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Some Fixed Point Theorems in Banach Spaces and its Applications

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General Introduction

Analysis is considered one of the mathematical sciences, it develops in these last years because of among some hypotheses and theorems, the most important is the theorems of fixed point.

The latter is a main tool that discovers the existence of solution in different types of equations especially in nonlinear analysis, so it has different applications in topology and in other sciences such as physics, chemistry, biology and the economy...etc.

The first appearance of the fixed point begins in the last 19th century, it has been used to find approximate and successive solutions and the existence of a single solution to the equations in particular the differential equations.

In 1922, Banach proved a fixed point theorem which insures under appropriate conditions, the existence and uniqueness of a fixed point.

This result of Banach is known as Banach's fixed point theorem or Banach contraction principle, many authors have extended, generalized and improved Banach's fixed point theorem in different ways.

In 1955, Krasnoselskii combined Banach's fixed-point theorem with that of Schauder and established a new fixed-point theorem that bore its name.

This Krasnoselskii theorem is captivating and has been the subject of several research articles and has many interesting applications in nonlinear analysis.

Throughout this period, it was observed that most researchers in fixed-point theory used contraction mappings.

In this work, we will present some fixed point theorems, common fixed point for expansive mappings under certain conditions in Banach spaces, on other hand we will apply these theories to obtain results in the existence and uniqueness of solution some equations, even with the study of certain hypotheses and the results related to them.

We have divided these ideas forming this memory into three chapters:

1) The first chapter contains a reminder on notation, several definitions and propitious as: metric spaces, complete spaces, Normed spaces, Banach spaces the campacity, contracting and expansive mappings, with some important theorems, Banach's principle theorem, Schauder's Theorem and the Ascoli- Arzela theorem.

2) The second chapter deals with the presentation and proof of many fixed point theories, the most important of which is the fixed point theory of the sum of two mappings, one of which is expansive and the other continuous and compact.

3) In the third chapter, we will apply the theorems of the fixed point,wich were studied in the second chapter to prove the existence and uniquness of solution to integral and differential equations with delay in Banach spaces.

Chapitre 1

Preliminaries

In this chapter, we will remember some basic definitions and theorems, around:

Metric spaces, complete spaces, normed spaces, Banach spaces, contractions and expansive mappings, Compactness, Banach's principle theorem and Schauder's theorem.

1.1 Metric spaces

Definition 1.1.1 *We call distance on a set* X *any application* d *defined on the product* $X^2 = X \times X$ *and with values in the set* $\mathbb{R}^+$ *of the positive reals:*

$$d : X \times X \longrightarrow \mathbb{R}^+,$$

checking the following properties:

1. $\forall x, y \in X,\ d(x, y) = 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$ *(symmety).*
3. $\forall x, y, z \in X,\ d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

Example 1.1.1 $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|,\ x, y \in X = \mathbb{R}^*$ *be a distence on* $\mathbb{R}^*$ *for:*

1. $d(x, y) = 0 \Longleftrightarrow \left|\frac{1}{x} - \frac{1}{y}\right| = 0 \Longleftrightarrow \frac{1}{x} = \frac{1}{y} \Longleftrightarrow x = y$
2. $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|(-1)\left(\frac{1}{y} - \frac{1}{x}\right)\right| = |-1|\left|\frac{1}{y} - \frac{1}{x}\right| = d(y, x)$
3. $d(x, z) = \left|\frac{1}{x} - \frac{1}{z}\right| = \left|\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}\right|$

$$\leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right|$$

$$= d(x, y) + d(y, z).$$

Then d is a distance on $\mathbb{R}^*$.

Remark 1 *A metric space is an ordered pair (X, d), where X is a set and d a metric.*

Example 1.1.2 *the set of positive real numbers $\mathbb{R}_*^+$ with distance function $d(x, y) = \left|\log\left(\frac{y}{x}\right)\right|$ is a metric space.*

1.1.1 The open sets, closed and bounded

Definition 1.1.2 (***Ball and sphere***) *Given a point $x_0 \in X$ and a real number $r > 0$, define:*

(a) $B(x_0, r) = \{x \in X, d(x, x_0) < r\}$ (Open ball).

(b) $\tilde{B}(x_0, r) = \{x \in X, d(x, x_0) \leq r\}$ (Closed ball).

(c) $S(x_0, r) = \{x \in X, d(x, x_0) = r\}$ (Sphere).

In all three cases, x_0 is called the center and r the radius.

Definition 1.1.3 (***Open set***) *A subset M of a metric space X is said to be open if it*

$$\forall x \in M, \ \exists \epsilon > 0, \quad \text{such that} \ \ B(x, \epsilon) \subset M.$$

Definition 1.1.4 (***Closed set***)*A subset M of X is said to be closed if its complement in X is open, that is, $M^c = X - M$ is open.*

Conclusion 1.1.1 *The reader will easily see from these definitions that an open ball is an open set and a closed ball is a closed set.*

Definition 1.1.5 (***Diameter,bounded set***)*The diameter $\delta(A)$ of a nonempty set A in a metric space (X, d) is defined to be*

$$\delta(A) = \sup_{x,y \in A} d(x, y).$$

A is said to be bounded if $\delta(A) < \infty$.

1.1.2 The sequences in metric spaces

Definition 1.1.6 (***Convergence of a sequence,limit***). *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in a metric space* (X,d) *is said to be convergent if :*

$$\forall\epsilon>0\ ;\ \exists N(\epsilon)\in\mathbb{N}^*:\forall n\geq N(\epsilon),\ d(x_n,x)<\epsilon$$

is called the limit of $(x_n)_{n\in\mathbb{N}}$ and we write

$$\lim_{n\to\infty}x_n=x.$$

We say that $(x_n)_{n\in\mathbb{N}}$ converges to x or has the limit x.

Definition 1.1.7 (***Cauchy sequences***) *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in a metric space* (X,d) *is said to be-Cauchy (or fundamental) if for every* $\epsilon>0$ *there is an* $N=N(\epsilon)$ *such that:*

$$d(x_m,x_n)<\epsilon\qquad \textit{for every}\quad m,n>N.$$

Remark 2 *all convergent sequence is Cauchy sequence.*

Proof. Let $x_n\longrightarrow x$, $\epsilon>0$, and $N(\epsilon)\in\mathbb{N}^*$ be such that $n\geq N(\epsilon)\Longrightarrow d(x_m,x)<\dfrac{\epsilon}{2}$, and $m,n>N(\epsilon)$.

Then $d(x_m,x)<\dfrac{\epsilon}{2}$ and $d(x_n,x)<\dfrac{\epsilon}{2}$ and the triangle inequality yields

$$d(x_m,x_n)\leq d(x_m,x)+d(x_n,x)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

■

Let U be an interval of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions with $f_n:U\longrightarrow\mathbb{R}^p$. Let $|.|$ be any norm from $\mathbb{R}^p$.

Definition 1.1.8 $\{f_n\}$*is uniformly bounded on* U *if there exists* $M>0$ *such that:*

$$|f_n(t)|\leq M\quad \textit{for all } n \textit{ and all } t\in U.$$

1.1.3 Complete spaces

Definition 1.1.9 *A metric space (X, d) is said to be complete if each Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X has a limit (converges).*

Example 1.1.3 *(The space $B[0,1]$) This space consists of all bounded real valued functions defined on $[0,1]$ with the distance $d(f,g)$ for $f, g \in B[0,1]$ taken as above:*

$$d(f,g) = \sup\{|f(t) - g(t)| \quad : t \in [0.1]\},$$

it is complete metric space.

Example 1.1.4 *(The metric transform ϕ) Let (M, d)be a metric space, define the metric space (M, d_ϕ) by taking for $x, y \in M$;*

$$d_\phi(x,y) = \phi(d(x,y)),$$

where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is increasing, concave downward,

$$\phi(at + (1-t)b) \geq t\phi(a) + (1-t)\phi(b)$$

and satisfies $\phi(0) = 0$.

It is complete metric space.

1.1.4 Continuous mappings

Definition 1.1.10 *(**Continuous mapping**)Let $X = (X, d_1)$ and $Y = (Y, d_2)$ be metric spaces.*

A mapping $T : X \longrightarrow Y$ is said to be continuous at a point $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that:

$$d(Tx, Tx_0) < \epsilon \quad \text{for all } x \text{ satisfying} \quad d(x, x_0) < \delta.$$

T is said to be continuous on X if it is continuous at every point $x \in X$.

Definition 1.1.11 *f is equicontinuous if for any $\epsilon > 0$ it exists $\delta > 0$ Such that :*

if $t_1, t_2 \in U$ and $|t_1 - t_2| \leq \delta$ so $|f(t_1) - f(t_2)| \leq \epsilon$.

Example 1.1.5 *Let* $f:[a,b] \longrightarrow \mathbb{R}$ *where* $|f'(t)| \leq k$, $\forall t \in]a,b[$ *and* $k > 0$.

We apply mean value theorem,

if $t_0 \in [a,b]$ *fixe* , $\forall t \in]a,b[: \exists c \in]t,t_0[$,

$$\begin{aligned} |f(t) - f(t_0)| &\leq |f'(c)|\,|t - t_0| \\ &\leq k\,|t - t_0| \end{aligned}$$

taking $\delta = \frac{\epsilon}{k}$

if

$$|t - t_0| \leq \delta = \frac{\epsilon}{k} \Longrightarrow |f(t) - f(t_0)| \leq \epsilon$$

so f *equicontinuous an* t_0.

Definition 1.1.12 *(Reciprocally continuous mappings) Let* f *and* g *be two self-mappings of Banach space* $(E, \|.\|)$.

Then f *and* g *are called reciprocally continuous if*

$$\lim_{n\to\infty} fg(x_n) = ft \quad \text{and} \quad \lim_{n\to\infty} gf(x_n) = gt,$$

whenever $(x_n)_{n\in\mathbb{N}}$ *is a sequence such that :*

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \quad \text{for some} \quad t \in E.$$

Remark 3 *If* f *and* g *are both continuous, then they are obviously reciprocally continuous, but the converse need not be true.*

Definition 1.1.13 *(weakly reciprocally continuous mappings) Let* f *and* g *be two self-mappings of a Banach space* $(E, \|\,.\,\|)$.

