People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Echahid Hamma Lakhdar University of El-Oued

Faculty of Technology

Domain: Science and Technology

Department: Hydraulic and Civil Engineering

Level: Bachelor’s Degree - Year 2 (Semester 3)

Course material for teaching the module

Waves and Vibrations

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Academic Year : 2025/2026

Abstract

This course material presents the fundamental principles of waves and vibrations for undergraduate students in engineering and applied sciences. The objective of the course is to develop a solid theoretical and analytical understanding of oscillatory systems and wave propagation phenomena commonly encountered in physical and engineering applications.

The material begins with an introduction to the Lagrangian formalism, providing the theoretical framework required to derive equations of motion for mechanical systems with single and multiple degrees of freedom. It then explores free and forced oscillations, including undamped and damped systems, resonance phenomena, and steady-state responses.

The course further investigates multi-degree-of-freedom systems, modal analysis, and coupled oscillators. These concepts are extended to the study of wave propagation, where the wave equation is derived and analyzed in one-dimensional and three-dimensional media.

Applications to vibrating strings, elastic waves in solids, and acoustic waves in fluids are presented, emphasizing practical engineering contexts such as structural dynamics, mechanical vibrations, and sound propagation.

Overall, this course provides students with the mathematical tools and physical insight necessary to analyze oscillatory systems and wave phenomena in engineering and scientific disciplines.

Keywords

Waves, Vibrations, Oscillations, Lagrangian Mechanics, Wave Propagation, Resonance, Harmonic Oscillator, Mechanical Systems, Elastic Waves, Acoustics, Engineering Physics.

Résumé

Ce support de cours présente les principes fondamentaux des ondes et vibrations destinés aux étudiants de licence en sciences et ingénierie. L'objectif principal est de fournir une compréhension théorique et analytique des systèmes oscillatoires et des phénomènes de propagation des ondes rencontrés en physique et en ingénierie.

Le cours commence par l'introduction du formalisme lagrangien, qui constitue un cadre mathématique permettant d'établir les équations du mouvement pour les systèmes mécaniques à un ou plusieurs degrés de liberté. Il aborde ensuite l'étude des oscillations libres et forcées, incluant les systèmes amortis et non amortis, les phénomènes de résonance et les régimes permanents.

Le contenu s'étend également à l'analyse des systèmes à plusieurs degrés de liberté, aux modes propres et aux oscillateurs couplés. Ces concepts sont ensuite appliqués à l'étude de la propagation des ondes, avec la dérivation de l'équation des ondes et son analyse dans différents milieux.

Enfin, le cours traite des cordes vibrantes, des ondes élastiques dans les solides et des ondes acoustiques dans les fluides, tout en mettant l'accent sur leurs applications en dynamique des structures et en ingénierie.

Mots-clés

Ondes, Vibrations, Oscillations, Mécanique Lagrangienne, Propagation des Ondes, Résonance, Oscillateur Harmonique, Physique Appliquée, Ondes Élastiques, Acoustique.

الملخص

يقدم هذا المقرر المبادئ الأساسية لـ الموجات والاهتزازات لطلبة المرحلة الجامعية في مجالات الهندسة والعلوم التطبيقية. يهدف المقرر إلى تزويد الطلبة بفهم نظري وتحليلي للأنظمة الاهتزازية وظواهر انتشار الموجات في الأنظمة الفيزيائية والهندسية.

يبدأ المقرر بتقديم الصياغة اللاغرانجية التي تمثل الإطار الرياضي لاشتقاق معادلات الحركة للأنظمة الميكانيكية ذات درجة حرية واحدة أو عدة درجات حرية. كما يتناول دراسة الاهتزازات الحرة والقسرية بما في ذلك الأنظمة المخمدة وغير المخمدة، وظاهرة الرنين والاستجابة في الحالة المستقرة.

كما يتم دراسة الأنظمة متعددة درجات الحرية وتحليل الأنماط الطبيعية للاهتزازات والأنظمة المقترنة. بعد ذلك ينتقل المقرر إلى دراسة انتشار الموجات من خلال اشتقاق معادلة الموجة وتحليلها في الأوساط المختلفة.

ويتضمن المقرر تطبيقات على الأوتار المهتزة والموجات المرنة في المواد الصلبة والموجات الصوتية في الموائع مع التركيز على التطبيقات الهندسية مثل ديناميكا المنشآت وانتشار الصوت.

الكلمات المفتاحية

الموجات، الاهتزازات، التذبذبات، ميكانيكا لاغرانج، انتشار الموجات، الرنين، المذبذب التوافقي، الموجات المرنة، الصوتيات، الفيزياء الهندسية.

Preface

The study of ***waves and vibrations*** is fundamental to understanding a wide range of physical and engineering phenomena—from the motion of mechanical structures to the propagation of sound and elastic waves. This course is designed to provide students with a strong foundation in the theoretical and practical aspects of oscillatory systems and wave behavior.

The course is structured into nine comprehensive chapters, each with specific objectives that build progressively toward a deeper understanding:

- **Chapter 1** introduces the **Lagrangian formalism**, enabling the formulation of equations of motion for mechanical systems with one or more degrees of freedom.
- **Chapter 2** explores **free oscillations** of single-degree-of-freedom systems, both undamped and damped, analyzing their natural response and energy characteristics.
- **Chapter 3** focuses on **forced oscillations**, introducing mechanical impedance, resonance, and the steady-state behavior under harmonic excitation.
- **Chapter 4** extends the analysis to **free oscillations in systems with two degrees of freedom**, discussing mode shapes and natural frequencies.
- **Chapter 5** covers **forced oscillations in two-degree-of-freedom systems**, with emphasis on resonance, anti-resonance, and coupled responses.
- **Chapter 6** presents general principles of **wave propagation**, deriving the wave equation and discussing group and phase velocity.
- **Chapter 7** applies wave theory to **vibrating strings**, analyzing standing waves, harmonic modes, and wave reflection/transmission.
- **Chapter 8** investigates **elastic waves in solids**, covering both longitudinal and transverse wave propagation in continuous media.
- **Chapter 9** concludes with **acoustic waves in fluids**, introducing sound propagation, acoustic impedance, energy transport, and the Doppler effect.

This course is intended for undergraduate students in physics, engineering, and applied sciences. A basic understanding of Newtonian mechanics and differential equations is assumed. The material aims to combine rigorous theory with practical insight, supported by examples, exercises, and illustrative applications

Contents

Chapter 1: Introduction to the Lagrangian Formalism

1.1 The Case of a Single Particle

1.1.1 Fundamentals of Lagrange’s Equations

Consider a particle constrained to move without friction along a curve that lies in the xOy plane. The constraints defining this curve reduce the particle's accessible space, limiting its motion to a surface described mathematically by two equations—one for the plane and another for the trajectory. Since these constraints restrict two of the three spatial degrees of freedom, the particle is left with only one degree of freedom, which we describe using a generalized coordinate q.

The particle's position vector $\vec{r}$ is thus expressed as a function of q, and its motion is governed by Newton’s second law. The infinitesimal work done by a force $\vec{F}$ during a small displacement $\delta\vec{r}$ can be related to the generalized coordinate variation δq by:

$$\delta\vec{r} = \frac{\partial\vec{r}}{\partial q}\delta q \Rightarrow \delta W = \vec{F}\cdot\frac{\partial\vec{r}}{\partial q}\delta q \tag{1.1}$$

This leads us to define the generalized force:

$$F_q = \frac{\delta W}{\delta q} = \vec{F}\cdot\frac{\partial\vec{r}}{\partial q} \tag{1.2}$$

Through kinetic energy expressions and vector calculus, one arrives at the Lagrange equation for a single-degree-of-freedom system:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}}\right) - \frac{\partial T}{\partial q} = F_q \tag{1.3}$$

Where $T = \frac{1}{2}mv^2$ is the kinetic energy of the particle.

1.1.2 Conservative Systems

If the forces acting on the system originate from a potential energy function $U(q)$, then $F_q = -\frac{\partial U}{\partial q}$.

Substituting this into the previous relation, we get:

$$\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial \dot{q}}\right) - \frac{\partial(T-U)}{\partial q} = 0 \qquad (1.4)$$

This introduces the Lagrangian function, defined as:

$$L = T - U \qquad (1.5)$$

and yields the well-known Lagrange equation for conservative systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad (1.6)$$

1.1.3 Velocity-Dependent Frictional Forces

When the particle is subject to viscous damping, the frictional force takes the form $\vec{f} = -\alpha\vec{v}$. This results in a generalized force:

$$F_q = -\beta\dot{q}, \text{ with } \quad \beta = \alpha\left(\frac{\partial\vec{r}}{\partial q}\right)^2 \qquad (1.7)$$

The Lagrange equation now includes this non-conservative force:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = -\beta q \qquad (1.8)$$

To handle energy dissipation analytically, we define the dissipation function:

$$D = \frac{1}{2}\beta\dot{q}^2 \tag{1.9}$$

Then the equation becomes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0 \tag{1.10}$$

1.1.4 Time-Dependent External Forces

In the presence of an external force that explicitly depends on time, and also of viscous damping, the full expression of the Lagrangian equation becomes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = F_{\text{ext}}(t) \tag{1.11}$$

This generalized form accommodates time-varying and dissipative interactions within the system.

1.2 Systems with Multiple Degrees of Freedom

For a system characterized by several generalized coordinates $q_1, q_2, \ldots, q_N$, the dynamics are captured by a set of N Lagrange equations, one for each coordinate:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = F_{\text{ext},q_i} \quad \text{for } i = 1, 2, \ldots, N \tag{1.12}$$

Each equation corresponds to the balance of energy and force in the direction of a particular generalized coordinate. The right-hand term captures the generalized external force associated with each q_i , and can be interpreted as the virtual work done by external influences when varying only q_i .

Exercises

Exercise 1 : Particle on a Circular Path

A particle of mass m moves without friction on a circular ring of radius R in the horizontal plane.

1. Express the constraint equation.
2. Identify the number of degrees of freedom.
3. Choose an appropriate generalized coordinate and write the kinetic energy.
4. Using the Lagrangian formalism, derive the equation of motion.

Exercise 2: Simple Pendulum

A mass m is suspended from a point O by a massless rigid rod of length ℓ. The pendulum oscillates in a vertical plane under gravity.

1. Determine the number of degrees of freedom.
2. Choose a suitable generalized coordinate and express the kinetic and potential energy.
3. Derive the equation of motion using Lagrange's equation.

Exercise 3: Particle in a Viscous Medium

A particle of mass m moves along a straight horizontal line in a fluid that exerts a damping force proportional to velocity, $\vec{f} = -\alpha\vec{v}$.

1. Write the Lagrangian and dissipation function.
2. Derive the Lagrange equation including the damping force.
3. Discuss the nature of the motion for different damping coefficients.

Exercise 4: Spring-Mass System

A mass M is attached to a horizontal spring with stiffness k, fixed at one end. The mass moves on a frictionless surface.

1. Identify the generalized coordinate.
2. Derive the kinetic and potential energies.
3. Obtain the Lagrangian and the equation of motion.

Exercise 5: External Time-Dependent Force

Consider the system in Exercise 4, but now the mass is subjected to an external force $F(t) = F_0 \cos(\omega t)$

1. Modify the Lagrangian to include the external force.
2. Use Lagrange's equation to derive the differential equation of motion.
3. Comment on the physical meaning of each term.

Exercise 6: Two-Degree-of-Freedom System

Two identical masses m are connected by three identical springs of stiffness k, placed in a line. The outer springs are fixed to walls. Assume horizontal motion only.

1. Define appropriate generalized coordinates.
2. Write expressions for kinetic and potential energies.
3. Derive the two coupled Lagrange equations.

Chapter 2: Free Oscillations of Single-Degree-of-Freedom Systems

2.1 Introduction

Mechanical systems often experience oscillatory motion when disturbed from an equilibrium position. In the absence of external forces and dissipation, these are termed free undamped oscillations. This chapter focuses on the theoretical foundations of such systems when modeled with one generalized coordinate—i.e., one degree of freedom (DOF).

Examples include:

- A mass attached to a linear spring sliding without friction.
- A simple pendulum undergoing small-angle oscillations.
- Vibrational modes of mechanical components such as beams or bars constrained in simple ways.

The objective is to derive the equations governing these motions, understand energy distribution, and characterize their natural frequency and amplitude behavior.

2.2 Free Undamped Oscillations

2.2.1 The Linear Harmonic Oscillator

Let us consider a particle of mass m connected to a spring with stiffness k, constrained to move along a line (say the x-axis). Its displacement from equilibrium is denoted by $q(t)$.

Applying Newton’s Second Law:

$$m\ddot{q} = -kq \quad \Rightarrow \quad \ddot{q} + \omega_0^2 q = 0 \tag{2.1}$$

Where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2.2}$$

This is the classical **harmonic oscillator** differential equation, whose general solution is:

$$q(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{2.3}$$

Or, in amplitude-phase form:

$$q(t) = C\cos(\omega_0 t + \phi) \tag{2.4}$$

Where:

- $C = \sqrt{A^2 + B^2}$ is the amplitude,
- ϕ is the **initial phase**, determined by initial conditions.

2.3 Kinetic and Potential Energy in Oscillations

A key property of undamped oscillators is conservation of mechanical energy. The total energy is the sum of kinetic and potential components:

2.3.1 Kinetic Energy:

$$T = \frac{1}{2}m\dot{q}^2 \tag{2.5}$$

This depends on the velocity of the particle.

2.3.2 Potential Energy:

$$U = \frac{1}{2}kq^2 \tag{2.6}$$

This is derived from the work done by the restoring spring force.

2.3.3 Total Mechanical Energy:

$$E = T + U = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 = \text{constant} \tag{2.7}$$

Throughout the motion, energy oscillates between kinetic and potential forms but the total remains fixed.

2.3.4 Generalized Coordinates and the Lagrangian Approach

In more abstract settings, we use a generalized coordinate q to describe the system. The Lagrangian is defined as:

$$L(q, \dot{q}) = T - U \tag{2.8}$$

For small oscillations around equilibrium, the kinetic and potential energies are approximated as:

$$T = \frac{1}{2}a(q)\dot{q}^2 \approx \frac{1}{2}a_0\dot{q}^2 \quad \text{and} \quad U = \frac{1}{2}b(q)q^2 \approx \frac{1}{2}b_0 q^2 \tag{2.9}$$

Where a_0 and b_0 are constants evaluated at the equilibrium position $q = 0$. Applying Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{2.10}$$

$$\Rightarrow a_0\ddot{q} + b_0 q = 0 \quad \Rightarrow \quad \ddot{q} + \omega_0^2 q = 0, \quad \text{with} \quad \omega_0 = \sqrt{\frac{b_0}{a_0}} \tag{2.11}$$

2.3.5 Characteristics of Free Undamped Oscillations

Period and Frequency:

a. **Period:** $T = \dfrac{2\pi}{\omega_0}$

b. **Frequency:** $f = \dfrac{1}{T} = \dfrac{\omega_0}{2\pi}$

- f : frequency in hertz (Hz)
- T : period in seconds (s)
- ω_0 : natural angular frequency in radians per second (rad/s)

Amplitude:

The amplitude C is determined by initial conditions $q(0)$ and $\dot{q}(0)$:

$$C = q(0)^2 + \left(\frac{\dot{q}(0)}{\omega_0}\right)^2 \tag{2.12}$$

Phase:

$$\phi = \arctan\left(-\frac{\omega_0 q(0)}{\dot{q}(0)}\right) \tag{2.13}$$

Velocity and Acceleration:

$$\dot{q}(t) = -C\omega_0 \sin(\omega_0 t + \phi), \quad \ddot{q}(t) = -C\omega_0^2 \cos(\omega_0 t + \phi) \tag{2.14}$$

These oscillations are **sinusoidal, periodic**, and **constant in amplitude.**

2.4 Damped Oscillations

In real systems, energy loss occurs due to friction or resistance (air, internal material damping, etc.). This is modeled as a damping force proportional to velocity:

$$F_d = -c\dot{q} \Rightarrow \ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0 \tag{2.15}$$

Where:

- $\delta = \dfrac{c}{2m}$ is the **damping factor**.

2.4.1 Solution Based on Damping Level

a) Overdamped System: $\delta > \omega_0$

No oscillation. The system returns slowly to equilibrium without crossing it.

$$q(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} \tag{2.16}$$

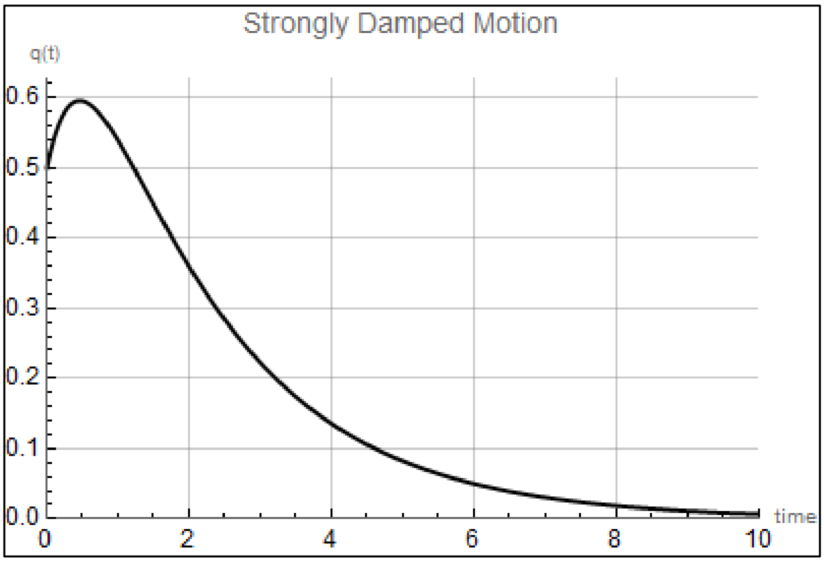


Figure 2.1: *Strong damping: evolution of* $q(t)$ *over time.*

b) Critically Damped System: $\delta = \omega_0$

Fastest return to equilibrium without oscillating.

$$q(t) = (A + Bt)e^{-\delta t} \tag{2.17}$$

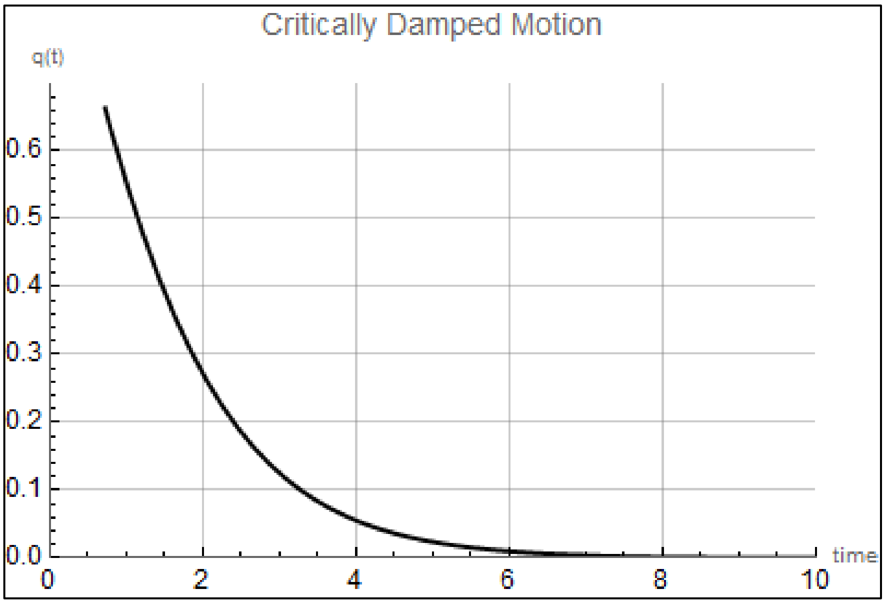


Figure 2.2: *Critical damping: evolution of* $q(t)$ *over time.*

c) Underdamped System: $\delta < \omega_0$

$$q(t) = Ce^{-\delta t}\cos(\omega_d t + \phi), \quad \text{where} \quad \omega_d = \sqrt{\omega_0^2 - \delta^2} \tag{2.18}$$

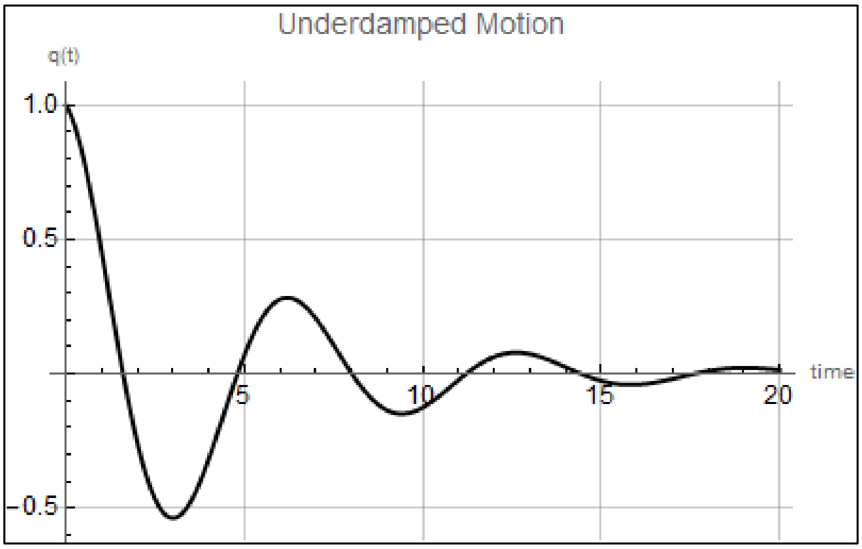


Figure 2.3: *Underdamped system: variation of* $q(t)$ *over time.*

Properties:

- Oscillation frequency decreases to ω_d ,

- Amplitude decays exponentially,
- Energy loss occurs at a rate $E(t)=E_0e^{-2\delta t}$.

2.4.2 Quality Factor and Logarithmic Decrement

- **Quality Factor (Q) :**

$$Q=\frac{2\delta}{\omega_0} \tag{2.19}$$

Describes the sharpness of resonance and damping effect.

- **Logarithmic Decrement (δ) :**

$$\Lambda=\ln\left(\frac{q(t+T_d)}{q(t)}\right)=\frac{2\pi\delta}{\omega_d} \tag{2.20}$$

Where $T_d=\dfrac{2\pi}{\omega_d}$ is the damped period.

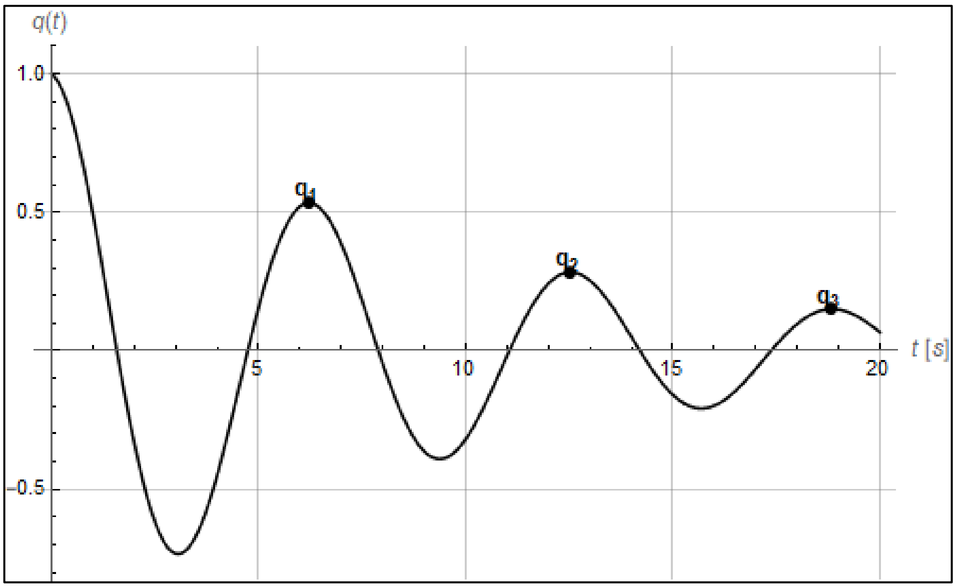


Figure 2.4: *Underdamped system: evolution of $q(t)$ over time with labelled successive peaks q_1, q_2, q_3 illustrating logarithmic decrement*

❖ Real-World Examples

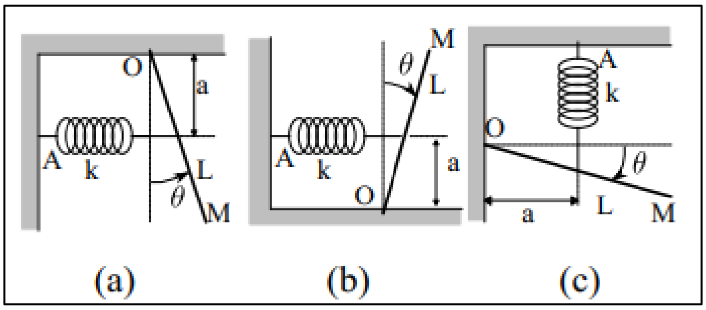
- A car’s suspension system should be underdamped for comfort but not underdamped enough to allow excessive bouncing.
- A door damper is typically critically damped to close quickly without slamming.
- Seismographs are designed to avoid resonance with earthquake frequencies.

Exercises

Exercise 1:

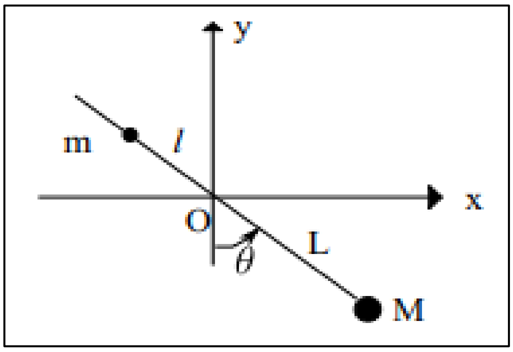
In the figures below, a homogeneous rod of mass M and length L oscillates without friction, in a vertical plane, around a fixed axis perpendicular to the plane of motion at point O.

1. What is the deformation of the spring at equilibrium, knowing that at this position $\theta = 0$?
2. Establish the differential equation of motion in the case of small-amplitude oscillations.
3. Under what condition can the system in figure (b) oscillate? What is the nature of the motion when this condition is not satisfied?



