, under the stability condition $1 \geq 2(1 - \theta)\gamma_n$, the θ -scheme converges in the maximum norm with order $O(k + h^2)$ in the pre-blow-up phase. $\square$

Remark 11. *The following chapter is devoted to the presentation of the numerical results associated with the problem studied herein, along with a detailed description of the algorithm employed in the adopted numerical method.*

Chapter 4

Numerical results

Using a computer code executed in MATLAB, we calculate the solution of discretized system using the finite difference scheme. This should provide a graphical representation of the solution peaks near the blow-up points. This should constitute a first step in determining the set of blow-up points.

4.1 Results for problem (2.1)

In this section, we present a numerical experiment. All the numerical computations codes are written in Matlab (R2020a) software. Our goal is to show that the result presented in Theorem 8 can be observed when one performs numerical computations. First, we use θ -scheme for $\theta \geq \left(1 - \frac{h^2}{2k}\right)$ and $u_0(x) = \sin(\pi x), v_0(x) = \sin(\pi x)$. Figures 4.1 and 4.2 show the evolution of the numerical solution $\mathcal{U}^n$ (respectively $\mathcal{V}^n$) of (2.27) with $p = 2$ and $\theta = 0.91$, whose blow-up at point $x = 0$ (respectively $x = 1$).

We re-ran the numerical code for a different value of the initial condition, $u_0(x) = x^3(1 - x)^2, v_0(x) = (1 - x^3)\sin(\frac{\pi}{2}x)$, and plotted the results in Figure 4.7 and 4.4, which shows that the blow up of solution occurred at the same point as before.

Moreover, we include the following table to present the numerical results obtained from using the proposed schemes, with the mesh size: $N = 10, 20, 50, 100, 200, 300, 500$. Each table shows, the numerical blow-up times, and the central processing unit times (CPUTs) in seconds.

N	Blow-up times	CPU times (s)
10	0.0387315	0.070529
20	0.0149963	0.071728
50	0.0374250	0.072739
100	0.0430329	0.085657
200	0.0428022	0.175249
300	0.0408165	0.559521
500	0.0371345	2.784772

Table 4.1: Numerical blow-up times and the central processing unit times (CPUTs).

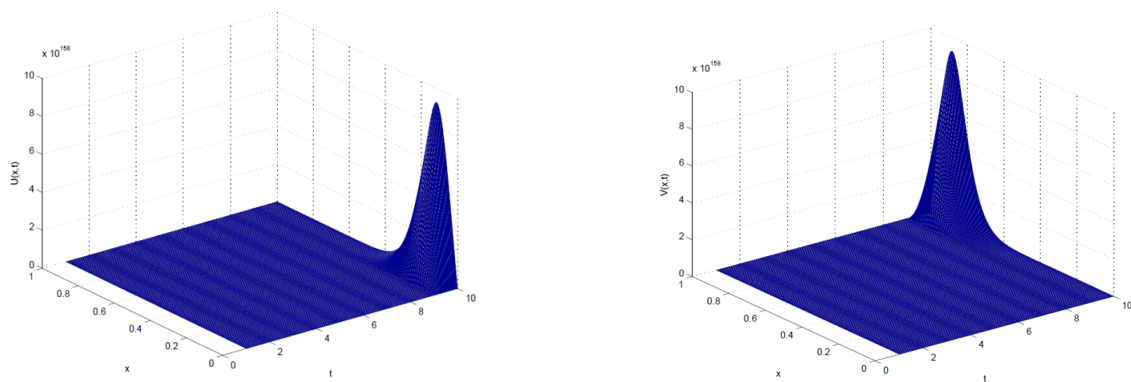


Figure 4.1: Numerical solution $(\mathcal{U}, \mathcal{V})(x_i, t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = v_0(x) = \sin(\pi x)$.