Then f *and* g *are called weakly reciprocally continuous if*

$$\lim_{n\to\infty} fg(x_n) = ft \quad \text{or} \quad \lim_{n\to\infty} gf(x_n) = gt,$$

whenever $(x_n)_{n\in\mathbb{N}}$ *is a sequence such that:*

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \quad \text{for some} \quad t \in E.$$

Remark 4 *If f and g are reciprocally continuous, then they are obviously weakly reciprocally continuous, but the converse need not be true.*

Definition 1.1.14 *(g-reciprocally continuous mappings) Two self-mappings f and g of a Banach space $(E, \|\ .\ \|)$ are called g-reciprocally continuous if*

$$\lim_{n\to\infty} ffx_n = ft \quad \textit{and} \quad \lim_{n\to\infty} gfx_n = gt,$$

whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence such that:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \quad \textit{for some} \quad t \in E.$$

Definition 1.1.15 *(g-weakly reciprocally continuous mappings) Two self-mappings f and g of a Banach space $(E, \|\ .\ \|)$ will be called g-weakly reciprocally continuous if*

$$\lim_{n\to\infty} ff(x_n) = ft \quad \textit{or} \quad \lim_{n\to\infty} gf(x_n) = gt,$$

whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence such that:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \quad \textit{for some} \quad t \in E.$$

Remark 5 *It may be observed that if f and g are both continuous then they are obviously g-reciprocally continuous but the converse is not true (see example).*

Example 1.1.6 *Let $E = [2, 20]$ and d be the usual metric on E. Define $f, g : E \longrightarrow E$ as follows*

$$fx = \begin{cases} 2 & \textit{if} \quad x = 2 \textit{ or } x > 5 \\ 6 & \textit{if} \quad 2 < x \leq 5 \end{cases}$$

$$gx = \begin{cases} 2 & \textit{if} \quad x = 2 \\ 7 & \textit{if} \quad 2 < x \leq 5 \\ (4x + 10)/15 & \textit{if} \quad x > 5 \end{cases}$$

Then f and g are g-reciprocally continuous but not continuous and not reciprocally continuous.

To see this let us consider the sequence $\left\{x_n = 5 + \frac{1}{n} : n > 1\right\}$ then

$$\lim_{n\longrightarrow\infty} fx_n = 2, \quad \textit{and} \quad \lim_{n\longrightarrow\infty} gx_n = \lim_{n\longrightarrow\infty} \left(2 + \frac{4}{15n}\right) = 2,$$

$$\lim_{n\longrightarrow\infty} ffx_n = 2 = f2, \quad and \quad \lim_{n\longrightarrow\infty} gfx_n = 2 = g2,$$

$$\lim_{n\longrightarrow\infty} fgx_n = \lim_{n\longrightarrow\infty} f\left(2+\frac{4}{15n}\right) = 6 \neq 2.$$
$$\lim_{x\overset{>}{\longrightarrow}2} f(x) = 6 \neq \lim_{x\overset{<}{\longrightarrow}2} f(x) = 2$$

1.2 Compactness

1.2.1 Compact metric spaces

Definition 1.2.1 *Let X be a metric space, we say that X is compact if every sequence of points of E has a convergent subsequence.*

1.2.2 Compact parts

Definition 1.2.2 *A subset M of a metric space (X,d) is said to be compact if any $(X_n)_{n\in N}$ of M admits a subsequence converging to a limit belonging to M.*

Example 1.2.1 *Any closed and bounded part of $\mathbb{R}$ is compact.*

Example 1.2.2 $\{\frac{1}{n}, n\in\mathbb{N}^*\}\cup\{0\}$ *in the metric space $(\mathbb{R},|.|)$, it is a bounded closed of a normed vector space of finite dimension.*

It is closed because convergent with values in this set is either stationary or converges to 0, since the points $\frac{1}{n}$ are isolated.

Definition 1.2.3 (***Relatively compact parts***) *X is relatively compact if every sequence of X admits a subsequence converging to a limit belonging to X ,That is to say, if the closure of X is compact.*

Example 1.2.3 *The relatively compact parts of $\mathbb{R}^n$ are the bounded parts.*

1.2.3 Compact mappings

Definition 1.2.4 *E and F two vector spaces normed and $T : E \longrightarrow F$ a linear mapping, T is said to be compact if ,*

1. *the image of each bounded set in E is relatively compact in F.*
2. *$T(B_E(0,1))$ is relatively compact in F.*
3. *for each sequence (x_n) bounded in E one can extract a subsequence (x_{n_k}) such that $T(x_{n_k})$ converges in F.*

Theorem 1.2.1 (***Arzela-Ascoli***)*If $\{f_n\}$is a sequence of real functions uniformly bounded and equicontinuous defined over an interval $[a,b]$, then the sequence admits a subsequence converging uniformly on $[a,b]$ to a continuous function.*

Example 1.2.4 *Let $f : ([0,1]) \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous application, consider the following integral equation:*

$$u(t) \longrightarrow \int_0^t f(s, u(s))\, ds; \qquad t \in [0,1],$$

Then the operator of Hammerstein

$$G : C([0,1]) \longrightarrow C([0,1])$$

$$u \longrightarrow Gu,$$

such that

$$Gu(t) = \int_0^t f(s, u(s))\, ds$$

is compact ,

assume the set $A = \{f \in C([0,1]), \|f\| \leq M\}$,Since f is continuous and bounded so:

$$\begin{aligned} |Gu(t)| &= \left| \int_0^t f(s, u(s))\, ds \right| \\ &\leq \int_0^t |f(s, u(s))|\, ds \\ &\leq M \int_0^t ds \\ &= Mt \\ &\leq M \quad \forall t \in [0,1]. \end{aligned}$$

Therefore G is bounded.

We will show G is equicontinuous, for all $t_1, t_2 \in [0,1]$ (suppose $t_1 < t_2$) we have:

$$\begin{aligned}
|Gu(t_1) - Gu(t_2)| &\leq \int_0^{t_1} |f(s,u(s))|\, ds - \int_0^{t_2} |f(s,u(s))|\, ds \\
&= \int_{t_1}^{t_2} |f(s,u(s))|\, ds \\
&\leq M \int_{t_1}^{t_2} ds \\
&\leq M |t_1 - t_2|.
\end{aligned}$$

So, for all $\epsilon > 0$, does it exist $\delta \leq \frac{\epsilon}{M}$, such that:

for all $t_1, t_2 \in [0,1] : |t_2 - t_1| \leq \delta$ so:

$$|Gu(t_1) - Gu(t_2)| \leq M\delta \leq \epsilon.$$

Hence the equicontinuity of G.

From the Ascoli-Arzela theorem, G is compact in A.

1.3 Banach spaces

1.3.1 Linear spaces

Let E denote any nonempty set that contains with each of its elements x and each real number λ a unique element $\lambda.x$, written λx, called a scalar multiple of x. Also assume that for each two elements $x, y \in E$ there exists a unique element $x + y \in E$ called the sum of x and y.

Definition 1.3.1 *The system $\{E, ., +\}$ is called a linear space or a vector space (over $\mathbb{R}$) if the following conditions are satisfied.*

Here, $x, y, z \in E$; $\lambda, \mu, 1 \in \mathbb{R}$.

(1) $x + y = y + x$, (2) $x + (y + z) = (x + y) + z$,

(3) $\lambda(x + y) = \lambda x + \lambda y$, (4) $x + y = x + z$ *implies* $y = z$,

(5) $(\lambda + \mu)x = \lambda x + \mu x$, (6) $\lambda(\mu x) = (\lambda\mu)x$,

(7) $1.x = x$.

1.3.2 Normed linear spaces

Definition 1.3.2 *(norm) A norm on a linear space E is a mapping $\|.\| : E \longrightarrow \mathbb{R}^{+}$ which satisfies for each $x, y \in E$; $\lambda \in \mathbb{R}$:*

(1) $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \, \|x\|$.
(3) $\|x + y\| \leq ||x|| + \|y\|$.

Definition 1.3.3 *A linear space with a norm called a normed linear space.*

Example 1.3.1 (*The space $C[0,1]$*) *This space consists of all countined real valued functions defined on $[0,1]$ with the norme $\|f\|$ for $f, g \in C[0,1]$ taken as above:*

$$\|f\| = \int_0^1 f(t)\, dt,$$

it is normed space.

Definition 1.3.4 (***Convex set***)*.A subset M of a vector space E is said to be convex if x,$y \in M$ implies:*

$$K = \{z \in E \mid z = \alpha x + (1-\alpha) y, \quad 0 \leq \alpha \leq 1\} \subset M.$$

Definition 1.3.5 *A Banach space is a normed linear space $(E, \|.\|)$ which is complete relative to the metric d defined above.*

Example 1.3.2 $l^{\infty}(M)$ *This is the space of all bounded real-valued functions $f : M \longrightarrow \mathbb{R}$ where M is a complete metric space and*

$$\|f\| = \sup_{x \in M} |f(x)| \, .$$

The completeness of $l^{\infty}(M)$ follows from the fact that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $l^{\infty}(M)$ then $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in M for each $x \in M$.

The function defined by $f(x) = \lim\limits_{n\to\infty} f_n(x), x \in M$, exists since M is complete.

It is quite easy to show that $\lim\limits_{n\to\infty} \|f_n - f\| = 0$.

Theorem 1.3.1 (***Schauder's Theorem***) [2]*Let K be a nonempty compact convex subset of a Banach space E, and suppose $f : K \longrightarrow K$ is continuous.*

Then f has at least one fixed point.

1.4 Contractive conditions

1.4.1 Contraction

Definition 1.4.1 *Let f be a self-mapping of a Banach space $(E, \| \,.\, \|)$.*

Then f is said to be contraction if there exists a real number $h < 1$ such that:

$$\| fx - fy \| \leq h \| x - y \| \quad \text{for all} \quad x, y \in E. \tag{1.4.1}$$

A contraction mapping is also known as Banach contraction.

If we replace the inequality (1.4.1) with strict inequality and $h = 1$, then f is called contractive (or strict contractive).

If (1.4.1) holds for $h = 1$, then f is called nonexpansive; and if (1.4.1) holds for fixed $h < \infty$, then f is called Lipschitz continuous .