Exercice 02:

A metronome is schematized in the figure below. The mass M is welded to the end of the rod. The position of the mass m on the rod can be adjusted. The rod is assumed to be massless; it can move without friction around point O. With the mass M at the bottom, we displace it by a small angle θ_0 and release it without initial velocity.



1. What condition(s) must the system satisfy for it to be able to oscillate?
2. Determine the expression for the period for small-amplitude oscillations.
3. Given that $M = 80\ g, m = 20\ g$, and $L = 4\ cm$, determine the distance ℓ so that the period of the metronome is 2 seconds.
4. To increase the oscillation period of the metronome, should the mass m be moved closer to or farther from point O?

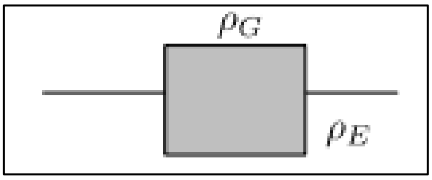
Exercice 03:

An iceberg with density ρ_G, modelled as a regular and homogeneous rectangular block of mass M, floats on water with constant density ρ_E. Its base area is S and its height is L.

Recall that the Archimedes' buoyant force exerted on an immersed object is:

$$\vec{P}_A = -\rho_E V \vec{g}$$

where V is the immersed volume, and $\vec{g}$ is the acceleration due to gravity.



1. At equilibrium, compute the immersed volume of the iceberg as a function of its total volume.

   Given:

   - Ice density: $\rho_G = 900\ kg/m^3$

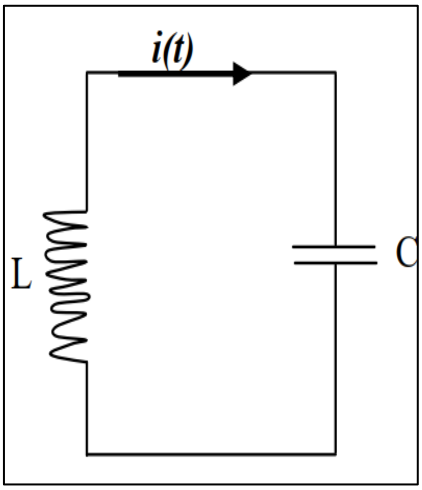
  - Water density: $\rho_E = 1000\ Kg / m^3$

2. The iceberg is displaced vertically by a distance h from its equilibrium position. Calculate the period of its oscillations, assuming friction is negligible.
   Do the numerical calculation for:

   Given $L = 150\ m,\ h = 2\ m,\ g = 9.8\ m/s^2$

Exercice 04:

The electrical circuit in the figure below consists of an inductor and a capacitor. Find the differential equation describing the motion (evolution) of the circuit.



Exercice 05:

The mechanical system in the figure below consists of a straight, homogeneous rod AD, with:

- Mass $M = 3\ kg$
- Length $L = 2\ m$

The rod can rotate in a vertical plane without friction around a fixed horizontal axis (Δ).

- The ends A and D of the rod are connected to a fixed support B_2 via two identical viscous dampers, each with damping coefficient α.

- The midpoint C of the rod is connected to another fixed-point B_1 by a spring with stiffness k .
- At equilibrium, the rod is horizontal.

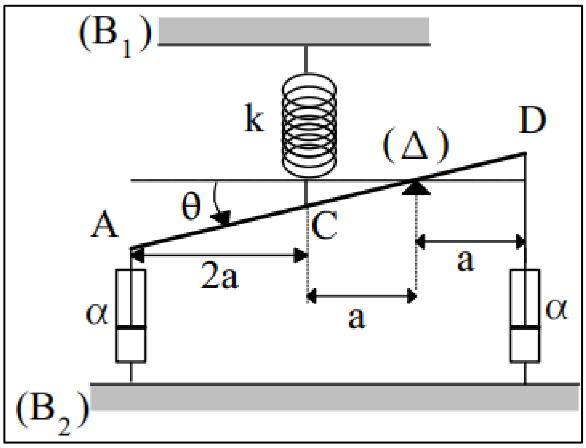
The rod is displaced from equilibrium by a small angle θ_0 and released without initial velocity.

It undergoes a damped oscillatory motion with a pseudo-period of 1 second.

It is observed that after 5 pseudo-periods, the amplitude is 20% of the initial amplitude.

Using this information, determine the numerical value of:

1. The damping coefficient α ,
2. The spring stiffness k .



Chapter 3: Forced Oscillations of Systems with One Degree of Freedom

3.1 Overview and Physical Context

In practical mechanical systems, external time-varying forces often act on structures or components—these may originate from engines, vibrating surfaces, or fluctuating loads. When such forces interact with systems that possess the capacity to oscillate (e.g., masses connected to springs), they induce forced vibrations. Unlike free vibrations that depend solely on the system's internal properties, forced oscillations are driven by external excitations. These are particularly important in engineering because they determine how a structure responds to its environment—whether it remains stable or enters a state of dangerous resonance.

3.2 Governing Equation of Motion

Consider a classical system consisting of a mass m, a spring with stiffness k, and a damper with damping coefficient c.

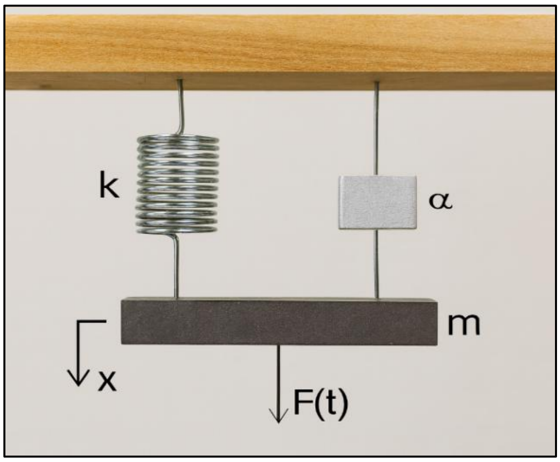


Figure 3.1: *Damped Mass–Spring System*

The mass is subject to a time-varying external force $F(t)$. Using Newton's second law or the Lagrangian formalism, the equation of motion can be written as:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t) \tag{3.1}$$

In the special case where the force is harmonic (i.e., sinusoidal), the forcing function becomes:

$$F(t)=F_0\cos(\Omega t) \tag{3.2}$$

Where:

- F_0 is the amplitude of the excitation,
- Ω is the forcing frequency,
- $x(t)$ is the displacement of the mass as a function of time.

This results in a second-order linear non-homogeneous differential equation:

$$\ddot{x}+2\delta\dot{x}+\omega_0^2x=\frac{F_0}{m}\cos(\Omega t) \tag{3.3}$$

Where:

$\delta=\dfrac{c}{2m}$ is the damping ratio,

- $\omega_0=\sqrt{\dfrac{k}{m}}$ is the natural angular frequency of the system.

This equation fully characterizes the dynamics of forced vibrations in linear damped systems.

3.3 General Solution: Transient and Steady-State Response

The complete response of the system is composed of:

1. **Transient response**: Dependent on initial conditions and typically decays over time due to damping.

2. **Steady-state response**: Long-term behaviour governed by the external force, oscillating at the frequency of excitation.

$$x(t) = x_{\text{transient}}(t) + x_{\text{steady}}(t) \tag{3.4}$$

As time progresses, the transient component—often involving exponential decay—becomes negligible. Thus, for large times:

$$x(t) \approx x_{\text{steady}}(t) \tag{3.5}$$

❖ Steady-State Analysis Using Complex Numbers

To find the steady-state solution, one efficient method is the use of complex exponentials. Assume:

$$x(t) = \Re\{Xe^{i\Omega t}\}, F(t) = \Re\{F_0 e^{i\Omega t}\} \tag{3.6}$$

Substituting into the governing equation yields a complex algebraic equation:

$$(-m\Omega^2 + ic\Omega + k)X = F_0 \tag{3.7}$$

Solving for the complex amplitude X, we get:

$$X = \frac{F_0}{k - m\Omega^2 + ic\Omega} \tag{3.8}$$

The magnitude of the steady-state response (amplitude) is:

$$|X| = \frac{F_0}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}} \tag{3.9}$$

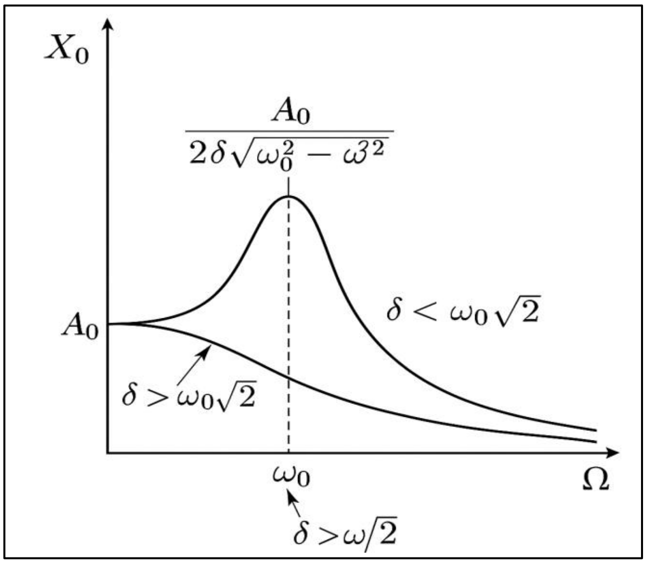


Figure 3.2: *Variation of Amplitude* X_0 *with Frequency* Ω

The phase shift ϕ between the force and the displacement is:

$$\tan(\phi) = \frac{c\Omega}{k - m\Omega^2} \tag{3.10}$$

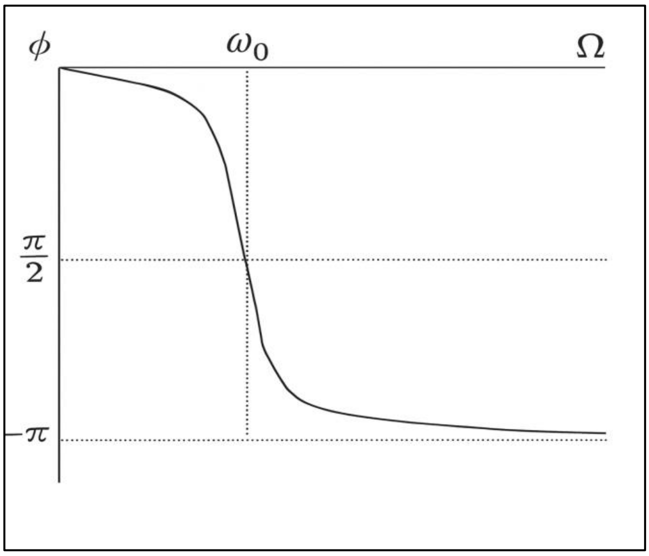


Figure 3.3: *Variation of Phase* ϕ *with Frequency* Ω.

❖ Resonance and Its Consequences

Resonance occurs when the frequency of the external force Ω approaches the natural frequency ω_0 of the system. In the absence of damping (i.e., $c=0$), the amplitude theoretically grows infinitely large:

$$\lim_{\Omega\to\omega_0} |X| \to \infty \tag{3.11}$$

In real systems, however, damping limits this growth. The peak amplitude still becomes very large if the damping is weak. The resonance frequency for a damped system is slightly less than ω_0:

$$\Omega_r = \sqrt{\omega_0^2 - 2\delta^2} \tag{3.12}$$

At resonance, the response is characterized by:

- Maximum amplitude,
- A phase shift of $\phi = -\dfrac{\pi}{2}$,
- Large energy transfer from the force to the system.

Engineering implication: Structures must be designed to avoid operating at or near their resonance frequencies unless controlled properly.

3.4 Mechanical Impedance

The concept of mechanical impedance Z plays a role similar to electrical impedance. It relates force to velocity:

$$Z(\Omega) = \frac{F(\Omega)}{v(\Omega)} = c + i\left(m\Omega - \frac{k}{\Omega}\right) \tag{3.13}$$

The magnitude of impedance affects how much energy is transmitted into the system. At resonance, impedance is minimized, leading to high response amplitudes.

❖ Power Absorption and Energy Transfer

The average power absorbed by the system is:

$$\bar{P} = \frac{1}{2} F_0 v_0 \cos(\phi) \tag{3.14}$$

Where:

- $v_0 = \Omega X_0$,
- ϕ is the phase difference.

Power absorption is maximal when $\phi = 0$, i.e., force and velocity are in phase—this occurs near resonance.

Table 3.1: *Summary of Key Parameters*

Parameter	Formula	Description
Natural frequency	$\omega_0 = \sqrt{\frac{k}{m}}$	Frequency of free undamped oscillations
Damping ratio	$\zeta = \frac{c}{2\sqrt{km}}$	Indicates how rapidly vibrations decay
Resonance frequency	$\Omega_r = \sqrt{\omega_0^2 - 2\delta^2}$	Frequency at which amplitude is maximum
Amplitude at resonance	$X_{max} = \frac{F_0}{2m\delta\sqrt{\omega_0^2 - \delta^2}}$	Max response amplitude for small damping

3.5 Applications and Examples

Forced vibration analysis is applied in:

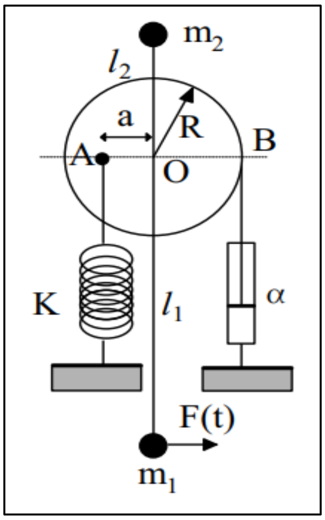
- Designing vibration isolators and shock absorbers in vehicles,
- Tuning musical instruments to avoid dissonant vibrations,

- Engineering buildings and bridges for earthquake resistance,
- Managing rotating machinery with unbalanced forces.

Exercises

Exercice 01

A homogeneous circular disk of mass M and radius R can oscillate without friction around its horizontal axis O.Two masses m_1 and m_2 are welded to the ends of a massless rod that is rigidly connected to the disk and passes through point O. The distances from m_1 and m_2 to the center are denoted ℓ_1 and ℓ_2, respectively.



A vertical spring with stiffness constant K has one fixed end, and its other end is connected to the disk at a point A, located at a distance a from point O. In the equilibrium position, the rod is vertical with m_1 at the bottom, and point A is at the same height as the center O.

The disk is subject to viscous damping at point B with damping coefficient α. Mass m_1 is subjected to a force $F(t) = F_0 \cos(\Omega t)$, perpendicular to the rod.

Given numerical values :

$M = 1kg$, $m_1 = m_2 = 0.1\text{kg}$, $K = 16\text{N/m}$, $R = 20\text{cm}$, $\ell_1 = 50\text{cm}$, $\ell_2 = 25\text{cm}$, $a = 10\text{cm}$, $g = 10\text{m/s}^2$, $\alpha = 7.25\times10^{-2}\text{kg/s}$

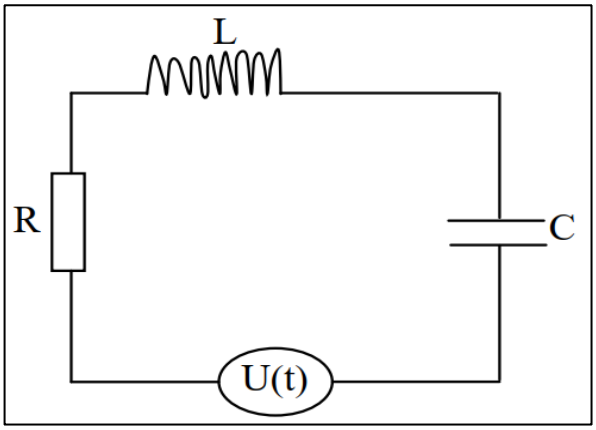
Questions :

1. Derive the differential equation of motion.
2. Find the steady-state (particular) solution of the system.
3. Calculate the quality factor Q of the system.
4. Determine the value of F_0 such that at resonance, the maximum amplitude is equal to $\frac{\pi}{30}\text{rad}$.

Exercise 02: RLC Electrical Oscillatory Circuit

Establish the differential equation for the circuit shown in the figure below:

- First in terms of current
- Then in terms of charge



Given parameters:

$R = 80\Omega$, $L = 10\text{H}$, $C = 0.005\text{F}$, $U_0 = 53\text{V}$,

$\Omega = 3\text{rad/s}$, Voltage source : $U(t) = U_0 \cos(\Omega t)$

1. Calculate:
   - The natural period T_0
   - The damping coefficient δ
2. Determine:
   - The solution of the transient regime
   - The pseudo-frequency ω_D
3. Determine:
   - The solution of the steady-state regime

Exercise 03: Inverted Pendulum with Springs and Damping

Consider the inverted pendulum shown in the given figure. The moment of inertia is given by $I = mL^2$. At equilibrium, the rod OC is vertical, and both springs are undeformed.

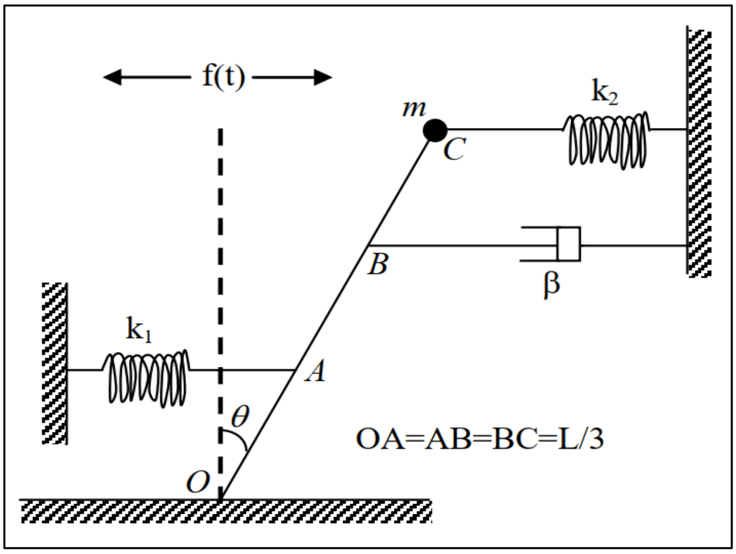
Given:

- $m = 0.2\text{kg}$, $k_1 = 9N/m$, $k_2 = 5N/m$, $\beta = 0.9\text{kg/s}$ (viscous damping coefficient), $L = 0.5\text{m}$
- External torque : $f(t) = \cos(2t)$

1. Derive the differential equation of motion of the system (for small oscillations).
2. Determine the following parameters:

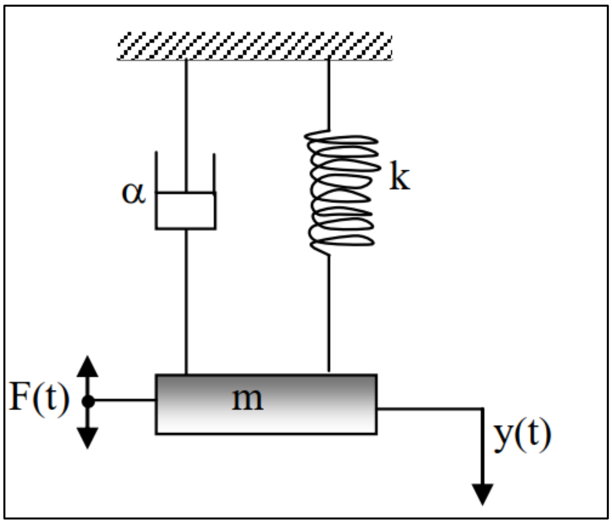
  - Natural frequency ω_0
  - Damped (pseudo) frequency ω_D
  - Logarithmic decrement δ_L

3. Find the steady-state solution (particular solution).
4. Find the transient solution (homogeneous solution).



Exercise 04: Forced Damped Oscillator (Mass-Spring-Damper System)

In the figure below there is, a mass m is attached to a spring of stiffness k and a damper with damping coefficient α. An external sinusoidal force is applied: $F(t) = F_0 \sin(\omega t)$



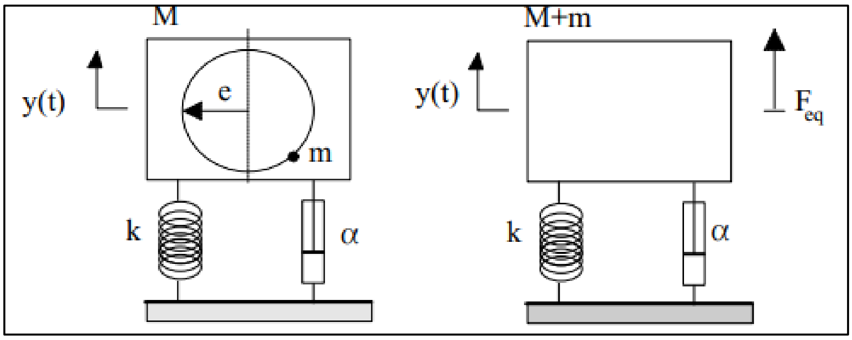
1. Derive the differential equation of motion for the forced damped system.
2. Find the general solution of the differential equation by computing:
    - The homogeneous solution (transient response)
    - The particular solution (steady-state response)

Exercise 05:

Rotating machines (electric motors, turbines, washing machines, etc.) can experience significant vibrations because the center of mass often does not coincide with the axis of rotation. To limit these vibrations, anti-vibration mounts are used, typically made of reinforced rubber. Due to their mechanical properties, these mounts can be modelled as a damper in parallel with a spring.

As an example, we will study the case of a washing machine (see Figure below). Let M be the mass of the machine. The rotating part consists of a drum of radius e, rotating at a constant angular speed Ω. The rotating mass is considered to be the clothing, with a total mass m.

For simplicity, it is assumed that the washing machine can only move vertically, described by a single coordinate y.



1. Derive the differential equation of motion for the vertical coordinate y.

2. Show that such a system is equivalent to the simplified diagram shown in Figure 2, and give the expression of the equivalent excitation force F_{eq}.

3. Under the assumption of low damping ($\delta \ll \omega_0$), plot and comment on the graph of the amplitude Y of the vertical displacement of the washing machine as a function of the rotation speed Ω.

4. Calculate the amplitude of the force transmitted to the ground at resonance.

Chapter 4 – Free Vibrations of Systems with Two Degrees of Freedom

4.1 Overview

Mechanical systems that require two independent coordinates to define their configuration are known as systems with two degrees of freedom.

Illustrative examples include:

- **Vertical Oscillators**: Two vertically moving masses m_1 and m_2, where the positions are determined by x_1 and x_2.
- **Rigid Body in a Plane**: A solid body of mass M constrained to move in a vertical plane. Its motion is characterized by two coordinates — typically one linear displacement and one angular rotation, such as x and θ.
- **Double Pendulum**: Two linked pendulums, where coordinates like θ_1 and θ_2 or x_1 and x_2 are used to describe the motion of the masses.

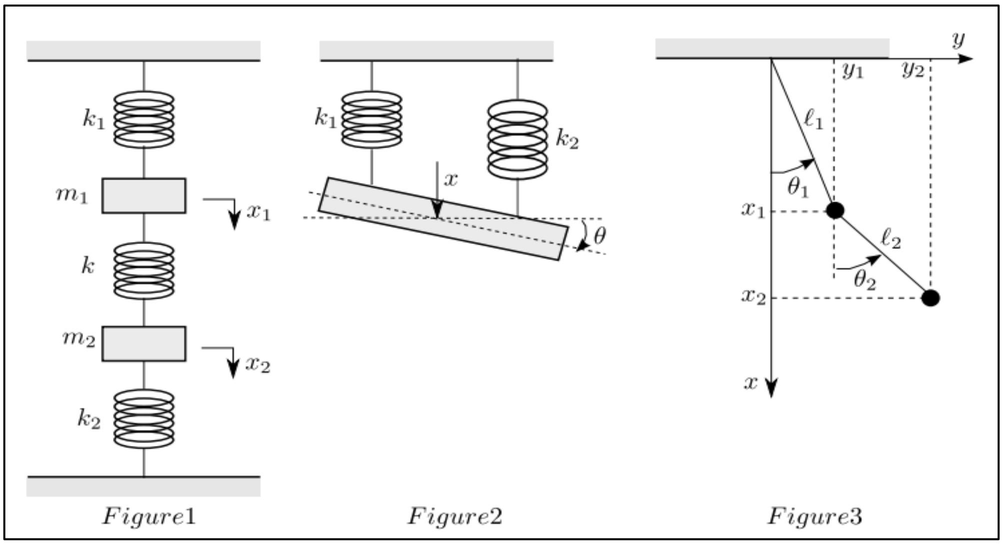


Figure 4.1: *Mechanical Models with Two Degrees of Freedom*

In general, such systems are described using generalized coordinates. The number of coordinates required equals the number of Lagrange equations needed — which is also the number of degrees of freedom.

Thus, for a system with two degrees of freedom, two Lagrangian equations must be formulated.

4.2 A Two-Mass Spring System

4.2.1 Model Description

Consider a model consisting of:

- Two masses, m_1 and m_2,
- Each attached to a fixed wall with springs k_1 and k_2,
- Connected to each other by a third spring of stiffness K (the coupling spring).