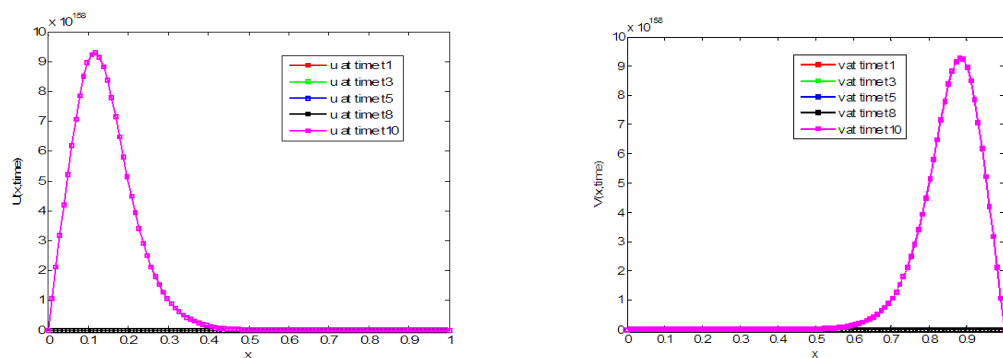


Figure 4.2: Numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = v_0(x) = \sin(\pi x)$.

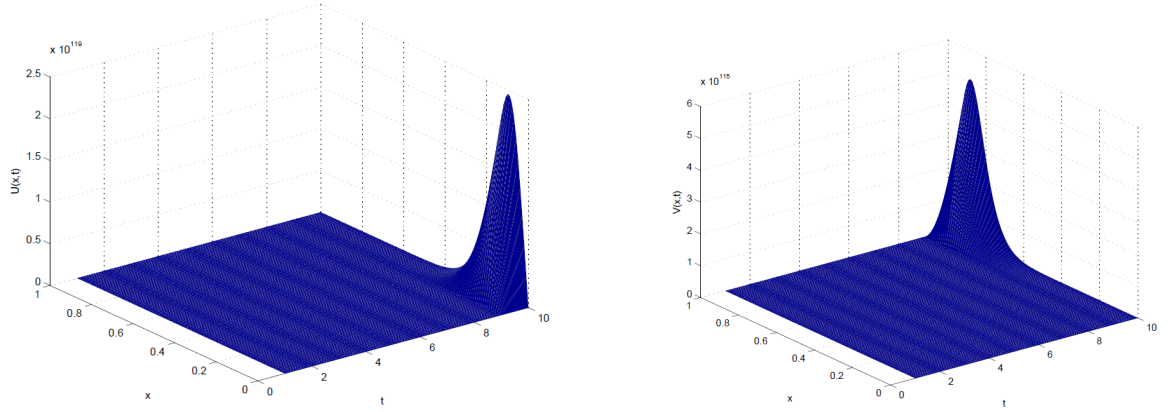


Figure 4.3: Numerical solution $(\mathcal{U}, \mathcal{V})(x_i, t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = x^3(1-x)^2$, $v_0(x) = (1-x^3)\sin(\frac{\pi}{2}x)$.

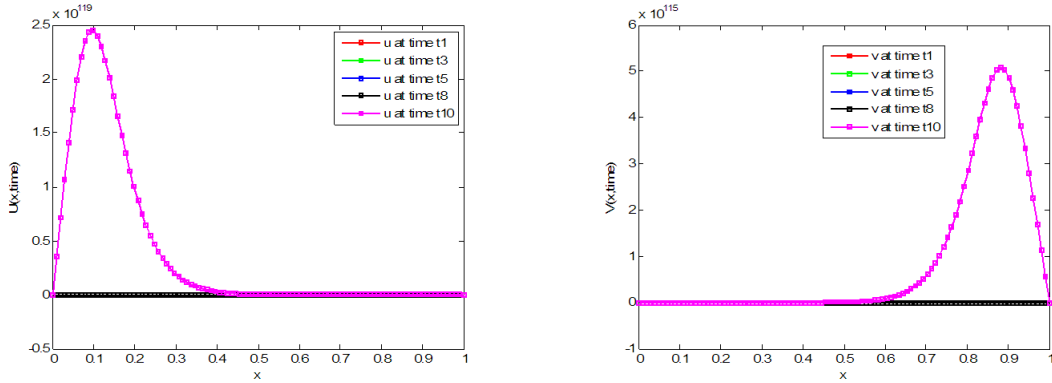


Figure 4.4: Numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $(\mathcal{U}_0)_i = (x_i^3(1-x_i)^2)$, $(\mathcal{V}_0)_i = (1-x_i^3)\sin(\frac{\pi}{2}x_i)$.

4.1.1 Point of Infinite Kinetics (PIK) or blow-up point

What does PIK usually mean?

