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Stability and decay of solutions for some wave equations with delay term

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الإهداء

بسم الله الرحمن الرحيم

الحمد لله الذي بنعمته تتم الصالحات، وبفضله وكرمه أتممتُ سنوات الدراسة، وتوجتُها بتخرجي هذا. اللهم لك الحمد على ما أوليت، ولك الشكر على ما أنعمت، فبفضلك وحدك تحقق هذا الانجاز

إهداء إلى أمي الغالية

إلى من كانت ولا تزال نبع الحنان، ومصدر الأمل، وملجأي في كل حين، أمي الغالية، أهديكِ هذا التخرج. أنتِ من زرع فيّ بذور النجاح، وسقيتها بحبك واهتمامك. فكل كلمة علمتها لي، وكل دعاء رفعتِه لي، كان له الأثر الكبير في وصولي إلى هذه اللحظة

إهداء إلى إخوتي الأعزاء

لى إخوتي الذين كانوا دومًا السند والرفيق، أهدى لكم هذا التخرج. لقد كنتم مصدر قوتي، ورفقاء دربي، فبفضل الله ثم بفضل دعمكم وتشجيعكم، استطعتُ تجاوز كل الصعاب. لكم مني كل الحب والامتنان

إلى كل من وقف بجانبي، وساهم في نجاحي، سواء بكلمة طيبة أو بمساعدة عملية، أقدم لكم جزيل الشكر والتقدير. إلى أساتذتي الكرام الذين أفادوني بعلمهم، وإلى أصدقائي الذين شاركوني لحظات الفرح والحزن، وإلى كل من كان له دور في مسيرتي، لكم مني أسمى آيات الشكر والعرفان.

أسأل الله أن يجعل هذا التخرج بداية لمستقبل مشرق، وأن يوفقني لما يحب ويرضى

شوية مبروكة

الإهداء

بسم الله والصلاة والسلام على رسول الله

لكل لحضه صبر , ولكل دمعة تعب , ولكل ابتسامة أمل , وفي كل حكاية قلوب كانت العون والنور إليهم أهدي ثمرة هذا الجهد امتنانًا و وفاءاً

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بشيري منار

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ملخص

تهدف هذه المذكرة إلى دراسة استقرار وتلاشي الحلول لمعادلات الموجة التي تحتوي شروط حدودية ذات تأخر زمني . تم في البداية عرض المفاهيم الأساسية كالفضئات المترية و المعياريّة، متبوعاً بدراسة متعمقة لمفهوم الاستقرار المنتظم والضعيف مع وجود تأخير زمني في الراجعة الديناميكية. كما يهدف الفصل الثاني إلى دراسة معادلة الموجة ذات تأخير في التحكم الديناميكي، بعد ذلك نبرهن أن المعادلة ذات تأخر زمني لها نفس معدل التلاشي النسبي كما في المعادلة العادية، مما يشير إلى أن تأثير التأخير لا يغيّر من سلوك التلاشي على المدى البعيد. أما الفصل الأخير توجهنا الي دراسة موجة شبه خطية ذات شروط حدودية، ونثبت في هذا الفصل نتيجة **التلاشي العام** الذي يشمل عدة حالات (التلاشي الآسي والحدودي كحالات خاصة)، دون فرض شروطا محددة على دالة الاسترخاء , وبهذا يعد هذا الفصل تعميما وتحسينا لمجموعة من النتائج السابقة.

كلمات مفتاحية: التأخر الزمني, التحكم الديناميكي, الاستقرار القوي, غياب الاستقرار المنتظم, التلاشي النسبي, التلاشي العام, التخميد الحدودي من نوع الذاكرة, معادلة الموجة, دالة الاسترخاء

Abstract

This thesis aims to investigate the stability and decay of solutions for wave equations involving a time-delay term. It begins with a presentation of fundamental mathematical concepts, such as metric and normed spaces, followed by a detailed analysis of strong and weak stability, particularly in the presence of delayed dynamic feedback. The second focuses on a wave equation with a delayed term in the dynamical control. It is shown that the delayed system exhibits the same asymptotic decay rate as the corresponding system without delay, indicating that the delay does not alter the long-term decay behaviour. But the last addresses a semi linear wave equation in a bounded domain with boundary damping of memory type. A general decay result is established, which encompasses both exponential and polynomial decay as special cases. This result does not require specific assumptions on the relaxation function, thereby extending and refining previous results in the literature.

Keywords:
time delay, dynamical control, strong stability, lack of uniform stability, rational decay. General decay, Memory type Boundary damping, Wave equation, Relaxation function,

Contents

Notations

$W^{m,p}(\Omega)$	The sobolev space of function on Ω derivatives up to order m in L^p
$H^m = W^{m,2}(\Omega)$	Sobolev Spaces ,
∂	The operator of patial differentiation ,
$D(A)$	The domain of the operator A ,
D^α	A partial derivative of multi-order α ,
$\sigma(A)$	The spectrum of operator A ,
$\lvert . \rvert$	The euclidean norm on $\mathbb{R}^d$
$N(A)$	The kernel of the operator A ,
$R(A)$	The range of the operator A ,
$\lVert . \rVert_x$	The norm on a normed space X ,
$\mathcal{L}(X)$	The space of bounded linear operators from X into X ,
X'	The dual space of X ,
$\mathrm{Re}\,\langle ., . \rangle$	The real part of the inner product ,
$C_0^\infty(\Omega)$	The test functions space ,
$(T(t))_{t\geq 0}$	A semigroup ,
$C(X, Y)$	The space of all continuous function from X into Y ,
$\Omega \subset \mathbb{R}^N$	A domain of $\mathbb{R}^N$,
$\partial\Omega = \Gamma$	The boundary of Ω ,
$L(\Omega)$	The space of measurable functions on the domain Ω
$L^p(\Omega)$	The space of functions whose absolute value raised to the power p is integrabl
$L_p^\infty(\Omega)$	The space of essentilly bounded function with weal-* convergence ,
$C(\Omega)$	The space of continuous function on the domain Ω ,
$C^\infty(\Omega)$	The space of infinitely differentiable functions on Ω ,
$C(\overline{\Omega})$	The space of coninuous functions on the closure of Ω ,
$B_r(x)$	The open ball of radius r centered at point x
$S_r(x)$	The sphere of radius r centered at point x ,
$\langle ., . \rangle$	The inner Product ,
$\rho(A)$	The resolvent set of the operator A ,
$\mathbb{R}^d$	The space of all ordered d-tuples of real numbers ,
$\Gamma(.)$	Fonction Gamma d'Euler.

Introduction

Many evolution equations coming from applications (in biology, electrical engineering and mechanics, ...) present a time-delay in the feedback law modeling mechanical time lag for instance [37] ,[28],[9]. In many cases, arbitrarily small delays in the feedback may destabilize the system, see for instance [51], [52], [53], [29], [23], [57] and [21]. Therefore the stability issue of systems with delay is of theoretical and practical importance. In domains such as control theory and biomechanics, wave equations with delay terms are frequently used to model systems in which the present state is dependent on previous states. Different methods have been recently developed to prove some exponential or polynomial decay of the energy of wave type equations with delay, we refer to [57], [58], [59], [14], [12], [36] and the references citing them.The stability and decay behavior of the system may become more complex when delay is introduced. The stability and decay of solutions for such equations are examined in this thesis, with an emphasis on the prerequisites that guarantee the intended asymptotic behaviors. We hope to develop criteria for exponential or polynomial decay by using mathematical tools such as Lyapunov functionals and energy methods, which will aid in the comprehension and management of delayed dynamical systems.

This thesis consists of three main chapters. In the first chapter of this memoir we devote a comprehensive review of the basic mathematical concepts and theories, which frome the theoretical foundation upon which we will later rely in studying the various aspects of this memorandum. In the second chapter, we study a wave equation set in a bounded domain (in any space dimension) with a dynamical control and prove that if the delay term is small enough, then the system has the same (polynomial) decay rate than the one without delay. Idea is to use a duality argument for a system without dynamical control is given by

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty), \\ \frac{\partial u}{\partial v}(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ \eta_t(x,t) - u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 \eta(x,t-\tau) = 0 \text{ 0n } \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0, \qquad u_t(x,0) = u_1 \quad \text{in } \Omega, \\ \eta(x,t-\tau) = f_0(x,t-\tau) \quad \text{on } \Gamma_N \times (0,\tau), \end{cases}$$

with respect to this equation throughout we will assume the following conditions:

i) $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a lipschitz boundary Γ.

ii) Γ is divided into two open parts Γ_D and Γ_N, i.e $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ with $meas\left(\Gamma_D\right) \neq 0$ and $meas\left(\Gamma_N\right) \neq 0$.

This chapter, is divided into three parts. In the first part we will prove that the equation is well posedness by using the semigroup theory, and then establish a strong stability result and, in the second part we will study the strong stability for this equation by using a general criteria of Arendt-Batty. Finally we show that if the our system without delay (i.e.,with $\beta_2 = 0$) is polynomially stable, then system (II.0.1) with delay inherits the same polynomial decay rate.

In the last chapter, we are concerned with the following problem

$$\begin{cases} u_{tt} - \Delta u + F\left(x, t, u, \nabla u\right) = 0 \quad \text{in } \Omega \times \mathbb{R}^+ \\ u\left(x, t\right) = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+ \\ u\left(x, t\right) = -\displaystyle\int_0^t g\left(t - s\right) \tfrac{\partial u}{\partial v}\left(s\right) ds \quad \text{on } \Gamma_1 \times \mathbb{R}^+ \\ u\left(0\right) = u_0, \quad u_t\left(0\right) = u_1 \quad \text{in } \Omega, \end{cases}$$

whith Ω is an open bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, υ is the unit outer normal vector, and $f \in C^1\left(\mathbb{R}^n\right)$ is a function satisfying

$$uf\left(u\right) \geq bF\left(u\right) \geq 0, \quad \text{for } b > 2, \quad F\left(u\right) = \int_0^u f\left(\xi\right) d\xi.$$

Recently, [40] Santos considered

$$\begin{cases} u_{tt} - \mu\left(t\right) u_{xx} = 0, \qquad \left(x, t\right) \in \left(0, 1\right) \times \mathbb{R}^+ \\ u\left(0, t\right) = 0, \quad u\left(1, t\right) = -\displaystyle\int_0^\tau g\left(t - s\right) \mu\left(s\right) u_x\left(1, s\right) ds, \qquad \forall t > 0 \\ u\left(0\right) = u_0, \qquad u_t\left(0\right) = u_1, \qquad x \in \left(0, 1\right), \end{cases}$$

where $\mu\left(t\right)$ is a nonincreasing function satisfying $\mu\left(t\right) \geq \mu_0 > 0$. By considering the resolvent kernel of $-g = g\left(0\right)$, the boundary condition takes the form

$$\mu\left(t\right) u_x\left(1, t\right) = -\tau\{u_t\left(1, t\right) + k\left(0\right) u\left(1, t\right) - k\left(t\right) u_0\left(1\right) + k'\left(t\right) * u\left(1, t\right)\}$$

where $\tau > 0$ is a constant and k is the resolvent kernel of $-g'/g_0$. He showed that the energy of the solution decays exponentially (polynomially) when k and k' decay exponentially (polynomially). This result has been later pushed to a nonlinear n-dimensional wave equation of Kirchhoff type by Santos et al. [41]. In that work, the authors established the existence of a global unique solution and showed, under the same conditions on k and k'; that the solution decays uniformly with the same rate of decay k. This latter result improves an earlier one by Park et al. [34]. A similar approach has been also used by Santos and Junior [42] to establish a similar result to a biharmonic wave equation supplemented by viscoelastic damping acting on a part of the boundary. Cavalcanti et al. [43] studied the existence and the uniform decay of solutions to a semilinear wave equation with a boundary damping of memory type and a nonlinear boundary source. Also, Rivera an Andrade [33] considered a one-dimensional nonlinear wave equation subject to a nonlinear boundary memory effect and showed that this effect is strong enough to guarantee global existence and uniform decay, at least for small initial data, provided that kernel decays exponentially (or polynomially).

Particularly, [45] Cavalcanti and Guesmia considered the following system

$$\begin{cases} u_{tt} - \Delta u + F(x,t,u,\nabla u) = 0 \text{ in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 \text{ on } \Gamma_0 \times \mathbb{R}^+ \\ u(x,t) = -\displaystyle\int_0^t g(t-s) \tfrac{\partial u}{\partial v}(s)\, ds \text{ on } \Gamma_1 \times \mathbb{R}^+ \\ u(0) = u_0, \quad u_t(0) = u_1 \text{ in } \Omega, \end{cases} \tag{a}$$

where Ω is an open bounded set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed, disjoint, with $meas(\Gamma_0) > 0$ and v is the unit outer normal vector. They established a more general decay result which depends on the value of u_0 on Γ_0 and the rate of the decay of k'. In their work, they treated the cases when k' is decaying exponentially or is k' decaying polynomially. When $u_0 = 0$ on Γ_0, they obtained the exponential and the polynomial decay as special cases. This result has been recently generalized to the case of a system of Timoshenko type by Messaoudi and Soufyane [60]. Cavalcanti et al. [44] studied a problem of the form

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)\, d\tau = 0, \quad \text{in } \Omega \times (0,\infty) \\ u = 0, \quad \text{on } \Gamma_0 \times (0,\infty) \\ \tfrac{\partial u}{\partial v} - \displaystyle\int_0^t g(t-\tau) \tfrac{\partial u}{\partial v}(\tau)\, d\tau + h(u_t) = 0, \quad \text{on } \Gamma_1 \times (0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{b}$$

for g,h specific functions and established uniform decay rate results under quite restrictive assumptions on both the damping function h and the kernel g. In fact, the function g had to behave exactly like e-mt and the function h had a polynomial behavior near zero. For more general assumptions on g and h, Cavalcanti et al. [61] proved the uniform stability of (b), provided that $g(0)$ and $\|g\|_{L^1(0,\infty)}$ are small enough. They also established explicit decay rate results for some special cases. This latter result of Cavalcanti et al. [61] has been recently improved by Messaoudi and Mustafa [48], where no growth assumption on h near zero has been imposed. Stabilization of wave equations or wave systems by frictional boundary damping has been studied by many researchers. Different mechanisms have been utilized to stabilize such systems and several decay and stability results have been obtained. In this regard we mention, among many others, the work of Alabau-Boussouira [18], Cavalcanti et al. [46], [47], Conrad and Rao [19], Gorain [22], Guesmia and Messaoudi [3], Komornik and Zuazua [64], Komornik [66], Komornik and Rao [67], Lasiecka [26], Lasiecka and Tataru [27], and Zuazua [16].

Chapter I

Preliminaire

This chapter aims to provide a review of the essential mathematical concepts and theories that from the theoretical foundation for the work presented in this thesis. These concepts will be relied upon later in addredding the various topics discussed.

I.1 Metric Space

Definition I.1.1 *[15] Let X be a non-empty set. A metric on X is function $d : X \times X \longrightarrow \mathbb{R}^+ \cup \{0\}$ which satisfies the following conditions*

$$d(x,y) > 0 \text{ for all } x,y \in X \quad \text{with } x \neq y; \tag{I.1.1}$$
$$d(x,y) = 0 \Longleftrightarrow x = y \quad \text{for all } x,y \in X; \tag{I.1.2}$$
$$d(x,y) = d(y,x) \text{ for all } x,y \in X \text{ (symetry)}; \tag{I.1.3}$$
$$d(x,y) \leq d(x,z) + d(z,y) \text{ for all } x,y,z \in X \text{ (traingle inequlity)}. \tag{I.1.4}$$

Here $d(x,y)$ is the distance between x and y; more over the ordered pair (X,d) is called a metric space.

Remark I.1.1 *[15]Applying n-times the inequality (I.1.4) for all $x, y, z_1, z_2, ..., z_n \in X$, we obtain*

$$\begin{aligned} d(x,y) &\leq d(x,z_1) + d(z_1,y) \\ &\leq d(x,z_1) + d(z_1,z_2) + d(z_2,y) \\ &\quad\vdots \\ &\leq d(x,z_1) + d(z_1,z_2) + ... + d(z_n,y) \end{aligned}$$

which is said to be the generlized traingle inequality.

Exercise I.1.1 *Let $C[a,b]$ be the set of continuous functions defined by*

$$C[a,b] = \{f \ \setminus \ f : [a,b] \longrightarrow \mathbb{R} \text{ and } f \text{ is continuous on } [a,b]\}.$$

show that the following functions

i) $d_1(f,g) = \int_a^b |f(x) - g(x)| \, dx$,

ii) $d_2(f,g) = (\int_a^b |f(x) - g(x)|^2 \, dx)^{\frac{1}{2}}$,

iii) $d_\infty(f,g) = \sup \{|f(x) - g(x)| : x \in [a,b]\}$, *defined on* $C[a,b]$ *are metric.*

Definition I.1.2 *[15]Let* (X,d) *be space,* $x_0 \in X$ *and* $r > 0$*. Then*

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

is said to be an open ball having radius r *with center* x_0*.*

$$\overline{B_r}(x_0) = \{x \in X : d(x, x_0) \leq r\}$$

is said to be a closed ball having radius r *with center* x_0*.*

$$S_r(x_0) = \{x \in X : d(x, x_0) = r\}$$

is said to be the boundary ball having radius r *with center* x_0*.*

Remark I.1.2 *Open ball, closed ball and the boundary of a ball having radius* r *with center* x_0 *are denoted by* $B_r(x_0) = B(x_0, r), \overline{B_r}(x_0) = \overline{B_r}(x_0, r)$ *and* $S_r(x_0) = S(x_0, r)$ *, respectively.*

Definition I.1.3 *A sequence* (x_n) *in a metric space* (X,d) *is said to convergy to a point* $x_0 \in X$ *if for* $\varepsilon > 0$ *there exists an integer* N *such that* $n \geq N$ *implies that* $\mathbb{R}$

$$d(x_n, x_m) < \varepsilon.$$

We say that the sequence (x_n) *is convergent and* x_0 *is the limit of* (x_n)*. In other words we write*

$$x_n \longrightarrow x_0 \quad or \quad \lim_{n \longrightarrow \infty} x_n = x_0.$$

On the other hand, if (x_n) *is not convergent, it is divergent.*

Definition I.1.4 *A sequence* (x_n) *in a metric space* (X,d) *is said to be a Cauchy sequence if for every* $\varepsilon > 0$ *there exists an integer* N *such that*

$$d(x_n, x_m) < \varepsilon$$

whenever $n \geq N$ *and* $m \geq M$ *. In other words,* (x_n) *is a Cauchy sequence if*

$$d(x_n, x_m) \longrightarrow 0 \quad as \quad n, m \longrightarrow \infty$$

or equivalently

$$\lim_{n,m \longrightarrow \infty} d(x_n, x_m) = 0$$

I.2 Normed Space

Definition I.2.1 *[15]Let X be a vector space over the real (or complex) numbers. A function,$\|.\|$, from X into the non-ngative real real numbers is called a norm on X if for all $x, y \in X$ and $\lambda \in \mathbb{R}$, the conditions $\|x\| > 0$ if $x \neq 0$ and $\|x + y\| \leq \|x\| + \|y\|$ are satisfied. The norm of x is denoted by $\|x\|$. Moreover, the vector space X with norm is called normed vector space or normed space .*

i) $\|x\| > o$ *if* $x \neq 0$,

ii) $\|x\| = 0 \Longleftrightarrow x = 0$,

iii) $\|\lambda x\| = |\lambda| \, \|x\|$,

iv) $\|x + y\| \leq \|x\| + \|y\|$,

Remark I.2.1 *$\|.\|$ is said to be a semi norm if only the conditions (n3) and (n4) are satisfied. Notice that for a semi norm the conditions $\|x\| = 0$ may be satisfied for non-zero x .*

Theorem I.2.1 *Let $(X, \|.\|)$ be a normed space. Then for all $x, y \in X$, the function defined by*

$$d(x, y) = \|x - y\|$$

is a metric on X. This metric is called normed metric.

