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Study of some fractional boundary value problems

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شُكْرٌ وَ عِرفَانٌ

الْحَمْدُ لِلَّهِ حَمْدًا كَثِيرًا طَيِّبًا مُبَارَكًا فِيهِ، كما يَنبغي لِجَلَالِ وَجْهِهِ وَعَظِيمِ سُلطانِهِ، على ما أَوْلانَا من نِعَمٍ لا تُعَدُّ، وتوفيقِهِ الذي لا يُحصى. والصلاةُ والسلامُ على خيرِ خلقِهِ محمّدٍ صلّى الله عليه وسلّم، وعلى آلِهِ وصحبِهِ أجمعين.

نتوجّه أوّلًا بخالصِ الشكرِ والثّناءِ إلى المولى عزّ وجلّ، الذي منَّ علينا بإتمامِ هذه المُذَكّرة، ووفّقنا لإنجازِها في أحسنِ الظّروف، فلَه الحمدُ في الأولى والآخرة، وله الشّكرُ دائمًا وأبدًا.

واقتداءً بحديثِ النبيِّ صلّى الله عليه وسلّم: "**مَنْ لَا يَشْكُرِ النَّاسَ لَا يَشْكُرِ اللَّهَ**" ، فإنّنا نتقدّم بجزيلِ الشكرِ والامتنانِ إلى أستاذنا المشرف الفاضل، الدكتور عبد اللطيف غندير عون، لما بذلته من جُهدٍ وتوجيهاتٍ قيّمة، ومتابعةٍ دقيقةٍ كان لها الأثرُ البالغُ في حُسنِ سيرِ هذا العملِ وإخراجِهِ في صورتِهِ النّهائية.

كما نتقدَّمُ بخالصِ الشكرِ والتقديرِ إلى الأستاذين بيلول السعيد و دودي نجاة، على تفضُّلِهما بقبولِ مناقشةِ هذه المذكرةوإثرائها.

وفي الختامِ، نرفعُ أسمى عباراتِ الشكرِ لكلِّ مَن ساهمَ، من قريبٍ أو بعيدٍ، في إنجازِ هذه المُذَكّرة. فجزاكُمُ الله عنّا كلَّ خيرٍ، ونسألُهُ أن يكتُبَ لكم الأجرَ والتوفيقَ والسّدادَ دائمًا.

إِهْدَاءٌ

أوَّلُ الحمدِ وآخرُه للهِ وحدَه، ولِما أحاطني به من عنايةٍ ورحمةٍ سبقت ضعفي، ما كنتُ لأعبرَ هذه الرحلة، ولا لأصلَ إلى هذا النور بعد طولِ العتمة. أحمدُك يا الله، لأنَّكَ كنتَ القُربَ حين بَعُدَ الكلُّ، والنَّاصرَ حين عزَّ النصير، وهادِيَّ حين تاهت بي الخُطى. كلُّ ما وصلت له هو من فضلِكَ، وكلُّ خطوةٍ قطعتُها كانت بمددِكَ ولطفِكَ، فلك الحمد كما ينبغي لجلالِ وجهِكَ، وعظيمِ سلطانِكَ، فلك الحمد ولك الشكر.

إلى من كَانَت لي نبضا في الْحيَاة، وَمَلَاذًا في كلّ الْأَوْقات، إِلَى أُمّي الحَبِيبَة، التي غَرَسَت فِيَّ بذور الْحبِّ والدُّعاء، أَهْدِيكِ ثَمَرَةَ هَذَا الْجُهْدِ، وَأَرْجُو أَنْ أكون عند حسنِ ظنّكِ دائمًا.

إلى من علّمني معنى الْقوّةِ والصَّبرِ، ورسم لي طريق الْكرامةِ... أبي الْعزيزِ.

إلى من أهداني القدر قلبه... زوجي قرة عيني ورفيق دربي, سندي الثابت الذي لا ولن يميل .

إِلَى رفيقَات دَرْبِي، مؤنسات الفؤاد مَنْ كن لي النُّورَ فِي عَتَمَةِ الطَّرِيقِ، وَالدَّفْعَ فِي لَحَظَات التَّرَدُّدِ... عفاف , شيماء و فاطمة شُكْرًا لِكونكن جزءًا مِن هَذا الطَّرِيقِ، وَنُورا فِي كُلِّ مَحَطَّةٍ.

إلى أستاذنا المشرف الفاضل, المعروف بسماحة عقله وتواضعه وأدبه وواسع علمه, وإلى كل من شجعني وساندني على الاستمرار.

انشراح محلي

إِهْدَاءُ

الحمدُ للهِ الَّذي وفَّقني لإتمامِ هذا العمل، ثمَّ أُهدي ثمرةَ جهدي وتخرُّجي إلى أعزِّ النَّاس على قلبي...

إلى أبي الغالي، رمزِ العطاءِ والتَّعبِ والصَّبر، الَّذي كان دائمًا سندي وفخري.

إلى أمِّي الحبيبة، مصدرِ الحنانِ والدُّعاءِ، الَّتي أنارت دربي بمحبَّتها ودعمها، فجزاها اللهُ عنِّي خيرَ الجزاءِ.

إلى إخوتي الأعزَّاء، الَّذين كانوا لي قوَّةً وسعادةً ورفقةً جميلةً طوالَ مشواري الدِّراسي.

إلى خطيبي، شكرًا لاحتوائك الدَّائم، ولدعمك الَّذي خفَّف عنِّي كلَّ تعب، ولكلماتك الَّتي كانت تمنحني الأملَ والقوَّة.

إلى زوجِ أختي الكريم، على دعمه وتشجيعه وطيبِ تعامله.

إلى أصدقائي الأوفياء، الَّذين شاركوني أجملَ اللَّحظاتِ وأصعبها، وكانوا خيرَ مُعينٍ وخيرَ رفقة.

إليكم جميعًا أُهدي هذا العملَ المتواضع، عربونَ محبَّةٍ ووفاءٍ وامتنان

الشيماء جديعي

الملخص

تتناول هذه المذكرة دراسة وجود ووحدانية الحلول لمسائل حدودية من رتبة كسرية حيث تم التعامل مع نوعين من المشتقات الكسرية هما مشتقة كابوتو ومشتقة ريمان-ليوفيل وذلك في إطار شروط حدية من نمط كوشي وديريكليه ونيومان على مجالات منتهية. تعتمد المنهجية المتبعة على تحويل المعادلة التفاضلية الكسرية إلى معادلة تكاملية مكافئة باستخدام دوال غرين المرتبطة بكل مسألة, ثم تطبيق نظريات النقطة الصامدة الكلاسيكية (باناخ, شاودر, شيفر, ليري-شاودر) لإثبات الوجود والوحدانية وقد تم تدعيم هذه النتائج بأمثلة تطبيقية في نهاية كل فصل.

الكلمات المفتاحية : معادلة تفاضلية من الرتبة الكسرية , مشتقة كابوتو, مشتقة ريمان-ليوفيل, الوجود, الوحدانية, نظرية النقطة الصامدة.

Abstract

This research investigates the existence and uniqueness of solutions to fractional-order boundary value problems, addressing two types of fractional derivatives: the Caputo derivative and the Riemann-Liouville derivative, under Cauchy, Dirichlet, and Neumann-type boundary conditions on finite intervals. The methodological approach consists of transforming the fractional differential equation into an equivalent integral formulation using the corresponding Green's functions, followed by the application of classical fixed-point theorems (Banach, Schauder, Schaefer, and Leray Schauder) to establish existence and uniqueness results, which are supported by illustrative examples provided at the end of each chapter.

boundary conditions, finite interval, fixed point theorem, Green's function

Keywords : Fractional order differential equation, Caputo derivative, Riemann-Liouville derivative, existence, uniqueness, fixed point theorem.

Résumé

Ce mémoire est consacré à l'étude de l'existence et de l'unicité des solutions de problèmes aux limites d'ordre fractionnaire, en considérant deux types de dérivées fractionnaires : la dérivée de Caputo et la dérivée de Riemann-Liouville, avec des conditions aux limites de type Cauchy, Dirichlet et Neumann sur des intervalles finis. L'approche adoptée consiste à transformer l'équation différentielle fractionnaire en une équation intégrale équivalente au moyen des fonctions de Green associées, puis à appliquer les théorèmes classiques de point fixe (Banach, Schauder, Schaefer, Leray-Schauder) pour établir les résultats d'existence et d'unicité, résultats qui sont illustrés par des exemples concrets en fin de chaque chapiitre.

Mots clés : Équation différentielle d'ordre fractionnaire,dérivée de Caputo, dérivée de Riemann-Liouville, existence, unicité, théorème du point fixe .

Contents

Notation

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$[a, b]$	the closed interval $a \leq x \leq b$
(a, b)	the open interval $a < x < b$
$(.)' = \frac{d(.)}{dt}$	the ordinary derivative with respect to t
a.e.	almost everywhere
i.e.	that is to say.

Spaces

$\Omega \subset \mathbb{R}^n$:	open set in $\mathbb{R}^n$
$\overline{\Omega}$:	closure of Ω
$\partial\Omega$:	boundary of Ω
$L^p(\Omega)$	$= \{$u measurable on Ω and $\int_\Omega |u|^p dx < \infty\}$, $1 \leq p < \infty$
$C(\overline{\Omega})$:	Space of continuous functions on Ω
$C^n(\overline{\Omega})$:	Space of n times continuously differentiable functions on Ω

In this work, we will specifically use the following spaces:

- $C([0,T],\mathbb{R})$, $T > 0$: Banach space of continuous functions $u : [0,T] \to \mathbb{R}$, equipped with the norm

$$\|u\|_\infty = \sup\{|u(t)|,\ t \in [0,T]\}.$$

- $L^1([0,T],\mathbb{R})$: Space of Lebesgue integrable functions $u : [0,T] \to \mathbb{R}$ equipped with the norm

$$\|u\|_1 := \|u\|_{L^1} = \int_0^T |u(t)| dt.$$

Introduction

Students of mathematics early encounter the differential operators

$$\frac{d.}{dx}, \frac{d^2.}{dx^2}, \frac{d^3.}{dx^3},$$

etc..., and some doubtless ponder whether it is necessary for the order of differentiation to be an integer. Why should there not be a $\frac{d^{\frac{1}{2}}.}{dx^{\frac{1}{2}}}$ operator, for instance? Or $\frac{d^{-1}.}{dx^{-1}}$ or even $\frac{d^{\sqrt{2}}.}{dx^{\sqrt{2}}}$? It is to these and related questions that the present work is addressed. It will come as no surprise to one versed in the calculus that the operator $\frac{d^{-1}.}{dx^{-1}}$ is nothing but an indefinite integral in disguise, but fractional orders of differentiation are more mysterious because they have no obvious geometric interpretation along the lines of the customary introduction to derivatives and integrals as slopes and areas. The reader who is prepared to dispense with a pictorial representation, however, will soon find that fractional order derivatives and integrals are just as tangible as those of integer order and that a new dimension in mathematics opens to him when the order α of the operator $\frac{d^{\alpha}}{dx^{\alpha}}$ becomes an arbitrary parameter. Nor is this a sterile exercise in pure mathematics many problems in the physical sciences can be expressed and solved succinctly by recourse to the fractional calculus.