Clearly, for the mapping f , the following obvious implications hold:

contraction $\Longrightarrow$contractive $\Longrightarrow$ nonexpansive $\Longrightarrow$ Lipschitz continuous

Example 1.4.1 *Consider the usual metric space $(\mathbb{R}, d)$. Define*

$$f(x) = \frac{x}{a} + b, \qquad \text{for all } x \in \mathbb{R}$$

$$\begin{aligned} d(fx, fy) &= \left| \frac{x}{a} + b - \frac{y}{a} - b \right| \\ &= \left| \frac{1}{a}(x - y) \right| \\ &\leq \left| \frac{1}{a} \right| |x - y| . \end{aligned}$$

Then, f is contraction on $\mathbb{R}$ if $a > 1$.

Example 1.4.2 *Consider the Euclidean metric space $(\mathbb{R}^2, d)$.Define*

$$f(x, y) = \left(\frac{x}{a} + b, \frac{y}{c} + d \right),$$

for all $(x, y) \in \mathbb{R}^2$.

Then, f is contraction on $\mathbb{R}^2$ if $a, c > 1$.

Definition 1.4.2 [12]*Let f be a self-mapping of a Banach space $(E, \| \,.\, \|)$.*

Then f is said to be ϕ-contractive if there exists a continuous mapping:

$$\phi : [0, \infty) \to [0, \infty)$$

with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$ such that:

$$\| fx - fy \| \leq \phi \left(\max \; \{\| x - y \|, \| x - fx \|, \| y - fy \|, [\| x - fy \| + \| fx - y \|] /2 \; \}\right)$$
for all $x, y \in E$.

Definition 1.4.3 *Let f be a self-mapping of a Banach space $(E, \| \,.\, \|)$.*

Then f is said to be ϕ-weakly contraction if there exists a continuous mapping:

$$\phi : [0, \infty) \to [0, \infty)$$

with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$ such that:

$$\| fx - fy \| \leq \| x - y \| - \phi(\| x - y \|) \quad \textit{for all} \;\; x, y \in E.$$

1.4.2 Contraction Mapping Principle

Theorem 1.4.1 (***Banach's theorem fixed point***) *Let (M, d)be a complete metric space and let $f : M \longrightarrow M$ be a contraction mapping.*

Then f has a unique fixed point x_0.

Proof. Let h be a contraction constant of the mapping f .

We will explicitly construct a sequence converging to the fixed point.

Let x_0 be an arbitrary but fixed element in X.

Define a sequence of iterates $(x_n)_{n \in \mathbb{N}}$ in X by

$$x_n = f(x_{n-1}) = (f^n(x_0)), \quad \text{for all} \;\; n \geq 1.$$

Since f is a contraction, we have:

$$d\left(x_n, x_{n+1}\right) = d\left(f(x_{n-1}), f(x_n)\right) \leq hd\left(x_{n-1}, x_n\right) \qquad \text{for any} \;\; n \geq 1.$$

Thus, we obtain

$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \quad$ for any $\; n \geq 1$.

Hence, for any $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) d(x_0, x_1) \\ &\leq \frac{h^n}{1-h} d(x_0, x_1). \end{aligned}$$

We deduce that $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence in a complete space X. Let $x_n \longrightarrow p \in X$. Now using the continuity of the map f, we get:

$$p = \lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} f(x_{n-1}) = f(p).$$

Finally, to show f has at most one fixed point in X, let p and q be fixed points of f. Then,

$$d(p, q) = d(f(p), f(q)) \leq hd(p, q).$$

Since $h < 1$, we must have $p = q$. ∎

1.5 Expansive mappings

Definition 1.5.1 *Let f be a self-mapping of a Banach space $(E, \| \,.\, \|)$.*

Then f is said to be expansive if there exists a real number $h > 1$ such that

$$\| fx - fy \| \geq h \| x - y \| \quad \textit{for all} \;\; x, y \in E. \tag{1.5.1}$$

Definition 1.5.2 *Let f be a self-mapping of a Banach space $(E, \| \,.\, \|)$.*

Then f is said to be ϕ-weakly expansive if there exists a continuous mapping:

$$\phi : [0, \infty) \to [0, \infty)$$

with $\phi(0) = 0$ and $\phi(t) > t$ for all $t > 0$ such that:

$$\| fx - fy \| \geq \| x - y \| + \phi(\| x - y \|) \quad \textit{for all} \quad x, y \in E.$$

Definition 1.5.3 *Let f and g be two self-mappings of a Banach space $(E, \| \,.\, \|)$.*

Then f is said to be a ϕ-weakly expansive with respect to $g : E \to E$ if there exists a continuous mapping:

$$\phi : [0, \infty) \to [0, \infty)$$

with $\phi(0) = 0$ *and* $\phi(t) > t$ *for all* $t > 0$ *such that:*

$$\| fx - fy \| \geq \| gx - gy \| + \phi(\| gx - gy \|) \quad \text{for all} \;\; x, y \in E.$$

1.6 Compatible mappings

Definition 1.6.1 *Let* f *and* g *be two self-mappings of a Banach space* $(E, \| \,.\, \|)$.

Then f *and* g *are said to be compatible if:*

$$\| fgx_n - gfx_n \| = 0,$$

whenever $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* E *such that :*

$$\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = t \quad \text{for some} \;\; t \in E.$$

Definition 1.6.2 *Let* f *and* g *be two self-mappings of a Banach space* $(E, \|.\|)$.

Then f *and* g *are called R-weakly commuting if there exists* $R > 0$ *such that :*

$$\| fgx - gfx \| \leq R \| fx - gx \| \quad \text{for all} \;\; x \in E.$$

Definition 1.6.3 *Let* f *and* g *be two self-mappings of a Banach space* $(E, \|.\|)$.

Then f *and* g *are called:*

(1) R-weakly commuting of type (A_g) *if there exists* $R > 0$ *such that:*

$$\| ffx - gfx \| \leq R \| fx - gx \| \quad \text{for all} \;\; x \in E.$$

(2) R-weakly commuting of type (A_f) *if there exists* $R > 0$ *such that:*

$$\| fgx - ggx \| \leq R \| fx - gx \| \quad \text{for all} \;\; x \in E.$$

Definition 1.6.4 *Let* f *and* g *be two self-mappings of a Banach space* $(E, \|.\|)$.

Then f *and* g *are called R-weakly commuting of type* (P) *if there exists* $R > 0$ *such that:*

$$\| ffx - ggx \| \leq R \| fx - gx \| \quad \text{for all} \;\; x \in E.$$

Chapitre 2

Some fixed point theorems in Banach spaces

This chapter is devoted to present some fixed point theorems for expansive mappings:

1) Fixed point theorems for expansive mapping under certain conditions.

2) Some fixed point theorems for the sum conditions of two mappings, where one is expansive and the other is completely continuous.

3) Two common fixed point theorems for two pairs of expansive mappings by using the concept of weakly reciprocal continuity combined with compatibility concept.

2.1 Fixed point theorem for expansive mapping

Theorem 2.1.1 *Let $(E, \|.\|)$ be Banach space and let M be a closed subset of E.*

Assume that the mapping $T : M \to E$ is expansive and $M \subset T(M)$, then there exists a unique point $x^* \in M$ such that $Tx^* = x^*$.

Proof. Since T is expansive, we have for all $x \neq y$,

$$\begin{aligned} \|Tx - Ty\| &\geq h\|x - y\| \\ &> \|x - y\| \\ &> 0 \end{aligned}$$

Such that : $Tx \neq Ty$

then T is one-to-one.

Hence, $T : M \to T(M)$ is onto, that the inverse of $T : M \to T(M)$ exists.

$$T^{-1} : T(M) \to M,$$

which, in view of the fact that $M \subset T(M)$, shows in particular, that

$$T^{-1}|_M : M \to M,$$

is a contraction and hence continuous, let $x = Tx_1$ and $y = Ty_1$,

$$\begin{aligned} \left\|T^{-1}x - T^{-1}y\right\| &= \left\|T^{-1}Tx_1 - T^{-1}Ty_1\right\| \\ &= \left\|TT^{-1}x_1 - TT^{-1}y_1\right\| \\ &\geq h\left\|T^{-1}x_1 - T^{-1}y_1\right\|, \end{aligned}$$

thus,

$$\|x_1 - y_1\| \geq h\left\|T^{-1}x_1 - T^{-1}y_1\right\|,$$

$$\left\|T^{-1}x_1 - T^{-1}y_1\right\| \leq \frac{1}{h}\|x_1 - y_1\|, \qquad \left(\frac{1}{h} < 1\right)$$

where $T^{-1}|_M$ denotes the restriction of the mapping T^{-1} to the set M.

Since, M is a closed subset of a banach space, then in view of the Banach Contraction Mapping Principle there exists an $x^* \in M$ such that:

$$T^{-1}x^* = x^*.$$

Clearly, x^* is also a fixed point of T.

For the uniqueness, suppose there exist another a fixed point y, by using expansive condition T, we get

$$\begin{aligned} \| x - y \| &= \| Tx - Ty \| \\ &\geq h \| x - y \| \\ &> \| x - y \| . \end{aligned}$$

Which is a contradiction , then x is unique ■

Corollary 2.1.2 *Assume that the mapping $T : E \to E$ is expansive and onto, then there exists a unique point $x^* \in E$ such that $Tx^* = x^*$.*

Corollary 2.1.3 *Let $T : E \to E$, assume that there exists a positive integer n such that T^n is expansive and onto, then there exists a unique point $x^* \in E$ such that $Tx^* = x^*$.*

Proof. *According to precedent corollary, there exists a unique point $x^* \in E$ such that:*

$$T^n x^* = x^*.$$

Which implies that Tx^* is a fixed point of T^n,

$$Tx^* = TT^n x^* = T^n T x^*.$$

In view of uniqueness, we have $Tx^* = x^*$.

Hence x^* is the unique fixed point of T . ■

2.2 Fixed point theorems for the sum of two mappings

Theorem 2.2.1 *Let $K \subset E$ be a nonempty closed convex subset, suppose that T and S map K into E such that:*

(i) S is continuous, $S(K)$resides in a compact subset of E.

(ii) T is an expansive mapping.

(iii) $z \in S(K)$ implies $K \subset T(K) + z$, where $T(K) + z = \{y + z \mid y \in T(K)\}$.

