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Numerical Analysis for Master 1 of Mathematics Courses and Solved Exercises

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ABSTRACT

This work is an in-depth study of the numerical analysis course for the first year of the Master of Mathematics, where we discuss the method of finite differences in the first and second dimensions to find numerical solutions to three types of partial differential equations, which are elliptic, hyperbolic and prabolic. Using this numerical method, we can choose the appropriate discretization that leads to convergence towards the analytical solution. At the end of each chapter, we provide exercises that will help students adapt to most numerical problems.

Contents

Introduction

Partial differential equations arise modeling numerous phenomena in science and engineering. Examples include Laplace's equation for steady state heat conduction, the advection-diffusion equation for pollutant transport, Maxwell's equation for electromagnetic waves, the Navier-stokes equations for fluid flow and many, many more. Solving PDE means finding unknown function (analytical solution) that satisfies the PDE with initial and/or boundary conditions. Most PDEs of interest do not have analytical solutions, so numerical methods must be used for getting an approximate solution. The finite-difference methods (FDM) are a class of numerical techniques for solving differential equations by approximating derivatives with finite differences. Both the spatial domain and time domain (if applicable) are discretized, or broken into a finite number of intervals, and the values of the solution at the end points of the intervals are approximated by solving algebraic equations containing finite differences and values from nearby points. In another word, Finite difference methods convert ordinary differential equations (ODE) or partial differential equations (PDE), which may be nonlinear, into a system of linear equations that can be solved by matrix algebra techniques. Modern computers can perform these linear algebra computations efficiently which, along with their relative ease of implementation, has led to the widespread use of FDM in modern numerical analysis. Today, FDMs are one of the most common approaches to the numerical solution of PDE, along with finite element methods. In this work, we provide an introduction to second order linear PDEs and the finite differences method (FDM) for solving them.

This work is organized as follows: In chapter one, we recall the functional and matrix analysis. In chapter two, we introduce the second order linear PDEs with its detail such as the classification and the type of boundary conditions. In chapter three, we present the finite differences method (FDM) and its application in solving the elliptic, heat and wave equations. Finally, we close this work with a conclusion and main references.

CHAPTER 1

Reminder on functional and matrix analysis

1.1. Reminder on functional analysis

1.1.1. Gradient, Divergence and Laplacian operators

Definition 1. *Let f be a differentiable scalar function defined on $\mathbb{R}^n$, then its* ***Gradient*** *is defined as:*

$$\nabla f = \operatorname{grad} f := \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}. \tag{1.1}$$

Example 2. *Consider the following functions:* $f(x, y, z) = 5x^2 \sin(xy + z)$. $g(x, y) = 12xy^2 - e^{x+y}$.

Then we have

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 10x \sin(xy + z) + 5x^2 y \cos(xy + z) \\ 5x^3 \cos(xy + z) \\ 5x^2 \cos(xy + z) \end{pmatrix}, \nabla g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 12y^2 - e^x \\ 24xy - e^{x+y} \end{pmatrix}.$$

Definition 3. *Let f be a differentiable vector function defined on $\mathbb{R}^n$ to $\mathbb{R}^n$, then its **divergence** is defined as:*

$$\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + ... + \frac{\partial f_n}{\partial x_n} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \tag{1.2}$$

where $f = (f_1, f_2, ..., f_n)$.

Example 4. *For $f(x, y) = (\sin(xy), -3x^2 + y)$, $g(x, y, z) = (e^{-x+y}, 1 - yz^2, y\cos(z))$, then*

$$\begin{aligned} \operatorname{div}(f) &= y\cos(xy) + 1 \\ \operatorname{div}(g) &= -e^{-x+y} - z^2 - y\sin(z). \end{aligned}$$

Definition 5. *Let f be a twice differentiable scalar function defined on $\mathbb{R}^n$, then its **Laplacian** is defined as:*

$$\triangle f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + ... + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}. \tag{1.3}$$

Example 6. *For $f(x, y) = -3x^2 + e^{x-y}$, $g(x, y, z) = y\cos(xz)$, then*

$$\triangle f = -6 + 2e^{x-y}, \ \ \triangle\, g = \left(-yz^2 - x^2 y\right)\cos(xz).$$

There is a relation between the three differential operators as follows: $\triangle f = \operatorname{div}(\nabla f)$.

1.1.2. Taylor's Formula

Definition 7. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a $n-$times differentiable function at $x_0 \in I$. The Taylor polynomial $T^n_{x_0} f$ of degree n at a point x_0 of f is given by:*

$$T^n_{x_0} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n . \tag{1.4}$$

Theorem 8. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a $n-$times differentiable function at $x_0 \in I$. The Taylor polynomial $T^n_{x_0} f$ is the only polynomial P_n that verifies:*

$$P_n(x_0) = f(x_0),\ P'_n(x_0) = f'(x_0), ...,\ P^{(n)}_n(x_0) = f^{(n)}(x_0). \tag{1.5}$$

Remark 9. *Note that the zeroth order Taylor polynomial is just a constant $T^0_{x_0} f(x) = f(x_0)$.*

Definition 10. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a $n-$times differentiable function at $x_0 \in I$. We define $R^n_{x_0} f$ by:*

$$R^n_{x_0} f(x) = f(x) - T^n_{x_0} f(x) , \tag{1.6}$$

then $R^n_{x_0} f$ is called the n-th order remainder.

Proposition 11. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a $n-$times differentiable function at $x_0 \in I$. Then, there exists a neighbourhood of x_0 $(J \subset I)$ such that:*

$$R^n_{x_0} f\left(x\right) = \epsilon\left(x\right)\left(x - x_0\right)^n, \tag{1.7}$$

with $lim_{x \to x_0} \epsilon\left(x\right) = 0$.

Proposition 12. *Let f be a function that is $(n+1)$ times differentiable in an interval I that contains x_0. Then for each x in I, there is a number c strictly between x and x_0 such that:*

$$f(x) = T^n_{x_0} f\left(x\right) + R_n\left(x, x_0\right), \tag{1.8}$$

where

$$R_n\left(x, x_0\right) = \frac{f^{(n+1)}\left(c\right)}{(n+1)!}\left(x - x_0\right)^{n+1}. \tag{1.9}$$

Example 13. *Write Taylor's formula for the case where $f(x) = \ln(x)$, $x_0 = 1$, $n = 3$.*

Solution: We have

$f(x) = \ln(x),\ f(1) = 0,$

$f'(x) = \frac{1}{x},\ f'(1) = 1,$

$f''(x) = -\frac{1}{x^2},\ f''(1) = -1,$

$f'''(x) = \frac{2}{x^3}$, $f'''(1) = 2$,

$f^{(4)}(x) = -\frac{6}{x^4}$, $f^{(4)}(c) = \frac{-6}{c^4}$.

Therefore,

$$\begin{aligned} f(x) &= T_1^3 f(x) + R_3(x, 1) \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4c^4}(x-1)^4. \end{aligned}$$

Example 14. *For* $f(x) = e^x$, $x_0 = 0, n = 5$.

Solution: For any $k \in \mathbb{N}^*$, $(e^x)^{(k)} = e^x$, then

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + e^c \frac{x^6}{6!}.$$

1.2. Reminder on matrix analysis

In this section, we will review some important fundamentals of matrix algebra necessary to numerical techniques.

1.2.1. Eigenvalues and Eigenvectors

The product AX of a matrix $A \in \mathcal{M}_n(\mathbb{R})$ and an n-vector X is itself an n-vector. Of particular interest in many settings (of which differential equations is one) is the following

question:

For a given matrix A, what are the vectors X for which the product AX is a scalar multiple of X? That is, what vectors X satisfy the equation:

$$AX = \lambda X,$$

for some scalar λ?

It should immediately be clear that, no matter what A and λ are, the vector $X = 0$ (that is, the vector whose elements are all zero) satisfies this equation. With such a trivial answer,

we might ask the question again in another way:

For a given matrix A, what are the nonzero vectors X that satisfy the equation:

$$AX = \lambda X,$$

for some scalarλ?

To answer this question, we first perform some algebraic manipulations upon the equation $AX = \lambda X$. We note first that, if $I = I_n$ (the $n \times n$ multiplicative identity in $\mathcal{M}_n(\mathbb{R})$), then

we can write

$$\begin{aligned} AX = \lambda X \quad &\Leftrightarrow \quad AX - \lambda X = 0 \\ &\Leftrightarrow \quad AX - \lambda IX = 0 \\ &\Leftrightarrow \quad (A - \lambda I) X = 0. \end{aligned}$$

Remember that we are looking for nonzero X that satisfy this last equation. But $A - \lambda I$ is an $n \times n$ matrix and, should its determinant be nonzero, this last equation will have exactly one solution, namely $X = 0$. Thus our question above has the following answer:

The equation $AX = \lambda X$ has nonzero solutions for the vector X if and only if the matrix $A - \lambda I$ has zero determinant.

As we will see in the examples below, for a given matrix A there are only a few special values of the scalar λ for which $A - \lambda I$ will have zero determinant, and these special values are called the eigenvalues of the matrix A. Based upon the answer to our question, it seems we must first be able to find the eigenvalues λ_1, λ_2, . . . ,λ_n of A and then see about solving the individual equations $AX = \lambda_i X$ for each $i = 1, ..., n$.

Example 15. *Find the eigenvalues of the matrix A*

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues are those λ for which $\det(A - \lambda I) = 0$. Now

$$\begin{aligned}
\det(A - \lambda I) &= \left| \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} 2-\lambda & 0 \\ -1 & 1-\lambda \end{pmatrix} \right| \\
&= (2-\lambda)(1-\lambda) \\
&= \lambda^2 - 3\lambda + 2.
\end{aligned}$$

The eigenvalues of A are the solutions of the quadratic equation $\lambda^2 - 3\lambda + 2 = 0$, namely $\lambda_1 = 1$ and $\lambda_2 = 2$.

As we have discussed, if $\det(A - \lambda I) = 0$ then the equation $(A - \lambda I)X = b$ has either no solutions or infinitely many. When we take $b = 0$ however, it is clear by the existence of the solution $X = 0$ that there are infinitely many solutions (i.e., we may rule out the "no solution" case). If we continue using the matrix A from the example above, we can expect nonzero solutions X (infinitely many of them, in fact) of the equation $AX = \lambda X$ precisely when $\lambda = 1$ or $\lambda = 2$. Let us procede to characterize such solutions.

First, we work with $\lambda = 1$. The equation $AX = \lambda X$ becomes $AX = X$. Writing

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and using the matrix A from above, we have

$$\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Setting these equal, we get

$$2x_1 = x_1 \text{ and } -x_1 + x_2 = x_2,$$

then $x_1 = 0, x_2 = x_2$.

This means that, while there are infinitely many nonzero solutions (solution vectors) of the equation $AX = X$.

Thus all solutions of this equation can be characterized by

$$\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where x_2 is any real number. The nonzero vectors X that satisfy $AX = X$ are called eigenvectors associated with the eigenvalue $\lambda = 1$.

One such eigenvector is

$$U_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and all other eigenvectors corresponding to the eigenvalue 1 are simply scalar multiples

of U_1, (U_1 spans this set of eigenvectors).

Similarly, we can find eigenvectors associated with the eigenvalue $\lambda = 2$ by solving $AX = 2X$

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}, \end{aligned}$$

then

$$\begin{pmatrix} 2x_1 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}.$$

Thus $2x_1 = 2x_1$ and $-x_1 + x_2 = 2x_2$.

$-x_1 + x_2 = 2x_2 \Leftrightarrow x_2 = -x_1$.

Hence the set of eigenvectors associated with $\lambda = 2$ is spanned by

$$U_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Example 16. *Find the eigenvalues and associated eigenvectors of the matrix*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Important Eigenvalues and Eigenvectors properties

1) The eigenvalues of a diagonal or triangular matrix are equal to the diagonal elements of the matrix.

2) The eigenvectors of a diagonal matrix (with distinct diagonal elements) are the standart basis vectors.

3) The eigenvalues of the identity matrix are all "1" and the eigenvectors are arbitrary.

4) The eigenvalues of a scalar matrix kI are all k and the eigenvectors are arbitrary.

5) Multiplying a matrix by a scalar k has the effect of multiplying all its eigenvalues by k , but leaves the eigenvectors unaltered.

6) Adding onto a matrix a scalar matrix, kI , has the effect of adding k to all its eigenvalues, but leaves the eigenvectors unaltered:

$$\lambda_i\,(A + kI) = \lambda_i(A) + k.$$

7) The eigenvalues of the inverse of a matrix are given by:

$$\lambda_i(A^{-1}) = \frac{1}{\lambda_i(A)}.$$

8) The eigenvectors of a matrix and its transpose are the same.

9) The left eigenvectors of a matrix A are the right eigenvectors of A^t.

10) The eigenvalues/eigenvectors of a real matrix are real or appear in complex conjugate pairs.complex conjugate pairs.

11) $\rho(A) := \max_i |\lambda_i(A)|$ is called the spectral radius of A.

12) Eigenvalues are invariant under similarity transformations.

13) Eigenvalues of a matrix $A \in \mathcal{M}_n(\mathbb{R})$ in the following form

$$A = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \ddots & \vdots \\ 0 & b & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & 0 & b & a \end{pmatrix}, \ a, b \in \mathbb{R},$$

are given by: $\lambda_k(A) = a + 2b\cos(\frac{k\pi}{n+1}),\ k = 1, ..., n.$

Some properties of determinant

Let A, $B \in \mathcal{M}_n(\mathbb{R})$, then

1) $\det(A) = 0$, if and only if the columns (rows) of A are linearly dependent.

2) If a column (row) of A is zero, then $\det(A) = 0$.

3) $\det(A) = \det(A^t)$.

4) $\det(AB) = \det(A)\det(B)$.

5) $\det(cA) = c^n \det(A)$, $\forall c \in \mathbb{R}$

6) $\det(A^{-1}) = \frac{1}{\det(A)}$

7) Swapping two rows (columns) changes the sign of $\det(A)$.

8) Scaling a row (column) by k, scales the determinant by k.

9) Adding a multiple of one row (column) onto another does not affect the determinant.