Definition I.2.2 *[15] Let $\|.\|_1$ and $\|.\|_2$ norms be given on the vector space X.*
For every $x \in X$ if there exists $c_1, c_2 > 0$ such that

$$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1 .$$

then $\|.\|_1$and $\|.\|_2$ are called equivalent norms.

Definition I.2.3 *As a result of this definition. We can deduce that different norms can determine the same topology. The following result is based on the fact that any two norms on a finite dimensional vector space determine the same topology.*

Theorem I.2.2 *All the norms defined on finite dimensional normed (vector) spaces are equivalent.*

As a consquence of Theorem, all the norms defined on finite dimensional normed spaces describe the same topology on these spaces. For instance, if a squence (x_n) in norm space X is convergent, bounded or Cauchy sequence with respect to the norm $\|.\|_1$ $(\|.\|_2)$, then it is also convergent, bounded or Cauchy sequence with respect to the norm $\|.\|_1$ $(\|.\|_2)$.

Definition I.2.4 *If every point sequence in A has a subsquence that converges to an element of A in X, then the subset of a normed space X is said to be compact.*

Definition I.2.5 *Let X and Y be normed spaces, and let $\|.\|_X$ and $\|.\|_Y$ be their respective norms. If every bounded subset of X maps into a comparatively compact subset of Y, then the continuous linear operator $L \in \mathcal{L}(X, Y)$ is considered compact. The Heine–Borel theorem for finite dimensions normed spaces might be seen as leading to the following outcome.*

Theorem I.2.3 *Assume that X is a finite dimensional normed space and that $Y \subset X$. If and only if Y is closed and bounded, then the set is compact.*

Remark I.2.2 ***Remark I.2.3*** *According to the theory of metric spaces, every convegent sequence is Cauchy. There re metric spaces and normed spaces where a Cauchy sequence does not converge, hence the oposite of this statement may not always be true. The completeness property of a normed space, which yields a Banach space, is given in the following. Only the basic characteristics of Bach spaces presented here. Additional information is available in the functional and real analysis books listed in the reference section.*

I.3 The Hölder and Young Inequalities

1) [63]Let (X, Γ, μ) be a measured space and let $1 \leq p, q, r \leq +\infty$ be real numbers such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

If $u \in L^p(X)$ and $v \in L^q(X)$ then $uv \in L^r(X)$ and,

$$\|uv\|_{L^r(X)} \leq \|u\|_{L^p(X)} \|v\|_{L^q(X)}.$$

2) Let $1 \leq p, q \leq +\infty$ and let $r \geq 1$ be such that,

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

If $u \in L^p(\mathbb{R}^d)$ then for almost all x in $\mathbb{R}^d$ the integral,

$$(u * v)(x) = \int_{\mathbb{R}^d} u(x - y)v(y)dy$$

is convergent, and $u * v$ belongs to $L^r(\mathbb{R}^d)$. Moreover we have,

$$\|u * v\|_{L^r(\mathbb{R}^d)} \leq \|u\|_{L^r(\mathbb{R}^d)} \|v\|_{L^r(\mathbb{R}^d)}.$$

I.4 Approximation of the Identity

1. [63] Let $\rho \in C^{\infty}(\mathbb{R}^d)$ be such that

$$\operatorname{supp}(\rho) \subset \left\{x \in \mathbb{R}^d : |x| \leq 1\right\}, \rho \geq 0, \int_{\mathbb{R}^d} \rho(x)dx = 1.$$

   For ε in $(0,1]$ set,$\rho_{\varepsilon}(x) = \varepsilon^{-d}\rho(\frac{x}{\varepsilon})$.

$$\lim_{\varepsilon \to 0}(\rho_{\varepsilon} * u) = u \quad \text{in} \quad L^p(\mathbb{R}^d).$$

   The family $[\rho_{\varepsilon}]_{\varepsilon \in (0,1]}$ is called an appoximation of the identity. Then we have,

2. for any $1 \leq p < +\infty$, if u belongs to $\in L^p(\mathbb{R}^d)$,then

$$\lim_{\varepsilon \to 0}(\rho_{\varepsilon} * u) = u \quad \text{in} \quad L^p(\mathbb{R}^d).$$

3. If $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ is a bounded uniformly continuous function, then

$$\lim_{\varepsilon \to 0}(\rho_{\varepsilon} * u) = u \quad \text{in} \quad L^{\infty}(\mathbb{R}^d).$$

4. For any $1 \leq p \leq +\infty$, if u belongs to $L^p(\mathbb{R}^d)$, then for almost all x in $\mathbb{R}^d$,

$$\lim_{\varepsilon \to 0}(\rho_{\varepsilon} * u)(x) = u(x).$$

5. For any $1 \leq p \leq +\infty$, if u belongs to $L^p(\mathbb{R}^d)$, then for any $\varepsilon \in (0,1]$, the function $\rho_{\varepsilon} * u$ belongs to $C^{\infty}(\mathbb{R}^d)$.

I.5 The Lebesgue Space $L^p(\Omega)$

I.5.1 Definition and Basic Properties of $L^p(\Omega)$

[55] Let p be a positive real number and let Ω be domain in $\mathbb{R}^n$. The class of all measurable functions u defined on Ω in indicated by $L^p(\Omega)$, for which

$$\int_{\Omega} |u(x)|^p \, dx < \infty. \tag{I.5.1}$$

The members of $L^p(\Omega)$ are therfore equivalence classes of measurable functions satisfying (I.5.1), where two functions are comparable if they are equal a.e.in Ω. We find functions in $L^p(\Omega)$ that are identical nearly everywhere in Ω. We disregard this distinction for simplicity and write $u = 0$ in $L^p(\Omega)$ if $u(x) = 0$ a.e.in Ω, and $u \in L^p(\Omega)$ if

u fulfills (I.5.1). If $u \in L^p(\Omega)$ and $c \in \mathbb{C}$, then $cu \in L^p(\Omega)$ is obviously true. We must demonstrate that if $u, v \in L^p(\Omega)$, then $u+v \in L^p(\Omega)$ in order to prove that $L^p(\Omega)$ is a vector space. This is a direct result of the inquity thar follows, which will also be helpful in the future.

Lemma I.5.1 *If $1 \leq p \leq \infty$ and $a, b \geq 0$, then*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \tag{I.5.2}$$

Lemma I.5.2 ***(The L_p Norm)*** *Now will confirm that the functional $\|.\|_p$ defined*

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p}$$

is a standard on $L^p(\Omega)$, given $1 < p < \infty$.(If $0 < p < \infty$, it is not a norm). When there is a possibility of domain confusion, we substitute $\|.\|_{p,\Omega}$ for $\|.\|_p$.

It is evident that if and only if $u = 0$ in $L^p(\Omega)$, then $\|u\|_p \geq 0$ and $\|u\|_p = 0$. Additionally

$$\|cu\|_p = |c| \, \|u\|_p \ , \qquad c \in \mathbb{C}.$$

Therefore, after we have confirmed the triangle inequality, we will have demonstrated that $\|.\|_p$ is norm on $L^p(\Omega)$.

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p \ ,$$

Minkowski's inequality is the name given to it. This requires Holder's inequality first.

Corollary I.5.1 *It is possible to apply Hölder's inquality to products of more than two functions. Assume that where $u_j \in L^q(\Omega), 1 \leq j \leq N$, and $p_j > 0, u = \prod_{j=1}^{N} u_j$. Soppose that $\sum_{j=1}^{N}(1/p_j) = 1/q$, then $u \in L^q(\Omega)$ and $\|u\|_q \leq \prod_{j=1}^{N} \|u_j\|_{p_j}$. By induction on $\mathbb{N}$, this follows from the prior corollary.*

Theorem I.5.1 ***(A Converse of Hölder's Inequality)*** *A mesurable function u is a member of $L^p(\Omega)$ if and only if*

$$\sup\left\{\int_\Omega |u(x)| \, v(x)dx : v(x) \geq 0 \ on \ \Omega, \|v\|_{p'} \leq 1\right\} \tag{I.5.3}$$

is finite, and then that supremum equals $\|u\|_p$.

Theorem I.5.2 ***(Minkowski's Inquality)*** *If* $1 \leq p \leq \infty$*, then*

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p \,. \tag{I.5.4}$$

Theorem I.5.3 ***(Minkowski's Inequality for Integals)*** *Assume* $1 < p < \infty$*. Assume that the function* $y \to \|f(.,y)\|_{p,\mathbb{R}^n}$ *belongs to* $L^1(\mathbb{R}^n)$*. that* f *is measurable on* $\mathbb{R}^n \times \mathbb{R}^n$*, and that* $f(.,y) \in L^p(\mathbb{R}^m)$ *for almost all* $y \in \mathbb{R}^n$*. Then,* $L^p(\mathbb{R}^m)$ *a contains the function* $x \to \int_{\mathbb{R}^n} f(x,y)dy$*.*

$$\left(\int_{\mathbb{R}^m}\left|\int_{\mathbb{R}^n} f(x,y)dy\right|^p dx\right)^{1/p} \leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} |f(x,y)|^p \, dx\right)^{1/p} dy.$$

that is,

$$\left\|\int_{\mathbb{R}^n} f(.,y)dy\right\|_{p,\mathbb{R}^m} \leq \int_{\mathbb{R}^n} \|f(.,y)\|_{p,\mathbb{R}^m} \, dy.$$

I.5.2 The Space $L^\infty(\Omega)$

If a constant K exists such that $|u(x)| \leq K$ a.e.on Ω, then a function u that is measurable on Ω is said to be basically limited on Ω. The essential supremum of $|u|$ on Ω, represented by ess $\sup_{x\in\Omega}|u(x)|$, is the maximum lower bound of such constants K. The vector space of all functions u that are fundamentally bounded on Ω is represented by $L^\infty(\Omega)$, with functions being once more recognized if they are equal a.e.on Ω. Verification of the functional $\|.\|_\infty$ defined by

$$\|u\|_\infty = \text{ess} \sup_{x\in\Omega} |u(x)|$$

is a norm on $L^\infty(\Omega)$. Moreover, Hölder's inequality (3) and its corollaries extend to cover the two cases $p = 1, p' = \infty$ and $p = \infty, p' = 1$.

I.6 The Sobolev Space $W^{m,p}(\Omega)$

I.6.1 Definitions and Basic Properties

The Sobolev Norms

We define a functional $\|.\|_{m,p}$ where m is a positive integer and $1 \leq p \leq \infty$, as follows:

$$\|u\|_{m,p} = (\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_p^p)^{\frac{1}{p}} \quad \text{if} \quad 1 \leq p < \infty, \tag{I.6.1}$$

$$\|u\|_{m,\infty} = \max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_\infty \quad \text{with } p = \infty \tag{I.6.2}$$

for any function u for which the right side makes sence , $\|.\|_p$ being, of course, the norm in $L^p(\Omega)$. In some situations where confusion of domains may occur we will use $\|u\|_{m,p,\Omega}$ in place of $\|u\|_{m,p}$. Evidently (I.6.1) or (I.6.2) defines a norm on any vector space of functions on which the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω.

Sobolev Spaces

For any positive integer m and $1 \leq p \leq \infty$ we consider three vector spaces on which $\|.\|_{m,p}$ is a norm:

a) $H^{m,p}(\Omega) \equiv$the completion of $\left\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\right\}$ with respect to the norm $\|.\|_{m,p}$.

b) $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}$

c) $W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

Equipped with the appropriate norm (1) or (2) thesΩe are called Sobolev Space over Ω. Clearly $W^{0,p}(\Omega) = L^p(\Omega)$, and if $1 \leq p \leq \infty$, $W_0^{0,p}(\Omega) = L^p(\Omega)$. For any m , we have the obvious chain of imbeddings

$$W_0^{m,p}(\Omega) \longrightarrow W^{m,p}(\Omega) \longrightarrow L^p(\Omega).$$

Theorem I.6.1 *[55] $W^{m,p}(\Omega)$ is a Banach space.*
Proof: *Let $\{u_n\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then $\{D^\alpha u\}$ is a Chauchy sequence in $L^p(\Omega)$ for $0 \leq |\alpha| \leq m$. Since $L^p(\Omega)$ is complet there exist functions u and $u_\alpha, 0 \leq |\alpha| \leq m$, such that $u_n \longrightarrow u$ and $D^\alpha u_n \longrightarrow u_\alpha$ in $L^p(\Omega)$ as $n \longrightarrow \infty$. Now $L^p(\Omega) \subset L^1_{loc}(\Omega)$ and so u_n determines a distribution $T_{u_n} \in \mathcal{D}'(\Omega)$. For any $\phi \in \mathcal{D}(\Omega)$ we have*

$$|T_{u_n}(\phi) - T_u(\phi)| \leq \int_\Omega |u_n(x) - u(x)|\,|\phi(x)|\,dx \leq \|\phi\|_{p'}\,\|u_n - u\|_p$$

by Hölder's inequality, where p' is the exponent conjugate to p. Therfre $T_{u_n}(\phi) \longrightarrow T_{u_n}(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$ as $n \longrightarrow \infty$. Similarly, $T_{D^\alpha u_n}(\phi) \longrightarrow T_{u_\alpha}(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$. It follows that

$$T_{u_\alpha}(\phi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\phi) = \lim_{n\to\infty} (-1)^{|\alpha|} T_{u_n}(D^\alpha \phi) = (-1)^{|\alpha|} T_u(D^\alpha \phi)$$

for every $\phi \in \mathcal{D}(\Omega)$. Thus $u_\alpha = D^\alpha u$ in the distributional sence on Ω for $0 \leq |\alpha| \leq m$, whence $u \in W^{m,p}(\Omega)$. Since $\lim_{n\to\infty} \|u_n - u\|_{m,p} = 0$, the space $W^{m,p}(\Omega)$ is complete.
■

Theorem I.6.2 *[55]$W^{m,p}(\Omega)$ is separable if $1 \leq p \leq \infty$, and is uniformly convex and reflexive if $1 \leq p \leq \infty$. In particular, $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product*

$$(u, v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v),$$

where $(u, v) = \int_\Omega u(x)\overline{v(x)}dx$ is the inner product on $L^2(\Omega)$.

I.6.2 Duality and the spaces $W^{-m,p'}(\Omega)$

In this part, the operator P, the spaces $L^p(\Omega^{(m)})$ and W, and the number N for fixed Ω, m and p. Additionally, for any functions u, v for which the right side makes sence,we define

$$\langle u, v \rangle = \int_\Omega u(x)v(x)dx$$

for any functions u, v that make sence on the right side.

Let's agree that p' always represents the conjugate exponent for given p :

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ p/(p-1) & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty \end{cases}$$

The Riesz Representation Theorem is first extended too the space $W^{m,p}(\Omega)$. The dual of $W_0^{m,p}(\Omega)$ with a subspace of $\mathcal{D}(\Omega)$ is then found lastly, we demonstrate that the completion of $L^{p'}(\Omega)$ with regard to a norm weaker than the typical $L^{p'}$norm may likewise be associated with the dual of $W_0^{m,p}(\Omega)$ if $1 < p < \infty$.

The Dual of $L^p(\Omega^{(m)})$

A unique $v \in L^{p'}(\Omega^{(m)})$ corresponds to each $L \in (L^p(\Omega^{(m)}))'$, where $1 < p < \infty$, so that for each $u \in L^p(\Omega^{(m)})$

$$L(u) = \int_{\Omega^{(m)}} u(x)v(x)dx = \sum_{|\alpha| \leq m} \int_{\Omega_\alpha} u_\alpha(x)v_\alpha(x)dx = \sum_{|\alpha| \leq m} \langle u_\alpha, v_\alpha \rangle,$$

where u_α and v_α are the restrictions of u and v, resectively, to Ω_α. Moreover, $\left\|L; (L^p(\Omega^{(m)}))'\right\| = \left\|v; L^{p'}(\Omega^{(m)}))'\right\|$. Thus $(L^p(\Omega^{(m)}))' = L^{p'}(\Omega^{(m)})$.

This is valid because $L^p(\Omega^{(m)})$ is, after all, an L^p space, albeit one defined on an unusual domain.

Theorem I.6.3 *[55] **(The Dual of** $\mathbf{W}^{m,p}(\Omega)$) Let $1 \leq p \leq \infty$. Then for every functional $L \in (W^{m,p}(\Omega))'$ there exist a function $v \in L'(W^{m,p}(\Omega))$ such that if he* *restriction of v to Ω_α is v_α, we have for all $u \in W^{m,p}(\Omega)$*

$$L(u) = \sum_{0\leq|\alpha|\leq m} \langle D^\alpha u, v_\alpha\rangle \tag{I.6.3}$$

Moreover

$$\|L;(W^{m,p}(\Omega))'\| = \inf \left\|v; L^{p'}(\Omega^{(m)})\right\| = \min \left\|v; L^{p'}(\Omega^{(m)})\right\|, \tag{I.6.4}$$

For any $u \in W^{m,p}(\Omega)$, (I.6.3) holds, and the infimum is taken over and reached on the set of all $v \in L^{p'}(\Omega^{(m)})$. If $1 < p < \infty$, the element $v \in L^{p'}(\Omega^{(m)})$ satisfying (I.6.3) and (I.6.4) is unique.