The concept of differentiation and integration to noninteger order is by no means new. Interest in this subject was evident almost as soon as the ideas of the classical calculus were known **Leibniz** [1] (1859) mentions it in a letter to **L'Hospital** [2] in 1695. The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by **Liouville**[3] (1832), **Riemann**[4] (1953), and Holmgren (1864), although **Euler**[5] (1730), **Lagrange** [6] (1772), and others made contributions even earlier.

It was Liouville (1832) who expanded functions in series of exponential and defined the qth derivative of such a series by operating term-by-term as though q were a positive integer. Riemann (1953) proposed a different definition that involved a definite integral and was applicable to power series with noninteger exponents. Evidently it was Griinwald and Krug who first unified the results of Liouville and Riemann. Grünwald (1867), disturbed by the restrictions of Liouville's approach, adopted as his starting point the definition of a derivative as the limit of a difference quotient and arrived at definite-integral formulas for the qth derivative. Krug (1890), working through **Cauchy's** [7] integral formula for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit while Liouville's definition, in which no distinguishable lower limit appeared, corresponded to a lower limit $-\infty$.

Parallel to these theoretical beginnings was a development of the applications of the fractional calculus to various problems. In a sense, the first of these was the discovery by **Abel**[8] (1823, 1825) in 1823 that the solution of the integral equation for the tautochrone could be accomplished via an integral transform, which, as we shall see, benefits from being written as a semi-derivative. A powerful stimulus to the use of fractional calculus to solve problems was provided by the development by **Boole**[9] (1844) of symbolic methods for solving linear differential equations with constant coefficients. The essence of Boole's idea is the formal expansion of an arbitrary function $f(D)$ of the differential operator as a power series and the solution of differential equations by formal inversion of such series. Boole's methods have subsequently been made rigorous for certain classes of functions and extended in many directions.

The operational calculus of **Heaviside**[10] (1892, 1893, 1920), developed by him to solve certain problems of electromagnetic theory, was an important next step in the application of generalized derivatives. Heaviside (1920) introduced fractional differentiation in his investigation of transmission line theory; this concept has been extended by Gemant (1936) for use in problems of elasticity. While Heaviside seemed to scorn the "wet blankets of rigorists," at least some theorists recognized the merit of his techniques, and attempted to justify them by acceptable mathematical standards.

In the present century notable contributions have been made to both the theory and application of the fractional calculus. Weyl (1917), Hardy (1917), Hardy and Littlewood (1925, 1928, 1932), Kober (1940), and Kuttner (1953) examined some rather special, but natural, properties of differintegrals of functions belonging to Lebesgue and Lipschitz classes. Erdelyi (1939, 1940, 1954) and Osier (1970) have given definitions of differintegrals with respect to arbitrary functions, and Post (1930) used difference quotients to define generalized differentiation for operators $f(D)$, where D denotes differentiation and/is a suitably restricted function. Riesz (1949) has developed a theory of fractional integration for functions of more than one variable. Erdelyi (1964, 1965) has applied the fractional calculus to integral equations and Higgins (1967) has used fractional integral operators to solve differential equations. Other applications include those to rheology (Scott Blair et ah, 1947; Sher- mergor, 1966; Scott Blair, 1947, 1950; Scott Blair and Caffyn, 1949; Graham, 1961), to electrochemistry (Belavine, 1964; Oldham, 1969; Oldham and Spanier, 1970; Grenness and Oldham, 1972), to chemical physics (Somorjai and Bishop, 1970), and to general transport problems (Oldham, 1973; Oldham and Spanier, 1972). The developments are far too numerous to give an exhaustive survey here, nor is this our purpose. The readers inter- ested in further references to the literature may consult the chronological bibliography which ends this section. Virtually no area of classical analysis has been left untouched by the fractional calculus. Indeed, could one expect less from the natural extension of perhaps the two most basic operations of mathematics differentiation and integration?. And in the following additional historical perspective to our subject for fractional calculus We sum it up in the following list.

In 1772, Lagrange's contribution in this direction is the law of exponents (indices) for operators of integer order:

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}y = \frac{d^{m+n}}{dx^{m+n}}y.$$

Later, when the theory of fractional calculus started, it became important to know whether this law held true if m and n were fractions.

Lacroix (1819), develops a formula for fractional differentiation for the nth derivative of v^n by induction. Then, he formally replaces n with the fraction $\frac{1}{2}$, and together with the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, he obtains

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

Abel (1823) was probably the first to give an application of fractional calculus. He used derivatives of arbitrary order to solve the tautochrone (isochrone) problem. The integral he worked with $\int_0^t (t-s)^{-\frac{1}{2}} f(s)ds$ is precisely of the same form that Riemann used to define fractional operations.

The first major study of fractional calculus starts with Liouville (1832). Where considered $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}e^{2x}$. In his work, some problems in mechanics and geometry are solved by the use of fractional operations.

Liouville (1834) continues work on the complementary function. He argues that if the differential equation $\frac{d^n y}{dx^n} = 0$ has a complementary solution, why shouldn't $\frac{d^u y}{dx^u} = 0$ have a complementary solution when u is arbitrary?.

Liouville (1835) gives the definition of a fractional derivative, as an infinite series: $\frac{d^u y}{dx^u} = \sum a_n e^{nx} n^u$, where u is any number, integer or fractional, positive or negative, real or imaginary.

Riemann (1847) sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$\frac{d^{-r}}{dt^{-r}}u(t)=\frac{1}{\Gamma(r)}\int_c^t (t-s)^{r-1}u(s)ds.$$

However, he saw fit to add a complementary function to the above definition. Today, this definition is in common use as a definition for fractional integration but with the complementary function taken to be identically zero, and the lower limit of integration c is usually zero.

Kober (1940) extended some results over a wider range where he deals with Mellin transforms, and also a uniqueness theorem for a solution to the equation

$$g(t)=\int_a^t (t-s)^{\alpha-1}f(s)ds.$$

M. Riesz (1949) used in his research (The Riemann-Liouville integral and the Cauchy Problem) various aspects of the fractional integral

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds.$$

B. Kuttner (1953) considers in his work (Some Theorems on Fractional Derivatives) relation between the integrals:

$$\frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f(s)ds$$

and

$$(-1)^n\frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_t^1 (s-t)^{n-\alpha-1}f(s)ds.$$

In addition to all of this research, in recent years we have noticed a major revolution in research on this topic by studying many types of differential equations with fractional orders, and this research is still ongoing to this day.

Fractional equations are natural generalization of the classical integer-order differential equations. They turn out to be very adequate for modeling dynamics of many processes involving complex systems that can found in science, engineering, aerodynamics, etc. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, electrical circuits, biology, and so on, and involves derivatives of fractional order.

Fractional derivatives turn out to be an excellent tool for the description of memory and hereditary properties of various materials and processes. They involve derivatives of fractional order which provide an excellent tool to describe memory and hereditary properties of various materials and processes.

This work is organized as follows.

In **first chapter** is devoted to presenting some basic tools from functional analysis and some classical fixed point theorems: **Banach**[11] contraction theorem, **Schaefer's**[12] fixed point theorem, **Schauder's**[13] fixed point theorem, **Leray**[14]-Schauder's nonlinear alternative fixed point theorem. We also collect some definitions and basic lemmas from fractional calculus (see [1], [10] for more details).

Second chapter is proposed some results on the existence and uniqueness of solutions for fractional-order differential equations of **Caputo**[15] type.

In **the third chapter**, we present some fractional differential equations of Riemann-Liouville type with Cauchy condition of fractional order $\alpha\in]0,1[$, **Dirichlet**[16] condition of fractional order $\alpha\in]1,2[$ and Dirichlet- **Neumann**[17] type of fractional order $\alpha\in]1,2[$.
For confirmation of the results in our work, we propsed some examples is provided to demonstrate the applicability of the theoretical results.

1

Preliminaries

Contents

In this chapter we introduce the tools, definitions, lemmas and theorems necessary for this memoir.

1.1 Basic concepts

1.1.1 Compact mapping

Let X and Y two spaces of Banach and $\Omega \subset X$ an open.

Definition 1.1.1: ([11]) A continuous map $f:\Omega \longrightarrow Y$ is said to be compact if $f(\overline{\Omega})$ is compact. It is said to be completely continuous if the image of any bounded is relatively compact.

Remark :

(a) Any compact application is completely continuous; The reverse is true if Ω is bounded.

(b) Any compact linear map is continuous; The converse is true if this map is of finite rank. (Rank is the dimension of the image space).

Proposition 1.1.1: ([11]) A map $f:X \longrightarrow Y$ is compact if and only if for any suite $(x_n)_{n\in\mathbb{N}}$ of X, we can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such as the suite $(f(x_{n_k}))_{k\in\mathbb{N}}$ converges in Y.

Definition 1.1.2: Let X be an ordered Banach space. It is said that a mapping T is increasing (resp.decreasing) if $T(x) \leq T(y)$ (resp.$T(x) \geq T(y)$), for all $x, y \in X$ with $x < y$.

Definition 1.1.3: Let X be Banach space with the norm $\| . \|$ and $T: X \rightarrow Y$ a mapping. we say that T is a contraction on X if there exists a real $k \in]0,1[$ such that, for all elements $x, y \in X$ we have $\| Tx - Ty \| \leq k \| x - y \|$.

1.1.2 Lebesgue dominated convergence theorem

Theorem 1.1: ([11])
Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfies
(i) $f_n(t) \to f(t)$ for almost every $t \in \Omega$,
(ii) there is a function $g \in L^1(\Omega)$ such that $|f_n(t)| \leq g(t)$ for all $n \in \mathbb{N}$ and for almost every $t \in \Omega$.
Then $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$.

1.1.3 Ascoli-Arzela theorem

Theorem 1.2: ([11])
Let E a compact metric space, F a complete metric space. A subset $\mathfrak{F} \subset C(E,F)$ is relatively compact, if and only if it satisfies the following two conditions:
1.For all $t \in E$, the set

$$\mathfrak{F}(t) = \{T(t);\, T(.) \in \mathfrak{F}\}$$

is relatively compact in$F$.
2. $\mathfrak{F}$ is equicontinuous i.e:
For all $\epsilon > 0$, there exists $\delta > 0$, such that for any, $T(.) \in \mathfrak{F}$ and for any, $t, s \in E$, with $d_E(t,s) \leqslant \delta$ we have $d_F(T(t), T(s)) \leqslant \epsilon$.

Theorem 1.3: (A special case of the Ascoli-Arzela theorem)
Let $\mathfrak{F} \subset C(J, \mathbb{R}^n)$ and J a bounded interval of $\mathbb{R}$. $\mathfrak{F}$ is relatively compact if:
1. $\mathfrak{F}$ is uniformly bounded i.e:

$$\exists M > 0 \text{ such that } |T(t)| \leqslant M, \text{ for any } T(.) \in \mathfrak{F}\, e\, t \text{ for any } t \in J.$$

2. $\mathfrak{F}$ is equicontinuous, i.e:
$\forall \epsilon > 0, \exists \delta > 0$, such that for any, $T(.) \in \mathfrak{F}$ and for any, $t, s \in J$, with $|t-s| \leqslant \delta$ we have $\|T(t) - T(s)\| \leqslant \epsilon$.