Then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.

Proof. Let $z \in S(K)$ define the mapping:

$$T + z : K \to E.$$

From (iii) the mapping $T + z : K \to T(K) + z$ is onto.

From (ii) the mapping $T + z : K \to T(K) + z$ is expansive.

Let $x, y \in K$,

$$\begin{aligned} \|(T+z)(x) - (T+z)(y)\| &= \|Tx + z - Ty - z\| \\ &= \|Tx - Ty\| \\ &\geq h\|x - y\|. \end{aligned}$$

So $T + z$ have a fixed point x^*, which implies that $Tx^* + z = x^*$.

i.e, the equation

$$Tx + z = x, \tag{2.2.1}$$

has a unique solution $x = \tau(z)$, such that:

$\tau : S(K) \to K$ for any $z_1, z_2 \in S(K)$, from

$$T(\tau(z_1)) + z_1 = \tau(z_1)$$

$$T(\tau(z_2)) + z_2 = \tau(z_2),$$

it follows that

$$T(\tau(z_1)) = \tau(z_1) - z_1,$$

$$T(\tau(z_2)) = \tau(z_2) - z_2,$$

$$\| T(\tau(z_1)) - T(\tau(z_2)) \| = \| z_2 - z_1 + \tau(z_1) - \tau(z_2) \|$$

$$\leq \| z_1 - z_2 \| + \| \tau(z_1) - \tau(z_2) \| .$$

Recalling that T is expansive, therefore, there exists a constant $h > 1$ with

$$\| T(\tau(z_1)) - T(\tau(z_2)) \| \geq h \| \tau(z_1) - \tau(z_2) \| .$$

Thus, we have

$$\| \tau(z_1) - \tau(z_2) \| \leq \frac{1}{h-1} \| z_1 - z_2 \| . \tag{2.2.2}$$

It follows from (2.2.2) that $\tau : S(K) \to K$ is continuous.

On the other hand, since $S : K \to S(K)$ is continuous on K, it implies that $\tau S : K \to K$ is also continuous and $\tau S(K)$ resides in a compact subset of E, (for a proof of this result in general metric spaces, see [13]).

By(Schauder Fixed Point Theorem), there exists $x^* \in K$, such that:

$$\tau(S(x^*)) = x^*.$$

From (2.2.1) we deduce that:

$$T(\tau(S(x^*))) + S(x^*) = \tau(S(x^*)) \qquad \text{i.e,} \quad Tx^* + Sx^* = x^* .$$

■

Lemma 2.2.1 *Let $(E, \| \, . \, \|)$ be a linear normed space, $M \subset E$.*

Assume that the mapping $T : M \to E$ is expansive with constant $h > 1$.

Then the inverse of $F := I - T : M \longrightarrow (I - T)(M)$ exists.

Hence,

$$\| F^{-1}x - F^{-1}y \| \leq \frac{1}{h-1} \| x - y \|, \qquad x, y \in F(M). \tag{2.2.3}$$

Proof. For each $x, y \in M$, we have

$$\begin{aligned} \| Fx - Fy \| &= \|x - Tx - y + Ty\| \\ &= \| (Ty - Tx) - (y - x) \| \\ &\geq \|Ty - Tx\| - \| x - y \| \\ &\geq h \| x - y \| - \| x - y \| \\ &= (h - 1) \| x - y \| \end{aligned}$$

so

$$\| Fx - Fy \| \geq (h - 1) \| x - y \|, \tag{2.2.4}$$

which shows that F is one-to-one, hence the inverse of $F : M \longrightarrow F(M)$ exists.

Now taking $x, y \in F(M)$, then $F^{-1}x, F^{-1}y \in M$, thus using $F^{-1}x, F^{-1}y$ substitute for x, y in (2.2.4), respectively, we obtain

$$\| F^{-1}x - F^{-1}y \| \leq \frac{1}{h-1} \| x - y \| .$$

■

Theorem 2.2.2 *Let $K \subset E$ be a nonempty closed convex subset, suppoes that $T : E \to E$ (or $T : K \to E$) and $S : K \to E$ such that:*

(i) S is continuous, and $S(K)$ resides in a compact subset of E.

(ii) T is an expansive map with constant $h > 1$.

(iii) $S(K) \subset (I - T)(E)$ and $[x = Tx + Sy, \; y \in K] \Longrightarrow x \in K$ $(or\ S(K) \subset (I - T)(K))$.

Then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.

Proof. For each $y \in K$, by (iii), there exists $x \in E$ such that:

$$x - Tx = Sy.$$

By Lemma (2.2.1) and (iii), we have $x = (I - T)^{-1} Sy \in K$.

Now, $S(K)$ resides in a compact subset of E, while $(I-T)^{-1}$ is continuous, and so $(I-T)^{-1} S(K)$ resides in a compact subset of the closed set K.

By Theorem schauder fixed point, $(I-T)^{-1} S$ has a fixed point $x^* \in K$ with

$$x^* = (I-T)^{-1} Sx^* = Sx^* + Tx^*.$$

■

Theorem 2.2.3 ([11] , *petryshyn) Let* $B_r = \{x \in E : \|x\| \leq r\}$, $\partial B_r = \{x \in E : \|x\| = r\}$, *if* $S : B_r \to E$ *is a completely continuous map and*

$$\|Sx\| \leq \|Sx - x\| \qquad \text{for each } x \in \partial B_r. \tag{2.2.5}$$

Then S *has at least one fixed point in* B_r.

Theorem 2.2.4 *Suppose that* $T : E \to E$ *and* $S : B_r \to E$ *such that:*

(i) S is continuous, and $S(B_r)$ resides in a compact subset of E.

(ii) T is an expansive map with constant $h > 1$.

(iii) $S(B_r) \subset (I-T)(E)$.

(iv) $\|Sx + T\theta\| \leq \frac{(h-1)}{2} \|x\|$ for each $x \in \partial B_r$, (where θ is zero element).

that there exists a point $x^* \in \partial B_r$ with $Sx^* + Tx^* = x^*$.

Proof. For each $x \in B_r$, by (iii) , we see that there exsists $y \in E$ such that:

$$y - Ty = Sx. \tag{2.2.6}$$

By lemma (2.2.1) ,we have

$$y = (I-T)^{-1} Sx := GSx \in E.$$

Again by Lemma 2.2.1 and (i), one can easily know that $GS : B_r \to E$ is completely continuous.

It is remained to check that (2.2.5) holds, indeed, for each $x \in B_r$, from (2.2.6), we obtain

$$T(GSx) + Sx = GSx,$$

which implies that

$$\|T(GSx) - T\theta\| \leq \|GSx\| + \|Sx + T\theta\|. \tag{2.2.7}$$

On the other hand, we have

$$\|T(GSx) - T\theta\| \geq h\|GSx\|. \tag{2.2.8}$$

From (2.2.7) and (2.2.8), we deduce that

$$\|GSx\| \leq \frac{1}{h-1}\|Sx + T\theta\|. \tag{2.2.9}$$

For any $x \in \partial B_r$, from (2.2.9) and (iv), we derive that

$$\begin{aligned} \|GSx\|^2 - (\|GSx\| - \|x\|)^2 &= \|x\|(2\|GSx\| - \|x\|) \\ &\leq \|x\|\left(\frac{2}{h-1}\|Sx + T\theta\| - \|x\|\right) \\ &\leq 0, \end{aligned}$$

thus:

$$\begin{aligned} \|GSx\| &\leq \|GSx\| - \|x\| \\ &\leq \|GSx - x\| \end{aligned}$$

which implies (2.2.5), so $GS := (I - T)^{-1}$ has at least one fixed point in B_r, i.e. exists a point $x^* \in \partial B_r$ with $Sx^* + Tx^* = x^*$. ∎

2.3 Common fixed point theorems for expansion mappings

Theorem 2.3.1 *Let f and g be two weakly reciprocally continuous self-mappings of a Banach space $(E, \| . \|)$ satisfying the following conditions:*

$$g(E) \subseteq f(E), \tag{2.3.1}$$

for any $x, y \in E$ and $q > 1$, we have that:

$$\| fx - fy \| \geq q \| gx - gy \| . \tag{2.3.2}$$

If f and g are either compatible or R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f) or R-weakly commuting of type (P), then f and g have a unique common fixed point.

Proof. Let x_0 be any point in E, since $g(E) \subseteq f(E)$, there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $g(x_n) = f(x_{n+1})$.

Define a sequence $(y_n)_{n\in\mathbb{N}}$ in E by:

$$y_n = g(x_n) = f(x_{n+1}). \tag{2.3.3}$$

Now, we will show that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E.

For proving this, from(2.3.3) and (2.3.2), we have

$$\begin{aligned} \| y_n - y_{n+1} \| &= \| gx_n - gx_{n+1} \| \\ &\leq \frac{1}{q} \| fx_n - fx_{n+1} \| \\ &= \frac{1}{q} \| y_{n-1} - y_n \| . \end{aligned}$$

Hence,

$$\begin{aligned} \| y_n - y_{n+1} \| &\leq \frac{1}{q} \| y_{n-1} - y_n \| \\ &\leq \frac{1}{q^2} \| y_{n-2} - y_{n-1} \| \leq ... \leq \frac{1}{q^n} \| y_0 - y_1 \| . \end{aligned}$$

Therefore, for all $n, m \in \mathbb{N}$, where $n < m$, we have

$$\| y_n - y_m \| \leq \| y_n - y_{n+1} \| + \| y_{n+1} - y_{n+2} \| + \| y_{n+2} - y_{n+3} \| + ... + \| y_{m-1} - y_m \|$$

$$\leq \left(\frac{1}{q^n} + \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + ... + \frac{1}{q^{m-1}} \right) \| y_0 - y_1 \|$$

$$\leq \left(\frac{1}{q^n} + \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + ... \right) \| y_0 - y_1 \|$$

$$= \frac{1}{q^{n-1}(q-1)} \| y_0 - y_1 \| \to 0 \qquad \text{as } n \to \infty.$$

Thus, $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E.

Since E is complete, there exists a point z in E such that $\lim_{n\to\infty} y_n = z$.

Therefore, by (2.3.3) we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_{n+1}) = z.$$

Suppose that f and g are compatible mappings.