Theorem I.6.4 ***(The Normed Dual of*** $W_o^{m,p}(\Omega)$***)*** *The dual space $(W_0^{m,p}(\Omega))'$ is isometrically isomorphic to he Banach space $W^{-m,p'}(\Omega)$, which is composed of those distributions $T \in \mathcal{D}(\Omega)$ that satisfy (I.8.1) and have norm*

$$\|T\| = \min\left\{\left\|v; L^{p'}(\Omega^{(m)})\right\| : v \text{ satisfies (I.8.1)}\right\}.$$

The isometric isomiphism results in this space being complete.

If $1 < p < \infty$, then $W_0^{m,p'}(\Omega)$ is clearly separable and reflexive.

Distributions T provided by (I.8.1) do not fully define continuous linear functionals on $W^{m,p}(\Omega)$ when $W_0^{m,p}(\Omega)$ is proper subset of $W^{m,p}(\Omega)$ due to their limits to $C_0(\Omega)$.

I.6.3 Approximation by Smooth Functions on Ω

$\left\{\phi \in C^\infty(\Omega) : \|\phi\|_{m,p} < \infty\right\}$ is dense in $W^{m,p}(\Omega)$ is what we want to demonstrate . The following existence theorem for ndlessly defferentiable partial functions of unity is necessary for his .

Theorem I.6.5 ***(Unity Partitions)*** *Let A be a set of open sets in W that cover A, or $A \subset \cup_{u\in\theta} U$, and let A be an abritrary subset of W then, a set of functions $\ell \in C_0^\infty(\mathbb{R}^n)$ having the following properties;*

i) *For every $\ell \in \Psi$ and every $x \in \mathbb{R}^n$, $o \leq \ell(x) \leq 1$.*

ii) *If $K \Subset A$, all but finitely many $\ell \in \Psi$ vanish identically on K.*

iii) *For every $\ell \in \Psi$ there exists $U \in \theta$ such that supp $(\ell) \subset U$.*

iv) *For every $x \in A$, we have $\sum_{\ell\in\Psi} \ell(x) = 1$.*

Such a collection Ψ is called a C^∞-partition of unity for A subordinate to θ.

I.6.4 Approximation by Smooth Functions on $\mathbb{R}^n$

Having shown that an element of $W^{m,p}(\Omega)$ can always be approximated by functions smooth on Ω we now ask whether the approximation can in fact be done with bounded functions having bounded derivatives of all orders, or at least of all orders up to and including at least m. That is, we are asking whether, for any values of $k \geq m$, the space $C^k(\overline{\Omega})$ is dence in $W^{m,p}(\Omega)$. The following example shows that the answer mav be negative.

I.7 Sobolev embedding theorems

When examining the regularity of a weak solution to a boundary value issue, for example, Sobolev embedding theorems are crucial.

I.7.1 The Rellich Kondrachov Theorems

Definition I.7.1 *[1] Let $V \subset W$, V and W be two Banach spaces. We write $V \hookrightarrow W$ if the spaces V is continuously embedded in W*

$$\|\nu\|_W \leq c\,\|\nu\|_V \qquad \forall \nu \in V. \tag{I.7.1}$$

We say the space V is compactly embedded in W and write $V \hookrightarrow\hookrightarrow W$, if (I.7.1) holds and each bounded sequence in V has a subsequence coonverging in W.

Remark I.7.1 *if $V \hookrightarrow W$, the functions in V are more smooth than the remaining functions in W. A simple example is $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$.*

Proofs of most parts of the following two theorems can be found in [39]. The first theorem is embedding of Sobolev spaces, and the second on compact embedding.

Theorem I.7.1 ***(continuously embedded)*** *Consider the Lipshitz domain $\Omega \in \mathbb{R}^d$. Then, the following claims are true.*

a) *If $k < d/p$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \leq p^*$, where p^* is defined by $1/p^* = 1/p - k/d$.*

b) *If $k = d/p$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q < \infty$.*

c) *If $k > d/p$, then*

$$W^{k,p}(\Omega) \hookrightarrow C^{k-[d/p]-1,\beta}(\Omega),$$

where

$$\beta = \begin{cases} [d/p] + 1 - d/p\ , & \text{if } d/p \neq \text{integer}, \\ \text{any positive number} < 1\ , & \text{if } d/p = \text{integer} \end{cases}$$

In the theorem, denotes the integer part of x, i.e., the largest integer less than or equal to x. We remark that in the one-dimensional case, with $\Omega = (a, b)$ a bounded interval, we have

$$W^{k,p}(a, b) \hookrightarrow C[a, b]$$

for any $k \geq 1, p \geq 1$.

Theorem I.7.2 ***(compactly embedded)*** *Consider the Lipschitz domain $\Omega \in \mathbb{R}^d$. Then, the following claims are true.*

a) *If $k < d/p$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $q < p^*$, where p^* is defined by $1/p^* = 1/p - k/d$.*

b) *If $k = d/p$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $q < \infty$.*

c) *If $k > d/p$, then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{k-[d/p]-1,\beta}(\Omega),$$

where $\beta \in (0, [d/p] + 1 - d/p)$.

How can we recall these outcomes ?. We use the case of theorem I.7.2. The functions from the space $W^{k,p}(\Omega)$ are smoother the larger the product kp. For this product, there is a crucial value d (the domain's dimension) such that a $W^{k,p}(\Omega)$ function is truly continuous if $kp > d$ or more specifically, represents a continuous function a.e.For an exponent p^* greater than p, a $W^{k,p}(\Omega)$ function belongs to $L^{p^*}(\Omega)$ when $kp < d$. Starting with the requirement $kp < d$, expressed as $1/p - k/d > 0$, we calculate the exponent p^*. The difference $1/p - k/d$ is then defined as $1/p^*$. Knowing if a $W^{k,p}(\Omega)$ function has continuous derivatives up to a specific order is typically helpful when $kp > d$. We start with

$$W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \qquad \text{if } k > d/p.$$

Then we apply this embedding result to derivatives of Sobolev functions, it is easy to see that

$$W^{k,p}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \qquad \text{if } k - l > d/p.$$

As some concrete examples, for a two-dimensional Lipschitz domain Ω, $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \;\; \forall\, 1 \leq q \leq \infty$ and $H^2(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$. So in particular, a sequence bounded in $H^1(\Omega)$ has a subsequence that converges in $L^2(\Omega)$, $C(\overline{\Omega})$. For a three-dimensional Lipschitz domain Ω, $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)\, \forall\, 1 \leq q < 6$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and $H^1(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$.

Theorem I.7.3 *A direct consequence of Theorem I.7.3 is the following compact embedding result.*

I.7.2 Traces theorem

$L^P(\Omega)$ spaces are used to define Sobolev spaces. Sobolev functions are therefore only uniquely defined in Ω. The boundary value of a Sobolev function appears to be poorly defined now that the boundary Γ has measure zero in $\mathbb{R}^d$. However, a Sobolev function's tace on the boundary can be defined so that, for a Sobolev function that is continuous up to the border, its trace and its boundary value coincide.

Theorem I.7.4 *[1] Let $\mathbb{R}^d$, $1 \leq p < \infty$, be a Lipschitz domain with Ω. A continuous linear operator $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ then has the following characteristics:*

a) $\gamma v = v|_\Gamma$ *if* $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

b) *For some constant* $c > 0, \|\gamma v\|_{L^p(\Gamma)} \leq c \|v\|_{W^{1,p}(\Omega)} \quad \forall v \in W^{1,p}(\Omega)$.

c) *The mapping $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is compact; i.e., for any bounded sequence $\{v_n\}$ in $W^{1,p}(\Omega)$, there is a subsequence $\{v_{n'}\} \subset \{v_n\}$ such that $\{\gamma v_{n'}\}$ is convergent in $L^p(\Gamma)$.*

Remark I.7.2 *i) The trace operator is denoted by γ, and the generalaized boundary value of v is denoted γv. The trace operator from $W^{1,p}(\Omega)$ to $L^p(\Gamma)$ is neither an injection nor a surjection. The range $\gamma(W^{1,p}(\Omega))$ is a positive order Sobolev space over the boundary, namely $W^{1-1/p,p}(\Omega)$, which is smaller than $L^p(\Gamma)$. Typically, the trace of $v \in H^1(\Omega)$ is represented by the same symbol v*

ii) If $v_n \to v$ in $H^1(\Omega)$, then $v_n \to v$ in $L^2(\Gamma)$.

Theorem I.7.5 *When we discuss weak formulations of boundary value problems later in this book, we need to use traces of the $H^1(\Omega)$ functions. These traces form the space $H^{1/2}(\Gamma)$; in other words,*

$$H^{1/2}(\Gamma) = \gamma(H^1(\Omega)).$$

Correspondingly, we can use the following as the norm for $H^{1/2}(\Gamma)$:

$$\|g\|_{H^{1/2}(\Gamma)} = \inf_{v \in H^1(\Omega)} \|v\|_{H^1(\Omega)} . \tag{I.7.2}$$

The ability to appropriately impose necessary boundary conditions in formulations is a prerequisite for understanding boundary value problems. Theorem I.7.3 is enough for the purpose because necessary boundary conditions for second-order boundary value issues only require function values on the border. The traces of partial derivatives on the border must be used for higher-order boundary value problems. For instance, all boundary requirements containing derivatives of order at most one are regarded as fundamental boundary conditions for fourth-order boundary value issues. Given that By differentiating the function's boundary value, we may get the tangential derivative of a function on the boundary; all we need to do is use the function's traces and normal *derivative.Let the outward unit normal to the boundary Γ of Ω be represented by $v = (v_1, ..., v_d)^T$. Remember that the classical normal derivative of v on the boundary if it is a member of $C^1(\Omega)$ is*

$$\frac{\partial v}{\partial \nu} = \sum_{i=1}^{d} \frac{\partial v}{\partial x_i} \nu_i.$$

I.7.3 Generalized Poincaré Inequality

Theorem I.7.6 *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^n$. Let $p \in [1, +\infty[$ and let $\mathcal{N}$ be a continuous seminorm on $W^{1,p}(\Omega)$; that is, a norm on the constant functions. Then there exists a constant $C > 0$ that depends only on Ω, N, p, such that*

$$\|u\|_{W^{1,p}(\Omega)} \leq C \left(\left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p} + \mathcal{N}(u) \right).$$

Proposition I.7.1 *[13] Let Ω be a bounded domain of class C^1; then there exists a constant $C_P > 0$ such that every u in $H_0^1(\Omega)$ satisfies*

$$\|u\|_{H^1(\Omega)} \leq C_P \|\nabla u\|_2 .$$

I.7.4 The Lax-Milgram Lemma

The Lax-Milgram Lemma is employed frequently in the study of linear elliptic boundary value problems. For a real Banach space V, let us first explore the relation between a linear operator $A : V \to V'$ and a bilinear form $a : V \times V \to \mathbb{R}$ related by

$$\langle Au, v \rangle = a(u, v) \quad \forall u, v \in V. \tag{I.7.3}$$

The bilinear form $a(.,.)$ is continuous if and only if there exixts $M > 0$ such that

$$|a(u, v)| \leq M \|u\| \|v\| \qquad \forall u, v \in V$$

Theorem I.7.7 *[1] there exixts a one-to-one correspondence between linear continuous operators $A : V \to V'$ and continuous bilinear forms $a : V \times V \to \mathbb{R}$, given by the formula (I.7.3).*

I.8 Weak Derivative

Definition I.8.1 *[15] Let α be a multi-index and the function $u \in L^1_{loc}(\Omega)$ be provided for each $\varphi \in C_0^\infty(\Omega)$ the relationship*

$$T_{v_\alpha}(\phi) = \langle \phi, v_\alpha \rangle = \int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega \varphi(x) v(x) dx \tag{I.8.1}$$

is met, the function $v \in L^1_{loc}(\Omega)$ is known as the $\alpha - th$ weak (generalized) derivative of u, and it is represented by the notation $v = D^\alpha u$.

A number of authors also use $v = v_\alpha = D^\alpha u$ to indicate the weak derivative. Only weak derivatives exist.

I.8.1 Properties of the Weak Derivative

We wrap off the chapter by presenting the following theorem, which allows one to compare the weak derivative with the classical one and determine some of its fundamental characteristics, especially from the first two propositions.

Theorem I.8.1 ***i)*** *A function has a weak derivative if it has a (classical) derivative. The opposite might not always be true, though. In other words, a function with a weak derivative might not have a (classical) derivative.*

ii) *The weak derivative is defined worldwide, whereas the classical derivative is pointwise.*

iii) *The weak derivative is linear. In other words, if $c_1, c_2 \in \mathbb{R}$ and $u_1, u_2 \in L^1_{loc}(\Omega)$, we obtain*

$$D^\alpha(c_1 u_1 + c_2 u_2) = c_1 D^\alpha(u_1) + c_2 D^\alpha(u_2).$$

iv) *If u has a weak derivative $v = D^\alpha u$ and if v has a weak derivative $w = D^\beta v = D^\beta(D^\alpha u)$, then*

$$D^{\beta+\alpha} u = D^\beta(D^\alpha u) = D^\beta v = w.$$

The fact that the sequence of differentiation is irrelevant for mixed derivatives is another essential characteristic of the weak derivative that sets it apart from the classical derivative. Stated otherwise, the weak derivative is independent of the differentiation order.

Remark I.8.1 *In the classical sense, a function must be continuous in order to be differentiable. However, the weak derivative does not need neither continuity nor differentiability of the function; integrability is sufficient.*

I.9 Green's Identities

To keep things simple, we look at $\mathbb{R}^3$ elements. Let $F = (F_1, F_2, F_3)$ be a vector in $\mathbb{R}^3$ and $u = u(x, y, z)$ be a function. The following lists some helpful notations.

- $\nabla u = \operatorname{grad} u = (u_x, u_y, u_z)$,
- $\nabla \cdot F = \operatorname{div} F = F_{1x} + F_{2y} + F_{3z}$,
- $\triangle u = \operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = u_{xx} + u_{yy} + u_{zz}$,
- $|\nabla u|^2 = u_x^2 + u_y^2 + u_z^2$,

- $\nabla \cdot \nabla = \nabla^2 = \triangle$.

Theorem I.9.1 ***(Green's First Identity)***[**15**] *For* $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$\int_\Omega v\triangle u dx = \int_{\partial\Omega} v\frac{\partial u}{\partial n} ds - \int_\Omega \nabla v \nabla u dx$$

where n *is the unit vector in the outward normal direction and* $\frac{\partial u}{\partial n} = n \cdot \nabla u$.

(Green's Second Identity). *For* $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$\int_\Omega (u\triangle v - v\triangle u)dx = \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds.$$

I.10 Some Semigroup arguments

Definition I.10.1 *Let* $\mathcal{B} \in \mathcal{L}(X)$ *be a bounded operator. The resolvent set* $\rho(\mathcal{B})$ *of* $\mathcal{B}$ *is the set of all* λ *in* $\mathbb{C}$ *for which* $(\mathcal{B} - \lambda I)^{-1}$ *exists and bounded*

$$\rho(\mathcal{B}) = \left\{\lambda \in \mathbb{C}; (\mathcal{B} - \lambda I)^{-1} \in \mathcal{L}(X)\right\}.$$

The spectrum of $\mathcal{B}$*, denoted by* $\sigma(\mathcal{B})$*, is the complement of the resolvent set, i.e.,*

$$\sigma(\mathcal{B}) = \mathbb{C} \backslash \rho(\mathcal{B}).$$

A complex number λ *is an eigenvalue of* $\mathcal{B}$ *if*

$$N(\mathcal{B} - \lambda I) \neq \{0\}.$$

Theorem I.10.1 *Let* X *be a Banach space and* $\mathcal{B} \in \mathcal{L}(X)$ *. If the operator* $\mathcal{B}$ *satisfies* $\|\mathcal{B}\| < 1$*, then* $I - \mathcal{B}$ *is invertible and its inverse is given by*

$$(I - \mathcal{B})^{-1} = \sum_{n=0}^{\infty} \mathcal{B}^n.$$

Definition I.10.2 ***Definition I.10.3*** [5] *Let* X *be a Banach space. A semigroup of bounded linear operators is a family of bounded linear operators* $T(t) \in \mathcal{L}(X)$*, which depend on a parameter* $0 \leq t \leq \infty$ *and that satisfies the following properties*

$T(0) = I$*, (*I *is the identity operator on* X*).*

$T(t + s) = T(t)T(s)$ *for every* $t, s \geq 0$ *(the semigroup property).*

A semigroup of bounded linear operators$T(t)$*is uniformly continuous if*

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

the infunitesinal generator of a semigroup $T(t)$ is the operator $\mathcal{B}$ defined on

$$D(\mathcal{B}) = \left\{ x \in X : \lim_{t \to o} \frac{T(t)x - x}{t} \quad exists \right\}$$

by

$$\mathcal{B}x = \lim_{t \to 0} \frac{T(t)x - x}{t}, \quad for \ \ x \in D(\mathcal{B}).$$

Definition I.10.4 ***(Maximal Monotone Operators)*** *[24] Let X be a Hilbert space,an unbounded linear operator $\mathcal{B} : D(\mathcal{B}) \subset X \to X$ is said to be monotone(accretive) if it satisfies*

$$\operatorname{Re} \langle \mathcal{B}v, v \rangle \geq 0, \forall v \in D(\mathcal{B})$$

In addition, the operator is called maximal monotone , if $R(I + \mathcal{B}) = X$ i.e,

$$\forall f \in X; \exists v \in D(\mathcal{B}), such\ that\ u + \mathcal{B}u = f.$$

Remark I.10.1 *If -$\mathcal{B}$ is monotone, we say that B is dissipative.*

Proposition I.10.1 ***Proposition I.10.2*** *[24] Let $\mathcal{B}$ be a maximal monotone operator on a Hilbert space. Then*
1) $D(\mathcal{B})$ is dense in X.
2) $\mathcal{B}$ is a closed operator.
3) For every $\lambda > 0$, $(I + \lambda\mathcal{B})$ is bijective from $D(\mathcal{B})$ into X, $(I + B)^{-1}$ is a bounded operator, and $\|(I + \lambda\mathcal{B})^{-1}\|_{\mathcal{L}(X)} \leq 1$.