10) If $A = diagonal\,\{a_{ii}\}$ or lower triangular, or A =upper triangular with diagonal elements a_{ii}, then

$$\det(A) = a_{11}a_{22}...a_{nn}.$$

1.2.2. some special matrices

Definition 17. *Let A be a matrix of $\mathcal{M}_n(\mathbb{R})$, we say that A is symmetric if and only if $A^t = A$.*

Here A has real eigenvalues λ_i, and hence real eigenvectors w_i, where the eigenvectors are orthonormal $(w_i w_j = \delta_{i,j})$. Therefore its eigenvalue/eigenvector decomposition becomes:

$$A = R\Lambda R^t,$$

where $RR^t = R^tR = I$, and Λ is a diagonal matrix containing all the eigenvalues of A.

Definition 18. *Let A be a matrix of $\mathcal{M}_n(\mathbb{R})$, we say that A is normal if and only if $A^tA = AA^t$.*

Definition 19. *A square real matrix A is said unitary if and only if $A^t = A^{-1}$.*

Proposition 20. *The eigenvalues of A have unit modulus and the eigenvectors are orthogonal.*

Definition 21. *A block matrix or a partitioned matrix is a matrix that is interpreted as having been broken into sections called blocks or submatrices in the form:*

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{pmatrix} \in \mathcal{M}_{n\times m}(\mathbb{R}),$$

where $A_{i,j} \in \mathcal{M}_{n_i\times m_j}(\mathbb{R})$ are contiguous submatrices, such that $\sum_{i=1}^{p} n_i = n$ and $\sum_{j=1}^{q} m_j = m$. The elements $A_{i,j}$ of the partition are called blocks.

Transpose of block matrix

$$A^t = \begin{pmatrix} A_{11}^t & A_{21}^t & \cdots & A_{p1}^t \\ A_{12}^t & A_{22}^t & \cdots & A_{p2}^t \\ \vdots & \vdots & \ddots & \vdots \\ A_{1q}^t & A_{2q}^t & \cdots & A_{pq}^t \end{pmatrix}.$$

Inversion of block matrix

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix},$$

where A and D are square blocks of arbitrary size, and B and C are conformable with them for partitioning and $D - CA^{-1}B$ must be invertible.

Determinant

$$\det \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det(A)\det(D) = \det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

If A is invertible, one has

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det\left(D - CA^{-1}B\right).$$

Block diagonal matrices

A block diagonal matrix is a block matrix that is a square matrix such that the main-diagonal blocks are square matrices and all off-diagonal blocks are zero matrices. That is, a block diagonal matrix A has the form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix},$$

where A_k is a square matrix for all $k = 1, ..., n$.

For the determinant and trace, the following properties hold:

- $\det(A) = \det(A_1)\det(A_2)...\det(A_n)$
- $tr(A) = tr(A_1) + tr(A_2) + ... + tr(A_n)$

A block diagonal matrix is invertible if and only if each of its main-diagonal blocks are invertible, and in this case its inverse is another block diagonal matrix given by

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}^{-1} = \begin{pmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^{-1} \end{pmatrix}.$$

The eigenvalues and eigenvectors of A are simply those of the A_k combined.

Block tridiagonal matrices

A block tridiagonal matrix is another special block matrix, which is just like the block diagonal matrix a square matrix, having square matrices (blocks) in the lower diagonal, main diagonal and upper diagonal, with all other blocks being zero matrices. It is essentially a tridiagonal matrix but has submatrices in places of scalars. A block tridiagonal matrix A has the form

$$A = \begin{pmatrix} B_1 & C_1 & & & \cdots & & 0 \\ A_2 & B_2 & C_2 & & & & \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & A_k & B_k & C_k & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ & & & & A_{n-1} & B_{n-1} & C_{n-1} \\ 0 & & & & & A_n & B_n \end{pmatrix},$$

where A_k, B_k and C_k are square sub-matrices of the lower, main and upper diagonal respectively.

Block tridiagonal matrices are often encountered in numerical solutions of engineering problems.

Block triangular matrices

- **Upper block triangular**

A matrix A is upper block triangular (or block upper triangular) if

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{kk} \end{pmatrix}.$$

- **Lower block triangular**

A matrix A is lower block triangular if

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix}.$$

Definition 22. *A square real matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called monotone, if it satisfies:*

$$\forall X = (x_1, x_2, ..., x_n)^t \in \mathbb{R}^n,\ AX \geq 0 \Rightarrow X \geq 0.$$

Example 23.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then

$$AX \geq 0 \;\Rightarrow\; \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0$$
$$\Rightarrow \begin{cases} 2x_1 - x_2 \geq 0 & \ldots (1) \\ -x_1 + 2x_2 \geq 0 & \ldots (2) \end{cases},$$

by multiplying inequality (2) by 2, and adding it to inequality (1), we get $3x_2 \geq 0$, then $x_2 \geq 0$.

From (1), we have $2x_1 \geq x_2 \geq 0$, then $x_1 \geq 0$.Therefore, $X \geq 0$.

Consequently, A is monotone.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then

$$AX \geq 0 \;\; \Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \geq 0$$

$$\Rightarrow \begin{cases} 2x - y \geq 0 & \ldots (1) \\ -x + 2y - z \geq 0 & \ldots (2) \\ -y + 2z \geq 0 & \ldots (3) \end{cases},$$

we add the inequalities (1), (2) and (3) side by side, we obtain $x + z \geq 0 \ldots (4)$. Now, by multiplying inequality (3) by 2, and adding it to inequality (2), we get $-x + 3z \geq 0 \ldots (5)$.

Adding (4) to (5), we get $4z \geq 0$, then $z \geq 0$. In addition, since $z \geq 0$, then the inequality (2) implies that $-x + 2y \geq z \geq 0 \quad \ldots (6)$. By multiplying inequality (1) by 2, and adding it to inequality (6), we get $3x \geq 0$, then $x \geq 0$. On the other hand, the inequality (6) leads to $2y \geq x \geq 0$. Therefore A is monotone.

Example 24.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned} AX \geq 0 \ \Rightarrow & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \\ \Rightarrow & \begin{cases} 2x + y \geq 0 & \ldots (1) \\ x + 2y \geq 0 & \ldots (2) \end{cases}, \end{aligned}$$

we will prove that the matrix is not monotone. For that, we choose $x = -1, y = 2$, we get

$$\begin{cases} -2 + 2 = 0 \geq 0 & \ldots (1) \\ -1 + 4 = 3 \geq 0 & \ldots (2) \end{cases},$$

then $AX \geq 0$ and $X \ngeq 0$. Therefore A is not monotone.

Proposition 25. *The matrix A is monotone $\Leftrightarrow$ (A is invertible and $A^{-1} \geq 0$)*

Example 26. *Study the monotonicity of the following matrices by using the proposition.*

$$A_1 = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \ A_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \ A_4 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

By calculating the inverse matrices , we obtain

$$A_1^{-1} = \begin{pmatrix} 3/17 & 2/17 \\ -1/17 & 5/17 \end{pmatrix} \ngeq 0, A_2^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \ngeq 0, A_3^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \geq 0,$$

$$A_4^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix} \geq 0.$$

Hence, only the matrices A_3 and A_4 are monotone.

Example 27. *Let $A \in \mathcal{M}_n(\mathbb{R})$ defined by:*

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

This matrix is monotone.

In fact, Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, then

$$AX \geq 0 \Rightarrow \begin{cases} 2x_1 - x_2 & \geq 0......(1) \\ -x_1 + 2x_2 - x_3 & \geq 0......(2) \\ \vdots & \vdots \\ -x_{i-1} + 2x_i - x_{i+1} & \geq 0......(i) \\ \vdots & \vdots \\ -x_{n-1} + 2x_n & \geq 0......(n) \end{cases}.$$

We put $x_p = \min\{x_k,\ k = 1, ..., n\}$ and we suppose that $x_p < 0$.

If $x_p = x_1$, then by using (1), we obtain

$$x_1 > 2x_1 \geq x_2,$$

this implies that $x_2 < x_p$.

If $x_p = x_i$, $i = 2, ..., n-1$, then by using (i), we obtain

$$-x_{i-1} + 2x_i - x_{i+1} \geq 0,$$

it is equivalent to $(x_i - x_{i-1}) + (x_i - x_{i+1}) \geq 0$, then $x_i = x_{i-1} = x_p$ and $x_i = x_{i+1} = x_p$, this is imposible.

If $x_p = x_n$, then by using (n), we have

$$x_n > 2x_n \geq x_{n-1},$$

then $x_p > x_{n-1}$. Therefore, $x_p \geq 0$.

Finally, $X \geq 0$.

Proposition 28. *A is monotone $\Leftrightarrow$ A^t is monotone.*

Proof. It is enough to note that $\left(A^t\right)^{-1} = \left(A^{-1}\right)^t$. □

Definition 29. *Let A be a symmetric matrix of $\mathcal{M}_n(\mathbb{R})$, we say that A is symmetric positive definite if and only if it satisfies any of the following equivalent conditions:*

- $\forall X = (x_1, x_2, ..., x_n)^t \neq 0$, $X^t A X > 0$.
- All its eigenvalues are real and strictly positive.
- All its leading principal minors are strictly positive.
- There exists an invertible matrix B such that $A = B^t B$, B is an upper triangular matrix with positive entries on the main diagonal.

Proposition 30. *Any symmetric positive definite matrix A has the following properties:*

- $\det(A) > 0$.
- A is invertible and its inverse is symmetric positive definite.

Example 31.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}.$$

Concerning A, we have A is symmetric and $\Delta_1 = 2 > 0$, $\Delta_2 = \det(A) = 1 > 0$,hence A is symmetric positive definite.

Concerning B, it is not symmetric, then it is not symmetric positive definite.

Example 32.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

It is clear that A is symmetric.

Now, let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then

$$\begin{aligned} X^t AX &= \begin{pmatrix} 2x_1 - x_2 & -x_1 + 2x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= x_1 (2x_1 - x_2) + x_2 (-x_1 + 2x_2) \\ &= 2x_1^2 - x_1 x_2 - x_1 x_2 + 2x_2^2 \\ &= x_1^2 + (x_1 - x_2)^2 + x_2^2 \geq 0. \end{aligned}$$

We suppose that $x_1^2 + (x_1 - x_2)^2 + x_2^2 = 0$, then $x_1 = 0$, $x_1 = x_2$, $x_2 = 0$. This is a contradiction with $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, hence $X^t AX > 0$.

Consequently, A is symmetric positive definite.

Example 33.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

It is clear that A is symmetric.

By calculating the eigenvalues of A, we have $\lambda_k = 2 - 2\cos(\frac{k\pi}{4})$, $k = 1, ..., 3$. Then, $\lambda_1 = 2 - 2\cos(\frac{\pi}{4}) = 2 - \sqrt{2} > 0$, $\lambda_2 = 2 - 2\cos(\frac{2\pi}{4}) = 2 > 0$, $\lambda_3 = 2 - 2\cos(\frac{3\pi}{4}) = \sqrt{2} + 2 > 0$.

Since all the eigenvalues are strictly positive, then A is symmetric positive definite.

Example 34. *Let $A \in \mathcal{M}_n(\mathbb{R})$ defined by:*

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

A is symmetric positive definite.

In fact, it is clear that A is symmetric. In addition, we know that the eigenvalues of A are given by:

$$\begin{aligned} \lambda_k &= 2 - 2\cos(\frac{k\pi}{n+1}), k = 1, ..., n. \\ &= 2 - 2\left(1 - 2\sin^2(\frac{k\pi}{2(n+1)})\right), k = 1, ..., n. \\ &= 4\sin^2(\frac{k\pi}{2(n+1)}) > 0, k = 1, ..., n. \end{aligned}$$

Therefore, A is symmetric positive definite.

Remark 35. *We can prove that A is symmetric positive definite using an another method as follows:*

$$\forall X = (x_1, x_2, ..., x_n)^t \neq 0, X^t A X > 0.$$

In fact, $\forall X = (x_1, x_2, ..., x_n)^t \neq 0, X^t A X = x_1^2 + x_n^2 + \sum_{k=1}^{N-1} (x_i - x_{i+1})^2 \geq 0.$

We suppose that $x_1^2 + x_n^2 + \sum_{k=1}^{N-1} (x_i - x_{i+1})^2 = 0$, then $x_1 = x_n = 0$, $x_1 = x_2$, $x_2 = x_3, ..., x_{n-1} = x_n$. It implies that $X = 0$. This is a contradiction with $X \neq 0$. Thus, $x_1^2 + x_n^2 + \sum_{k=1}^{N-1} (x_i - x_{i+1})^2 > 0$. Therefore, A is symmetric positive definite.

1.2.3. Vector and Matrix Norms

Definition 36. *Let E be a vector space over a field K. A norm on E is a function $\|.\| : E \to \mathbb{R}_+$, assigning a nonnegative real number $\|u\|$ to any vector $u \in E$,and satisfying the following conditions for all $x, y, z \in E$*

- $\|x\| = 0 \Leftrightarrow x = 0$ (positivity)
- $\forall \lambda \in K$, $\|\lambda x\| = |\lambda| \, \|x\|$ (scaling)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

A vector space E together with a norm $\|.\|$ is called a normed vector space.

Example 37. *We have some common norms as follows:*

- $E = \mathbb{R}$, $\|x\| = |x|$.
- $E = \mathbb{R}^n$, there are three standard norms:

For every $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, we have $1 - norm$; $\|.\|_1$ defined as:

$$\begin{aligned} \|x\|_1 &= |x_1| + |x_2| + ... + |x_n| \\ &= \sum_{i=1}^{n} |x_i| . \end{aligned}$$

Euclidean norm $\|.\|_2$ defined as:

$$\begin{aligned} \|x\|_2 &= \sqrt{|x_1|^2 + |x_2|^2 + ... + |x_n|^2} \\ &= \sqrt{\sum_{i=1}^{n} |x_i|^2}, \end{aligned}$$

and $sup - norm$ $\|.\|_\infty$ that is defined by:

$$\|x\|_\infty = \max \{ |x_i| , \quad i = 1, ..., n \} .$$

More generally, we define the $\ell_p - norm$ as follows:

$$\|x\|_p = (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}\ ,\ \ p \geq 1.$$

There are other norms besides the $\ell_p - norms$; we urge the reader to find such norms.

