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Theme

On the class of m-isometric operators

Submitted by: Ben aoun Rania
Slimani Safa

Committee in jury

Mansour Abdelouahab	PROF	President	Univ. El Oued
Messaoud Guesba	MCA	Supervisor	Univ. El Oued
Ben Ali Brahim	MCA	Examiner	Univ. El Oued

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Contents

List of Notations

H, H_1, H_2	:	Hilbert Spaces.
$\mathbb{C}$	:	Space of complex mumbers.
l_2	:	Space of infinte square summable sequences.
$C(\Omega)$	:	Space of complex valued functions that are continuous on Ω.
$\mathcal{L}(H)$	:	Banach algebra of bounded linear operator on H.
$R(T)$	:	Range of T.
$N(T)$	:	Null space of T.
$M \oplus N$	:	Direct sum of M and N.
$\overline{\mathbb{D}}$	=	$\{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$.
$\partial\mathbb{D}$	=	$\{z \in \mathbb{C} : \lvert z\rvert = 1\}$.
$M^{\perp}$	:	Orthogonal complement of M.
T^*	:	Adjoint of T.
$\lvert T\rvert$	=	$(T^*T)^{\frac{1}{2}}$.
$\lVert \cdot \rVert$	:	Norm.
$\langle , \rangle$	:	Inner product.
$\sigma(T)$	:	Spectrum of T.
$\sigma_p(T)$	:	Point Spectrum of T.
$\sigma_c(T)$	:	Continuous Spectrum.
$\sigma_r(T)$	:	Residual Spectrum.
$\sigma_{cp}(T)$	:	Compression Spectrum.
σ_{ap}	:	Approximate Spectrum.
$\Pi(T)$	:	Set of eigenvalues of finite multiplicity.
$\Pi_{00}(T) = \phi$	:	the set of eigen values with finite multiplicity.

INTRODUCTION

Spectral theory is one of the branches of functional analysis, which can be described as trying to "classify" linear operators.

In order to unterstand its importance, we shall give a brief history of fuctional analysis.

Functional analysis is the branch of mathematics where vector spaces and operators (functions) on them are in focus. In linear algebra, the focus is on finite dimensional vector spaces over any field of scalars and the functions are viewed as matrices with scalar entries, but in functional analysis the vector spaces are infinite dimensional and not all operators can be represented by matrices.

Functional analysis has its origin in the theory of ordinary and partial differontial equations which was nsed to solve several phiysical problems, which included the work of Joseph Fourier $(1768 - 1830)$ on the theory of heat in which he wrote differential equations as integral equations. His work triggered not only the development of trigonometric series, which required mathematicians to consider what is a function and the meaning of convergence, this concieved lebesgue Integral which could accomodate broader functions compared to the classical Reimavvian Integral. It also gave birth to the spectral theory which is a central concept of functionnal analysis.

Spectral theory can be described as trying to "classify" all linear operators which was motivated by the need to solve the linear equations $T(v) = w$ between Hilbert spaces. It was introduced by David Hilbert during his initial formulation of Hilbert spaces theory. The restriction to a Hilbert space occurs since Hilbert space are distinguished among Banach spaces by being closely linked to plane Euclidean geometry which is the correct description of our universe at many scales.

This research contains three chapters as follows:

In Chapter 1 We shall give basic definitions and concepts in operator theory, especially properties of bounded linear operators and some classes of Hilbert space operators.

In Chapter 2 We shall give a definition of m-isometric operator on Hilbert spaces and study some properties of this class of operators and we give a new definition of the notion of m-invertibility and explore some basic properties.

In Chapter 3 we shall give describe the spectral picture of m-isometry and we shall look at the spectrum of T and its partitions, the numerical range, maximal generalized invers and the Weyl spectrum of m-isometries and 2-isometries.

Chapter 1

OPERATORS IN TOPOLOGICAL VECTOR SPACES

1.1 Banach Spaces

Définition 1.1.1

Let X be a vector space over $\mathbb{C}$. A norm $\|\cdot\|$ on X is a real valued function on X which has the following properties.

(i) $\|x\| \geq 0, \|x\| = 0$ implies $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\|x\|, \; \forall \alpha \in \mathbb{C}$.

(iii) $\|x + y\| \leq \|x\| + \|y\|,$ (triangle inequality).

The space X, together with $\|\cdot\|$, is called a normed linear space.

Example 1.1.1

Let $C([a, b])$ denote the set of continuous complex valued functions defined on the interval $a \leq x \leq b$. With the usual definitions of addition and scalar multiplication of functions, $C([a, b])$ is a vector space over $\mathbb{C}$ and

$$\|f\| = \max_{t \in [a,b]} |f(t)|,$$

defines a norm on this space.

1.2 Hilbert Spaces

Définition 1.2.1

Let X be a vector space over $\mathbb{C}$. An inner product on X is a complex valued vectors, $\langle , \rangle$, defined on $X \times X$ with the following properties hold:

(i) $\forall x \in X, \quad \langle x, x \rangle \geq 0.$

(ii) $\forall x \in X, \quad \langle x, x \rangle = 0 \Leftrightarrow x = 0.$

(iii) $\forall x, y \in X, \quad \overline{\langle x, y \rangle} = \langle y, x \rangle.$

(iv) $\forall x, y, z \in X, \quad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$

(v) $\forall x, y \in X, \quad \forall \lambda \in \mathbb{C}, \quad \langle \lambda x, y \rangle = \lambda \langle x, y \rangle.$

X with an inner product, is $\langle \ , \ \rangle$ called an inner product space.

Example 1.2.1

l_2 is a Hilbert space with inner product defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}, \tag{1.1}$$

where $\quad x = (x_i, x_2, \ldots x_n, \ldots) \in l_2 \quad$ and $\quad y = (y_1, y_2, \ldots, y_n, \ldots) \in l_2.$
Since $\quad x, y \in l_2, \quad \sum_{i=1}^{\infty} |x_i|^2 < \infty \quad$ and $\quad \sum_{i=1}^{\infty} |y_i|^2 < \infty.$
We have

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{1/2} < \infty. \tag{1.2}$$

From (1.1) we obtain the norm defined by

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

Using the metric induced by the norm, for l_2,

Theoreme 1.2.1 *[8]*

For any element $x, y \in H$ the followning properties hold:

(i) $|\langle x, y\rangle| \leq \|x\|\|y\|$, (Schwartz Inequality).

(ii) $\|x+y\| \leq \|x\|+\|y\|$, (Triangle inequality).

(iii) $\|x+y\|^2+\|x-y\|^2=2\left(\|x\|^2+\|y\|^2\right)$, (Parallelogram law).

1.3 Norm spaces

Définition 1.3.1

A norm on a $\mathbb{K}$ -espace voctoriel E is an application:

$$\|\ \| : \begin{array}{l} E \longrightarrow \mathbb{R}_+ \\ x \longmapsto \|x\| \end{array}$$

having the following properties:

(i) for all $x \in E$ no zero, we have $\|x\| \neq 0$

(ii) for all $x \in E$ and all $\lambda \in \mathbb{K}$, on a $\|\lambda x\| = |\lambda|\|x\|$;

(iii) for all $x, y \in E$, we $|x+y\| \leq \|x\|+\|y\|$ (convexity inequality).

Space E, provided with the norm $\|\ \|$, is said to be normed space or normed vector space (or K-normed vector space, if we want to specify the K). We often note such a space ($E, \|\ \|$). We deducted from in property (iii) that we have $\|0\| = 0$.

Example 1.3.1

Let $E = M_n(\mathbb{R})$, if A is in E, we put $A = (a_{ij})_{1\leq i,j\leq n}$ so

$$\|\ A\ \|_\infty = \sup_{1\leq i,j\leq n} |a_{ij}|,$$

Sets a norm on E.

1.4 Linear Spaces

Définition 1.4.1

Let E be a set of elements of a certain nature satisfying the following axioms:

(i) E is an additive abelian group.

This means that if x and $y \in E$ then their sum $x + y$ also belongs to the same set E, where the operation of addition satisfies the following axioms:

(a) $x + y = y + x$ (commutativity);

(b) $x + (y + z) = (x + y) + z$ (associativity)

(c) There exists a uniquely defined element θ, such that $x + \theta = x$ for any x in E

(d) For every element $x \in E$ there exists a unique element $(-x)$ of the same space, such that $x + (-x) = \theta$

(e) The element θ is said to be the null element or zero element of E and the element $-x$ is called the inverse element of x

(ii) A scalar multiplication is said to be defined if for every $x \in E$, for any scalar λ(real or complex) the element $\lambda x \in E$ and the following conditions are satisfied:

(a) $\lambda(\mu x) = \lambda\mu x$ (associativity)

(b) $$\left.\begin{aligned} \lambda(x + y) &= \lambda x + \lambda y \\ (\lambda + \mu)x &= \lambda x + \mu x \end{aligned}\right\} \quad \text{(distributivity)}$$

(c) $1 \cdot x = x$

The set E satisfying the axioms (i) and (ii) is called a linear or vector space.

Example 1.4.1

Real line $\mathbb{R}$: The set of all real numbers for which the ordinary additions and multiplications are taken as linear operations, is a real linear space $\mathbb{R}$.

1.5 Properties of bounded linear operators

Throughout this section H,H_1,H_2 denote Hilbert spaces over $\mathbb{C}$ and $\mathcal{L}(H)$ denotes the Banach algebra of bounded operators on H.

Définition 1.5.1

An application T which maps H_1 into H_2 is called a linear operator if for all $x, y \in H_1$ and λ a complex number

(i) it is additive, that is,

$$T(x+y) = Tx + Ty$$

(ii) it is homogeneous, that is,

$$T(\lambda x) = \lambda Tx$$

Example 1.5.1

1. The zero mapping $\mathcal{O} : \mathrm{X} \to \mathrm{Y}$ defined as

   $$\mathcal{O}(\mathrm{x}) = 0_y \quad \text{for all } \mathrm{x} \in \mathrm{X}$$

   Hence $\mathcal{O}$ is a linear operator on Y.

2. The identity mapping $\mathcal{I} : \mathrm{X} \to \mathrm{X}$ defined as

   $$\mathcal{I}(\mathrm{x}) = \mathrm{x} \quad \text{for all } \mathrm{x} \in \mathrm{X}$$

   Hence $\mathcal{I}$ is a linear operator on X.

Définition 1.5.2 *[8]*

The linear operator $T : H_1 \to H_2$ is called bounded if;

$$\sup_{\|x\| \le 1} \|Tx\| < \infty$$

The norm of T, written $\|T\|$ is given by;

$$\|T\| = \sup_{\|x\| \le 1} \|Tx\|$$

Thus an operator is a bounded linear transformation of a non-zero complex Hilbert space into itself.

Example 1.5.2

Let $S_r : l_2 \to l_2$ be defined by

$$S_r(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots,)$$

S_r is called the forward shift operator. S_r is linear and $\|S_r x\| = \|x\|, \quad x \in l_2$.
In particular , $\|S_r\| = 1$.