Now, by the weak reciprocal continuity of f and g, we obtain

$$\lim_{n\to\infty} fg(x_n) = fz \quad \text{or} \quad \lim_{n\to\infty} gf(x_n) = gz.$$

Let $\lim_{n\to\infty} fg(x_n) = fz$, then the compatibility of f and g gives

$$\lim_{n\to\infty} \| fg(x_n) - gf(x_n) \| = 0,$$

that is, $\lim_{n\to\infty} \| gf(x_n) - fz \| = 0$

Hence,

$$\lim_{n\to\infty} gf(x_n) = fz.$$

From (2.3.3), we get

$$\lim_{n\to\infty} gf(x_{n+1}) = \lim_{n\to\infty} gg(x_n) = fz.$$

Therefore, from (2.3.2), we get

$$\| gz - ggx_n \| \leq \frac{1}{q} \| fz - fgx_n \| .$$

Taking the limit as $n \to \infty$, we get

$$\| gz - fz \| \leq \frac{1}{q} \| fz - fz \| = 0.$$

Hence, $fz = gz$.

Again, the compatibility of f and g implies the commutativity at a coincidence point.

Hence,

$$gfz = fgz = ffz = ggz.$$

Using (2.3.2), we obtain

$$\| gz - ggz \| \leq \frac{1}{q} \| fz - fgz \|$$

$$= \frac{1}{q} \parallel gz - ggz \parallel .$$

Which proves that $gz = ggz$.

We also get $gz = ggz = fgz$ and then gz is a common fixed point of f and g.

Next, suppose that $\lim_{n\to\infty} gf(x_n) = gz$.

The assumption $g(E) \subseteq f(E)$ implies that $gz = fu$ for some $u \in E$ and therefore,

$$\lim_{n\to\infty} gf(x_n) = fu.$$

The compatibility of f and g implies that

$$\lim_{n\to\infty} fg(x_n) = fu.$$

By virtue of (2.3.3), we have

$$\lim_{n\to\infty} gf(x_{n+1}) = \lim_{n\to\infty} gg(x_n) = fu.$$

Using (2.3.2), we get

$$\parallel gu - ggx_n \parallel \leq \frac{1}{q} \parallel fu - fgx_n \parallel .$$

Taking the limit as $n \to \infty$, we get

$$\parallel gu - fu \parallel \leq \frac{1}{q} \parallel fu - fu \parallel = 0.$$

Then we get $fu = gu$.

The compatibility of f and g yields

$$fgu = ggu = ffu = gfu.$$

Finally, using (2.3.2), we obtain

$$\parallel gu - ggu \parallel \leq \frac{1}{q} \parallel fu - fgu \parallel$$

$$= \frac{1}{q} \parallel gu - ggu \parallel,$$

that is, $gu = ggu$.

We also have $gu = ggu = fgu$ and gu is a common fixed point of f and g.

Now, suppose that f and g are R-weakly commuting of type (A_f).

Now, the weak reciprocal continuity of f and g implies that

$$\lim_{n\to\infty} fg(x_n) = fz \quad \text{or} \quad \lim_{n\to\infty} gf(x_n) = gz.$$

Let us first assume that $\lim_{n\to\infty} fg(x_n) = fz$.

Then the R-weak commutativity of type (A_f) of f and g yields

$$\| ggx_n - fgx_n \| \leq R \| fx_n - gx_n \|,$$

and therefore

$$\lim_{n\to\infty} \| ggx_n - fz \| \leq R \| z - z \| = 0.$$

This proves that $\lim_{n\to\infty} gg(x_n) = fz$.

Again, using (2.3.2), we get

$$\| gz - ggx_n \| \leq \frac{1}{q} \| fz - fgx_n \| .$$

Taking the limit as $n \to \infty$, we get

$$\| gz - fz \| \leq \frac{1}{q} \| fz - fz \| = 0.$$

Hence, we get $fz = gz$.

Again, by using the R-weak commutativity of type (A_f), we have

$$\begin{aligned} \| ggz - fgz \| &\leq \frac{1}{q} \| gz - fz \| \\ &= \frac{1}{q} \| fz - fz \| = 0. \end{aligned}$$

This yields $ggz = fgz$.

Therefore,

$$ffz = fgz = gfz = ggz.$$

Using (2.3.2), we get

$$\begin{aligned} \| gz - ggz \| &\leq \frac{1}{q} \| fz - fgz \| \\ &= \frac{1}{q} \| gz - ggz \|, \end{aligned}$$

that is, $gz = ggz$.

Then we also get $gz = ggz = fgz$ and gz is a common fixed point of f and g.

Similar proof works in the case where

$$\lim_{n\to\infty} gf(x_n) = gz.$$

Suppose that f and g are R-weakly commuting of type(A_g) .

Again, as done above, we can easily prove that fz is a common fixed point of f and g.

Finally, suppose that f and g are R-weakly commuting of type (P).

The weak reciprocal continuity of f and g implies that

$$\lim_{n\to\infty} fg(x_n) = fz \quad \text{or} \lim_{n\to\infty} gf(x_n) = gz.$$

Let us assume that $\lim_{n\to\infty} fg(x_n) = fz$.

Then the R-weak commutativity of type (P) of f and g yields

$$\| ffx_n - ggx_n \| \leq R \| fx_n - gx_n \| .$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \| ffx_n - ggx_n \| \leq R \| z - z \| = 0,$$

that is,

$$\lim_{n\to\infty} \| ffx_n - ggx_n \| = 0.$$

Using (2.3.1) and (2.3.3), we have

$$fgx_{n-1} = ffx_n \to fz \quad \text{as} \quad n \to \infty,$$

which gives $ggx_n \to fz$ as $n \to \infty$.

Also, using (2.3.2), we get

$$\| gz - ggx_n \| \leq \frac{1}{q} \| fz - fgx_n \| .$$

Taking the limit as $n \to \infty$, we get

$$\| gz - fz \| \leq \frac{1}{q} \| fz - fz \| = 0.$$

Hence, $fz = gz$.

Again, by using the R-weak commutativity of type (P)

$$\| ffz - ggz \| \leq R \| fz - gz \| = 0.$$

This yields $ffz = ggz$.

Therefore

$$ffz = fgz = gfz = ggz.$$

Using (2.3.2), we get

$$\begin{aligned} \| gz - ggz \| &\leq \frac{1}{q} \| fz - fgz \| \\ &= \frac{1}{q} \| gz - ggz \| . \end{aligned}$$

This proves that $gz = ggz$.

Hence, $gz = ggz = fgz$ and gz is a common fixed point of f and g.

Similar proof works in the case where

$$\lim_{n\to\infty} gf(x_n) = gz.$$

Uniqueness of the common fixed point theorem follows easily in each of the four cases by using (2.3.2). ■

Theorem 2.3.2 *Let f and g be two weakly reciprocally continuous self-mappings of a Banach space $(E, \| . \|)$ satisfying*

$(C1)$ $gE \subset fE$,

$(C3)$ there exists a continuous mapping $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) > t$ for all $t > 0$ such that:

$$\| fx - fy \| \geq \| gx - gy \| + \phi (\| gx - gy \|) \quad \text{for all } \ x, y \in E.$$

If f and g are compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be any point in E, since $gE \subset fE$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $gx_n = fx_{n+1}$.

Define a sequence $(y_n)_{n\in\mathbb{N}}$ in E by

$$y_n = gx_n = fx_{n+1}. \tag{2.3.4}$$

Moreover, we assume that if $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then there is nothing to prove.

Now, we assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$.

From $(C3)$, we have

$$\begin{aligned} \| \quad & y_n - y_{n-1} \| = \| fx_{n+1} - fx_n \| \\ \geq \quad & \| gx_{n+1} - gx_n \| + \phi \| gx_{n+1} - gx_n \| \\ = \quad & \| y_{n+1} - y_n \| + \phi \left(\| y_{n+1} - y_n \| \right), \end{aligned} \tag{2.3.5}$$

that is,

$$\| y_n - y_{n-1} \| > \| y_{n+1} - y_n \| .$$

Hence the sequence $\left(\| y_{n+1} - y_n \| \right)_{n \in \mathbb{N}}$ is strictly decreasing and bounded below.

Thus there exists $r \geq 0$ such that:

$$\lim_{n \longrightarrow \infty} \| y_{n+1} - y_n \| = r.$$

Letting $n \to \infty$ in (2.3.5), we get $r \geq r + \phi(r)$, which is a contradiction.

Hence we have $r = 0$.

Therefore

$$\lim_{n \longrightarrow \infty} \| y_{n+1} - y_n \| = 0. \tag{2.3.6}$$

Now, we will show that $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

Let $(y_n)_{n \in \mathbb{N}}$ is not a Cauchy sequence.

So there exists an $\epsilon > 0$ and the subsequences $\left(y_{m(k)}\right)_{m \in \mathbb{N}}$ and $\left(y_{n(k)}\right)_{n \in \mathbb{N}}$ of$(y_n)_{n \in \mathbb{N}}$ such that minimal $n(k)$ in the sense that $n(k) > m(k) > k$ and $\| y_{m(k)} - y_{n(k)} \| > \epsilon$.

Therefore

$$\| y_{m(k)} - y_{n(k)-1} \| \leq \epsilon.$$

By the triangular inequality, we have

$$\epsilon < \| y_{m(k)} - y_{n(k)} \|$$

$$\leq \| y_{m(k)} - y_{m(k)-1} \| + \| y_{m(k)-1} - y_{n(k)-1} \| + \| y_{n(k)-1} - y_{n(k)} \|$$

$$\leq \| y_{m(k)} - y_{m(k)-1} \| + \| y_{m(k)-1} - y_{m(k)} \| + \| y_{m(k)} - y_{n(k)-1} \| + \| y_{n(k)-1} - y_{n(k)} \|$$

$$\leq 2 \parallel y_{m(k)} - y_{m(k)-1} \parallel + \epsilon + \parallel y_{n(k)-1} - y_{n(k)} \parallel .$$

Letting $k \to \infty$ in the above inequality and using (2.3.6), we get

$$\lim_{k \longrightarrow \infty} \parallel y_{m(k)} - y_{n(k)} \parallel = \lim_{k \longrightarrow \infty} \parallel y_{m(k)-1} - y_{n(k)-1} \parallel = \epsilon. \tag{2.3.7}$$

From $(C3)$, we have

$$\parallel y_{m(k)-1} - y_{n(k)-1} \parallel = \parallel fx_{m(k)} - fx_{n(k)} \parallel$$

$$\geq \parallel gx_{m(k)} - gx_{n(k)} \parallel + \phi \parallel gx_{m(k)} - gx_{n(k)} \parallel$$
$$= \parallel y_{m(k)} - y_{n(k)} \parallel + \phi \left(\parallel y_{m(k)} - y_{n(k)} \parallel \right) .$$

Letting $k \to \infty$ and using (2.3.7), we get $\epsilon \geq \epsilon + \phi(\epsilon)$, which is a contradiction since $\phi(\epsilon) > \epsilon$.