Let $x_1(t)$ and $x_2(t)$ denote the displacements of the masses from equilibrium

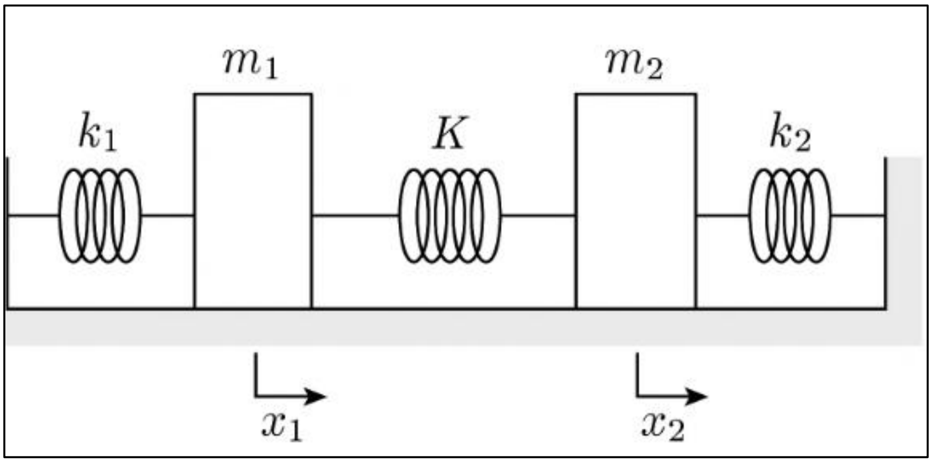


Figure 4.2: *Mass–Spring System in Translational Motion*

❖ **Energy Analysis**

- ***Kinetic Energy***

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \tag{4.1}$$

- ***Potential Energy*** *:*

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} K (x_1 - x_2)^2 \tag{4.2}$$

Expanding the coupling term :

$$U = \frac{1}{2}(k_1 + K) x_1^2 + \frac{1}{2}(k_2 + K) x_2^2 - K x_1 x_2 \tag{4.3}$$

Lagrangian:

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \left[\frac{1}{2}(k_1 + K) x_1^2 + \frac{1}{2}(k_2 + K) x_2^2 - K x_1 x_2 \right] \tag{4.4}$$

❖ Equations of Motion

Using the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2 \tag{4.5}$$

We obtain two coupled equations:

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + K) x_1 - K x_2 &= 0 \\ m_2 \ddot{x}_2 + (k_2 + K) x_2 - K x_1 &= 0 \end{aligned} \tag{4.6}$$

These are second-order linear differential equations, and because they are coupled, the motion of each mass influences the other.

❖ Two-Degree-of-Freedom Torsional System

Now consider a system involving rotational motion — for instance, two disks mounted on a common shaft and connected by torsional springs.

Let:

- $J_1, \quad J_2$: moments of inertia of the two disks,
- $C_1, \quad C_2$: torsional spring constants connecting each disk to a fixed wall,
- C : torsional stiffness of the spring coupling the two disks,

- $\theta_1(t), \quad \theta_2(t)$: angular displacements of the disks as generalized coordinates.

Kinetic Energy

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 \tag{4.7}$$

Potential Energy

$$U = \frac{1}{2}C_1\theta_1^2 + \frac{1}{2}C_2\theta_2^2 + \frac{1}{2}C(\theta_1 - \theta_2)^2 \tag{4.8}$$

Expanding:

$$U = \frac{1}{2}(C_1 + C)\theta_1^2 + \frac{1}{2}(C_2 + C)\theta_2^2 - C\theta_1\theta_2 \tag{4.9}$$

- **Equations of Motion**

Using the Lagrangian method again, we obtain:

$$\begin{aligned} J_1\ddot{\theta}_1 + (C_1 + C)\theta_1 - C\theta_2 = 0 \\ J_2\ddot{\theta}_2 + (C_2 + C)\theta_2 - C\theta_1 = 0 \end{aligned} \tag{4.10}$$

This is structurally the same as the translational system but applied to rotation, with:

  - $m \leftrightarrow J$,
  - $k \leftrightarrow C$,
  - $x \leftrightarrow \theta$.

4.2.2 Modal Analysis – Normal Modes and Natural Frequencies

We now analyze the system to find:

- Its natural frequencies of vibration,
- The corresponding mode shapes (normal modes).

We assume each generalized coordinate oscillates harmonically:

$$x_1(t) = X_1\cos(\omega t), \quad x_2(t) = X_2\cos(\omega t) \tag{4.11}$$

Substituting into the equations of motion leads to a homogeneous algebraic system:

$$\begin{aligned}[-\omega^2 m_1+(k_1+K)]X_1-KX_2=0\\ [-\omega^2 m_2+(k_2+K)]X_2-KX_1=0\end{aligned} \tag{4.12}$$

This system can be expressed as:

$$\begin{bmatrix} k_1+K-m_1\omega^2 & -K \\ -K & k_2+K-m_2\omega^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.13}$$

For non-trivial solutions $(X_1, X_2)\neq(0,0)$, the determinant of the coefficient matrix must be zero:

$$\begin{vmatrix} k_1+K-m_1\omega^2 & -K \\ -K & k_2+K-m_2\omega^2 \end{vmatrix}=0 \tag{4.14}$$

This gives the characteristic equation, whose solutions are ω_1^2 and ω_2^2 , the squared natural frequencies**.**

Once the frequencies are known, the ratios $\frac{X_1}{X_2}$ can be determined from either original equation. These define the mode shapes, which are constant geometric patterns of vibration.

There are always two distinct solutions (if damping is negligible):

- **Mode 1**: Typically, both masses move in the same direction (in-phase),
- **Mode 2**: The masses move in opposite directions (out-of-phase).

One of the key properties of the natural modes in undamped mechanical systems is orthogonality. This means that the eigenvectors (mode shapes) associated with different natural frequencies are orthogonal relative to both the mass matrix and the stiffness matrix.

Let $Q^{(1)}$ and $Q^{(2)}$ be the eigenvectors corresponding to the natural frequencies ω_1 and ω_2.

Then:

$$\left(Q^{(1)}\right)^T AQ^{(2)}=0, \quad \left(Q^{(1)}\right)^T KQ^{(2)}=0 \tag{4.15}$$

This orthogonality is fundamental in simplifying the dynamic analysis of multi-degree-of-freedom systems, especially when using modal superposition. It ensures that when the system vibrates in one mode, the energy remains confined to that mode without transferring to others.

When a system is given an arbitrary initial displacement or velocity, the response is a superposition of the two normal modes. That is:

$$\begin{aligned} x_1(t) &= X_1^{(1)}\cos(\omega_1 t+\phi_1)+X_1^{(2)}\cos(\omega_2 t+\phi_2) \\ x_2(t) &= X_2^{(1)}\cos(\omega_1 t+\phi_1)+X_2^{(2)}\cos(\omega_2 t+\phi_2) \end{aligned} \tag{4.16}$$

Here:

$\omega_1,\quad \omega_2$: natural frequencies,

$\phi_1,\quad \phi_2$: phase constants,

$X_1^{(1)},\quad X_2^{(1)},\quad X_1^{(2)},\quad X_2^{(2)}$: modal amplitudes.

The particular shape of the resulting motion depends entirely on the initial conditions. This linear combination results in beating phenomena and modulated vibration patterns when both modes are excited.

❖ Energy Distribution and Mode Decoupling

In the absence of damping, each mode conserves its own total energy (sum of kinetic and potential). When the system vibrates in a single mode :

- The motion is entirely confined to that mode,
- There is no transfer of energy between modes,
- The response can be analyzed as if the system had only one degree of freedom.

This decoupling simplifies analysis and is a major reason modal decomposition is powerful in vibration studies.

❖ Visualization and Physical Examples

Several typical motion patterns can arise:

- In one mode, both masses move in the same direction,
- In the second mode, the two masses move oppositely.

If the system is excited with a combination of these modes, the motion may appear complex and non-periodic, depending on the frequency ratio ω_1 / ω_2. If this ratio is rational, the motion is periodic. If irrational, the response is quasi-periodic.

4.3 Coupled Oscillators

Consider two identical masses m connected by springs of stiffness k and a coupling spring of stiffness K. Assuming symmetry:

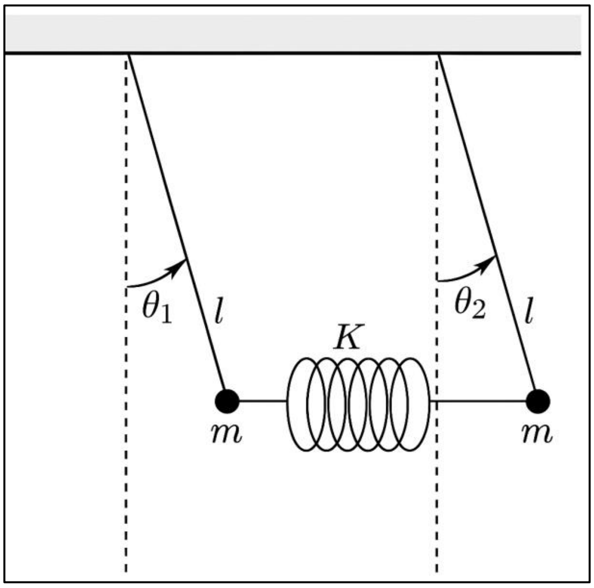


Figure 4.3: *Coupled Pendulum Oscillators*

- The **natural frequencies** are derived from the characteristic polynomial.
- The **mode shapes** are :
  - Mode 1: $x_1 = x_2$ (in-phase),

- Mode 2: $x_1 = -x_2$ (out-of-phase).

This simple example illustrates how symmetry leads to orthogonal and easily interpretable modes.

Systems with two degrees of freedom reveal richer behaviors than single-degree systems:

- Multiple natural frequencies,
- Distinct mode shapes,
- Superposition effects,
- Mode interaction and decoupling.

Understanding these dynamics lays the foundation for analyzing more complex systems like multi-mass chains, buildings, machines, and structures.

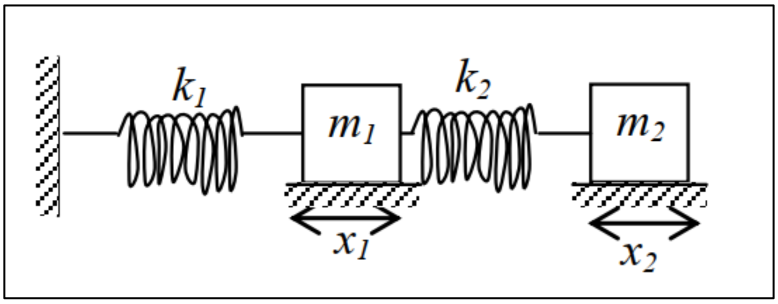
Exercises

Exercise 01

We consider the free oscillations of the two-degree-of-freedom system shown below:

1) Calculate the kinetic and potential energies of the system.
2) For $k_1 = k_2 = k$ and $m_1 = m = \frac{m_2}{2}$, and using Lagrange's equation, establish the differential equations of motion.

Then, deduce the natural angular frequencies (normal mode frequencies) of the system.



Exercise 02:

We consider the setup in Figure given below.

Two identical cylinders (mass M, radius R, and moment of inertia $J = \frac{1}{2}MR^2$) roll without slipping on a horizontal surface.

Let θ_1 and θ_2 be the rotation angles of these two cylinders with respect to their respective equilibrium positions.

At rest ($\theta_1 = \theta_2 = 0$), the springs are undeformed.

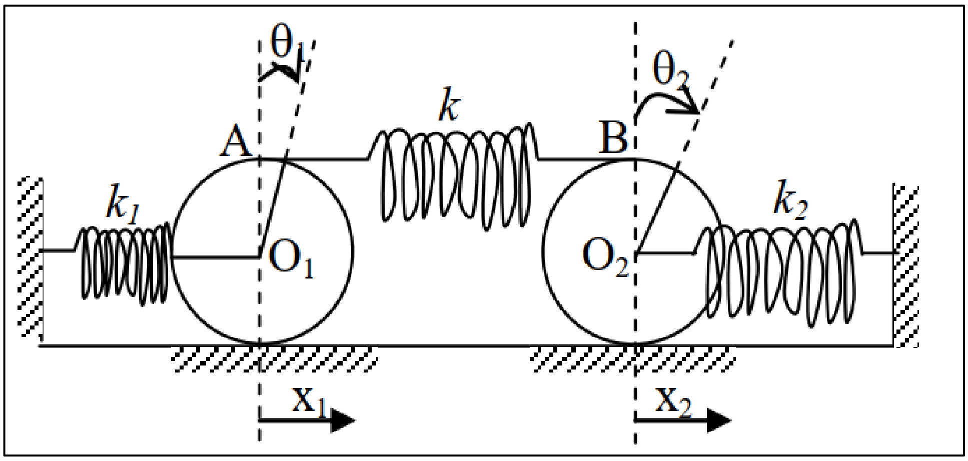
1. Establish the Lagrangian of the system as a function of x_1 and x_2. (Here, x_1 and x_2 are likely the translational displacements of the centers of the two cylinders.)
2. Assume $k_1 = k_2 = k' \neq k$. Find the equations of motion. (This suggests the two end springs have stiffness k', and there's a coupling spring between the masses with stiffness k.)
3. Deduce the natural angular frequencies (normal mode frequencies).

4. Rewrite the Lagrangian in the following form:

$$L = \frac{3}{4} M \left[\dot{x}_1^2 + \dot{x}_2^2 - \omega_0^2 \left(x_1^2 + x_2^2 - 2K x_1 x_2 \right) \right]$$

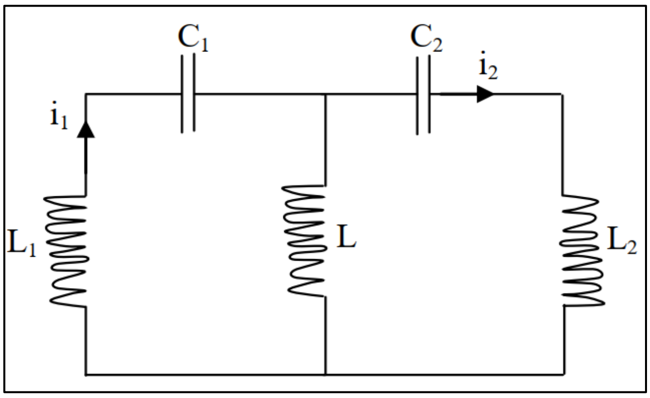
   From this, deduce the expressions for ω_0^2 and K.

5. K is the coupling coefficient. Show that it varies between two limiting values.
6. Give the physical meaning of ω_0 by comparing it with the natural frequencies found in question 3. Deduce the effect of the coupling k on the natural frequencies.



Exercise 03:

We consider the electrical system of the following figure given below, made of two coupled oscillating circuits connected by a mutual inductance L, as follows:

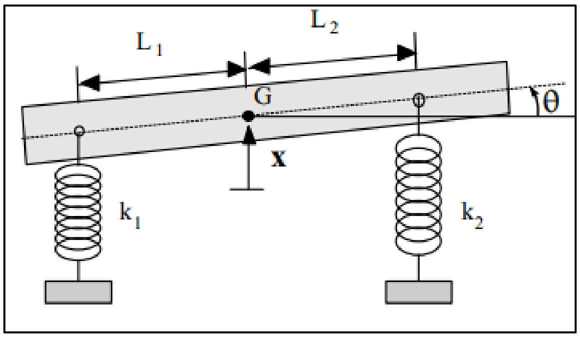


1. Establish the two differential equations for the two loops:
   - First in terms of the currents i_1 and i_2,
   - Then in terms of the charges q_1 and q_2.
2. For $C_1 = C_2 = C$ and $L_1 = L_2 = L$, find the natural angular frequencies (normal mode frequencies) of the system.
3. Deduce the general solutions.

Exercise 04:

We consider the vehicle and suspension system shown in the figure (without dampers). We assume the springs remain vertical. The mass of the vehicle is mm, and its moment of inertia about a horizontal axis D passing through the center of gravity G and perpendicular to the plane of the figure is J_0.

- The vertical displacement of the center of gravity from its equilibrium is denoted by x (heave or bounce).
- The rotation angle θ (pitch) that the chassis makes with the ground, around axis D, is assumed to be small.
- Roll (side inclination) is neglected.



We are given the following values:

- Mass of the vehicle: $m = 1000\ \text{kg}$
- Distance between the front axle and the center of gravity G: $L_1 = 1\ \text{m}$
- Distance between the rear axle and the center of gravity G: $L_2 = 1.5\ \text{m}$
- Stiffness of the front spring: $k_1 = 18\ \text{kN/m}$
- Stiffness of the rear spring: $k_2 = 18\ \text{kN/m}$
- Moment of inertia of the vehicle: $J_0 = mr^2$, with $r = 0.9\ \text{m}$

1. Determine the natural angular frequencies (normal modes) of the system, as well as the amplitude ratio for each mode.
2. Write the solutions $x(t)$ and $\theta(t)$.
3. a) What condition must be met in order to decouple the motions x and θ?

b) What are the resulting natural frequencies of heave f_P and pitch f_T?

Exercise 05:

We consider the following mechanical system, which includes a horizontal bar of negligible mass that can pivot without friction around an axis passing through its center.

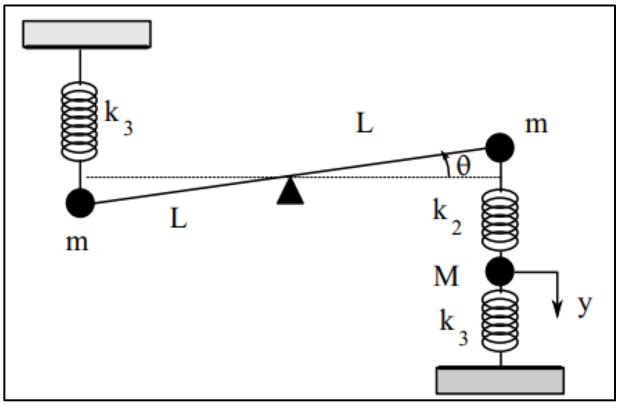
We are given:

- $M = 2m$
- $k_1 = k_2 = k_3 = k$

1. Establish the equations governing the small oscillations of the system.

2. Find the natural angular frequencies (normal mode frequencies) and the amplitude ratios for the different modes.

3. Write the general solutions $y(t)$ and $\theta(t)$.



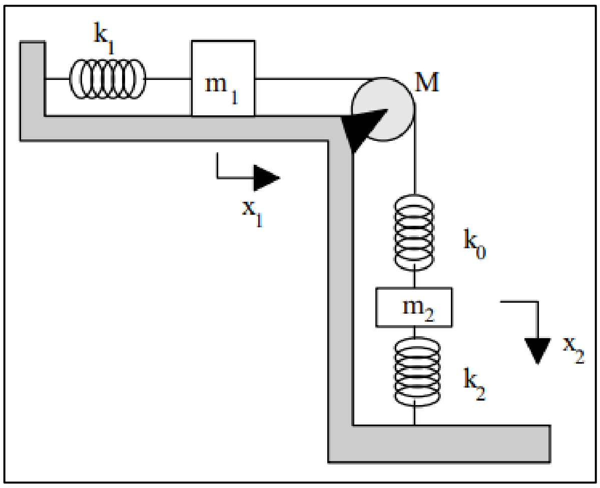
Exercise 06:

In the figure below, M and R represent the mass and radius of the pulley, respectively. x_1 and x_2 represent the displacements of the two masses from their equilibrium positions.
We are given:

- $M = 2(m_2 - m_1)$

- $m_2 = m$
- $k_0 = k_1 = k_2 = k$

1. Write the Lagrangian of the system.
2. Determine the natural frequencies (eigenfrequencies) and the amplitude ratio of each mode as functions of m and k.



Chapter 5 – Forced Oscillations of Two-Degree-of-Freedom Systems

5.1 Lagrange Equations

Consider a mechanical system with two degrees of freedom, subjected to:

- Forces derived from a potential (conservative),
- Viscous damping forces,
- External forces that vary with time.

Let the generalized coordinates be q_1 and q_2. The Lagrange equations for such a system are:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} = F_{q_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} = F_{q_2} \end{aligned} \tag{5.1}$$

Where:

- $L = T - U$ is the Lagrangian, defined as kinetic energy minus potential energy,
- D is the dissipation function due to damping,
- F_{q_1} and F_{q_2} are generalized forces corresponding to coordinates q_1 and q_2.

Each generalized force is the partial derivative of the virtual work δW done by external forces with respect to the corresponding coordinate, holding the other coordinate fixed:

$$F_{q_1} = \left.\frac{\delta W_1}{\delta q_1}\right|_{\delta q_2 = 0} \tag{5.2}$$

$$F_{q_2} = \left.\frac{\delta W_2}{\delta q_2}\right|_{\delta q_1 = 0} \tag{5.3}$$

5.2 Mass-Spring-Damper System

We now study a symmetric two-degree-of-freedom system where an external horizontal force $F(t)$ is applied only to the first mass. The system includes:

- Masses m,
- Springs with stiffness k and coupling stiffness K,
- (Possibly) damping elements.

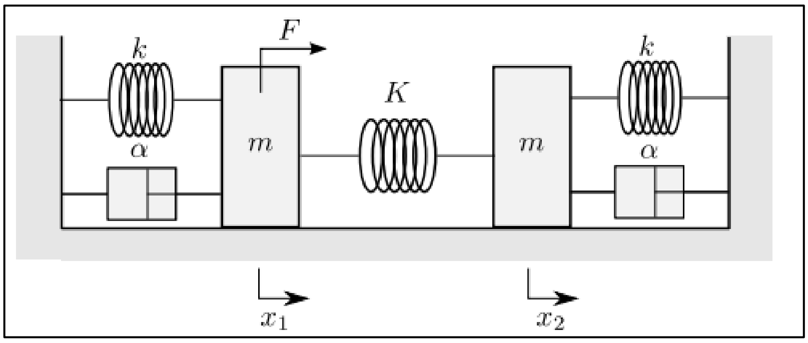


Figure 5.1: *Forced Vibrations of a Two-Degree-of-Freedom Mechanical System.*

5.2.1 Differential Equations

The coupled differential equations of motion are:

$$\begin{aligned} m\ddot{x}_1 + (k+K)x_1 - Kx_2 &= F(t) \\ m\ddot{x}_2 + (k+K)x_2 - Kx_1 &= 0 \end{aligned} \tag{5.4}$$

5.2.2 Steady-State Sinusoidal Response

We look for a steady-state solution when the system is forced sinusoidally. Assume:

$$\begin{aligned} x_1(t) &= X_1 \cos(\omega t + \phi_1) \\ x_2(t) &= X_2 \cos(\omega t + \phi_2) \end{aligned} \tag{5.5}$$

To simplify, we use complex notation:

$$x_1(t) = \Re\left\{\tilde{X}_1 e^{j\omega t}\right\},\ \ x_2(t) = \Re\left\{\tilde{X}_2 e^{j\omega t}\right\},\ \ F(t) = \Re\left\{\tilde{F} e^{j\omega t}\right\} \tag{5.6}$$

Substituting into the system yields a set of algebraic equations:

$$\begin{aligned} \left[(k + K - m\omega^2) + j\omega b\right]\tilde{X}_1 - K\tilde{X}_2 &= \tilde{F} \\ \left[(k + K - m\omega^2) + j\omega b\right]\tilde{X}_2 - K\tilde{X}_1 &= 0 \end{aligned} \tag{5.7}$$

In the negligible damping case (i.e., $b \approx 0$), the system simplifies further.

The solutions are:

$$X_1 = \frac{F_0}{m(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)},\ \ X_2 = \frac{KF_0}{m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \tag{5.8}$$

Where ω_1 and ω_2 are the natural frequencies.

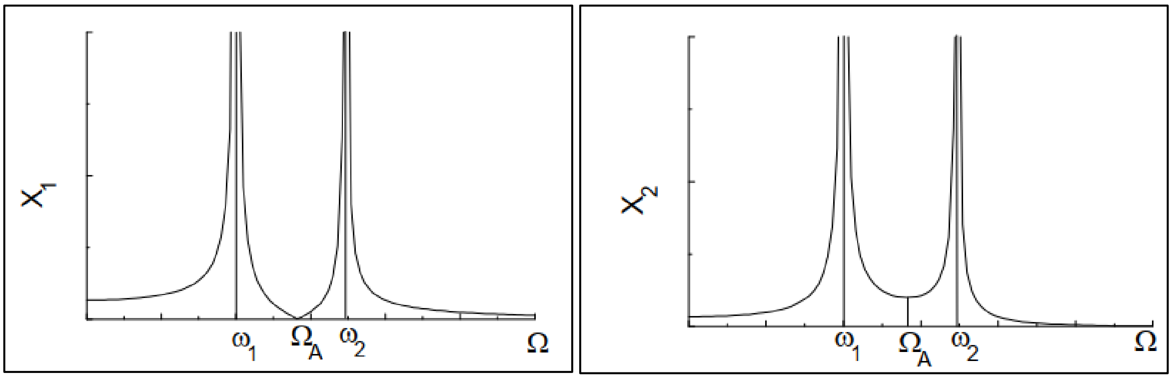


Figure 5-2: *Amplitude Responses* X_1 *and* X_2 *vs Excitation Frequency as a Function of* Ω.

5.3 Mechanical Impedance

The notion of impedance helps simplify the response analysis of vibrating systems under sinusoidal excitation.

Let's consider a mass-spring-damper system:

- A force $f(t) = F_0 \cos(\omega t)$ is applied to a mass m connected to:
    - A spring of stiffness k,
    - A damper with coefficient b.

The equation of motion is:

$$m\ddot{x} + b\dot{x} + kx = f(t) \tag{5.9}$$

Assuming a complex sinusoidal response:

$$x(t) = \Re\left\{\tilde{X} e^{j\omega t}\right\} \tag{5.10}$$

We substitute into the equation:

$$\left(-m\omega^2 + j\omega b + k\right)\tilde{X} = \tilde{F} \tag{5.11}$$

We define mechanical impedance $Z(j\omega)$ as the complex ratio between force and velocity:

$$Z(j\omega) = \frac{\tilde{F}}{j\omega\tilde{X}} = \frac{-m\omega^2 + j\omega b + k}{j\omega} \tag{5.12}$$

This generalizes the system's dynamic response into a compact, frequency-dependent form.

5.4 Application: Vibration Absorber (Dynamic Damper)

We apply the theory to a real-world system — a mass-spring assembly designed to suppress vibration:

- A primary mass m_1 is connected to the ground by a spring k_1,
- A secondary mass m_2 is attached to m_1 by another spring k_2,
- A force $f(t) = F_0 \cos(\omega t)$ acts on the primary mass.