1.6 Adjoint operators

Définition 1.6.1

Suppose $T \in \mathcal{L}(H_1, H_2)$. For each $y \in H_2$, the functional $f_y(x) = \langle Tx, y\rangle$ is a bounded linear functional. Hence the Riesz representation theorem guarantees the existence of a unique $y^* \in H_1$ such that for all $x \in H_1$

$$\langle Tx, y\rangle = f_y(x) = \langle x, y^*\rangle$$

This gives rise to an operator $T^* : H_2 \to H_1$ defined by $T^* y = y^*$. Thus for all $x \in H_1$

$$\langle Tx, y\rangle = \langle x, y^*\rangle = \langle x, T^* y\rangle$$

Example 1.6.1

1. We have $\langle Ix, y\rangle = \langle x, I^* y\rangle$, or $\langle Ix, y\rangle = \langle x, y\rangle$, for all $x, y \in H$ so

$$I^* = I$$

2. Let

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x, 0)$$

Let $Y = (y_1, y_2)$ and $X = (x_1, x_2)$

$$\begin{aligned}\langle Tx, y\rangle &= \langle (x_1, 0), (y_1, y_2)\rangle \\ &= x_1 y_1.\end{aligned} \tag{1.3}$$

We have $\langle Tx, y\rangle = \langle T^*x, y\rangle$

Let $T^*y = Z = (z_1, z_2)$

$$\begin{aligned} \langle T^*x, y\rangle &= \langle (x_1, x_2), (z_1, z_2)\rangle \\ &= x_1z_1 + x_2z_2, \end{aligned} \tag{1.4}$$

by (1.3) and (1.4) we get $z_1 = y_1; z_2 = 0.$

So $T^*y = (y_1, 0)$

T^* is defined by: $T^*(x, y) = (x, 0)$.

3. One considered operator shift $S : \ell^2(\mathbb{C}) \longrightarrow \ell^2(\mathbb{C})$ defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$

   Let $(x_n)_n$ and $(y_n)_n$ in $\ell^2(\mathbb{C})$.so

$$\begin{aligned} \langle S^*x_n, y_n\rangle &= \langle x_n, Sy_n\rangle \\ &= \langle (x_1, x_2, \ldots), (0, y_1, y_2, \ldots)\rangle \\ &= x_2\bar{y}_1 + x_3\bar{y}_2 + \ldots \\ &= \langle (x_2, x_3, \ldots), (y_1, y_2, \ldots)\rangle \end{aligned}$$

   So S^* is defined by

$$S^*(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

4. Let $K \in \mathcal{L}(H_1, H_2)$ be the operator of finite rank defined by $Kx = \sum_{j=1}^{n} \langle x, u_j\rangle v_j, u_j \in H_1, v_j \in H_2$. Then for all $x \in H_1$ and $y \in H_2$

$$\langle Kx, y\rangle = \sum_{j=1}^{n} \langle x, u_j\rangle \langle v_j, y\rangle = \left\langle x, \sum_{j=1}^{n} \langle y, v_j\rangle u_j \right\rangle$$

   Hence

$$K^*y = \sum_{j=1}^{n} \langle y, v_j\rangle u_j.$$

Properties of Adjoint of on operator

Proposition 1.6.1

If T and S are in $\mathcal{L}(H_1, H_2)$, then

(i) $T^{**} = T$.

(ii) $\|T^*\| = \|T\|$.

(iii) $(T + S)^* = T^* + S^*$.

(iv) $(\alpha T)^* = \bar{\alpha} T^*, \alpha \in \mathbb{C}$.

(v) If D is in $\mathcal{L}(H_2, H_3)$, where H_3 is a Hilbert space, then $(DT)^* = T^* D^*$.

(vi) If T has an inverse T^{-1} then T^* has an inverse and $(T^*)^{-1} = (T^{-1})^*$.

Proof

(i)

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, (T^*)^*x\rangle} = \langle (T^*)^*x, y\rangle$$

Thus, by the definition of the adjoint of $T^*, T^{**}x = Tx$

Hence $\forall x \in H_1 : \quad T^{**} = T$.

(ii) $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|x\| \, \|T^*Tx\|$

Or, $\sup_{x\neq\theta} \frac{\|Tx\|}{\|x\|} \leq \sup_{Tx\neq\theta} \frac{\|T^*Tx\|}{\|Tx\|}$.

Hence, $\|T\| \leq \|T^*\|$

Similarly, considering $\|T^*x\|^2$ we can show that $\|T^*\| \leq \|T\|$

Hence, $\|T\| = \|T^*\|$, showing that T^* is bounded.

(iii) Given $x \in H_1$ and $y \in H_2$

$$\langle (T + S)x, y\rangle = \langle Tx, y\rangle + \langle Sx, y\rangle = \langle x, T^*y + S^*y\rangle$$

Therefore, by definition, $(T + S)^*y = T^*y + S^*y$.

(iv) $\langle \alpha Tx, y \rangle = \alpha \langle x, T^* y \rangle = \langle x, \bar{\alpha} T^* y \rangle$. Hence $(\alpha T)^* y = \bar{\alpha} T^* y$

(v) For $v \in H_3$

$$\langle DTx, v \rangle = \langle Tx, D^* v \rangle = \langle x, T^* D^* v \rangle .$$

Hence $(DT)^* v = T^* D^* v$.

(vi) Let A mapping H into H have an inverse T^{-1}

$$\begin{aligned} \left\langle T^{-1}x, y \right\rangle &= \left\langle x, \left(T^{-1}\right)^* y \right\rangle = \overline{\langle y, T^{-1}x \rangle} = \overline{\langle AA^{-1}y, A^{-1}x \rangle} \\ &= \overline{\langle A^{-1}y, T^* T^{-1}x \rangle} = \overline{\langle y, \left(T^{-1}\right)^* T^* T^{-1}x \rangle} \\ &= \left\langle \left(T^{-1}\right)^* T^* T^{-1}x, y \right\rangle \end{aligned}$$

Thus,

$$\left(T^{-1}\right)^* T^* = I \tag{1.5}$$

Agin $\left\langle T^* \left(T^{-1}\right)^* x, y \right\rangle = \left\langle \left(T^{-1}\right)^* x, Ty \right\rangle = \langle x, T^{-1}Ay \rangle = \langle x, y \rangle$ Thus

$$T^* \left(T^{-1}\right)^* = I \tag{1.6}$$

Hence it follows from (1.3) and (1.4) that $(T^*)^{-1} = \left(T^{-1}\right)^*$.

Proposition 1.6.2

For all $x \in H_1$. Thus, by definition,

$$T^*(\alpha u + \beta v) = \alpha T^* u + \beta T^* v, \quad \forall \alpha, \beta \in \mathbb{C}.$$

Corollary 1.6.1

Let T be an operator. Then

(i) $\|T^*T\| = \|TT^*\| = \|T\|^2$

(ii) $T^*T = 0$ if and only if $T = 0$

Proof

(i) We have

$$
\begin{aligned}
\|T^*T\| &\leq \|T^*\|\,\|T\| = \|T\|^2 \\
\|T^*T\| &= \sup_{\|x\|\leq 1} \|T^*T(x)\| \\
&= \sup_{\|x\|\leq 1, \|y\|\leq 1} |\langle T^*A(x), y\rangle| \\
&\geq \sup_{\|x\|\leq 1} |\langle T^*T(x), x\rangle| \\
&= \sup_{\|x\|\leq 1} |T(x), T(x)| \\
&= \|T\|^2
\end{aligned}
$$

Therefore $\|T^*T\| = \|T\|^2$.

(ii) Obvious by (i).

1.7 Self-Adjoint operators

Définition 1.7.1 *[19]*

A bounded linear operator T is said to be self-adjoint if it is equal to its adjoint, i.e., $T = T^*$. Self-adjoint operators on a Hilbert space H are also called Hermitian.

• A linear (not necessarily bounded) operator T with domain $\mathcal{D}(T)$ dense in H is said to be symmetric, if for all $x, y \in \mathcal{D}(T)$ the equality

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

holds. If T is unbounded, it follows from that

$$y \in \mathcal{D}(T) \Longrightarrow y \in \mathcal{D}\left(T^*\right)$$

Hence $\mathcal{D}(T) \subseteq \mathcal{D}\left(T^*\right)$. In other words, $T \subseteq T^*$ or T^* is an extension of T for A bounded, $\mathcal{D}(T) = \mathcal{D}\left(T^*\right) = H$. For, $T = T^*$ and $\mathcal{D}(T)$ dense in H, T is called self-adjoint.

Example 1.7.1

1. Let the matrix T of $M_2 \in (\mathbb{R})$ such as

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   We notice $T = T^*$, so T is self-adjoint.

2. Consider the operator T defined on $L^2(\mathbb{R})$ by

$$(Tx)(t) = e^{-|t|}x(t)$$

   Is a self-adjoint bound operator, because

$$\langle Tx, y\rangle = \int_{-\infty}^{+\infty} e^{-|t|}x(t)\overline{y(t)}dt = \int_{-\infty}^{+\infty} x(t)\overline{e^{-|t|}y(t)}dt = \langle x, Ty\rangle$$

Remark 1.7.1

Given T and S a self-adjoint operators, then

(i) $\alpha T + \beta S$ is self-adjoint where $\alpha, \beta \in \mathbb{R}$.

(ii) TS is self-adjoint if T and S are respectively self-adjoint and $TS = ST$.

(iii) If T is any operator T^*T , TT^* and $(T + T^*)$ are self-adjoint .

Theoreme 1.7.1

An operator $T \in \mathcal{L}(H)$ is self adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in H$.

Proof

If $T = T^*$, then for $x \in H$

$$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

Thus $\langle Tx, x\rangle$ is real suppose $\langle Tu, u\rangle$ is real for all $u \in H$. Then for all $x, y \in H$ and $\lambda \in \mathbb{C}$

$$\langle T(x + \lambda y), (x + \lambda y)\rangle = \langle x + \lambda y, T(x + \lambda y)\rangle$$

which, together with the assumption, implies that

$$\bar{\lambda}(Tx, y) + \lambda\langle Ty, x\rangle = \bar{\lambda}\langle x, Ty\rangle + \lambda\langle y, Tx\rangle$$

Hence

$$\Im\lambda\langle Ty, x\rangle = \Im\lambda\langle y, Tx\rangle$$

Taking $\lambda = 1$ and $\lambda = i$, it follows that $\langle Ty, x\rangle = \langle y, Tx\rangle$. Thus $T = T^*$.

Remark 1.7.2

Let $T \in \mathcal{L}(H)$ a self-adjoint

$$\langle Tx, x\rangle = 0, \forall x \in H \Rightarrow T = 0.$$

1.8 Some classes of Hilbert space operators

Positive operator

Linear operator T on a Hilbert space H is said to be positive (which is noted $T \geqslant 0$) if

$$\forall x \in H, \quad \langle Tx, x\rangle \geq 0.$$

Example 1.8.1

1. The identity operator and the null operator are positive.

2. $$\begin{aligned} A : L^2([0,1]) &\longrightarrow L^2([0,1]) \\ f &\longrightarrow xf \\ \langle Af, f\rangle &= \langle xf(x), f(x)\rangle, \quad \forall f \in L^2([0,1]) \\ &= \int_0^1 xf(x)\overline{f(x)}dx \\ &= \int_0^1 x|f(x)|^2 dx \geqslant 0. \end{aligned}$$

Inverse operator

Définition 1.8.1

An operator $T \in \mathcal{L}(H_1, H_2)$ is called invertible is there exists an operator $T^{-1} \in \mathcal{L}(H_2, H_1)$. such that:

$$T^{-1}Tx = x \quad \text{for all} \quad x \in H_1$$
$$TT^{-1}y = y \quad \text{for all} \quad y \in H_2$$

The operalor T^{-1} is called the inverse of T.

Isometry operator

An operator $T \in \mathcal{L}(H)$ is isometric if $\|Tx - Ty\| = \|x - y\|$ for all $x, y \in H$; this is equivalent to the condition $\|Tx\| = \|x\|$ for all $x \in H$.