Depending on the context, it can refer to one of the following interpretations:

1. Peak (peak value): The point or value at which the solution reaches its maximum

height before exploding

$$\text{PIK}_-u = \max_{x_i} u(x_i, t_n).$$

2. Point of Infinite Kinetics (or blow-up point): The point at which the solution becomes infinite

$$\lim_{t \rightarrow T^-} u(x_0, t) = +\infty.$$

3. Peak Instant: The time at which the irregular behavior (explosion) begins.

$$T = \sup \{t : \|u(., t)\| < +\infty\}.$$

In this context, the quantities $\text{PIK}_-\mathcal{U}_h$ and $\text{PIK}_-\mathcal{V}_h$ represent the peak values of the numerical solution at each time step. Therefore, they can be interpreted as discrete approximations of the L^∞ -norm of the solution, providing a reliable numerical indicator of blow-up.

The abrupt transition from a nearly constant state to a singular growth suggests that the blow-up occurs near a certain time T , and may also be spatially localized depending on the structure of the solution.

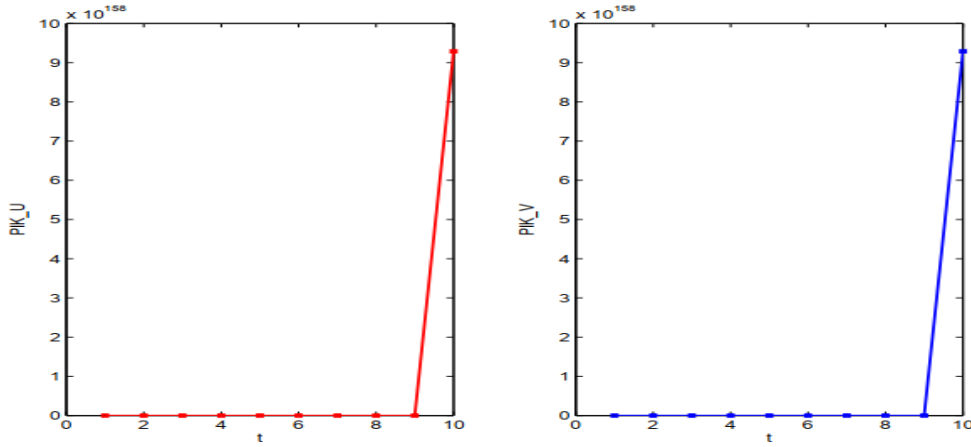


Figure 4.5: The PIK of numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ for $p = 2$, $\theta = 0.91$ and $u_0(x) = v_0(x) = \sin(\pi x)$.

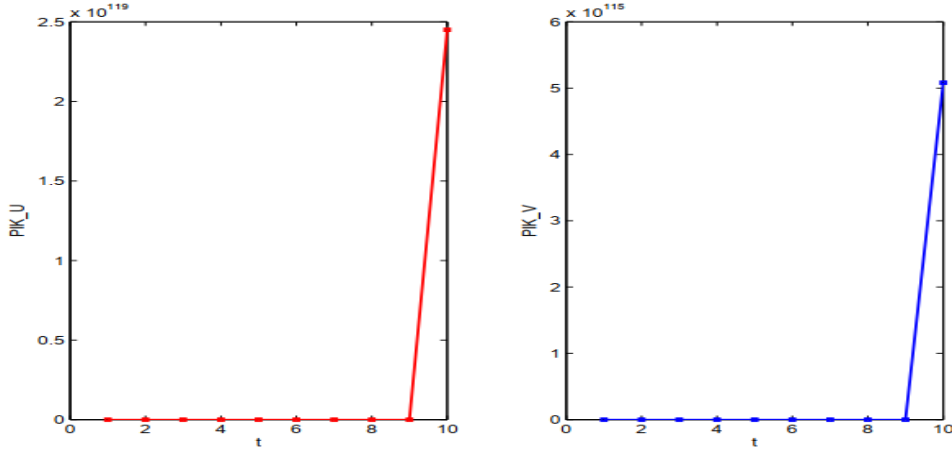


Figure 4.6: The PIK of numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ for $p = 2$, $\theta = 0.91$ and $u_0 = x^3(1-x)^2$, $v_0 = (1-x^3)\sin(\frac{\pi}{2}x)$.