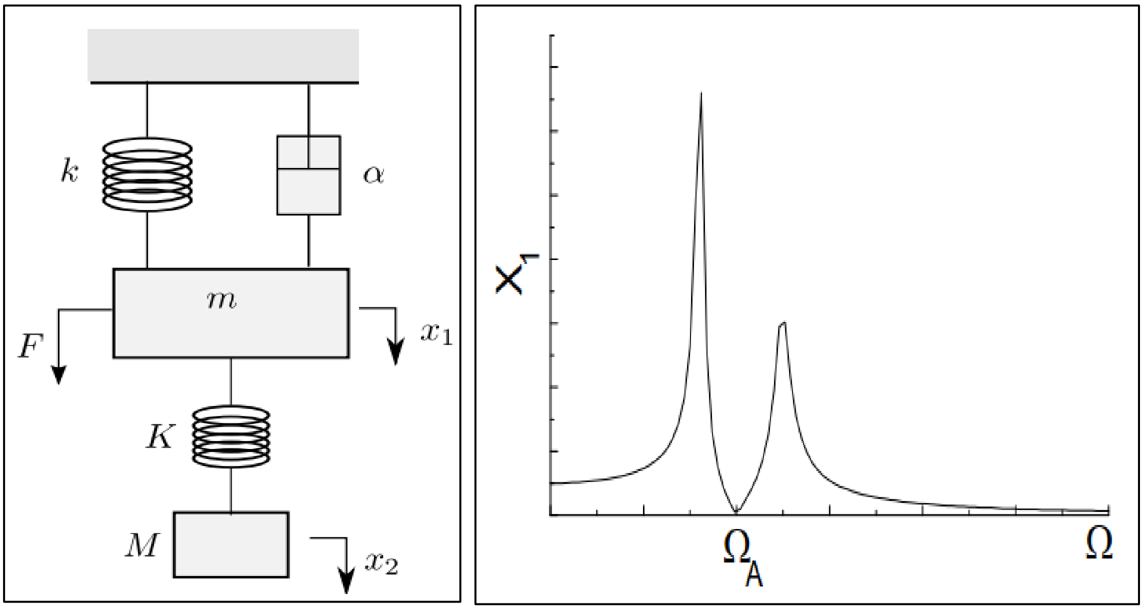


Figure 5.3: a- *Vibration Absorber*. **b-** *Variation of* X_1 *as a Function of* Ω.

The coupled system:

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= F_0\cos(\omega t) \\ m_2\ddot{x}_2 + k_2x_2 - k_2x_1 &= 0 \end{aligned} \tag{5.13}$$

We seek steady-state sinusoidal solutions:

$$x_1(t) = X_1\cos(\omega t), \quad x_2(t) = X_2\cos(\omega t) \tag{5.14}$$

Substituting yields:

$$\begin{aligned} (-m_1\omega^2 + k_1 + k_2)X_1 - k_2X_2 &= F_0 \\ -k_2X_1 + (-m_2\omega^2 + k_2)X_2 &= 0 \end{aligned} \tag{5.15}$$

Solving gives:

$$X_1 = \frac{(k_2 - m_2\omega^2)F_0}{\Delta} \tag{5.16}$$

And

$$X_2 = \frac{k_2 F_0}{\Delta} \tag{5.17}$$

Where:

$$\Delta = (k_2 - m_2\omega^2)(k_1 + k_2 - m_1\omega^2) - k_2^2 \tag{5.18}$$

- ✓ **Antiresonance Effect**

If the excitation frequency ω is chosen such that:

$$k_2 = m_2\omega^2 \tag{5.19}$$

Then the numerator of X_1 becomes zero → the primary mass does not move at all**.**

This is the principle of a dynamic vibration absorber: a properly tuned secondary mass cancels the motion of the primary structure.

This chapter has demonstrated that:

- Two-degree-of-freedom systems respond in complex ways to external forcing,
- Modal analysis helps identify key frequency responses and resonance risks,
- Tools like impedance and vibration absorbers offer practical means to manage and reduce unwanted motion.

Exercises

Exercise 1:

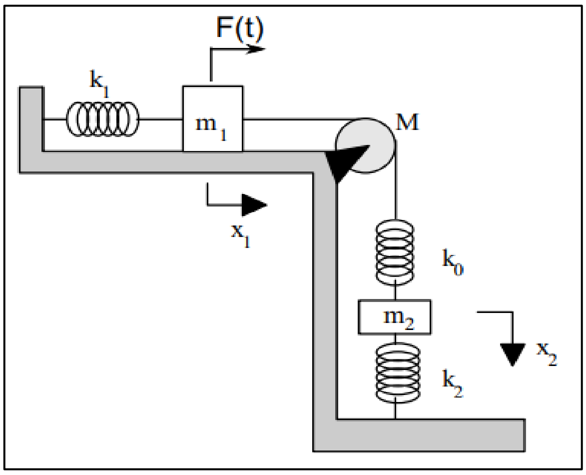
In the figure below, M and R represent the mass and radius of the pulley, respectively. x_1 and x_2 denote the displacements of the two masses from their equilibrium positions.

We are given:

$M = 2(m_2 - m_1)$, with $m_2 = m$, and $k_0 = k_1 = k_2 = k$

The system is to be studied in steady-state regime with:

$F(t) = F_0 \cos(\omega t)$



1. Calculate the input impedance.
2. Determine the velocities $\dot{x}_1(t)$ and $\dot{x}_2(t)$.
3. (a) At what angular frequency ω\omega does mass m_1 remain stationary?

   (b) In that case, what is the amplitude of oscillation of mass m_2?
4. (a) At which angular frequencies are the velocity of mass m_1 and the force $F(t)$ in phase?

   (b) By comparing these angular frequencies with those calculated in Exercise 06 of the previous chapter, deduce the phase shift of $\dot{x}_2$ with respect to $F(t)$ for each of these frequencies.

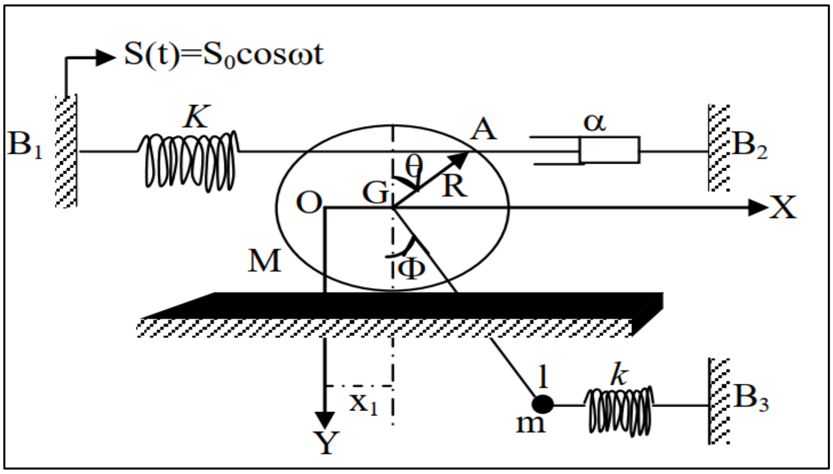
Exercise 02

In the oscillating system shown in the figure, the cylinder is homogeneous, with mass M and radius R. This cylinder is connected at point A to a base B_1 via a spring of stiffness K, and this base moves sinusoidally with amplitude S_0 and angular frequency ω. The cylinder is also connected to a fixed base B_2 by a damper with damping coefficient α. It rolls without slipping on a horizontal plane.

A massless rod of length l is attached at one end and can oscillate without friction around the axis of the cylinder. A point mass m is attached at the other end of the rod and is connected to a fixed base B_3 via another spring of stiffness k. At equilibrium, the rod is vertical, and the cylinder axis G lies at the origin O of the coordinate system. It is assumed that the springs are unstressed at equilibrium.

Let :

- ϕ be the angle of rotation of the rod from the vertical,
- θ be the rotation angle of the cylinder,
- $x_2 = l\phi$, and $x_1 = R\theta$



We consider small-amplitude oscillations.

Given: $3M = 2m, \quad 4K = k = \frac{mg}{l}$

1. Show that the Lagrangian of the system can be written in the form:

$$L = m\dot{x}_1^2 + m\dot{x}_1\dot{x}_2 + \frac{1}{2}m\dot{x}_2^2 - kx_1^2 - kx_1x_2 - kx_2^2 + \frac{1}{2}kx_1s - \frac{1}{8}ks^2$$

2. Determine the differential equations in $x_1(t)$ and $x_2(t)$. Show that the system is equivalent to a forced system subjected to a sinusoidal force $F(t)$, and determine the amplitude F_0 of this force.
3. Express the equations of motion in terms of the velocities $\dot{x}_1(t)$ and $\dot{x}_2(t)$.
4. a. Determine $\dot{x}_1(t)$ and $\dot{x}_2(t)$ for the angular frequency $\omega = \omega_0 = \sqrt{\frac{k}{m}}$.

   Deduce the behavior of the system at this resonance frequency.

   b. Determine the input impedance of the system, defined by $Z_e / \dot{x}_1$, at this frequency.

5. a. Determine $\dot{x}_1(t)$ and $\dot{x}_2(t)$ for the angular frequency $\omega = \omega_1 = \sqrt{\frac{2k}{m}}$.

   Deduce the behavior of the system at this frequency.

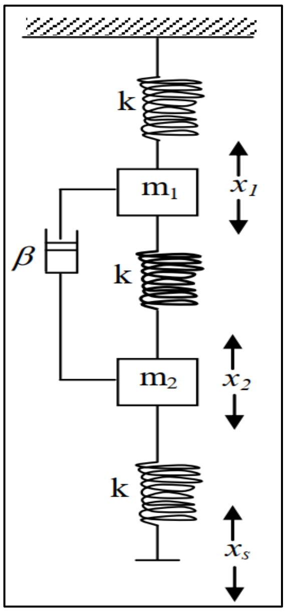
   b. Determine the input impedance of the system, defined by $Z_e / \dot{x}_1$, at this frequency.

Exercice 03

We consider the oscillatory system shown in the figure below.

1. Establish the differential equations for the two masses m_1 and m_2, with x_1 and x_2 being their respective dynamic amplitudes.
2. Give the equivalent electrical circuit of the system by:

   First establishing the equations in terms of charges q_1 and q_2



Then expressing them in terms of currents i_1 and i_2.

3. Assuming $m_1 = m_2 = m$, find the expressions of the complex amplitudes of the steady-state solutions x_1 and x_2, given:

$$F(t) = k \cdot x_s = a \cdot \exp(i\Omega t)$$

4. Now, assume there is no damping ($\beta = 0$):

a) For which angular frequency Ω does mass m_2 remain stationary?

b) What are the resonance frequencies of the system?

c) Now consider the unforced case $F(t) = 0$. Find:

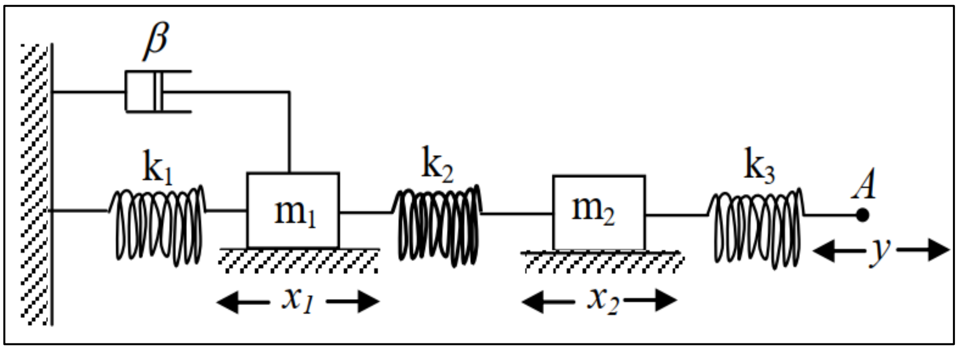
- The natural angular frequencies (associated with the normal modes)
- The modal transformation matrix (also called the passage matrix)
- The general solution of the system.

Exercice 04:

A vibratory excitation force F of amplitude y, where:

$$F = k_3 \cdot y$$

is applied at point A of the mechanical oscillatory system shown in that figure.



Let x_1 and x_2 be the resulting dynamic displacements of masses m_1 and m_2 relative to their equilibrium positions.

1. Determine the equations of motion of masses m_1 and m_2.

   Deduce the corresponding system of differential equations.
2. Establish the analogous electrical differential equations, first in terms of:
   - Charges q_1 and q_2, then
   - currents i_1 and i_2.

   Deduce the electrical equivalent circuit diagram for the mechanical system.
3. Assume :

   $m_1 = m_2 = m$

   $k_1 = k_2 = k_3 = k$

   Given that:

   $$F = k \cdot y = a \cdot \exp(i\Omega t)$$

   - Provide the differential system in complex amplitudes $\overline{X}_1$ and $\overline{X}_2$ corresponding to the steady-state solutions x_1 and x_2.
   - If $\beta = 0$ (no damping), determine the angular frequency Ω for which mass m_2 remains stationary.
4. Now assume $\beta = 0$ and the unforced case $a = 0$:

   Find:
   - The natural frequencies (associated with the vibrational modes)
   - The modal transformation matrix (also called the passage matrix)
   - The general solutions of the system.

Exercise 05:

1. Establish the differential equations governing the behavior of the systems shown in Figures 1 and 2.

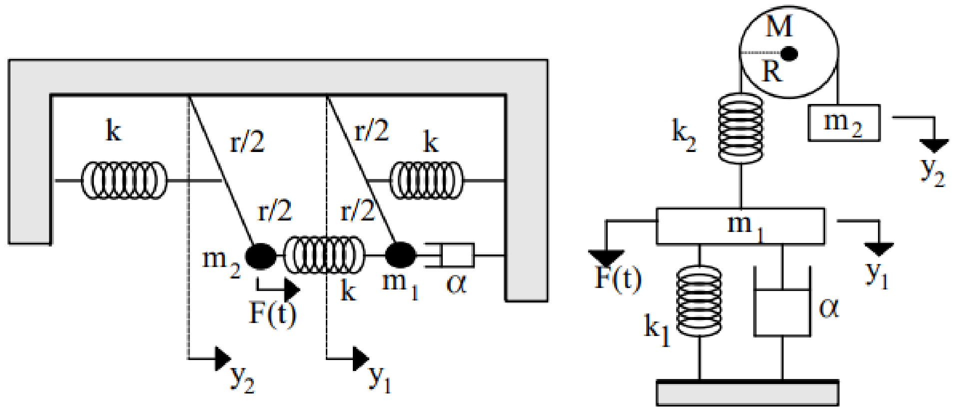


figure 1 *figure 2*

2. For the steady-state sinusoidal regime, use the following conditions:

- For Figure 1 :

$$\omega = \sqrt{\frac{5k}{4m_1} + \frac{g}{r}}, \quad \text{and} \quad m_1 = m_2$$

- For Figure 2 :

$$\omega = \sqrt{\frac{k_2}{m}}, \quad \text{with} \quad m = \frac{M}{2} + m_2$$

➢ Calculate:

  a. the input impedance of each system;
  b. The following quantities :

- $\dot{y}_1$: velocity of the first mass
- $\dot{y}_2$: velocity of the second mass
- y_1 : displacement of the first mass
- y_2 : displacement of the second mass

Chapter 06 – General Concepts of Propagation Phenomena

6.1 One-Dimensional Propagation

6.1.1 Propagation Equation

In the vibration phenomena studied in the previous chapters, we focused on systems where the physical quantities depended on only one variable — time.

Now, we shift our attention to a broader range of phenomena described by functions that depend on both time t and a spatial variable, such as x.

These types of phenomena are governed by partial differential equations. One of the most fundamental of these is known as the:

- **D'Alembert equation**,
- Or the **wave equation**,
- or more generally, the **one-dimensional propagation equation**.

Its mathematical form is:

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 s}{\partial t^2} \tag{6.1}$$

Here:

- $s(x,\ t)$: represents the physical quantity being propagated (e.g., displacement, pressure),
- V : is the propagation speed of the wave, which has the units of velocity.

6.1.2 Solution of the Propagation Equation – D’Alembert’s Method

To solve the one-dimensional wave equation, we introduce a change of variables that simplifies the equation’s structure.

We define:

$$\xi = t + \frac{x}{V}, \quad \eta = t - \frac{x}{V} \tag{6.2}$$

These new variables, called characteristic variables, represent directions in the $x-t$ plane along which information propagates at speed V.

Using the chain rule, we express partial derivatives with respect to x and t in terms of ξ and η, which transforms the wave equation into a simpler form:

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 s}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 s}{\partial \xi \partial \eta} = 0 \tag{6.3}$$

This equation indicates that s is the sum of two independent functions:

$$s(x,t) = f\left(t - \frac{x}{V}\right) + g\left(t + \frac{x}{V}\right) \tag{6.4}$$

Where:

- f : represents a wave propagating to the **right** (positive x-direction),
- g : represents a wave propagating to the **left** (negative x-direction).

This elegant solution, known as **D’Alembert’s solution**, shows that the general behaviour of a wave can be understood as the superposition of two waves traveling in opposite directions, each without distortion.

- **Properties of Particular Solutions:** $F\left(t - \frac{x}{V}\right)$ and $G\left(t + \frac{x}{V}\right)$

Let’s examine the behaviour of a specific solution:

$$s(x,t) = F\left(t - \frac{x}{V}\right) \tag{6.5}$$

We assume the boundary conditions are such that the second wave component $G\left(t + \frac{x}{V}\right)$ is identically zero throughout. This simplifies the solution to a single traveling wave moving to the right.

Let’s consider the following:

- At time t_1 , a point located at position x_1 on the medium has a displacement:

$$s(x_1, t_1) = F\left(t_1 - \frac{x_1}{V}\right) \tag{6.6}$$

- Now, at a later time $t_2 > t_1$, we want to find the position x_2 where the displacement is still the same:

$$s(x_2, t_2) = s(x_1, t_1) \tag{6.7}$$

So, we require:

$$F\left(t_2 - \frac{x_2}{V}\right) = F\left(t_1 - \frac{x_1}{V}\right) \tag{6.8}$$

Since the function F is the same on both sides, this implies:

$$t_2 - \frac{x_2}{V} = t_1 - \frac{x_1}{V} \quad \Rightarrow \quad x_2 = x_1 + V(t_2 - t_1) \tag{6.9}$$

This equation shows that the same value of the wave function moves from x_1 to x_2 in time $t_2 - t_1$, which means the wave travels to the right at a constant speed V, without changing shape.

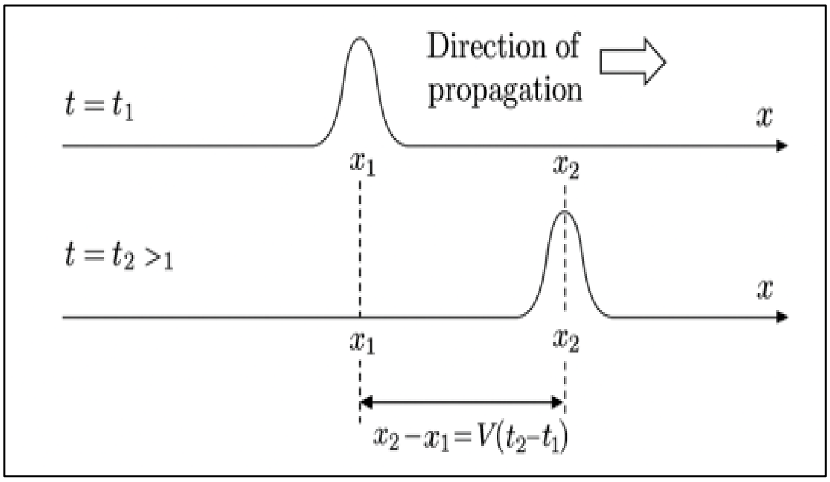


Figure 6.1: *Forward-Traveling Wave in the* x *-Direction:* $F\left(t-\frac{x}{V}\right)$

- ❖ **Properties of the Solution** $G\left(t+\frac{x}{V}\right)$

Now let’s study the wave solution:

$$s(x,t)=G\left(t+\frac{x}{V}\right) \tag{6.10}$$

This represents a wave traveling to the left, i.e., in the direction of decreasing x, again assuming that the other component $F(t-\frac{x}{V})$ is zeroed out due to boundary conditions.

We proceed similarly :

- Take a point at position x_1 at time t_1, with displacement:

$$s(x_1,t_1)=G\left(t_1+\frac{x_1}{V}\right) \tag{6.11}$$

- We now want to determine the location x_2 at a later time $t_2 > t_1$ where the displacement remains the same:

$$s(x_2,t_2) = s(x_1,t_1) \Rightarrow G\left(t_2 + \frac{x_2}{V}\right) = G\left(t_1 + \frac{x_1}{V}\right) \tag{6.12}$$

So, we must have:

$$t_2 + \frac{x_2}{V} = t_1 + \frac{x_1}{V} \quad \Rightarrow \quad x_2 = x_1 - V(t_2 - t_1) \tag{6.13}$$

This confirms that the wave moves to the left at a constant speed V, again without deformation — just like the F wave, but in the opposite direction.

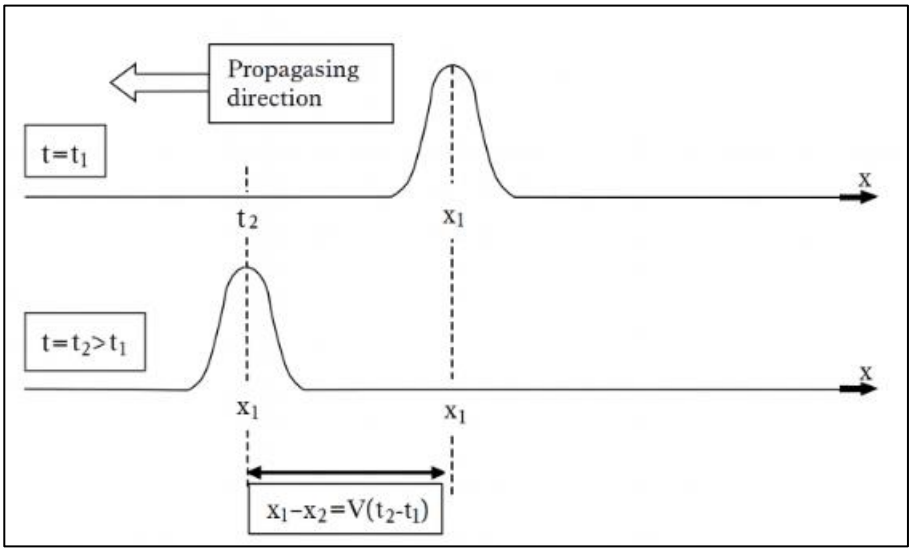


Figure 6.2: *Onde progressive dans le sens des x décroissants :* $G\left(t + \frac{x}{V}\right)$

In summary:

- $F\left(t - \frac{x}{V}\right)$: rightward-propagating wave
- $G\left(t + \frac{x}{V}\right)$: leftward-propagating wave

Each part of the wave maintains its shape and travels at constant velocity in its respective direction.

6.1.3 Sinusoidal Traveling Wave

Let's consider a progressive wave that propagates along the positive direction of the x-axis.

Suppose the point located at $x = 0$ (the origin) is subjected to a sinusoidal vibration, described by:

$$s(0,t) = A\cos(\omega t) \tag{6.14}$$

Where:

- A is the amplitude of the vibration,
- ω is the angular frequency (in radians per second),
- t is time.

We seek a wave function $s(x,\ t)$ that satisfies the one-dimensional wave equation and replicates this boundary condition at $x = 0$. Based on what we've learned from D'Alembert's solution, the wave propagating in the positive x-direction can be written as:

$$s(x,t) = A\cos\left(\omega\left(t - \frac{x}{V}\right)\right) \tag{6.15}$$

This equation represents a sinusoidal wave traveling to the right, with:

- Angular frequency ω,
- Propagation speed V,
- Wavelength $\lambda = \dfrac{2\pi V}{\omega}$.

This form shows how the same wave shape shifts in space over time. Each point on the wave oscillates in a sinusoidal fashion, and the entire wave moves steadily forward with constant speed V.

6.1.4 Superposition of Two Sinusoidal Traveling Waves

- **Case: Two Waves with the Same Frequency Moving in the Same Direction**

Let's examine the case where two sinusoidal waves:

- Have the same frequency ω,
- Travel in the same direction (say, to the right),
- Have amplitudes S_1 and S_2,
- Have initial phase angles ϕ_1 and ϕ_2,

We write them as:

$$s_1(x,t) = S_1 \cos\left(\omega\left(t - \frac{x}{V}\right) + \phi_1\right) \tag{6.16}$$

$$s_2(x,t) = S_2 \cos\left(\omega\left(t - \frac{x}{V}\right) + \phi_2\right) \tag{6.17}$$

Since the two waves have the same argument $\omega(t - \frac{x}{V})$, they can be added algebraically. The resulting wave is also a sinusoidal wave moving in the same direction, with:

$$s(x,t) = s_1(x,t) + s_2(x,t) = S \cos\left(\omega\left(t - \frac{x}{V}\right) + \phi\right) \tag{6.18}$$

Where:

- S is the resultant amplitude, calculated using vector addition (phasor addition) of the two original waves:

$$S = \sqrt{S_1^2 + S_2^2 + 2S_1S_2\cos(\phi_1 - \phi_2)} \tag{6.19}$$

- ϕ is the resultant phase shift, determined by the relative phases and amplitudes of the original waves.

In essence, two sinusoidal waves of the same frequency and direction superpose to form another sinusoidal wave of the same frequency and direction — but with new amplitude and phase depending on how the originals align.

❖ Superposition of Two Sinusoidal Waves Traveling in Opposite Directions

Now consider two sinusoidal waves:

- With the same frequency ω,
- Traveling in opposite directions,
- With equal amplitude A,
- One moving to the right, and one to the left.

We write:

$$s_1(x,t) = A\cos\left(\omega\left(t-\frac{x}{V}\right)\right) \quad \text{(rightward wave)} \tag{6.20}$$

$$s_2(x,t) = A\cos\left(\omega\left(t+\frac{x}{V}\right)\right) \quad \text{(leftward wave)} \tag{6.21}$$

The resultant wave is the sum of these two:

$$s(x,t) = s_1 + s_2 = A\left[\cos\left(\omega\left(t-\frac{x}{V}\right)\right)+\cos\left(\omega\left(t+\frac{x}{V}\right)\right)\right] \tag{6.22}$$

Using the trigonometric identity:

$$\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{6.23}$$

We get:

$$s(x,t) = 2A\cos\left(\frac{\omega x}{V}\right)\cos(\omega t) \tag{6.24}$$

This result is no longer a traveling wave. Instead, it represents a standing wave:

- The spatial variation is described by $\cos\left(\frac{\omega x}{V}\right)$, which defines the positions of nodes (where the wave amplitude is always zero) and antinodes (where amplitude is maximal).

- The time variation is described by $\cos(\omega t)$, meaning every point on the string oscillates in sync — no wave is moving along the string.