Theoreme 1.8.1 *[7]*

The following conditions on an operator T are equivalent:

(i) T is isometry

(ii) $T^*T = I$

(iii) $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$

Proof

(i) implies (ii): $\|Tx\| = \|x\|, \forall x \in H$.
Hence $(\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 = \|x\|^2 = \langle x, x\rangle = \langle Ix, x\rangle)$.

(ii) implies (iii): $\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle = \langle Ix, y\rangle = \langle x, y\rangle$.

(iii) implies (i): $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, x\rangle = \|x\|^2$.
If T is isometric, it is clear from the relation $\|Tx - Ty\| = \|x - y\|$ that T is injective.

If T is isometric, it is clear from the relation $\|Tx - Ty\| = \|x - y\|$ that T is injective.

Example 1.8.2

The shift operator S is an isometry, because

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$
$$S^*(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

We have

$$S^*S(x_1, x_2, \ldots) = S(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

So,

$$S^*S = I.$$

Unitary operators

Définition 1.8.2

An operator T on a Hilbert space H is said to be a unitary operator if T is an isometry operator from H onto H.

Theoreme 1.8.2

An operator T on a Hilbert space H is a unitary operator if and only if

$$T^*T = TT^* = I$$

Hence a has an inverse and $T^{-1} = T^*$.

Proof

Since T is unitary if and only if T is an isometry operator from H onto H,
$T^*T = I$ and for any $x \in H$, there exists $y \in H$ such that $Ty = x$, and $T^*x = T^*Ty = y$ so that

$$\|T^*x\| = \|y\| = \|Ty\| = \|x\|$$

Thus, T^* is isometry and $TT^* = (T^*)^* T^* = I$

Conversely, if $T^*T = TT^* = I$, then T is isometry and for any $x \in H, x = TT^*x \in R(T)$, where $R(T)$ means the range of T, and so U is an isometry operator from H onto H

Example 1.8.3

Let $T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then $T^* = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Therefore, $TT^* = T^*T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus T is unitary.

Lemma 1.8.1

Let $T \in \mathcal{L}(H)$ be unitary then;

(i) T has an inverse T^{-1} which is unitary.

(ii) The adjoint operator coincides with its inverse..

Example 1.8.4

1. Let $a(t)$ be a Lebesgue measurable function on $[a, b]$ such that $|a(t)| = 1$ almost everyw-here. The operator T defined on $L_2[a, b]$ by $(Tf)(t) = a(t)f(t)$ is unitary.

2. Let $H = L^2[0, 1]$ define an operator T on H by:

$$Tf(t) = f(1-t), t \in [0, 1]$$

We calculate the operator T

$$\langle Tf(t), g(t)\rangle = \langle f(1-t), g(t)\rangle = \int_0^1 f(1-t)\overline{g(t)}dt$$

We take $y = 1 - t \Longrightarrow dy = -dt$, so

$$\int_0^1 f(y)\overline{g(1-y)} = \langle f(y), T^*g(y)\rangle$$

Then

$$T^*g(y) = g(1-y)$$

That is to say,

$$T^*f(t) = f(1-t)$$

Now we can easily verify that T is a unit operator, such that

$$TT^*f(t) = Tf(1-t) = f(t)$$
$$T^*Tf(t) = T^*f(1-t) = f(t)$$

So

$$TT^* = T^*T = I.$$

Normal operators

Définition 1.8.3

An operator $T \in \mathcal{L}(H)$ is said to be normal if $T^*T = TT^*$.

Remark 1.8.1

Every unitary operator is normal; so is every self-adjoint operator.
An isometric operator is normal if and only if it is unitary.

Theoreme 1.8.3 *[7]*

Let $T \in \mathcal{L}(H)$. The following conditions on an operator T are equivalent:

(i) T is normal

(ii) T^* is normal

(iii) $\|T^*x\| = \|Tx\|$ for all $x \in H$

Proof

The equivalence of (i) and (ii) results from $T^{**} = T$.

(i) implies (iii): $\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$.

(ii) implies (i): $\langle T^*Tx, x\rangle = \|Tx\|^2 = \|T^*x\|^2 = \langle TT^*x, x\rangle$ for all x.

Chapter 2

m-ISOMETRIC OPERATORS ON HILBERT SPACES

In this chapter, we present two parts in the first part, we introduce the class of m-isometric operators in Hilbert spaces and we develop some basic properties of this class. In the second part, we give a new definition of the notion of m-invertibility and explore some basic properties.

2.1 m-Isometric Operators

Définition 2.1.1

An operator $T \in \mathcal{L}(H)$ is an $m-i$sometry if it satisfies.

$$\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix} T^{*k}T^{k} = 0, \tag{2.1}$$

for some positive integer $m > 0$.

The stndy of m -isometries originated from the study of bounded linear transformations T on a Hilbert space which satisfy

$$\sum_{k=0}^{m}(-1)^{k}\begin{pmatrix} m \\ k \end{pmatrix} T^{*m-k}T^{k} = 0 \tag{2.2}$$

for some positive integer $m > 0, T$ is said to be m-symmetric.

Angler et.al in [1] studied the properties of m-isometries and some of the basic properties included isometry is an $(m+1)-$ isometry, that m-isometries are bounded below and that their spectrum lies in the closed unit disc.

Example 2.1.1

The identity operator is an m-isometry for every $m \in N$.
We have:

$$\sum_{K=0}^{m}(-1)^k \begin{pmatrix} m \\ K \end{pmatrix} (I^*)^{m-k} I^{m-k} = \sum_{K=0}^{m}(-1)^k \begin{pmatrix} m \\ K \end{pmatrix} I = 0.$$

Example 2.1.2

Let $H = \mathbb{C}^2$ be equipped with the norm $||(x,y)\|^2 = |x|^2 + |y|^2$, and consider the operator

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(H), \tag{2.3}$$

then by direct computation, we see that

$$(T^*)^3 T^3 - 3(T^*)^2 T^2 + 3(T^*)T - I = 0$$

$$T^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ (T^*)^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},\ (T^*)^2 T^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},\ (T^*)^3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},\ (T^*)^3 T^3 = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}, (T^*)T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

So,

$$\begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix} - 3 \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} + 3 \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, T is a 3-isemetry but not a 1 -isometry.

Remark 2.1.1

(i) 1-isometry is an isometry.

(ii) Every isometry is an m-isometry.

Proposition 2.1.1

Every isometry is a 2-isometry.

Proof

Suppose $T \in \mathcal{L}(H)$ is an isometry then, $T^*T = I$

Therefore

$$(T^*)^2T^2 - 2T^*T + I$$

$$= T^*T^*TT - 2I + I$$

$$= T^*IT - I = 0$$

It follows from (2.1) that T is a 2-isometry.

Remark 2.1.2

As a result we have the inclusion

unitary $\subset$ isometry $\subset$ 2 - isometry

Therefore if both T and T^* are 2-isometries then T is invertible arid so must be unitary. In particular if T is an invertible 2-isometry, then T is an isometry. In general an $(m+1)$-isometry is m-isometry.

Remark 2.1.3

Let $T \in \mathcal{L}(H)$, then

$$\|T^m(x)\|^2 - \binom{m}{1}\left\|T^{m-1}(x)\right\|^2 + \ldots + (-1)^m\|x\|^2 = 0, \forall x \in H$$

Lemma 2.1.1

Let $T \in \mathcal{L}(H)$ be an m-isometry. Then

(i) T is bounded below.

(ii)

$$||T|| \geq \gamma(T) \geq \frac{1}{\sqrt{m}\,(1+||T||^2)^{\frac{m-1}{2}}} \tag{2.4}$$

Proof

(i) Let $x \in H$, since T is in $e_m(H)$, it follows that

$$\begin{aligned} ||x||^2 &\leq \left(\|T^{m-1}\|^2 + \begin{pmatrix} m \\ j \end{pmatrix} \|T^{m-2}\|^2 + ... + (\begin{matrix} m \\ m-1 \end{matrix}) \right) \|T(x)\|^2 \\ &= C(m,T)\|T(x)\|^2 \end{aligned} \tag{2.5}$$

(ii) Let $x \in H$. Since T is in $e_m(H)$, it follows that

$$\begin{aligned} \|x\|_2 &\leq \|T^m(x)\|^2 + \begin{pmatrix} m \\ 1 \end{pmatrix} \|T^{m-1}(x)\|^2 + ... + \begin{pmatrix} m \\ m-1 \end{pmatrix} \|T^m(x)\|^2 \\ &\leq \left\|T^{m-1}\right\|^2 \|T(x)\|^2 + \begin{pmatrix} m \\ 1 \end{pmatrix} \left\|T^{m-2}\right\|^2 \|T(x)\|^2 + ... + \begin{pmatrix} m \\ m-1 \end{pmatrix} \|T(x)\|^2 \\ &\leq \left(\left(\|T\|^2\right)^{m-1} + \begin{pmatrix} m \\ 1 \end{pmatrix} \left(\|T\|^2\right)^{m-2} + ... + \begin{pmatrix} m \\ 1 \end{pmatrix} \right) \|T(x)\|^2 \\ &\leq m \sum_{j=0}^{m-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \left(\|T\|^2\right)^{m-1-j} \|T(x)\|^2 \\ &\leq m \left(1 + \| T \|^2\right)^{m-1} \| T(x) \|^2 . \end{aligned}$$

We deduce that

$$d^2(x, N(T)) = \| x \|^2 \leq m \left(1 + \|T \|^2\right)^{m-1} \|T(x)\|^2 \tag{2.6}$$

Consequently

$$\gamma(T) \geq \frac{1}{\sqrt{m}\,(1+|T|^2)^{\frac{m-1}{2}}} \tag{2.7}$$

which is the decired result.

Lemma 2.1.2 *[19]*

Let $T \in \mathcal{L}(H)$ be a 2 -isometry, then

(i)

$$||T(x)||^2 \geq \frac{n-1}{n}\|x\|^2,\ n \geq 1,\ x \in H \tag{2.8}$$

(ii)

$$\|T(x)\| \geq \|x\|,\ x \in H \tag{2.9}$$

(iii)

$$\|T^n(x)\|^2 + (n-1)\|x\|^2 = n\|T(x)\|^2,\ x \in H,\ n = 0, 1, 2, \dots \tag{2.10}$$

(iv)

$$\lim_{n\to+\infty} \|T^n(x)\||^{\frac{1}{n}} = 1,\ \text{for } x \in H,\ x \neq 0 \tag{2.11}$$

Theoreme 2.1.1 *[16]*

A power bounded 2-isometry is an isometry.

Proof

Let T be n power bounded 2-isometry. Then there exists constant $C > 0$ such that

$$\|T^n\| \leq C, \quad n = 1, 2, \tag{2.12}$$

From Lemma $1.1.2.$ (iii) we obtain

$$n \sup_{\|x\|=1} \|T(x)\|^2 \leq \sup_{\||x\||=1} \|T^n(x)\|^2 + (n-1) \tag{2.13}$$

so

$$\|T\|^2 \leq \frac{C}{n} + \frac{n-1}{n} \tag{2.14}$$

We find

$$\|T\| \leq 1 \tag{2.15}$$

and therefore we conclude that

$$\|x\| \leq \|T(x)\| \leq \|x\|.$$

Theoreme 2.1.2

A power of a 2-isometry is again a 2-isometry.