I.10.1 The Lumer Philips Theorem

Definition I.10.5 *[6] If there is a $x^* \in F(x)$ such that $\operatorname{Re} \langle Ax, x^* \rangle \leq 0$ for each $x \in D(A)$,then a linear operator A is dissipative. The following is a helpful description of dissipative operators.*

Theorem I.10.2 ***(Lumer-Phillips)*** *Let A be a linear operator in X with the dense domain $D(A)$.*

1) *If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitsimal generator of a C_0 semigroup of contractions on X.*

2) *If A is the infinitesimal generator of a C_0 semigroup of contractions on X the $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, $R \langle Ax, x^* \rangle \leq 0$.*

I.10.2 The Hille-Yosida Theorem

Consider a C_0 semigroup $T(t)$, for $t > 0$, there are constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$. $T(t)$ is said to be uniformly bounded if $M = 1$, and a C_0 semigroup of contractions if $\omega = 0$. Characterizing the infinitesimal generators of C_0 semigroup of contractions is the focus of this section. In order for an opperator A to be the infinitesimal generator of a C_0 semiggroup of contractions, some conditions on the resolvent's behavior are provided.

Recall that if A is a linear, not necessarily bounded, operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible,i.e.,$(\lambda I - A)^{-1}$ is a bounded linear operator in X . The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

Theorem I.10.3 *[6] **(Hill-Yosida)** A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, if and only if*

1) *A is closed and $\overline{D(A)} = X$.*

2) *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}. \tag{I.10.1}$$

I.10.3 Operator m-dissipative

Definition I.10.6 *[6] A dissipative operator A is referred to as m-dissipative if $R(I - A) = X$.*

If A dissipative so is μA for all $\mu > 0$ and therefore if A is m-dissipative then $R(\lambda I - A) = X$ for every $\lambda > 0$. In terms of m-dissipative operators the Lumer-Philips theorem can be restated as: A densely defined operator A is the infinitesimal generator of a C_0 semigroup of contractions if and only if it is m-dissipative.

The main result of this section is the following perturbation theorem for m-dissipative operators.

I.11 Stability and Hyperbolicity for Semigroups

We first examine the stability of strongly continuous semigroups $(T(t))_{t\geq 0}$, one of the many intriguing forms of asymptotic behavior. This means that as $t \to \infty$, the operators $T(t)$ ought to converge to zero. However, we must discriminate between several definitions of convergence, as is to be expected in infinite-dimensional spaces.

I.11.1 Stability concepts

Definition I.11.1 *A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called*

- *strongly exponentially stable, provided that* $\varepsilon > 0$.

$$\lim_{t\to\infty} e^{\varepsilon t} \|T(t)\| = 0, \tag{I.11.1}$$

- *uniformly stable if*

$$\lim_{t\to\infty} \|T(t)\| = 0, \tag{I.11.2}$$

- *strongly stable if*

$$\lim_{t\to\infty} \|T(t)x\| = 0 \qquad \textit{for all } x \in X, \tag{I.11.3}$$

- *weakly stable if*

$$\lim_{t\to\infty} \langle T(t)x, x'\rangle = 0 \qquad \textit{for all } x \in X \textit{ and } x' \in X'. \tag{I.11.4}$$

In the case A is not compact, then the classical methods such as Lasalle's invariance principle [18] or the spectral decomposition theory of SzNagy-Foias, Foguel and Benchimol [13, 8, 33] are not applicable. in this case. Thus, we will use a Stability Theorem of Arendt-Batty (given below) to study the strong stability of our system

Theorem I.11.1 ***(Stability Theorem)***[69] *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflixive space X. Denote by A the generator of $(T(t))_{t\geq 0}$ and by $\sigma(A)$ the spectrum of A. If $\sigma(A) \cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then $\lim_{t\to\infty} T(t)x = 0$ for all $x \in X$.*

Theorem I.11.2 *On a Banach space X, let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup with generator A. If*

$P\sigma(A') \cap i\mathbb{R} = \emptyset$ and
$\sigma(A) \cap i\mathbb{R}$ is countable,
then $(T(t))_{t\geq 0}$ is strongly stable, i.e.,

$$\lim_{t\to\infty} T(t)x = 0 \qquad \text{for all } x \in X. \tag{I.11.5}$$

Corollary I.11.1 *On a Banach space X, let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator A. The following claims are identical if $(T(t))_{t\geq 0}$ is substantially compact for the strong operator topology. $(T(t))_{t\geq 0}$ is strongly stable. $P\sigma(A) \cap i\mathbb{R} = \emptyset$.*

I.11.2 Lyapunov direct method

Theorem I.11.3 *Suppose that there exist constant $a, b, c, r > 0, p \geq 1$ and a C^1 function $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that,*

i) $\forall x, \|x\| \leq r, \;\; a\|x\|^p \leq V(x,t) \leq b\|x\|^p, \forall t \geq 0.$

ii) $V_t(t,x) \leq -c\|x\|^p, \forall t \geq 0, \forall x, \|x\| \leq r.$

Then, the equilibrium point of the equation $x_t = f(t,x)$ is exponentially stable.

I.12 Perturbation by Compact Operator

I.12.1 Fredholm Operator

An invertible operator is a basic illustration of a fredholm operator $J : E \to F$; in this case $N(t) = \{0\}, R(T) = F$, and so $\alpha(T) = \beta(T) = 0$. Conversely, $\alpha(T) = \beta(T) = 0$ implies that J is invertible. We shall now charactirize the Fredholm operators, that is, the operators $T \in \Phi$ such that $\alpha(T) = \beta(T), i(T) = 0$.

Definition I.12.1 *A* ***Fredholm operator*** *is one whose kernel is finite-dimensional and whose image has finite codimension. The* ***index*** *of a fredholm operator is the difference*

$$index(T) := \dim \ker T - codim \operatorname{Im} T.$$

A Fredholm operator $T : X \to Y$ gives rise to decompositions

$$X = \ker T \oplus M, \quad Y = im\ T \oplus N,$$

for some closed linear subspace M, N. The restricted operator $R : M \to im\ T, x \longmapsto Tx$ is then bijective and continuous, and thus an isomorphism by the open mapping theorem.

Theorem I.12.1 *[35] **(Riesz-Schauder)** If $T = J + V$, where J is invertible and V is compact, then T is fredholm, that is $\alpha(T) = \beta(T) < \infty$.*

Chapter II

Well posedness and decay rates for the wave equation with delay term on the dynamical control

Contents

In this chapter, we study a wave equation set in a bounded domain (in any space dimension) with a dynamical control and prove that if the delay term is small enough, then the system with delay has the same (polynomial) decay rate than the one without delay. The main (and simple) idea is to use a duality argument already used in [72] in a fully different context and in [20] for a system without dynamical control. Hence our purpose is to generalize these results to a system with delayed dynamical control.

Let us shortly describe the distributed parameter systems that we will analyse. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a lipschitz boundary Γ. We assume that Γ is divided into two open parts Γ_D and Γ_N, i.e. $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ with meas $(\Gamma_D) \neq 0$ and meas $(\Gamma_N) \neq 0$.

In this domain Ω, we consider the following wave equation with delay term on the dynamical control

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty), \\ \frac{\partial u}{\partial v}(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ \eta_t(x,t) - u_t(x,t) + \beta_{1\eta}(x,t) + \beta_{2\eta}(x,t-\tau) = 0 \text{ 0n } \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega, \\ \eta(x,t-\tau) = f_0(x,t-\tau) \quad \text{on } \Gamma_N \times (0,\tau), \end{cases} \tag{II.0.1}$$

Where $v(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial v}$ is the normal derivative of u. Besides η denotes the dynamical control, $\tau > 0$ is the time delay, β_1 and β_2 are positive constants and the initial data (u_0, u_1, f_0) belong to a suitable space (precisely described below). The damping of the system is made via the indirect damping mechanism on Γ_N

Our first goal is to show that this system is well-posed and is strongly stable under the following assumption

$$\beta_2 < \beta_1. \tag{II.0.2}$$

Afterwards, we show that this system is not exponentially stable.Finally we show that if the system (II.0.1) without delay (i.e.,with $\beta_2 = 0$) is polynomially stable, then system (II.0.1) with delay inherits the same polynomial decay rate.

The next is organized as follows; the first section is devoted to the well-posedness and strong stability of problem (II.0.1) but, in the second section we establish the non uniform stability, and finally in third section we prove the rational stability.

II.1 Well posedness and strong stability

In this section we will give the well posedness for the problem (II.0.1) by using the semigroup theory, and then establish a strong stability result.

II.1.1 Well posedness

Let us set

$$z(x,\rho,t) = \eta(x,t-\tau\rho), \; x \in \Gamma_N, \quad \rho \in (0,1), \; t \in (o,+\infty).$$

The problem (II.0.1) is now equivalent to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty) \ , \\ \tau z_t(x,p,t) + z_\rho(x,\rho,t) = 0 \text{ in } \Gamma_N \times (0,1) \times (0,+\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty), \\ \partial_v u(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ \eta_t(x,t) - u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 z\left(x,1,t\right) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ z(x,0,t) = \eta(x,t) \quad \text{on } \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0, u_t\left(x,0\right) = u_1 \text{ in } \Omega, \\ \eta\left(x,t\right) = \eta_0 \text{ on } \Gamma_N \\ z\left(x,\rho,0\right) = f_0\left(x,-\rho\tau\right) \quad \text{on} \quad \Gamma_N \times (0,\tau) \end{cases} \tag{II.1.1}$$

We set

$$\mathcal{U} = (u, u_t, \eta, z)^T.$$

Then we have

$$\mathcal{U}_t = (u_t, u_{tt}, \eta_t, z_t)^T = \left(u_t, \Delta u, \left(u_t - \beta_1 \eta - \beta_2 z\left(.,1\right)\right)_{|\Gamma_N}, -\tau^{-1} z_\rho\right)^T.$$

Therefore problem (II.1.1) can be rewritten in an abstract framework:

$$\begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U}, \\ \mathcal{U}\left(0\right) = \left(u_0, u_1, \eta_0, f_0\left(., -.\tau\right)\right)^T \end{cases} , \tag{II.1.2}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\left(u,v,\eta,z\right)^T = \left(v, \Delta u, \gamma\left(v - \beta_1 \eta - \beta_2 z\left(.,1\right)\right), -\tau^{-1} z_\rho\right)^T,$$

with domain

$$\mathcal{D}\left(\mathcal{A}\right) = \left\{ \left(u,v,\eta,z\right)^T \in D\left(\Delta\right) \times H^1_{\Gamma_D}\left(\Omega\right) \times L^2\left(\Gamma_N\right) \times L^2\left(\Gamma_N \times H^1\left(0,1\right)\right) \middle| \begin{array}{l} \partial_v u = -\eta \\ z\left(.,0\right) = \eta \end{array} \quad \text{on } \Gamma_N \right\},$$

where

$$H^1_{\Gamma_D}\left(\Omega\right) = \left\{u \in H^1\left(\Omega\right), u = 0 \text{ on } \Gamma_D\right\},$$

γ is the trace mapping from $H^1\left(\Omega\right)$ into $L^2\left(\Gamma_N\right)$, and $D\left(\Delta\right)$ is the "maximal" domain of the Laplace operator

$$D\left(\Delta\right) = \left\{u \in H^1_{\Gamma_D}\left(\Omega\right) \middle| \Delta u \in L^2\left(\Omega\right), \text{ and } \partial_v u \in L^2\left(\Gamma_N\right)\right\}.$$

Let us now introduce the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_D}\left(\Omega\right) \times L^2\left(\Omega\right) \times L^2\left(\Gamma_N\right) \times L^2\left(\Gamma_N \times (0,1)\right)$$

with the norm

$$\left\|(u,v,\eta,z)^T\right\|_{\mathcal{H}}^2=\|\nabla u\|_{L^2(\Omega)^n}^2+\|v\|_{L^2(\Omega)^n}^2+\|\eta\|_{L^2(\Gamma_N)}^2+\varsigma\|z\|_{L^2(\Gamma_N\times(0,1))}^2$$

and the natural associated inner product

$$\left\langle\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\z^*\end{pmatrix}\right\rangle_H=\int_\Omega(\nabla u.\nabla\bar{u}^*+v\bar{v}^*)\,dx+\int_{\Gamma_N}\eta\bar{\eta}^*d\Gamma+\zeta\int_{\Gamma_N}\int_0^1 z(x,\rho)\,\bar{z}^*(x,\rho)\,d\rho,$$

where, here and below, ζ is a fixed constant satisfying

$$\tau\beta_1<\zeta<\tau(2\beta_1-\beta_2) \tag{II.1.3}$$

which always exists due to the assumption (II.0.2).

Proposition II.1.1 *The operator $\mathcal{A}$ defined above is m-dissipative.*

Proof: By Green's formula and Cauchy-Schwarz's inequality, we see that

$$\Re\langle\mathcal{AU},\mathcal{U}\rangle_H=\Re\left\langle\begin{pmatrix}v\\\Delta u\\\gamma(v-\beta_1\eta-\beta_2 z(.,1))\\-\tau^{-1}z_\rho\end{pmatrix},\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix}\right\rangle_H$$

$$=\Re\int_\Omega\nabla v.\nabla\bar{u}dx+\int_\Omega\Delta u\bar{v}dx+\int_{\Gamma_N}(v-\beta_1\eta-\beta_2 z(.,1))\,\bar{\eta}d\Gamma-\zeta\tau^{-1}\int_{\Gamma_N}\int_0^1 z_\rho(x,\rho)\,\bar{z}(x,\rho)\,d\rho d\Gamma$$

$$=-\Re\int_{\Gamma_N}\eta\bar{v}d\Gamma+\Re\int_{\Gamma_N}(v-\beta_1\eta-\beta_2 z(.,1))\bar{\eta}d\Gamma-\zeta\tau^{-1}\int_{\Gamma_N}\left[\frac{1}{2}\mid z(x,\rho)\mid^2\right]_0^1 d\Gamma$$

$$=-\beta_1\int_{\Gamma_N}|\eta|^2 d\Gamma-\beta_2\Re\int_{\Gamma_N}z(x,1)\,\bar{\eta}d\Gamma-\frac{\zeta\tau^{-1}}{2}\int_{\Gamma_N}z(x,1)^2 d\Gamma+\frac{\zeta\tau^{-1}}{2}\int_{\Gamma_N}|\eta|^2 d\Gamma$$

$$\le-\beta_1\int_{\Gamma_N}|\eta|^2 d\Gamma+\beta_2\int_{\Gamma_N}|z(.,1)\,\bar{\eta}|\,d\Gamma-\frac{\zeta\tau^{-1}}{2}\int_{\Gamma_N}\left|z(x,1)^2\right|+\frac{\zeta\tau^{-1}}{2}\int_{\Gamma_N}|\eta|^2\ d\Gamma$$

$$\le-\beta_1\int_{\Gamma_N}|\eta|^2 d\Gamma+\frac{\beta_2}{2}\int_{\Gamma_N}|z(x,1)|^2 d\Gamma+\frac{\beta_2}{2}\int_{\Gamma_N}|\eta|^2 d\Gamma-\frac{\zeta\tau^{-1}}{2}\int_{\Gamma_N}|z(x,1)|^2 d\Gamma+\frac{\zeta\tau^{-1}}{2}\int_{\Gamma_N}|\eta|^2 d\Gamma.$$

This proves that

$$\Re\langle\mathcal{AU},\mathcal{U}\rangle_H\le\Lambda_1\left\|\eta^2\right\|_{L^2(\Gamma_N)}+\Lambda_2\|z(.,1)\|_{L^2(\Gamma_N)}^2, \tag{II.1.4}$$

Which

$$\Lambda_1=-\beta_1+\frac{\beta_2}{2}+\frac{\zeta}{2}\tau^{-1}\ ,\quad\Lambda_2=\frac{\beta_2}{2}-\frac{\zeta}{2}\tau^{-1} \tag{II.1.5}$$

that are negative constants due to (II.1.3). We therefore deduce that $\mathcal{A}$ is dissipative.

Now, we will show that the operator $\lambda I - \mathcal{A}$ is surjective for all $\lambda \geqslant 0$.

Given $F = (f, g, h, k)^T \in \mathcal{H}$, we look for $U = (u, v, \eta, z)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})\, U = F$$

that is

$$\begin{cases} \lambda u - v = f \\ \lambda v - \Delta u = g \\ \lambda \eta - (v - \beta_1 \eta - \beta_2 z(.,1)) = h \\ \lambda z + \tau^{-1} z_\rho = k. \end{cases} \tag{II.1.6}$$

Assuming for the moment that we have found η and z with appropriate regularity such that

$$z(.,0) = \eta. \tag{II.1.7}$$

Then using (II.1.7) we deduce from the fourth equation of (II.1.6) that

$$z(.,\rho) = \eta e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho} \int_0^\rho k(.,\sigma)\, e^{\lambda\tau\sigma} d\sigma \in L^2(\Gamma_N \times (0,1))\,; \tag{II.1.8}$$

consequently

$$z(.,1) = \eta e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 k(.,\sigma)\, e^{\lambda\tau\sigma} d\sigma. \tag{II.1.9}$$

Eliminating v from the first identity in (II.1.6), namely

$$v = \lambda u - f \tag{II.1.10}$$

and using (II.1.9) in the third identity of (II.1.6), we find

$$\eta = \lambda C_\lambda u - \tilde{f}_\lambda \quad \text{on} \quad \Gamma_N \tag{II.1.11}$$

where for shortness we set

$$\begin{cases} C_\lambda = \frac{1}{\lambda + \beta_1 + \beta_2 e^{-\lambda\tau}}, \\ \tilde{f}_\lambda = C_\lambda \left(f + \beta_2 \tau e^{-\lambda\tau} \int_0^1 k(.,\sigma)\, e^{\lambda\tau\sigma} d\sigma - h \right). \end{cases} \tag{II.1.12}$$

It follows that u verifies

$$\begin{cases} -\Delta u + \lambda^2 u = g + \lambda f \quad \text{in } \Omega \\ u = o \quad \text{on} \quad \Gamma_D \\ \partial_v u + \lambda C_\lambda u = \tilde{f}_\lambda \quad \text{on} \quad \Gamma_N. \end{cases} \tag{II.1.13}$$

Multiplying the first equation of (II.1.13) by $\varphi \in H^1_{\Gamma_D}(\Omega)$ and integrating formally by parts we get

$$a_\lambda (u, \varphi) = L_\lambda (\varphi), \forall \varphi \in H^1_{\Gamma_D} (\Omega), \tag{II.1.14}$$

where the bilinear and continuous form $a_\lambda$is given by

$$a_\lambda = (u, \varphi) = \int_\Omega \nabla u \nabla \varphi dx + \lambda^2 \int_\Omega u\varphi dx + C_\lambda \lambda \int_{\Gamma_N} u\varphi d\Gamma$$

while the linear form $L_\lambda$is

$$L_\lambda (\varphi) = \int_\Omega (g + \lambda f) \varphi dx + \int_{\Gamma_N} \tilde{f}_\lambda \varphi d\Gamma.$$

Since $a_\lambda$is clearly strongly coerciveon $H^1_{\Gamma_D} (\Omega) \times H^1_{\Gamma_D} (\Omega)$ and L_λ is continuous on $H^1_{\Gamma_D} (\Omega)$ the Lax-Milgram theorem ensures that the problem (II.1.14) admits a unique weak solution $u \in H^1_{\Gamma_D} (\Omega)$.