This phenomenon occurs frequently in systems with fixed boundaries, such as vibrating strings or air columns in musical instruments.

6.1.5 Phase Velocity

We now analyze how fast a specific part of a sinusoidal wave — such as a crest or a trough — moves through space. This speed is called the phase velocity.

Let's take a sinusoidal traveling wave propagating in the positive x-direction, written as:

$$s(x,t) = A\cos(\omega t - kx) \tag{6.25}$$

Where:

- A is the wave's amplitude (maximum displacement),
- ω is the angular frequency (radians per second),
- k is the wave number (radians per meter),
- x is the position, and
- t is the time.

This expression represents a periodic disturbance moving to the right.

The quantity:

$$\phi(x,t) = \omega t - kx \tag{6.26}$$

is the phase of the wave at point x and time t. A specific value of the phase corresponds to a recognizable feature of the wave (e.g., a crest when $\phi = 0$, or a trough when $\phi = \pi$).

To follow such a feature, we hold the phase constant, i.e.:

$$\omega t - kx = \phi_0 \tag{6.27}$$

for some constant ϕ_0.

Differentiating both sides with respect to time:

$$\omega - k\frac{dx}{dt} = 0 \quad \Rightarrow \quad \frac{dx}{dt} = \frac{\omega}{k} \tag{6.28}$$

The speed at which the phase propagates is called the phase velocity v_p :

$$v_p = \frac{dx}{dt} = \frac{\omega}{k} \tag{6.29}$$

This tells us that the position of any constant phase point — like a crest or trough — moves with velocity v_p . In a purely sinusoidal wave, the whole waveform moves at this speed.

We can express v_p using more familiar quantities:

$$\omega = 2\pi f, \quad k = \frac{2\pi}{\lambda} \quad \Rightarrow \quad v_p = \frac{\omega}{k} = f\lambda \tag{6.30}$$

Where:

- f is the frequency (Hz),
- λ is the wavelength (m),
- v_p is the phase velocity (m/s).

This relation is fundamental in wave physics and applies to all types of periodic waves: sound waves, water waves, electromagnetic waves, etc.

- **Important Notes on Phase Velocity**

- The phase velocity describes how fast the shape of the wave moves — it does not describe the movement of energy or particles in the medium.
- In some media (especially dispersive ones), different frequency components travel at different phase velocities, leading to distortion.
- In non-dispersive media (like an ideal string under tension), all components travel at the same v_p , and the wave maintains its shape.

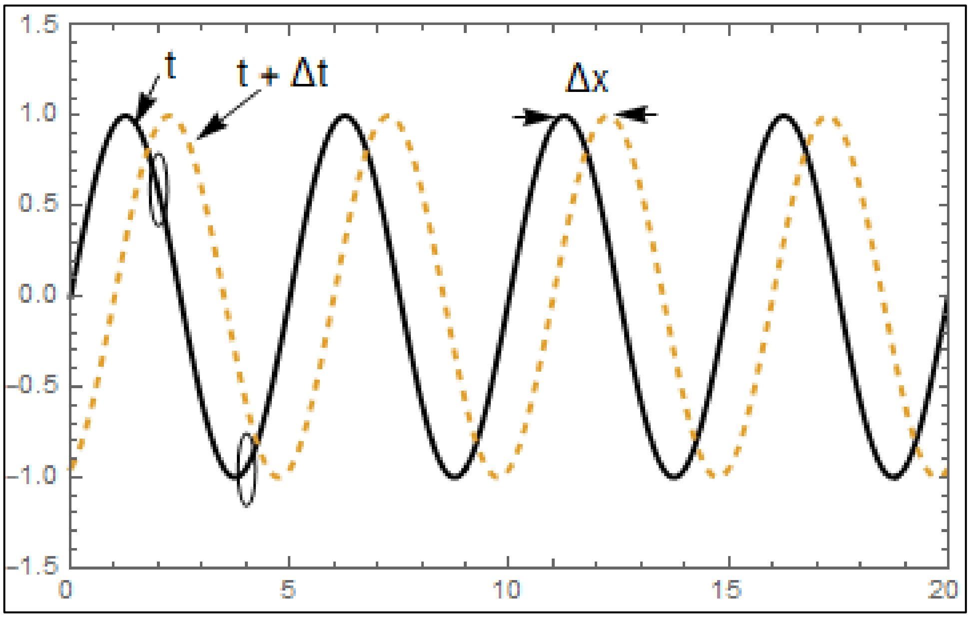


Figure 6.3: *Progressive Wave with Sinusoidal Form*

6.1.6 Group Velocity

The phase velocity V_ϕ — the speed at which a single wave crest travels — is not always the most meaningful or observable velocity when analyzing real wave motion.

In reality, most waves are not perfectly sinusoidal and do not extend infinitely. Instead, physical waves often appear as short bursts or wave packets, commonly called "pulses" or "wave groups".

These pulses are made up of multiple frequency components — in other words, they are composite waves, formed by the superposition of several sinusoidal waves with slightly different frequencies and wave numbers.

The speed at which the overall pulse or energy envelope moves is called the group velocity, denoted V_G. This velocity represents the propagation speed of the signal or the information carried by the wave.

Now, consider two cases :

1. **Non-dispersive Medium**:

   If the phase velocity $v_p = \frac{\omega}{k}$ is independent of frequency (i.e., the medium is non-dispersive), then all wave components of different frequencies travel at the same speed.

   Result: The group velocity V_G is equal to the phase velocity.

2. **Dispersive Medium**:

   If the phase velocity depends on frequency, meaning that different frequency components travel at different speeds, the medium is said to be dispersive.

   Result: The group velocity V_G is different from the phase velocity V_ϕ, and the shape of the wave packet can change as it propagates.

To better understand group velocity, consider the superposition of two sinusoidal waves with:

- Close (but not identical) frequencies: ω_1 and ω_2,
- Corresponding wave numbers: k_1 and k_2,
- Same amplitude A,
- Propagating in the same direction.

At $x = 0$, the combined wave might be written as:

$$s(0,t) = A\cos(\omega_1 t) + A\cos(\omega_2 t) \tag{6.31}$$

Using the trigonometric identity:

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \tag{6.32}$$

We get:

$$s(0,t) = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right) \tag{6.33}$$

This represents a carrier wave at average frequency $\frac{\omega_1+\omega_2}{2}$ modulated by an envelope at a slower rate $\frac{\omega_1-\omega_2}{2}$. The envelope forms the pulse, which is the observable part of the wave — and its speed of motion is the group velocity.

For waves with continuously varying frequency, the group velocity is given by the derivative:

$$V_G = \frac{d\omega}{dk} \tag{6.34}$$

Where:

- ω is the angular frequency,
- k is the wave number,
- $\frac{d\omega}{dk}$ is the slope of the dispersion relation $\omega(k)$.

In a **non-dispersive medium**, this derivative is constant, and $V_G = V_\phi$.

In a **dispersive medium**, the slope varies, and $V_G \neq V_\phi$.

- **Phase velocity**: Speed of crests, troughs, or individual points of constant phase.
- **Group velocity**: Speed of the overall shape of the pulse or energy transmission.

In optics, acoustics, and quantum mechanics, understanding the difference between these two is essential for analyzing wave packets, signal transmission, and even particle behavior.

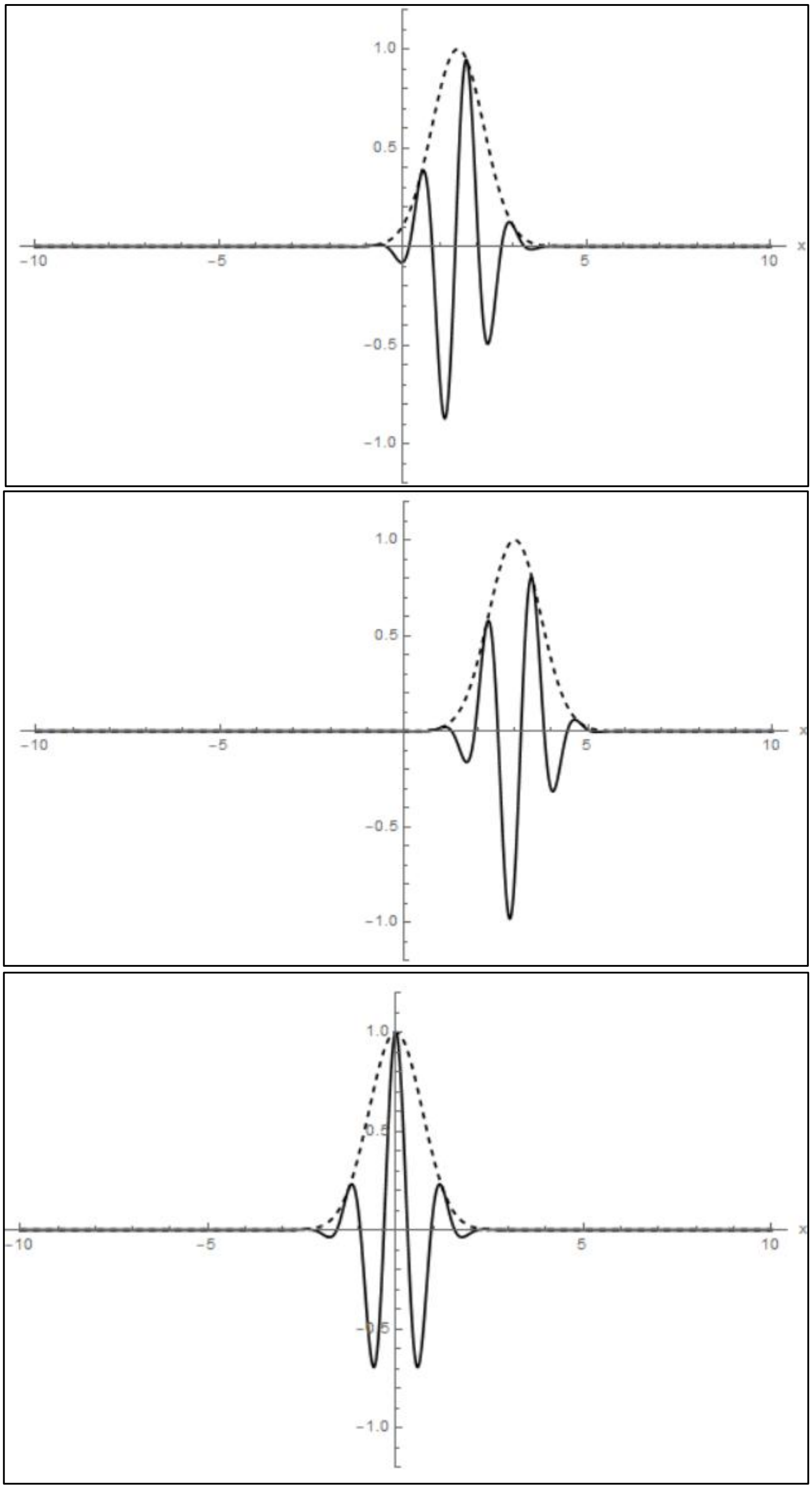


Figure 6.4: *Dynamics of a Propagating Wave Packet.*

6.1.7 Vector Waves

Up until now, the function $s(x,\ t)$ has represented a scalar quantity, such as the displacement along a single axis or pressure variation. However, many physical phenomena involve vector fields, where each point in space and time are associated with a vector instead of a scalar.

These vector-based wave phenomena are governed by similar propagation equations, but now expressed in vector form.

Let's denote the wave quantity as a vector field $\vec{A}(x,t)$. In a one-dimensional spatial medium (along the x-axis), the vector wave equation becomes:

$$\frac{\partial^2 \vec{A}}{\partial x^2}-\frac{1}{V^2}\frac{\partial^2 \vec{A}}{\partial t^2}=0 \tag{6.35}$$

This equation means that each component of the vector field $\vec{A}$ — namely $A_x,\ A_y,\ A_z$ — behaves as an independent scalar wave, satisfying the same type of propagation equation.

Explicitly, the equation applies separately to each component:

$$\begin{gathered}\frac{\partial^2 A_x}{\partial x^2}-\frac{1}{V^2}\frac{\partial^2 A_x}{\partial t^2}=0\\ \frac{\partial^2 A_y}{\partial x^2}-\frac{1}{V^2}\frac{\partial^2 A_y}{\partial t^2}=0\\ \frac{\partial^2 A_z}{\partial x^2}-\frac{1}{V^2}\frac{\partial^2 A_z}{\partial t^2}=0\end{gathered} \tag{6.36}$$

Thus, each component of the vector field propagates through space and time just like a scalar wave, with waveforms of the form:

$$f(t-\frac{x}{V}) \quad \text{and} \quad g(t+\frac{x}{V}) \tag{6.37}$$

These represent rightward and leftward traveling waves for each component, respectively.

✓ **Applications and Physical Examples**

- In **electromagnetism**, both the electric field $\vec{E}$ and magnetic field $\vec{B}$ are vector quantities that propagate according to Maxwell's equations, which reduce to wave equations in free space.
- In **mechanical systems** like elastic solids, the displacement of a particle can occur in three dimensions, meaning the displacement vector $\vec{u}(x,t)$ must be described by vector wave equations.
- In **fluid dynamics**, vorticity and velocity fields often exhibit vectorial wave behavior.

Although the wave is vectorial in nature, the analysis techniques used for scalar waves — such as D'Alembert's solution, phase velocity, and superposition — apply individually to each vector component.

6.2 Propagation in Three-Dimensional Space

6.2.1 The Propagation Equation

In the previous sections, we considered wave propagation along a single spatial dimension. However, in real-world situations, waves often travel in three-dimensional space. This requires generalizing the wave equation to multiple spatial coordinates.

In a Cartesian coordinate system $(x,\ y,\ z)$, the three-dimensional wave equation is written as:

$$\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}=\frac{1}{V^2}\frac{\partial^2 s}{\partial t^2} \tag{6.38}$$

This is a partial differential equation describing the evolution of a scalar wave function $s(x,\ y,\ z,\ t)$ in space and time.

- The left-hand side represents the spatial variation of the wave — the second derivatives with respect to x, y, and z capture the curvature of the wavefront in each direction.
- The right-hand side reflects the temporal acceleration of the wave field.

The spatial part on the left can be compactly written using the Laplacian operator ∇^2:

$$\nabla^2 s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \tag{6.39}$$

So, the wave equation becomes:

$$\nabla^2 s = \frac{1}{V^2}\frac{\partial^2 s}{\partial t^2} \tag{6.40}$$

This is the standard form of the scalar wave equation in three dimensions.

6.2.2 Sinusoidal Plane Progressive Wave

- **Definition**

A sinusoidal (or harmonic) plane progressive wave propagating in space is defined as a wave that travels in a fixed direction, without changing its shape, and with wavefronts that are infinite parallel planes perpendicular to the direction of motion.

Let’s denote:

- $\vec{u}$: a unit vector indicating the direction of propagation.

Then the wave function is given by:

$$s(\vec{r},t) = A\cos(\omega t - \vec{k}\cdot\vec{r} + \phi_0) \tag{6.41}$$

Where:

- A : amplitude of the wave,
- ω : angular frequency,

- $\vec{k}$: the wave vector, defined by $\vec{k} = \frac{2\pi}{\lambda}\vec{u}$, and points in the direction of propagation,
- $\vec{r}$: position vector $(x,\ y,\ z)$,
- ϕ_0 : initial phase.

✓ **Key Properties**

- The term $\vec{k}\cdot\vec{r}$ (scalar product) represents the projection of the position vector onto the propagation direction.
- The phase of the wave $\phi = \omega t - \vec{k}\cdot\vec{r} + \phi_0$ remains constant on each plane perpendicular to $\vec{u}$.
- Since the phase depends only on the distance along $\vec{u}$, the wave is said to be plane — all points lying on the same plane perpendicular to $\vec{u}$ experience the same oscillation at the same time.

This wave model is idealized and used to represent wave motion in:

- **Optics** (light waves far from their source),
- **Acoustics** (sound waves in uniform media),
- **Electromagnetic theory** (radio waves, microwaves),
- **Quantum mechanics** (wavefunctions of free particles).

Although no real wave is truly infinite in extent, plane waves are excellent approximations for waves over small regions of space or far from their source.

➢ **Dispersion Relation**

To find whether the sinusoidal plane wave is a valid solution of the wave equation, we substitute the expression:

$$s(\vec{r},t) = A\cos(\omega t - \vec{k}\cdot\vec{r} + \phi_0) \tag{6.42}$$

into the three-dimensional wave equation:

$$\nabla^2 s = \frac{1}{V^2}\frac{\partial^2 s}{\partial t^2} \tag{6.43}$$

By doing this, we derive a relationship between the wave vector $\vec{k}$ and the angular frequency ω— this relationship is known as the dispersion relation.

✓ Mathematical Form of the Dispersion Relation

$$k = k(\omega) \tag{6.44}$$

In the case of an ideal, non-dispersive medium, where all frequency components travel at the same speed V, the dispersion relation becomes:

$$|\vec{k}| = \frac{\omega}{V} \tag{6.45}$$

Or in scalar form:

$$k = \frac{\omega}{V} \tag{6.46}$$

Where:

- $k = |\vec{k}|$ is the magnitude of the wave vector,
- ω is the angular frequency,
- V is the phase velocity of the wave in the medium.

This relation guarantees that the wave function is a valid solution to the wave equation. It also tells us how frequency and wavelength are linke**d** in the propagation medium.

✓ In Dispersive Media

If the medium is dispersive, the velocity V depends on ω\omega, and the dispersion relation becomes more complex:

$$k = f(\omega) \quad \text{or} \quad \omega = f^{-1}(k) \tag{6.47}$$

In this case, different frequency components travel at different phase velocities, which leads to distortion of wave packets and varying group velocity.

➢ Wave Surface (Equiphasic Surface)

A wave surface, also known as an equiphase surface, is defined as the set of all points in space where the wave function $s(\vec{r},t)$ has the same value at the same time. In other words, it's the surface along which the phase of the wave remains constant.

Let's determine the wave surface that passes through a point M_0 at a specific time t. The condition for any other point M in space to lie on the same wave surface is:

$$\omega t - \vec{k}\cdot\vec{r} = \omega t - \vec{k}\cdot\vec{r}_0 \qquad (6.48)$$

Which simplifies to:

$$\vec{k}\cdot(\vec{r}-\vec{r}_0) = 0 \qquad (6.49)$$

This equation defines a plane perpendicular to the wave vector $\vec{k}$. It means all points $\vec{r}$ that lie on this surface are at the same phase in the wave's oscillation at that instant t.

✓ Interpretation

- The wave surface is a plane for a plane wave — hence the name.
- These surfaces move in space as time progresses, perpendicular to the wave vector.
- The distance between two successive wave surfaces (where the phase differs by 2π) is the wavelength λ.

So, in a plane sinusoidal wave, wave surfaces are parallel planes, uniformly spaced, and they move forward in the direction of propagation.

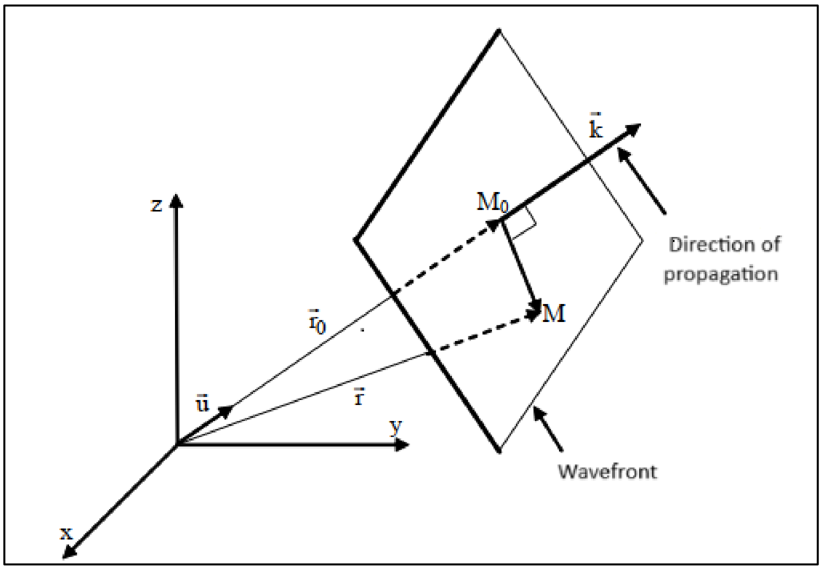


Figure 6.5: *Representation of a Plane Wavefront.*

➢ Polarization

When dealing with a plane progressive sinusoidal wave represented by a vector quantity $\vec{A}(\vec{r},t)$, the orientation of this vector relative to the wave surface (or direction of propagation) defines the type of wave and its polarization state.

✓ Longitudinal Waves

- If the vector $\vec{A}$ is always parallel to the direction of propagation (i.e., it points in the same direction as the wave moves), or equivalently, perpendicular to the wavefronts, the wave is called a longitudinal wave.
- In this case, the oscillations occur along the direction of travel.
- **Example**: Sound waves in air — the air particles vibrate back and forth in the same direction the wave is moving.

✓ Transverse Waves

- If the vector $\vec{A}$ lies within the wave surface (i.e., perpendicular to the direction of propagation), the wave is a transverse wave.
- In this case, the oscillations occur in directions orthogonal to the wave's motion.

Transverse waves can be further classified based on how the tip of the vector $\vec{A}$ moves over time:

- **Linear Polarization**:
    - The vector oscillates in a fixed straight-line direction.
    - The wave maintains a constant orientation of vibration.
    - **Example**: Light polarized through a polarizing filter.
- **Circular Polarization**:
    - The vector's tip traces a circular path over time.
    - The magnitude remains constant, but the direction continuously rotates at a constant rate.
    - Occurs when two perpendicular linear components have the same amplitude and a 90° phase shift.
- **Elliptical Polarization**:
    - The vector describes an elliptical trajectory.
    - A general case where the two perpendicular components have different amplitudes or a non-90° phase shift.
    - This is the most general form of polarization; linear and circular are just special cases.

Summary Table

Orientation of $\vec{A}$	**Wave Type**	**Polarization Type**
Parallel to propagation	Longitudinal	N/A
Perpendicular to propagation	Transverse	Linear, Circular, Elliptical

Exercises

Exercise 1 : **One-Dimensional Sinusoidal Wave**

A sinusoidal progressive wave travels along the x-axis with a speed of $v = 200\text{m/s}$. Its expression is:

$$s(x,t) = 0.05\cos(20\pi t - \pi x)$$

1. What is the frequency f of the wave?
2. What is the wavelength λ?
3. Verify that the dispersion relation $v = \lambda f$ holds.
4. What is the displacement s at position $x = 2\text{m}$ and time $t = 0.1\text{s}$?

Exercise 2: **Group Velocity vs Phase Velocity**

Two waves of equal amplitude but slightly different frequencies are superimposed:

$$s_1(x,t) = A\cos(\omega_1 t - k_1 x), \quad s_2(x,t) = A\cos(\omega_2 t - k_2 x)$$

with:

- $\omega_1 = 1000\text{rad/s}, \omega_2 = 1100\text{rad/s}$,
- $k_1 = 4.9\text{rad/m}, k_2 = 5.2\text{rad/m}$

1. Calculate the average phase velocity.
2. Determine the group velocity $v_g = \dfrac{\Delta\omega}{\Delta k}$.
3. Explain the physical meaning of the difference between phase and group velocity.

Exercise 3: **Polarization of a Plane Electromagnetic Wave**

A plane electromagnetic wave propagates in vacuum along the z -axis. The electric field is given by:

$$\vec{E}(z,t) = E_0\left[\cos(\omega t - kz)\vec{e}_x + \sin(\omega t - kz)\vec{e}_y\right]$$

1. What is the nature of the wave's polarization? Justify.
2. What is the direction of propagation?
3. Write the expression for the associated magnetic field $\vec{B}(z,t)$, assuming the wave propagates in free space.

Chapter 7– Vibrating Strings

7.1 Wave Equation

Let us consider a tight, perfectly straight string stretched along the horizontal x-axis. The string is assumed to be infinitely long to simplify boundary effects.

We are interested in the propagation of a small disturbance or vibration that moves along the string. Assume this disturbance occurs in the vertical direction, along the y-axis.

To derive the wave equation governing this motion, we analyze the mechanics of a small portion of the string.

- Let T represent the tension force applied to the string (constant throughout).
- Consider a small segment of the string located at position x, with length Δx.
- The mass of this segment is:

$$\Delta m = \mu \Delta x \tag{7.1}$$

Where:

- μ is the linear mass density of the string (mass per unit length), expressed in kg/m.

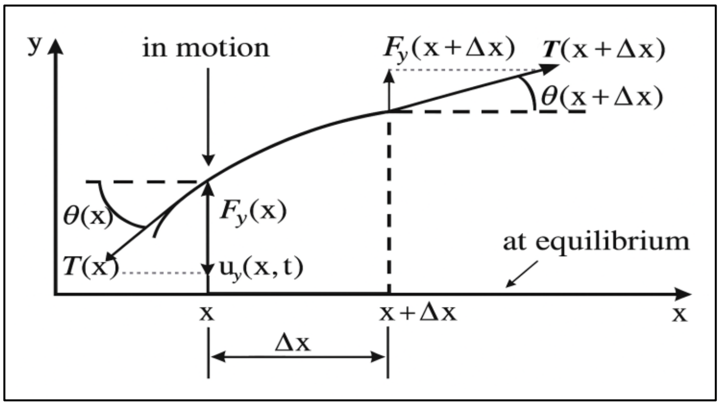


Figure 7.1: *Transverse Vibrations of a String*

Our goal is to apply Newton's second law to this small segment, using the tension forces at both ends and the resulting acceleration, to derive the equation of motion for transverse oscillations — leading to the classical wave equation for a vibrating string.

When the string is disturbed from its equilibrium position, the segment we observe is no longer straight — it exhibits a slight curvature. We analyze small-amplitude oscillations, meaning the displacements are small enough for certain mathematical simplifications to hold.

Let the transverse displacement be described by the vector:

$$\vec{u}(x,t) = u(x,t)\vec{e}_y \tag{7.2}$$

That is, the displacement is entirely in the vertical y-direction.