Proof

Let T be a 2 -isometry. We prove the assertion by using the mathematical induction. since T is a 2 -isometry, the result is true for $n=1$. Now assume that the result is true for $n=k$,

$$T^{*2k}T^{2k} - 2T^{*k}T^{k} + I = 0 \tag{2.16}$$

then

$$\begin{aligned}
& T^{*2(k+1)}T^{2(k+1)} - 2T^{*k+1}T^{k+1} + I \\
= & T^{*2}\left(T^{*2k}T^{2k}\right)T^{2} - 2T^{*k+1}T^{k+1} + I \\
= & T^{*2}\left(2T^{*k}T^{k} - I\right)T^{2} - 2T^{*k+1}T^{k+1} + I \quad \text{(by (1.16))} \\
= & 2T^{*k+2}T^{k+2} - T^{*2}T^{2} - 2T^{*k+1}T^{k+1} + I \\
= & 2T^{*k}\left(T^{*2}T^{2} - T^{*}T\right)T^{k} - T^{*2}T^{2} + I \\
= & 2T^{*k}\left(T^{*}T - I\right)T^{k} - T^{*2}T^{2} + I \quad (T \text{ is a } 2-isometry) \\
= & \left(2T^{*k+1}T^{k+1} - 2T^{*k}T^{k}\right) - T^{*2}T^{2} + I \\
= & 2\left(T^{*2}T^{2} - T^{*}T\right) - T^{*2}T^{2} + I \quad \text{(by (1.16))} \\
= & T^{*2}T^{2} - 2T^{*}T + I \\
= & 0.
\end{aligned}$$

This shows that the result is true for $n=k+1$: thus T^{n} is a 2 -isometry for each n

It is well known and obvious that a unilateral weighted shift is an isometry iff all its weights lie on the unit circle. In the next result, we obtain a necessary and sufficient condition under which a non-isometric unilateral weighted shift is a 2 -isometry.

Proposition 2.1.2 *[19]*

Let $T \in \mathcal{L}(H)$. If T is an m-isometry, then

$$\gamma\left(T^{*}T\right) > 0.$$

Proof

Let $x \in H$, by using the inequality

$$\|T^*T(x)\|\|x\| \geq \langle T^*T(x), X\rangle = \|T(x)\|^2 \geq \gamma(T)^2\|x\|^2, \tag{2.17}$$

we deduce that

$$\|T^*T(x)\| \geq \gamma(T)^2\| x\| \geq \gamma(T)^2 d\left(x, N\left(T^*T\right)\right). \tag{2.18}$$

Hence,

$$\gamma\left(T^*T\right) \geq \gamma(T)^2 > 0. \tag{2.19}$$

Lemma 2.1.3

Let $T \in \mathcal{L}(H)$, where X is a Hilbert space. If T is a 2-isometry, then

$$\left(T^*\right)^{k+1} T^{k+1} = \left(T^*\right)^k T^k - T^*T - I, \text{ for all } k = 0, 1, 2, ... \tag{2.20}$$

In particular, the sequence of operators

$$\left(\left(T^*\right)^{k+1} T^{k+1} - \left(T^*\right)^k T^k\right)_{\mathrm{k}\geq 0} \tag{2.21}$$

is positive.

Proof

We prove the assertion by induction. Since T is a 2 -isometry, the result is true for $k = 0$ and $k = 1$. Now assume that the result in true for k, i.e.

$$\left(T^*\right)^k T^k - \left(T^*\right)^{k-1} T^{k-1} = T^*T - I. \tag{2.22}$$

Then,

$$T^*\left(\left(T^*\right)^k T^* - \left(T^*\right)^{k-1} T^{k-1}\right) T = T^*\left(T^*T - I\right) T = T^*T - I. \tag{2.23}$$

Hence, the reault.

On the other hand, for $x \in H$ we have

$$\left\langle\left(\left(T^*\right)^{k+1} T^{k+1} - \left(T^*\right)^k T^k\right)(x), x\right\rangle = \|Tx\|^2 - \|x\|^2 \geq 0. \tag{2.24}$$

Définition 2.1.2

Let $T \in \mathcal{L}(H)$. T is said to be:

(i) power bounded if there exists some constant $C > 0$ such that

$$\|T^n\| \leq C \text{ for all } n \in \mathbb{N}. \tag{2.25}$$

(ii) a spectraloid operator if

$$r(T) = |W(T)|. \tag{2.26}$$

The next result in the case of Hilbert spacen in shown in [theorem 2.1.1]

2.2 m-Invertibility

Définition 2.2.1

For $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(H)$ is called:

(i) m-left invertible if there exists an operator $S \in \mathcal{L}(H)$ for which

$$S^m T^m - \begin{pmatrix} m \\ 1 \end{pmatrix} S^{m-1} T^{m-1} + \begin{pmatrix} m \\ 2 \end{pmatrix} S^{m-2} T^{m-2} + \ldots + (-1)^m I = 0. \tag{2.27}$$

In this case S is called an m-left inverse of T.

(ii) m -right invertible if there exists an operator $S \in \mathcal{L}(H)$ for which

$$T^m S^m - \begin{pmatrix} m \\ 1 \end{pmatrix} T^{m-1} S^{m-1} + \begin{pmatrix} m \\ 2 \end{pmatrix} T^{m-2} S^{m-2} + \ldots + (-1)^m I = 0. \tag{2.28}$$

In this case S is called an m-right inverse of T.

Example 2.2.1

Let T be the bounded linear operator on the Banach space $X = l^2_{\mathbb{C}}$ given by

$$T\left(x_1, x_2, \ldots\right) = \left(0, x_1, x_2, \ldots\right). \tag{2.29}$$

If we consider the operator S an $l^2_{\mathbb{C}}$ given by

$$S\left(x_1, x_2, \ldots\right) = \left(x_2, x_3, \ldots\right), \tag{2.30}$$

then it is ensy to see that

$$\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} S^{m-j}T^{m-j} = 0. \tag{2.31}$$

Therefore T is m -left invertible.

Example 2.2.2

Let $H = \mathbb{C}^2$ be equipped with the norm $\|(x, y)\| = |x| + |y|$, and consider the operator

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(H). \tag{2.32}$$

Then,

$$\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} S^{m-j}T^{m-j} = 0 \tag{2.33}$$

with

$$S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(H). \tag{2.34}$$

Thus, T is m-left and m-right invertible for all m.

Remark 2.2.1

In genral, if S in an m-right (m-left) inverse of T, then S is not unique For example if we consider the operator $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ given in Example 2.2.2 , we can see that T satisfies

$$T^4 - 2T^2 + I = 0, \tag{2.35}$$

and

$$T^2 - 2T + I = 0, \tag{2.36}$$

then $S = T$ and $S = I$.

Remark 2.2.2

Every right invertible operator is m -right invertible and every isft invertible operator in m-left invertible.

Notational 2.2.1

For $T \in \mathcal{L}(H)$, we denote by

$$\Im_l^m(T) = \{s \in \mathcal{L}(H) : \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} S^{m-j} T^{m-j} = 0\}, \tag{2.37}$$

and

$$\Im_r^m(T) = \{s \in \mathcal{L}(H) : \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T^{m-j} S^{m-j} = 0\}. \tag{2.38}$$

Remark 2.2.3

In general,

$$\Im_l^m(T) \cap \Im_r^m(T) \neq \emptyset. \tag{2.39}$$

For example if we consider nilpotent operator T of order p $(T = T + I - I)$, then it is easy to see that

$$T + I \in \Im_l^m(T) \cap \Im_r^m(T), \text{ for } m \geq p. \tag{2.40}$$

Remark 2.2.4

Every m-isometric operator on Hilbert space is m-left invertible.

Remark 2.2.5

Let $T \in \mathcal{L}(H)$ and $S \in \mathcal{T}_l^m(T)$ its m-left inverse, we have

$$\|S\| \, \|T\|(1 + \|S\| \, \|T\|)^{m-1} \geq \frac{1}{m}. \tag{2.41}$$

Remark 2.2.6

Let $T \in \mathcal{L}(H)$ and $S \in \Im_l^2(T)$ its 2 -left inverse, we have

$$S(ST - I)T = ST - I, \tag{2.42}$$

and we deduce from this equality that

$$\|S\| \, \|T\| \geq 1. \tag{2.43}$$

Corollary 2.2.1 *[19]*

Let $T \in \mathcal{L}(H)$ have an m-left inverse $S \in \Im_l^m(T)$, then

$$\gamma(T) \geq \frac{1}{m\|S\|(1+\|S\|\|T\|)^{m-1}}. \tag{2.44}$$

$$\gamma(S) \geq \frac{1}{m\|T\|(1+\|S\|\|T\|)^{m-1}}. \tag{2.45}$$

Proof

Let $x \in H$, we have

$$\begin{aligned}
\|x\| &\leq \|S\| \left(\|S\|^{m-1}\|T\|^{m-1} + \begin{pmatrix} m \\ 1 \end{pmatrix} \|S\|^{m-2}\|T\|^{m-2} + \dots \begin{pmatrix} m \\ m-1 \end{pmatrix} \right) \|T(x)\| \\
&\leq m||S|| \left(\sum_{j=0}^{m-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \|S\|^{m-1-j}\|T\|^{m-1-j} \right) \|T(x)\| \\
&\leq m\|S\| \left(1+\|S\|\|T\|\right)^{m-1} \|T(x)\|.
\end{aligned} \tag{2.46}$$

It follows that

$$d(x, N(T)) = \|x|\| \leq m\|S\|(1+\|S|| \ \|T\|)^{m-1}\|T(x)\|, \tag{2.47}$$

consequently,

$$\gamma(T) \geq \frac{1}{m\|S\|(1+\|S\| \ \|T\|)^{m-1}}. \tag{2.48}$$

On the other hand let $f \in H$, we have

$$\begin{aligned}
\|f\| &\leq \left(\|T^*\|^{m-1}||S^*\|^{m-1} \begin{pmatrix} m \\ 1 \end{pmatrix} \|T^*\|^{m-2}\|S^*\|^{m-1} + \dots + \begin{pmatrix} m \\ m-1 \end{pmatrix} \right) \|S^*(f)\| \\
&\leq m\|T^*\| \left(\sum_{j=0}^{m-1} \|T^*\|^{m-1-j}\|S^*\|^{m-1-j} \right) \|S^*(f)\| \\
&\leq \|T^*\| \left(1+\|T^*\| \ \|S^*\|\right)^{m-1} \|S^*(f)\|.
\end{aligned} \tag{2.49}$$

By using the fact that

$$\gamma\left(S^*\right) = \gamma(S), \tag{2.50}$$

and

$$\|T^*\| = \|T\|, \tag{2.51}$$

we deduce that

$$\gamma(S) \geq \frac{1}{m\|T\| \,(1+\|S\|\,\|T\|)^{m-1}}. \tag{2.52}$$

Remark 2.2.7

We have the following inequnlity for every $T \in \mathcal{L}(H)$:

$$\gamma(T) \geq \sup\left\{\frac{1}{m\|S\|(1+\|S\|\,\|T\|)^{m-1}},\ S \in \Im_l^m(T)\right\}. \tag{2.53}$$

Remark 2.2.8

In general,the inequality in (2.53) may be strict, as can be shown by the following example.