Moreover, considering test function $\varphi \in \mathcal{D}(\Omega)$, we recover the first identity of (II.1.13) in $\mathcal{D}'(\Omega)$ that is in distributions sense; consequently $\Delta u \in L^2 (\Omega)$ since f and g belong to $L^2 (\Omega)$. Coming back to (II.1.14), and again applying Green's formula, we find that

$$\partial_v u + \lambda C_\lambda u = \tilde{f}_\lambda \quad \text{on} \quad \Gamma_N.$$

Further since $\tilde{f}_\lambda$ and u belong to $L^2 (\Gamma_N)$, we deduce that u belongs to $D(\Delta)$. Furthermore, from (II.1.6) and (II.1.13) we define v and η by (II.1.10) and (II.1.11) respectively, which gives the regularity $v = \lambda u - f \in H^1_{\Gamma_D} (\Omega) \subset L^2 (\Omega)$, $\eta \in L^2 (\Gamma_N)$ and the boundary condition $\partial_v u = -\eta$ on Γ_N. Finally we have found $U \in \mathcal{D}(\mathcal{A})$ which verifies (II.1.6). This shows that the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and then generates a C_0-semigroup of contractions in $\mathcal{H}$. Thanks to Lumer-Phillips' theorem, problem (II.0.1) is well posed. ■

II.1.2 Strong stability

The main result of this subsection is the following.

$$\lim_{t \to 0} \left\| e^{t\mathcal{A}} U_0 \right\|_{\mathcal{H}} = 0, \text{ for any } U_0 \in \mathcal{D}(\mathcal{A})$$

Since it is easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle [65] or the spectral decomposition theory of SzNagy-Foias, Foguel and Benchimol [56], [11], [7] are not applicable in this case. Then we will study the strong stability of system (II.0.1) by using a general criteria of Arendt-Batty [68]. Following this criteria, the C_0-semigroup of contractions $e^{t\mathcal{A}}$ isstronglystableif $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis, where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$.

First of all, let us look for the spectrum on the imaginary axis, and next analyse the behaviour of the resolvent on the imaginary axis.

Lemma II.1.1 *for all $\lambda \in \mathbb{R}$ we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\};$$

more precisely, there is no eigenvalue of $\mathcal{A}$ on the imaginary axis.

Proof: Let us set $U = (u, v, \eta, z,)^T$. Given $\lambda \in \mathbb{R}$, $U \in \ker(i\lambda I - \mathcal{A})$. It follows that

$$i\lambda U - \mathcal{A}U = 0; \tag{II.1.15}$$

which is equivalent to

$$\begin{cases} i\lambda u - v = 0 \quad \text{in} \quad \Omega \\ i\lambda v - \Delta u = 0 \quad \text{in} \quad \Omega \\ i\lambda \eta - v + \beta_1 \eta + \beta_2 z(.,1) = 0 \quad \text{on} \quad \Gamma_N \\ i\lambda z + \tau^{-1} z_\rho = 0 \quad \text{on} \quad \Gamma_N \times (0,1). \end{cases} \tag{II.1.16}$$

Recalling the dissipativity property (II.1.4)of $\mathcal{A}$, we get

$$0 = \Re\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \leq \Lambda_1 \|\eta\|^2_{L^2(\Gamma_N)} + \Lambda_2 \|z(.,1)\|^2_{L^2(\Gamma_N)} \leq 0, \tag{II.1.17}$$

and as Λ_1 and Λ_2 from (II.1.5), are negative constants, we deduce that

$$\begin{cases} \eta = 0 \quad \text{on} \quad \Gamma_N \\ z = 0 \quad \text{on} \quad \Gamma_N \times (0,1). \end{cases} \tag{II.1.18}$$

Combining (II.1.18) and the third equation of (II.1.16) gives

$$v = 0 \qquad \text{on} \qquad \Gamma_N. \tag{II.1.19}$$

Using now (II.1.19) in the first equation of (II.1.16) it follows that

$$v = 0 \qquad \text{on} \qquad \Gamma. \tag{II.1.20}$$

Furthermore, $U \in \mathcal{D}(\mathcal{A})$ implies that

$$\partial_v u = -\eta = 0 \quad \text{on} \quad \Gamma_N. \tag{II.1.21}$$

Combining the first and the second equation of (II.1.16) and using (II.1.20) and (II.1.21) we get

$$\begin{cases} \Delta u + \lambda^2 u = 0 \quad \text{in} \quad \Omega \\ u = 0 \quad 0\text{n} \quad \Gamma \\ \partial_v u = 0 \quad \text{on} \quad \Gamma_N. \end{cases} \tag{II.1.22}$$

Thanks to Holmgren's theorem (see [31]), the unique solution of system (II.1.22) is $u \equiv 0$. Finally we conclude that $U = 0$, the proof of lemma II.1.1 is thus completed. ■

Lemma II.1.2 *Equipped with the norm*

$$\|u\|_{D(\Delta)} = \|\nabla u\|_{L^2(\Omega)^n} + \|\Delta u\|_{L^2(\Omega)} + \|\partial_v u\|_{L^2(\Gamma_N)},$$

$D(\Delta)$ *is compactly embedded into* $H^1_{\Gamma_D}(\Omega)$

Proof: Let $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $D(\Delta)$ or equivalently

$$\|\nabla u_k\|_{L^2(\Omega)^n} + \|\Delta u_k\|_{L^2(\Omega)} + \|\partial_v u_k\|_{L^2(\Gamma_N)} \leq C, \forall k \in \mathbb{N}, \tag{II.1.23}$$

for a positive constant C. Then by Kondrachov embedding theorem and a trace theorem, there exist $u \in H^1_{\Gamma_D}(\Omega)$ and a subsequence, still denoted by $(u_k)_{k\in\mathbb{N}}$ for shortness such that

$$u_k \to u \quad \text{strongly in} \quad L^2(\Omega) \quad \text{as } k \to \infty, \tag{II.1.24}$$

$$u_k \to u \quad \text{strongly in} \quad L^2(\Gamma_N) \quad \text{as } k \to \infty. \tag{II.1.25}$$

Now by setting $f_k := \Delta u_k$ in Ω and $g_k := \partial_v u_k$ on Γ_N, we notice that

$$\int_\Omega |\nabla(u_k - u_\varrho)|^2 dx = -\int_\Omega (f_k - f_\varrho)\overline{(u_k - u_\varrho)}dx + \int_{\Gamma_N}(g_k - g_\varrho)\overline{(u_k - u_\varrho)}dx \ .$$

and by Cauchy-Schwarz's inequality and (II.1.23) we deduce that

$$\int_\Omega |\nabla(u_k - u_\varrho)|^2 dx \leq 2C\left(\|u_k - u_\varrho\|_{L^2(\Omega)} + \|u_k - u_\varrho\|_{L^2(\Gamma_N)}\right).$$

By (II.1.24) and (II.1.25), this right-hand side tends to zero as k goes to infinity and we deduce that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{\Gamma_D}(\Omega)$, hence the conclusion. ■

Lemma II.1.3 *For all* $\lambda \in \mathbb{R}$ *we have*

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof: Since we have already shown that $\mathcal{A}$ is surjective, it suffices to treat the case $\lambda \leq 0$. Hence let $\lambda \in \mathbb{R}^*$ and $F(f,g,h,k)^T \in \mathcal{H}$. Then we look for $U = (u,v,\eta,z)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(i\lambda I - \mathcal{A})U = F. \tag{II.1.26}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f \quad \text{in} \quad \Omega \\ i\lambda v - \Delta u = g \quad \text{in} \quad \Omega \\ i\lambda\eta - v + \beta_1\eta + \beta_2 z\,(.,1) = h \quad \text{on} \quad \Gamma_N \\ i\lambda z - \tau^{-1} z_\rho = k \quad \text{on} \quad \Gamma_N \times (0,1)\,. \end{cases} \tag{II.1.27}$$

Proceeding as in the proof of Proposition 2.1 we arrive to problem (II.1.13) with λ replaced by i_λ, its variational formulation being (II.1.14) with λ replaced by i_λ. Since $a_{i\lambda}$ is no more coercive in $H^1_{\Gamma_D}(\Omega)$, we use a compact perturbation argument. Namely we introduce the (unbounded) operator $\mathcal{A}_{i\lambda}$ from $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ to $\mathcal{H}$ by

$$\mathcal{A}_{i\lambda}\,(u,v,\eta,z)^T = \left(-v, -\Delta u, i\lambda\eta - v + \beta_1\eta + \beta_2 z\,(.,1)\,, i\lambda z + \tau^{-1}z_\rho\right)^T, \forall\,(u,v,\eta,z)^T \in \mathcal{D}(\mathcal{A})\,.$$

We first show that $\mathcal{A}_{i\lambda}$ is an isomorphism from $\mathcal{D}(\mathcal{A})$ to $\mathcal{H}$. Indeed given $F = (f,g,h,k)^T \in \mathcal{H}$, we look for $U = (u,v,\eta,z)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$\mathcal{A}_{i\lambda} U = F,$$

or equivalently,

$$\begin{cases} -v = f \quad \text{in} \quad \Omega \\ -\Delta u = g \quad \text{in} \quad \Omega \\ i\lambda\eta - v + \beta_1\eta + \beta_2 z\,(.,1) = h \quad \text{on} \quad \Gamma_N \\ i\lambda z + \tau^{-1} z_\rho = k \quad \text{on} \quad \Gamma_N \times (0,1)\,. \end{cases} \tag{II.1.28}$$

This directly yields $v = -f$, and as in the proof of Proposition 2.1, we find

$$z\,(.,\rho) = \eta e^{-i\lambda\tau\rho} + \tau e^{-i\lambda\tau\rho} \int_0^\rho k\,(.,\sigma)\, e^{i\lambda\tau\sigma} d\sigma.$$

Eliminating v and $z(\cdot,1)$ in the third identity of (II.1.28), we find

$$\eta = -\tilde{f}_{i\lambda} \quad \text{on} \quad \Gamma_N, \tag{II.1.29}$$

where $\tilde{f}_{i\lambda}$ was defined in (II.1.12). It follows that u verifies

$$\begin{cases} -\Delta u = g \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \Gamma_D \\ \partial_v u = \tilde{f}_{i\lambda} \quad \text{on} \quad \Gamma_N. \end{cases} \tag{II.1.30}$$

Since $g \in L^2(\Omega)$ and $\tilde{f}_{i\lambda}$ belongs to $L^2(\Gamma_N)$, it is well-known that a unique solution u exists in $H^1_{\Gamma_D}(\Omega)$, with the regularity

$$\Delta u \in L^2(\Omega)\,, \text{and } \partial_v u \in L^2(\Gamma_N)\,.$$

This furnishes a unique solution $(u, v, \eta, z)^T \in \mathcal{D}(\mathcal{A})$ of (II.1.28)
Now we notice that for $(u, v, \eta, z)^T \in \mathcal{D}(\mathcal{A})$, we have

$$((i\lambda I - \mathcal{A}) - \mathcal{A}_{i\lambda})(u, v, \eta, z)^T = (i\lambda u, i\lambda v, 0, 0)^T .$$

By Lemma II.1.2 and again the Kondrachov embedding theorem, we deduce that $(i\lambda I - \mathcal{A}) - \mathcal{A}_{i\lambda}$ is a compact operator from $D(\mathcal{A})$ (equipped with the graph norm) into $\mathcal{H}$, because (using the inequality $(a+b+c)^2 \leq 4(a^2+b^2+c^2)$ valid for all real numbers a, b, c)

$$\begin{aligned} \|u\|_{D(\Delta)} + \|\nabla v\|_{L^2(\Omega)^n} &= \|\nabla u\|_{L^2(\Omega)^n} + \|\Delta u\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Gamma_N)} + \|\nabla v\|_{L^2(\Omega)^n} \\ &\leq 2\left\|(u, v, \eta, z)^T\right\|_{\mathcal{H}} + \left\|\mathcal{A}(u, v, \eta, z)^T\right\|_{\mathcal{H}}. \end{aligned}$$

As $\mathcal{A}_{i\lambda}$ is an isomorphism, we deduce that $i\lambda I - \mathcal{A}$ is a Fredholm operator of index zero. As Lemma II.1.1 guarantees its injectivity, we deduce that it is also surjective. ■

Theorem II.1.1 *For any $U_0 \in \mathcal{H}$, the C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$.*

$$\lim_{t\to 0} \left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} = 0.$$

Proof: Lemma II.1.3 ensures that there is no eigenvalue of $\mathcal{A}$ on the imaginary axis, while Lemma II.1.1 guarantees that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \phi$.That achieves the proof of Theorem II.1.1. ■

II.2 Lack of exponential stability

In this section, we will show that the system (II.0.1) is not exponential stable. Our argument is based on a frequency domain approach for exponential stability (see Huang [17] and Prüss [32]), more precisely on the next result.

Lemma II.2.1 *A C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ of contractions on a Hilbert space $\mathcal{H}$ isexponentially stable, namely satisfies*

$$\left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} \leq Ce^{-\omega t}\|U_0\|_{\mathcal{H}} \quad \forall\, U_0 \in \mathcal{H},\ \forall\, t \geq 0, \tag{II.2.1}$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{A}) \supset \{i\beta, \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{II.2.2}$$

and

$$\sup \left\|(i\beta - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{II.2.3}$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator$\mathcal{A}$.

The main result of current section is the following.

Theorem II.2.1 *The system (II.0.1) is not exponentially stable in the energy space $\mathcal{H}$.*

Proof: As the condition (II.2.2) is guaranteed by Lemma II.1.1, it suffices now to built a contradiction following some techniques used in [70], [38], [8]. More precisely, we prove that the condition (II.2.3) is not satisfied in the sense that there exist a positive real number C and some sequences of $\lambda_n \in i\mathbb{R}$,$U_n = (u_n, v_n, \eta_n, z_n)^T \in D(\mathcal{A})$, and $F_n = (f_n, g_n, h_n, k_n)^T \in \mathcal{H}$, with $n \in \mathbb{N}$, such that

$$(\lambda_n - \mathcal{A}) U_n = F_n \ ,n \in \mathbb{N}, \tag{II.2.4}$$

$$\|U_n\|_{\mathcal{H}} \geq C, \forall n \in \mathbb{N}, \tag{II.2.5}$$

and

$$\lim_{n \to +\infty} \|F_n\|_{\mathcal{H}} = 0. \tag{II.2.6}$$

Following [8], theorem 3.1, let $(\mu_n^2)_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the Laplacian with Dirichlet boundary condition on Γ_D and Robin boundary condition on Γ_N repeated according to their mulitplicity) and let φ_n be its associated normalized eigenfunction, more precisely, solution of

$$\begin{cases} -\Delta\varphi_n = \mu_n^2\varphi_n \ \text{ in } \ \Omega, \\ \varphi_n = 0 \ \text{ on } \ \Gamma_D, \\ \partial_v\varphi_n + \varphi_n = 0 \ \text{ on } \ \Gamma_N, \end{cases} \tag{II.2.7}$$

with

$$\|\varphi_n\|_{L^2(\Omega)} = 1$$

Without loss of generality, we can assume that all μ_n are positive. Then for all $n \in N$, we chose

$$\lambda_n = i\mu_n, \quad u_n = \frac{\varphi_n}{\lambda_n}, \quad v_n = \varphi_n, \quad z_n(.,\rho) = \eta_n e^{-i\mu_n\tau\rho} \text{ and } \eta_n = \frac{\varphi_n}{\lambda_n}.$$

With this choice, we easily check that $(u_n, v_n, \eta_n, z_n)^T$ belongs to $D(\mathcal{A})$ since by the Robin condition on Γ_N in (II.2.7) and the definition of z_n , we have

$$z_n(.,0) = \eta_n = -\partial_v u_n = 0 \ \text{ on } \ \Gamma_N.$$

Further, we see that $(u_n, v_n, \eta_n, z_n)^T$ is solution of

$$\begin{cases} \lambda_n u_n - v_n = 0 \\ \lambda_n v_n - \Delta u_n = 0 \\ \lambda_n \eta_n - (v_n - \beta_1 \eta_n - \beta_2 z_n(.,1)) = h_n, \\ \lambda_n z_n + \tau^{-1} z_{n,\rho}(.,\rho) = 0, \end{cases} \tag{II.2.8}$$

where

$$h_n = \frac{\beta_1 + \beta_2 e^{-i\mu_n T}}{i\mu_n} \varphi_{n.}$$

Equivalently this means that $(u_n, v_n, \eta_n, z_n)^T$ is solution of (II.2.4) with $f_n = g_n = k_n = 0$ and h_n defined above. Now we notice that

$$\|U_n\|^2_{\mathcal{H}} = \|\nabla u_n\|^2_{L^2(\Omega)^n} + \|v_n\|^2_{L^2(\Omega)} + \|\eta_n\|^2_{L^2(\Gamma_N)} + \varsigma \|z_n\|^2_{L^2(\Gamma_N \times (0,1))} \geq \|v_n\|^2_{L^2(\Omega)} = \|\varphi_n\|^2_{L^2(\Omega)} = 1,$$

which proves (II.2.5)with $C = 1$. Further straightforward computations yield

$$\|F_n\|^2_{\mathcal{H}} = \|h_n\|^2_{L^2(\Gamma_N)} \leq \frac{(|\beta_1| + |\beta_2|)^2}{|\mu_n|^2} \|\varphi_n\|^2_{L^2(\Gamma_N)}.$$

But using the trace estimate of interpolation type (see [49], Theorem 1.5.1.10 or [71],Theorem 1.4.4]), there exists a positive constant C_{tr} (independent of n) such that

$$\int_{\Gamma_N} |\varphi_n(x)|^2 \, d\sigma(x) \leq C_{tr} \|\varphi_n\|_{H^1(\Omega)} \|\varphi_n\|_{L^2(\Omega)} \leq C_{tr} \left(1 + \mu_n^2\right)^{\frac{1}{2}}.$$