Because the slope of the string is small, we can approximate:

$$\sin(\theta)|_x \approx \tan(\theta)|_x \approx \frac{\partial u}{\partial x}\Big|_x \quad \text{and} \quad \sin(\theta)|_{x+\Delta x} \approx \tan(\theta)|_{x+\Delta x} \approx \frac{\partial u}{\partial x}\Big|_{x+\Delta x} \tag{7.3}$$

These approximations :

- Neglect stretching of the string,
- Assume the tension T remains constant along the string.

Now let's calculate the net vertical force acting on the small segment between x and $x+\Delta x$:

- The downward force at point x is approximately:

$$F(x,t) = T \cdot \frac{\partial u}{\partial x}\Big|_x \tag{7.4}$$

- The upward force at point $x+\Delta x$ is:

$$F(x+\Delta x,t) = T \cdot \frac{\partial u}{\partial x}\Big|_{x+\Delta x} \tag{7.5}$$

- The net vertical force on the segment is:

$$R = F(x+\Delta x,t) - F(x,t) = T\left[\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_{x}\right] = T\cdot\frac{\partial^2 u}{\partial x^2}\cdot\Delta x \tag{7.6}$$

The total vertical force R must equal the mass of the segment times its vertical acceleration:

- The mass of the segment: $\Delta m = \mu\Delta x$
- Acceleration: $\frac{\partial^2 u}{\partial t^2}$

So :

$$T\cdot\frac{\partial^2 u}{\partial x^2}\cdot\Delta x = \mu\cdot\Delta x\cdot\frac{\partial^2 u}{\partial t^2} \tag{7.7}$$

Divide both sides by Δx :

$$\mu\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} \tag{7.8}$$

We define the wave speed V as:

$$V = \sqrt{\frac{T}{\mu}} \tag{7.9}$$

Thus, the final wave equation for the vibrating string becomes:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{7.10}$$

This is the classical wave equation, describing transverse waves propagating along a stretched string. The speed V depends only on the tension T and the mass per unit length μ of the string.

7.2 Harmonic Progressive Waves

7.2.1 Definition

A harmonic progressive wave (or sinusoidal wave) traveling along the x-axis is defined by the following real-valued function:

$$u(x,t) = U_0 \cos(\omega t - kx) \tag{7.11}$$

Where:

- U_0 is the amplitude of the wave (maximum displacement),
- ω is the angular frequency (in radians per second),
- k is the wave number (in radians per meter),
- t is time, and x is the position along the string.

Alternatively, using complex notation, the wave can be expressed as:

$$u(x,t) = U_0 e^{i(\omega t - kx)} \tag{7.12}$$

This form is especially useful in mathematical analysis, even though the physical displacement corresponds to the real part of the expression.

The wave number k is defined as:

$$k = \frac{\omega}{V} = \frac{2\pi}{\lambda} \tag{7.13}$$

Where:

- V is the wave speed,
- λ is the wavelength.

This relation ties together the wave's frequency, speed, and spatial repetition.

7.2.2 Force at a Point

In the context of a vibrating string, the force at a point refers to the vertical component (along the y-axis) of the tension force that the left segment of the string exerts on the right segment at that specific location.

Mathematically, this vertical force is expressed as:

$$F(x,t) = -T\frac{\partial u}{\partial x} \tag{7.14}$$

Where:

- T is the constant tension in the string,
- $\frac{\partial u}{\partial x}$ is the slope of the string at point x, giving the direction of the force.

For a sinusoidal progressive wave described by the complex expression:

$$u(x,t) = U_0 e^{i(\omega t - kx)} \tag{7.15}$$

The vertical force becomes:

$$F(x,t) = -T\cdot\frac{\partial u}{\partial x} = -T\cdot(-ikU_0 e^{i(\omega t - kx)}) = ikTU_0 e^{i(\omega t - kx)} \tag{7.16}$$

The velocity of a particle on the string (its time derivative) is:

$$\dot{u}(x,t) = \frac{\partial u}{\partial t} = i\omega U_0 e^{i(\omega t - kx)} \tag{7.17}$$

From the two expressions above:

$$F(x,t) \propto e^{i(\omega t - kx)} \tag{7.18}$$

$$\dot{u}(x,t) \propto e^{i(\omega t - kx)} \tag{7.19}$$

We see that the force and the particle velocity are in phase. This means:

- They reach their maxima and minima at the same time,
- There is no phase shift between the applied force and the motion it induces.

This is a key feature of harmonic progressive waves in ideal strings.

7.2.3 Impedance

In wave mechanics on a string, the impedance at a point is defined as the ratio of the complex amplitude of the force to the complex amplitude of the particle velocity at that point.

Mathematically, this is expressed as:

$$Z(x) = \frac{F_y}{\dot{u}_y} \tag{7.20}$$

Where:

- F_y is the vertical force at point x,
- $\dot{u}_y$ is the vertical velocity of the string element at point x.

For a harmonic progressive wave, substituting the earlier expressions for force and velocity:

$$F_y = ikTU_0 e^{i(\omega t - kx)} \tag{7.21}$$

$$\dot{u}_y = i\omega U_0 e^{i(\omega t - kx)} \tag{7.22}$$

Taking the ratio:

$$Z(x) = \frac{ikTU_0}{i\omega U_0} = \frac{kT}{\omega} \tag{7.23}$$

Now using the relation $k = \omega / V$, we get:

$$Z(x) = \frac{T}{V} \tag{7.24}$$

Since the wave speed $V = \sqrt{T / \mu}$, this simplifies to:

$$Z(x) = \mu V \tag{7.25}$$

This constant value is called the characteristic impedance of the string, denoted:

$$Z_c = \sqrt{\mu T} = \mu V \tag{7.26}$$

It depends only on :

- μ, the linear mass density,
- T, the tension in the string.

For a plane progressive wave, the impedance is the same at every point on the string:

$$Z(x) = Z_c \quad \forall x \tag{7.27}$$

This uniformity plays a critical role in how waves transmit through or reflect from boundaries and different media.

7.3 Free Oscillations of a Finite-Length String

Let us consider a string of length L, fixed at both ends, located at positions $x = 0$ and $x = L$. We aim to find solutions to the wave equation for this system.

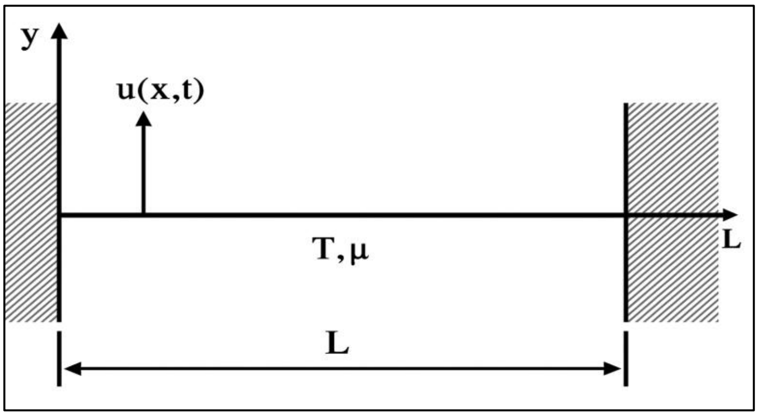


Figure 7.2: *String of Length L Fixed at Both Ends.*

We look for a solution of the form:

$$u(x,t) = g(x)\cdot f(t) \tag{7.28}$$

Substituting into the standard wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{7.29}$$

gives:

$$\frac{1}{g}\frac{d^2 g}{dx^2} = \frac{1}{V^2}\cdot\frac{1}{f}\frac{d^2 f}{dt^2} \tag{7.30}$$

Since the left side depends only on x and the right side only on t, both sides must be equal to a constant. To ensure bounded, physical solutions, we take this constant to be negative, say $-k^2$.

Letting $\omega = kV$, we then get two ordinary differential equations:

$$\frac{d^2 g}{dx^2} = -k^2 g \quad \text{and} \quad \frac{d^2 f}{dt^2} = -\omega^2 f \tag{7.31}$$

These equations have sinusoidal solutions:

- In time :

$$f(t) = A\cos(\omega t) + B\sin(\omega t) \tag{7.32}$$

- In space:

$$g(x) = C\cos(kx) + D\sin(kx) \tag{7.33}$$

Combining both:

$$u(x,t) = [A\cos(\omega t) + B\sin(\omega t)]\cdot[C\cos(kx) + D\sin(kx)] \tag{7.34}$$

Since the string is fixed at both ends, we impose:

$$u(0,t)=0 \Rightarrow C=0 \tag{7.35}$$

$$u(L,t)=0 \Rightarrow \sin(kL)=0 \Rightarrow kL=n\pi \Rightarrow k=\frac{n\pi}{L} \tag{7.36}$$

with $n=1,2,3,\ldots$

Hence, the allowed waveforms must satisfy:

$$u(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos\left(\omega_n t\right)+b_n\sin\left(\omega_n t\right)\right]\sin\left(\frac{n\pi x}{L}\right) \tag{7.37}$$

Where:

$$k_n=\frac{n\pi}{L} \tag{7.38}$$

$$\omega_n=k_nV=\frac{n\pi V}{L} \tag{7.39}$$

- ω_n are the natural frequencies (also called eigenfrequencies)

Suppose at $t = 0$, the string has:

- Initial displacement: $u\left(x,\ 0\right) = u_0\left(x\right)$
- Initial velocity: $\dot{u}(x,0)=v_0(x)$

Then:

$$u_0(x)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad v_0(x)=\sum_{n=1}^{\infty}-\omega_n b_n\sin\left(\frac{n\pi x}{L}\right) \tag{7.40}$$

To find a_n and b_n, use Fourier's method:

Multiply both sides of the equations by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from 0 to L. Using orthogonality:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{L}{2} & \text{if } m = n \end{cases} \tag{7.41}$$

This leads to:

$$a_n = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{7.42}$$

$$b_n = -\frac{2}{\omega_n L}\int_0^L v_0(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{7.43}$$

These coefficients fully determine the time evolution of the string's vibration.

7.4 Reflection and Transmission

7.4.1 Reflection and Transmission Between Two Semi-Infinite Strings

Consider two semi-infinite strings joined at point $x = 0$. Each string has a different linear mass density:

- μ_1 for the string on the left,
- μ_2 for the string on the right.

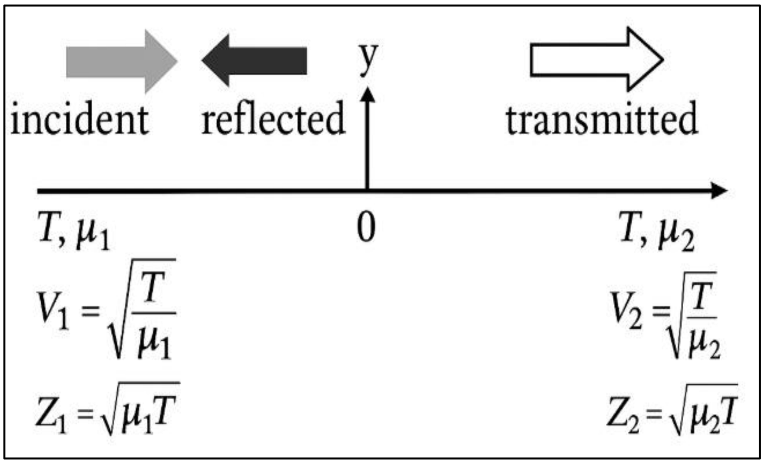


Figure 7.3: *Wave Reflection and Transmission Between Two Semi-Infinite Strings.*

An incident wave originating from the left (coming from $x = -\infty$) travels toward the junction at $x = 0$. At this junction, the wave splits:

- A reflected wave travels back into the first string,
- A transmitted wave continues into the second string.

To find the amplitudes of the reflected and transmitted waves, we apply continuity conditions at the junction:

- Continuity of displacement,
- Continuity of force.

These give us the reflection coefficient R_u and the transmission coefficient T_u, defined as:

$$R_u = \frac{U_R}{U_i}, \quad T_u = \frac{U_T}{U_i} \tag{7.44}$$

Where:

- U_i is the amplitude of the incident wave,
- U_R is the amplitude of the reflected wave,
- U_T is the amplitude of the transmitted wave.

Let :

- $Z_1 = \sqrt{\mu_1 T}$ be the characteristic impedance of the first string,
- $Z_2 = \sqrt{\mu_2 T}$ be the same for the second string.

Then:

$$R_u = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad T_u = \frac{2Z_1}{Z_1 + Z_2} \tag{7.45}$$

These formulas describe how much of the wave is reflected or transmitted depending on the impedance mismatch between the two strings.

7.4.2 Reflection on an Arbitrary Impedance

Now consider a single semi-infinite string (mass density μ, tension T) which is terminated at $x = 0$ by a mechanical impedance Z_T. An incident harmonic wave travels along the string from the left, and is partially reflected at the termination point.

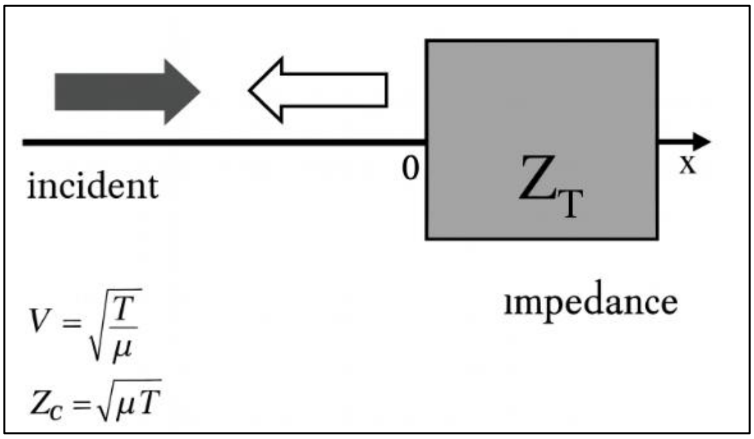


Figure 7.4: *Wave Propagation in a Semi-Infinite String Terminated with Impedance* Z_T.

The total displacement at any point x is given by:

$$u(x,t) = U_i e^{i(\omega t - kx)} + U_R e^{i(\omega t + kx)} \tag{7.46}$$

Where:

- U_i: amplitude of the incident wave,

- U_R : amplitude of the reflected wave.

From this, we compute:

- Particle velocity :

$$\dot{u}(x,t) = \frac{\partial u}{\partial t} = i\omega\left(U_i e^{i(\omega t - kx)} + U_R e^{i(\omega t + kx)}\right) \tag{7.47}$$

- Vertical force from tension :

$$F(x,t) = -T\frac{\partial u}{\partial x} = -ikT\left(U_i e^{i(\omega t - kx)} - U_R e^{i(\omega t + kx)}\right) \tag{7.48}$$

At the termination point $x = 0$, we apply the impedance condition:

$$Z_T = \frac{F(0,t)}{\dot{u}(0,t)} \tag{7.49}$$

Substituting the expressions at $x = 0$, we find the reflection coefficient:

$$R_u = \frac{U_R}{U_i} = \frac{Z_c - Z_T}{Z_c + Z_T} \tag{7.50}$$

Where:

- $Z_c = \sqrt{\mu T}$ is the characteristic impedance of the string,
- Z_T is the termination impedance.

This formula shows that complete reflection occurs if $Z_T \to \infty$ (e.g., a fixed end), while no reflection occurs when $Z_T = Z_c$, indicating perfect impedance matching.

Exercises

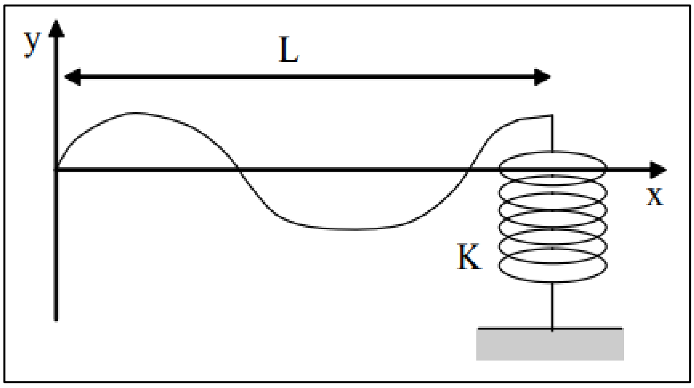
Exercise 1:

Consider two semi-infinite strings connected at the point $x = 0$. The first string, extending from $-\infty$ to 0, has a linear mass density μ_1, while the second string, extending from 0 to $+\infty$, has a linear mass density $\mu_2 = 0.25\mu_1$. A sinusoidal wave is incident from the left ($x < 0$) and propagates toward the junction in the positive x-direction. At the junction point $x = 0$, the wave undergoes partial reflection. The amplitude of the incident wave is denoted U_0, and its angular frequency is ω.

1. Determine the reflection coefficient at the boundary $x = 0$.
2. Demonstrate that the total wave in the first string results in an amplitude that oscillates between a maximum and a minimum value, denoted $U_{\max}$ and $U_{\min}$, respectively.
   (a) Derive the expression for $U_{\max}$ in terms of U_0, and calculate the positions of the maxima as a function of the wavelength λ.
   (b) Derive the expression for $U_{\min}$ in terms of U_0, and calculate the positions of the minima as a function of the wavelength λ.
   (c) Compute the standing wave ratio (SWR) for the system.

Exercise 2:

A string of length L is stretched between two points located at $x = 0$ and $x = L$. The end at $x = 0$ is fixed, while the end at $x = L$ is attached to a spring of stiffness K. The tension in the string is T. At equilibrium, the string lies horizontally, and its weight can be neglected. We are interested in studying sinusoidal standing transverse waves of angular frequency ω. The wave propagation speed is V.



1. Write the expression for the transverse displacement $u_y(x,\ t)$ at any point xx along the string and at any time t.
2. By applying the boundary condition at $x\ =\ 0$, show that the displacement can be expressed in the form:

$$u_y(x,t) = Ae^{i\omega t} f(x)$$

where $f(x)$ is a function to be explicitly determined.

3. Formulate the boundary condition at the point $x\ =\ L$.
4. Show that the allowed angular frequencies (eigenfrequencies) must satisfy the condition:

$$\tan\left(\frac{\omega L}{V}\right) = C \cdot \frac{\omega L}{V}$$

where C is a constant to be specified. Propose a method for solving this equation graphically or numerically.

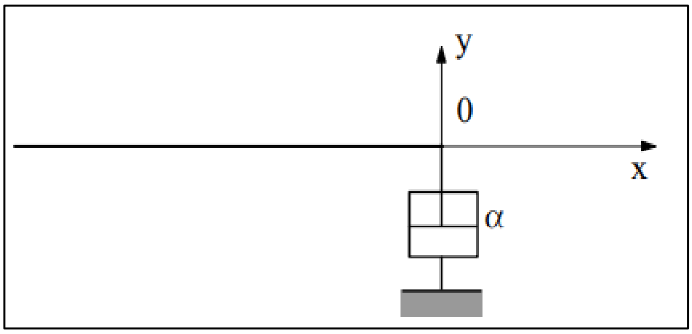
5. Determine the first three eigenfrequencies ω_1, ω_2, and ω_3 in the limiting case where $\frac{KL}{T} \to \infty$.

Exercise 3:

Consider a homogeneous string with a linear mass density μ, stretched horizontally under a tension T that is much greater than the weight of the string, allowing us to neglect gravity. The string is infinitely long but is terminated at $x=0$ by a damper characterized by a viscous damping coefficient α. A transverse incident wave arrives from $-\infty$ and propagates along the string in the positive x-direction. The incident wave is given by the transverse displacement:

$$u_i(x,t)=U_i e^{i(\omega t-kx)}$$

where k is the wave number (the magnitude of the wave vector).



1. Determine the expression for the reflection coefficient R (in terms of displacement amplitude) at $x=0$. What are the magnitude and phase (argument) of R in the special case where $\alpha \le \sqrt{\mu T}$?
2. Write the expression for the total displacement $u(x,\ t)$ at any point along the string, in terms of the problem parameters and the reflection coefficient R.
3. Show that the total displacement at any point on the string can be expressed as the sum of a traveling wave and a standing wave, and that it takes the form:

$$u(x,t)=U_p e^{i(\omega t-kx)}+U(x)e^{i\omega t}$$

Here, U_p and $U(x)$ are real-valued amplitudes corresponding to the traveling and standing components, respectively.

Compute the expressions of:

- U_p in terms of R and U_i,
- $U(x)$ in terms of R, U_i, k, and x.

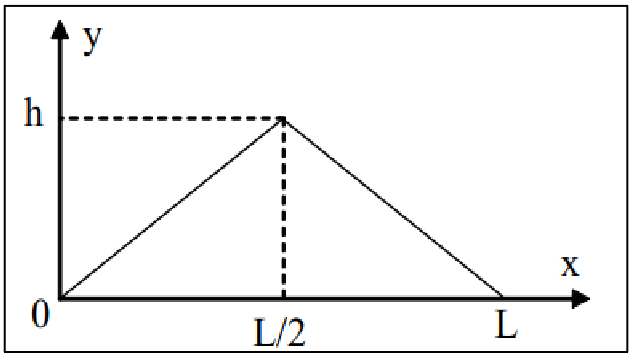
4. Determine the behavior of $u(x,\ t)$ in the following special cases:

   (a) When $\alpha = \sqrt{\mu T}$,

   (b) When $\alpha = 0$.

Exercise 4:

A string of length L and total mass m is stretched between two fixed points under a tension T. At the initial moment $t = 0$, the string is plucked at its midpoint and displaced vertically by a height h from its horizontal equilibrium position, then released with zero initial velocity. The initial displacement profile $u(x,0)$ of the string is illustrated in the figure below.



Task:

- Establish the expression for the transverse displacement $u(x,t)$ of the string at any time t.

Chapter 8 – Elastic Waves in Solids

8.1 Elastic Properties of Solids

8.1.1 Strain (Deformation)

Let us consider a deformable, continuous, and isotropic solid—in the form of a straight rod with a small cross-section, meaning its lateral dimensions are much smaller than its length ℓ. Suppose one end of the rod is rigidly fixed, while the other end is subject to a tensile force F applied along its axis.

As a result of this tensile force, the rod elongates. This deformation remains as long as the tensile force continues to act.

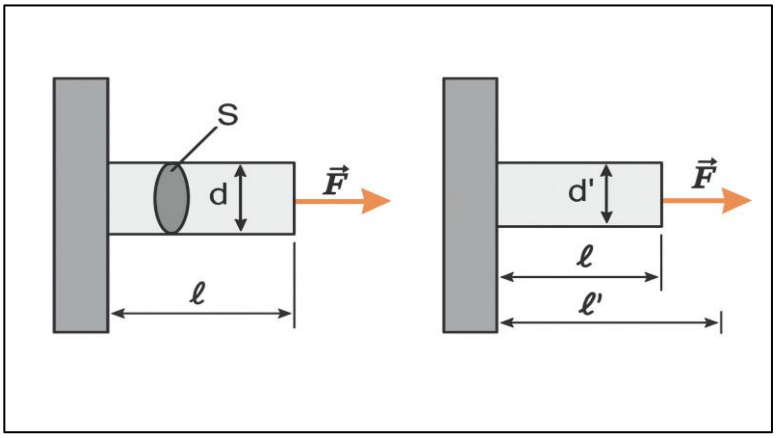


Figure 8.1: *Axial Elongation of a Rod Subjected to Tension.*

The relative elongation (also called engineering strain) is defined as:

$$\frac{\delta\ell}{\ell} = \frac{\ell_0 - \ell}{\ell} \tag{8.1}$$

Where:

- ℓ is the original (undeformed) length of the rod,

- ℓ_0 is the length after deformation.

However, since not all parts of the rod may stretch equally, it is important to define a local strain—that is, a measure of deformation that varies along the length of the rod. This is typically expressed as a function of position x, referring to each point along the rod.

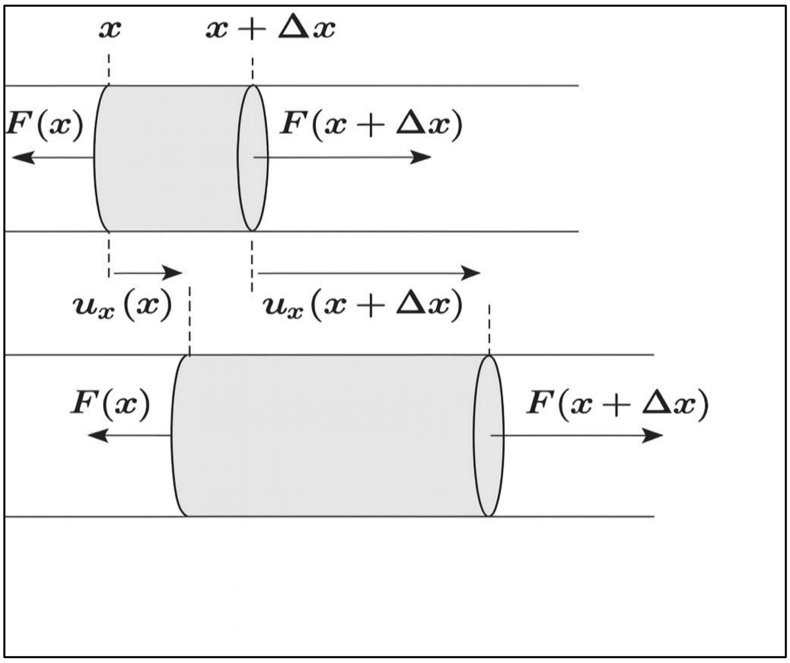


Figure 8.2: *Local Structural Deformation in a Bar.*

To analyze deformation more precisely, consider a small segment of the rod initially located between positions x and $x+\Delta x$.

When the tensile force is applied, each point in this segment shifts:

- The point at x moves to $x + u_x(x)$
- The point at $x+\Delta x$ moves to $x+\Delta x+u_x(x+\Delta x)$

So, the new length of the segment becomes:

$$\Delta x_0 = [x+\Delta x+u_x(x+\Delta x)]-[x+u_x(x)] \tag{8.2}$$

If Δx is very small, we can use a first-order Taylor expansion:

$$u_x(x+\Delta x) \approx u_x(x) + \frac{\partial u_x}{\partial x} \cdot \Delta x \tag{8.3}$$

Then, the relative elongation becomes:

$$\frac{\Delta x_0 - \Delta x}{\Delta x} = \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} \tag{8.4}$$

Taking the limit as $\Delta x \to 0$, we define the strain ε as:

$$\varepsilon = \frac{\partial u_x}{\partial x} \tag{8.5}$$

This quantity measures the local deformation and is dimensionless, as it compares two lengths.