1.2 Auxiliary functions

1.2.1 Gamma Function

Definition 1.2.1: The Gamma function Γ is defined by the following integral:

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} \, dt, \quad \alpha > 0$$

Proposition 1.2.1: An important property of the Gamma function is the following recurrence relation:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0$$

Proof :
We have $\Gamma(\alpha+1) = \int_0^{+\infty} t^{\alpha} e^{-t} \, dt$.
An integration by parts aplied to the definition of gamma function:

$$\Gamma(\alpha+1) = \left[-t^{\alpha} e^{-t}\right]_0^{+\infty} + \alpha \int_0^{+\infty} t^{\alpha-1} e^{-t} \, dt = 0 + \alpha\Gamma(\alpha) = \alpha\Gamma(\alpha).$$

■

Proposition 1.2.2:
⋆ Since $\Gamma(1) = \displaystyle\int_0^{+\infty} e^{-t} \, dt = 1$, we have $\Gamma(n+1) = n\Gamma(n) = n! \quad n \in \mathbb{N}^*$ because:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots \times 2 \times 1 \times \Gamma(1) = n!.$$

⋆ We have $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$, and $\forall n \in \mathbb{N}$, $\Gamma\left(n + \frac{1}{2}\right) = \dfrac{\sqrt{\pi}(2n)!}{4^n n!}$

Proof : Indeed, we prove it by recurrent:

for $n = 0$: $\Gamma\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}(0)!}{4^0 0!} = \sqrt{\pi}$

Suppose that $\Gamma\left(n + \dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}(2n)!}{4^n n!}$ and let's prove :

$\Gamma\left(n + 1 + \dfrac{1}{2}\right) = \Gamma\left(n + \dfrac{3}{2}\right) = \dfrac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}$

$$\begin{aligned}\Gamma\left(n+1+\frac{1}{2}\right) &= \Gamma\left(n+\frac{1}{2}+1\right)\\ &= \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\\ &= \left(n+\frac{1}{2}\right)\frac{\sqrt{\pi}(2n)!}{4^n n!}\\ &= \frac{(2n+1)\sqrt{\pi}(2n)!}{2\times 4^n n!}\\ &= \frac{(2n+1)(2n+2)\sqrt{\pi}(2n)!}{2(2n+2)\times 4^n n!}\\ &= \frac{\sqrt{\pi}(2n+2)!}{2\times 2(n+1)\times 4^n n!}\\ &= \frac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}.\end{aligned}$$

■

1.2.2 Beta Functions

Definition 1.2.2: For $\alpha>0, \gamma>0$, the Euler Beta function is defined by

$$\beta(\alpha,\gamma)=\int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt.$$

Proposition 1.2.3: ([1]) The Beta function is linked to the Gamma function by the following relationship:

$$\beta(\alpha,\gamma)=\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \quad \alpha>0, \gamma>0$$

Then,we have

$$\beta(\alpha,\gamma)=\beta(\gamma,\alpha)$$

1.3 Fractional integral

1.3.1 The fractional integral on an interval

Let h be a continuous function on the interval $[a,b]$.
We consider the integral

$$I^1_{a^+}h(t)=\int_a^t h(s)ds.$$

$$I^2_{a^+}h(t)=\int_a^t d\tau\int_a^\tau h(s)ds.$$

An integration by parts:

$$u=\int_a^\tau h(s)ds \longrightarrow u'=h(\tau)$$

$$v'=d\tau \longrightarrow v=\tau$$

$$I_{a^+}^2 h(t) = \left[\tau \int_a^\tau h(s)ds\right]_a^t - \int_a^t \tau \times h(\tau)\tau$$
$$= t\int_a^t h(s)ds - a\int_a^a h(s)ds - \int_a^t \tau \times h(\tau)d\tau = \int_a^t (t-\tau)h(\tau)d\tau$$
$$I_{a^+}^2 h(t) = \int_a^t (t-\tau)h(\tau)d\tau.$$

More generally the n^{th} iteration of the operator I can be written

$$\begin{aligned} I_{a^+}^n h(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 ... \int_a^{\tau_{n-1}} h(\tau_n)d\tau_n \\ &= \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} h(\tau)d\tau, \quad n \in \mathbb{N}^*. \end{aligned}$$

Since the generalization of the factorial by the Gamma function: $\Gamma(n) = (n-1)!$,Riemann's account could make sense even when n takes a non-integer value,it was natural to define fractional integration as follows.

1.3.2 Fractional integral of Riemann-Liouville

Definition 1.3.1: If $h \in C([a,b]), \alpha \in \mathbb{R}_+^*$ the integral

$$I_{a^+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} h(\tau)d\tau,$$

is called fractional integral of Riemann-Liouville of order α.

Example

calculate $I_{0^+}^\alpha t^\mu$, for $\mu > -1$

$$I_{0^+}^\alpha t^\mu = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^\mu d\tau \quad \text{for} \quad \alpha > 0, \mu > -1.$$

Let's make the change $\tau = st, d\tau = tds$, we have

$$\begin{aligned} I_{0^+}^\alpha t^\mu &= \frac{1}{\Gamma(\alpha)}\int_0^1 (t-st)^{\alpha-1}(st)^\mu t ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}(s)^\mu (t)^\mu t ds \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1+\mu+1}\int_0^1 (1-s)^{\alpha-1}s^\mu ds \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha+\mu} B(\mu+1,\alpha) \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha+\mu}\frac{\Gamma(\mu+1)\Gamma(\alpha)}{\Gamma(\alpha+\mu+1)}. \end{aligned}$$

So, $I_{0^+}^\alpha t^\mu = \dfrac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}, \quad \alpha > 0, \mu > -1.$

Properties 1 *([1]) we have the following properties:*

$(P_1)\quad I^0_{0^+}h(t)=h(t).$

$(P_2)\quad I^\alpha_{0^+}I^\beta_{0^+}=I^\beta_{0^+}I^\alpha_{0^+}=I^{\alpha+\beta}_{0^+},\quad \alpha>0,\beta>.0$

1.4 Fractional derivative

1.4.1 Riemann-Liouville fractional derivative

Definition 1.4.1: For a given function f defined on the interval $[a,b]$, the Riemann-Liouville fractional derivative of order $\alpha>0$ is defined by:

$$^{RL}D^\alpha_{a^+}h(t)=\frac{d^n}{dt^n}I^{n-\alpha}_{a^+}h(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{h(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,$$

where $n=[\alpha]+1$
($[\alpha]$ is the integer part of α and $n-1\leq\alpha<n$).

In particular:
- If $\alpha=0$, so

$$^{RL}D^0_{a^+}h(t)=I^0_{a^+}h(t)=h(t).$$

- If $\alpha=n\in\mathbb{N}$, so

$$^{RL}D^n_{a^+}h(t)=h^{(n)}(t).$$

- If $0<\alpha<1$, so $n=1$ hence

$$^{RL}D^\alpha_{a^+}h(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dt^n}\int_a^t(t-\tau)^{-\alpha}h(\tau)d\tau.$$

The derivative of $f(t)=t^\mu$ in the Riemann-Liouville sense:

Let $\alpha>0,\mu>-1$ and $f(t)=t^\mu$, so $^{RL}D^\alpha_{0^+}t^\mu=\dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}$.

In fact, $\quad ^{RL}D^\alpha_{0^+}t^\mu=\dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dt^n}\displaystyle\int_0^t(t-\tau)^{n-\alpha-1}\tau^\mu d\tau,\quad n=[\alpha]+1.$

By changing the variable $\tau=st$, we will have:

$$\begin{aligned}^{RL}D^\alpha_{0^+}t^\mu&=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}t^{n+\mu-\alpha}\int_0^1(1-s)^{n-\alpha-1}s^\mu ds\\&=\frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha+\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}B(n-\alpha,\mu+1)\\&=\frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha+\mu+1)}{\Gamma(\mu-\alpha+1)}\times\frac{\Gamma(n-\alpha)\Gamma(\mu+1)}{\Gamma(n-\alpha+\mu+1)}t^{\mu-\alpha}\\&=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}.\end{aligned}$$

In particular, if $\mu=0$ and $\alpha>0$, then the Riemann-Liouville fractional derivative of a constant is in general non-zero. In fact, we have

$$\begin{aligned}^{RL}D^\alpha_{a^+}C&={}^{RL}D^\alpha_{a^+}Ct^0\\&=C\frac{\Gamma(1)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}\\&=\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}.\end{aligned}$$

Lemma 1.1 ***(Composition with the fractional integral):*** *For $\alpha > 0$ and $h \in L^1[a,b]$, we have*

$$^{RL}D^{\alpha}_{a^+} I^{\alpha}_{a^+} h(t) = h(t).$$

This property means that the fractional derivation operator in the sense of Riemann-Liouville is a left inverse of the fractional integration operator in the sense of Riemann-Liouville of the same order.

Proof : We have for $n = [\alpha]+1$,

$$\begin{aligned} ^{RL}D^{\alpha}_{0^+} I^{\alpha}_{a^+} h(t) &= \frac{d^n}{dt^n}(I^{n-\alpha}_{a^+} I^{\alpha}_{a^+} h(t)) \\ &= \frac{d^n}{dt^n}(I^{n}_{a^+} h(t)) \\ &= h(t). \end{aligned}$$

In general, $^{RL}D^{\alpha}_{a^+} I^{\gamma}_{a^+} h(t) = I^{\gamma-\alpha}_{a^+} h(t), \quad \gamma > \alpha > 0, h \in L^1[a,b].$ ■

Lemma 1.2 *([1]) Let $\alpha > 0, h \in L^1[0,b]$,then*
$I^{\alpha}_{0^+}(^{RL}D^{\alpha}_{0^+})h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$, *for some $c_i \in \mathbb{R}, i = 1,2,...,n$ and $n = [\alpha]+1$.*

Corollary 1.4.1 *Let $\alpha > 0$ and $h \in L^1[a,b]$,then the equation $^{RL}D^{\alpha}_{0^+} h(t) = 0$ has the solutions*
$h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$,
such that $n = [\alpha]+1$ and $c_1, c_2, ..., c_n$ are constant.

- *If* $0 < \alpha < 1$, $D^{\alpha}_{0^+} h(t) = 0 \Longrightarrow h(t) = c\,t^{\alpha-1}, \quad c \in \mathbb{R}.$
- *If* $1 < \alpha < 2$, $D^{\alpha}_{0^+} h(t) = 0 \Longrightarrow h(t) = c\,t^{\alpha-1} + d\,t^{\alpha-2}, \quad c,d \in \mathbb{R}.$

1.4.2 Caputo fractional derivative

The definition of the Riemann-Liouville type fractional derivation played an important role in the development of the theory of fractional derivatives and integrals because of their applications in pure mathematics (solution of integer order differential equations,definition of new classes function,summation of series,etc.).However,modern technology requires some revision of the well-known pure mathematical approach. Mathematical modeling based on rheological models naturally leads to differential equations of fractional order, and to the need to formulate the initial conditions of such equations.The applied problems require definitions of fractional derivatives authorizing the use of physically interpretable initial conditions, which contain f(a);f(b),etc...Despite the fact that initial value problems with such initial conditions can be solved mathematically , the solution of this problem was proposed by M.Caputo (in the sixties) in his definition which he adapted with Mainardi in the structure of the theory of viscoelastics: Therefore we introduces a fractional derivative which is more restrictive than that of Riemann-Liouvile.
Let [a,b] be a finite interval of $\mathbb{R}$.