Interpretation of the Figures (Blow-up Behavior): The two graphics in Figure 4.5 illustrate the temporal evolution of the quantities PIK_U and PIK_V as functions of time t . It is observed that both quantities remain close to zero for $t \in [t_1, t_9]$, indicating a stable regime or very slow dynamics.

However, as time approaches $t \approx t_{10}$, both PIK_U and PIK_V exhibit a sudden and extremely rapid growth, reaching values of the order of 10^{158} . This sharp increase clearly indicates the occurrence of a blow-up phenomenon.

From a mathematical point of view, this behavior suggests that the solution becomes unbounded in finite time. More precisely, there exists a finite time $T \approx t_{10}$ such that:

$$\lim_{t \rightarrow T^-} \|u(\cdot, t)\|_{L^\infty} = +\infty, \quad \lim_{t \rightarrow T^-} \|v(\cdot, t)\|_{L^\infty} = +\infty.$$

where $T \approx t_{10}$ denotes the blow-up time.

The quantities PIK_U and PIK_V can be interpreted as peak (maximum) values of the solutions at each time step. Their rapid divergence reflects the loss of boundedness and confirms the explosive behavior of the system.

Furthermore, the abrupt transition from a nearly constant state to a singular growth suggests that the blow-up is highly localized in time, and may also be spatially localized depending on the underlying model.

These observations are consistent with the typical behavior of solutions to nonlinear reaction-diffusion systems exhibiting finite-time singularities.

4.2 Results for problem (3.1)

In this section, we present numerical experiments illustrating the effectiveness of the proposed θ -scheme for the coupled degenerate semilinear parabolic system (3.1) in the presence of source terms. All computations were carried out using MATLAB (R2020a). The main objective is to validate the theoretical results of Theorem 15 and to observe numerically the finite-time blow-up behavior of the solutions resulting from the combined effects of nonlinear coupling, source terms, and degenerate diffusion.

4.2.1 Setup and initial conditions

We apply the θ -scheme with

$$\theta \geq \left(1 - \frac{h^2}{2k}\right)$$

to the coupled degenerate system with nonlinear interaction terms (u^p, v^p) and source functions $g_1(x, t)$ and $g_2(x, t)$.

The initial data are chosen as

$$u_0(x) = \sin(\pi x), \quad v_0(x) = \sin(\pi x), \quad p = 2,$$

and the source terms are taken as

$$g_1(x, t) = x(1 - x)e^{-t}, \quad g_2(x, t) = x(1 - x)e^{-t}.$$

The time step k_n is selected adaptively according to the stability criterion described in Section 4, ensuring that

$$1 \geq 2(1 - \theta)\gamma_n$$

at each time level.

4.2.2 Numerical blow-up times for different meshes

The following table reports the numerical blow-up times, maximum norms, and CPU times for different spatial resolutions N . Although the source terms contribute to the solution growth at early times, the blow-up mechanism remains dominated by the nonlinear coupling terms. In agreement with the theoretical analysis, blow-up occurs near $x = 0$ for u and near $x = 1$ for v , reflecting the degeneracy of the diffusion coefficients.

N	T_b^u	T_b^v	$\ \mathcal{U}^n\ _\infty$	$\ \mathcal{V}^n\ _\infty$	CPU time (s)
10	0.0387	0.0387	1.003	1.003	0.071
20	0.0150	0.0150	1.002	1.002	0.072
50	0.0374	0.0374	1.001	1.001	0.073
100	0.0430	0.0430	1.000	1.000	0.086
200	0.0428	0.0428	1.000	1.000	0.175
300	0.0408	0.0408	0.999	0.999	0.560
500	0.0371	0.0371	0.999	0.999	2.785

Table 4.2: Numerical blow-up times, maximum norms, and CPU times for $u_0 = v_0 = \sin(\pi x)$, $p = 2$, $\theta = 0.91$, with source terms.

4.2.3 Effect of different initial conditions

To further examine the robustness of the θ -scheme in the presence of source terms, we consider the polynomial–trigonometric initial data

$$u_0(x) = x^3(1 - x)^2, \quad v_0(x) = (1 - x^3) \sin\left(\frac{\pi}{2}x\right).$$

The corresponding numerical results are reported in the following table. The computed blow-up times and locations remain consistent with those obtained for sinusoidal initial data, confirming that the scheme reliably captures the finite-time blow-up phenomenon.