Hence $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X.

Since X is complete, there exists a point $z \in E$ such that $\lim_{n \longrightarrow \infty} y_n = z$.

Therefore, by (2.3.4), we have

$$\lim_{n \longrightarrow \infty} y_n = \lim_{n \longrightarrow \infty} gx_n = \lim_{n \longrightarrow \infty} fx_{n+1} = z.$$

Suppose that f and g are compatible mappings.

Now, by the weak reciprocal continuity of f and g, we obtain

$$\lim_{n \longrightarrow \infty} fgx_n = fz \quad \text{or} \quad \lim_{n \longrightarrow \infty} gfx_n = gz.$$

Let $\lim_{n \longrightarrow \infty} fgx_n = fz$.

Then the compatibility of f and g gives

$$\lim_{n \longrightarrow \infty} \parallel fgx_n - gfx_n \parallel = 0.$$

Hence $\lim_{n \longrightarrow \infty} gfx_n = fz$.

Now, we claim that $fz = gz$.

Let $fz \neq gz$.

From (2.3.4), we get

$$\lim_{n \longrightarrow \infty} gfx_{n+1} = \lim_{n \longrightarrow \infty} ggx_n = fz.$$

Therefore from $(C3)$, we get

$$\| fz - fgx_n \| \geq \| gz - ggx_n \| + \phi\left(\| gz - ggx_n \|\right).$$

Letting $n \to \infty$, we get

$$\| fz - fz \| \geq \| gz - fz \| + \phi\left(\| gz - fz \|\right)$$

$$> 2 \| gz - fz \| .$$

which is a contradiction.

Hence $fz = gz$.

Again the compatibility of f and g implies the commutativity at a coincidence point.

Hence

$$gfz = fgz = ffz = ggz.$$

Using $(C3)$, we obtain

$$\| gz - ggz \| = \| fz - fgz \|$$

$$\geq \| gz - ggz \| + \phi\left(\| gz - ggz \|\right).$$

which implies that $gz = ggz$.

Also we get $gz = ggz = fgz$ and so gz is the common fixed point of f and g.

Next, suppose that $\lim\limits_{n\longrightarrow\infty} gfx_n = gz$.

Since $gE \subset fE$, there exists $u \in E$ such that $gz = fu$ and therefore $\lim\limits_{n\longrightarrow\infty} gfx_n = fu$.

The compatibility of f and g implies that $\lim\limits_{n\longrightarrow\infty} fgx_n = fu$.

Now, we claim that $fu = gu$.

Let $fu \neq gu$.

By virtue of(2.3.4), we have

$$\lim_{n\longrightarrow\infty} gfx_{n+1} = \lim_{n\longrightarrow\infty} ggx_n = fu.$$

From (C3), we have

$$\| fu - fgx_n \| \geq \| gu - ggx_n \| + \phi\left(\| gu - ggx_n \|\right).$$

Letting $n \to \infty$, we get

$$\| fu - fu \| \geq \| gu - fu \| + \phi\left(\| gu - fu \|\right).$$

Which is a contradiction, hence $fu = gu$.

Again the compatibility of f and g implies the commutativity at a coincidence point.

Hence

$$gfu = fgu = ffu = ggu.$$

Finally, using $(C3)$, we obtain

$$\begin{aligned} \| gu - ggu \| &= \| fu - fgu \| \\ &\geq \| gu - ggu \| + \phi\left(\| gu - ggu \|\right). \end{aligned}$$

Which implies that $gu = ggu$.

Also, we get $gu = ggu = fgu$ and so gu is the common fixed point of f and g.

For the uniqueness, let v and w , where $(v \neq w)$ be two common fixed points of f and g.

From $(C3)$, we have

$$\begin{aligned} \| v - w \| &= \| fv - fw \| \\ &\geq \| gv - gw \| + \phi\left(\| gv - gw \|\right) \\ &= \| v - w \| + \phi\left(\| v - w \|\right). \end{aligned}$$

Which implies that $v = w$.

Hence f and g have a unique common fixed point this completes the proof. ■

Chapitre 3

Applications

The aim of this chapter is to apply certain theorems which were given in second chapter : in the study of the existence and uniqueness for non linear integral equation with delay and non linear differential equation with delay.

3.1 Differential equations with delay

3.1.1 Definitions

Let $r > 0$ be a given real, we denote by $C([a,b], \mathbb{R}^d)$ the Banach space of continous functions defined over the interval $[a,b]$ with values in $\mathbb{R}^d$ with the topology of the convergence uniform.

For $[a,b] = [-r, 0]$ we pose

$$C_0 = C\left([-r,0], \mathbb{R}^d\right),$$

and we denote the norm of an element ϕ of C_0 by

$$\|\phi\| = \sup\{|\phi(\theta)| : \ -r \leq \theta \leq 0\}.$$

Where $\|.\|$ is a norm in $\mathbb{R}^d$.

Definition 3.1.1 *Let $t_0 \in \mathbb{R}$ and $L \geq 0$, let $x \in C\left([t_0 - r, t_0 + L], \mathbb{R}^d\right)$ and $t \in [t_0, t_0 + L]$, we define a new function x_t, an element of C_0, by*

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-r, 0].$$

Remark 1 *For any fixed* t*, the function* x_t *is obtained by considering the restriction of the function* x *on the interval* $[t-r,t]$*, translated on* $[-r,0]$.

Definition 3.1.2 *Let* U *be an open set of* $\mathbb{R}\times C_0$ *and* $f : U \longrightarrow \mathbb{R}^d$ *a continuous function.*

We called The functional differential equation with delay (EDFR) on U *a relation of the form*

$$\dot{x}(t) = f(t, x_t). \tag{3.1.1}$$

Where point " · "represents the right derivation.

Remark 2 *1. An application such as* f*, defined on a set of functions, is sometimes designated under the name of functional instead of function.*

2. The reference to equation (3.1.1)*, which is a functional differential equation, as being an* $(EDFR)$ *emphasizes the fact that only the present and the past of* x *intervene in the determination of* $\dot{x}$.

Definition 3.1.3 *Let* x *be a function of* $I\subset\mathbb{R}$ *in* $\mathbb{R}^d$.

1-We say that x *is a solution of equation* (3.1.1) *if there exists* $t_0\in\mathbb{R}$ *and* $L>0$*such that:*

$$x \in C\left([t_0 - r, t_0 + L[\,, \mathbb{R}^d\right), \qquad (t, x_t) \in U,$$

and x *verifies the relation* (3.1.1) *for all* $t\in[t_0,t_0+L[$.

2-For $t_0\in\mathbb{R}$ *and* $\phi\in C_0$ *given,* x *is said solution of the problem with initial value*

$$\dot{x} = f(t, x_t),\ t \geq t_0, \qquad x_{t_0} = \phi. \tag{3.1.2}$$

If there exists $L>0$ *such that* x *is a solution of* (3.1.1) *on* $[t_0-r,t_0+L[$ *and* $x_{t_0}=\phi$.

3-For $t_0\in\mathbb{R}$ *and* $\phi\in C_0$ *given, the solution of problem* (3.1.2) *is said to be unique if two Solutions coincide where they are simultaneously defined.*

3.2 Application to integral equation with delay

Consider the following nonlinear integral equation with delay,

$$x(t) = x^3(t-\tau) + \sigma(t)\,x(t-\tau) + p(t) + \int_{-\infty}^{t} f(t-s)\,g(x(s))\,ds, \tag{3.2.1}$$

where $\tau > 0$ is a constant, σ and p are continuous periodic functions on $\mathbb{R}$ with period $T > 0, f$ and g are continous functions on $\mathbb{R}$.

We make the following assumptions:

(H_1) $\sigma_0 := \inf_{t\in\mathbb{R}} \sigma(t) > 1$.

(H_2) $\int_{-\infty}^{t} |f(t-s)|\, ds < \infty$ and $\int_{-\infty}^{t} |f'(t-s)|\, ds < \infty,\ t \in \mathbb{R}$.

(H_3) there exists a $R > 0$ such that:

$$R^3 + \left(\sigma^0 - 1\right) R + \|P\| \geq M M_g,$$

where $\sigma^0 := \sup_{t\in\mathbb{R}} \sigma(t)$, $\|P\| := \sup_{t\in\mathbb{R}} |p(t)|$,

$M := \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} |f(t-s)|\, ds$, $M_g := \max_{|x|\leq R} |g(x)|$.

Let $E = \{x \in C(\mathbb{R}, \mathbb{R}) \,/\, x(t+T) = x(t),\ t \in \mathbb{R}\}$ be a Banach space with the supremum norm

$$\|x\| = \max_{t\in[0,T]} |x(t)|,$$

and $k = \{x \in E\ ;\ \|x\| \leq R\}$be a nonempty bounded and closed convex subset of E.

Theorem 3.2.1 *Suppose that* (H_1) *and* (H_3) *are satisfied, then the integral equation* (3.2.1) *has a* $T-$*periodic solution.*

Proof. We set

$$(Tx)(t) = x^3(t-\tau) + \sigma(t)\, x(t-\tau) + p(t),$$

and

$$(Sy)(t) = \int_{-\infty}^{t} f(t-s)\, g(x(s))\, ds.$$

Thus (3.2.1) is equivalent to the fixed point problem:

$$x = Tx + Sx.$$

Now, we check that all the hypotheses of theorem (2.2.1) are satisfied.

Obviously, T maps K into E.