Example 2.2.3

Let $H = \mathbb{C}^2$ with the norm $\|(x,y)\| = |x| + |y|$, and consider the operator $T \in \mathcal{L}(H)$ defined by

$$T(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right). \tag{2.54}$$

T in 2 -left invertible with

$$\gamma(T) = \|T\| = \frac{1}{2}, \tag{2.55}$$

and by using the inequalities (12.44) and (2.45), we deduce that for all $S \in \Im_l^2(T)$

$$\gamma(T) = \frac{1}{2} > \frac{1}{4} \geq \frac{1}{2\|S\|(1+\|S\|\,\|T\|)}, \tag{2.56}$$

and

$$\sup\left\{\frac{1}{2\|S\|(1+\|S\|\,\|T\|)}, S \in \Im_I^2(T)\right\} \leq \frac{1}{4} < \frac{1}{2}. \tag{2.57}$$

Chapter 3

SPECTRUM OF AN m-ISOMETRIC

3.1 Classification of the spectrum

Définition 3.1.1 *[8]*

A complex number λ is said to be a regular values of an operator T if the operator $T - \lambda I$ is an invertible.

The resolvent set denoted by $\rho(T)$ is the set of regular values of T .
The set of all those λ , which are not regular values of T is called the Spectrum of the operator T and is denoted by $\sigma(T)$.

Définition 3.1.2 *[8]*

The point spectrum denoted by $\sigma_p(T)$, is the set of all eigenvalues of T

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) \neq 0\}.$$

the continuous spectrum denoted by $\sigma_c(T)$ is defined as follows :
$\lambda \in \sigma_c(T)$ if $\lambda \in \sigma(T) / \sigma_p(T)$ and $R(T - \lambda I)$ is dense in H

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C} = (T - \lambda I)^{-1} \text{is unbounded and } \overline{R(T - \lambda I)} = H\right\}.$$

Example 3.1.1

On $(L_2[0,1])$ define $T : L_2[0,1] \to L_2[0,1]$ whith $Tx = t \cdot x(t)$, then

$$\sigma(T) = \sigma_c(T) = [0,1]$$

The residual spertrum is the set ; $\sigma_r(T) = \{\lambda \in C : (T - \lambda I)^{-1}\}$ exists and $R(T - \lambda I) \neq H$. From the definition it follows that;

$$\sigma_r = \sigma_p\left(T^*\right)^* \diagup \sigma_p T$$

Proposition 3.1.1 *$\sigma(T)$ is a closed set.*

Proposition 3.1.2 *[8]*

$\sigma\left(T\right) = \sigma_p\left(T\right) \cup \sigma_c\left(T\right) \cup \sigma_r\left(T\right)$ holds, where $\sigma_p\left(T\right), \sigma_c(T), \sigma_r\left(T\right)$ are mutually disjoint parts of $\sigma(T)$.

Définition 3.1.3

The compression spectrum $\sigma_{cp}(T) = \{\lambda \in \mathbb{C} : \overline{R\left(T - \lambda I\right)} \subset H\}$.
The set $\sigma_{ap}(T)$ of all complex numbers λ suth that there existe a sequence of unit vectors x_n suth that $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$ is said to be in approximat poit spectrum.

Proposition 3.1.3 *[5]*

$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{cp}(T)$ holds, where $\sigma_{ap}(T)$ and $\sigma_{cp}(T)$, are not necessarily disjoint parts of the spectrum. Also, $\sigma(T) = \sigma_r(T) \cup \sigma_{ap}(T)$ holds.

Définition 3.1.4 *[8]*

The Weyl spectrum denoted by $w(T) = \{\lambda \in \mathbb{C}, (T - \lambda I)^{-1}\}$ is not Weyl.

Lemma 3.1.1 *[11]*

The weyl theorem holds for 2-isometries.

Définition 3.1.5

Lel Ω be a non-empty set, the smallest convex set containing Ω denoted by conv(Ω) is known as the convex hull of Ω.

Définition 3.1.6

(i) The entire spectrum of self-adjoint operator T is con-fined between its bounds

$$M = \sup_{||x||=1} |\langle Tx, x\rangle| \quad \text{and} \quad m = \inf_{||x||=1} |\langle Tx, x\rangle|.$$

(ii) The bounds M and m of every self-adjoint operator T belong to its spretrum.

Proof

Supposes $\lambda \notin [m, M]$ and $\lambda < m$ since $\langle Tx, x\rangle > m$, we have $\langle((T - \lambda I)x, x)\rangle \geq m - \lambda$
But according to Shwartz inquality

$$|\langle(T - \lambda I)x, x\rangle| \leq \|(T - \lambda)x\|\|x\|$$
$$\Rightarrow \|(T - \lambda I)x\| \geq m - \lambda$$
$$\Rightarrow \lambda \notin \sigma(T)$$

similaly for $\lambda > M$
By definition $\|T\| = M \Rightarrow \exists : x_n \in H$ for which

$$\langle Tx_n, x_n\rangle \to M$$
$$\to Tx_n \to Mx_n$$

Therefore

$$0 \leq \|Tx_n - Mx_n\|^2 = \|(T - MI)x_n\|^2$$
$$= \langle(T - MI) \cdot x_n, (T - MI)x_n\rangle$$
$$= \|Tx_n\|^2 - 2M\langle Tx_n, x_n\rangle + M^2$$
$$\leq 2M^2 - 2M\langle Tx_n, x_n\rangle \to 0$$
$$\Rightarrow M \in \sigma(T)$$

Similarly for $m \in \sigma(T)$.

Theoreme 3.1.1

Let $T \in \mathcal{L}(H)$, then

$$\sigma\left(T^{*}T\right)=\left(\sigma\left(T^{*}T\right)|_{N(T)} \cup \left(\sigma\left(T^{*}T\right)|_{N(T)^{\perp}}\right).$$

Proof

From $H = N(T) \bigoplus N(T)^{\perp}$, relative to this decomposition and since both $N(T)$ and $N(T)^{\perp}$ are invariant under $T^{*}T$, we have

$$T^{*}T=\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix},$$

where $E_1 : N(T) \to N\left(T\right)$, and $E_2 : N(T)^{\perp} \to N(T)^{\perp}$

thus $\sigma\left(T^{*}T\right)=\left(\sigma\left(T^{*}T'\right)|_{N(T)} \cup \left(\sigma\left(T^{*}T\right)|_{N(T)^{\perp}}\right).$

Remark 3.1.1 *[11]*

Since $T^{*}T$ is a positive self adjoint operator

$$\|T^{*}T\|=\|T\|^{2}, \sigma\left(T^{*}T\right) \subseteq \left[0, \|T\|^{2}\right].$$

Définition 3.1.7 ***(The numerical range)***

For an operator T, the numerical range $W\left(T\right)$ of T is a subset of the complex plane, given by ;

$$W(T)=\{\langle Tx, x\rangle, x \in H; \|x\|=1\}.$$

The following properties of the numerical range are well known;

(i) $W(aT+bI)=aW(T)+b$

(ii) $W(B \bigoplus C)=\{\operatorname{conv} W(B) \cup W(C)\}$

(iii) $W(T)$ is a convex set (Hausdorff-Toeplitz).

(iv) $W(T)$ is bounded.

(v) $W(T)$ is closed if $\dim(H) < \infty$

Définition 3.1.8 *[8]*

The numerical radius $w(T)$ of T is defined by;

$$W(T) = \{\sup |\lambda| : \lambda \in W(T)\}$$

The crawford number $c(T)$ of T defined by ;

$$c(T) = \inf\{|\lambda| : \lambda \in W(T)\}$$

The spectral radius $r(T)$ of an operator T is defined by;

$$r(T) = \sup \{|\lambda| : \lambda \in \sigma(T)\}$$

which is smallest circle on the complex plane $\mathbb{C}$ which cantains the spectrum of T .

Définition 3.1.9

The essential numerical range $W_c(T)$ is defined as;

$$W_c(T) = \cap \overline{W(T + K)}$$

K compact.
Let $T = (T_1, T_2, \ldots T_n)$ be an n-luple of operators acting on H. The joint numerical range of T is defined as,

$$W_j(T) = (\langle T_1 x, x\rangle, \langle T_2 x, x\rangle, \ldots \langle T_n x; x; \rangle)$$

Proposition 3.1.4

Let $T \in \mathcal{L}(H)$ then $\sigma_p(T) \subseteq W(T)$.

Proof

Suppose

$$\lambda \in \sigma_p(T) \Rightarrow \exists x \neq 0 \in H : \lambda x = Tx$$

Therefore

$$\lambda = \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle$$

$$\Rightarrow \lambda \in W(T)$$

Therefore,

$$\sigma_p \subseteq W(T).$$

Corollary 3.1.1

$\sigma_p\,(T) \cup \sigma_r(T) \subseteq W\,(T)$

Proof

Suppose

$$\lambda \in \sigma_p(T) \Rightarrow \lambda \in W(T)$$

If $\lambda \in \sigma_r(T)$ then $\bar{\lambda} \in \sigma_p\,(T^*) \Rightarrow \lambda \in W\left(T^T\right)$, since $\sigma_p\,(T^*)^* \not\subset \sigma_p\,(T')$.

Hence $\quad \sigma_p(T) \cup \sigma_r(T) \subseteq W(T)$.

Proposition 3.1.5

Let $T \in \mathcal{L}(H)$ then $\sigma(T) \subseteq \overline{W(T)}$.

Proof

Recall :

$$\sigma(T) = \sigma_r(T) \cup \sigma_{ap}(T)$$

Suppose

$$\lambda \in \sigma_{ap}(T)$$

$$\Rightarrow 0 \leq |\lambda - \langle Tx_n, x_n\rangle|$$

$$= |\langle (T - \lambda I)x_n, x_n\rangle|$$

$$\leq \|(T - \lambda I) \cdot x_n\|\,\|x_n\| \to 0$$

as

$$n \to \infty$$

$$\Rightarrow \lambda \in \overline{W(T)}$$

Therefore,

$$\sigma_{ap}(T) \subseteq \overline{W(T)}$$

$$\Rightarrow \sigma(T) \subseteq \overline{W(T)}.$$

Theoreme 3.1.2 *[8]*

Let $T \in \mathcal{L}(H)$.Then

$$\text{conv}(\sigma(T)) \subseteq \overline{W(T)}.$$

This follows from the proposition then above.

Définition 3.1.10

An operator T $\in B(H)$ is said to be;

(i) convexoid if $\overline{W(T)} = \text{conv}(\sigma(T))$.

(ii) Normaloid if $r(T) = ||T||$.

(iii) spectraloid if $w\left(T\right) = r\left(T\right)$.

Définition 3.1.11

For $T \in \mathcal{L}(H)$, the minimum modulus is defined by the number

$$\gamma(T) = \inf\left\{||Tx||; ||x|| = 1, x \in N(T)^{\perp}\right\},$$

$$\gamma\left(T\right) = \infty \quad \text{if} \quad T = 0.$$

$\gamma\left(T\right) > 0$ implies injectivity of T, the converse does not hold true in general.

Définition 3.1.12

The Maximal Generalized Inverse of T , denoted by T^{+} , is a unique linear operator with domain $D\left(T^{+}\right) = R(T) \oplus^{\perp} R(T)^{\perp}$ and $N\left(T^{+}\right) = R\left(T\right)^{\perp}$ satisfying the following properties,

(i) $R(T) \subseteq D\left(T^{+}\right)$

(ii) $R\left(T^{+}\right) \subset D\left(T\right)$

(iii) $T^{+}Tx = P_{\overline{R(T^{+})}}x$, for all $x \in D(T)$

(iv) $TT^{+}y = P_{\overline{R(T)}}y$, for all $y \in D(T^{+})$

In general, an matrix T has many generalized inverses unless $m - n$ and T is invertible. It is possible to add conditions to the definition of a generalized invers so that there is always a unique generalizecl inverse:
T^{+} is called a Moore-Penrose inverse of T if it satisfies;

(i) $TT^{+}T = T$ and $T^{+}TT^{+} = T^{+}T$

(ii) $(TT^{+})^{*} = TT^{+}$ and $(T^{+}T)^{*} = T^{+}T$

Proposition 3.1.6

For $T \in \mathcal{L}(H)$, the following statements are equivalent.