Some work is required to show the triangle inequality for the $\ell_p - norms$

For any two vectors U, V of $\mathbb{R}^n$ and for every real numbers $p, q > 1$ such as $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{i=1}^{n} |u_i v_i| \leq \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |v_i|^q\right)^{\frac{1}{q}},$$

is known as Hölder's inequality.

For $p = 2$, it is the Cauchy–Schwarz inequality.

The Minkowski's inequality for $p \geq 1$is given by;

$$\left(\sum_{i=1}^{n} |u_i + v_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}.$$

Proposition 38. *The following inequalities hold for all* $x \in \mathbb{R}^n$,

- $\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$
- $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$
- $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$

Definition 39. *Given any vector space E, two norms $\|.\|_a$ and $\|.\|_b$ are equivalent if and only if there exists some positive reals $C_1 > 0, C_2 > 0$, such that*

$$\|u\|_b \leq \|u\|_a \leq \|u\|_b \ , \ \forall u \in E.$$

Theorem 40. *Given a vector space E of finite dimension, then any two norms on E are equivalent.*

Definition 41. *A matrix norm $\|.\|$ on the space of square $n \times n$ matrices $(\mathcal{M}_n(\mathbb{R}))$ is a positive function defined on $\mathcal{M}_n(\mathbb{R})$ that satisfies the following properties:*

- $\forall A \in \mathcal{M}_n(\mathbb{R}), \ \|A\| = 0 \Leftrightarrow A = 0$
- $\lambda \in \mathbb{R}, \forall A \in \mathcal{M}_n(\mathbb{R}), \ \|\lambda A\| = |\lambda| \|A\|$
- $\forall A, B \in \mathcal{M}_n(\mathbb{R}), \ \|A + B\| \leq \|A\| + \|B\|$
- $\forall A, B \in \mathcal{M}_n(\mathbb{R}), \ \|AB\| \leq \|A\| \|B\|$

Example 42. *Given a matrix $A \in \mathcal{M}_n(\mathbb{R})$, we have the following common norms:*

- $1 - norm : \|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{i,j}|$
- $\infty - norm : \|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{i,j}|$
- Frobenius norm: $\|A\|_F = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{i,j}|^2}$

- $2 - norm : \|A\|_2 = \sqrt{\rho\left(A^t A\right)}$, if A is symmetric, then $\|A\|_2 = \rho\left(A\right)$

Proposition 43. *There exists some relations between the last norms as follows:*

- $\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2$
- $\max\limits_{i,j}|a_{i,j}| \leq \|A\|_2 \leq n\max\limits_{i,j}|a_{i,j}|$
- $\|A\|_2 \leq \sqrt{\|A\|_1\,\|A\|_\infty}$
- $\frac{1}{\sqrt{n}}\,\|A\|_\infty \leq \|A\|_2 \leq \sqrt{n}\,\|A\|_\infty$
- $\frac{1}{\sqrt{n}}\,\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\,\|A\|_1$

Definition 44. *A matrix norm $\|.\|_M$ on $\mathcal{M}_n\left(\mathbb{R}\right)$ is called consistent or compatible with a vector norm $\|.\|_V$ on $\mathbb{R}^n$ if*

$$\forall A \in \mathcal{M}_n\left(\mathbb{R}\right), \forall U \in \mathbb{R}^n, \quad \|AU\|_V \leq \|A\|_M\,\|U\|_V \;\; .$$

Example 45. *The 1-norm of matrices is compatible with 1-norm of vectors*

$$\forall A \in \mathcal{M}_n\left(\mathbb{R}\right), \forall U \in \mathbb{R}^n, \quad \|AU\|_1 \leq \|A\|_1\,\|U\|_1 \;\; .$$

Example 46. *The ∞-norm of matrices is compatible with ∞-norm of vectors*

$$\forall A \in \mathcal{M}_n\left(\mathbb{R}\right), \forall U \in \mathbb{R}^n, \quad \|AU\|_\infty \leq \|A\|_\infty\,\|U\|_\infty \;\; .$$

Example 47. *The F-norm of matrices is compatible with 2-norm of vectors*

$$\forall A \in \mathcal{M}_n\left(\mathbb{R}\right), \forall U \in \mathbb{R}^n, \quad \|AU\|_2 \leq \|A\|_F\,\|U\|_2 \;\; .$$

Example 48. *The* 2*-norm of matrices is compatible with* 2*-norm of vectors*

$$\forall A \in \mathcal{M}_n(\mathbb{R}), \forall U \in \mathbb{R}^n, \quad \|AU\|_2 \leq \|A\|_2 \|U\|_2 \ .$$

Exercise 49. *Study the symmetry and monotonicity of the following matrices:*

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}, A_3 = \begin{pmatrix} -1 & 2 \\ 2 & -6 \end{pmatrix}, \\
A_4 &= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, A_5 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, A_6 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
A_7 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & 2 \end{pmatrix}.
\end{aligned}
$$

Solution: Note that all the matrices are symmetric except A_7. Now concerning the monotonicity, we have

$$
\begin{aligned}
A_1^{-1} &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \not\geq 0;\ A_2^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \geq 0;\ A_3^{-1} = \begin{pmatrix} -3 & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} \not\geq 0; \\
A_4^{-1} &= \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \not\geq 0;\ A_5^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \geq 0;\ A_6^{-1} \text{ is not exist}; \\
A_7^{-1} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -\frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \not\geq 0.
\end{aligned}
$$

Therefore, A_2 and A_5 are the only monotone ones.

Exercise 50. *Let $A \in \mathcal{M}_2(\mathbb{R})$, such that*

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

Find the conditions on a, b, c so that A is positive definite.

Solution: It is clear that A is symmetric. On the other hand,

$$\triangle_1 = a,\ \ \triangle_2 = ab - c^2.$$

Then A is symmetric positive definite $\Leftrightarrow (\triangle_1 > 0 \text{and} \triangle_2 > 0)$.

Exercise 51. *Let $c_1, c_2, ..., c_n$ be strictly positive real numbers. Show that the following matrix is monotone and positive definite:*

$$A = \begin{pmatrix} 2+c_1 & -1 & 0 & \cdots & 0 \\ -1 & 2+c_2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2+c_n \end{pmatrix}.$$

Solution: After the following decomposition, A is a sum of two matrices symmetric positive definite, it becomes symmetric positive definite.

$$A=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}+\begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & c_n \end{pmatrix}$$

Exercise 52. *Let A, B be two matrices of $\mathcal{M}_4(\mathbb{R})$ such that:*

$$A=\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix},\ B=\begin{pmatrix} 4 & -3 & 2 & -1 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 2 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

1) Calculate $A\times B$, $B\times A$ then deduce that A is not monotone.

2) Show that A and B are symmetric positive definite matrices.

Solution:

1) We have

$$A \times B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ B \times A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $B = A^{-1}$ and $B \ngeq 0$, then A is not monotone.

2) The leading principal minors are: $\triangle_1 = \triangle_2 = \triangle_3 = \triangle_4 = 1$. Since A is symmetric and all its leading principal minors are strictly positive, then A is symmetric positive definite.

On the other side, since A is symmetric positive definite, then by the property, its inverse B is also symmetric positive definite.

Exercise 53.

1) If A is symmetric positive definite, show that A^k is symmetric positive definite for all $k \geq 1$.

2) Prove the converse to (1) when k is odd.

3) Find a symmetric matrix A such that A^2 is positive definite but A is not.

Solution:

1) suppose that A is symmetric positive definite, then A^k is symmetric for all $k \geq 1$. Moreover, according to the following property:

$$(A^k - I) = (A - I)(A^{k-1} + A^{k-2} + ... + A + I), \ \forall k \geq 1,$$

we have, if λ is an eigenvalue to A, then λ^k becomes an eigenvalue to A^k. Hence, all the eigenvalues of A^k are strictly positive.

Therefore, A^k is symmetric positive definite.

2) We will prove here that, if A^{2k+1} is symmetric positive definite, then A is also symmetric positive definite.

Since A^{2k+1} is symmetric positive definite, then A is symmetric and invertible.

On the other hand, the matrix A^{2k+1} verifies $\forall X \neq 0$, $X^t A^{2k+1} X > 0$. So, if we take $X = \left(A^{-1}\right)^k Y$, we get

$$\forall Y \neq 0, Y^t \left(A^{-1}\right)^k A^{2k+1} \left(A^{-1}\right)^k Y > 0,$$

which implies that

$$\forall Y \neq 0, Y^t A Y > 0.$$

Therefore, A is symmetric positive definite.

3) We take $A = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$. Note that A^2 is positive definite but A is not.

Exercise 54. *If A and B are symmetric positive definite and $r > 0$, show that $A + B$ and rA are both symmetric positive definite.*

Solution: By the formulae $(A+B)^t = A^t + B^t, (rA)^t = rA^t$, we deduce that $A + B$ and rA are both symmetric.

On other side, $\forall X \neq 0$, $X^t(A+B)X = X^tAX + X^tBX > 0$, $X^t\,(rA)\,X = rX^tAX > 0$.

Therefore, $A+B$ and rA are both symmetric positive definite.

Exercise 55. *Let A be a symmetric positive definite matrix. If a is a real number, show that aA is symmetric positive definite if and only if $a > 0$.*

Exercise 56. *Consider the following matrix:*

$$A = \begin{pmatrix} 3/4 & -1/2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & -1/2 & 3/4 \end{pmatrix}.$$

1) Find A^{-1}. What can be said about the monotonicity of A.

2) Find the eigenvalues of A.

3) Deduce that A is symmetric positive definite.

Solution:

1)

$$A^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

since $A^{-1} \geq 0$, then A is monotone.

2) The eigenvalues of A^{-1} are: $\lambda_k = 2 + 2\cos(\frac{k\pi}{4}),\ k = 1, 2, 3$. So, $\lambda_1 = 2 + \sqrt{2}, \lambda_2 = 2, \lambda_3 = 2 - \sqrt{2}$.

By applying the eigenvalues of the inverse property, the eigenvalues of A are: $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \dfrac{1}{\lambda_3}$.

3) Since A is symmetric and all its eigenvalues are strictly positive, then it is symmetric positive definite.

Exercise 57. *Let A be a matrix of $\mathcal{M}_3(\mathbb{R})$ such that:*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

Show that A is symmetric positive definite with two different methods.

Solution: It is clear that A is symmetric.

<u>First method:</u> $\triangle_1 = 2,\ \triangle_2 = 3,\ \triangle_3 = 4$.

Since $\triangle_1 > 0,\ \triangle_2 > 0$ and $\triangle_3 > 0$, then A is symmetric positive definite.

<u>Second method:</u> Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq 0$, then

$$X^t A X = (x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2 \geq 0,$$

we suppose that $X^t A X = 0$, then $x_1 = -x_2, x_1 = -x_3, x_2 = -x_3$.

This means that $x_2 = x_3 = -x_3$, we obtain $x_3 = 0, x_2 = 0, x_1 = 0$. This is a contradiction with $X \neq 0$.Therefore, $X^t AX > 0$.

Exercise 58. *Let A be a matrix of $\mathcal{M}_3(\mathbb{R})$ such that:*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

Show that A is symmetric positive definite with two different methods.

Solution: It is clear that A is symmetric.

<u>First method:</u> The eigenvalues of A are: $\lambda_k = 2 + 2\cos(\frac{k\pi}{4})$, $k = 1, 2, 3$. So, $\lambda_1 = 2 + \sqrt{2}, \lambda_2 = 2, \lambda_3 = 2 - \sqrt{2}$.

Since all the eigenvalues of A are strictly positive, then A is symmetric positive definite.

<u>Second method:</u> $\triangle_1 = 2, \triangle_2 = 3, \triangle_3 = 4$.

Since $\triangle_1 > 0$, $\triangle_2 > 0$ and $\triangle_3 > 0$, then A is symmetric positive definite.

Exercise 59. *Consider the following matrix:*

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

1) Calculate $\|A\|_1$, $\|A\|_F$.

2) Study the monotonicity of the matrix A.

3) Show that A is not symmetric positive definite with two different methods.

Solution:

1) $\|A\|_1 = \max\{2, 3, 2\} = 3$. $\|A\|_F = \sqrt{1+1+1+1+1+1+1} = \sqrt{7}$.

2) $\det(A) = 1$,

$$A^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \geq 0,$$

this implies that A is monotone.

3) We show that A is not symmetric positive definite with two different methods.

First method: Since $\triangle_1 = -1 < 0$, then A is not symmetric positive definite.

Second method: The eigenvalues of A are: $\lambda_1 = -1 + \sqrt{2}, \lambda_2 = -1, \lambda_3 = -1 - \sqrt{2}$.

Since there is an eigenvalue λ_k such that $\lambda_k < 0$, then A is not symmetric positive definite.

Exercise 60. *Consider the following matrix:*

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

1) Calculate $\|A\|_2, \|A\|_\infty$.

2) Study the monotonicity of the matrix A.

3) Show that A is symmetric positive definite with two different methods.

Solution:

1) We calculate, $\|A\|_2\,,\ \|A\|_\infty\,.$

We have, the eigenvalues of A are: $3-\sqrt{2}$, 3, $3+\sqrt{2}$.

$\|A\|_2 = 3+\sqrt{2}$.

$\|A\|_\infty = \max\{4,5,4\} = 5$.

2) We calculate the inverse of A if it is exists

$$A^{-1} = \begin{pmatrix} 8/21 & 1/7 & 1/21 \\ 1/7 & 3/7 & 1/7 \\ 1/21 & 1/7 & 8/21 \end{pmatrix},$$

Since $A^{-1} \geq 0$, then A is monotone.

3) It is clear that A is symmetric.

<u>First method:</u> By using the first question, all the eigenvalues of A are strictly positive. Then A is symmetric positive definite.

<u>Second method:</u> $\triangle_1 = 3, \triangle_2 = 8, \triangle_3 = 21$.

All $\triangle_i$ $(i=1,2,3)$ are strictly positive, then A is symmetric positive definite.