Definition 1.4.2: The fractional derivative of order $\alpha > 0$ of Caputo of a function h defined on $[a,b]$ is given by

$$^{C}D^{\alpha}_{a^+} h(t) = ^{RL}D^{\alpha}_{a^+}\left(h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{k!}(t-a)^k\right),$$

such that $n = [\alpha]+1$.

Remark :

- If $\alpha = 0$, then

$$^{C}D^{0}_{a^+} h(t) = h(t) - h(a).$$

- If $0<\alpha<1$, then
$$^{C}D_{a^+}^{\alpha}h(t)={}^{RL}D_{a^+}^{\alpha}(h(t)-h(a)).$$

- $^{C}D_{a^+}^{\alpha}h(t)={}^{RL}D_{a^+}^{\alpha}h(t)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, n=[\alpha]+1.$

According to the previous relationship:
$^{C}D_{a^+}^{\alpha}h(t)={}^{RL}D_{a^+}^{\alpha}h(t)\Leftrightarrow h(a)=h'(a)=...=h^{(n-1)}(a)=0.$

Theorem 1.4
Let $\alpha>0$ and $n=[\alpha]+1$,If $f\in C^n[a,b]$,then:

$$(i)\ If\,\alpha\notin\mathbb{N}\quad {}^{C}D_{a^+}^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}h^{(n)}(\tau)d\tau,$$
$$=I_{a^+}^{n-\alpha}D_{a^+}^{\alpha}h(t).$$

(ii) If $\alpha=n\in\mathbb{N}$, $\quad {}^{C}D_{a^+}^{\alpha}h(t)=h^{(n)}(t).$

Proof : According to the definition, we have

$$\begin{aligned}^{C}D_{a^+}^{\alpha}h(t)&={}^{RL}D_{a^+}^{\alpha}\left(h(t)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right)\\&=\frac{d^n}{dt^n}I_{a^+}^{n-\alpha}\left(h(t)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right)\\&=\frac{d^n}{dt^n}\int_a^t\frac{1}{\Gamma(n-\alpha)}(t-\tau)^{n-\alpha-1}\left(h(\tau)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(\tau-a)^k\right)d\tau.\end{aligned}$$

Integrating by parts, we will have

$$u=h(\tau)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(\tau-a)^k\longrightarrow u'=\frac{d}{d\tau}\left(h(\tau)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(\tau-a)^k\right)$$

$$v'=\frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)}\longrightarrow v=-\frac{(t-\tau)^{n-\alpha}}{(n-\alpha)\Gamma(n-\alpha)}=-\frac{(t-\tau)^{n-\alpha}}{\Gamma(n-\alpha+1)}$$

$$\begin{aligned}\int_a^t\frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(h(\tau)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(\tau-a)^k\right)d\tau&=-\left[\frac{(t-\tau)^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(h(\tau)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(\tau-a)^k\right)\right]_a^t\\&+\frac{1}{\Gamma(n-\alpha+1)}\int_a^t(t-\tau)^{n-\alpha}\frac{d}{d\tau}\left(h(\tau)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(\tau-a)^k\right)d\tau\\&=I_{a^+}^{n-\alpha+1}\frac{d}{dt}\left(h(t)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right)\\&=I_{a^+}^{n-\alpha+n}\frac{d^n}{dt^n}\left(h(t)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right)\\&=I_{a^+}^{n}I_{a^+}^{n-\alpha}\frac{d^n}{dt^n}\left(h(t)-\sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right)\\&=I_{a^+}^{n}I_{a^+}^{n-\alpha}\frac{d^n}{dt^n}h(t),\end{aligned}$$

because $\dfrac{d^n}{dt^n}\left(\sum_{k=0}^{n-1}\dfrac{h^{(k)}(a)}{k!}(t-a)^k\right)=0.$

Thus,

$$\begin{aligned}{}^C D^{\alpha}_{a^+}h(t) &= \frac{d^n}{dt^n} I^n_{a^+} I^{n-\alpha}_{a^+}\frac{d^n}{dt^n}h(t)\\ &= I^{n-\alpha}_{a^+}\frac{d^n}{dt^n}h(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}h^{(n)}(\tau)d\tau.\end{aligned}.$$

Example

Let $0<\alpha<1$ and $h(t)=t^{\mu},\mu>-1$ then ${}^C D^{\alpha}_{0^+}h(t)=\dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}$

In fact,

$$\begin{aligned}{}^C D^{\alpha}_{0^+}t^{\mu} &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}(\tau^{\mu})'d\tau.\\ &= \frac{\mu}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\tau^{\mu-1}d\tau.\end{aligned}$$

let's make the change $\tau=st, d\tau=tds$, we have

$$\begin{aligned}{}^C D^{\alpha}_{0^+}t^{\mu} &= \frac{\mu}{\Gamma(1-\alpha)}\int_0^1 t^{-\alpha}(1-s)^{-\alpha}s^{\mu-1}t^{\mu-1}t\,ds\\ &= \frac{\mu}{\Gamma(1-\alpha)}t^{-\alpha+\mu}\int_0^1 (1-s)^{-\alpha}s^{\mu-1}ds\\ &= \frac{\mu}{\Gamma(1-\alpha)}t^{\mu-\alpha}B(1-\alpha,\mu)\\ &= \frac{\mu}{\Gamma(1-\alpha)}t^{\mu-\alpha}\frac{\Gamma(1-\alpha)\Gamma(\mu)}{\Gamma(1-\alpha+\mu)}\\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}.\end{aligned}$$

In particular, the caputo derivative of a constant is zero:
${}^C D^{\alpha}_{a^+}C=0,\quad C$ a constant.
$\left({}^C D^{\alpha}_{a^+}C=\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-\tau)^{n-\alpha-1}c^{(n)}d\tau=0\right).$

Lemma 1.3 *([16]) Let $\beta>\alpha>0$ and $h\in L^1[a,b]$, then*

$${}^C D^{\alpha}_{a^+}I^{\beta}_{a^+}h(t)=I^{\beta-\alpha}_{a^+}h(t).$$

- If $\beta=\alpha$

$${}^C D^{\alpha}_{a^+}I^{\alpha}_{a^+}h(t)=h(t).$$

Lemma 1.4 *([16]) Let $\alpha>0, h\in L^1[a,b]$, then*

$$I^{\alpha}_{0^+}({}^C D^{\alpha}_{0^+})h(t)=h(t)+c_0+c_1t+c_2t^2+........+c_{n-1}t^{n-1}$$

for some $c_i\in\mathbb{R}, i=0,1,2,......,n-1,\quad n=[\alpha]+1.$

Corollary 1.4.2 *Let $\alpha > 0$ and $h \in L^1[a,b]$, then the equation ${}^C D_{0^+}^{\alpha} h(t) = 0$ has the solutions :*
$h(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1}$
for some $c_i \in \mathbb{R}, i = 0,1,2,......, n-1, \quad n = [\alpha]+1$
- *If $0 < \alpha < 1$, ${}^c D_{0^+}^{\alpha} h(t) \Rightarrow h(t) = c, \quad c \in \mathbb{R}$*
- *If $1 < \alpha < 2$,^{c} ${}^c D_{0^+}^{\alpha} h(t) \Rightarrow h(t) = c + dt, \quad c, d \in \mathbb{R}$*

1.4.3 Comparison between the fractional derivative in the sense of Caputo and that of Riemann-Liouville

- The main advantage of the Caputo approach is that the initial conditions of fractional differential equations with Caputo derivatives accept the same form as for integer differential equations, i.e., contain the limiting values of the derivatives of integer order of unknown functions in lower bound $X = a$
- Another difference between the definition of Riemann-Liouville and that of Caputo is that derivative of a constant is zero by Caputo on the other hand by Riemann-Liouville it is $\dfrac{c}{\Gamma(1-\alpha)} t^{-\alpha}$.

1.5 Some fixed point theorems

In this section, we will cite some fixed point theorems (Banach contraction, nonlinear alternative of Leray-Schauder type Schaefer, Schauder type) used in the resolution of differential equations by the approach of fixed point.

1.5.1 Banach fixed point theorem

Theorem 1.5: ([15])
Let X be a Banach space and $T : X \to X$ an mapping. If T is a contraction on X ($\exists k \in]0,1[\forall x, y \in X$ such that $\| Tx - Ty \| \leq k \| x - y \|$) then T admits a unique fixed point in X.

1.5.2 Schaefer's fixed point theorem

Theorem 1.6: ([15])
Let X be a Banach space and U be convex set such that $0 \in U$. Let T be an operator defined in X such that $T : U \to U$ is completely continuous.
If the set $\Omega = \{x \in U : \lambda T x = x,$ for some $\lambda \in]0,1[\}$ is bounded, then T has at least one fixed point.

1.5.3 Schauder's fixed point theorem

Theorem 1.7: ([15])
Let C be a convex, closed, bounded, non-empty subset of a Banach space X and $T : C \to C$ it's a compact mapping.Then T admits at least one fixed point.

1.5.4 Leray-Schauder nonlinear alternative theorem

Theorem 1.8: ([15])
Let C be a convex subset of a Banach space and Ω an open subset of C with $0 \in \Omega$. Then every completely continuous map $N : \overline{\Omega} \to C$ satisfies at least one of the following two properties:
$(A1)$ N has a fixed point in $\overline{\Omega}$, or
$(A2)$ There is an $x \in \partial\Omega$ and $\lambda \in (0, 1)$ with $x = \lambda N x$.

Remark :

Hypothesis $(A2)$ indicates that the set $S = \{x \in \Omega : x = \lambda \mathrm{T} x; \lambda \in [0,1]\}$ is not bounded. So to apply the nonlinear alternative of Leray-Schauder, one can just show that T is completely continuous by checking the following a priori estimate:

$$\exists R > 0, \forall x \in E, \forall \lambda \in [0,1] : (x = \lambda \mathrm{T} x \Longrightarrow \|x\| < R)$$

to assert that T admits at least one fixed point in $B(0, R)$.

2

Boundary value problems for fractional-order differential equations of Caputo type

Contents

2.1 Introduction

We propose in this chapter some results on the existence and uniqueness of solutions to fractional differential equations. We studied in the first section Cauchy problem of order $0 < \alpha < 1$. On the call to the appropriate fixed point theorem of Banach to prove the result listed in this chapter.