For any $x, y \in K$, we get:

$$T : K \to E \quad \text{is expansive.}$$

$$\begin{aligned}
\|Tx-Ty\| &= \left\|x^3(t-\tau)+\sigma(t)x(t-\tau)+p(t)-y^3(t-\tau)+\sigma(t)y(t-\tau)+p(t)\right\| \\
&= \left\|\sigma(t)(x-y)+\left(x^3-y^3\right)\right\|,
\end{aligned}$$

we have two cases.

the first case, if $x \geq y$ so $x^3-y^3 \geq 0$ we obtien :

$$\begin{aligned}
\|Tx-Ty\| &\geq \|\sigma(t)(x-y)\| \\
&\geq \sup_{t\in\mathbb{R}}\sigma(t)\|x-y\| \\
&= \sigma^0\|x-y\|.
\end{aligned}$$

For the second case, if $x \leq y$, we get:

$$\begin{aligned}
\|Tx-Ty\| &= \left\|\sigma(t)(y-x)+\left(y^3-x^3\right)\right\| \\
&= \left\|-\left(\sigma(t)(x-y)+\left(x^3-y^3\right)\right)\right\| \\
&= \left\|\sigma(t)(x-y)+\left(x^3-y^3\right)\right\| \\
&\geq \|\sigma(t)(x-y)\| \\
&\geq \sigma^0\|x-y\|.
\end{aligned}$$

Next, if $y \in K$ then $\|y\| \leq R$ and

$$\begin{aligned}
\|Sy\| &\leq \left\|\int_{-\infty}^{t}|f(t-s)g(y(s))|\,ds\right\| \\
&\leq MM_g
\end{aligned} \tag{3.2.2}$$

and for $y \in E=\{y\in C(\mathbb{R},\mathbb{R})\,/\,y(t+T)=y(t),\ t\in\mathbb{R}\}$ so :

$$\begin{aligned}
(Sy)(t+T) &= \int_{-\infty}^{t+T} f(t+T-s)g(y(s))\,ds \\
&= \int_{-\infty}^{t} f(t-s)g(y(s))\,ds \\
&= (Sy)(t).
\end{aligned} \tag{3.2.3}$$

Remark 1 (***Libinitz integral rule***)*In calculus, Leibniz's rule for differentiation under the integral sign, named after Gottfried Leibniz, states that for an integral of the form:*

$$\int_{a(x)}^{b(x)} f(x,t)\,dt,$$

then for $-\infty < a(x), b(x) < +\infty$ *the derivative of this integral is expressible as:*

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x)).\frac{d}{dx}b(x) - f(x,a(x)).\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt.$$

Where the partial derivative indicates that inside the integral, only the variation of $f(.,t)$ with t is considered in taking the derivative.

Notice that if $a(x)$ and $b(x)$ are constants rather than functions of x, we have a special case of Leibniz's rule:

$$\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt.$$

Together with (3.2.2) and (3.2.3) illustrate that S maps K into E and $S(K)$ is uniformly bounded.

For $y \in K$, differentiating$(Sy)(t)$ with respect to t yields:

$$(Sy)'(t) = f(0)\,g(y(t)) + \int_{-\infty}^{t} f'(t-s)\,g(y(s))\,ds. \tag{3.2.4}$$

Now, we get by (3.2.4) that

$$\begin{aligned} \left\|(Sy)'(t)\right\| &= \left\| f(0)\,g(y(t)) + \int_{-\infty}^{t} f'(t-s)\,g(y(s))\,ds \right\| \\ &\leq \|f(0)\,g(y(t))\| + \left\| \int_{-\infty}^{t} f'(t-s)\,g(y(s))\,ds \right\| \\ &\leq f(0)\,M_g + M'M_g \\ &= (f(0)+M')\,M_g, \end{aligned} \tag{3.2.5}$$

where $M' := \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} |f'(t-s)|\,ds$.

The estimate (3.2.5) implies that $S(K)$ is an equicontinuous subset of E.

Then using the Arzela Ascoli Theorem, we obtain that S is a compact mapping.

In view of the supremum norm and the continuity of g, it is easy to see that S is continuous.

Therefore, $S : K \longrightarrow E$ is completely continuous, i.e, (i) of theorem (2.2.1) is satisfied.

Finally, it remains to check that (iii) of theorem (2.2.1) is also satisfied.

Fixing any

$$z_0(t) = (Sx_0)(t) \in S(K), \quad \text{for each } \ x \in K,$$

we have to seek $y \in K$ such that:

$$(Ty)(t) + z_0(t) = x(t). \tag{3.2.6}$$

Assume that (3.2.6) holds, we then deduce that

$$|(Ty)(t)| \leq \|x\| + \|z_0\| \leq R + MM_g. \tag{3.2.7}$$

From the expression of T, for each $x \in K$, we have

$$|(Ty)(t)| \leq R^3 + \sigma^0 R + \|p\|. \tag{3.2.8}$$

From (H3), (3.2.7) and (3.2.8), we obtain that

$$x(t) - z_0(t) \in T(K).$$

Therefore, there exists $y \in K$ solving (3.2.6), i.e, (iii) of theorem (2.2.1) holds.

Hence the Eq (3.2.1) has a T -periodic solution. ■

Corollary 3.2.2 *Suppose that g is continuous and bounded, and (H_1) and (H_2) hold, then the Eq (3.2.1) has a T -periodic solution.*

By the same token, we can prove the following result.

We modify the above assumptions as follows:

(H_1') *there exists an $r > 0$ such that $r^2 + \sigma^0 \geq 2$, where $\sigma^0 := \sup\limits_{t\in\mathbb{R}} \sigma(t)$, and*

$$|x| \leq r \Longrightarrow |g(x)| \leq r - \|p\| \quad \textit{where} \quad \|p\| := \sup_{t\in\mathbb{R}} |p(t)|.$$

(H_2')

$$\int_{-\infty}^{t} |f(t-s)|\, ds \leq 1 \qquad \textit{and} \qquad \int_{-\infty}^{t} |f'(t-s)|\, ds < \infty, \ t \in \mathbb{R}.$$

Theorem 3.2.3 *Suppose that (H_1), (H_1') and (H_2') are satisfied, then the integral equation (3.2.1) has a T -periodic solution.*

Proof. We set

$$(Tx)(t) = x^3(t-\tau) + \sigma(t)\, x(t-\tau),$$

and

$$|(Sy)(t)| = p(t) + \int_{-\infty}^{t} f(t-s)\, g(y(s))\, ds.$$

Thus (3.2.1) is equivalent to the fixed point problem:

$$x = Tx + Sx.$$

We also verify that all the hypotheses of theorem (2.2.1) are satisfied.

First, it is obvious that T maps K into E and is expansive, here $K = \{x \in E \mid \|x\| \leq r\}$.

Secondly, if $y \in K$ then $\|y\| \leq r$ and from (H_2'), we get

$$\begin{aligned} \|Sy\| &= \left\| p(t) + \int_{-\infty}^{t} f(t-s)\, g(y(s))\, ds \right\| \\ &\leq \|p(t)\| + \left\| \int_{-\infty}^{t} f(t-s)\, g(y(s))\, ds \right\| \\ &\leq \|p\| + (r - \|p\|) \\ &= r, \end{aligned} \tag{3.2.9}$$

and a change of variable yields that

$$\begin{aligned} (Sy)(t+T) &= p(t+T) + \int_{-\infty}^{t+T} f(t+T-s)\, g(y(s))\, ds \\ &= p(t) + \int_{-\infty}^{t} f(t-s)\, g(y(s))\, ds \\ &= (Sy)(t), \end{aligned}$$

further more

$$\begin{aligned} \left\|(Sy)'(t)\right\| &= \left\| p'(t) + f(0)\, g(y(t)) + \int_{-\infty}^{t} f'(t-s)\, g(y(s))\, ds \right\| \\ &\leq \|p'\| + \|f(0)\, g(y(t))\| + \left\| \int_{-\infty}^{t} f'(t-s)\, g(y(s))\, ds \right\| \\ &\leq \|p'\| + f(0)(r - \|p\|) + M'(r - \|p\|). \end{aligned}$$

So $S : K \longrightarrow K$, it is not difficult to check that S maps K into an equicontinuous subset of E.

The continuity of S on K is also readily verified.

Therefore, $S : K \longrightarrow K$ is completely continuous,

thus (i) of theorem (2.2.1) holds.

Finally, it is left to check that (iii) of theorem (2.2.1) is satisfied.

Fixing any

$$z_0(t) = (Sx_0)(t) \in S(K), \quad \text{for each } \ x \in K,$$

we have to seek $y \in K$ such that:

$$(Ty)(t) + z_0(t) = x(t). \tag{3.2.10}$$

From $\left(H_1'\right)$, this implies that there exists $y \in K$ solving (3.2.10), i.e, (iii) of theorem (2.2.1) holds.

Hence, the Eq (3.2.1) has a T -periodic solution. ■

Corollary 3.2.4 *Suppose that g is continuous and bounded, and (H_1) and $\left(H_2'\right)$ hold.*

Then the Eq (3.2.1) has a T -periodic solution.

3.3 Application to differential equations with delay

Considers the following differential equation:

$$x'(t) = a(t)\,x(t) + g(t,x). \tag{3.3.1}$$

Define $P_T = \{\varphi \in C(\mathbb{R},\mathbb{R}) : \varphi(t+T) = \varphi(t)\}$, where $C(\mathbb{R},\mathbb{R})$ is the space of all real valued continuous functions on $\mathbb{R}$.

Then P_T is a Banach space when endowed with the supremum norm

$$\|x(t)\| = \sup_{t\in[0,T]} |x(t)| = \sup_{t\in\mathbb{R}} |x(t)|$$

The next Lemma is essential for the construction of our mapping that is required for the application of theorem (2.2.1).

To have a well behaved mapping we must assume that

$$\int_0^t a(s)\,ds < \infty.$$

Lemma 3.3.1 *If $x \in P_T$, then $x(t)$ is a solution of equation (3.3.1) if and only if*

$$x(t) = x(t)\, e^{\int_{t-T}^{t} a(s)ds} + \int_{t-T}^{t} g(u, x(u))\, e^{\int_{u}^{t} a(s)ds} du. \tag{3.3.2}$$

Proof. Let $x \in P_T$ be a solution of (3.3.1).