Inserting this estimate in the previous one, we find

$$\|F_n\|^2_{\mathcal{H}} \leq \frac{C_{tr} (|\beta_1| + |\beta_2|)^2 (1 + \mu_n^2)^{\frac{1}{2}}}{|\mu_n|^2} \to 0 \text{ as } n \to \infty. \tag{II.2.9}$$

Hence (II.2.6) is also satisfied. This shows that the resolvent of $\mathcal{A}$ is not uniformly bounded on the imaginary axis. ■

II.3 Rational stabilization result

In this section we will prove the rational stability of problem (II.0.1) using again a frequency domain approach, the rational stability of problem (II.0.1) without delay, and a duality argument. First of all we recall the following result due to Borichev and Tomilov [4]:

Theorem II.3.1 *Let $\mathcal{A}$ be the generator of a C_0-semigroup of bounded operators on a*

Hilbert space $\mathcal{H}$. Assume that

$$i\mathbb{R} \in \rho(\mathcal{A}). \tag{II.3.1}$$

Let l be a positive real number. Then we have the polynomial decay

$$\left\|e^{t\mathcal{A}}U_0\right\| \leq \frac{C}{t^{1/l}} \|U_0\|_{D(\mathcal{A})}, t > 0, U_0 \in D(\mathcal{A}), \tag{II.3.2}$$

if and only if

$$\lim_{|\lambda|\to+\infty} \sup \frac{1}{|\lambda^t|} \left\|(i\lambda - \mathcal{A})^{-1}\right\| < \infty. \tag{II.3.3}$$

Problem (II.0.1) without delay is the following one

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty), \\ \frac{\partial u}{\partial v}(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ \eta_t(x,t) - u_t(x,t) + \beta_1 \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0, \quad u_t(x,t) = u_1 \text{ in } \Omega, \\ \eta(x,0) = \eta_0 \quad \text{on } \Gamma_N. \end{cases} \tag{II.3.4}$$

This problem is well-posed in

$$\mathcal{H}_0 := H^1_{\Gamma_D}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N) \tag{II.3.5}$$

endowed with the norm

$$\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0} := \|\nabla u\|^2_{(L^2(\Omega))^n} + \|v\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Gamma_N)}. \tag{II.3.6}$$

The generator of its semigroup is $\mathcal{A}_0$ defined by

$$\mathcal{A}_0(u,v,\eta)^T := (v, \Delta u, \gamma(v, \beta_1\eta))^T \tag{II.3.7}$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{(u,v,\eta)^T \in \left(H^2(\Omega) \cap H^1_{\Gamma_D}(\Omega)\right) \times H^1_{\Gamma_D}(\Omega) \times L^2(\Gamma_N) : \partial_v u + \eta = 0 \text{ on } \Gamma_N\right\}. \tag{II.3.8}$$

The main result of this section is the following:

Theorem II.3.2 *Assume that the semigroup generated by $\mathcal{A}_0$ in $\mathcal{H}_0$ decays polynomially, namely*

$$\left\|e^{t\mathcal{A}_\prime}U_0\right\| \leq \frac{C}{t^{1/l}} \|U_0\|_{\mathcal{D}(\mathcal{A}_\prime)}, \forall U_0 \in \mathcal{D}(\mathcal{A}_0), \forall t > 0. \tag{II.3.9}$$

for some positive real number l. Then the semigroup of system (II.0.1) inherits the *same polynomial decay, namely*

$$\left\|e^{t\mathcal{A}}U_0\right\| \leq \frac{C}{t^{1/l}}\left\|U_0\right\|_{\mathcal{D}(\mathcal{A})}, \forall U_0 \in \mathcal{D}\left(\mathcal{A}\right), \forall t > 0. \tag{II.3.10}$$

Proof: As the condition (II.3.1) is already checked in lemma II.1.1, we only need to check the condition (II.3.3). So according to the above theorem, we will establish that for any $\lambda \in \mathbb{R}$ and $F = (f, g, h, k)^T \in \mathcal{H}$, the solution $U = (u, v, \eta, z)^T \in \mathcal{D}(\mathcal{A})$ of

$$\left(i\lambda I - \mathcal{A}\right)U = F \tag{II.3.11}$$

satisfies

$$\|U\|_{\mathcal{H}} \leq C\left(1 + |\lambda|^l\right)\|F\|_{\mathcal{H}}; \tag{II.3.12}$$

where $C$is positive constant (independent of λ and F). Now we consider the solution $U^* = (u^*, v^*, \eta^*)^T$ of

$$(i\lambda I - \mathcal{A}_0)\begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix} = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \tag{II.3.13}$$

that due to our assumption satisfies

$$\left\|(u^*, v^*, \eta^*)^T\right\|_{\mathcal{H}_0} \leq C_* |\lambda|^l \left\|(u, v, \eta)^T\right\|_{\mathcal{H}_0} \tag{II.3.14}$$

where C_* is a positive constant. Note further that the dissipativeness of $\mathcal{A}_0$ directly yields

$$\beta_1 \int_{\Gamma_N} |\eta^*|^2 d\Gamma \leq \Re\left(\left(i\lambda I - \mathcal{A}_0\right)U^*, U^*\right)_{\mathcal{H}_0} \leq \left\|(u, v, \eta)^T\right\|_{\mathcal{H}_0} \|U^*\|_{\mathcal{H}_0}. \tag{II.3.15}$$

On the other hand the system (II.3.13) can be equivalently written as

$$\begin{cases} i\lambda u^* - v^* = u \ \text{ on } \ \Omega \\ i\lambda v^* - \Delta u^* = v \ \text{ on } \ \Omega \\ i\lambda\eta^* - v^* + \beta_1\eta^* = \eta \ \text{ on } \ \Gamma_N. \end{cases} \tag{II.3.16}$$

Integrating by parts and using (II.3.16) we get

$$
\begin{aligned}
&\left\langle (i\lambda I - \mathcal{A}) \begin{matrix} u \\ v \\ \eta \\ z \end{matrix}, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} i\lambda u - v \\ i\lambda v - \Delta u \\ i\lambda\eta - v + \beta_1\eta + \beta_2 z\,(.,1) \\ i\lambda z + \tau^{-1} z_\rho \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ 0 \end{pmatrix} \right\rangle \\
&= \int_\Omega \nabla\,(i\lambda u - v)\,.\nabla\overline{u^*}dx + \int_\Omega (i\lambda v - \Delta u)\,\overline{v^*}dx + \int_{\Gamma_N} (i\lambda\eta - v + \beta_1\eta + \beta_2 z\,(.,1))\,\overline{\eta^*}d\Gamma \\
&= i\lambda\int_\Omega \nabla u.\nabla\overline{u^*}dx - \int_\Omega \nabla v.\nabla\overline{v^*}dx + i\lambda\int_\Omega v\overline{v^*}dx - \int_\Omega \Delta u.\nabla\overline{v^*}dx \\
&\qquad\qquad + \int_{\Gamma_N} (\lambda\eta - v + \beta_1\eta + \beta_2 z\,(.,1))\,\overline{\eta^*}d\Gamma \\
&= i\lambda\int_\Omega \nabla u.\nabla\overline{u^*}dx + \int_\Omega \Delta\overline{u^*}v dx + \int_{\Gamma_N} \overline{\eta^*}v d\Gamma + i\lambda\int_\Omega v\overline{v^*}dx + \int_\Omega \nabla u.\nabla\overline{u^*}dx + \int_{\Gamma_N} \eta\overline{v^*}d\Gamma \\
&\qquad\qquad + \int_{\Gamma_N} (i\lambda\eta - v + \beta_1\eta + \beta_2 z\,(.,1))\,\overline{\eta^*}d\Gamma \\
&= \int_\Omega \nabla u.\nabla\overline{(-i\lambda u^* + v^*)}dx + \int_\Omega v\overline{(-i\lambda v^* + \Delta u^*)\,dx} + \int_{\Gamma_N} \eta\overline{(-i\lambda\eta^* + v^* - \beta_1\eta^*)}d\Gamma \\
&\qquad\qquad + \int_{\Gamma_N} (2\beta_1\eta + \beta_2 z\,(.,1))\,\overline{\eta^*}d\Gamma \\
&= -\int_\Omega |\nabla u|^2\,dx - \int_\Omega |v|^2\,dx - \int_{\Gamma_N} |\eta|^2\,d\Gamma + \int_{\Gamma_N} (2\beta_1\eta + \beta_2 z\,(.,1))\,\overline{\eta^*}d\Gamma.
\end{aligned}
$$

Recalling (II.3.11) and using (II.3.6), the above relation can be rewritten as

$$
\left\|(u, v, \eta)^T\right\|^2_{\mathcal{H}_0} = -\left\langle F, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ 0 \end{pmatrix} \right\rangle + \int_{\Gamma_N} (2\beta_1\eta + \beta_2 z\,(.,1))\,\overline{\eta^*}d\Gamma. \tag{II.3.17}
$$

Applying Cauchy-Schwarz's and Young's inequalities, we get

$$
\left\|(u, v, \eta)^T\right\|^2_{\mathcal{H}_0} \le \|F\|_{\mathcal{H}} \left\|(u^*, v^*, \eta^*)^T\right\|_{\mathcal{H}_0} + \frac{4\beta_1^2}{\varepsilon}\|\eta\|^2_{L^2(\Gamma_N)} + \frac{\beta_2^2}{\varepsilon}\|z\,(.,1)\|^2_{L^2(\Gamma_N)} + \varepsilon\|\eta^*\|^2_{L^2(\Gamma_N)}, \tag{II.3.18}
$$

for all $\varepsilon < 0$. Furthermore, using the dissipativity property (II.1.4) of $\mathcal{A}$, we have

$$
\Re\left\langle (i\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle = -\Re\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \ge -\Lambda_1\|\eta\|^2_{L^2(\Gamma_N)} - \Lambda_2\|z\,(.,1)\|^2_{L^2(\Gamma_N)};
$$

where we recall that Λ_1 and Λ_2 are negative constants. Then Cauchy-Schwarz's inequality guarantees that

$$\|\eta\|^2_{L^2(\Gamma_N)} + \|z(.,1)\|^2_{L^2(\Gamma_N)} \leq (\min\{-\Lambda_1, -\Lambda_2\})^{-1} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{II.3.19}$$

This estimate in (II.3.18) yields

$$\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0} \leq \|F\|_{\mathcal{H}} \left\|(u^*,v^*,\eta^*)^T\right\|_{\mathcal{H}_0} + \frac{C}{\varepsilon}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \varepsilon\|\eta^*\|^2_{L^2(\Gamma_N)};$$

for all $\varepsilon > 0$, and a positive constant C independent of ε. At this stage we make use of (II.3.14) and (II.3.15) to find

$$\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0} \leq C_* \|F\|_{\mathcal{H}} |\lambda|^l \left\|(u,v,\eta)^T\right\|_{\mathcal{H}_0} + \frac{C}{\varepsilon}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \frac{C_* |\lambda|^l \varepsilon}{\beta_1}\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0},$$

or all $\varepsilon > 0$, and a positive constant Cindependent of ε and λ. Using again Young's nequality in the first term of this right-hand side, we arrive at

$$\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0} \leq \frac{C_*^2}{4\delta}\|F\|^2_{\mathcal{H}} + \delta|\lambda|^{2l}\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0} + \frac{C}{\varepsilon}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \frac{C_*|\lambda|^l\varepsilon}{\beta_1}\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0},$$

for all $\varepsilon, \delta > 0$.Now by fixing ε and δ so that $\frac{C_*|\lambda|^l\varepsilon}{\beta_1} = \frac{1}{4}$ and $\delta|\lambda|^{2l} = \frac{1}{4}$, we get

$$\left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0} \leq C_1\left(|\lambda|^{2l}\|F\|^2_{\mathcal{H}} + |\lambda|^l\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\right), \tag{II.3.20}$$

with a positive constant C_1 independent of λ.

Now by its definition, we have

$$\|U\|^2_{\mathcal{H}} = \left\|(u,v,\eta)^T\right\|^2_{\mathcal{H}_0} + \zeta\|z\|^2_{L^2(\Gamma_N\times(0,1))}, \tag{II.3.21}$$

therefore it remains to estimate the L^2-norm of z. But it is easy to check that (compare with (II.1.8))

$$z(.,\rho) = \eta e^{-i\lambda\tau\rho} + \tau e^{-i\lambda\tau\rho}\int_0^\rho k(.,\sigma)\, e^{i\lambda\tau\rho} d\sigma,$$

hence we have

$$\|z\|^2_{L^2(\Gamma_N\times(0,1))} \leq \|\eta\|_{L^2(\Gamma_N)} + \tau\|k\|_{L^2(\Gamma_N\times(0,1))}.$$

By (II.3.19), we obtain

$$\|z\|^2_{L^2(\Gamma_N\times(0,1))} \leq 2(\min\{-\Lambda_1,-\Lambda_2\})^{-1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + 2\tau^2\|F\|^2_{\mathcal{H}}.$$

Inserting this estimate in (II.3.21) and using (II.3.20), we obtain

$$\|U\|_{\mathcal{H}}^2 \leq C_2 \left(\left(1+|\lambda|^{2l}\right) \|F\|_{\mathcal{H}}^2 + \left(1+|\lambda|^{l}\right) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \right),$$

with a positive constant C_2 independent of λ. Applying Young's inequality, we obtain (II.3.12). The proof of Theorem II.3.2 is thus completed. ■

To the best of our knowledge the polynomial decay (II.3.9) for system (II.3.4) is not true in a general setting. Since it holds under some geometrical assumptions (described below), our previous result allows to get the same decay rate for our system (II.0.1).

Corollary II.3.1 *In addition to the assumptions from section* 1*, assume that there exists* $x_0 \in \mathbb{R}^n$ *such*

$$\begin{aligned} (x-x_0).\nu(x) &> 0, \quad \forall x \in \overline{\Gamma_N}, \\ (x-x_0).\nu(x) &\leq 0, \quad \forall x \in \overline{\Gamma_D}. \end{aligned}$$

Then (II.3.10) holds with $l=2$.

Proof: With our additional assumptions, the results from Theorem 3.2 of [8] (see also [38] and Theorem 2.1 of [2] if $n=1$) guarantee that (II.3.9) holds with $l=2$, hence the conclusion directly follows from Theorem II.3.2 ■

Corollary II.3.2 *In addition to the assumptions from section 1, suppose that one of the following assumptions holds:*

1) *the boundary of* Ω*is of class* C^∞ *and* Γ_N *satisfies the Geometric Control Condition (GCC). Recall [10] that the GCC can be formulated as follows:* Γ_N *satisfies the Geometric Control Condition if there exists* $T>0$ *such that every geodesic traveling at speed one issued from* Ω *at time* $t=0$ *intersects* Γ_N *before time* Γ,

2) *the boundary of* Ω *is of class* C^∞ *and there exists a vector field* $h \in C^2\left(\bar{\Omega}, \mathbb{R}^n\right)$ *such that*

$$h(x).\nu(x) \leq 0, \quad \forall x \in \overline{\Gamma_D},$$

and the symetrized Jacobian matrix $J_h + J_h^T$ *is strictly positive definite in* $\overline{\Omega}$.

3) *the boundary of* Ω *is of class* C^2 *and there exists a vector field* $h \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$ *such that*

$$\begin{aligned} h(x).\nu(x) &> 0, \quad \forall x \in \overline{\Gamma_N}, \\ h(x).\nu(x) &\leq 0, \quad \forall x \in \overline{\Gamma_D}, \end{aligned}$$

and the symetrized Jacobian matrix $J_h + J_h^T$ *is strictly positive definite in* Ω. *Then (II.3.9) holds with* $l=3$.

Proof: First we notice that by [[10], §5] in case 1., [25], Theorem 1.2 in case 2., and [30], Theorem 1 in case 3., the wave equation with standard damping on Γ_N:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \ \text{ in } \ \Omega \times (0,+\infty), \\ u(x,t) = 0 \ \text{ on } \ \Gamma_D \times (0,+\infty), \\ \frac{\partial u}{\partial v}(x,t) + u_t(x,t) = 0 \ \text{ on } \ \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \ \text{ in } \Omega, \end{cases} \tag{II.3.22}$$

is exponentially stable. Hence we apply [72], Proposition 2.2 with $B = -\beta_1$ and $C = -1$ to deduce that (II.3.9) holds with $l = 3$. Again the conclusion directly follows from TheoremII.3.2 ■

Corollary II.3.3 *In addition to the assumptions from section 1, assume that* $\Omega = (0,1)^2$ *is the unit square of* $\mathbb{R}^2$ *and* $\Gamma_N = \{1\} \times (0,1)$*. Then (II.3.10) holds with* $l = 5$*.*

Proof: Due to [72], Remark 2.6 and the arguments stated in the proof of the previous Corollary, (II.3.9) holds with $l = 5$, hence the conclusion directly follows from Theorem II.3.2 ■

Chapter III

Decay solution and boundary control of memory type for some wave equation

Contents

In this chaper we will study a wave equation, in a bounded domain, where the memory-type damping is acting on a part of the boundary

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0 \text{ in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 \text{ on } \Gamma_0 \times \mathbb{R}^+ \\ u(x,t) = -\int_0^t g(t-s) \frac{\partial u}{\partial v}(s)\, ds \text{ on } \Gamma_1 \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{III.0.1}$$

With the following hypothesis

H1) Ω is an open bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and υ is the unit outer normal vector,

H2) $f \in C^1(\mathbb{R}^n)$ is a function satisfying

$$uf(u) \geq bF(u) \geq 0, \quad \text{for } b > 2, \quad F(u) = \int_0^u f(\xi)\, d\xi \tag{III.0.2}$$

with

$$F(u) \leq d|u|^p, \quad \forall u \in \mathbb{R}, \tag{III.0.3}$$

for some constant $d > 0$ and $p \geq 1$ such that $(n-2)\,p \leq n$.

H3) The partition Γ_0 and Γ_1 are closed, disjoint, with $meas\,(\Gamma_0) > 0$ and satisfying

$$\begin{aligned} \Gamma_0 &= \{x \in \partial\Omega : \nu.m\,(x) \leq 0\} \\ \Gamma_1 &= \{x \in \partial\Omega : \nu.m\,(x) > 0\}. \end{aligned} \tag{III.0.4}$$

where $m\,(x) = x - x^0$, for some $x^0 \in \mathbb{R}^n$.

Finally, we establish a general decay result, from which the usual exponential and polynomial decay rates are only special cases. Our work allows certain relaxation functions which are not necessarily of exponential or polynomial decay and, therefore, generalizes and improves earlier results in the literature.