2.2 Cauchy problem of fractional order $\alpha \in [0,1[$

Consider the following problem

$$ {}^{C}D_{0^+}^{\alpha} u(t) = f(t, u(t)), t \in [0,\delta], \delta > 0, \tag{2.1}$$

$$ u(0) = u_0, \tag{2.2}$$

such that $0 < \alpha < 1$ and $f : [0,\delta] \times \mathbb{R} \to \mathbb{R}$ a given function, $u_0 \in \mathbb{R}$.
we say that $u \in C^1([0,\delta],\mathbb{R})$ is said solution (2.1)-(2.2) if u satisfied (2.1) and (2.2).

Lemma 2.1 *Let $h : [0,\delta] \to \mathbb{R}$ a continuous function then the linear problem*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha} u(t) & = h(t), t \in [0,\delta], \\ u(0) & = u_0, \end{cases}$$

admits only one solution given by $u(t) = u_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau.$

Proof : By Lemma(1.4) and from ${}^{C}D_{0^+}^{\alpha} u(t) = h(t)$, we have

$$I_{0^+}^{\alpha}({}^{C}D_{0^+}^{\alpha} u(t)) = I_{0^+}^{\alpha} h(t).$$

i.e. $$u(t) = c + I^{\alpha}_{0^+} h(t).$$

So $$u(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau.$$

From $u(0) = u_0$ we get $c = u_0$

so, $u(t) = u_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau.$ ■

Remark :

$u \in C^1([0,\delta],\mathbb{R})$ is solution of problem (2.1)-(2.2) if and only if $u \in C([0,\delta],\mathbb{R})$ is solution of integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau. \tag{2.3}$$

Our first result on the problem (2.1)-(2.2) is based on the Banach contraction principle.
We introduce the following conditions:
(H_1) The function $f : [0,\delta] \times \mathbb{R} \to \mathbb{R}$ is continuous.
(H_2) $\exists\, \varphi \in C([0,\delta],\mathbb{R}_+)$ such that $|f(t,u_1(t)) - f(t,u_2(t))| \le \varphi(t)|u_1(t) - u_2(t)|$,

$\forall t \in [0,\delta], \forall u_1, u_2 \in C([0,\delta],\mathbb{R})$.

(H_3) $\dfrac{\delta^{\alpha}}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,\delta]} \varphi(t) < 1.$

Theorem 2.1
Suppose (H_1), (H_2), (H_3) are satisfied, then the problem (2.1)-(2.2) admits only one solution.

Proof : Considering the operator T

$$T : C([0,\delta],\mathbb{R}) \longrightarrow C([0,\delta],\mathbb{R}),$$

$$(Tu)(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau.$$

The problem (2.1)-(2.2) has a solution u if and only if u solves the operator equation $u = Tu$.
We will prove the existence and uniqueness of a fixed point of T. For this we verify that the operator T satisfies the theorem of Banach.
To do this we show that T is contraction with respect to norm $\|\,.\,\|_{\infty}$.
Let $u_1, u_2 \in C([0,\delta],\mathbb{R})$, and $t \in [0,\delta]$,

$$\begin{aligned}
|(Tu_1)(t) - (Tu_2)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau,u_1(\tau)) - f(\tau,u_2(\tau))| d\tau,\\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi(\tau)|u_1(\tau) - u_2(\tau)| d\tau,\\
&\le \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,\delta]} \varphi(t) \| u_1 - u_2 \|_{\infty} \int_0^t (t-\tau)^{\alpha-1} d\tau,\\
&\le \frac{\delta^{\alpha}}{\Gamma(\alpha+1)} \sup_{t\in[0,\delta]} \varphi(t) \| u_1 - u_2 \|_{\infty}.
\end{aligned}$$

This implies $\| Tu_1 - Tu_2 \|_{\infty} \leqslant \dfrac{\delta^{\alpha}}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,\delta]} \varphi(t) \| u_1 - u_2 \|_{\infty}$.

As $\dfrac{\delta^{\alpha}}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,\delta]} \varphi(t) < 1$, T is contraction.

According to theorem of Banach of fixed point we conclude that the problem (2.1)-(2.2) has only one solution. ■

Example

Consider the following problem:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}} u(t) = \dfrac{1}{2(t+1)(1+(u(t))^2)}, t \in [0,1] \\ u_0 = 0 \end{cases} \tag{2.4}$$

with $0 < \alpha = \frac{1}{2} < 1$.
In this case, we pose

$$f(t, u(t)) = \frac{1}{2(t+1)(1+(u(t))^2)}.$$

So we have
(H_1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.
(H_2) Let $u_1, u_2 \in C([0,1], \mathbb{R})$, $t \in [0,1]$

$$\begin{aligned} |f(t,u_1(t)) - f(t,u_2(t))| &= \frac{1}{2(t+1)} \left| \frac{1}{1+(u_1(t))^2} - \frac{1}{1+(u_2(t))^2} \right| \\ &= \frac{1}{2(t+1)} \times \frac{|(u_2(t))^2 - (u_1(t))^2|}{(1+(u_1(t))^2)(1+(u_2(t))^2)} \\ &= \frac{1}{2(t+1)} \times \frac{|u_1(t)+u_2(t)|}{(1+(u_1(t))^2)(1+(u_2(t))^2)} |u_1(t) - u_2(t)| \\ &\leq \frac{1}{2(t+1)} |u_1(t) - u_2(t)|. \end{aligned}$$

Let's put $\varphi(t) = \dfrac{1}{2(t+1)}$, we have $\sup\limits_{t\in[0,1]} \varphi(t) = \dfrac{1}{2}$ and $\dfrac{\delta^\alpha}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,1]} \varphi(t) = \dfrac{1}{\Gamma(\frac{3}{2})} \times \dfrac{1}{2} = \dfrac{1}{\sqrt{\pi}} < 1$ with $\Gamma\left(\dfrac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}$.
By Theorem 2.2, we deduce that the problem (2.4) has only one solution.

Our second result is based on Schauder's fixed point theorem.
We introduce the following hypotheses:
(H_4) It exists $\phi \in C([0,\delta], \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ Continuous and increasing functions such that

$$| f(t, u(t)) | \leq \phi(t) \psi(| u(t) |), \forall t \in [0,\delta], \forall u \in C([0,\delta], \mathbb{R}).$$

(H_5) $\exists R > 0$ such that $| u_0 | + \dfrac{\delta^\alpha \psi(R)}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,\delta]} \phi(t) \leq R$.

Theorem 2.2:
If $(H_1), (H_4), (H_5)$ are satisfied. Then the problem (2.1)-(2.2) admits at least one solution.

Proof : To be the operator T defined in the previous Theorem 2.2.
Let $C = \{u \in C([0,\delta], \mathbb{R}), \| u \|_\infty \leq R\}$ such that R it's the constant given by (H_5).
We have C is a closed, bounded, convex of $C([0,\delta], \mathbb{R})$
Claim 1. $TC \subset C$.
Let $u \in C$ and $t \in [0,\delta]$, we have

$$\begin{aligned}
|(Tu)(t)| &\leq |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,u(\tau))|\,d\tau,\\
&\leq |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\phi(\tau)\psi(|u(\tau)|)d\tau,\\
&\leq |u_0| + \frac{1}{\alpha\Gamma(\alpha)}\psi(R)\delta^{\alpha}\sup_{t\in[0,\delta]}\phi(t),\\
&= |u_0| + \frac{1}{\Gamma(\alpha+1)}\psi(R)\delta^{\alpha}\sup_{t\in[0,\delta]}\phi(t) \leq R.
\end{aligned}$$

This is implies $\| Tu \|_{\infty} \leq R$.

So $Tu \in C$

Claim 2. T **is a compact.**

To do this just show that TC is relatively compact.

(a) TC is bounded, because $TC \subset C$ and C is bounded.

(b) TC is equicontinuous:

Let $u \in C$ and $t_1, t_2 \in [0,\delta]$ with $t_1 \leq t_2$

$$\begin{aligned}
|(Tu)(t_1)-(Tu)(t_2)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau - \int_0^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\right|,\\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau - \int_0^{t_1}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\right.\\
&\left. + \int_0^{t_1}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau - \int_0^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\right|,\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}||f(\tau,u(\tau))|\,d\tau\\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}|(t_2-\tau)^{\alpha-1}||f(\tau,u(\tau))|\,d\tau,\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}|\phi(\tau)\psi(|u(\tau)|)d\tau\\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}\phi(\tau)\psi(|u(\tau)|)d\tau,\\
&\leq \frac{\psi(R)}{\Gamma(\alpha)}\sup_{t\in[0,\delta]}\phi(t)\left(\int_0^{t_1}|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}|\,d\tau + \int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau\right),\\
&\longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2.
\end{aligned}$$

So TC is relatively compact.

According to Theorem of Schauder of fixed point we conclude that the problem (2.1)-(2.2) has at least one solution. ■

Example

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}u(t) = \dfrac{1+\sqrt{|u(t)|}}{e^t}, t\in[0,1] \\ u_0 = 0 \end{cases} \tag{2.5}$$

with $0 < \alpha = \frac{1}{2} < 1$
In this case, we pose

$$f(t,u(t)) = \frac{1+\sqrt{|u(t)|}}{e^t}.$$

So we have
(H_1) $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous.
(H_4) Let $u\in C([0,1],\mathbb{R})$, $t\in[0,1]$, $|f(t,u(t))| \le \frac{1}{e^t}(1+\sqrt{|u(t)|})$.
Let's put $\phi(t) = \frac{1}{e^t}$, $\psi(t) = 1+\sqrt{t}$ until the condition (H_4) is met.
(H_5) We choose $R=4$, in order to see that $|\, u_0\,| + \frac{\delta^\alpha \psi(R)}{\Gamma(\alpha+1)} \sup_{t\in[0,\delta]} \phi(t) = \frac{6}{\sqrt{\pi}} \le 4 = R$.
By Theorem 2.2, we deduce that the problem (2.5) has at least one solution.

The following result is based on the fixed point Theorem of Schaefer.
We introduce the following condition.
(H_6) $\lim_{r\to+\infty} \frac{r}{\psi(r)} = +\infty$.

Theorem 2.3
If $(H_1),(H_4),(H_6)$ are satisfied. Then the problem (2.1)-(2.2) has at least one solution.