We rewrite (3.3.1) in the form

$$x'(t) - a(t)\, x(t) = g(t, x).$$

Next we multiply both sides of the resulting equation with $e^{-\int_0^t a(s)ds}$,

$$x'(t)\, e^{-\int_0^t a(s)ds} - a(t)\, x(t)\, e^{-\int_0^t a(s)ds} = g(t, x(t))\, e^{-\int_0^t a(s)ds}.$$

Note that,

$$\left(x(t)\, e^{-\int_0^t a(s)ds}\right)' = g(t, x(t))\, e^{-\int_0^t a(s)ds}.$$

Now, we integrate from $t-T$ to t get:

$$\int_{t-T}^{t} \left(x(u)\, e^{-\int_0^u a(s)ds}\right)' du = \int_{t-T}^{t} g(u, x(u))\, e^{-\int_0^u a(s)ds} du.$$

So

$$x(t)\, e^{-\int_0^t a(s)ds} - x(t-T)\, e^{-\int_0^{t-T} a(s)ds} = \int_{t-T}^{t} g(u, x(u))\, e^{-\int_0^u a(s)ds} du.$$

Using the fact that $x(t+T) = x(t)$ We get:

$$x(t) = x(t)\, e^{\int_{t-T}^{t} a(s)ds} + \int_{t-T}^{t} g(u, x(u))\, e^{\int_{u}^{t} a(s)ds} du.$$

We can verify that $x(t)$ given by (3.3.2) satisfies Eq(3.3.1), Since each step in the above work is reversible, the proof is complete. ■

First we note that for $T \in [0, t]$ and $u \in [t-T, t]$ we have

$$e^{\int_u^t a(s)ds} \leq e^{\left|\int_0^t a(s)ds\right|} \leq \sup_{t \in \mathbb{R}} e^{\left|\int_0^t a(s)ds\right|} := \alpha. \tag{3.3.3}$$

Let J be a positive constant, define the set $M_j = \{x \in P_T : \|x(t)\| \leq j\}$

Obviously, the set M_j is a bounded and convex subset of the Banach space P_T .

Let the mapping $A : M_j \longrightarrow P_T$ be defined by:

$$(Ax)(t) = \int_{t-T}^{t} g(u, x(u))\, e^{\int_u^t a(s)ds} du, \quad t \in \mathbb{R}. \tag{3.3.4}$$

Similarly we define the mapping $B : M_j \longrightarrow P_T$ by:

$$(By)(t) = y(t)\, e^{\int_{t-T}^{t} a(s)ds}, \quad t \in \mathbb{R}. \tag{3.3.5}$$

It is clear from (3.3.4) and (3.3.5) that Ax and By are T -periodic in t.

Lemma 3.3.2 *Suppose there exists a function $\xi \in P_T$ such that:*

$$|g(t, x(t))| \leq \xi(t) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in M_j. \tag{3.3.6}$$

Then the mapping A, defined by (3.3.4), is continuous in $x \in M_j$.

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of functions in M_j such that $x_i \longrightarrow x$ as $i \to \infty$.

By (3.3.3),(3.3.6), and the the continuity of g, the dominated convergence theorem yields,

$$\begin{aligned}
\lim_{i\to\infty} \|Ax_i - Ax\| &= \lim_{i\to\infty} \left\| e^{\int_u^t a(s)ds} \left(\int_{t-T}^{t} g(u, x_i(u)) - g(u, x(u))\, du \right) \right\| \\
&\leq \alpha \lim_{i\to\infty} \int_{t-T}^{t} |g(u, x_i(u)) - g(u, x(u))|\, du \\
&\leq \alpha \int_{t-T}^{t} \lim_{i\to\infty} |g(u, x_i(u)) - g(u, x(u))|\, du \\
&= 0.
\end{aligned}$$

This shows the continuity of the mapping A. ■

In the next example, we display such a function satisfying (3.3.6).

Example 3.3.1 *If we assume that $g(t,x)$satisfies the Lipschitz condition in x, i.e., there is a positive constant k such that:*

$$|g(t, z) - g(t, w)| \leq k\,|z - w| \quad \text{for } z, w \in P_T, \tag{3.3.7}$$

then for $x \in P_T$, we obtain the following

$$\begin{aligned}
|g(t, x(t))| &= |g(t, x(t)) - g(t, 0) + g(t, 0)| \\
&\leq |g(t, x(t)) - g(t, 0)| + |g(t, 0)| \\
&\leq kj + |g(t, 0)|.
\end{aligned}$$

In this case we may choose ξ as

$$\xi(t) = kj + |g(t,0)|.$$

In the next two results we assume that for all $t \in \mathbb{R}$ and $x \in M_j$.

$$\int_{t-T}^{t} \xi(u)\, e^{\int_u^t a(s)ds} du \leq j\ , \qquad \forall t \in \mathbb{R}. \tag{3.3.8}$$

Lemma 3.3.3 *Suppose (3.3.6) and (3.3.8), then A is continuous in $x \in M_j$ and maps M_j into a compact subset of M_j.*

Proof. Let $x \in M_j$, continuity of A for $x \in M_j$ follows from Lemma (2.2.1).

Now, by (3.3.6) and (3.3.8) we have

$$\begin{aligned} |(Ax)(t)| &= \left| \int_{t-T}^{t} g(u, x(u))\, e^{\int_u^t a(s)ds} du \right| \\ &\leq \left| \int_{t-T}^{t} \xi(u)\, e^{\int_u^t a(s)ds} du \right| \\ &< J. \end{aligned}$$

Thus, $Ax \in M_j$.

Let $x_i \in M_j$,$i = 1, 2, ...$ then from the above discussion we conclude that

$$\|Ax_i\| \leq j\ , \quad i = 1, 2, ...$$

This shows $A(M_j)$ is uniformly bounded.

It remins to show that $A(M_j)$ is equicontinuous.

Since ξ is continuous and $T-$periodic, (3.3.6) and the differentiation of (3.3.4) with respect to t yield

$$\begin{aligned} \left|(Ax_i)'(t)\right| &= \left| g(t, x(t)) \left(1 - e^{\int_{t-T}^t a(s)ds}\right) - a(t) \int_{t-T}^{t} g(u, x_i(u))\, e^{\int_u^t a(s)ds} du \right| \\ &\leq \xi(t) \left(1 - e^{\int_{t-T}^t a(s)ds}\right) + \left| a(t) \int_{t-T}^{t} g(u, x_i(u))\, e^{\int_u^t a(s)ds} du \right| \\ &\leq N\xi(t) + \|a\| \|Ax_i\| \\ &\leq L, \end{aligned}$$

for some positive constant L.

Thus the estimation on $(Ax_i)'(t)$ implies that $A(M_j)$ is equicontinuous, Then the Arzela–Ascoli theorem yields the compactness of the mapping A. ■

Lemma 3.3.4 *suppose* $h = \inf\limits_{t\in[0,T]} e^{\int_{t-T}^{t} a(s)ds}$ *where* $h > 1$ *so* B *is expansive.*

Proof. $\forall x, y \in M_j$:

$$\begin{aligned} \|(Bx)(t) - (By)(t)\| &= \left\| x(t)\, e^{\int_{t-T}^{t} a(s)ds} - y(t)\, e^{\int_{t-T}^{t} a(s)ds} \right\| \\ &= \left| e^{\int_{t-T}^{t} a(s)ds} \right| \|x(t) - y(t)\| \\ &\geq h \|x(t) - y(t)\| . \end{aligned}$$

So B expansive . ■

Then equation (3.3.1) can be put in the form

$$x(t) = Ax(t) + Bx(t),$$

with B is a expansive of constant $h > 1$ and A is a compact mapping.

Now, from (iii) of theorem (2.2.1), fixing any $z_0(t) = (Ax_0)(t) \in A(M_j)$, for each $x \in M_j$.

We have to seek $y \in M_j$ such that:

$$By(t) + z_0(t) = x(t). \tag{3.3.9}$$

Assume that (3.3.9) holds, we then deduce that:

$$\begin{aligned} |By(t)| &\leq \|x\| + \|z_0\| \\ &\leq j + j = 2j. \end{aligned} \tag{3.3.10}$$

From(3.3.10), we obtain that:

$$x(t) - z_0(t) \in B(M_j).$$

Therefore, there exists $y \in M_j$ solving (3.3.9) for each $x \in M_j$.

So, $M_j \subset T(M_j) + z$, i.e (iii) of theorem (2.2.1) holds.

Hence, the equation (3.3.1) has a $T-$periodic solution.

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الملخص

في هذا العمل، قمنا بدراسة مجموعة من النظريات تتعلق بالنقطة الثابتة وميزنا ثلاث حالات وهي: نظريات النقطة الثابتة للتطبيقات الموسعة، ونظريات النقطة الثابتة لمجموع تطبيقين أحدهما موسع، بالإضافة إلى نظريات النقطة الثابتة المشتركة للتطبيقات التوسيعية. وفي الأخير طبقنا إحدى هاته النظريات على المعادلات التفاضلية بتأخر والمعادلات التكاملية بتأخر لإثبات وجود ووحدانية الحل.

كلمات مفتاحية: فضاء بناخ - النقطة الثابتة - التطبيقات التوسيعية - التطبيقات المقلصة.

Abstract

In this work, we have studied a some of theorems related to fixed point and we have distinguished three cases: fixed point theorems for expansive mappings, and fixed point theorems for a sum of two mappings, one of which is expansive. in addition, common fixed point theorems for expansion mappings. In the latter, we applied one of these theorems on differential equations with a delay and integral equations with a delay to prove the existence and uniqueness of solution.

Keywords: Banach space - fixed point - expansive mappings - contraction mappings.

Résumé

Dans ce travail, nous avons étudié un ensemble de théories liées au point fixe et nous avons distingué trois cas : théorèmes de points fixes pour des applications expansive, et des théorèmes de points fixes pour un somme de deux applications, dont l'une est expansive. En plus des théories communes des points fixes pour les applications d'expansion. Dans ce dernier cas, nous avons appliqué une de ces théories sur les équations différentielles retardées et les équations intégrales retardées pour prouver l'existence et l'unité de la solution.

Mots clés : Espace de Banach - point fixe - applications expansive - applications contraction.