(i) $\mathrm{R}(T)$ is closed .

(ii) $\mathrm{R}(\mathrm{T}^{*})$ is closed.

(iii) $\gamma(T) > 0$

(iv) T^{+} is bounded.

(v) $\gamma(T) = \gamma(T^{*})$

(vi) Let $\lambda \in (0, \infty)$. Then $\lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{+})$
If T^{-1} exists, then $0 \neq \lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{-1})$

Lemma 3.1.2 *[11]*

The spectrum of 2-isometry is the closed unit disc provided it is non-unitary.

Theoreme 3.1.3

Let T be a bounded linear operator on H Then the follwing results hold.

(i) $\sigma(T)/\{0\} = \sigma(T_0)/\{0\}$.

(ii) $\sigma(T^{+})/\{0\} = \sigma(T_0)^{-1}/\{0\}$.

Proof

(i) consider the operator T is self-adjoint, it is reducible by $N(T)$, that is

$$T(N(T)) \subseteq N(T) \quad \text{and} \quad T\left(N(T)^{\perp}\right) \subseteq N(T)^{\perp}$$

By lemma,

$$\sigma(T) = \sigma\left(T|_{N(T)}\right) \cup \sigma\left(T|_{N(T)^{\perp}}\right)$$

That is

$$\sigma(T) = \{0\} \cup \sigma\left(T_0\right)$$

Hence

$$\sigma(T)/\{0\} = \sigma\left(T_0\right)/\{0\}$$

(ii) since T^{+} is self-adjoint, it is reducible by $R(T)^{\perp}$

(i) implies that,

$$\sigma\left(T^{+}\right)/\{0\} = \sigma\left(T_0\right)^{-1}/\{0\}.$$

Proposition 3.1.7

Let $T \in \mathcal{L}(H)$ be a positive operator. Then the following results hold.

(i) T^{+} is positive.

(ii) $\sigma(T)/\{0\} = \sigma(T_0)/\{0\}$ where $T_0 = T|_{N(T)}$.

(iii) $\sigma(T^{+})/0 = \sigma(T_0^{-1})/0$.

Proof

(i) Let $y = Tv + u$ where $u \in N\left(T\right)^{\perp}$ and $v \in R(T)^{\perp}$

Since

$$D(T^{+}) = R\left(T\right)\left(\theta\right)^{\perp} R\left(T\right)^{\perp}$$

we have

$$\langle T^+y, y\rangle = \langle T^+y, Tu + v\rangle$$
$$= \langle T^+y, Tu\rangle + \langle T^+y, v\rangle$$
$$= \langle T^+y, Tu\rangle \left(\text{since } R(T)^\perp = N\left(T^+\right), \langle T^+y, v\rangle = 0\right)$$
$$= \left\langle P|_{\overline{R(T)}}\, y, u\right\rangle = \langle Tu, u\rangle \geq 0$$
$$\Rightarrow T^+ \text{is positive.}$$

(ii) Since T is self-adjoint, it is reducible by $N(T)$

i.e

$$T(N(T)) \subseteq N(T)$$

and

$$T\left(N(T)^\perp\right) \subseteq N(T)^\perp$$

by theorem $.2.2.9$ we have

$$\sigma(T) = \sigma(T)|_{N(T)\cup\sigma\left(T|_{N(T)^\perp}\right)}$$

i.e

$$\sigma(T) = \{(0\} \cup \sigma\left(T_0\right)$$

hence

$$\sigma(T)/\{0\} = \sigma\left(T_o\right)/\{0\}.$$

(iii) Since T^+ is self-adjoint,it is reducible by

$$N\left(T^+\right) = R(T)^\perp$$

since

$$T^+|_{R(T)} = T_0^{-1},$$

(ii) implies that

$$\sigma\left(T^+\right)/\{0\} = \sigma\left(T_0^{-1}\right)/\{0\}.$$

Theoreme 3.1.4

Let $T \in \mathcal{L}(H)$ be a positive operator and

$$d(T) = \inf\{|\lambda| : \lambda \in \sigma(T)/\{0\}\} = d\left(0, \sigma\left(T\right)/\{0\}\right)$$

Then $\gamma\left(T\right) = d\left(T\right).$

Proof

Case 1 : $\gamma\left(T\right) > 0$ if $\gamma\left(T\right) > 0$, then $R(T)$ is closed in this case T^{-1} and T^{+} are bounded self adjoint operators with

$$\|T_0^{-1}\| = \|T^{+}\| = \frac{1}{\gamma(T)}$$

hence:

$$\gamma(T) = \frac{1}{\|T^{+}\|}$$

$$= \left(\sup\left\{|\mu| : \mu \in \sigma\left(T_0^{-1}\right)\right\}\right)^{-1}$$

$$= \left(\sup\left\{(\lambda)^{-1} : 0 \neq \lambda \in \sigma\left(T_o\right)\right\}\right)^{-1}$$

$$= \inf\left\{|\lambda| : 0 \neq \lambda \in \sigma\left(T_o\right)\right\} = d\left(T\right)$$

Case 2: $\gamma(T) = 0$

Since T^{+} is positive,

$\gamma\left(T\right) = 0 \Rightarrow T^{+}$ is unbounded $\Rightarrow \sigma\left(T^{+}\right)$ is unbounded.

$\Rightarrow$ for all $n = 1, 2, 3, \ldots, \exists \lambda_n \in \sigma(T^{+})$

such that

$$\lambda_n > n \Rightarrow \frac{1}{\lambda} \in \sigma\left(T\right)$$

and

$$\frac{1}{\lambda_n} \to 0$$

as

$$n \to \infty$$

Hence

$$d(T) = 0$$

Theoreme 3.1.5

Suppose $T \in \mathcal{L}(H)$ is a positive operator and 0 is an isolated spertral valued of T. Then 0 is an eigenvalue.

Proof

Since 0 is an isolated spectral value, $d(T) > 0$ then by proposition 3.1.6

$$\gamma(T) > 0 \Rightarrow R(T)$$

is closed.

if

$$0 \in \sigma_p(T),$$

then

$$N(T) = \{0\} \Rightarrow R(T) = H$$

making T one to one and onto , hence invertible , a contradiction.

3.2 Spectral Properties of m-Isometries

Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \overline{\mathbb{D}} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\partial\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$

Theoreme 3.2.1 *[19]*

Let $T \in \mathcal{L}(X)$ be an m-isometry, then the approximate point spectrum of T lies in the unit circle. Thus, either $\sigma(T) \subset \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$.

Proof

Let $\lambda \in \sigma_{ap}(T)$ then there exists a sequence $(x_n)_n \in X$ with $\|x_n\| = 1$ such that

$$(T - I)(x_n) \to 0 \text{ if } n \longrightarrow \infty$$

Then

$$\left(T^{m-j} - \lambda^{m-j}\right)(x_n) \longrightarrow 0 \quad \text{for } j = 0, 1, \ldots, m$$

The hypothesis that T is an m -isometry yields that

$$0 = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(\left\|T^{m-j}(x_n)\right\|^2\right.$$

and when $n \longrightarrow +\infty$ we have

$$0=\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}\left(|\lambda|^{m-j}\right)^{2}=\left(1-|\lambda|^{2}\right)^{m}$$

and so $|\lambda|=1$. As

$$\partial\sigma(T)\subset\sigma_{ap}(T)\subset\partial\mathbb{D},\sigma(T)\subset\partial\mathbb{D}$$

or

$$\sigma(T)=\overline{\mathbb{D}}.$$

Remark 3.2.1

Let T be an m -isometry and let $\lambda\in\mathbb{D}$. since $\sigma_{le}\subset\sigma_{ap}(T)\subset\partial\mathbb{D}$ the operator $T-\lambda$ is semi-Frodholm. since $\sigma_{p}(T)\subset\partial\mathbb{D}$, the Fredholm index of $T-\lambda$, ind $(T-\lambda)$, is nonpositive.

Lemma 3.2.1 *[19]*

If $T\in\mathcal{L}(H)$ is an m -isometry, then the approximate point spectrum lies in the unit circle. Thus, either $\sigma(T)\subseteq\partial\mathbb{D}$ or $\sigma(T)=\mathbb{D}^{-}$. In particular, T is injective and $\operatorname{ran}(T)$ is closed.

Proof

If $\lambda\in$ C is in the approximate point spectrum of T, then there exist unit vectors $x_{\ell}\in\mathcal{H}$ such that $(T-\lambda)x_{\ell}\rightarrow 0$. By induction, for each integer $k\geq 1$, $\left(T^{k}-\lambda^{k}\right)x_{\ell}\rightarrow 0$ Thus

$$\begin{aligned}
0&=\left\langle\left(\sum_{k=0}^{m}(-1)^{\text{m-k}}\begin{pmatrix}m\\k\end{pmatrix}T^{*k}T^{k}\right)x_{\ell},x_{\ell}\right\rangle\\
&=\sum_{k=0}^{m}(-1)^{\mathrm{m}-k}\begin{pmatrix}m\\k\end{pmatrix}\left(T^{k}x_{\ell},T^{k}x_{\ell}\right)\\
&\rightarrow\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix}m\\k\end{pmatrix}|\lambda|^{2k}\\
&=\left(|\lambda|^{2}-1\right)^{m}
\end{aligned}$$

and so $|\lambda|=1$. since $\partial\sigma(T)\subseteq\sigma_{ap}(T)\subseteq\partial\mathbb{D},\sigma(T)\subseteq\partial D$ or $\sigma(T)=\mathbb{D}^{-}$. Since $0\neq\sigma_{ap}(\tau)$ T is injective and $\operatorname{ran}(T)$ is closed. This establishes Lemma 2.1.3

Before continuing, we make the following remarks concerning the spectral picture of an m -isometry T, Fix $\lambda \in \mathbb{D}$, since $\sigma_{\ell e}(T) \subseteq \sigma_{ap}(T) \subseteq \partial\mathbb{D}, T-\lambda$ is semj-Fredholm. since $\sigma_p(T) \subseteq \partial\mathbb{D}$, the Fredholm index of $T-\lambda, \operatorname{ind}(T-\lambda)$, Is nonpositive, Finally, since $\sigma_{\ell e}(T) \subseteq \partial\mathbb{D}$, the function from D into $\mathbb{Z} \cup \{\pm\infty\}$ given by $\lambda \mapsto \operatorname{ind}(T-\lambda)$ is constant.

Corollary 3.2.1

Let $T \in \mathcal{L}(\mathcal{H})$. If both T and T^* are m -isometries, then $\sigma(T) \subseteq \partial\mathbb{D}$.

Proof

We argue by contradiction. By Lemma 3.2.1 , if $\sigma(T) \nsubseteq \partial\mathbb{D}$, then $\sigma(T) = \mathbb{D}^-$. Since Lemma 3.2.1 also implies that $0 \notin \sigma_{ap}(T)$, we see that $(\operatorname{ran} T)^- \neq H$. Hence $0 \in \sigma_{ap}(T^*)$ contradicting Lemma 3.2.1 and the fact that T^* is an m -isometry. Thus, there are two cases: If T is an m -isometry ejther T is invertible and $\sigma(T) \subseteq \partial\mathbb{D}$ or T is not invertible and $\sigma(T) = \mathbb{D}^-$. In fact both of these cases cannot occur for all m.