Proof : Let the operator be T defined in the previous Theorem 2.2.
As in the proof of the previous Theorem 2.2, we got that the operator T is compact. So in order to the operator T to be completely continuous, it's sufficient to prove that is continuous.
- T is continuous on C
Let $(u_n)\subset C$ be a sequence converges to u^* .
So

$$|\,(Tu_n)(t)-(Tu^*)(t)\,| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,|\,f(\tau,u_n(\tau))-f(\tau,u^*(\tau))\,|\,d\tau.$$

From lebesgue dominated convergence theorem and from continuity of f, we have
$|\,(Tu_n(t)-(Tu^*)(t)\,| \longrightarrow 0$ as $n\longrightarrow\infty$.
Now consider the subset:
$\Omega = \{u\in C,\ \lambda Tu = u,$ for some $\lambda\in]0,1[\}$.
Let $u\in\Omega$ and $t\in[0,\delta]$, we have

$$\begin{aligned}
|u(t)| &= \lambda|(Tu)(t)|,\\
&\le |(Tu)(t)|,\\
&\le |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi(\tau)\psi(|u(\tau)|)d\tau,\\
&\le |u_0| + \frac{\delta^\alpha\psi(\|\,u\,\|_\infty)}{\Gamma(\alpha+1)} \sup_{t\in[0,\delta]} \varphi(t).
\end{aligned}$$

This is implies $\qquad \dfrac{\|\,u\,\|_\infty}{|u_0| + \dfrac{\delta^\alpha\psi(\|\,u\,\|_\infty)}{\Gamma(\alpha+1)} \sup_{t\in[0,\delta]} \varphi(t)} \le 1.$
If $\|\,u\,\|_\infty\to\infty$, so we while have a contradiction with (H_6).
Then it exists $\rho>0$ such that $\|\,u\,\|_\infty \le \rho$ it's who gives Ω is bounded.
From Schaefer's Theorem of fixed point, the operator T has at least one fixed which represents a solution to the problem (2.1)-(2.2).
■

The following result is based on the nonlinear alternative of Leray-Schauder.
We introduce the following condition:

(H_7) $\exists M > 0$ such that $\dfrac{M}{|u_0| + \dfrac{\delta^\alpha \psi(M)}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,\delta]} \phi(t)} > 1.$

Theorem 2.4
If $(H_1),(H_4),(H_7)$ are satisfied. Then the problem (2.1)-(2.2) has at least one solution.

Proof : Let to be the operator T defined in the previous Theorem 2.2.
Assume that the open set $U = \{u \in C([0,\delta],\mathbb{R}),\ \parallel u \parallel_\infty \leq M\}$
As in the proof of the previous Theorem 2.2, we got that the operator T is continuous and compact, so it's completely continuous.
If it exists $u \in \partial U$ and $\lambda \in [0,1]$ such that $u = \lambda T u$, then

$$\begin{aligned} |u(t)| &= \lambda |(Tu)(t)|, \\ &\leq |(Tu)(t)|, \\ &\leq |u_0| + \frac{\delta^\alpha \psi(\parallel u \parallel_\infty)}{\Gamma(\alpha+1)} \sup_{t\in[0,\delta]} \phi(t). \end{aligned}$$

This is implies $\dfrac{\parallel u \parallel_\infty}{|u_0| + \dfrac{\delta^\alpha \psi(\parallel u \parallel_\infty)}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,\delta]} \phi(t)} \leq 1.$

Which means this from (H_7),$\parallel u \parallel_\infty \neq M$, but $\parallel u \parallel_\infty = M$ because $u \in \partial U$,
this is a contradiction.
From Theorem of Leray-Schauder of fixed point, the operator T has at least one point fixed which represents a solution to the problem (2.1)-(2.2). ■

3

Riemann-Liouville type fractional order differential equations

Contents

3.1 Introduction

We present in this chapter some results on the existence and uniqueness of solutions to fractional differential equations of Riemann-Liouville type. We studied in the first section Cauchy problem of order $0<\alpha<1$. In the second section we discuss the boudary value problem of Dirichlet type, then we study a Dirichlet-Neumann type problem and Cauchy problem in the case $1<\alpha<2$.

3.2 Cauchy problem of fractional order $\alpha \in]0,1[$

Suppose the following problem

$$^{RL}D^{\alpha}_{0^+}u(t) = f(t,u(t)), t \in [0,\delta], (\delta > 0), \tag{3.1}$$

$$\lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) = u_0, \tag{3.2}$$

such that $0<\alpha<1$ and $f:[0,\delta]\times\mathbb{R}\to\mathbb{R}$ a given fonction, $u_0\in\mathbb{R}$.
Suppose the Banach space[16] $X = \left\{ u \in C((0,\delta],\mathbb{R}), \lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) \text{ exists} \right\}$ with the norm:

$$\| u \|_X = \sup_{t\in[0,\delta]} t^{1-\alpha}|u(t)|.$$

We say that $u \in X$ is said solution (3.1)-(3.2) if u satisfied (3.1) and (3.2).

Lemma 3.1 *Let $h:[0,\delta]\to\mathbb{R}$ a continuous fonction.*
Then the linear problem

$$\begin{cases} ^{RL}D^{\alpha}_{0^+}u(t) = h(t), t \in [0,\delta] \\ \lim_{t \overset{>}{\to} 0} t^{1-\alpha}u(t) = u_0 \end{cases}$$

Has only one solution given by

$$u(t)=u_0t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau)d\tau.$$

Proof : By lemma 1.2 and from ${}^{RL}D_{0^+}^{\alpha}u(t)=h(t)$, we have

$$I_{0^+}^{\alpha}\left({}^{RL}D_{0^+}^{\alpha}u(t)\right)=I_{0^+}^{\alpha}h(t)$$

it implies,

$$u(t)=c\,t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau)d\tau.$$

By the condition (3.2) we get $c=u_0$.
Thus

$$u(t)=u_0t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau)d\tau$$

Now, Considering the following operator $T:X\longrightarrow X$ defined by:

$$(Tu)(t)\quad=u_0t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau.$$

Then we have the following equivalent: $u\in X$ is solution of problem (3.1)-(3.2) if and only if $u\in X$ satisfies the operational equation $u=Tu$ this means that u is a fixed point of T.
■

Our first result on the problem (3.1)-(3.2) is based on the Banach fixed point theorem.
We introduce the following conditions:
$(H1)$ The fonction $f:[0,\delta]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous.
$(H2)$ $\exists k>0$ such that

$$\begin{aligned}|t^{\alpha-1}f(t,u_1(t))-t^{\alpha-1}f(t,u_2(t))| &\le k|u_1(t)-u_2(t)|,\\ &\forall t\in[0,\delta],\quad \forall u_1,u_2\in X.\end{aligned}$$

$(H3)$ $0<k<\dfrac{\Gamma(\alpha+1)}{\delta}$.

Theorem 3.1
Suppose (H1) - (H3) are satisfied. Then the problem (3.1)-(3.2) admits only one solution.

Proof : We will prove the existence and uniqueness of a fixed point of T. For this we verify that the operator T satisfies the Banach's fixed point theorem. To do this we show that T is contraction with respect to norm $\|\,.\,\|_X$.
Let $u_1,u_2\in X$ and $t\in[0,\delta]$,

$$\begin{aligned}|(Tu_1)(t)-(Tu_2)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|f(\tau,u_1(\tau))-f(\tau,u_2(\tau)|d\tau,\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{1-\alpha}\tau^{\alpha-1}|f(\tau,u_1(\tau))-f(\tau,u_2(\tau)|d\tau,\\ &\le \frac{k}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{1-\alpha}|u_1(\tau)-u_2(\tau)|d\tau,\end{aligned}$$

so

$$t^{1-\alpha}|(Tu_1)(t)+(Tu_2)(t)| \leq \frac{\delta^{1-\alpha}k}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{1-\alpha}|u_1(\tau)-u_2(\tau)|d\tau.$$

Thus

$$\| Tu_1 - Tu_2 \|_X \leq \frac{\delta k}{\Gamma(\alpha+1)} \| u_1 - u_2 \|_X .$$

From $(H3)$. $0 < \dfrac{\delta k}{\Gamma(\alpha+1)} < 1$. This means that T is contraction. According to Theorem of Banach, we conclude that the problem (3.1)-(3.2) has only one solution. ■

Example

Consider the following problem:

$$^{RL}D_{0^+}^{\frac{1}{2}} u(t) = \frac{t^{\frac{1}{2}} e^{2t}|u(t)|}{5(e^{2t}+1)(1+|u(t)|)}, t \in [0,1] \tag{3.3}$$

$$\lim_{t \overset{>}{\to} 0} t^{\frac{1}{2}} u(t) = 0. \tag{3.4}$$

Let $f(t,u(t)) = \dfrac{t^{\frac{1}{2}} e^{2t}|u(t)|}{5(e^{2t}+1)(1+|u(t)|)}$.
So we have
$(H1)$ $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.
$(H2)$ Let $u_1, u_2 \in X, t \in [0,1]$,

$$\begin{aligned} |t^{-\frac{1}{2}} f(t,u_1(t)) - t^{-\frac{1}{2}} f(t,u_2(t))| &= \frac{t^{-\frac{1}{2}} t^{\frac{1}{2}} e^{2t}}{5(e^{2t}+1)} \left| \frac{|u_1(t)|}{1+|u_1(t)|} - \frac{|u_2(t)|}{1+|u_2(t)|} \right|, \\ &= \frac{e^{2t}}{5(e^{2t}+1)} \frac{||u_1(t)|-|u_2(t)||}{(1+|u_1(t)|)(1+|u_2(t)|)}, \\ &\leq \frac{1}{5}|u_1(t)-u_2(t)|. \end{aligned}$$

In this case, we choose $k = \dfrac{1}{5}$ as achieved

$$(H3)\, 0 < k < \frac{\Gamma(\alpha+1)}{\delta} \quad \left(0 < \frac{1}{5} < \frac{\Gamma\left(\frac{3}{2}\right)}{1} \text{ with } \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}\right).$$

By Theorem 3.2, we deduce that the problem (3.3)-(3.4) has only one solution.

Our second result id based on Schawder's fixed point theorem.
Insufficient the following hypotheses:

$$(H4) \qquad \exists \phi \in C([0,\delta],\mathbb{R}_+) \text{ and } \varphi : \mathbb{R}_+ \to \mathbb{R}_+$$

continuous and increasing functions such that

$$|f(t, t^{\alpha-1}u(t))| \leq \varphi(t)\phi(|u(t)|) \forall t \in [0,\delta], \forall u \in C([0,\delta],\mathbb{R}).$$

$(H5)$ $\quad \exists R > 0$ such that

$$|u_0| + \frac{\delta \phi(R)}{\Gamma(\alpha+1)} \sup_{0 \leq t \leq \delta} \varphi(t) \leq R.$$

Theorem 3.2

If $(H1)$, $(H4)$, $(H5)$ are satisfied. Then the problem (3.1)-(3.2) has at least one solution.