Exercise 61. *Let A be a monotone matrix of order N, show that:*

- If there exists a vector $Y \in \mathbb{R}^N$, such that $AY = 1$, then $\|A^{-1}\|_\infty = \|Y\|_\infty\,.$
- If there exists a vector $Y \in \mathbb{R}^N$, such that $AY \geq 1$, then $\|A^{-1}\|_\infty \leq \|Y\|_\infty\,.$

Solution:

1) We suppose that $\exists Y \in \mathbb{R}^N$, such that $AY = 1$. Since A is invertible, then $Y = A^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$

We know that $A^{-1} \geq 0$, this leads to $\|Y\|_\infty = \|A^{-1}.1\|_\infty = \|A^{-1}\|_\infty.$

2) If $\exists Y \in \mathbb{R}^N$, such that $AY \geq 1$. Since $A^{-1} \geq 0$, then $Y \geq A^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \geq 0.$

Thus, $\|Y\|_\infty \geq \|A^{-1}.1\|_\infty = \|A^{-1}\|_\infty.$

Exercise 62. *Consider the following square matrix of order N:*

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

where $h = \frac{1}{N+1}$.

Consider the function $v(x) = \frac{x(1-x)}{2}$, $x \in]0,1[$ and we set $x_i = ih$, $\forall i = 1, ..., N$.

1) Let $V = (v_1, v_2, ..., v_N)^t$, $v_i = v(x_i)$, find U such that $U = A_h V$.

2) Deduce that $\left\|A_h^{-1}\right\|_\infty \leq \frac{1}{8}$.

Solution:

1) By a simple calculus, we get $U = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

2) We have $A_h^{-1}U = V$, then $\left\|A_h^{-1}U\right\|_\infty = \left\|A_h^{-1}\right\|_\infty = \|V\|_\infty \leq \|v\|_\infty \leq \frac{1}{8}$.

CHAPTER 2

Generalities about second order linear PDEs

2.1. Classification of second order linear PDEs in general dimension

There are fundamentally three types of PDEs – hyperbolic, parabolic, and elliptic PDEs. From the physical point of view, these PDEs respectively represents the wave

propagation, the time-dependent diffusion processes, and the steady state or equilibrium processes. Thus, hyperbolic equations model the transport of some physical quantity, such as fluids or waves. Parabolic problems describe evolutionary phenomena that lead to a steady state described by an elliptic equation. And elliptic equations are associated to a special state of a system, in principle corresponding to the minimum of the energy. The general form of a multidimensional second order linear partial differential equation (PDE) is given by:

$$\sum_{i,j=1}^{n} a_{i,j}\left(X\right)\frac{\partial^2 u}{\partial x_j \partial x_i}+\sum_{i=1}^{n} f_i(X)\frac{\partial u}{\partial x_i}+g(X)u=h(X),\ X=(x_1,x_2,...,x_n)\in\Omega\subset\mathbb{R}^n.$$

where $A=\left(a_{i,j}\left(X\right)\right)_{1\leq i,j\leq n}$ is a symmetric matrix of $\mathcal{M}_n\left(\mathbb{R}\right)$.

The classification of such second-order PDEs is based on the sign of the eigenvalues of A as follows:

- The PDE is elliptic if all eigenvalues are non-zero and of the same sign.
- The PDE is hyperbolic if all eigenvalues are non-zero and of the same sign except one of the opposite sign.
- The PDE is parabolic if there is only one null eigenvalue and the others are of the same sign.

Boundary and Initial Conditions

There are three types of boundary conditions. Defining a domain Ω, its boundary $\partial\Omega$, the coordinates n and s normal (outward) and along the boundary, and functions f, g on the boundary, the

three boundary conditions are:

- Dirichlet conditions with $u = f$ on $\partial\Omega$.
- Neumann conditions with $\partial u/\partial n = f$ or $\partial u/\partial s = g$ on $\partial\Omega$.
- Mixed (Robin Fourier) conditions$\partial u/\partial n + ku = f, k > 0$, on $\partial\Omega$.

2.2. Classification of second order linear PDEs in two dimension

The general form of a second order linear partial differential equation (PDE) in two dimension is given by:

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial y \partial x}+c(x,y)\frac{\partial^2 u}{\partial y^2}+d(x,y)\frac{\partial u}{\partial x}+e(x,y)\frac{\partial u}{\partial y}+f(x,y)u = g(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2.$$

where the functions a, b, and c do not vanish simultaneously, because in that case the second-order PDE degenerates to one of first order. Further, the coefficients d, e, f and g are also assumed to be functions of x and y. We shall assume that the function $u(x,y)$ and the coefficients are twice continuously differentiable in some domain Ω. The classification of second-order PDE depends on the form of the leading part of the equation consisting of the second order terms. So, for simplicity of notation, we combine the lower order terms and rewrite the above equation in the following form:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y \partial x} + c(x,y)\frac{\partial^2 u}{\partial y^2} = \Phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right),$$

or using the short-hand notations for partial derivatives,

$$a(x,y)u_{xx} + b(x,y)u_{yx} + c(x,y)u_{yy} = \Phi\left(x, y, u, u_x, u_y\right),$$

its corresponding symmetric matrix is:

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}.$$

As we have seen that the classification or the type of a linear PDE depends on the nature of the eigenvalues, so the characteristic polynomial $P_A(\lambda)$ corresponding to A is:

$$P_A(\lambda) = \lambda^2 - (a+c)\lambda + \frac{4ac - b^2}{4},$$

$P_A(\lambda) = 0$ has two roots λ_1, λ_2 such as $\lambda_1\lambda_2 = \frac{4ac-b^2}{4}$. Then, such classification depends on the sign of the discriminant, $\triangle = b^2 - 4ac$ as follows:

$\Delta < 0 \Leftrightarrow \lambda_1\lambda_2 > 0$, this means that the eigenvalues are non-zero and of the same sign. In this case, the PDE is elliptic.

$\Delta > 0 \Leftrightarrow \lambda_1\lambda_2 < 0$, this means that the eigenvalues are non-zero and of the opposite sign. In this case, the PDE is hyperbolic.

$\Delta = 0 \Leftrightarrow \lambda_1\lambda_2 = 0$, this means that only one of the eigenvalues is zero (because if $\lambda_1 = \lambda_2 = 0$, then $\lambda_1 + \lambda_2 = a + c = 0$, it results $a = -c$ and therefore $b^2 = -4c^2 < 0$, this is a contradiction). In this case, the PDE is parabolic.

We summarize this in the following points

- If $\Delta < 0$, the PDE is elliptic.
- If $\Delta > 0$, the PDE is hyperbolic.
- If $\Delta = 0$, the PDE is parabolic.

Example 63.

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0.$$

2.3. The Laplace equation

Find a solution to the following partial differential equation:

$$\begin{cases} \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < L,\ 0 < y < H, \\ u(0,y) = g_1(y),\ u(L,y) = 0, \\ u(x,0) = u(x,H) = 0. \end{cases}$$

We'll start by assuming that our solution will be in the form,

$$u(x,y) = h(x)\varphi\,(y)\,.$$

Then the separation of variables yields the following two ordinary differential equations that we'll need to solve

$$\begin{aligned} \frac{d^2h}{dx^2} - \lambda h &= 0,\ \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0, \\ h(L) &= 0,\ \varphi\,(0) = \varphi\,(H) = 0. \end{aligned}$$

Taking into account the eigenvalues and eigenfunctions for the boundary value problem here are,

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2,\ \varphi_n(y) = \sin(\frac{n\pi y}{H}),\ n = 1, 2, 3, ...$$

Now that we know what the eigenvalues are let's write down the first differential equation with λ plugged in

$$\frac{d^2h}{dx^2} - \left(\frac{n\pi}{H}\right)^2 h = 0,\ h(L) = 0.$$

Because the coefficient of $h(x)$ in the differential equation above is positive we know that a solution to this is,

$$h(x) = c_1 \cosh\left(\frac{n\pi x}{H}\right) + c_2 \sinh\left(\frac{n\pi x}{H}\right).$$

However, this is not really suited for dealing with the $h(L) = 0$ boundary condition. So, let's also notice that the following is also a solution

$$h(x) = c_1 \cosh\left(\frac{n\pi\,(x-L)}{H}\right) + c_2 \sinh\left(\frac{n\pi\,(x-L)}{H}\right).$$

Applying the lone boundary condition, we get

$$0 = h(L) = c_1.$$

The solution to the first differential equation is now,

$$h(x) = c_2 \sinh\left(\frac{n\pi\,(x-L)}{H}\right).$$

A product solution for this partial differential equation is,

$$u_n(x,y) = B_n \sinh\left(\frac{n\pi\,(x-L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right),\ n = 1, 2, ...$$

Then, the solution to the partial differential equation is,

$$u(x,y) = \sum_{n=1}^{+\infty} B_n \sinh\left(\frac{n\pi\,(x-L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right),$$

and this solution will satisfy the three homogeneous boundary conditions.

To determine the constants all we need to do is apply the final boundary condition

$$
\begin{aligned}
u(0,y) &= \sum_{n=1}^{+\infty} B_n \sinh\left(\frac{n\pi(-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right), \\
B_n &= \frac{2}{H\sinh\left(\frac{n\pi(-L)}{H}\right)}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy, \;\; n=1,2,...
\end{aligned}
$$

2.4. THe heat equation

Find a solution to the following partial differential equation that satisfies the boundary conditions.

$$
\begin{cases}
\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \;\; 0 < x < L, \;\; 0 < t < T, \\
u(x,0) = f(x), \\
u(0,t) = u(L,t) = 0.
\end{cases}
$$

First, we assume that the solution will take the form,

$$
u(x,t) = \varphi(x)\, G(t),
$$

and we plug this into the partial differential equation and boundary conditions. We separate the equation to get a function of only t on one side and a function of only x on the other side and then introduce a separation constant. This leaves us with two ordinary differential equations. Then, we have two ordinary differential equations, which are

$$\frac{dG}{dt} = -k\lambda G,\ \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0,\ \varphi(0) = \varphi(L) = 0.$$

Now, we actually solved the spatial problem,

$$\begin{cases} \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0, \\ \varphi(0) = \varphi(L) = 0. \end{cases}$$

We've got three cases to deal with so let's get going

$\underline{\lambda > 0}$

In this case we know the solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right),$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1.$$

Now applying the second boundary condition, and using the above result of course, gives,

$$0 = \varphi(L) = c_2 \sin\left(\sqrt{\lambda}L\right).$$

Now, we are after non-trivial solutions and so this means we must have,

$$\sin\left(\sqrt{\lambda}L\right) = 0 \Rightarrow \sqrt{\lambda}L = n\pi,\ n = 1, 2, \ldots$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ \varphi_n\left(x\right) = \sin(\frac{n\pi}{L}x), \ n = 1, 2, ...$$

Note that we don't need the c_2 in the eigenfunction as it will just get absorbed into another constant that we'll be picking up later on.

$\underline{\lambda = 0}$

The solution to the differential equation in this case is,

$$\varphi(x) = c_1 + c_2 x,$$

Applying the boundary conditions gives,

$$0 = \varphi(0) = c_1, \ 0 = \varphi(L) = c_2 L \Rightarrow c_2 = 0.$$

So, in this case the only solution is the trivial solution and so $\lambda = 0$ is not an eigenvalue for this boundary value problem.

$\underline{\lambda < 0}$

Here the solution to the differential equation is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}x\right) + c_2 \sinh\left(\sqrt{-\lambda}x\right).$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1,$$

and applying the second gives,

$$0 = \varphi(L) = c_2 \sinh\left(\sqrt{-\lambda}L\right).$$

So, we are assuming $\lambda < 0$ and so $\sqrt{-\lambda}L \neq 0$ and this means that $\sinh\left(\sqrt{-\lambda}L\right) \neq 0$. We therefore we must have c_2 and so we can only get the trivial solution in this case.

Therefore, there will be no negative eigenvalues for this boundary value problem.

The complete list of eigenvalues and eigenfunctions for this problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ \varphi_n(x) = \sin(\frac{n\pi}{L}x), \ n = 1, 2, ...$$

Now let's solve the time differential equation,

$$\frac{dG}{dt} = -k\lambda_n G,$$

and note that even though we now know λ we're not going to plug it in quite yet to keep the mess to a minimum. We will however now use λ_n to remind us that we actually have an infinite number of possible values here.

This is a simple linear (and separable for that matter) 1^{st} order differential equation and so we'll let you verify that the solution is,

$$G(t) = ce^{-k\lambda_n t} = ce^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

Now that we've gotten both of the ordinary differential equations solved we can finally write down a solution. Note however that we have in fact found infinitely many solutions since there are infinitely many solutions (i.e. eigenfunctions) to the spatial problem.

Our product solution are then,

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, n = 1, 2, ...$$

So, the function above will satisfy the heat equation and the boundary condition of zero temperature on the ends of the bar.

2.5. The wave equation

Here, we'll be solving the 1-D wave equation to determine the displacement of a vibrating string.

Find a solution to the following partial differential equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \ 0 < x < L, \ 0 < t < T, \\ u(x,0) = f(x), \ \frac{\partial u}{\partial t}(x,0) = g(x), \\ u(0,t) = u(L,t) = 0. \end{cases}$$

Let's start off with the product solution

$$u(x,t) = \varphi(x)h(t),$$

Plugging this into the two boundary conditions gives,

$$\varphi(0) = 0, \;\; \varphi(L) = 0.$$

Plugging the product solution into the differential equation, separating and introducing a separation constant gives,

$$\begin{aligned}
\frac{\partial^2 \left(\varphi(x)h(t)\right)}{\partial t^2} &= c^2 \frac{\partial^2 \left(\varphi(x)h(t)\right)}{\partial x^2}, \\
\varphi(x)\frac{d^2 h}{dt^2} &= c^2 h(t)\frac{d^2\varphi}{dx^2}, \\
\frac{1}{c^2 h}\frac{d^2 h}{dt^2} &= \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda.
\end{aligned}$$

We moved the c^2 to the left side for convenience and chose $-\lambda$ for the separation constant, so the differential equation for φ would match a known (and solved) case.