Corollary 3.2.2

Let $T \in \mathcal{L}(X)$, where X is a Hilbert space. If T is an m-isometry, then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

Proof

Let λ and μ be distinct eigenvalues of T. Suppose $Tx = \lambda x$ and $Ty = \mu y$

Then

$$
\begin{aligned}
0 &= \sum_{j-0}^{m} (-1)^j \binom{m}{j} \left\|T^{m-j}(x+y)\right\|^2 \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left\|\lambda^{m-j}x + \mu^{m-j}y\right\|^2 \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left\|\left(|\lambda|^2\right)^{m-j}\right\| \|x\|^2 + 2Re\left(\sum_{j-0}^{m} (-1)^j \binom{m}{j} (\lambda\bar{\mu})^{m-j}\langle x, y\rangle\right) \\
&+ \sum_{j=0}^{m} (-1)^j \binom{m}{j} |\mu|^{m-j}\|y\|^2 \\
&= \left(1-|\lambda|^2\right)^m \|x\|^2 + 2Re\left(\left(1-\frac{\lambda}{\mu}\right)^m \langle x, y\rangle\right) + \left(1-|\mu|^2\right)^m \|y\|^2 \\
&= \operatorname{Re}\left(\left(1-\frac{\lambda}{\mu}\right)^m \langle x, y\rangle\right).
\end{aligned}
$$

Replacing y by iy we obtain

$$
Im\left(\left(1-\frac{\lambda}{\mu}\right)^m \langle x, y\rangle\right) = 0.
$$

Hence

$$\left(1-\frac{\lambda}{\mu}\right)^m \langle x, y\rangle = 0.$$

since $\lambda \neq \mu$, we deduce that

$$\langle x, y\rangle = 0.$$

which proves the assertion.

Theoreme 3.2.2 *[10]*

Let T be an m-isometry. Then the following assertions hold.

(i) If z is an approximate eigenvalue of T, then $z \in \mathbf{T}$.

(ii) If T is invartible, then $\sigma(T) \subset \mathbf{T}$.

(iii) If T is not invertible, then $\sigma(T) = \{z : |z| \leq 1\}$.

Theoreme 3.2.3

For an m-isometry T the following statements hold.

(i) if a is an eigenvalue of T, then $\bar{a}$ is an eigenvalue of T^*.

(ii) Eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

(iii) If a is an approximate eigenvalues of T, then $\bar{a}$ is an approximate eigenvalue of T^*.

(iv) if a, b are distinct approximate eigenvalues of T, and x_n, y_n sequences of unit vectors such that $(T-a)x_n \to 0$, and$(T-b)y_n \to 0$, then $(x_n, y_n) \to 0$.

Proof

(i) Let $Tx = ax(x \neq 0)$. Then it holds

$$0 = \left(\sum_{j=0}^{m}(-1)^j_m C_j T^{*m-j/T^{m-j}}\right) x = (aT^* - 1)^m x$$

since $|a| = 1$, we have $(T^* - \bar{a})^m x = 0$ and hence $\bar{a}$ is an eigenvalue of T^*.

(ii) Let a, b be distinct eigenvalues of T and x, y be corresponding eigenvectors. It holds

$$0 = \left\langle \left(\sum_{j=0}^{m} (-1)^j_m C_j T^{*(m-j)} T^{m-j} \right) x, y \right\rangle = (a\bar{b} - 1)^m \langle x, y \rangle$$

Since $a \neq b$ and $|b| = 1, a - b = (a\bar{b} - 1)b$ and $a\bar{b} - 1 \neq 0$, Hence , $\langle x, y \rangle = 0$

(iii) Let $\{x_n\}$ be a sequence of unit vectors such that $\lim(T - a)x_n = 0$. Then since

$$\left(\sum_{j=0}^{m} (-1)^j_m C_1 T^{(m-j)} T^{m-j} \right) x_n = 0$$
$$0 = \lim_{n\to\infty} \left(\sum_{j=0}^{m} (-1)^j_m C_j T^{(m-j) T^{m-j}} \right) x_n = \lim_{n\to\infty} (aT^* - 1)^m x_n$$

Therefore, since $|a| = 1$

$$\lim_{n\to\infty} d^m (T^* - \bar{a})^m x_n = 0$$

and $\bar{a}$ is an approximate eigenvalue of T^* .

(iv) Since

$$0 = \lim_{n\to\infty} \left(\left[\sum_{j=0}^{m} (-1)^j_m C_j T^{(m-j) T^{m-j}} \right) x_n y_n \right) = (a\bar{b} - 1)^m \lim_{n\to\infty} (x_n, y_n)\,,$$

we have $\langle x_n, y_n \rangle \to 0$. An operator T is said to have the single valued extension property if, for every open subset $\mathcal{U}$ of $\mathbf{C}$, an analytic function $f : \mathcal{U} \to \mathcal{H}$ satisfies $(T - \lambda) f(\lambda) = 0 \quad \forall \lambda \in \mathcal{U}$, then $f(\lambda) = 0 \quad \forall \lambda \in \mathcal{U}$. Uchiyama and Tanahashi [7] studied the spectral condition (iv) in theorem 3.2.2 and proved that if T has the spectral condition (iv), then T has the single valued extension property. Hence we have the following result.

Corollary 3.2.3

Let $T \in \mathcal{L}(X)$ be an m -isometry, then $r(T) = 1$.

Proof

From Theorem 3.2.1. the approximate point spectrum of T is contained in the boundary of the unit disc and since from [8] we know that the convex envelopes of all spectra coincide. Thus, again it follows that the spectral radius of T .

Lemma 3.2.2 *[19]*

Let $T \in \mathcal{L}(X)$. If T possesses an m -left inverse $S \in \Im_{\iota}^{\mathrm{m}}(T)$, then we have the following properties.

(i)

$$0 \notin \sigma_{\mathrm{ap}}(T)$$

(ii) If $\lambda \in \sigma_{\mathrm{ap}}(T)$, then

$$\frac{1}{\lambda} \in \sigma_{ap}(S)$$

(iii) If $\lambda \in \sigma_p(T)$, then

$$\frac{1}{\lambda} \in \sigma_p(S)$$

Proof

(i) Suppose contrary to our claim that $0 \in \sigma_{\mathrm{ap}}(T)$, there exists a sequence $(x_n)_n \in X$ with $\|x_n\| = 1$ such that $Tx_n \to 0$ if $n \to +\infty$. Then

$$S^{m-j}T^{m-j}x_n \to 0 \text{ for } j = 0, 1, \ldots, m-1 \text{ and } S \in \mathcal{L}(X)$$

The hypothesis that T possesses an m-left inverse yields for $S \in \Im_{\iota}^{m}(T)$ that

$$x_n = (-1)^m \left(\sum_{j=0}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} S^{m-j}T^{m-j}(x_n) \right) \to 0$$

which is impossible.

(ii) Let $\lambda \in \sigma_{ap}(T)$ then there exists a sequence $(x_n)_n \in X$ with $\|x_n\| = 1$ and

$$(T - \lambda I_X)\, x_n \to 0, n \to +\infty$$

Then

$$\left(S^{m-j}T^{m-j} - \lambda^{m-j}S^{m-j}\right)(x_m) \longrightarrow 0 \quad \text{for } j = 0, 1, \ldots, m \quad \text{and } S \in \mathcal{L}(X)$$

The hypothesis that T possesses an m -left inverse yields for $S \in \Im_{t}^{m}(T)$ that

$$\begin{aligned} &\textstyle\sum_{j=0}^{m}(-1)^j \binom{m}{j} \left(S^{m-j}T^{m-j} - \lambda^{m-j}S^{m-j}\right)(x_n) \\ &= -\textstyle\sum_{j=0}^{m}(-1)^j \binom{m}{j} \left(\lambda^{m-j}S^{m-j}\right)(x_n) \longrightarrow 0 \end{aligned}$$

We find that

$$\left(S - \frac{1}{\lambda}\right)^m (x_n) \to 0$$

It follows that

$$\frac{1}{\lambda} \in \sigma_{ap}(S)$$

(iii) The argument is similar to the one given in (ii).

3.3 Spectral Properties of 2-Isometries

Theoreme 3.3.1

Let $T \in \mathcal{L}(H)$ be a non-unitary 2-isometry, then $\sigma_{ap}(T)$ lies on the unit circle $\partial\mathbb{D}$.

Proof

Assume T is a 2-isometry and let $\lambda \in \sigma_{ap}(T) \Rightarrow \exists x_n \in H \|x_n\| = 1$

Such that

$$\|(T - \lambda I)\| \, \|x_n\| \to 0$$

as $n \to \infty$

By induction,

$$\left\|\left(T^k - \lambda^k I\right)\right\| \, \|x_n\| \to 0$$

as $n \to \infty$

Therefore

$$\begin{aligned} 0 &= \left\langle \sum_k^2 (-1)^{2-k} \begin{pmatrix} 2 \\ k \end{pmatrix} T^{*k} T^k x_n, x_n \right\rangle \\ &= \sum_k^2 (-1)^{2-k} \begin{pmatrix} 2 \\ k \end{pmatrix} \left\langle T^{*k} T^k x_n, x_n \right\rangle \\ &= \sum_k^2 (-1)^{2-k} \begin{pmatrix} 2 \\ k \end{pmatrix} \left\langle \lambda^k x_n, \lambda^k x_n \right\rangle \\ &= \sum_k^2 (-1)^{2-k} \begin{pmatrix} 2 \\ k \end{pmatrix} \mid \lambda^k \mid = (\mid \lambda^2 \mid -1)^2 \Rightarrow \mid \lambda \mid = 1 \end{aligned}$$

Hence the result.

Theoreme 3.3.2

let $T \in \mathcal{L}(H)$ be a non-unitary 2-isometry, then unit disc $\overline{\mathbb{D}}$.

Proof

since

$$\partial(\sigma(T)) \subseteq \sigma_{ap}(T),$$

then from the lemma above we have

$$\sigma_{ap}(T) \subseteq \sigma(T) \ldots\ldots\ldots \text{(i)}$$

If $\lambda \notin \sigma_{ap}(T)$, then $\exists \epsilon > 0$ sucla that $\|Ty - \lambda y\| \geq c\|y\|$, for all $y \in H$ with

$$\|y\| \leq 1$$

If

$$y \perp R(T - \lambda I)$$

then

$$0 = \langle (T - \lambda I)x, y \rangle = \left\langle x, \left(T^* - \bar{\lambda}\right) y \right\rangle$$

and therefore

$$T^*y - \bar{\lambda}y = 0$$

It follows that

$$R(T - \lambda I) = 0.$$

So that

$$H = \overline{R(T - \lambda I)}$$

ie

$$T - \lambda I$$

has bounded inverse so

$$\lambda \neq \sigma(T) \Rightarrow \sigma(T) \subseteq \sigma_{ap}(T)......(ii)$$

From (i) and (ii) equality holds. Since $\sigma_{ap}(T)$ lies on the unit circle $\partial\mathbb{D}$, it follows that $\sigma(T) = \sigma_{ap}(T) =$ the closed unit disc $\mathbb{D}$.

Corollary 3.3.1

Let T be a non-unitary 2-isometry, then $1 \in \sigma\left(T^*T\right)$.

Proof

Suppose

$$1 \notin \sigma\left(T^*T\right) \Rightarrow A = \left(T^*T - I\right)$$

is invertible.