- If $\varepsilon > 0$, the material is locally **stretched**.
- If $\varepsilon < 0$, it is locally **compressed**.

8.1.2 Average Stress

As the rod elongates under an applied force F (normal to the cross-section of area S), we define the average stress as:

$$\tau = \frac{F}{S} \tag{8.6}$$

This represents a force per unit area, with units of Pascals (Pa) or N/m^2.

Note: In general, F may vary along the length of the rod, making τ a function of x.

8.1.3 Hooke’s Law

In the linear elastic regime (small stresses and small deformations), the strain is proportional to the applied stress. This relationship is given by Hooke’s Law:

$$\varepsilon = \frac{1}{E} \cdot \tau = \frac{1}{E} \cdot \frac{F}{S} \quad \text{or equivalently} \quad \tau = E \cdot \varepsilon \tag{8.7}$$

Where:

- E is the Young's modulus, a material property that quantifies stiffness,
- It has the same units as stress: Pascals (Pa).

8.1.4 Poisson's Ratio

When the rod stretches along the longitudinal axis (x -axis), it also narrows in the transverse directions.

Let d be the original lateral dimension of the rod, and d_0 be the dimension after stretching. Then:

$$\frac{\Delta d}{d} = \frac{d_0 - d}{d} = -\nu \cdot \frac{\Delta \ell}{\ell} \tag{8.8}$$

Here, ν is the Poisson's ratio—a dimensionless, positive number that quantifies the amount of lateral contraction per unit longitudinal extension. For most metals, $\nu < 0.5$.

8.1.5 Hooke's Law for Shear Forces

When a tangential (shear) force F is applied to a surface of area S on a solid bar (instead of a normal force), the bar does not stretch in length. Instead, it undergoes angular deformation, causing the originally vertical edges to rotate by a small angle α .

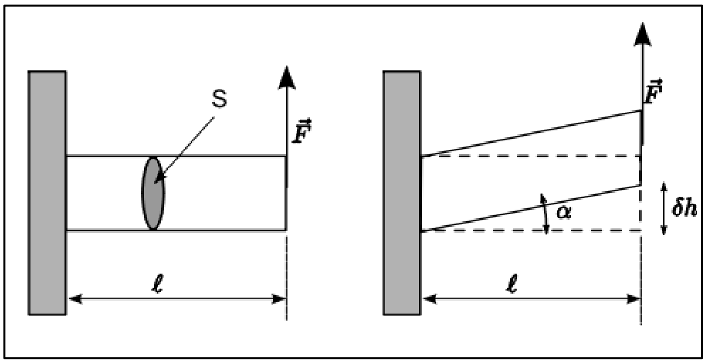


Figure 8.3: *Shear Loading Applied to a Rod.*

In the linear elastic regime (small angles α), the deformation angle is proportional to the applied shear stress. Hooke's law for shear then becomes:

$$\frac{F}{S} = G \cdot \alpha \approx G \cdot \tan(\alpha) = G \cdot \frac{\delta h}{\ell} \tag{8.9}$$

Where:

- G is the shear modulus, characterizing the material's resistance to shearing,
- δh is the horizontal shift of the top surface,
- ℓ is the vertical height of the sample.

Like Young's modulus, G has units of N/m^2 ($Pascals$).

Table 8.1: *Mechanical Properties of Common Materials*

Material	**Density** $\rho\,(kg/m^3)$	**Young's Modulus** E (N/m^2)	**Shear Modulus** G (N/m^2)
Steel	7.8×10^3	2.2×10^{11}	0.9×10^{11}
Iron	7.85×10^3	2.0×10^{11}	0.8×10^{11}
Aluminum	2.7×10^3	0.7×10^{11}	0.26×10^{11}
Copper	8.93×10^3	1.22×10^{11}	0.42×10^{11}

8.2 Longitudinal Plane Waves

8.2.1 Wave Propagation Equation

We now focus on longitudinal plane waves traveling through a solid rod.

Consider a small section of the rod between positions x and $x+\Delta x$, with cross-sectional area S. The forces acting are:

- $F(x)$ from the left segment at x
- $F(x+\Delta x)$ from the right segment at $x+\Delta x$

These forces cause displacements $u_x(x)$ and $u_x(x+\Delta x)$ in the x-direction. The displacement function u_x depends on both space x and time t.

If the material density is ρ, then the mass of this small element is:

$$\Delta m = \rho S \Delta x \tag{8.10}$$

Using Newton's second law for this element:

$$\rho S \Delta x \cdot \frac{\partial^2 u_x}{\partial t^2} = F(x+\Delta x) - F(x) \tag{8.11}$$

For small Δx, we expand $F(x+\Delta x)$ via Taylor series:

$$F(x+\Delta x) \approx F(x) + \frac{\partial F}{\partial x} \cdot \Delta x \tag{8.12}$$

Substituting, we get:

$$\rho S \Delta x \cdot \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial F}{\partial x} \cdot \Delta x \tag{8.13}$$

Canceling Δx, we obtain:

$$\rho S \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial F}{\partial x} \tag{8.14}$$

From Hooke's Law, the force at a section is related to local strain:

$$F = SE\varepsilon = SE \frac{\partial u_x}{\partial x} \tag{8.15}$$

Differentiating:

$$\frac{\partial F}{\partial x} = SE \cdot \frac{\partial^2 u_x}{\partial x^2} \tag{8.16}$$

Substituting back:

$$\rho S \frac{\partial^2 u_x}{\partial t^2} = SE \cdot \frac{\partial^2 u_x}{\partial x^2} \Rightarrow \frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E} \frac{\partial^2 u_x}{\partial t^2} = 0 \tag{8.17}$$

This is the one-dimensional wave equation (d'Alembert's form), describing the propagation of a longitudinal elastic wave with speed:

$$V = \sqrt{\frac{E}{\rho}} \tag{8.18}$$

The wave variable is the displacement vector $\vec{u}$ aligned with the direction of propagation, classifying this as a longitudinal wave.

8.2.2 Harmonic Progressive Waves

❖ Definition

For a harmonic (sinusoidal) plane wave propagating through a solid, the particle displacement can be expressed in complex notation as:

$$u_x(x,t) = U_0 e^{i(\omega t - kx)} \tag{8.19}$$

Where:

- $k = \frac{\omega}{V} = \frac{2\pi}{\lambda}$ is the wave number (with λ the wavelength),
- ω is the angular frequency,
- V is the wave speed.

The force component in the x-direction exerted on a point x by the left-hand segment of the bar is:

$$F_x(x,t) = -SE\frac{\partial u_x}{\partial x} = ikEu_x(x,t) = ikEU_0 e^{i(\omega t - kx)} \tag{8.20}$$

❖ Mechanical Impedance

The mechanical impedance at any point is defined as the ratio of the complex amplitude of the force to the complex amplitude of particle velocity:

$$Z(x) = \frac{F_x}{\dot{u}_x} \tag{8.21}$$

Where:

- The particle velocity is $\dot{u}_x = \frac{\partial u_x}{\partial t} = i\omega u_x(x,t)$

Substituting gives :

$$Z(x) = \frac{ikEu_x}{i\omega u_x} = \frac{kE}{\omega} = S\rho V = S\sqrt{\rho E} \tag{8.22}$$

This leads to the characteristic impedance of the medium:

$$Z_c = \sqrt{\rho E} = \rho V \tag{8.23}$$

And the mechanical impedance of the bar becomes:

$$Z(x) = S\,Z_c \tag{8.24}$$

Notably, in a uniform plane progressive wave, the impedance is independent of position. However, in more complex scenarios—such as reflections—the local impedance can vary along the length of the medium.

8.2.3 Reflection and Transmission

- **At the Junction of Two Semi-Infinite Rods**

Consider two semi-infinite rods with the same cross-sectional area S, but different densities ρ_1 and ρ_2, joined at $x = 0$. A longitudinal wave originating from the left rod (as $x \to -\infty$) reaches the interface and is partially reflected and transmitted.

By enforcing the continuity of displacement and mechanical stress at the boundary, the reflection and transmission coefficients are defined as:

$$R_u = \frac{U_R}{U_i}, \quad T_u = \frac{U_T}{U_i} \tag{8.25}$$

Where:

- U_i, U_R, and U_T are the displacement amplitudes of the incident, reflected, and transmitted waves, respectively.

These are expressed in terms of the impedances $Z_1 = S\sqrt{\rho_1 E_1}$ and $Z_2 = S\sqrt{\rho_2 E_2}$ of the two rods:

$$R_u = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad T_u = \frac{2Z_1}{Z_1 + Z_2} \tag{8.26}$$

❖ Reflection from a Mass Termination

Consider a semi-infinite rod terminated at $x = L$ by a point mass M. When a harmonic wave reaches this boundary, it is reflected.

The total wave in the rod is a combination of incident and reflected components:

$$u(x,t) = U_i e^{i(\omega t - kx)} + U_R e^{i(\omega t + kx)} \tag{8.27}$$

Applying Newton’s second law at $x = L$, where the mass experiences acceleration:

$$M \frac{\partial^2 u}{\partial t^2}\Big|_{x=L} = ES \frac{\partial u}{\partial x}\Big|_{x=L} \tag{8.28}$$

Solving leads to the reflection coefficient:

$$R_u = \frac{U_R}{U_i} = \frac{\sqrt{\rho E} - i\omega M / S}{\sqrt{\rho E} + i\omega M / S} \tag{8.29}$$

This shows that:

- The magnitude of the reflection coefficient is 1 → indicating total reflection,

- This is typical when the terminal impedance is purely imaginary, such as with a discrete mass.

8.2.4 Free Oscillations of a Bar

Let's consider a bar of length L, with one end ($x = 0$) rigidly fixed and the other end ($x = L$) free. If this bar experiences an initial longitudinal deformation, it will support longitudinal waves traveling in both directions—toward increasing and decreasing x. These waves reflect completely at both ends, resulting in an interference pattern from multiple overlapping waves.

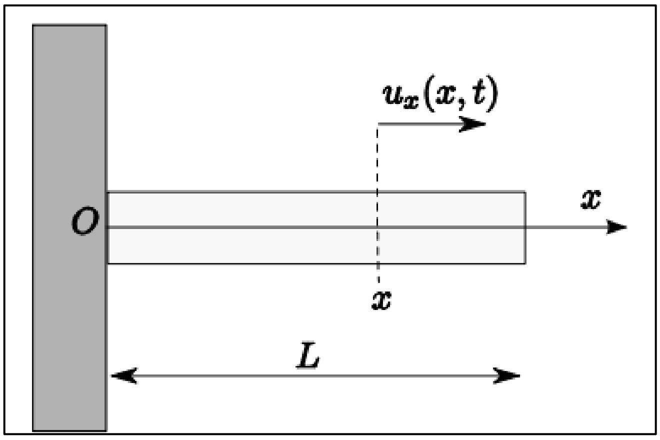


Figure 8.4: *Natural Vibrations of a Uniform Rod.*

To describe this situation, we look for a general solution to the wave equation of the form:

$$u_x(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)} \tag{8.30}$$

- $k = \dfrac{\omega}{V}$ is the wave number, derived from the dispersion relation.
- A and B are complex constants representing the amplitudes of waves traveling in the positive and negative xx-directions.

The boundary conditions at the ends of the bar are:

1. Fixed end at $x = 0$: $u_x(0, t) = 0$
2. Free end at $x = L$: $\dfrac{\partial u_x}{\partial x}\Big|_{x=L} = 0$ (no force exerted)

From the first boundary condition:

$$u_x(0,t) = Ae^{i\omega t} + Be^{i\omega t} = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \quad (8.31)$$

Substituting into the original expression:

$$u_x(x,t) = A\left(e^{-ikx} - e^{ikx}\right)e^{i\omega t} = -2iA\sin(kx)e^{i\omega t} \quad (8.32)$$

Now applying the second boundary condition:

$$\frac{\partial u_x}{\partial x}\Big|_{x=L} = -2ikA\cos(kL)e^{i\omega t} = 0 \Rightarrow \cos(kL) = 0 \quad (8.33)$$

This gives the allowed values of k (i.e. the spatial resonance modes):

$$k_n = \frac{(2n+1)\pi}{2L}, \quad n = 0,1,2,\ldots \quad (8.34)$$

The corresponding natural angular frequencies (or eigenfrequencies) are:

$$\omega_n = k_n V = \frac{(2n+1)\pi V}{2L} \quad (8.35)$$

Thus, the general solution for the longitudinal displacement is:

$$u_x(x,t) = \sum_{n=0}^{\infty} -2iA_n \sin(k_n x)e^{i\omega_n t} \quad (8.36)$$

Switching to the real-valued form (by taking the real part):

$$u_x(x,t) = \sum_{n=0}^{\infty}\left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right]\sin(k_n x) \quad (8.37)$$

Here:

- a_n and b_n are real coefficients related to the complex amplitudes A_n by:

$$-2iA_n = a_n - ib_n \quad (8.38)$$

And:

$$k_n = \frac{(2n+1)\pi}{2L}, \quad \omega_n = \frac{(2n+1)\pi V}{2L} \tag{8.39}$$

To determine the coefficients a_n and b_n, we apply initial conditions:

- Initial displacement: $u(x,\ 0) \ = u_0(x)$
- Initial velocity: $\dot{u}(x,0) = v_0(x)$

Expanding:

$$u_0(x) = \sum_{n=0}^{\infty} a_n \sin(k_n x), \quad v_0(x) = \sum_{n=0}^{\infty} \omega_n b_n \sin(k_n x) \tag{8.40}$$

To isolate the coefficients, we use the Fourier sine series orthogonality:

$$\int_0^L \sin\left(\frac{(2m+1)\pi x}{2L}\right)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \end{cases} \tag{8.41}$$

This yields the formulas:

$$a_n = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx, \tag{8.42}$$

$$b_n = \frac{2}{\omega_n L}\int_0^L v_0(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx \tag{8.43}$$

8.2.5 Forced Oscillations of a Finite-Length Bar

Let’s consider a bar of length L with:

- One end at $x \ = \ 0$ rigidly fixed,
- The other end at $x \ = \ L$ subjected to a sinusoidal force:

$$F(t) = F_0 \cos(\Omega t) \tag{8.44}$$

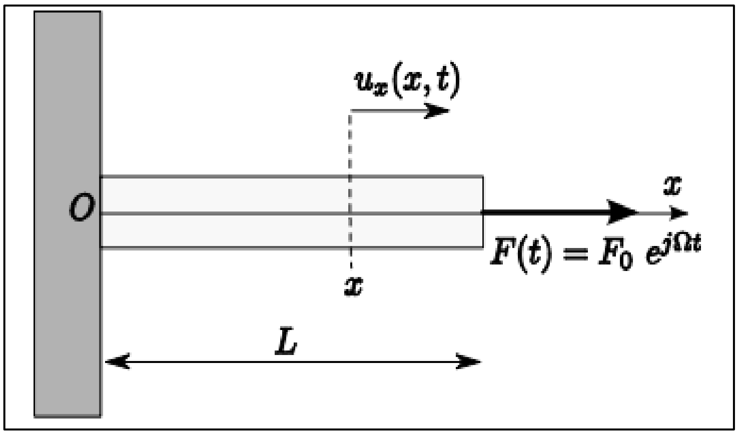


Figure 8.5: *Dynamic Response of a Finite Rod under Forced Oscillations.*

In complex notation, the boundary conditions become:

- $F_x(x = 0,\ t) = 0$ (no force at the fixed end),
- $F_x(x = L, t) = F_0 e^{i\Omega t}$ (driven force at free end).

Since the bar is finite in length, the steady-state displacement $u_x(x,\ t)$ of particles takes the form:

$$u_x(x,t) = Ae^{i(\Omega t - Kx)} + Be^{i(\Omega t + Kx)} \tag{8.45}$$

with $K = \dfrac{\Omega}{V}$ as the wave number, and $V = \sqrt{E/\rho}$ is the wave speed.

The force along the x-direction is:

$$F_x = ES\frac{\partial u_x}{\partial x} = -2iKES\left[Ae^{i(\Omega t - Kx)} - Be^{i(\Omega t + Kx)}\right] \tag{8.46}$$

Applying the boundary conditions:

- At $x = 0$: $F_x(0,t) = 0 \Rightarrow A = B$
- At $x = L$: $F_x(L,t) = F_0 e^{i\Omega t}$

This leads to :

$$A = \frac{F_0 e^{iKL}}{2iKES\cos(KL)}, \quad B = \frac{F_0 e^{-iKL}}{2iKES\cos(KL)} \tag{8.47}$$

The complex solution for the displacement becomes:

$$u(x,t) = U(x)e^{i\Omega t} \tag{8.48}$$

with amplitude:

$$U(x) = \frac{F_0 \sin(Kx)}{KES\cos(KL)} \tag{8.49}$$

This describes a stationary wave pattern, with nodes (points of zero displacement) and antinodes (maximum displacement) located at positions defined in the following table:

Table 8.1: *Positions of Nodes and Antinodes in the Displacement Profile*

Nodes (displacement $= 0$)	**Antinodes (maximum displacement)**
$x_n = L - \frac{(2n+1)\lambda}{4}$	$x_n = L - \frac{n\lambda}{2}$
$U_{\min} = 0$	$U_{\max} = \frac{F_0}{KES\cos(KL)}$

Resonance occurs when the excitation frequency Ω matches one of the bar's natural frequencies:

$$\Omega = \omega_n = \frac{(2n+1)\pi V}{2L} \tag{8.50}$$

At resonance, the denominator $\cos(KL) \to 0$, causing the amplitude $U_{\max} \to \infty$, theoretically producing infinite response at the antinodes.

8.3 Transverse Elastic Waves in a Solid Bar

We now analyze transverse elastic waves in a solid bar.

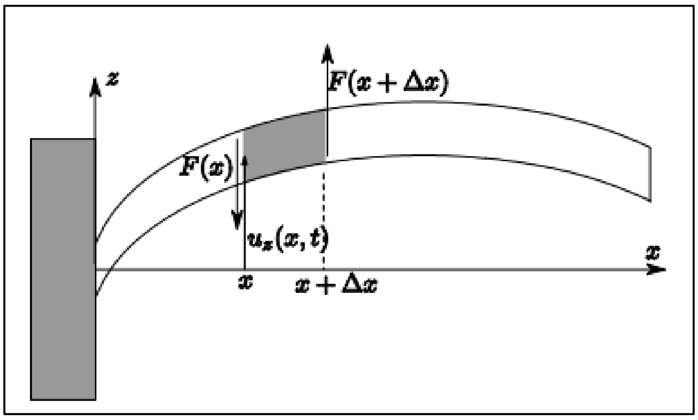


Figure 8.6: *Propagation of transverse waves along a solid rod*

- The bar is subject to a shear force at its end, acting parallel to the cross-section S.
- Each cross-section moves vertically (along z) but not horizontally, resulting in transverse displacement $u_z(x, t)$.
- u_z depends on position xx; otherwise, the entire bar would shift uniformly.

This setup causes shear deformation, where each small slice of the bar (of thickness Δx) experiences opposing tangential forces $F(x)$ and $F(x+\Delta x)$.

There exists an analog of Hooke's Law for shear stress and strain:

$$\frac{F}{S} = G\frac{\partial u_z}{\partial x} \tag{8.51}$$

where:

- G: shear modulus (modulus of rigidity),
- $\frac{\partial u_z}{\partial x}$: shear strain,
- $\frac{F}{S}$: shear stress.

From Newton's second law, the net force on the slice is:

$$F(x+\Delta x)-F(x)=\frac{\partial F}{\partial x}\Delta x=\rho S\Delta x\frac{\partial^2 u_z}{\partial t^2} \tag{8.52}$$

From Hooke's Law, differentiate:

$$\frac{\partial F}{\partial x}=SG\frac{\partial^2 u_z}{\partial x^2} \tag{8.53}$$

Substituting gives the wave equation:

$$\frac{\partial^2 u_z}{\partial x^2}-\frac{\rho}{G}\frac{\partial^2 u_z}{\partial t^2}=0 \tag{8.54}$$

This is again a standard wave propagation equation, now for shear waves, propagating at speed:

$$V=\sqrt{\frac{G}{\rho}} \tag{8.55}$$

The shear force component applied by the left part of the bar is:

$$F_z=-SG\frac{\partial u_z}{\partial x} \tag{8.56}$$

The mechanical impedance of the bar for transverse waves is:

$$Z=\frac{F_z}{\dot{u}_z}=SZ_c \tag{8.57}$$

where:

- $Z_c=\rho V=\rho\sqrt{\frac{G}{\rho}}=\sqrt{\rho G}$ is the characteristic impedance of the material.

8.4 Linear Chain Model of Elastic Wave Propagation in Solids

8.4.1 Microscopic Modeling

- A solid crystal can be modeled as a 1D chain of identical atoms (mass M) connected by identical springs (stiffness K, rest length a), representing atomic bonds.
- Each atom can move along the x-axis with small displacements $u_n(t)$ from its equilibrium position $x_n = na$.
- The total energy includes:
    - Kinetic energy: $T = \frac{1}{2}\sum M\dot{u}_n^2$
    - Potential energy: $U = \frac{1}{2}K\sum(u_{n+1} - u_n)^2$

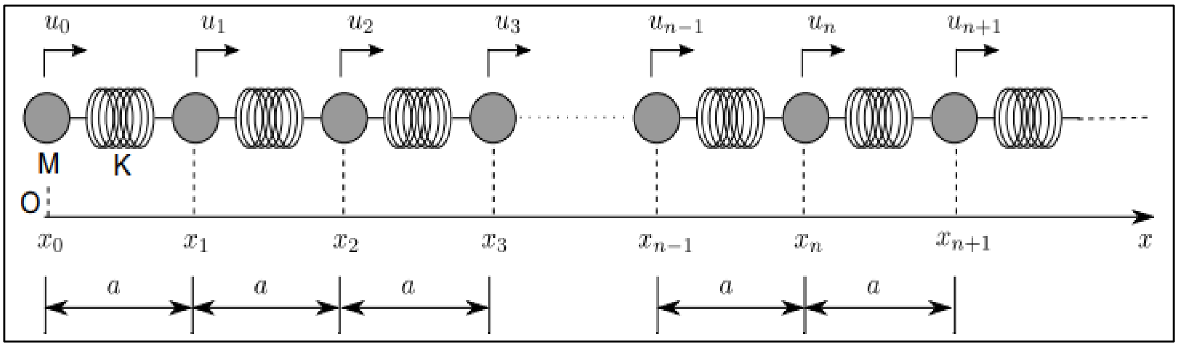


Figure 8.7: *Model of a one-dimensional atomic chain.*

Using Lagrange's equation, the motion for each atom n is:

$$M\ddot{u}_n = -K(2u_n - u_{n-1} - u_{n+1}) \tag{8.58}$$

8.4.2 Sinusoidal Solution (Wave Regime)

- Suppose the atom at $n = 0$ oscillates sinusoidally: $u_0 = a_0 e^{i\omega t}$
- The solution for any atom is: $u_n = a_0 e^{i(\omega t - kna)}$
- Plug into the equation of motion to get the dispersion relation:

$$\omega = 2\sqrt{\frac{K}{M}}\sin\left(\frac{ka}{2}\right) \tag{8.59}$$

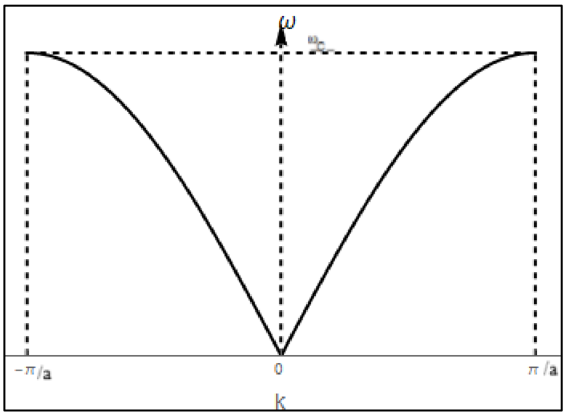


Figure 8.8: *Dispersion diagram*

- This shows that frequency ω depends non-linearly on wavenumber k : this is dispersion.
- Maximum frequency (cutoff) occurs at $k = \frac{\pi}{a}$:

$$f_c = \frac{1}{\pi}\sqrt{\frac{K}{M}} \tag{8.60}$$

- The shortest wavelength allowed: $\lambda_c = 2a$ (twice atomic spacing).
- For small k (long wavelengths): $\omega \approx v_0 k$, where:

$$v_0 = a\sqrt{\frac{K}{M}} \tag{8.61}$$

is the low-frequency wave speed.

8.4.3 Continuous Medium Approximation

- When $\lambda \gg a$, the system can be approximated as a continuous medium:

$$\frac{\partial^2 u}{\partial t^2} = v_0^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad v_0 = a\sqrt{\frac{K}{M}} \tag{8.62}$$

- This is the classical wave equation, valid for low-frequency elastic waves in solids.

✓ **Key Takeaways**

- A crystal behaves like a chain of coupled oscillators, and waves propagate through atomic vibrations.
- For small k, the wave behaves like in a continuous medium with constant phase velocity.
- The dispersion relation limits the range of transmittable frequencies and wavelengths.
- The cutoff frequency is very high (in the THz range), so most practical elastic waves (in MHz–GHz) behave as if the solid were continuous.

Exercises

Exercise 1:

A bar of length L is rigidly fixed at $x = 0$, while its other end at $x = L$ is free.

1. Show that only odd harmonics are allowed for this configuration.
2. Determine the fundamental frequency of the bar if it is made of steel and has a length $L = 0.5\mathrm{m}$.
3. If a static force is applied to the free end of the bar, causing it to be displaced by a distance h , show that, when the force is suddenly removed and the bar begins to vibrate, the amplitudes of the different harmonics of vibration are given by:

$$A_n = \frac{8h}{n\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

4. Determine these amplitudes if the static force is $F_S = 5000\mathrm{N}$ and the cross-sectional area of the bar is $s = 0.00005\mathrm{m}^2$.

Exercise 2:

A steel bar with a cross-sectional area of $0.0001\mathrm{m}^2$ and a length of $0.25\mathrm{m}$ is free to move at $x = 0$ and is loaded at the other end ($x = 0.25\mathrm{m}$) by a mass of $0.15\mathrm{kg}$.