Proof : To be the operator T defined in the previous Theorem 3.2.
Let $C = \{u \in C(X), \| u \|_X \le R\}$
C is a closed, bounded, convex of X.
Step 1.$TC \subset C$:
Let $u \in C$, and $t \in [0,\delta]$ we have

$$\begin{aligned}|(Tu)(t)| &\le t^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,u(\tau))|d\tau,\\ &\le \delta^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,\tau^{\alpha-1}\tau^{1-\alpha}u(\tau))|d\tau,\\ &\le \delta^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)\phi(\tau^{1-\alpha}|u(\tau)|)d\tau,\\ &\le \delta^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha)}\phi(R)\sup_{0\le t\le\delta}\varphi(t)\int_0^t (t-\tau)^{\alpha-1}d\tau,\\ &\le \delta^{\alpha-1}|u_0| + \frac{1}{\Gamma(\alpha+1)}\phi(R)\delta^{\alpha}\sup_{0\le t\le\delta}\varphi(t).\end{aligned}$$

Thus

$$\begin{aligned}t^{1-\alpha}|Tu(t)| &\le \delta^{1-\alpha}\delta^{\alpha-1}|u_0| + \frac{\delta^{1-\alpha}}{\Gamma(\alpha+1)}\phi(R)\delta^{\alpha}\sup_{0\le t\le\delta}\varphi(t),\\ &\le |u_0| + \frac{\delta\phi(R)}{\Gamma(\alpha+1)}\sup_{0\le t\le\delta}\varphi(t),\\ &\le R.\end{aligned}$$

So $Tu \in C$.
Step 2. T is continuous:
Let $(u_n) \subset C$ be a sequence converges to u. Then we get

$$\begin{aligned}|(Tu_n)(t)-(Tu)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,u_n(\tau))-f(\tau,u(\tau))|d\tau,\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,\tau^{\alpha-1}\tau^{1-\alpha}u_n(\tau))-f(\tau,\tau^{\alpha-1}\tau^{1-\alpha}u(\tau))|d\tau.\end{aligned}$$

Thus

$$\delta^{1-\alpha}|(Tu_n)(t)-(Tu)(t)| \le \frac{\delta^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,\tau^{\alpha-1}\tau^{1-\alpha}u_n(\tau))-f(\tau,\tau^{\alpha-1}\tau^{1-\alpha}u(\tau))|d\tau.$$

From Lebesgue dominated convergence theorem and from the continuity of f, we have

$$|(Tu_n)(t)-(Tu)(t)| \to 0 \text{ as } n \to +\infty.$$

Hence the continuity of T.
Step 3. T is compact:
To do this just show that TC is relatively compact.
(a). TC is bounded, because $TC \subset C$ and C is bounded.
(b). TC is equicontinuous:

Let $u \in C$ and $t_1, t_2 \in [0,\delta]$ with $t_1 \leq t_2$ we have,

$$\begin{aligned}
|(Tu)(t_1)-(Tu)(t_2)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau-\int_0^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\right|,\\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau-\int_0^{t_1}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\right.\\
&\quad\left.+\int_0^{t_1}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau-\int_0^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\right|,\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}||f(\tau,\tau^{\alpha-1}\tau^{1-\alpha}u(\tau)|d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}|f(\tau,\tau^{\alpha-1}\tau^{1-\alpha}u(\tau)|d\tau,\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}|\varphi(\tau)\phi(\tau^{1-\alpha}|u(\tau)|)d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}\varphi(\tau)\phi(\tau^{1-\alpha}|u(\tau)|)d\tau,\\
&\leq \frac{\phi(R)}{\Gamma(\alpha)\sup\limits_{0\leq t\leq T}\varphi(t)}\left(\int_0^{t_1}|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}|d\tau\right.\\
&\quad\left.+\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau\right)\longrightarrow 0 \text{ as } t_1\longrightarrow t_2.
\end{aligned}$$

So TC is relatively compact.
According to theorem of Schauder, we conclude that the problem (3.1)-(3.2) has at least one solution.
■

The following résultat is based on the fixed point theorem of Schaefer.
We introduce the following condition:

$$(H6) \qquad \lim_{r\to+\infty}\frac{r}{\phi(r)}=+\infty$$

Theorem 3.3
If $(H1)$, $(H3)$, $(H6)$ are satisfied, then the problem (3.1)-(3.2) has at least one solution.

Proof : Let to be the operator T defined in the previous Theorem 1.
As in the proof of the previous Theorem 3.2, we get that the operator T is continuous and compact, so it's completely continuous.
Now consider the subsets:

$$Z=\left\{u\in X, \| u \|_X \leq \rho\right\}$$

with $\rho > 0$ and

$$\Omega=\{u\in Z, \lambda Tu=u, \text{ for some } \lambda\in]0,1[\}$$

Let $u \in \Omega$ and $t \in [0,\delta]$ we have

$$\begin{aligned}
|u(t)| &= \lambda|(Tu)(t)|,\\
&\leq |(Tu)(t)|,\\
&\leq t^{\alpha-1}|u_0|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\varphi(\tau)\phi(\tau^{1-\alpha}|u(\tau)|)d\tau.
\end{aligned}$$

Therefore

$$t^{1-\alpha}|u(t)| \leq |u_0| + \frac{\delta\phi(\| u \|_X}{\Gamma(\alpha+1)} \sup_{0\leq t\leq\delta} \varphi(t),$$

so

$$\| u \|_X \leq |u_0| + \frac{\delta\phi(\| u \|_X)}{\Gamma(\alpha+1)} \sup_{0\leq t\leq\delta} \varphi(t).$$

Thus

$$\frac{\| u \|_X}{|u_0| + \dfrac{\delta\phi(\| u \|_{X)}}{\Gamma(\alpha+1)} \sup\limits_{0\leq t\leq\delta} \varphi(t)} \leq 1$$

If $\| u \|_X \to +\infty$, then we will have a contradiction with $(H6)$.
Hence it exists $\rho_0 > 0$ such that $\| u \|_X \leq \rho_0$ it's who gives Ω is bounded.
From theorem of Schaefer, the operator T has at least one fixed point which represents a solution to the problem ((3.1)-(3.2).
■

The following result is based on the nonlinear alternative of Learay-Schauder fixed point theorem. For this we introduce the following condition:

$$(H7) \qquad \exists\delta > 0 \text{ such that } \frac{k}{|u_0| + \dfrac{T\phi(\| u \|_X)}{\Gamma(\alpha+1)} \sup\limits_{0\leq t\leq T} \varphi(t)} > 1.$$

Theorem 3.4
If $(H1)$, $(H3)$, $(H7)$ are satisfied, then the problem (3.1)-(3.2) has at least one solution.

Proof : Let to be operator T defined in the previous Theorem 3.2.
Assume that the open set

$$W = \{u \in X, \| u \|_X < \delta\}.$$

As in the proof of the previous Theorem 3.2 we get that the operator T is continuous.
If it exists $u \in \partial W$ and $\lambda \in [0,1]$ such that $u = \lambda T u$ then

$$t^{1-\alpha}|u(t)| \leq |u_0| + \frac{T\phi(\| u \|_X}{\Gamma(\alpha+1)} \sup_{0\leq t\leq T} \varphi(t),$$

so

$$\| u \|_X \leq |u_0| + \frac{T\phi(\| u \|_X}{\Gamma(\alpha+1)} \sup_{0\leq t\leq T} \varphi(t).$$

Thus

$$\frac{\| u \|_X}{|u_0| + \dfrac{T\phi(\| u \|_{X)}}{\Gamma(\alpha+1)} \sup\limits_{0\leq t\leq T} \varphi(t)} \leq 1.$$

Which means this from $(H7)$, $\| u \|_X \neq \delta$ but $\| u \|_X = \delta$ because $u \in \partial W$ this is a contradiction.
From Theorem of Leray-Schauder, the operator T has at least one fixed point which represents a solution to the problem (3.1)-(3.2). ■

3.3 Boundary value problem with Dirichlet condition of fractional order $\alpha \in]1,2[$

Consider the following problem :

$$^{RL}D^{\alpha}_{0^+} u(t) = f(t, u(t)), t \in [0,T], (T > 0) \tag{3.5}$$

$$\lim_{t \overset{>}{\to} 0} t^{2-\alpha} u(t) = u_0, \ \lim_{t \overset{<}{\to} T} t^{2-\alpha} u(t) = u_T, \tag{3.6}$$

such that $1 < \alpha < 2$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ a given function u_0, $u_b \in \mathbb{R}$.
Suppose the Banach space

$$Y = \left\{ u \in C([0,T],\mathbb{R}), \lim_{t \overset{>}{\to} 0} t^{2-\alpha} u(t), \lim_{t \overset{<}{\to} b} t^{2-\alpha} u(t) \text{ exist} \right\}.$$

Lemma 3.2 *let* $h : [0,T] \to \mathbb{R}$ *a continuous function.*
Then the linear problem

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha} u(t) = h(t), \ t \in [0,T] \\ \lim_{t \overset{>}{\to} 0} t^{2-\alpha} u(t) = u_0, \quad \lim_{t \overset{<}{\to} T} t^{2-\alpha} u(t) = u_T, \end{cases},$$

has only one solution given by

$$u(t) = u_0 t^{\alpha-1} + \frac{u_T - u_0}{T} t^{\alpha-1} + \int_0^T G(t,\tau) h(\tau) d\tau,$$

such that G *is the Green's function defined by*

$$G(t,\tau) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-\tau)^{\alpha-1} - t^{\alpha-1} T^{1-\alpha} (T-\tau)^{\alpha-1} & if \quad 0 \le \tau \le t \le T, \\ -t^{\alpha-1} T^{1-\alpha} (T-\tau)^{\alpha-1} & if \quad 0 \le t \le \tau \le T. \end{cases}$$

Proof : By lemma 1.2 and from ${}^{RL}D_{0^+}^{\alpha} u(t) = h(t)$, $1 < \alpha < 2$ we have

$$I_{0^+}^{\alpha} ({}^{RL}D_{0^+}^{\alpha} u(t)) = I_{0^+}^{\alpha} h(t),$$

i.e.

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau.$$

So

$$t^{2-\alpha} u(t) = c_1 t + c_2 + \frac{1}{\Gamma(\alpha)} t^{2-\alpha} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau$$

with c_1, c_2 are constants.
From $\lim_{t \overset{>}{\to} 0} t^{2-\alpha} u(t) = u_0$ we obtain $c_2 = u_0$ and by $\lim_{t \overset{<}{\to} T} t^{2-\alpha} u(t) = u_T$ we get

$$c_1 T + c_2 + \frac{1}{\Gamma(\alpha)} T^{2-\alpha} \int_0^T (T-\tau)^{\alpha-1} h(\tau) d\tau = u_T.$$

This gives

$$c_1 = \frac{u_T}{T} - \frac{u_0}{T} - \frac{1}{\Gamma(\alpha)} T^{1-\alpha} \int_0^T (T-\tau)^{\alpha-1} h(\tau) d\tau.$$

Hence

$$\begin{aligned} u(t) \ &= \frac{u_T}{T} t^{\alpha-1} - \frac{u_0}{T} t^{\alpha-1} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} T^{1-\alpha} \int_0^T (T-\tau)^{\alpha-1} h(\tau) d\tau \\ &+ u_0 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau, \end{aligned}$$

which can be written in the from

$$
\begin{aligned}
u(t) &= u_0 t^{\alpha-2} + \frac{u_T - u_0}{T} t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t t^{\alpha-1} T^{1-\alpha} (T-\tau)^{\alpha-1} h(\tau) d\tau \\
&- \frac{1}{\Gamma(\alpha)} \int_t^T t^{\alpha-1} T^{\alpha-1} (T-\tau)^{\alpha-1} h(\tau) d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau, \\
&= u_0 t^{\alpha-2} + \frac{u_T - u_0}{T} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} - t^{\alpha-1} T^{1-\alpha} (T-\tau)^{\alpha-1} h(\tau) d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_t^T - t^{\alpha-1} T^{1-\alpha} (T-\tau)^{\alpha-1} h(\tau) d\tau, \\
&= u_0 t^{\alpha-2} + \frac{u_T - u_0}{T} t^{\alpha-1} + \int_0^T G(t,\tau) h(\tau) d\tau.
\end{aligned}
$$

■

Our first result for the problem (3.5)-(3.6) is based on the principle of Banach contraction. We introduce the following condition:

$$
(H8) \qquad G^* k T^{2-\alpha} < 1,
$$

such that $G^* = \sup_{(t,\tau)\in[0,T]\times[0,T]} |G(t,\tau)|$ and k as in $(H3)$.