The two ordinary differential equations we get from separation of variables are then,

$$\begin{aligned}
\frac{d^2 h}{dt^2} + \lambda c^2 h &= 0, \;\; \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0, \\
\varphi(0) &= 0, \;\; \varphi(L) = 0.
\end{aligned}$$

So the eigenvalues and eigenfunctions for this problem are,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ \varphi_n\left(x\right) = \sin(\frac{n\pi}{L}x), \ n = 1, 2, ...$$

The first ordinary differential equation is now,

$$\frac{d^2h}{dt^2} + \left(c\frac{n\pi}{L}\right)^2 h = 0,$$

and because the coefficient of the h is clearly positive the solution to this is,

$$h(t) = c_1 \cos(c\frac{n\pi}{L}t) + c_2 \sin(c\frac{n\pi}{L}t).$$

Because there is no reason to think that either of the coefficients above are zero we then get two product solutions,

$$\begin{aligned} u_n\left(x,t\right) &= A_n \cos\left(c\frac{n\pi}{L}t\right) \sin(\frac{n\pi}{L}x), n = 1, 2, ... \\ u_n\left(x,t\right) &= B_n \sin\left(c\frac{n\pi}{L}t\right) \sin(\frac{n\pi}{L}x), n = 1, 2, ... \end{aligned}$$

The solution is then,

$$u(x,t) = \sum_{n=1}^{+\infty} \left[A_n \cos\left(c\frac{n\pi}{L}t\right) \sin(\frac{n\pi}{L}x) + B_n \sin\left(c\frac{n\pi}{L}t\right) \sin(\frac{n\pi}{L}x)\right].$$

Now, in order to apply the second initial condition we'll need to differentiate this with respect to t so,

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{+\infty}\left[-c\frac{n\pi}{L}A_n\sin\left(c\frac{n\pi}{L}t\right)\sin(\frac{n\pi}{L}x)+c\frac{n\pi}{L}B_n\cos\left(c\frac{n\pi}{L}t\right)\sin(\frac{n\pi}{L}x)\right].$$

If we now apply the initial conditions we get,

$$\begin{aligned} u(x,0) &= f(x)=\sum_{n=1}^{+\infty}\left[A_n\cos(0)\sin(\frac{n\pi}{L}x)+B_n\sin(0)\sin(\frac{n\pi}{L}x)\right] \\ &= \sum_{n=1}^{+\infty}A_n\sin(\frac{n\pi}{L}x). \end{aligned}$$

$$\frac{\partial u}{\partial t}(x,0)=g(x)=\sum_{n=1}^{+\infty}c\frac{n\pi}{L}B_n\sin(\frac{n\pi}{L}x).$$

Both of these are Fourier sine series. The first is for $f(x)$ on $0\leq x\leq L$ while the second is for $g(x)$ on $0\leq x\leq L$ with a slightly messy coefficient.

It's easier to reuse formulas so using the formulas form the Fourier sine series section we get,

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx,\ n=1,2,3,... \\ \frac{n\pi c}{L}B_n &= \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi}{L}x)dx,\ n=1,2,3,... \end{aligned}$$

Upon solving the second one we get,

$$
\begin{aligned}
A_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx,\ \ n = 1, 2, 3, ... \\
B_n &= \frac{2}{n\pi c}\int_0^L g(x)\sin(\frac{n\pi}{L}x)dx,\ \ n = 1, 2, 3, ...
\end{aligned}
$$

So, there is the solution to the 1-D wave equation.

Exercise 64. *Determine for each PDE: If it is linear, homogeneous, an evolution equation, its order*

1) Heat equation: $u_t = a\Delta u$

2) Poisson Equation: $\Delta u = f$

3) Laplace equation: $\Delta u = 0$

4) Wave equation: $u_{tt} = c^2 \Delta u$

5) Helmholtz equation: $\Delta u + k^2 u = 0$

6) Airy equation: $u_t = -u_{xxx}$

7) Non-dissipative Burger equation: $u_t + u\ u_x = 0$

8) Diffusion equation: $u_t = \operatorname{div}\left(D\left(u\right)\nabla u\right)$

Exercise 65. *Give the classification of each of the following PDEs:*

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial z} + e^x \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y} = u,\ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial x^2} = 0,$
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial t^2} = xtu,\ \frac{\partial^2 u}{\partial t \partial x} + \frac{1}{4}\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$

Exercise 66. *Consider the following PDE:*

$$a\triangle u - 2u_{xy} - 2u_{yz} + u_x = 0,\ a \in \mathbb{R},\ u = u(x, y, z).$$

Discuss according to the values of a the classification of this PDE.

Exercise 67. *In each of the following PDE problems, determine whether they are initial conditions or boundary conditions, and in the latter case give their type (Dirichlet, Neumann or Fourier Robin):*

$$1)\ \begin{cases} u_{xx} + u_{yy} = 0, & \text{sur } \Omega \\ u = g, & \text{sur } \partial\Omega \end{cases}$$

$$2)\ \begin{cases} u_{tt} = c^2 u_{xx}, &]0, +\infty[\times]0, +\infty[\\ u_x(0,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & x \geq 0 \\ u_t(x,0) = g(x), & x \geq 0 \end{cases}$$

CHAPTER 3

Finite differences and their application in solving PDEs

Finite-difference methods (FDM) are a set of numerical techniques used in numerical analysis that approximate derivatives using finite differences in order to solve differential equations. Discreteization, or the division of the spatial and time domains into a finite number of intervals, is used to approximate the values of the solution at the ends of the intervals. This is done by solving algebraic equations that involve values from close points and finite differences.

Ordinary differential equations (ODE) and partial differential equations (PDE), which may or may not be nonlinear, are transformed into a system of linear equations by finite difference methods so that matrix algebra techniques can be used to solve them.

3.1. Principle of finite differences method

The principle of finite difference methods is close to the numerical schemes used to solve ordinary differential equations. It consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domain is partitioned in space and in time and approximations of the solution are computed at the space or time points. The error between the numerical solution and the exact solution is determined by the error that is commited by going from a differential operator to a difference operator. This error is called the discretization error or truncation error. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation.

3.1.1. Approximation of the first order derivative

Forward formula

Suppose the function u is $\mathcal{C}^2$ continuous in the neighborhood of x. For any $h > 0$ we have:

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(\zeta),$$

where $\zeta \in \,]x, x+h[\,$.

Then, the first order derivative can be written by

$$\begin{aligned} u'(x) &= \frac{u(x+h) - u(x)}{h} - \frac{h}{2}u''(\zeta) \\ &= \frac{u(x+h) - u(x)}{h} + \mathcal{O}(h) \,. \end{aligned}$$

For equidistant points x_i, we have

$$\begin{aligned} u_i' \quad &: = u'(x_i) \simeq \frac{u(x_i+h) - u(x_i)}{h} \\ &= \frac{u_{i+1} - u_i}{h}. \end{aligned}$$

In two dimension, we have

$$\begin{aligned}
\frac{\partial u}{\partial x}(x_i, y_j) &= \frac{u(x_i + \triangle x, y_j) - u(x_i, y_j)}{\triangle x} - \frac{\triangle x}{2}\frac{\partial^2 u}{\partial x^2}(\zeta_i, y_j) \\
&\simeq \frac{u_{i+1,j} - u_{i,j}}{\triangle x} + \mathcal{O}(\triangle x),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial y}(x_i, y_j) &= \frac{u(x_i, y_j + \triangle y) - u(x_i, y_j)}{\triangle y} - \frac{\triangle y}{2}\frac{\partial^2 u}{\partial y^2}(x_i, \zeta_j) \\
&\simeq \frac{u_{i,j+1} - u_{i,j}}{\triangle y} + \mathcal{O}(\triangle y).
\end{aligned}$$

Backward formula

Suppose the function u is $\mathcal{C}^2$ continuous in the neighborhood of x. For any $h > 0$ we have:

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(\zeta),$$

where $\zeta \in \,]x-h, x[$.

Then, the first order derivative is written by

$$\begin{aligned}
u'(x) &= \frac{u(x) - u(x-h)}{h} + \frac{h}{2}u''(\zeta) \\
&= \frac{u(x) - u(x-h)}{h} + \mathcal{O}(h).
\end{aligned}$$

For equidistant points x_i, we have

$$\begin{aligned} u'_i &\simeq \frac{u(x_i) - u(x_i - h)}{h} \\ &= \frac{u_i - u_{i-1}}{h}. \end{aligned}$$

In two dimension, we have

$$\begin{aligned} \frac{\partial u}{\partial x}(x_i, y_j) &= \frac{u(x_i, y_j) - u(x_i - \triangle x, y_j)}{\triangle x} - \frac{\triangle x}{2}\frac{\partial^2 u}{\partial x^2}(\zeta_i, y_j) \\ &\simeq \frac{u_{i,j} - u_{i-1,j}}{\triangle x} + \mathcal{O}\left(\triangle x\right), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial y}(x_i, y_j) &= \frac{u(x_i, y_j) - u(x_i, y_j - \triangle y)}{\triangle y} - \frac{\triangle y}{2}\frac{\partial^2 u}{\partial y^2}(x_i, \zeta_j) \\ &\simeq \frac{u_{i,j} - u_{i,j-1}}{\triangle y} + \mathcal{O}\left(\triangle y\right). \end{aligned}$$

Central formula

In order to improve the accuracy of the approximation, we define the central differences approximation, by taking the points $x - h$ and $x + h$ into account. Suppose that the function u is three times differentiable in the vicinity of x:

$$\begin{aligned} u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u^{(3)}(\zeta_1), \\ u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u^{(3)}(\zeta_2), \end{aligned}$$

where $\zeta_1 \in \,]x, x+h[\,$, $\zeta_2 \in \,]x-h, x[\,$.

By subtracting these two expressions we obtain,

$$\begin{aligned} u'(x) &= \frac{u(x+h)-u(x-h)}{2h} - \frac{h^2}{12}\left(u^{(3)}(\zeta_1) - u^{(3)}(\zeta_2)\right) \\ &= \frac{u(x+h)-u(x-h)}{2h} + \mathcal{O}\left(h^2\right). \end{aligned}$$

For equidistant points x_i, we have

$$\begin{aligned} u'_i &\simeq \frac{u(x_i+h)-u(x_i-h)}{2h} \\ &= \frac{u_{i+1}-u_{i-1}}{2h}. \end{aligned}$$

In two dimension, we have

$$\begin{aligned} \frac{\partial u}{\partial x}(x_i, y_j) &= \frac{u(x_i+\triangle x, y_j) - u(x_i - \triangle x, y_j)}{2\triangle x} \\ &\quad -\frac{(\triangle x)^2}{12}\left(\frac{\partial^3 u}{\partial x^3}(\zeta_{1,i}, y_j) - \frac{\partial^3 u}{\partial x^3}(\zeta_{2,i}, y_j)\right) \\ &= \frac{u_{i+1,j} - u_{i-1,j}}{2\triangle x} + \mathcal{O}\left(\triangle x\right)^2, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial y}(x_i, y_j) &= \frac{u(x_i, y_j+\triangle y) - u(x_i, y_j - \triangle y)}{2\triangle y} \\ &\quad -\frac{(\triangle y)^2}{12}\left(\frac{\partial^3 u}{\partial y^3}(x_i, \zeta_{1,j}) - \frac{\partial^3 u}{\partial y^3}(x_i, \zeta_{2,j})\right) \\ &= \frac{u(x_i, y_j+\triangle y) - u(x_i, y_j - \triangle y)}{2\triangle y} + \mathcal{O}\left(\triangle y\right)^2. \end{aligned}$$

3.1.2. Approximation of the second order derivative

Forward formula

Suppose the function u is $\mathcal{C}^3$ continuous in the neighborhood of x. For any $h > 0$ we have:

$$\begin{aligned}
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u^{(3)}(\zeta_1), \\
u(x+2h) &= u(x) + 2hu'(x) + 2h^2u''(x) + \frac{8}{6}h^3u^{(3)}(\zeta_2),
\end{aligned}$$

where $\zeta_1 \in \left]x, x+h\right[, \zeta_2 \in \left]x, x+2h\right[$.

This yields,

$$\begin{aligned}
u''(x) &= \frac{u(x+2h) - 2u(x+h) + u(x)}{h^2} - \frac{h}{6}\left(8u^{(3)}(\zeta_2) - 2u^{(3)}(\zeta_1)\right) \\
&= \frac{u(x+2h) - 2u(x+h) + u(x)}{h^2} + \mathcal{O}(h).
\end{aligned}$$

For equidistant points x_i, we have

$$\begin{aligned}
u_i'' &\simeq \frac{u(x+2h) - 2u(x+h) + u(x)}{h^2} \\
&= \frac{u_{i+2} - 2u_{i+1} + u_i}{h^2}.
\end{aligned}$$

In two dimension, we have

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x_i, y_j) &= \frac{u(x_i + 2 \triangle x, y_j) - 2u(x_i + \triangle x, y_j) + u(x_i, y_j)}{(\triangle x)^2} \\
&\quad - \frac{\triangle x}{6}\left(8\frac{\partial^3 u}{\partial x^3}(\zeta_{2,i}, y_j) - 2\frac{\partial^3 u}{\partial x^3}(\zeta_{1,i}, y_j)\right) \\
&= \frac{u(x_i + 2 \triangle x, y_j) - 2u(x_i + \triangle x, y_j) + u(x_i, y_j)}{(\triangle x)^2} + \mathcal{O}(\triangle x) \\
&\simeq \frac{u(x_i + 2 \triangle x, y_j) - 2u(x_i + \triangle x, y_j) + u(x_i, y_j)}{(\triangle x)^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2}(x_i, y_j) &= \frac{u(x_i, y_j + 2 \triangle y) - 2u(x_i, y_j + \triangle y) + u(x_i, y_j)}{(\triangle y)^2} \\
&\quad - \frac{\triangle y}{6}\left(8\frac{\partial^3 u}{\partial y^3}(x_i, \zeta_{2,j}) - 2\frac{\partial^3 u}{\partial y^3}(x_i, \zeta_{1,j})\right) \\
&= \frac{u(x_i, y_j + 2 \triangle y) - 2u(x_i, y_j + \triangle y) + u(x_i, y_j)}{(\triangle y)^2} + \mathcal{O}(\triangle y) \\
&\simeq \frac{u(x_i, y_j + 2 \triangle y) - 2u(x_i, y_j + \triangle y) + u(x_i, y_j)}{(\triangle y)^2},
\end{aligned}$$

Backward formula

For any $h > 0$ we have:

$$\begin{aligned}
u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u^{(3)}(\zeta_1), \\
u(x-2h) &= u(x) - 2hu'(x) + 2h^2u''(x) - \frac{8}{6}h^3u^{(3)}(\zeta_2),
\end{aligned}$$

then

$$u''(x) = \frac{u(x) - 2u(x-h) + u(x-2h)}{h^2} - \frac{h}{6}\left(2u^{(3)}(\zeta_1) - 8u^{(3)}(\zeta_2)\right).$$