Here, we are concerned with the following problem

Remark III.0.1 *An example of a function satisfying III.0.2 and III.0.3 is*

$$f\,(u) = |u|^{\gamma-2}\,u, \gamma > 2.$$

This work is divided into three sections. In Section 2 we state, without proof, an existence result of solutions to system (III.0.1) and present some material needed for the proof of our main result. In particular, we establish some relations between the relaxation function $g$and the corresponding resolvent kernel k similar to, but more general than, those usually found in the literature. In Section 3 our main result is stated and proved.

III.1 Some notations and results

In this section we introduce some notations and discuss the existence of solutions to system (III.0.1). First, we exploit (III.0.1)$_3$ to estimate the boundary term $\frac{\partial u}{\partial v}$.

Defining the convolution product operator by

$$(g * \varphi)\,(t) = \int_0^t g\,(t-s)\,\varphi\,(s)\,ds$$

and differentiating equation (III.0.1)$_3$, we obtain

$$\frac{\partial u}{\partial v} + \frac{1}{g\,(0)}\left(g^{'} * \frac{\partial u}{\partial v}\right) = -\frac{1}{g\,(0)}u_t \quad \text{on } \Gamma_1 \times \mathbb{R}^+.$$

Applying Volterra's inverse operator, we get

$$\frac{\partial u}{\partial v} = -\frac{1}{g\,(0)}\,(u_t + k * u_t) \quad \text{on } \Gamma_1 \times \mathbb{R}^+,$$

where the resolvent kernel k satisfies

$$k + \frac{1}{g(0)}\left(g^{'} * k\right) = -\frac{1}{g\left(0\right)}g^{'} \quad \text{on } \Gamma_1 \times \mathbb{R}^+.$$

Denoting by $\eta = \frac{1}{g(0)}$, we arrive at

$$\frac{\partial u}{\partial v} = -\eta\left(u_t + k\left(0\right)u - k\left(t\right)u^0 + k^{'} * u\right) \quad \text{on } \Gamma_1 \times \mathbb{R}^+. \tag{III.1.1}$$

Reciprocally, if $u_0 = 0$ on Γ_1,(III.1.1) implies (III.0.1)$_3$.

Since we are interested in relaxation functions of more general decay, we would like to know if the resolvent kernel k, involved in (III.1.1), inherits some properties of the relaxation function involved in (III.0.1)$_3$. The following Lemma answers this question.

Let $h = [0, +\infty) \to \mathbb{R}_+$ be continuous. Let k be its resolvent, that is

$$k\left(t\right) = h\left(t\right) + \left(k * h\right)\left(t\right). \tag{III.1.2}$$

It is well known that k is continuous and positive (see [45]).

Lemma III.1.1 *Suppose that*

$$h\left(t\right) \leq c_0 e^{-\int_0^t \gamma(\zeta)d\zeta}$$

where $\gamma : [0, +\infty) \to \mathbb{R}_+$, *is a nonincreasing function satisfying for some positive constant* $\varepsilon < 1$,

$$c_1 = \int_0^{+\infty} e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta} ds < \frac{1}{c_0}.$$

Then k *satisfies*

$$k\left(t\right) \leq \frac{c_0}{1 - c_0 c_1} e^{-\varepsilon\int_0^t \gamma(\zeta)d\zeta}.$$

Proof: Let $\delta\left(t\right) = \varepsilon\gamma\left(t\right)$ and denote by

$$k\left(t\right) = k\left(t\right)e^{\int_0^t \delta(\zeta)d\zeta}, H\left(t\right) = h\left(t\right)e^{\int_0^t \delta(\zeta)d\zeta}.$$

By multiplying (III.1.2) by $e^{\int_0^t \delta(\zeta)d\zeta}$, we obtain

$$\begin{aligned} k\left(t\right) &= H\left(t\right) + \int_0^t [e^{\int_0^t \delta(\zeta)d\zeta} e^{-\int_0^{t-s} \delta(\zeta)d\zeta} k\left(t-s\right)h\left(s\right)]ds \\ &= H\left(t\right) + \int_0^t [e^{\int_{t-s}^t \delta(\zeta)d\zeta} e^{-\int_0^s \gamma(\zeta)d\zeta} k\left(t-s\right)e^{\int_0^s \gamma(\zeta)d\zeta} h\left(s\right)]ds \\ &\leq c_0 + c_0 \sup_{0\leq r\leq t} k\left(r\right)\int_0^t e^{-\int_0^s [\gamma(\zeta) - \varepsilon\gamma(\zeta+t-s)]d\zeta} ds. \end{aligned}$$

By using the fact that γ is nonincreasing we arrive at

$$\begin{aligned} k(t) &\leq c_0 + c_0 \sup_{0\leq r\leq t} k(r) \int_0^t e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta} ds \\ &\leq c_0 + c_0 \sup_{0\leq r\leq T} k(r) \int_0^{+\infty} e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta} ds, \quad \forall t \leq T. \end{aligned}$$

which gives

$$\begin{aligned} \sup_{0\leq r\leq T} k(r) &\leq c_0 + c_0 \sup_{0\leq r\leq T} k(r) \int_0^{+\infty} e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta} ds \\ &\leq c_0 + c_1 c_0 \sup_{0\leq r\leq T} k(r) \end{aligned}$$

Consequently,

$$\sup_{0\leq r\leq T} k(r) \leq \frac{c_0}{1-c_0c_1}, \quad \forall T > 0$$

hence

$$k(t) \leq \frac{c_0}{1-c_0c_1}.$$

Therefore

$$k(t) \leq \frac{c_0}{1-c_0c_1} e^{-\varepsilon\int_0^t \gamma(\zeta)d\zeta}.$$

■

Remark III.1.1 *The result of [45] is is only a special case.*

Corollary III.1.1 *Suppose that*

$$h(t) \leq c_0 e^{-\gamma t}$$

if $\gamma > c_0$. *then there exists a positive constant* $\varepsilon < 1$ *such that*

$$k(t) \leq \beta e^{-\varepsilon\gamma t} \tag{III.1.3}$$

where $\beta > 0$ *is a constant*

Proof: It is easy to verify that

$$\int_0^{+\infty} e^{-(1-\varepsilon)\gamma s} ds = \frac{1}{(1-\varepsilon)\gamma} < \frac{1}{c_0}$$

if ε is chosen small. Thus (III.1.3) is a direct result of the lemma. ■

Corollary III.1.2 *Suppose that*

$$h(t) \leq \frac{c_0}{(1+t)^p}$$

for $c_0 < p-1$. *Then, there exists a positive constant* $\varepsilon < 1$ *such that :*

$$k(t) \leq \frac{\beta}{(1+t)^{\varepsilon p}} \tag{III.1.4}$$

where$\beta > 0$ *is a constant.*

Proof: We take $\gamma(\zeta) = \frac{p}{1+\zeta}$. It is easy to verify that

$$e^{-(1-\varepsilon)p\int_0^s \frac{d\zeta}{1+\zeta}} = \frac{1}{(1+\zeta)^{(1-\varepsilon)p}}$$

and

$$\int_0^{+\infty} e^{-(1-\varepsilon)p\int_0^s \frac{d\zeta}{1+\zeta}} ds = \int_0^{+\infty} \frac{1}{(1+\zeta)^{(1-\varepsilon)p}} ds = \frac{1}{(1-\varepsilon)p-1} < \frac{1}{c_0}$$

if ε is chosen small. Thus (III.1.4) is a direct result of the lemma.. ■

Example III.1.1 *If we take*

$$\gamma(\zeta) = a\zeta^p, \quad -1 < p < 0$$

and assume that

$$h(t) \leq c_0 e^{-\frac{a}{p+1}t^{p+1}}$$

then with an appropriate choice of $a > 0$*, one can easily see that, for some positive constant* $\varepsilon < 1$*;*

$$c_1 = \int_0^{+\infty} e^{-\frac{a(1-\varepsilon)}{p+1}s^{p+1}ds} < \frac{1}{c_0}.$$

Consequently, we get

$$k(t) \leq \beta e^{-\frac{\varepsilon a}{p+1}t^{p+1}}.$$

Based on Lemma III.1.1, we will use the boundary relation (III.1.1) instead of (III.0.1)$_3$.

Let us define

$$(g \circ \varphi)(t) := \int_0^t g(t-s)\left|\varphi(t) - \varphi(s)\right|^2 ds \tag{III.1.5}$$

and

$$(g \odot \varphi)(t) := \int_0^t g(t-s)(\varphi(t) - \varphi(s))ds. \tag{III.1.6}$$

Using the inequality of Hölder, we have

$$\left|(g \odot \varphi)(t)\right|^2 \leq \left(\int_0^t |g(s)|\, ds\right)(|g| \circ \varphi)(t). \tag{III.1.7}$$

Lemma III.1.2 *(see [40]-[42], [45]). If $g, \varphi \in C^1(R^+)$ then*

$$(g * \varphi)\varphi_t = -\frac{1}{2}g(t)|\varphi(t)|^2 + \frac{1}{2}g' \circ \varphi - \frac{1}{2}\frac{d}{dt}\left(g \circ \varphi - \left(\int_0^t g(s)\,ds\right)|\varphi(t)|^2\right). \quad \text{(III.1.8)}$$

The well-posedness of system (III.0.1) is presented in the following theorem

Theorem III.1.1 *Let $k \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $u_0 \in (H^2(\Omega) \cap V)$, and $u_1 \in V$ with*

$$\frac{\partial u_0}{\partial v} + \eta u_0 = 0 \text{ on } \Gamma_1. \text{ and } V = \{v \in H^1(\Omega) : v = 0 on \Gamma_0\}. \quad \text{(III.1.9)}$$

Assume that (III.0.2)–(III.0.4) hold. Then there exists a unique strong solution u of system (III.0.1) such that

$$\begin{aligned} u &\in L^\infty\left(\mathbb{R}^+; H^2(\Omega) \cap V\right), u_t \in L^\infty\left(\mathbb{R}^+; L^2(\Omega)\right) \\ u_t &\in L^\infty\left(\mathbb{R}^+; V\right), u_{tt} \in L^\infty\left(\mathbb{R}^+; L^2(\Omega)\right) \end{aligned}$$

Proof: This theorem can be proved, using the Galerkin method and following exactly the procedure of [41],[42]. ■

III.2 Some Lemmas

In this section we study the asymptotic behavior of the solutions of system (III.0.1) when the resolvent kernel k satisfy

$$k(0) > 0, k(t) \geq 0, k'(t) \leq 0, k''(t) \geq \gamma(t)\left(-k'(t)\right) \quad \text{(III.2.1)}$$

Where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the following conditions

$$\gamma(t) > 0, \gamma'(t) \geq 0, \text{and} \int_0^{+\infty} \gamma(t)\,dt = +\infty. \quad \text{(III.2.2)}$$

Example III.2.1 *Let*

$$k(t) = \frac{e^{-t}}{(1+t)^a}, t > 0, a > 0.$$

Direct computations show that

$$k''(t) = \gamma(t)\left(-k'(t)\right)$$

with

$$\gamma(t) = 1 + \frac{a}{t+1+a} + \frac{a(a+1)}{(t+1)(t+1+a)}.$$

By multiplying Eq. (III.0.1)$_1$ by u_t and integrating over Ω; using integration by parts, the boundary conditions, and (III.1.8), one can easily find that the first order energy of system (III.0.1) is given by (see Lemma III.2.1 and its proof)

$$E(t) := \frac{1}{2}\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) dx + \int_\Omega F(u)\,dx + \frac{\eta}{2}\int_{\Gamma_1}\left(k(t)|u|^2 - k' \circ u\right) d\Gamma_1. \quad \text{(III.2.3)}$$

Assume that we have

$$\lim_{t\to\infty} k(t) = 0. \quad \text{(III.2.4)}$$

Remark III.2.1 *Assumption (III.2.4) can be replaced by $\|k\|_\infty$ small enough as in [45].*

The main idea is to construct a Lyapunov functional $\mathcal{L}(t)$ equivalent to $E(t)$. To do this we use the multiplier techniques. The proof of Theorem III.3.1 will be achieved with the help of two lemmas.

Lemma III.2.1 *Under the assumptions of Theorem III.3.1, the energy of the solution of (III.0.1) satisfies*

$$\frac{dE}{dt} \leq -\frac{\eta}{2}\int_{\Gamma_1}|u_t|^2 d\Gamma_1 + \frac{\eta}{2}\int_{\Gamma_1}k^2(t)|u_0|^2 d\Gamma_1 - \frac{\eta}{2}\int_{\Gamma_1}k'' \circ u d\Gamma_1. \quad \text{(III.2.5)}$$

Proof: Multiplying Eq. (a)$_1$ by u_t and integrating by parts over Ω, we obtain

$$\frac{d}{2dt}\int_\Omega \left[|u_t|^2 + |\nabla u|^2 + 2F(u)\right] dx = \int_{\Gamma 1}\frac{\partial u}{\partial v}u_t d\Gamma_1.$$

Using (III.1.1), and Lemma III.1.2, we obtain the desired result. ■

Remark III.2.2 *a) If $u_0 = 0$ on Γ_1 .Then $E(t) \leq E(0)$.*

b) If $u_0 \neq 0$ on Γ_1, then

$$E(t) \leq E(0) + \frac{\eta}{2}\int_{\Gamma_1}|u_0|^2 d\Gamma_1 \int_0^t k^2(t)\,dt.$$

Lemma III.2.2 *Under the assumptions of Theorem III.3.1, the solution of (III.0.1) satisfies*

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega (2m.\nabla u + (n-\varepsilon_0)u)u_t dx\right] \\
&\leq \int_{\Gamma_1} m.\nu |u_t|^2 d\Gamma_1 - \varepsilon_0\int_\Omega \frac{\partial u}{\partial \nu}(2m.\nabla u + (n-\varepsilon_0)u)\,d\Gamma_1 \\
&- \int_{\Gamma_1} m.\nu |\nabla u|^2 d\Gamma_1 - (1-\varepsilon_0)\int_\Omega |\nabla u|^2 dx - [(b-2)n - \varepsilon_0 b]\int_\Omega F(u), \\
&\forall t \geq 0.
\end{aligned}$$

for some $0 < \varepsilon_0 < 1$.

Proof: We multiply Eq.(III.0.1)$_1$ by $2m.\nabla u + (n-\varepsilon_0)\,u$ we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega (2m.\nabla u + (n-\varepsilon_0)\,u)\,u_t dx\right] \\
&= \int_\Omega 2u_t m.\nabla u_t dx + (n-\varepsilon_0)\int_\Omega |u_t|^2\,dx + \int_\Omega 2\,(\Delta u)\,m.\nabla u d \\
&+ (n-\varepsilon_0)\int_\Omega u\Delta u dx - \int_\Omega (2m.\nabla u + (n-\varepsilon_0)\,u)\,f(u)dx.
\end{aligned}$$

By integrating by parts and using (III.0.2), (III.0.4), and the relation $\operatorname{div}(m) = n$ we get

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega (2m.\nabla u + (n-\varepsilon_0)\,u)\,u_t dx\right] \\
&\leq \int_{\Gamma_1} (m.\nu)\,|u_t|^2\,d\Gamma_1 - \varepsilon_0\int_\Omega |u_t|^2\,dx + \int_{\Gamma_1}\frac{\partial u}{\partial \nu}\,(2m.\nabla u + (n-\varepsilon_0)\,u)\,d\Gamma_1 \\
&- \int_{\Gamma_1} (m.\nu)\,|\nabla u|^2\,d\Gamma_1 - (1-\varepsilon_0)\int_\Omega |\nabla u|^2\,dx \\
&- [(b-2)\,n - \varepsilon_0 b]\int_\Omega F(u)\,dx - \int_{\Gamma_1} (m.\nu)\,F d\Gamma_1.
\end{aligned}$$

By recalling (III.0.4), the proof of Lemma III.2.2 is completed. ■

III.3 Decay of solutions of system (III.0.1)

In this section we establish a general decay theorem for our system (III.0.1)

Theorem III.3.1 *Under the hypothesis (III.0.2)-(III.0.4),(III.2.1), (III.2.2) and (III.2.4) Then for any $(u_0, u_1) \in (V \times L^2(\Omega))$ and for some t_0 large enough, we have, $\forall t \geq t_0$,*

$$E(t) \leq cE(0)\,e^{-a\int_0^t \gamma(s)ds}, \quad \text{if } u_0 = 0 \text{ on } \Gamma_1. \tag{III.3.1}$$

Otherwise,

$$E(t) \leq c\left\{E(0) + \left(\int_{\Gamma_1} |u^0|^2\,d\Gamma_1\right)\int_0^t k^2(s)\left[1 + e^{a\int_{t_0}^s \gamma(\zeta)d\zeta}\right]ds\right\} e^{-a\int_0^t \gamma(s)ds}, \tag{III.3.2}$$

where a is a fixed positive constant and c is a generic positive constant.