Theorem 3.5
Suppose that $(H1)$, $(H3)$, $(H8)$, are hold. Then the problem (3.5)-(3.6) has only one solution.

Proof : Considering the operator $T : X \to X$ defined by

$$
(Tu)(t) = u_0 t^{\alpha-2} + \frac{u_b - u_0}{T} t^{\alpha-1} + \int_0^T G(t,\tau) f(\tau, u(\tau)) d\tau
$$

We prove that T is contraction:
Let $u, v \in X$ and $t \in [0,T]$ we have

$$
\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\leq \int_0^T |G(t,\tau)| |f(\tau,u(\tau)) - f(\tau,v(\tau))| d\tau, \\
&\leq \int_0^T |G(t,\tau)| \tau^{1-\alpha} \tau^{\alpha-1} |f(\tau,u(\tau)) - f(\tau,v(\tau))| d\tau, \\
&\leq G^* k \int_0^T \tau^{\alpha-1} |u(\tau) - v(\tau)| d\tau, \\
&\leq G^* k T \parallel u - v \parallel_X .
\end{aligned}
$$

So

$$
t^{1-\alpha} |(Tu)(t) - (Tv)(t)| \leq G^* k T^{2-\alpha} \parallel u - v \parallel_X,
$$

i.e.

$$
\parallel Tu - Tv \parallel_X \leq G^* k T^{2-\alpha} \parallel u - v \parallel_X .
$$

As $G^* k T^{2-\alpha} < 1$ then T is contraction according to theorem of Banach, we conclude that the problem has only one solution. ■

3.4 Boundary value problem with Dirichlet-Neumann type of fractional order $\alpha \in]1,2[$

Suppose the following problem

$$
{}^{RL}D^{\alpha}_{t^+} u(t) = f(t, u(t)), t \in [0,T], (T > 0), \tag{3.7}
$$

$$
u(0) = 0, \quad u'(T) = 0, \tag{3.8}
$$

such that $1 < \alpha < 2$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ a given function.

Lemma 3.3 *Let $h:[0,T]\to\mathbb{R}$ a continuous function, then the linear problem*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha} u(t) = h(t), \ t \in [0,T], \\ u(0)=0, \quad u'(T)=0. \end{cases}$$

has only one solution given by

$$u(t) = -\frac{1}{T^{\alpha-1}\Gamma(\alpha)} t^{\alpha-1} \int_0^T (T-\tau)^{\alpha-2} h(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau)d\tau,$$

such that G is the Green's function defined by

$$G(t,\tau) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-\tau)^{\alpha-1} - \frac{1}{T^{\alpha-2}} t^{\alpha-1}(T-\tau)^{\alpha-2} & if \quad 0 \le \tau \le t \le T \\ -\frac{1}{T^{\alpha-2}} t^{\alpha-1}(T-\tau)^{\alpha-2} & if \quad 0 \le t \le \tau \le T \end{cases}$$

Proof : By lemma 1.2 and from ${}^{RL}D_{0^+}^{\alpha} u(t) = h(t)$, we have $I^{\alpha}\left({}^{RL}D_{0^+}^{\alpha} u(t)\right) = I_{0^+}^{\alpha} h(t)$ which gives

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau)d\tau,$$

and

$$u'(t) = (\alpha-1)c_1 t^{\alpha-2} + (\alpha-2)c_2 t^{\alpha-3} + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-2} h(\tau)d\tau,$$

such that c_1, c_2 are constants.

From $u(0)=0$ we obtain $c_2=0$ and by $u'(T)=0$ we have $c_1 = -\frac{1}{T^{\alpha-2}\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-2} h(\tau)d\tau.$

So

$$u(t) = -\frac{1}{T^{\alpha-2}\Gamma(\alpha)} t^{\alpha-1} \int_0^T (b-\tau)^{\alpha-2} h(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau)d\tau.$$

Which can be written in the form

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left((t-\tau)^{\alpha-1} - \frac{1}{T^{\alpha-2}} t^{\alpha-1}(T-\tau)^{\alpha-1}\right) h(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_t^T -\frac{1}{T^{\alpha-2}} t^{\alpha-1}(T-\tau)^{\alpha-1} h(\tau)d\tau,$$

the latter is written in the form

$$u(t) = \int_0^T G(t,\tau)h(\tau)d\tau.$$

■

Now,we present an result for the problem (3.5)-(3.6) is based on the principle of Banach contraction. Firstly, we introduce the following condition:

$$(H9) \qquad 0 < G^* k < 1.$$

Theorem 3.6
Suppose $(H1)$, $(H2)$, $(H9)$ are satisfied. Then the problem (3.7)-(3.8) has only one solution.

Proof : Considering the operator $T: C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ defined by:

$$(Tu)(t) = \int_0^T G(t,\tau) f(\tau, u(\tau))d\tau.$$

We show that T is contraction:
Let $u, v \in C([0,T],\mathbb{R})$, $t \in [0,T]$ we get

$$\begin{aligned} |(Tu)(t)-(Tv)(t)| &\leq \int_0^T |G(t,\tau)||f(\tau,u(\tau))-f(\tau,v(\tau))|d\tau, \\ &\leq k\int_0^T |G(t,\tau)||u(\tau)-v(\tau)|d\tau, \\ &\leq kG^* \parallel u-v \parallel_\infty . \end{aligned}$$

this implies $\parallel Tu-Tv \parallel_\infty \leq kG^* \parallel u-v \parallel_\infty$.
Since $kG^*<1$ so T is contraction.
According to theorem of Banach, the problem (3.7)-(3.8) has only one solution. ■

Conclusion and Perspectives

In this work, we dealt with the existence and uniqueness of solutions to fractional differential equations associated with Cauchy type, Dirichlet type, Neumann type conditions.

Firstly, we introduced the concepts of fractional calculus and some basic tools from functional analysis and some classic fixed point theorems.

In the different chapters of this research, to study the proposed problems, we transformed each problem into an operator equation, in this case we sometimes use the Green function associated with the imposed problem with applying the fixed point theory to prove the results provided. On the other hand we choose sufficient conditions so that the operators associated with the proposed problems satisfy the appropriate fixed point theory to demonstrate our results.

We hope that this research will benefit researchers studying in the specialty of differential equations, and that they will have support in other research that is an extension of this research.

Appendices

List of Cited Mathematicians

1 . Gottfried Wilhelm Leibniz (1646–1716) :German philosopher and mathematician, co-developer of classical differential and integral calculus. He is credited with the symbolic formulation of this discipline and was the first to discuss the possibility of differentiation of non-integer order, as recorded in his correspondence in 1695.

2 .Guillaume de l'Hôpital (1661–1704) : French mathematician, best known for l'Hôpital's rule for evaluating indeterminate limits. His textbook on infinitesimal calculus was among the first in the field.

3 . Joseph Liouville (1809–1882) : French mathematician. He proposed an early definition of fractional derivatives by expanding functions into exponential series and operating term-by-term, considering fractional orders as the natural extension of integer orders. He also contributed to proving the existence of transcendental numbers.

4 . Bernhard Riemann (1826–1866) : German mathematician. He proposed an alternative definition based on a definite integral with a variable lower limit (c), making it applicable to power series with non-integer exponents. His name is attached to the Riemann Hypothesis, one of the most famous unsolved problems in mathematics.

5 .Leonhard Euler (1707–1783) : Swiss mathematician, one of the most prolific and influential in history, contributing to nearly all branches of mathematics. He was the first to use the gamma function (Γ) to generalise the factorial, which later became an essential tool in defining fractional integrals and derivatives.

6 .Joseph-Louis Lagrange (1736–1813) :Italian-French mathematician, formulated analytical mechanics and contributed to the theory of equations and calculus. He is noted for formulating the law of exponents for integer-order operators.

7 .Augustin-Louis Cauchy (1789–1857) :French mathematician, who established the rigorous formulations of limits, continuity, and derivatives, and founded complex analysis. His Cauchy integral formula was later used by Krug to unify the definitions of fractional differentiation.

8 .Niels Henrik Abel (1802–1829) : Norwegian mathematician. He is regarded as the first to provide an application of fractional calculus by solving the tautochrone (isochrone) problem, thereby ending the debate on the first practical use of this field.

9 .George Boole (1815–1864) : English mathematician, renowned for developing Boolean Algebra. In the context of fractional calculus, he was a pioneer of symbolic methods for solving differential equations, treating the differential operator as an algebraic symbol. His ideas formed the direct foundation upon which Heaviside later built operational calculus, in which fractional powers of operators were employed.

10 .Oliver Heaviside (1850–1925) : Self-taught British electrical engineer and physicist. He reformulated Maxwell's equations, invented the step function that bears his name, and developed operational calculus in which he made use of fractional differentiation. His practical approach was both admired and rejected by academic circles.

11 .Stefan Banach (1892–1945) :Polish mathematician, one of the founders of modern functional analysis. The Banach fixed-point theorem (Contraction Mapping Theorem) is a fundamental result that guarantees the existence and uniqueness of solutions in complete metric spaces, and is an essential tool in the theory of differential equations.

12 .Helmut H. Schaefer (1925–2005) : German mathematician, known for contributions to topological vector spaces and functional analysis. Schaefer's fixed-point theorem bears his name; it is an extension of Banach's theorem used in nonlinear analysis to prove the existence of solutions.

13 .Juliusz Paweł Schauder (1899–1943) :Polish mathematician, member of the Lvov School of Mathematics. He is famous for Schauder's fixed-point theorem, which generalises Banach's theorem to compact operators. This theorem is a central tool in proving existence of solutions to nonlinear differential equations.

14 .Jean Leray (1906–1998):French mathematician, who made fundamental contributions to partial differential equations and aerodynamic theory. Together with Schauder, he developed the Leray-Schauder nonlinear alternative, an important principle in nonlinear analysis used to prove existence of solutions when trivial possibilities are excluded.

15 .Michele Caputo (born 1927): Italian geophysicist who proposed the Caputo fractional derivative. The main advantage of this definition is that it employs classical-type boundary conditions (values of the function and its integer-order derivatives at the starting point), making it particularly suitable for physical and engineering applications.

16 .Johann Peter Gustav Lejeune Dirichlet (1805–1859) : German mathematician, known for his work in number theory and series. The Dirichlet condition is a boundary condition that prescribes the values of the function itself on the boundary of the domain; it is appropriate for fractional differential equations of order $\alpha \in]1,2[$.

17 .Carl Gottfried Neumann (1832–1925) :German mathematician and physicist, son of physicist Franz Ernst Neumann. He formulated the Neumann condition, which prescribes the value of the normal derivative of the function on the boundary. This condition is often used together with the Dirichlet condition to formulate mixed boundary-value problems (Dirichlet–Neumann type).

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