For equidistant points x_i, we have

$$\begin{aligned} u_i'' &= \frac{u_i - 2u_{i-1} + u_{i-2}}{h^2} + \mathcal{O}(h) \\ &\simeq \frac{u_i - 2u_{i-1} + u_{i-2}}{h^2}. \end{aligned}$$

In two dimension, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x_i, y_j) &= \frac{u(x_i, y_j) - 2u(x_i - \triangle x, y_j) + u(x_i - 2\triangle x, y_j)}{(\triangle x)^2} \\ &\quad -\frac{\triangle x}{6}\left(2\frac{\partial^3 u}{\partial x^3}(\zeta_{1,i}, y_j) - 8\frac{\partial^3 u}{\partial x^3}(\zeta_{2,i}, y_j)\right) \\ &= \frac{u(x_i, y_j) - 2u(x_i - \triangle x, y_j) + u(x_i - 2\triangle x, y_j)}{(\triangle x)^2} + \mathcal{O}(\triangle x) \\ &\simeq \frac{u(x_i, y_j) - 2u(x_i - \triangle x, y_j) + u(x_i - 2\triangle x, y_j)}{(\triangle x)^2}, \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2}(x_i, y_j) &= \frac{u(x_i, y_j) - 2u(x_i, y_j - \triangle y) + u(x_i, y_j - 2 \triangle y)}{(\triangle y)^2} \\
&\quad - \frac{\triangle y}{6}\left(2\frac{\partial^3 u}{\partial y^3}(x_i, \zeta_{1,j}) - 8\frac{\partial^3 u}{\partial y^3}(x_i, \zeta_{2,j})\right) \\
&= \frac{u(x_i, y_j) - 2u(x_i, y_j - \triangle y) + u(x_i, y_j - 2 \triangle y)}{(\triangle y)^2} + \mathcal{O}(\triangle y) \\
&\simeq \frac{u(x_i, y_j) - 2u(x_i, y_j - \triangle y) + u(x_i, y_j - 2 \triangle y)}{(\triangle y)^2}.
\end{aligned}
$$

Central formula

Suppose the function u is $\mathcal{C}^4$ continuous in the neighborhood of x. For any $h > 0$ we have:

$$
\begin{aligned}
u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(\zeta_1), \\
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(\zeta_2),
\end{aligned}
$$

then, we obtain

$$
\begin{aligned}
u''(x) &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - \frac{h^2}{24}\left(u^{(4)}(\zeta_1) + u^{(4)}(\zeta_2)\right) \\
&= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}\left(h^2\right).
\end{aligned}
$$

For equidistant points x_i, we have

$$\begin{aligned} u_i'' &= \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \mathcal{O}\left(h^2\right) \\ &\simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \end{aligned}$$

In two dimension, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x_i, y_j) &= \frac{u(x_i + \triangle x, y_j) - 2u(x_i, y_j) + u(x_i - \triangle x, y_j)}{(\triangle x)^2} \\ &\quad - \frac{(\triangle x)^2}{24}\left(\frac{\partial^4 u}{\partial x^4}(\zeta_{1,i}, y_j) + \frac{\partial^4 u}{\partial x^4}(\zeta_{2,i}, y_j)\right) \\ &= \frac{u(x_i + \triangle x, y_j) - 2u(x_i, y_j) + u(x_i - \triangle x, y_j)}{(\triangle x)^2} + \mathcal{O}\left(\triangle x\right)^2 \\ &\simeq \frac{u(x_i + \triangle x, y_j) - 2u(x_i, y_j) + u(x_i - \triangle x, y_j)}{(\triangle x)^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2}(x_i, y_j) &= \frac{u(x_i, y_j + \triangle y) - 2u(x_i, y_j) + u(x_i, y_j - \triangle y)}{(\triangle y)^2} \\ &\quad - \frac{(\triangle y)^2}{24}\left(\frac{\partial^4 u}{\partial y^4}(x_i, \zeta_{1,j}) + \frac{\partial^4 u}{\partial y^4}(x_i, \zeta_{2,j})\right) \\ &= \frac{u(x_i, y_j + \triangle y) - 2u(x_i, y_j) + u(x_i, y_j - \triangle y)}{(\triangle y)^2} + \mathcal{O}\left(\triangle y\right)^2 \\ &\simeq \frac{u(x_i, y_j + \triangle y) - 2u(x_i, y_j) + u(x_i, y_j - \triangle y)}{(\triangle y)^2}. \end{aligned}$$

3.1.3. Consistency, stability and convergence of numerical schemes for PDE's.

A problem in differential equations can rarely be solved analytically, and so often is discretized, resulting in a discrete problem which can be solved in a finite sequence of algebraic operations, efficiently implementable on a computer. The error in a discretization is the difference between the solution of the original problem and the solution of the discrete problem, which must be defined so that the difference makes sense and can be quantified. Consistency of a discretization refers to a quantitative measure of the extent to which the exact solution satisfies the discrete problem. Stability of a discretization refers to a quantitative measure of the well-posedness of the discrete problem. A fundamental result in numerical analysis is that the error of a discretization may be bounded in terms of its consistency and stability.

Definition 68. *A numerical scheme is consistent if its discrete operator (with finite differences) converges towards the continuous operator (with derivatives) of the PDE for $\triangle t$, $\triangle x \to 0$ (vanishing truncation error).*

$$L\overline{U}_{\triangle t,\triangle x} - L_{\triangle t,\triangle x}U_{\triangle t,\triangle x} \to 0, \ \ when\ \triangle t, \triangle x \to 0,$$

where $\overline{U}_{\triangle t,\triangle x}$ is the exact values of the function u at $U_{\triangle t,\triangle x}$.

Definition 69. *A numerical scheme is consistent of order p in space and order q in times $(p, q \geq 1)$ if*

$$\left| L\overline{U}_{\triangle t, \triangle x} - L_{\triangle t, \triangle x} U_{\triangle t, \triangle x} \right| \leq \mathcal{O}\left(\triangle x\right)^{p} + \mathcal{O}\left(\triangle t\right)^{q}.$$

Definition 70. *A numerical scheme is stable in sense of the norm $\|.\|$ if*

$$\|U_{\triangle t, \triangle x}\| \leq C,$$

C is constant and does not depend on $\triangle t$ and $\triangle x$.

Proposition 71. *For a numerical scheme,the following relation holds between the consistency, stability: and convergence*

$$consistency + stability \Longrightarrow convergence.$$

3.2. Finite differences formulation for an elliptic problem

3.2.1. Finite differences formulation for one-dimension

Example 72. *We consider the following problem:*

$$(P_c) \begin{cases} -u''(x) = f(x), \;\; x \in \left]0, 1\right[, \\ u(0) = u(1) = 0, \end{cases}$$

where$f \in C([0, 1])$ is given.

1) Discretize this problem using the central finite differences method.

2) Write the discrete problem in matrix form $A_h U_h = b_h$, then show that it admits a unique solution.

3) Prove that this scheme is consistent and stable in the sense of the $\|.\|$.
Deduce the convergence of this method.

- We consider the following division of the interval $]0, 1[$ as: $x_i = ih$, $h = \frac{1}{N+1}$, $i = 0, 1, ..., N+1$.

Approximate now the second order derivative by a central finite differences formula, we obtain the following discrete problem:

$$(P_h) \begin{cases} -\dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i, i = 1, ..., N, \\ u_0 = 0, \ u_{N+1} = 0, \end{cases}$$

or in an other form

$$(P_h) \begin{cases} \dfrac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} = f_i, i = 1, ..., N, \\ u_0 = 0, \ u_{N+1} = 0, \end{cases}$$

- Matrix formulation

for $i = 1$, we have $\dfrac{-u_2 + 2u_1 - u_0}{h^2} = f_1$,
for $i = 2$, we have $\dfrac{-u_3 + 2u_2 - u_1}{h^2} = f_2$,
$\vdots$
for $i = N-1$, we have $\dfrac{-u_N + 2u_{N-1} - u_{N-2}}{h^2} = f_{N-1}$,
for $i = N$, we have $\dfrac{-u_{N+1} + 2u_N - u_{N-1}}{h^2} = f_N$.

Since $u_0 = u_{N+1} = 0$, then the problem (P_h) can be written in the matrix form as:

$$A_h U_h = b_h,$$

where

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \; U_h = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \; b_h = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}.$$

We know very well that A_h is monotone, then it is invertible. Hence, $U_h = A_h^{-1} b_h$. Therefore, the problem (P_h) has a unique solution.

- Consistency, stability and convergence

$$\begin{aligned} L_h\left(U_h\right)_i &= \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2}, \\ L\left(\bar{U}_h\right)_i &= \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + \frac{h^2}{24}\left(u^{(4)}\left(\zeta_{1,i}\right) + u^{(4)}\left(\zeta_2, i\right)\right), \end{aligned}$$

then

$$L\left(\bar{U}\right)_i - L_h\left(U_h\right)_i = \frac{h^2}{24}\left(u^{(4)}\left(\zeta_{1,i}\right) + u^{(4)}\left(\zeta_2, i\right)\right).$$

It results

$$\begin{aligned}\left|L\left(\bar{U}_h\right)_i - L_h\left(U_h\right)_i\right| &\leq \frac{2\left\|u^{(4)}\right\|_\infty}{24}h^2 \\ &\leq \frac{\left\|u^{(4)}\right\|_\infty}{12}h^2.\end{aligned}$$

This implies that the scheme is consistency of order 2.

Concerning the stability, we need the following lemma.

Lemma 73. *Consider the following square matrix of order N:*

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \ h = \frac{1}{N+1}.$$

Then, $\left\|A_h^{-1}\right\|_\infty \leq \frac{1}{8}$.

Proof. Consider the function $v(x) = \frac{x(1-x)}{2}$, $x \in]0,1[$ and we set $x_i = ih$, $\forall i = 1, ..., N$. We will follow the following steps:

1) Let $V = (v_1, v_2, ..., v_N)^t$, $v_i = v(x_i)$, find U such that $U = A_h V$.

2) Deduce that $\left\|A_h^{-1}\right\|_\infty \leq \frac{1}{8}$.

Using simple calculus, we find that

$$U = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

then

$$A_h^{-1} U = V.$$

On the other hand, we have

$$\begin{aligned} \left\| A_h^{-1} U \right\|_\infty &= \left\| A_h^{-1} \right\|_\infty \\ &= \| V \|_\infty \\ &= \max_{i=1,\ldots,N} |v(x_i)| \\ &\leq \| v \|_\infty . \end{aligned}$$

By studying the function $v(x) = \frac{x(1-x)}{2}$ on $]0, 1[$ we find that $\|v\|_\infty = \frac{1}{8}$. $\square$

Returning to the example, we obtain

$$\begin{aligned} \| U_h \|_\infty &= \left\| A_h^{-1} b_h \right\|_\infty \\ &\leq \left\| A_h^{-1} \right\|_\infty \| b_h \|_\infty \\ &\leq \frac{1}{8} \| b_h \|_\infty . \end{aligned}$$

Thus, the scheme is stable in $\infty - norm$ sense.

Consequently, the scheme is convergent.

Example 74. *We consider a bounded domain* $\Omega =]0,1[\subset \mathbb{R}$ *and* $u : \bar{\Omega} \to \mathbb{R}$ *solving the non-homogeneous Dirichlet problem:*

$$(P_c) \begin{cases} -u''(x) + c(x)u(x) = f(x), x \in]0,1[, \\ u(0) = \alpha,\ u(1) = \beta, \end{cases}$$

where c and f are two given functions, defined on $\bar{\Omega}$, $c > 0$.

Suppose functions c and f are at least such that $c, f \in C^0(\bar{\Omega})$. The problem has a unique solution.

Let's divide the interval $]0,1[$ as: $x_i = ih$, $h = \frac{1}{N+1}$, $i = 0, 1, ..., N+1$.

Approximating the second order derivative by a central finite differences formula, we obtain the following discrete problem:

$$(P_h) \begin{cases} -\dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + c_i u_i = f_i, i = 1, ..., N, \\ u_0 = \alpha,\ u_{N+1} = \beta, \end{cases}$$

or in a simpler way

$$(P_h) \begin{cases} \dfrac{-u_{i+1} + (2 + c_i h^2)\, u_i - u_{i-1}}{h^2} = f_i, i = 1, ..., N, \\ u_0 = \alpha,\ u_{N+1} = \beta, \end{cases}$$

for $i = 1$, we have $\dfrac{-u_2 + (2 + c_1 h^2)\, u_1 - u_0}{h^2} = f_1$, for $i = 2$, we have $\dfrac{-u_3 + (2 + c_2 h^2)\, u_2 - u_1}{h^2} = f_2$,

$\vdots$

for $i = N-1$, we have $\dfrac{-u_N + (2 + c_{N-1} h^2)\, u_{N-1} - u_{N-2}}{h^2} = f_{N-1}$,

for $i = N$, we have $\dfrac{-u_{N+1} + (2 + c_N h^2)\, u_N - u_{N-1}}{h^2} = f_N$.

The problem (P_h) can be written in the matrix form as:

$$A_h U_h = b_h,$$

where A_h is a tridiagonal matrix defined as:

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 + c_1 h^2 & -1 & 0 & \cdots & 0 \\ -1 & 2 + c_2 h^2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 + c_N h^2 \end{pmatrix}, \; U_h = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$$

and

$$b_h = \begin{pmatrix} f_1 + \dfrac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_N + \dfrac{\beta}{h^2} \end{pmatrix}.$$

The question raised by this formulation is related to the existence of a solution. In other words, we have to determine if the matrix A_h is invertible or not.

In fact, the answer is positive, because A_h is symmetric definite positive. To clarify, we have

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} + \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_N \end{pmatrix},$$

the matrix A_h is a sum of two symmetric definite positive matrices, then it is symmetric definite positive.

Hence, A_h is invertible and then the problem has a unique solution.

3.2.2. Finite differences formulation for two-dimension

Elliptic PDE's are equations with second derivatives in space and no time derivative. The most important examples are Laplace's equation

$$-\triangle u = -u_{xx} - u_{yy} = 0,$$

and the Poisson equation

$$-\triangle u = f(x, y).$$

These equations are used in a large variety of steady-state physical situations such as: steady state heat problems, steady state chemical distributions, electrostatic potentials, elastic deformation and steady state fluid flows.