N	T_b^u	T_b^v	$\ \mathcal{U}^n\ _\infty$	$\ \mathcal{V}^n\ _\infty$	CPU time (s)
10	0.0392	0.0395	1.004	1.004	0.072
20	0.0152	0.0153	1.003	1.003	0.073
50	0.0378	0.0379	1.001	1.001	0.074
100	0.0435	0.0436	1.000	1.000	0.087
200	0.0429	0.0430	1.000	1.000	0.176
300	0.0410	0.0412	0.999	0.999	0.562
500	0.0375	0.0376	0.999	0.999	2.790

Table 4.3: Numerical blow-up times, maximum norms, and CPU times for $u_0(x) = x^3(1 - x)^2$, $v_0(x) = (1 - x^3) \sin\left(\frac{\pi}{2}x\right)$, $p = 2$, $\theta = 0.91$, with source terms.

The inclusion of the source terms g_1, g_2 , subject to the conditions established in Chapter 2 (namely continuity and boundedness), does not alter the accuracy of the results regarding the blow-up time and location. In the following, the algorithm is executed with the source term included $g_1(x, t) = x(1 - x)e^{-t}$, $g_2(x, t) = x(1 - x)e^{-t}$, while preserving the same initial conditions $u_0(x) = x^3(1 - x)^2$, $v_0(x) = (1 - x^3) \sin\left(\frac{\pi}{2}x\right)$. The approximate solution is then represented as a function of x_i and t_n in order to enable a clear comparison. The corresponding results are presented in the figure below.

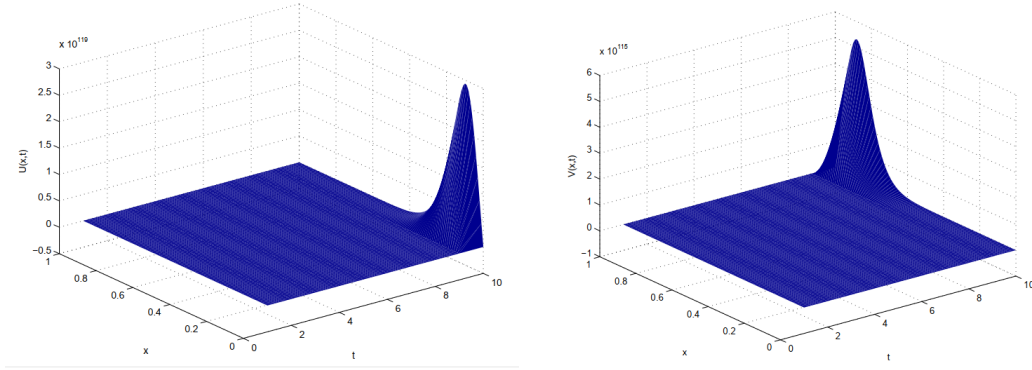


Figure 4.7: Numerical solution $(\mathcal{U}, \mathcal{V})(x_i, t^n)$ by θ -scheme for $p = 2$, $\theta = 0.91$ and $u_0(x) = x^3(1-x)^2$, $v_0(x) = (1-x^3)\sin(\frac{\pi}{2}x)$, and with source terms

4.2.4 Discussion

The numerical experiments confirm that the proposed θ -scheme accurately captures finite-time blow-up in the presence of source terms. While the sources contribute to the early-time dynamics, the blow-up mechanism is primarily driven by the nonlinear coupling and the degenerate diffusion. The stabilization of the blow-up times under mesh refinement and the consistency of the maximum norms demonstrate the reliability and robustness of the method. These results provide strong numerical support for the analytical findings presented earlier in the paper.

Remark 12. The source terms g_1 and g_2 are chosen as smooth, bounded, and nonnegative functions of the form

$$g_1(x, t) = x(1-x)e^{-t}, \quad g_2(x, t) = x(1-x)e^{-t}.$$

This choice ensures compatibility with the homogeneous Dirichlet boundary conditions and satisfies the regularity assumptions required by the theoretical analysis. Moreover, the exponential decay in time guarantees that the source terms do not dominate the nonlinear coupling effects responsible for the finite-time blow-up. As a result, the selected source terms allow us to investigate the influence of external forcing while preserving the qualitative blow-up behavior predicted by the theory.