1. Calculate the fundamental frequency of longitudinal vibrations of the bar.
2. Determine the position along the bar where it should be clamped to cause the least disturbance to its fundamental mode of vibration.
3. When the bar vibrates in its fundamental mode, what is the ratio of the amplitude of displacement at the free end to that at the loaded end (the end with the mass)?
4. What is the frequency of the first overtone (first harmonic) of this bar?

Exercise 3:

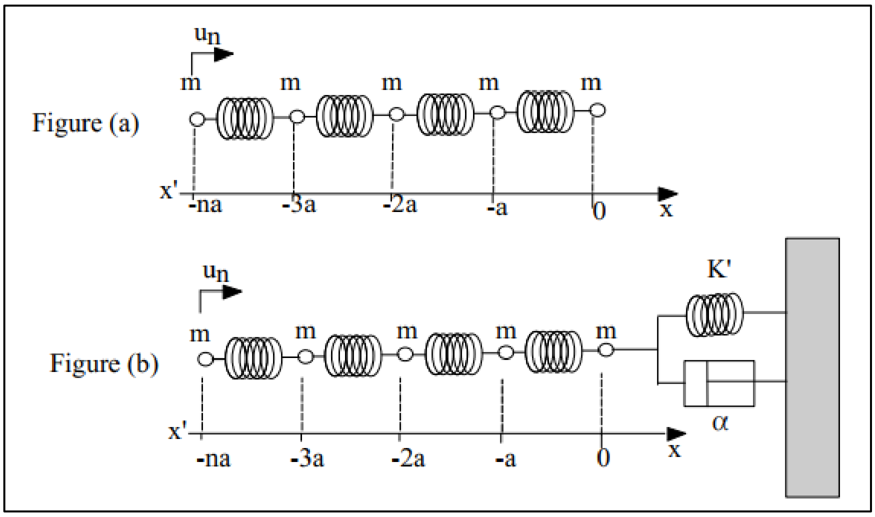
A mass $m = 2\text{kg}$ is suspended from a steel wire with a length $\ell = 1.0\text{m}$ and a cross-sectional area $s = 0.00001\text{m}^2$.

1. Calculate the fundamental frequency of vertical oscillations of the mass, assuming it behaves as a simple harmonic oscillator.
2. Calculate the fundamental frequency of vertical oscillations for the system composed of a vibrating bar, fixed at one end and loaded with a mass at the other end.
3. Show that, for small values of $k\ell < 0.2$, the equation obtained in the previous question can be approximated as:

$$\omega_0 = \sqrt{\frac{s}{m}}$$

Exercise 4:

Consider a linear chain of identical particles, each of mass mm. At equilibrium, the particle of order nn is located along the xx-axis at the position $x_n = n \cdot a$, where x_n may be zero or negative (see Figure a). The particles are connected to one another by identical springs of stiffness K, with rest length a. Let u_n denote the displacement of the n-th particle from its equilibrium position.



1. Write the equation of motion for the n-th particle.
2. Assuming solutions of the form of plane sinusoidal traveling waves:

$$u_n = U_0 e^{i(\omega t - kna)}$$

where U0U_0 is a real constant, determine the dispersion relation $\omega = \omega(k)$.

3. Suppose the particle at position $x_0 \ = \ 0$ is fixed. A wave is sent from the left to the right, described by

$$u_n = U_0 e^{i(\omega t - kna)}$$

Determine the amplitude and phase ϕ of the reflected wave.

4. Now, instead of being fixed, the particle at $x = 0$ is attached to a fixed base via a spring of stiffness K_0 and a damper with viscous damping coefficient α (see Figure b). The constants K_0 and α are chosen such that no wave is reflected.

   (a) Write the equation of motion for the mass located at $x = 0$.

   (b) A wave is again sent from the left as before. Determine the values of K_0 and α such that the mass at $x = 0$ vibrates only under the influence of the incident wave, with no reflected wave.

Chapter 9 – Acoustic Waves in Fluids

9.1 Introduction

- Acoustic waves are elastic waves that propagate in fluids (liquids and gases).
- These waves involve small perturbations in pressure and density, moving through the fluid due to local compressions and rarefactions.
- Notation used:
    - x : position at equilibrium
    - $u_x(x,\ t)$: particle displacement along x
    - ρ_0 : fluid density at equilibrium
    - P : instantaneous pressure, P_0 : equilibrium pressure
    - $p = P - P_0$: acoustic pressure (overpressure)
    - c : wave speed

Particles in this context are tiny fluid elements—small enough to consider local uniformity, but large enough to treat as continuous media.

9.2 Wave Equation Derivation

❖ ***Newton's Second Law (Force Balance)***

- Consider a small fluid slice between x and $x+\Delta x$.
- Pressure forces on its boundaries:
    - At x: $F_x = S\,P(x,\ t)$
    - At $x+\Delta x$: $F_{x+\Delta x} = -SP(x+\Delta x, t)$
- Net force (to first order using Taylor expansion):

$$\Delta F_x = -S\frac{\partial p}{\partial x}\Delta x \tag{9.1}$$

- Newton’s law

$$\rho_0 S\Delta x\frac{\partial^2 u_x}{\partial t^2} = \Delta F_x \Rightarrow \rho_0\frac{\partial^2 u_x}{\partial t^2} = -\frac{\partial p}{\partial x} \tag{9.2}$$

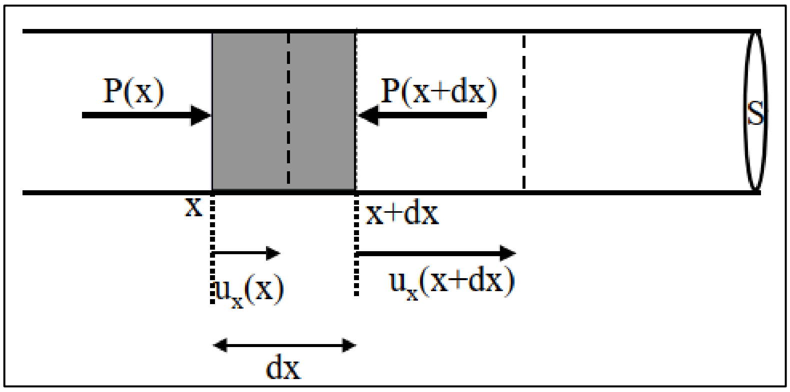


Figure 9.1: *Transmission of a sound wave*

❖ *Compressibility and Pressure Relation*

- The volumetric strain (relative volume change) :

$$\theta = \frac{\partial u_x}{\partial x} \tag{9.3}$$

- The pressure-strain relation (Hooke's law for fluids):

$$p = -\kappa\theta = -\kappa\frac{\partial u_x}{\partial x} \tag{9.4}$$

where κ is the bulk modulus, and $\chi = \frac{1}{\kappa}$ is the compressibility.

❖ *Acoustic Wave Equation*

Combining the two key equations:

- Newton’s law :

$$\rho_0 \frac{\partial^2 u_x}{\partial t^2} = -\frac{\partial p}{\partial x} \tag{9.5}$$

- Pressure-strain relation :

$$p = -\kappa \frac{\partial u_x}{\partial x} \tag{9.6}$$

Gives the wave equation :

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{V^2} \frac{\partial^2 p}{\partial t^2} = 0 \tag{9.7}$$

where the speed of sound is:

$$V = \sqrt{\frac{\kappa}{\rho_0}} = \sqrt{\frac{1}{\rho_0 \chi}} \tag{9.8}$$

Acoustic waves in fluids follow the classical wave equation and are governed by the balance between pressure forces and inertial response, along with the fluid’s compressibility.

9.3 Speed of Sound

- In fluids, sound waves propagate *adiabatically*.
- Using the adiabatic law $Pv^\gamma = \text{constant}$, we derive:

$$V = \sqrt{\frac{\gamma P_0}{\rho_0}} \tag{9.9}$$

- For air at 20°C, using $\gamma = 1.4,\ P_0 = 10^5\,\text{Pa},\ \text{and}\ \rho_0 = 1.29\text{kg/m}^3$, we find:

$$V \approx 330\text{m/s} \tag{9.10}$$

- In a perfect gas, this becomes:

$$V = \sqrt{\frac{\gamma RT}{M}} \tag{9.11}$$

which shows that sound speed increases with temperature.

9.4 Sinusoidal Plane Waves in Fluids

9.4.1 Definition

- A sinusoidal pressure wave :

$$p(x,t) = p_0 \cos(\omega t - kx) \quad \text{with } k = \frac{\omega}{V} \tag{9.12}$$

- The particle displacement and velocity:

$$u(x,t) = \frac{p_0}{i\omega\rho_0 V} e^{i(\omega t - kx)}, \quad \dot{u}(x,t) = \frac{p_0}{\rho_0 V} e^{i(\omega t - kx)} \tag{9.13}$$

→ Velocity and pressure are in phase.

9.4.2 Acoustic Impedance

- Defined as the ratio :

$$Z(x) = \frac{p}{\dot{u}} = \rho_0 V \tag{9.14}$$

→ Same at all points for a progressive wave.

9.4.3 Acoustic Energy

- ❖ *Kinetic energy density* :

$$E_c = \frac{1}{2}\rho_0 \dot{u}^2 \tag{9.15}$$

- ❖ *Potential energy density :*

$$E_p = \frac{1}{2\kappa} p^2 \tag{9.16}$$

- *Total energy density :*

$$E = \frac{1}{2}\rho_0 \dot{u}^2 + \frac{1}{2\kappa} p^2 \tag{9.17}$$

→ For sinusoidal waves, average energy:

$$\langle E \rangle = \frac{p_0^2}{2\rho_0 V^2} \tag{9.18}$$

- ***Acoustic Intensity***

- Instantaneous intensity :

$$I(t) = EV = \frac{p_0^2}{\rho_0 V}\cos^2(\omega t - kx) \tag{9.19}$$

- Average intensity :

$$\langle I \rangle = \frac{p_0^2}{2\rho_0 V} = \frac{p_0^2}{2Z_c} \tag{9.20}$$

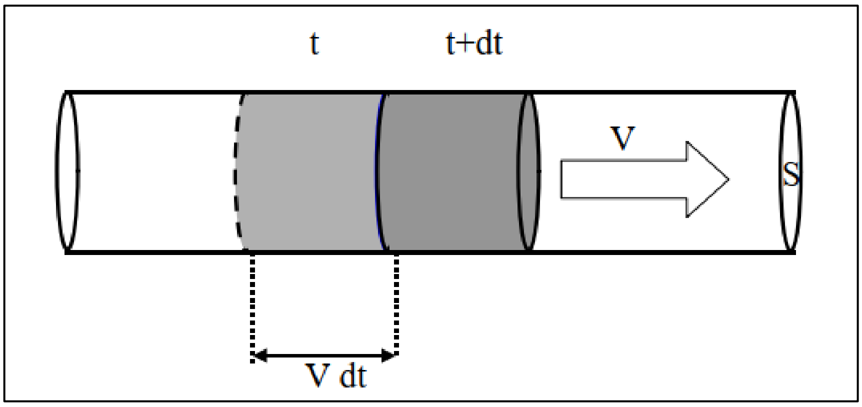


Figure 9.2: *Energy flow*

- Sound level in dB :

$$N_{\text{dB}} = 10\log_{10}\left(\frac{I}{I_0}\right), \quad I_0 = 10^{-12}\,\text{W/m}^2 \tag{9.21}$$

→ ***Examples:***

- Threshold of hearing (0 dB): $p_0 = 2.9\times10^{-5}\,\text{Pa}$
- Threshold of pain (130 dB): $p_0 = 91\text{Pa}$

9.5 Reflection and Transmission at Boundaries

- At a boundary between two fluids with acoustic impedances Z_1 and Z_2, a wave is partly:
  - ***Reflected***: $R_P = \dfrac{Z_2 - Z_1}{Z_2 + Z_1}$
  - ***Transmitted***: $T_P = \dfrac{2Z_2}{Z_2 + Z_1}$

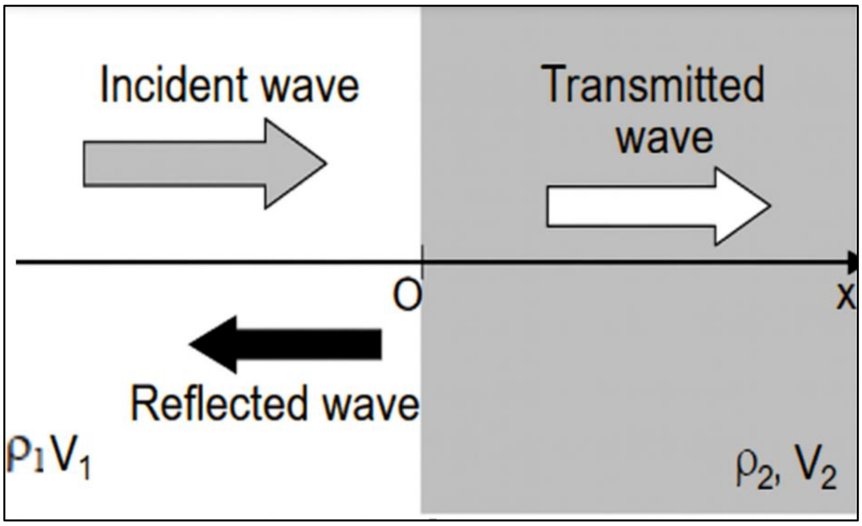


Figure 9.3: *Wave bounce at a fluid boundary*

- Coefficients for velocity and displacement:

$$R_u = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad T_u = \frac{2Z_1}{Z_1 + Z_2} \tag{9.22}$$

- Acoustic intensity reflection/transmission :

$$\alpha_R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2, \quad \alpha_T = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2} \tag{9.23}$$

9.6 Doppler Effect

9.6.1 General Formula

- Frequency observed when source and receiver are moving:

$$\nu = \nu_0 \cdot \frac{1+\frac{v_2}{V}\cos\theta_2}{1+\frac{v_1}{V}\cos\theta_1} \tag{9.24}$$

  - $v_1,\ v_2$: source and receiver speeds
  - θ_1, θ_2 : angles between velocity and wave direction

9.6.2 Approximation for Low Speeds

- If $v_1, v_2 \ll V$, then:

$$\nu \approx \nu_0 \left(1 - \frac{v\cos\theta}{V}\right) \tag{9.25}$$

  - $v = v_1 - v_2$, relative speed
  - θ : angle between relative velocity and propagation

Source and receiver ***approaching***: frequency increases

Source and receiver ***receding***: frequency decreases

Exercises

Exercise 1:

Under standard temperature and pressure conditions, air is characterized by an equilibrium mass density of $\rho = 1.21\,\text{kg/m}^3$ and a ratio of specific heats (adiabatic index):

$$\gamma = \frac{c_p}{c_v} = 1.402\,.$$

- Calculate the speed of sound propagation in air, as well as its specific acoustic impedance under these conditions (temperature $T = 20^{\circ}C$, pressure $P_0 = 10^5\,\text{N/m}^2$).

Exercise 2:

Consider a plane acoustic wave of frequency $f = 24\,\text{kHz}$ propagating in water with a speed $c = 1480\,\text{m/s}$. The wave transports an average power of $P = 1\,\text{W}$, uniformly distributed across a circular cross-section of diameter $D = 40\,\text{cm}$, perpendicular to the direction of propagation.

1. Calculate the acoustic intensity. What is the sound intensity level in decibels (dB), relative to the reference intensity level $I_0 = 10^{-12}\,\text{W/m}^2$?
2. Calculate the following amplitude quantities of the acoustic wave:
    - The acoustic pressure amplitude $p_{\max}$,
    - The particle velocity amplitude $v_{\max}$,
    - The particle displacement amplitude $\xi_{\max}$.
3. Compare the results above to those that would be obtained if the wave were propagating in air instead of water.

Exercise 3:

A cylindrical pipe of constant cross-sectional area S is filled with a gas of mass density ρ, in which acoustic waves can propagate at speed V. At one end of the pipe (at $x=0$), a flat piston oscillates along the axis Ox of the pipe with a sinusoidal displacement of amplitude U_0 and angular frequency ω. At the other end of the pipe (at $x=L$), the pipe is closed with a rigid wall.

1. Show that the displacement can be written in the form:

$$u(x,t)=U(x)e^{i(\omega t+\varphi)}$$

   where $U\left(x\right)$ is a real function. Give the expression for the amplitude $U\left(x\right)$ and the phase φ of the resulting wave.

2. Deduce the positions and values of the maxima and minima of the absolute amplitude $\left|U\left(x\right)\right|$.
3. Determine the resonance frequencies. What is the displacement amplitude at the antinodes (points of maximum displacement) under resonance conditions?
4. In the case where $L=\dfrac{13\lambda}{12}$ where λ is the wavelength, plot the variation of $\left|U\left(x\right)\right|$ as a function of x.

   Scale: $1\text{cm}=\dfrac{\lambda}{12}$.

   How many antinodes (maxima) and nodes (minima) are observed?

Exercise 4:

A plane sinusoidal acoustic wave of angular frequency ω and amplitude A_1 propagates along the Ox-axis in a fluid medium of mass density ρ_1. The wave is normally incident on a second fluid medium (mass density ρ_2) of thickness L_2, which is deposited on a perfectly rigid solid (see figure).

Let V_1 and V_2 be the speeds of sound in the two fluid media, and let k_1 and k_2 be the corresponding wave numbers.

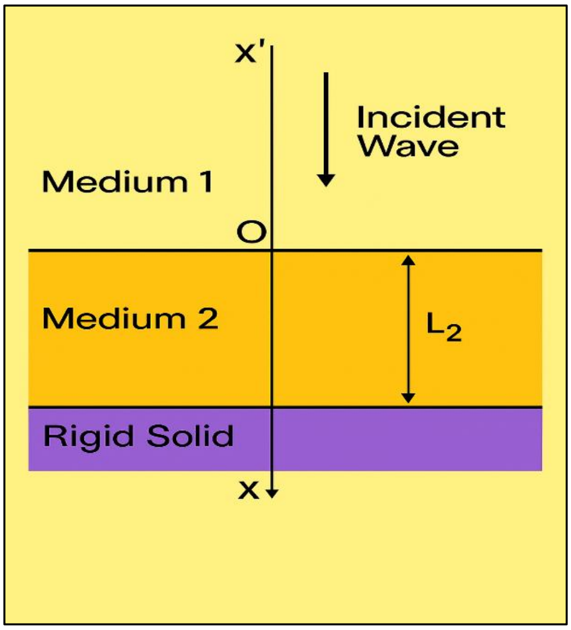
The acoustic pressure in the two media is expressed as:

- In medium 1 :

$$p_1(x,t) = A_1 e^{i(\omega t - k_1 x)} + A_1^0 e^{i(\omega t + k_1 x)}$$

- In medium 2 :

$$p_2(x,t) = A_2 e^{i(\omega t - k_2 x)} + A_2^0 e^{i(\omega t + k_2 x)}$$



1. Write the expressions for the particle velocities $\dot{u}_1(x,t)$ and $\dot{u}_2(x,t)$ in media 1 and 2, respectively.
2. Apply the boundary conditions of continuity of pressure and continuity of particle velocity at $x = 0$. Deduce two relations among the coefficients A_1, A_1^0, A_2, and A_2^0.
3. Apply the boundary condition of continuity of particle velocity at $x = L_2$. Deduce a relation between the coefficients A_2 and A_2^0.
4. Determine the reflection coefficient: $R = \dfrac{A_1^0}{A_1}$ and give both its magnitude and phase.

5. Measuring the Speed V_2:

   a. What condition must be met so that the pressure in medium 2 exhibits a pressure node at $x=0$ and only one antinode (pressure maximum)?
   b. Suppose the first pressure node in medium 1 is located at position $x = -L_1$. Calculate the speed of sound V_2 in medium 2 in terms of V_1, L_1, and L_2.
   c. The experiment is conducted under the above conditions, with:

- Medium 1: Water ($V_1 = 1500\,\mathrm{m/s}$)
- Medium 2 : Glycerin

  Measured values :
- $L_1 = 7.5\,\mathrm{mm}$
- $L_2 = 4.95\,\mathrm{mm}$

a) Calculate the speed of sound V_2 in glycerin.

Appendices

Appendix A – Second-Order Linear Differential Equations

A.1 Introduction

Many mechanical and wave phenomena are governed by **second-order linear differential equations with constant coefficients**. This annex recalls their main forms and solutions, serving as a quick reference for problems encountered throughout the course.

A.2 Homogeneous Equation

General form:

$$a\ddot{y} + b\dot{y} + cy = 0$$

Dividing by a:

$$\ddot{y} + 2\delta\dot{y} + \omega_0^2 y = 0$$

where:

$$\delta = \frac{b}{2a}, \quad \omega_0 = \sqrt{\frac{c}{a}}$$

Case 1 – Overdamped ($\delta > \omega_0$):

$$y(t) = A_1 e^{(-\delta + \sqrt{\delta^2 - \omega_0^2})t} + A_2 e^{(-\delta - \sqrt{\delta^2 - \omega_0^2})t}$$

No oscillation; amplitude decays exponentially.

Case 2 – Critically damped ($\delta = \omega_0$):

$$y(t) = (A_1 + A_2 t)e^{-\delta t}$$

Fastest non-oscillatory return to equilibrium.

Case 3 – Underdamped ($\delta < \omega_0$):

$$y(t) = Ae^{-\delta t}\cos(\omega_A t + \phi)$$

with:

$$\omega_A = \sqrt{\omega_0^2 - \delta^2}$$

Oscillatory decay.

A.3 Nonhomogeneous Equation

Form:

$$\ddot{y} + 2\delta\dot{y} + \omega_0^2 y = f(t)$$

General solution :

$$y(t) = y_h(t) + y_p(t)$$

where y_h is the homogeneous solution and y_p is a particular solution.

Example – Harmonic Forcing:

If $f(t) = F_0\cos(\Omega t)$,

$$y_p(t) = Y_0\cos(\Omega t - \varphi)$$

with:

$$Y_0 = \frac{F_0}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\delta\Omega)^2}}, \quad \varphi = \arctan\left(\frac{2\delta\Omega}{\omega_0^2-\Omega^2}\right)$$

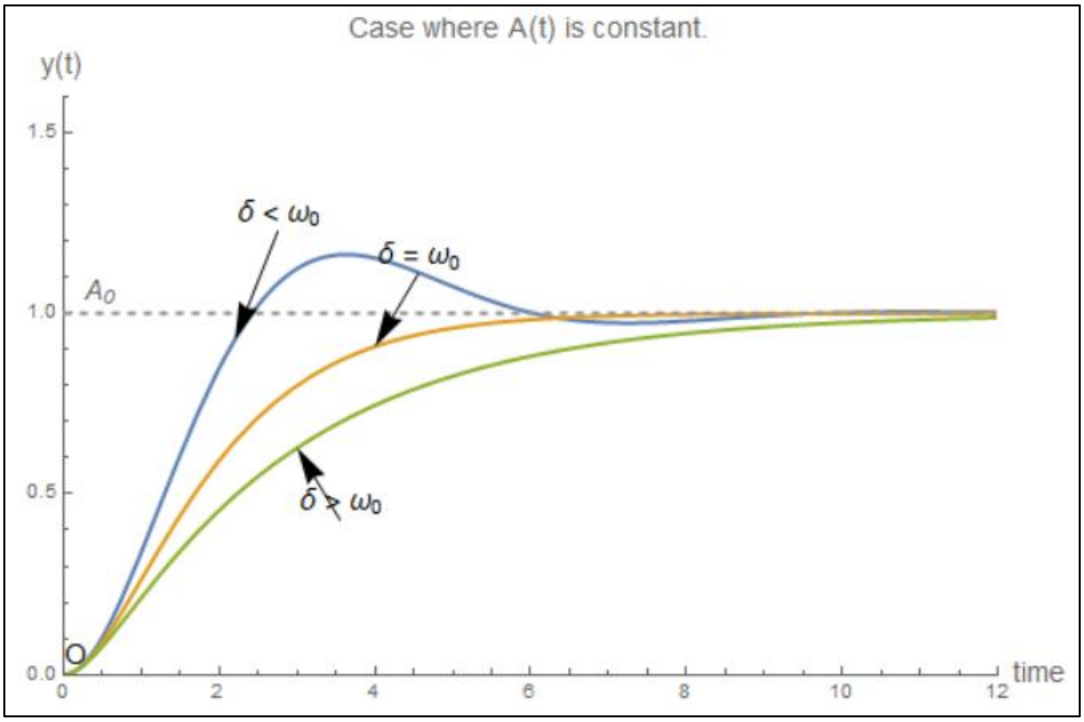


Figure: *Case in which* $A(t)$ *is constant.*

Appendix B – Mechanical–Electrical Analogies

Mechanical oscillators can be modeled using electrical circuit analogies, allowing the use of powerful circuit-analysis techniques in vibration problems.

Mechanical Quantity	Force–Voltage Analogy	Force–Current Analogy
Mass m	Inductance L	Capacitance C
Spring stiffness k	Inverse capacitance $1/C$	Inverse inductance $1/L$
Damping coefficient b	Resistance R	Conductance $G=1/R$
Force F	Voltage V	Current I
Displacement x	Charge q	Flux linkage λ

Such analogies are widely used in acoustics, structural dynamics, and mechatronics.

Appendix C – Fourier Analysis and Wave Decomposition

C.1 Introduction

Oscillatory signals are often composed of multiple frequency components. **Fourier analysis** allows us to represent complex periodic signals as sums of simple sinusoids, each characterized by an amplitude, frequency, and phase.

C.2 Fourier Series for Periodic Signals

Any periodic function $f\left(t\right)$ of period T can be written as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) \right]$$

with coefficients:

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi n}{T} t\right) dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi n}{T} t\right) dt$$

C.3 Fourier Transform for Nonperiodic Signals

For nonperiodic signals, the Fourier transform is:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

This gives the **spectrum** of $f\left(t\right)$, showing its frequency content.

C.4 Application to Vibrations and Waves

- **Vibrations:** Fourier analysis separates the contributions of different normal modes in a complex motion.
- **Waves:** It allows decomposition of a wave packet into monochromatic components, enabling study of dispersion, group velocity, and pulse spreading.
- **Signal Processing:** Useful for analyzing transient responses in mechanical and acoustic systems.

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