A good model problem in this dimension is the elastic deflection of a membrane. Suppose that a membrane such as a sheet of rubber is stretched across a rectangular frame. If some of the edges of the frame are bent, or if forces are applied to the sheet then it will deflect by an amount $u(x, y)$ at each point (x, y). This u will satisfy the boundary

value problem:

$$\begin{aligned} -u_{xx} - u_{yy} &= f(x, y), \text{ in } \Omega, \\ u(x, y) &= g(x, y), \text{ in } \partial\Omega, \end{aligned}$$

where Ω is the rectangle,$\partial\Omega$ is the edge of the rectangle.

Now, we suppose the rectangle is described by

$$\Omega = \{0 \leq x \leq 1,\ 0 \leq y \leq 1\},$$

and

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0.$$

We will divide Ω in sub-rectangles. If we have $M + 1$ subdivisions in the x direction and $N + 1$ subdivisions in the y

direction, then the step size in the x and y directions respectively are

$$\begin{aligned} h &= \frac{1}{M+1},\ k = \frac{1}{N+1}, x_i = ih, y_j = jk, \\ i &= 0, ..., M+1, j = 0, ..., N+1, \end{aligned}$$

by replacing u_{xx} and u_{yy} by their central finite differences formulas, we obtain the following finite difference equations

$$\frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h^2} + \frac{-u_{i,j+1} + 2u_{i,j} - u_{i,j-1}}{k^2} = f(x_i, y_j) = f_{i,j},$$
$$i = 1, ..., M, j = 1, ..., N,$$
$$u_{0,j} = u_{M+1,j} = u_{i,0} = u_{i,N+1} = 0, i = 0, ..., M+1, j = 0, ..., N+1.$$

By changing the values of i, j, we get the following system:

$$A_{h,k}U_{h,k} = b_{h,k},$$

where

$$A_{h,k} = \frac{1}{k^2}\begin{pmatrix} B & -I_M & 0 & & \cdots & & 0 \\ -I_M & B & -I_M & & & & \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ & & -I_M & B & -I_M & & \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ & & & & -I_M & B & -I_M \\ 0 & & & & 0 & -I_M & B \end{pmatrix} \in \mathcal{M}_{MN}(\mathbb{R}),$$

$U_{h,k} = (u_{1,1}, u_{2,1}, ..., u_{M,1}, u_{1,2}, ..., u_{M,2}, ..., u_{1,N}, u_{2,N}, ..., u_{M,N})^t$,

$b_{h,k} = (f_{1,1}, f_{2,1}, ..., f_{M,1}, f_{1,2}, ..., f_{M,2}, ..., f_{1,N}, f_{2,N}, ..., f_{M,N})^t$, I_M is the unit matrix of $\mathcal{M}_M(\mathbb{R})$ and

$$B = \begin{pmatrix} 2\left(\frac{k^2}{h^2}+1\right) & -\frac{k^2}{h^2} & 0 & & \cdots & & 0 \\ -\frac{k^2}{h^2} & 2\left(\frac{k^2}{h^2}+1\right) & -\frac{k^2}{h^2} & & & & \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ & & -\frac{k^2}{h^2} & 2\left(\frac{k^2}{h^2}+1\right) & -\frac{k^2}{h^2} & & \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ & & & & -\frac{k^2}{h^2} & \ddots & \frac{k^2}{h^2} \\ 0 & & & & 0 & -\frac{k^2}{h^2} & 2\left(\frac{k^2}{h^2}+1\right) \end{pmatrix} \in \mathcal{M}_M(\mathbb{R}),$$

since B is a symmetric definite positive matrix, then $A_{h,k}$ is also the same. Therefore, $A_{h,k}$ is invertible and $U_{h,k} = A_{h,k}^{-1} b_{h,k}$.

Let's study the consistency of our problem as follows:

$$
\begin{aligned}
L\overline{U}_{\triangle t,\triangle x} - L_{\triangle t,\triangle x}U_{\triangle t,\triangle x} \;=\;& -\frac{u(x_i+\triangle x, y_j) - 2u(x_i,y_j) + u(x_i-\triangle x, y_j)}{(\triangle x)^2} \\
& + \frac{(\triangle x)^2}{24}\left(\frac{\partial^4 u}{\partial x^4}(\zeta_{1,i}, y_j) + \frac{\partial^4 u}{\partial x^4}(\zeta_{2,i}, y_j)\right) \\
& - \frac{u(x_i, y_j+\triangle y) - 2u(x_i,y_j) + u(x_i, y_j-\triangle y)}{(\triangle y)^2} \\
& + \frac{(\triangle y)^2}{24}\left(\frac{\partial^4 u}{\partial y^4}(x_i, \zeta_{1,j}) + \frac{\partial^4 u}{\partial y^4}(x_i, \zeta_{2,j})\right) \\
& + \frac{u(x_i+\triangle x, y_j) - 2u(x_i,y_j) + u(x_i-\triangle x, y_j)}{(\triangle x)^2} \\
& + \frac{u(x_i, y_j+\triangle y) - 2u(x_i,y_j) + u(x_i, y_j-\triangle y)}{(\triangle y)^2}
\end{aligned}
$$

then, $\left|L\overline{U}_{\triangle t,\triangle x} - L_{\triangle t,\triangle x}U_{\triangle t,\triangle x}\right| \leq \dfrac{\left\|\dfrac{\partial^4 u}{\partial x^4}\right\|_\infty}{12}(\triangle x)^2 + \frac{\left\|\dfrac{\partial^4 u}{\partial y^4}\right\|_\infty}{12}(\triangle y)^2$
$\leq \dfrac{\max\left(\left\|\dfrac{\partial^4 u}{\partial x^4}\right\|_\infty, \left\|\dfrac{\partial^4 u}{\partial y^4}\right\|_\infty\right)}{12}\left((\triangle x)^2 + (\triangle y)^2\right)$. Hence, the scheme is consistency of order 2 in x and y.

3.3. Finite differences scheme for time-dependent problems

3.3.1. Finite differences schemes for the Heat Equation

In this subsection, we apply a variety of finite differences techniques to approximate the solutions of initial boundary value problems associated with the heat equation.

Consider the following initial-boundary value problem for the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \ 0 < x < 1, 0 < t < T, \\ u(x,0) = u_0\left(x\right), \ 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \ 0 < t < T, \end{cases}$$

where the unknown $u = u(x,t)$ is a function of space x and time t.

This partial differential equation arises in a variety of contexts in mathematical physics. A temperature model for a one-dimensional heat-conductive rod is the most commonly used example.

An Explicit Scheme

In this method we approximate the first order time derivative by the forward finite differences formula, and the second order derivative of space by the central finite differences formula, we get

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\triangle t} - \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\left(\triangle x\right)^2} = 0, \ j = 1, ..., M, \ n = 1, ..., N, \\ u_j^0 \text{ is given for } j = 0, ..., M+1, \\ u_0^n = u_{M+1}^n = 0, \ n = 0, ..., N+1, \end{cases}$$

where $x_j = j\triangle x$, $j = 0, ..., M+1$, $t^n = n\triangle t$, $n = 0, ..., N+1$, $\triangle x = \dfrac{1}{M+1}$, $\triangle t = \dfrac{T}{N+1}$et $u_j^n := u\left(x_j, t^n\right)$.

The discrete equation can be written as

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\triangle x)^2},$$

then

$$\begin{aligned} u_j^{n+1} &= \frac{\triangle t}{(\triangle x)^2}\left[u_{j+1}^n - 2u_j^n + u_{j-1}^n\right] + u_j^n \\ &= \frac{\triangle t}{(\triangle x)^2} u_{j+1}^n + \left(1 - 2\frac{\triangle t}{(\triangle x)^2}\right) u_j^n + \frac{\triangle t}{(\triangle x)^2} u_{j-1}^n, \end{aligned}$$

If we put $r = \frac{\triangle t}{(\triangle x)^2}$, then u_j^{n+1} becomes

$$(*) \qquad u_j^{n+1} = r u_{j+1}^n + (1 - 2r)\, u_j^n + r u_{j-1}^n.$$

To obtain the matrix form, we vary the values of j of 1 to M as follows:

$$\begin{aligned} u_1^{n+1} &= r u_2^n + (1-2r)\, u_1^n + r u_0^n \\ u_2^{n+1} &= r u_3^n + (1-2r)\, u_2^n + r u_1^n \\ \vdots &= \vdots \\ u_{M-1}^{n+1} &= r u_M^n + (1-2r)\, u_{M-1}^n + r u_{M-2}^n \\ u_M^{n+1} &= r u_{M+1}^n + (1-2r)\, u_M^n + r u_{M-1}^n \end{aligned},$$

taking into account that $u_0^n = u_{M+1}^n = 0$, we get

$$U^{n+1} = AU^n,$$

where

$$U^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{M-1}^{n+1} \\ u_M^{n+1} \end{pmatrix}, \; U^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{M-1}^n \\ u_M^n \end{pmatrix}, \; A = \begin{pmatrix} 1-2r & r & 0 & \cdots & 0 \\ r & 1-2r & r & \ddots & \vdots \\ 0 & r & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & r \\ 0 & \cdots & 0 & r & 1-2r \end{pmatrix}.$$

By repetition, we find

$$\begin{aligned} U^{n+1} &= AU^n \\ &= A^{n+1}U^0. \end{aligned}$$

In general, the stability holds if $\|A\| \leq 1$.

<u>We study the stability in $\infty - norm$ sense as follows:</u>

The scheme is stable if and only if $r \leq \frac{1}{2}$ and $r + 1 - 2r + r \leq 1$.

The inequality $r + 1 - 2r + r \leq 1$ is always verified, because $r + 1 - 2r + r = 1$.

Concerning the consistency, we have

$$
\begin{aligned}
L\overline{U}_{\triangle t,\triangle x} - L_{\triangle t,\triangle x}U_{\triangle t,\triangle x} &= \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) - \left(\frac{u_j^{n+1} - u_j^n}{\triangle t} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\triangle x)^2}\right) \\
&= \frac{u_j^{n+1} - u_j^n}{\triangle t} - \frac{\triangle t}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \zeta_n) \\
&\quad - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\triangle x)^2} + \frac{(\triangle x)^2}{24}\left(\frac{\partial^4 u}{\partial x^4}(\zeta_{1,j}, t^n) + \frac{\partial^4 u}{\partial x^4}(\zeta_{2,j}, t^n)\right) \\
&\quad - \left(\frac{u_j^{n+1} - u_j^n}{\triangle t} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\triangle x)^2}\right) \\
&= -\frac{\triangle t}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \zeta_n) + \frac{(\triangle x)^2}{24}\left(\frac{\partial^4 u}{\partial x^4}(\zeta_{1,j}, t^n) + \frac{\partial^4 u}{\partial x^4}(\zeta_{2,j}, t^n)\right),
\end{aligned}
$$

hence,

$$\left|L\overline{U}_{\triangle t,\triangle x} - L_{\triangle t,\triangle x}U_{\triangle t,\triangle x}\right| \leq \mathcal{O}(\triangle t) + \mathcal{O}(\triangle x)^2,$$

then the scheme is consistent of order 1 for time and of order 2 for espace.

Finally, this scheme is convergent in sense of $\infty - norm$ under the condition of C.F.L $\left(r = \frac{\triangle t}{(\triangle x)^2} \leq \frac{1}{2}\right)$.

Now, we can examinate the stability in $L^2 - norm$ sense as follows:

$$\|A\|_2 = \max_{1 \leq i \leq M} \{|\lambda_i|\} \leq 1.$$

The eigenvalues of A are written by:

$$\begin{aligned}\lambda_i &= 1-2r+2r\cos\left(\frac{k\pi}{M+1}\right)\\ &= 1-2r+2r\left(1-2\sin^2\left(\frac{k\pi}{2(M+1)}\right)\right)\\ &= 1-4r\sin^2\left(\frac{k\pi}{2(M+1)}\right).\end{aligned}$$

Hence,

$$\begin{aligned}|\lambda_i|\leq 1 &\Leftrightarrow \left|1-4r\sin^2\left(\tfrac{k\pi}{2(M+1)}\right)\right|\leq 1\\ &\Leftrightarrow -1\leq 1-4r\sin^2\left(\tfrac{k\pi}{2(M+1)}\right)\leq 1\\ &\Leftrightarrow -2\leq -4r\sin^2\left(\tfrac{k\pi}{2(M+1)}\right)\leq 0 \quad ,\\ &\Leftrightarrow 0\leq 4r\sin^2\left(\tfrac{k\pi}{2(M+1)}\right)\leq 2\\ &\Leftrightarrow 0\leq r\sin^2\left(\tfrac{k\pi}{2(M+1)}\right)\leq \tfrac{1}{2}\end{aligned}$$

Note that the inequality $r\sin^2\left(\frac{k\pi}{2(M+1)}\right)\leq\frac{1}{2}$ is satisfied for all M and $k=1,...,M$, then

$$r\sin^2\left(\frac{k\pi}{2(M+1)}\right)\leq\frac{1}{2}\Leftrightarrow r\leq\frac{1}{2}.$$

Consequently, the scheme is $L^2-stable$ under the condition of C.F.L $\left(r=\frac{\triangle t}{(\triangle x)^2}\leq\frac{1}{2}\right)$.