Proof: Now, we introduce the Lyapunov functional. So, for $N > 0$ large enough, let

$$£ (t) = NE (t) + \int_{\Omega}[(2m.\nabla u + (n - \varepsilon_0) u) u_t]dx. \tag{III.3.3}$$

It is a routine calculation to check that, for N large, we have

$$\frac{N}{2}E (t) \leq £ (t) \leq 2NE (t) . \tag{III.3.4}$$

Applying Young's inequality and Poincaré's inequality to the boundary integral we have, for $\varepsilon > 0$,

$$\int_{\Gamma_1} \frac{\partial u}{\partial \nu} (2m.\nabla u + (n - \varepsilon_0) u) d\Gamma_1 \leq \varepsilon \int_{\Omega} |\nabla u|^2 dx + \varepsilon \int_{\Gamma_1} m.\nu |\nabla u|^2 d\Gamma_1 + C_\varepsilon \int_{\Gamma_1} \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma_1.$$

By rewriting the boundary condition (III.1.1) as

$$\frac{\partial u}{\partial \nu} = -\eta_1 \left(u_t + k (t) u - k (t) u_0 - k' \odot u\right) \quad \text{on } \Gamma_1 \times \mathbb{R}^+$$

and combining all the above relations, we arrive at

$$\begin{aligned} £' (t) \leq & - \left(\frac{N\eta}{2} - C_\varepsilon - m.\nu\right) \int_{\Gamma_1} |u_t|^2 d\Gamma_1 + \frac{N\eta}{2} k^2 (t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1 \\ & - (1 - \varepsilon) \int_{\Gamma_1} (m.\nu) |\nabla u|^2 d\Gamma_1 - \frac{\eta N}{2} \int_{\Gamma_1} k'' \circ u d\Gamma_1 - \left(1 - \varepsilon_0 - \varepsilon - C_\varepsilon k^2 (t)\right) \int_{\Omega} |\nabla u|^2 dx \\ & - \varepsilon_0 \int_{\Omega} |u_t|^2 dx - [(b - 2) n - \varepsilon_0 b] \int_{\Omega} F (u) dx + C_\varepsilon \int_{\Gamma_1} \left|k' \odot u\right|^2 d\Gamma_1 + C_\varepsilon k^2 (t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1. \end{aligned} \tag{III.3.5}$$

At this point, we take

$$\varepsilon = \varepsilon_0 < \min\{\frac{1}{4}, \frac{(b - 2) n}{b}\}.$$

Once ε is fixed (hence C_ε), we pick N large enough so that

$$\frac{N\eta}{2} - C_\varepsilon - \max_{\Gamma_1} |m.\nu| > 0.$$

By using the fact that $\lim_{t\to\infty} k (t) = 0$, Poincaré's inequality, and (III.1.7), we arrive at

$$£' (t) \leq -\alpha E (t) + \beta k^2 (t) \int_{\Gamma_1} \left|u^0\right|^2 d\Gamma_1 - \frac{\eta N}{2} \int_{\Gamma_1} k'' \circ u d\Gamma_1 - C \int_{\Gamma_1} k' \circ u d\Gamma_1$$

$$\leq -\alpha E(t) + \beta \int_{\Gamma_1} k^2(t) |u_0|^2 d\Gamma_1 - C \int_{\Gamma_1} k' \circ u d\Gamma_1, \quad \forall t \geq t_0 \tag{III.3.6}$$

for some t_0 large enough and some positive constants α, β and C.

We multiply both sides of (III.3.6) by $\gamma(t)$ to get

$$\gamma(t) \pounds'(t) \leq -\alpha\gamma(t) E(t) + \beta\gamma(t) \int_{\Gamma_1} k^2(t) \left|u^0\right|^2 d\Gamma_1 - \gamma(t) C \int_{\Gamma_1} k' \circ u d\Gamma_1, \quad \forall t \geq t_0.$$

A simple calculation, using the fact that $\gamma(t)$ is nonincreasing, yields

$$\gamma(t) \pounds'(t) \leq -\alpha\gamma(t) E(t) + \beta\gamma(t) \int_{\Gamma_1} k^2(t) |u_0|^2 d\Gamma_1 + c \int_{\Gamma_1} k'' \circ u d\Gamma_1, \quad \forall t \geq t_0.$$

By using (III.2.5), we easily see that

$$\gamma(t) \pounds'(t) \leq -\alpha\gamma(t) E(t) + c \int_{\Gamma_1} k^2(t) |u_0|^2 d\Gamma_1 - cE'(t), \quad \forall t \geq t_0$$

which yields

$$\gamma(t) \pounds'(t) + cE'(t) \leq -\alpha\gamma(t) E(t) + c \int_{\Gamma_1} k^2(t) |u_0|^2 d\Gamma_1, \quad \forall t \geq t_0$$

or

$$\frac{d}{dt}(\gamma(t) \pounds(t) + cE(t)) - \gamma'(t) \pounds(t) \leq -\alpha\gamma(t) E(t) + c \int_{\Gamma_1} k^2(t) |u_0|^2 d\Gamma_1, \quad \forall t \geq t_0. \tag{III.3.7}$$

Again using the fact that $\gamma(t)$ is nonincreasing and setting

$$F(t) = \gamma(t) \pounds(t) + cE(t) \sim E(t) \tag{III.3.8}$$

estimate (III.3.7) gives

$$F'(t) \leq -\alpha\gamma(t) F(t) + c \int_{\Gamma_1} k^2(t) |u_0|^2 d\Gamma_1, \quad \forall t \geq t_0. \tag{III.3.9}$$

Case 1: If $u_0 = 0$ on Γ_1, then (III.3.9) reduces to

$$\frac{dF}{dt} \leq -\alpha\gamma(t) F(t), \quad \forall t \geq t_0.$$

A simple integration over (t_0, t) yields

$$F(t) \leq F(t_0) e^{-\alpha \int_{t_0}^{t} \gamma(s) ds}, \quad \forall t \geq t_0.$$

By using (III.3.8), then we obtain for some positive constant c

$$E(t) \leq cE(t_0)\, e^{-\alpha \int_{t_0}^{t} \gamma(s)ds}, \quad \forall t \geq t_0$$

using Remark III.2.2, then we get

$$E(t) \leq cE(0)\, e^{\alpha \int_{t}^{t_0} \gamma(s)ds}\, e^{-\alpha \int_{0}^{t} \gamma(s)ds}, \quad \forall t \geq t_0.$$

Thus, the estimate (III.3.1) is proved.

Case 2: If $u_0 \neq 0$ on Γ_1, then (III.3.9) gives

$$\frac{d}{dt}F'(t) \leq -\alpha\gamma(t)\, F(t) + C_1 k^2(t), \quad \forall t \geq t_0. \tag{III.3.10}$$

where

$$C_1 = c\int_{\Gamma_1} |u_0|^2\, d\Gamma_1.$$

In this case we introduce

$$H(t) := F(t) - C_1 e^{-\alpha \int_{t_0}^{t} \gamma(s)ds} \int_{t_0}^{t} k^2(s)\, e^{\alpha \int_{t_0}^{s} \gamma(\zeta)d\zeta}\, ds. \tag{III.3.11}$$

A simple differentiation of H, using (III.3.10), leads to

$$H'(t) \leq -\alpha\gamma(t)\, H(t), \quad \forall t \geq t_0.$$

Again a simple integration over (t_0, t) yields

$$H(t) \leq H(t_0)\, e^{-\alpha \int_{t_0}^{t} \gamma(s)ds}, \quad \forall t \geq t_0.$$

wich implies

$$F(t) \leq \left(F(t_0) + C_1 \int_{t_0}^{t} k^2(s)\, e^{\alpha \int_{t_0}^{s} \gamma(\zeta)d\zeta}\, ds \right) e^{-\alpha \int_{t_0}^{t} \gamma(s)ds}, \quad \forall t \geq t_0.$$

Using (III.3.8) and Remark III.2.2, then we obtain for some positive constant c

$$E(t) \leq c \left\{ E(0) + C_1 \int_0^t k^2(s) \left[\frac{\eta}{2C_1} + e^{\alpha \int_{t_0}^{s} \gamma(\zeta) d\zeta} \right] ds \right\} e^{\alpha \int_0^{t_0} \gamma(s) ds} e^{-\alpha \int_0^t \gamma(s) ds}, \quad \forall t \geq t_0. \tag{III.3.12}$$

This complete the proof of Theorem III.3.1. ■

Remark III.3.1 ***i)*** *Estimates (III.3.1) and (III.3.2) are also true for* $t \in [0, t_0]$ *by virtue of continuity and boundedness of* $E(t)$ *and* $\gamma(t)$*.*

ii) *This result generalizes and improves the results of [40],[41],[42],[45]. In particular, it allows kernels which satisfy*

$$k'' \geq a\left(-k'\right)^{1+q},$$

for $0 < q < 1$ *instead of the usual assumption* $0 < q < 1/2$*. It suffices to take, for example,*

$$k(t) = 1/(1+t)^{\nu},$$

for $\nu > 0$*. Direct computations yield*

$$k''(t) = c\left(-k'(t)\right)^{1+1/(1+\nu)}.$$

It is clear that $0 < 1/(1+\nu) < 1$*, for* $\nu > 0$*.*

iii) *Note that the exponential and the polynomial decay estimates, given in early works [40]-[42], [45] are only particular cases of (III.0.1). More precisely, we obtain exponential decay for* $\gamma(t) \equiv a$ *and polynomial decay for*

$$\gamma(t) = a(1+t)^{-1},$$

where $a > 0$ *is a constant.*

iv) *We note that our result also holds for the system (a), studied by Cavalcanti and Guesmia [43]. We only considered (III.0.1) for simplicity.*

Bibliography

[1] A. Stuart, K. Ord, Kendall's advanced theory of statistics, distribution theory, John Wiley & Sons (2010)

[2] A. Wehbe. Rational energy decay rate for a wave equation with dynamical control. Appl. Math. Lett., 16(3):357–364, 2003.

[3] A. Guesmia, S.A. Messaoudi, On the boundary stabilization of a compactly coupled system of nonlinear wave equations, Int. J. Evol. Equ. 1 (3) (2005) 211–224.

[4] A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. Math. Ann., 347:455–478, 2010.

[5] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[6] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Vol. 44, Springer Science & Business Media (2012)

[7] B. Sz-Nagy and C. Foias. Analyse Harmonique des opérateurs de l'espace de Hilbert. Masson Paris, 1967.

[8] B. Rao, L. Toufayli, and A. Wehbe. Stability and controllability of a wave equation with dynamical boundary control. Math. Control Relat. Fields, 5(2):305–320, 2015.

[9] C. Abdallah , P.Dorato,J.Benitez-Read , and R.Byrne . Delayed positive feedback can stabilize oscillatory systems . In ACC' 1993 ,pages 3106–3107. San Francisco ,1993 .

[10] C.Bardos , G.Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary . SIAM J. Control Optim., 30(5):1024–1065,1992.

[11] C. D. Benchimol. A note on weak stabilization of contraction semigroups. SIAM J. Control optim.,16:373–379,1978.

[12] C. Pignotti. A note on stabilization of locally damped wave equations with time delay. Systems Control Lett., 61(1):92–97, 2012.

[13] Demengel, Françoise. Functional spaces for the theory of elliptic partial differential equations. Springer, 2012.

[14] E.M. Ait Benhassi ,K .Ammari , S.Boulite , and L.Maniar . Feedback stabilization of a class of evolution equations with delay . J.Evol. Equ.,9 (1):103–121,2009 .

[15] E. Pişkin, B. Okutmuştur, An introduction to Sobolev spaces, Bentham Science Publishers 2021.

[16] E. Zuazua, Uniform stabilization of the wave equation by nonlinear boundary feedback, SIAM J. Control. Optim 28 (1990) 466–477.

[17] F. L. Huang. Strong asymptotic stability of linear dynamical systems in banach spaces. J. Berlin, 35:585–603, 1985.

[18] F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, Appl. Math. Optim. 51 (1) (2005) 61–105.

[19] F. Conrad, B. Rao, Decay of solutions of wave equations in a star-shaped domain with nonlinear boundary feedback, Asympt. Anal. 7 (1993) 159–177.

[20] G. Bayili , A. Ben Aissa , and S. Nicaise. Same decay rate of second order evolution equations with or without delay . Systems Control Lett .,141,1 2020 . 104700,8 pp.

[21] G. Q. Xu, S. P. Yung, and L. K. Li. Stabilization of wave systems with input delay in the boundary control. ESAIM Control Optim. Calc. Var., 12(4):770–785 (electronic), 2006.

[22] G.C. Gorain, Boundary stabilization of nonlinear vibrations of a flexible structure in a bounded domain in Rn, J. Math. Anal. Appl. 319 (2) (2006) 635–650.

[23] H. Logemann, R. Rebarber, and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. SIAM J. Control Optim., 34(2):572–600, 1996.

[24] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.

[25] I. Lasiecka and R. Triggiani. Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions. Appl. Math. Optim., 25(2):189–224, 1992.

[26] I. Lasiecka, Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary, J. Differential Equations 79 (1989) 340–381.

[27] I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping, Differential Integral Equations 6 (3) (1993) 507–533.

[28] I. H. Suh and Z. Bien. Use of time delay action in the controller design. IEEE Trans.Automat. Control., 25:600–603, 1980.

[29] J. Hale and S. Verduyn Lunel. Introduction to functional differential equations, volume 99 of Applied Mathematical Sciences. Springer-Verlag, New York, 1993.

[30] J. Lagnese. Decay of solutions of wave equations in a bounded region with boundary dissipation. J. Differential Equations, 50(2):163–182, 1983.

[31] J.-L. Lions. Contrôlabilité exacte, perturbations et stabilisation de systémes distribués. Tome 1, volume 8 of Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris, 1988. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[32] J. Pruss. On the spectrum of c0-semigroups. Trans. Amer. Math. Soc., 284:847–857,1984.

[33] J.E. Muñoz Rivera, D. Andrade, Exponential decay of nonlinear wave equation with a viscoelastic boundary condition, Math. Methods Appl. Sci. 23 (1) (2000) 41–61.

[34] J.Y.Park, J.J. Bae, Il Hyo Jung, Uniform decay for wave equation of Kirchhoff type with nonlinear boundary damping and memory term, Nonlinear Anal.50 (2002) 871–884.

[35] J.T. Oden, L. Demkowicz, Applied functional analysis, Chapman and Hall/CRC (2017)

[36] K.Ammari and S. Nicaise . Stabilization of elastic systems by collocated feedback ,volume 2124 of Lecture Notes in Mathematics . Springer , Cham,2015.

[37] K. P. Hadeler. Delay equations in biology. In Functional differential equations and approximation of fixed points (Proc. Summer School and Conf., Univ. Bonn, Bonn, 1978), volume 730 of Lecture Notes in Math., pages 136–156. Springer, Berlin, 1979.

[38] L. Toufayli. Stabilisation polynomiale et contrôlabilité exacte des équations des ondes par des contrôles indirects et dynamiques. PhD thesis, Universit ´e de Strasbourg, 2013.

[39] L. C. Evans, Partial Differential Equations, American Mathematical Society, 1998

[40] M.L. Santos, Asymptotic behavior of solutions to wave equations with a memory conditions at the boundary, Electron. J. Differ. Equ. 78 (2001) 1–11

[41] M.L. Santos, J. Ferreira, D.C. Pereira, C.A. Raposo, Global existence and stability for wave equation of Kirchhoff type with memory condition at the boundary, Nonlinear Anal. 54 (2003) 959–976.

[42] M.L. Santos, F. Junior, A boundary condition with memory for Kirchhoff plates equations, Appl. Math. Comput. 148 (2) (2004) 475–496.

[43] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.S. Prates Filho, J.A. Soriano, Existence and uniform decay of solutions of a degenerate equation with nonlinear boundary memory source term, Nonlinear Anal. 38 (1999) 281–294.

[44] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.S. Prates Filho, J.A. Soriano, Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, Differential Integral Equations 14 (1) (2001) 85–116.

[45] M.M. Cavalcanti, A. Guesmia, General decay rates of solutions to a nonlinear wave equation with boundary conditions of memory type, Differential Integral Equations 18 (5) (2005) 583–600.

[46] M.M. Cavalcanti, V.N. Domingos Cavalcanti, M.L. Santos, Existence and uniform decay rates of solutions to a degenerate system with memory conditions at the boundary, Appl. Math. Comput. 150 (2004) 439–465.

[47] M.M. Cavalcanti, V.N. Domingos Cavalcanti, I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping—source interaction, J. Differential Equations 236 (2) (2007) 407–459.

[48] M.M. Cavalcanti, V.N. Domingos Cavalcanti, P. Martinez, General decay rate estimates for viscoelastic dissipative systems, Nonlinear Anal. 68 (1) (2008) 177–193.

[49] P. Grisvard. Elliptic problems in nonsmooth domains, volume 24 of Monographs and Studies in Mathematics. Pitman, Boston–London–Melbourne, 1985.

[50] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston,1985

[51] R. Datko. Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. SIAM J. Control Optim., 26(3):697–713, 1988.

[52] R. Datko. Two examples of ill-posedness with respect to time delays revisited. IEEE Trans. Automat. Control, 42(4):511–515, 1997.

[53] R. Datko, J. Lagnese, and M. P. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. SIAM J. Control Optim., 24(1):152–156, 1986.

[54] R. Rebarber and S. Townley. Robustness with respect to delays for exponential stability of distributed parameter systems. SIAM J. Control Optim., 37(1):230–244,1999.

[55] R.A. Adams, J.J.F. Fournier, Sobolev spaces, Vol. 140, Elsevier (2003)

[56] S. R. Foguel. Powers of contraction in hilbert space. Pacific J. Math., 13(1):551–561,1963.

[57] S. Nicaise and C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim.,45(5):1561–1585 (electronic), 2006.

[58] S. Nicaise and J. Valein. Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks. Netw. Heterog. Media, 2(3):425–479 (electronic), 2007.

[59] S. Nicaise and J. Valein. Stabilization of second order evolution equations with unbounded feedback with delay. ESAIM Control Optim. Calc. Var., 16(2):420–456, 2010.

[60] S.A. Messaoudi, A. Soufyane, Boundary stabilization of solutions of a nonlinear system of Timoshenko type, Nonlinear Anal. 67 (2007) 2107–2121.

[61] S.A. Messaoudi, M.I. Mustafa, On the control of solutions of viscoelastic equations with boundary feedback, Nonlinear Anal. RWA 10 (2009) 3132–3140.

[62] S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delayterm in the boundary or internal feedbacks, SIAM J. Control Optim. 45, 1561–1585(2006).

[63] T. Alazard, C. Zuily, Tools and problems in partial differential equations, Springer (2020)

[64] V. Komornik, E. Zuazua, A direct method for the boundary stabilization of the wave equation, J. Math. Pure Appl. 69 (1990) 33–54.

[65] V. Komornic. Exact Controllability and Stabilization. The Multiplier Method, volume 36. Research in Applied Mathematics, 1994.

[66] V. Komornik, On the nonlinear boundary stabilization of the wave equation, Chen. Ann. Math 14B (2) (1993) 153–164.

[67] V. Komornik, B. Rao, Boundary stabilization of compactly coupled wave equations, Asympt. Anal. 14 (1997) 339–359.

[68] W. Arendt and C.J.K. Batty . Tauberian theorems and stability of one-parameter semigroups. Trans. Amer. Math. Soc., 306(2):837–853, 1988.

[69] W. Arendt, C.J.K. Batty, Tauberian theorems and stability of one-parameter semigroups, Trans. Amer. Math. Soc. 306 (2) (1988) 837–852

[70] Z. Liu and B. Rao. Frequency domain approach for the polynomial stability of a system of partially damped wave equations. J. Math. Anal. Appl., 335(2):860–881,2007.

[71] Z. Liu and S. Zheng. Semigroups associated with dissipative systems, volume 398 of Chapman & Hall/CRC Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[72] Z.Abbas and S.Nicaise . The multidimensional wave equation with generalized acoustic boundary conditions II:polynomial stability . SIAM J. Control Optim ., 53(4):2582–2607,2015.