From the definition of a 2-isometry we have

$$T^{*2}T^2 - T^*T = T^*T - I$$
$$\Rightarrow T^*\left(T^*T - I\right)T = T^*T - I$$
$$\Rightarrow T^*AT = A$$
$$\Rightarrow \sigma\left(T^*AT\right) = \sigma(A)$$

which implies that T is similar to an isometry and so must be isometry. This contradicts our assumption that $1 \in \sigma\left(T^*T\right)$.

Theoreme 3.3.3 *[8]*

Let T be a non-unitary 2-isometry. Then

(i) $z \in \sigma_{ap}(T) \Rightarrow z^* \in \sigma_{ap}(T^*)$

(ii) $z \in \sigma_p(T) \Rightarrow z^* \in \sigma_p(T^*)$

(iii) Eigenvectors of corresponding to distinct eigenvalues are orthgonal.

Theoreme 3.3.4 *[11]*

Let T be a 2 -isometric. Then the numerical range of T is equal to [-1,1] provided it is non-unitary.

Proof

Let T be non-unitary, 2 -isometric.

We know that the numerical range $W(T)$ of an operator T is a convex set in the complex plane. The convex hull of $W(T)$ is the smallest convex set containing $W(T)$

$$\overline{W(T)} = \text{ Convex hull of } \sigma(T)$$

By lemma 3.1.2 , we obtain

$$\sigma(T) = \{z : |z| \leq 1\}$$

using this concept, we have

$$\overline{W(T)} = \text{ Convex hull of } [-1, 1]$$
$$= [-1, 1]$$

since -1 and 1 are eigen values of T, this belongs to $W(T)$. Hence $W(T) = [-1, 1]$

Theoreme 3.3.5 *[11]*

Let T be a 2-isometric. Then the Weyl spectrum of T is equal to [-1,1] if T is non-unitary.

Proof

Given that T is 2 -isometric and non-unitary. By lemma 3.1.2

$$\sigma(T) = \{z : |z| \leq 1\}$$

This shows that $\Pi_{00}(T) = \phi$. By lemma 3.1.1 we have

$$\omega(T) = \sigma(T) \sim \Pi_{00}(T)$$
$$\omega(T) = \sigma(T) \text{ (since, } \Pi_{00}(T) = \phi)$$
$$\Rightarrow \omega(T) = \sigma(T) = \{z : |z| \leq 1\}$$
$$\Rightarrow \omega(T) = [-1, 1]$$

Hence the proof.

Theoreme 3.3.6

Let T be the 2-isometric operator on H. Then $\Pi(T) = \sigma(T)$ holds if T is non -unitary.

Proof

Given that T is 2 -isometric, non-unitary. By lemma 3.1.2,

$$\sigma(T) = [-1, 1]$$

Since $\Pi(T)$ is a subset of the unit circle from [1]. Thus we obtain that

$$\Pi(T) \subset \sigma(T)$$

Claim: $\sigma(T) \subset \Pi(T)$

If $\lambda \notin \Pi(T)$, then there exists $\epsilon > 0$ such that

$$\|Ty - \lambda y\| \geq \epsilon\|y\|, \text{ for all } y \in H \text{ with } \|y\| \leq 1$$

and since $\|(T-\lambda)y\| = \|(T-\lambda)^*y\|$, holds for any y

$$\|(T-\lambda)^*y\| = \|(T-\lambda)y\| \geq \epsilon\|y\|, \quad \forall y \in H$$

ie.,

$$\|(T-\lambda)^*y\| \geq \epsilon\|y\| \quad \forall y \in H,$$
$$\left\|T^*y - \bar{\lambda}y\right\| \geq \epsilon\|y\|, \quad \forall y \in H$$

In order to prove $\lambda \notin \sigma(T)$

we have only to show $\overline{R(T-\lambda)} = H$, ie., $(T-\lambda)$ has a bounded inverse.

If $y \perp R(T-\lambda)$ then we show $y = 0$

In fact,

$$0 =< (T-\lambda)x, y >=< x, T^*y - \bar{\lambda}y >, \text{ for } y \in H$$

and therefore,

$$T^*y - \bar{\lambda}y = 0$$

It follows that $y = 0$. So that $\overline{R(T-\lambda)} = H$. ie., $T-\lambda$ has a bounded inverse so $\lambda \notin \sigma(T)$.

Therefore

$$\sigma(T) \subset \Pi(T)$$

From (3.1) and (3.4), we get

$$\sigma(T) = \Pi(T)$$

Hence $\sigma(T) = \Pi(T)$, if T is non-unitary.

Corollary 3.3.2

The weyl spectrum of a 2-isometry is the closed unit disc.

Corollary 3.3.3

From the theorem 3.3.4, 3.3.5, 3.3.6 and lemma 3.1.2, it is identified that spectrum, weyl spectrum, numercial range and approximate point spectrum are equal to $[-1, 1]$ if T is nonunitary 2-isometric operator.

Theoreme 3.3.7

Let T be a bounded self adjoint operator on H. Then T has dense range if and only if T is 2 -isometric operator on H.

Proof

Assume that T has dense range. clearly $R(T)$ is closed.

Then

$$H = R(T) \oplus R(T)^{\perp}$$

since, T is self adjoint

$$R(T)^{\perp} = N(T^*) = N(T) = \{0\}$$
$$N(T) = N(T^*T) = \overline{R(T^*T)}^{\perp} = \overline{R(T)}^{\perp} = \{0\}$$

Then

$$R(T^*T)^{\perp} = \{0\}$$

it follows that

$$\Rightarrow \overline{R(T^*T)} = H$$

Thus T^*T has dense range. Then

$$\Rightarrow T^*Tx = x, \quad \forall x \in H.$$

Therefore $\qquad T^*T = I \quad \forall x \in H.$

Hence T is a 2 -isometric operator.

Conversely, assume that T is 2 -isometric operator. ie.

$$T^{*2}T^2 - 2T^*T + I = 0$$

since T is self adjoint $T^*T = I$. Therefore, $T^*Tx = x, \quad \forall x \in H$

Clearly $N(T^*T) = \{0\}$ and since $H = N(T^*T) \oplus N(T^*T)^{\perp}$, we have, $N(T^*T)^{\perp} = H$

$$N(T^*T)^{\perp} = \overline{R(T^*T)} = \overline{R(T^*)} = H$$
$$\overline{R(T^*)} = \overline{R(T)} = H$$

Hence $R(T)$ has dense range.

Theoreme 3.3.8

Let T be the non-zero 2 -isometric self adjoint operator on H Then 0 is not an accumulation point of $\sigma\left(T^{*} T\right)$ of $T^{*} T$.

Proof

Given that T is the non-zero self-adjoint 2 -isometric operator on H By Theorem 3.3.7, T has dense range in H. ie.

$$\overline{R(T)}=H$$

It follows that $R(T)$ is closed. By using Hilbert space theory

$$\begin{gathered}H=R(T) \oplus R(T)^{\perp} \\ N(T)^{\perp}=N\left(T^{*} T\right)^{\perp}=\overline{R\left(T^{*} T\right)}=\overline{R\left(T^{*}\right)}=\overline{R(T)}=H\end{gathered}$$

Consider the operator

$$A: T^{*} T \mid N(T)^{\perp}: \quad N(T)^{\perp} \rightarrow N(T)^{\perp}$$

Clearly A is injective and $\overline{R\left(T^{*} T\right)}=R\left(T^{*} T\right)=N(T)^{\perp}$, the following conditions are equivalent:

(i) $R(T)$ is closed $R\left(T^{*} T\right)$ is closed

(ii) A is bijective

(iii) $0 \notin \sigma(A)$

(iv) There exists $r>0$ such that $\sigma(A) \subseteq\left[r,\|T\|^{2}\right]$

and in this case, we have

$$\begin{aligned} \sigma\left(T^{*} T\right) &=\sigma\left(T^{*} T \mid N(T)\right) \cup \sigma\left(T^{*} T \mid N(T)^{\perp}\right) \\ & \subseteq\{0\} \cup\left[r,\|T\|^{2}\right] . \end{aligned}$$

It follows that 0 is not an accumulation point of $\sigma\left(T^{*} T\right)$ of $T^{*} T$.

Theoreme 3.3.9

Let T be the selfadjoint 2-isometric operator on H. Then 0 is not an accumulation point of $\sigma(T)$ of T.

Proof

Given that T is the selfadjoint 2-isometric operator on H.

By theorem 3.3.7

$\overline{R(T)} = H$ and hence range of T is closed.

Consider the operator T is selfadjoint, it is reducible by $N(T)$, that is

$$T(N(T)) \subseteq N(T), T\left(N(T)^{\perp}\right) \subseteq N(T)^{\perp}$$

By theorem 3.1.3

$$\sigma\left(T^{+}\right)/\{0\} = \sigma\left(T_0\right)^{-1}/\{0\}$$

Let

$$\begin{aligned} d(T) &= \inf\{|\lambda| : \lambda \in \sigma(T)/\{0\}\} \\ &= d(0, \sigma(T)/\{0\}) \end{aligned}$$

claim: $\gamma(T) = d(T)$ In this case T_0^{-1} and T^{+} are bounded selfadjoint operators with

$$||T_0^{-1}|| = ||T^{+}|| = \frac{1}{\gamma(T)}$$

Hence

$$\begin{aligned} \gamma(T) &= \frac{1}{\|T^{+}\|} \\ &= \frac{1}{\sup\left\{|\mu| : \mu \in \sigma\left(T^{+}\right)\right\}} \\ &= \frac{1}{\sup\left\{|\mu| : \mu \in \sigma\left(T_0^{-1}\right)\right\}} \\ &= \frac{1}{\sup\left\{\left(\frac{1}{|\lambda|}\right) : 0 \neq \lambda \in \sigma\left(T_0\right)\right\}} \\ &= \underset{=}{\inf} \inf\left\{|\lambda| : 0 \neq \lambda \in \sigma\left(T_0\right)\right\} \\ &= d(T) \end{aligned}$$

It follows that 0 is not an accumxulation point of $\sigma(T)$ of T.

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Abstract

In this memoir we studied the class of m-isometric operators and the study of spectral properties of this class and related operators on a Hilbert gave basic results on the "structure" of an m-isometry, Since this class of operators has not be studied extensively, we would like to suggest possible areas that can be investigated in future.

Keywords : Operator , m-isometric , Spectral , Hilbert space.

Résumé

Dans ce mémoire nous avons étudié la classe d'opérateurs m-isométriques et l'étude des propriétés spectrales des m-isométries et des opérateurs associés sur un Hilbert a donné des résultats de base sur la "structure" d'une m-isométrie. Puisque cette classe d'opérateurs n'a pas été étudiée largement, nous aimerions suggérer des domaines possibles qui peuvent être étudiés à l'avenir.

Mots clés : Opérateur, m-isométriques, Spectrale, Espace Hilbert.

ملخص

في هذه المذكرة درسنا فئة المؤثرات m متساوي القياس، وكما قمنا بدراسة الخصائص الطيفية لهذا النوع من المؤثرات والمؤثرات المرتبطة بها على فضاءات هيلبرت ، وأعطينا النتائج الأساسية لبنية هذا النوع، كذلك لم يتم دراسة هذه الفئة من المؤثرات على نطاق واسع ، ونود اقتراح المجالات المحتملة التي يمكن دراستها في المستقبل.

الكلمات المفتاحية : مؤثر، متساوي القياس، طيفي، فضاء هيلبرت .