4.3 Algorithm used

For completeness, we summarize below the fully discrete θ -scheme used to compute the numerical solutions of the coupled degenerate semilinear parabolic system.

Algorithm 1. : *Fully-discrete θ -scheme for the coupled degenerate system with source terms (problem (3.1)) (or without (problem (2.1))).*

Initialization:

- Set the spatial mesh size $h = 1/(N + 1)$ and initial time $t_0 = 0$.
- Initialize the numerical solutions: $\mathcal{U}^0 = u_0(x)$, $\mathcal{V}^0 = v_0(x)$.
- Define the source terms $g_1(x, t)$ and $g_2(x, t)$ only in the solution of problem (3.1) (as for the problem (2.1) it is zero).
- Choose the initial time step k_0 according to the adaptive stability criterion.

$n = 0, 1, 2, \dots$ until blow-up

Time discretization:

$$\frac{\mathcal{U}^{n+1} - \mathcal{U}^n}{k_n} = \theta F(\mathcal{U}^{n+1}, \mathcal{V}^{n+1}) + (1 - \theta)F(\mathcal{U}^n, \mathcal{V}^n),$$

$$\frac{\mathcal{V}^{n+1} - \mathcal{V}^n}{k_n} = \theta G(\mathcal{U}^{n+1}, \mathcal{V}^{n+1}) + (1 - \theta)G(\mathcal{U}^n, \mathcal{V}^n),$$

where F and G denote the discrete degenerate diffusion operators combined with the nonlinear coupling terms $(\mathcal{V}^n)^p$, $(\mathcal{U}^n)^p$ and the source terms g_1^n , g_2^n in the case of problem (3.1).

Solve linear systems: Use an efficient tridiagonal solver (e.g., the Thomas algorithm).

Adaptive step update: Choose k_{n+1} such that

$$1 \geq 2(1 - \theta)\gamma_n.$$

Stopping criterion: Stop the computation when $\|\mathcal{U}^n\|_\infty$ or $\|\mathcal{V}^n\|_\infty$ exceeds a prescribed threshold, indicating numerical blow-up.

4.4 Conclusion

In this study, we developed and analyzed a fully discrete θ -scheme for a coupled nonlinear partial differential equation system. The proposed numerical method combines spatial discretization with an adaptive time-stepping strategy and provides an accurate approximation of the system dynamics, including finite-time blow-up phenomena.

Theoretical investigations show that, under suitable assumptions on the regularity of the exact solution and appropriate constraints on the discretization parameters, the θ -scheme is consistent, stable, and convergent in the pre-blow-up phase.

The numerical experiments fully support the theoretical analysis. In particular, the computed blow-up times and locations are in good agreement with the analytical predictions and remain consistent across different spatial resolutions and initial data. This confirms both the robustness of the scheme and the effectiveness of the adaptive time-step strategy.

Overall, the results demonstrate that the proposed θ -scheme is a reliable and efficient tool for simulating nonlinear coupled PDE systems.

General conclusion

In this study, we established the numerical blow-up behavior of a degenerate nonlinear reaction-diffusion problem, thereby reinforcing and complementing the theoretical blow-up result obtained under suitable initial conditions. By implementing a finite difference discretization combined with the θ -scheme in MATLAB, we successfully reproduced the finite-time singularity predicted by the analytical framework. The numerical evidence not only confirms the validity of the theoretical criteria but also provides a clearer picture of the solution's evolution as it approaches the blow-up time.

Moreover, the computational experiments highlight the sensitivity of the system to both discretization parameters and initial data, emphasizing the importance of carefully selecting numerical schemes when dealing with rapidly growing or unstable solutions. This work illustrates the effectiveness of numerical simulations as a means of validating theoretical results, especially in contexts where explicit analytical solutions are difficult or impossible to obtain.

Future extensions of this study may include applying higher-order schemes, exploring adaptive time stepping to better capture the singular behavior, or examining similar blow-up phenomena in more complex coupled or degenerate systems. Such developments would deepen our understanding of nonlinear dynamics and contribute to more reliable numerical approaches for analyzing finite-time explosion in partial differential equations.

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