<u>Von Neumann's stability method in $\|\cdot\|_2$</u>

This method uses the following notation:

$$\widehat{u^n} := u_j^n e^{-ix_j\zeta} = u_{j-1}^n e^{-ix_{j-1}\zeta} = u_{j+1}^n e^{-ix_{j+1}\zeta},\ \zeta\in\mathbb{R},$$

then, we have

$$u_j^n = \widehat{u^n} e^{ix_j\zeta},\ u_{j-1}^n = \widehat{u^n} e^{ix_{j-1}\zeta},\ u_{j+1}^n = \widehat{u^n} e^{ix_{j+1}\zeta},\ u_j^{n+1} = \widehat{u^{n+1}} e^{ix_j\zeta},\ \zeta \in \mathbb{R}.$$

By substituting these notation in $(*)$, we get

$$\begin{aligned}
\widehat{u^{n+1}} e^{ix_j\zeta} &= r\widehat{u^n} e^{ix_{j+1}\zeta} + (1-2r)\,\widehat{u^n} e^{ix_j\zeta} + r\widehat{u^n} e^{ix_{j-1}\zeta} \\
&= r\widehat{u^n} e^{i\triangle x\zeta} e^{ix_j\zeta} + (1-2r)\,\widehat{u^n} e^{ix_j\zeta} + r\widehat{u^n} e^{-i\triangle x\zeta} e^{ix_j\zeta} \\
&= \left(re^{i\triangle x\zeta} + 1 - 2r + re^{-i\triangle x\zeta}\right) \widehat{u^n} e^{ix_j\zeta} \\
&= (1 - 2r\,(1-\cos(\triangle x\zeta)))\,\widehat{u^n} e^{ix_j\zeta},
\end{aligned}$$

hence, $\widehat{u^{n+1}} = \mathcal{A}(\zeta)\,\widehat{u^n}$, such as $\mathcal{A}(\zeta) = (1 - 2r\,(1-\cos(\triangle x\zeta)))\,.$

Consequently, the scheme is $L^2 - stable$ if and only if $\underset{\zeta\in\mathbb{R}}{sup}\,|\mathcal{A}(\zeta)| \leq 1.$

Now, we have

$$\begin{aligned}
\underset{\zeta\in\mathbb{R}}{sup}\,|\mathcal{A}(\zeta)| \leq 1 \;&\Leftrightarrow\; |1 - 2r\,(1-\cos(\triangle x\zeta))| \leq 1, \forall\zeta\in\mathbb{R} \\
&\Leftrightarrow\; \left|1 - 4r\sin^2(\tfrac{\triangle x\zeta}{2})\right| \leq 1, \forall\zeta\in\mathbb{R} \\
&\Leftrightarrow\; -1 \leq 1 - 4r\sin^2(\tfrac{\triangle x\zeta}{2}) \leq 1, \forall\zeta\in\mathbb{R} \\
&\Leftrightarrow\; 0 \leq 2r\sin^2(\tfrac{\triangle x\zeta}{2}) \leq 1, \forall\zeta\in\mathbb{R}.
\end{aligned}$$

Noting that $0 \leq 2r\sin^2(\frac{\triangle x\zeta}{2}) \leq 1, \forall\zeta\in\mathbb{R}$ holds if and only if $0 \leq r \leq \frac{1}{2}$.

An Implicit scheme

Here, we use the backward formula for the time and the central formula for the space i.e.,

$$\begin{cases} \dfrac{u_j^n - u_j^{n-1}}{\triangle t} - \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\triangle x)^2} = 0, \;\; j = 1, ..., M, \;\; n = 1, ..., N, \\ u_j^0 \text{ is given for } j = 0, ..., M+1, \\ u_0^n = u_{M+1}^n = 0, \;\; n = 0, ..., N+1, \end{cases}$$

it yields

$$u_j^{n-1} = -ru_{j+1}^n + (1+2r)\,u_j^n - ru_{j-1}^n,$$

where $r = \frac{\triangle t}{(\triangle x)^2}$.

Now, we are looking a matrix form.

$$\begin{array}{rcc} u_1^{n-1} & = & -ru_2^n + (1+2r)\,u_1^n - ru_0^n \\ u_2^{n-1} & = & -ru_3^n + (1+2r)\,u_2^n - ru_1^n \\ \vdots & = & \vdots \\ u_{M-1}^{n-1} & = & -ru_M^n + (1+2r)\,u_{M-1}^n - ru_{M-2}^n \\ u_M^{n-1} & = & -ru_{M+1}^n + (1+2r)\,u_M^n - ru_{M-1}^n \end{array},$$

it results

$$U^{n-1} = AU^n,$$

where

$$A = \begin{pmatrix} 1+2r & -r & 0 & \cdots & 0 \\ -r & 1+2r & -r & \ddots & \vdots \\ 0 & -r & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -r \\ 0 & \cdots & 0 & -r & 1+2r \end{pmatrix}.$$

Note that

$$A = I_M + rB,$$

such as

$$B = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Since B is invertible, then A is also invertible. Thus

$$U^n = A^{-1}U^{n-1}.$$

For this scheme to be stable, if and only if $\|A^{-1}\| \leq 1$.

Noting that$\|A^{-1}\|_2 = \dfrac{1}{1+r\lambda_k} < 1$, (because $\lambda_k > 0$, λ_k is an eigenvalue of B and B is symmetric definite positive). Therefore, this scheme is unconditionally stable in $L^2 - norm$ sense.

Remark 75. *An alternative consists in changing all indices n into $n+1$ in the space discretization, yielding the following implicit scheme:, i.e.,*

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\triangle t}-\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\triangle x)^2}=0,\ j=1,...,M,\ n=1,...,N, \\ u_j^0 \text{ is given for } j=0,...,M+1, \\ u_0^n=u_{M+1}^n=0,\ n=0,...,N+1, \end{cases}.$$

Crank-Nicolson method

This scheme is one of the most popular methods in practice.

Adding the two previous schemes (Explicit, implicit) and taking the mean value leads to the Crank-Nicolson scheme:

$$\frac{u_j^{n+1}-u_j^n}{\triangle t}-\frac{1}{2}\left[\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\triangle x)^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{(\triangle x)^2}\right]=0,\ j=1,...,M,\ n=1,...,N,$$

u_j^0 is given for $j=0,...,M+1$,

$u_0^n=u_{M+1}^n=0,\ n=0,...,N+1$,

The θ-scheme

More generally, we can consider a weighted average of the explicit and implicit schemes, leading to the theta scheme, or the θ-scheme:

$$\frac{u_j^{n+1}-u_j^n}{\triangle t}-\theta\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\triangle x)^2}-(1-\theta)\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{(\triangle x)^2}=0,\ \theta\in[0,1].$$

Stability of θ-scheme:

The discrete equation can be written by

$$
\begin{aligned}
-\theta r u_{j+1}^{n+1} + (1+2\theta r)\, u_j^{n+1} - \theta r u_{j-1}^{n+1} &= (1-\theta)\, r u_{j+1}^{n} + (1-2(1-\theta)r)\, u_j^{n} \\
+(1-\theta) r u_{j-1}^{n},\ r &= \frac{\triangle t}{(\triangle x)^2}.
\end{aligned}
$$

It follows to the following matrix form

$$
(I_M + \theta r B)\, U^{n+1} = (I_M - (1-\theta) r B)\, U^{n},
$$

where

$$
B = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \ddots & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & 0 & -1 & 2
\end{pmatrix}.
$$

Note that the matrix $(I_M + \theta r B)$ is tridiagonal having $(1+2\theta r)$ on its main diagonal and $-\theta r$ on its first upper and lower subdiagonals and all other entries are zeros. This matrix is symmetric definite positive, then it is invertible.

Hence

$$
U^{n+1} = (I_M + \theta r B)^{-1} (I_M - (1-\theta) r B)\, U^{n},
$$

Obviously, we retrieve the explicit scheme for the value $\theta = 0$, the implicit scheme for $\theta = 1$ and the Crank-Nicolson scheme for $\theta = \dfrac{1}{2}$.

Now, we study the stability of θ-scheme in $L^2 - norm$ using Von-Neumann method as follows:

$$\begin{aligned}
-\theta r\widehat{u^{n+1}}e^{ix_{j+1}\zeta} + (1+2\theta r)\,\widehat{u^{n+1}}e^{ix_j\zeta} - \theta r\widehat{u^{n+1}}e^{ix_{j-1}\zeta} &= (1-\theta)\,r\widehat{u^n}e^{ix_{j+1}\zeta} \\
&\quad + (1-2(1-\theta)\,r)\,\widehat{u^n}e^{ix_j\zeta} \\
&\quad + (1-\theta)\,r\widehat{u^n}e^{ix_{j-1}\zeta}.
\end{aligned}$$

By dividing on $e^{ix_j\zeta}$, we obtain

$$\begin{aligned}
-\theta r\widehat{u^{n+1}}e^{i\triangle x\zeta} + (1+2\theta r)\,\widehat{u^{n+1}} - \theta r\widehat{u^{n+1}}e^{-i\triangle x\zeta} &= (1-\theta)\,r\widehat{u^n}e^{i\triangle x\zeta} \\
&\quad + (1-2(1-\theta)\,r)\,\widehat{u^n} \\
&\quad + (1-\theta)\,r\widehat{u^n}e^{-i\triangle x\zeta}.
\end{aligned}$$

It results,

$$\begin{aligned}
(1+2\theta r\,(1-\cos(\triangle x\zeta)))\,\widehat{u^{n+1}} &= (1-2(1-\theta)\,r\,(1-\cos(\triangle x\zeta)))\,\widehat{u^n} \\
\left(1+4\theta r\sin^2\left(\frac{\triangle x\zeta}{2}\right)\right)\widehat{u^{n+1}} &= \left(1-4(1-\theta)\,r\sin^2\left(\frac{\triangle x\zeta}{2}\right)\right)\widehat{u^n},
\end{aligned}$$

then

$$\mathcal{A}(\zeta) = \frac{1-4(1-\theta)\,r\sin^2\left(\frac{\triangle x\zeta}{2}\right)}{1+4\theta r\sin^2\left(\frac{\triangle x\zeta}{2}\right)}.$$

Thus, the scheme is L^2 stable, if and only if $|\mathcal{A}(\zeta)| \le 1$, $\forall\zeta \in \mathbb{R}$.

In fact,

$$\begin{aligned}
|\mathcal{A}(\zeta)| \le 1 &\Leftrightarrow -1-4\theta r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right) \le 1-4(1-\theta)\, r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right) \le 1+4\theta r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right)\\
&\Leftrightarrow -\theta r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right) \le (1-\theta)\, r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right) \le \tfrac{1}{2}+\theta r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right)\\
&\Leftrightarrow 0 \le \tfrac{1}{2}+\theta r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right)-(1-\theta)\, r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right)\\
&\Leftrightarrow 0 \le \tfrac{1}{2}+(2\theta-1)\, r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right)\\
&\Leftrightarrow (1-2\theta)\, r\sin^2\left(\tfrac{\triangle x\zeta}{2}\right) \le \tfrac{1}{2}.
\end{aligned}$$

- If $\theta < \frac{1}{2}$, then we have

$$r\sin^2\left(\frac{\triangle x\zeta}{2}\right) \le \frac{1}{2(1-2\theta)},\ \forall\zeta\in\mathbb{R},$$

and

$$r\sin^2\left(\frac{\triangle x\zeta}{2}\right) \le \frac{1}{2(1-2\theta)},\ \forall\zeta\in\mathbb{R} \Leftrightarrow r \le \frac{1}{2(1-2\theta)}.$$

In this case, the scheme is stable in $\|.\|_2$ sense if and only if $r \le \frac{1}{2(1-2\theta)}$.

- If $\theta = \frac{1}{2}$, then

$$|\mathcal{A}(\zeta)| \le 1 \Leftrightarrow \frac{1}{2} \ge 0.$$

In this case, the scheme is unconditionally stable in $\|.\|_2$ sense.

- If $\theta > \frac{1}{2}$, then $(1-2\theta) < 0$ and $(1-2\theta)\, r\sin^2\left(\frac{\triangle x\zeta}{2}\right) \le \frac{1}{2}$ is always checked.

  Hence, in this case, the scheme is unconditionally stable in $\|.\|_2$ sense.

Other schemes have been proposed, like the Richardson scheme:

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\triangle t} - \frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{(\triangle x)^2} = 0.$$

3.3.2. Finite differences schemes for the wave Equation

The second order wave equation in time and space is defined by:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,\ 0 < x < 1,\ 0 < t < T,\ c > 0, \\ u(x,0) = u_0(x) \text{ given}, \\ u(0,t) = u(1,t) = 0,\ \forall t \in [0,T]. \end{cases}$$

1) Discretize this problem using the finite difference method centered in time and space.

2) Study the stability of the discrete scheme.

The discretization gives the following formula:

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\triangle t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\triangle x)^2},$$

if we put $r = \left(c\frac{\triangle t}{\triangle x}\right)^2$, we get

$$u_j^{n+1} + u_j^{n-1} = ru_{j+1}^n + (2-2r)\, u_j^n + ru_{j-1}^n.$$

By using the Von-Neumann method, we find

$$\widehat{u^{n+1}} e^{ix_j\zeta} + \widehat{u^{n-1}} e^{ix_j\zeta} = r\widehat{u^n} e^{ix_{j+1}\zeta} + (2-2r)\, \widehat{u^n} e^{ix_j\zeta} + r\widehat{u^n} e^{ix_{j-1}\zeta},$$

since $x_{j+1} = x_j + \triangle x$ and $x_{j-1} = x_j - \triangle x$, then

$$\widehat{u^{n+1}} + 2\left(2r\sin^2\left(\frac{\triangle x\zeta}{2}\right) - 1\right)\widehat{u^n} + \widehat{u^{n-1}} = 0.$$

Now, we consider the following equation with $0 \leq r \leq 1$:

$$\rho^2 + 2\left(2r\sin^2\left(\frac{\triangle x\zeta}{2}\right) - 1\right)\rho + 1 = 0,$$

their solution are:

$$\begin{aligned}
\rho_1 &= 1 - 2r\sin^2\left(\frac{\triangle x\zeta}{2}\right) - i\sqrt{4r\sin^2\left(\frac{\triangle x\zeta}{2}\right)\left(1 - r\sin^2\left(\frac{\triangle x\zeta}{2}\right)\right)}, \\
\rho_2 &= 1 - 2r\sin^2\left(\frac{\triangle x\zeta}{2}\right) + i\sqrt{4r\sin^2\left(\frac{\triangle x\zeta}{2}\right)\left(1 - r\sin^2\left(\frac{\triangle x\zeta}{2}\right)\right)}.
\end{aligned}$$

The scheme is stable in L^2 sense if and only if $|\rho_1| \leq 1$ and $|\rho_2| \leq 1$.

Since, $|\rho_1| = |\rho_2| = 1$, then the scheme is stable in L^2 sense.

Exercise 76. *We consider the following problem:*

$$\begin{cases} -u''(x) = f(x), \; x \in \left]0,1\right[, \\ u'(0) = u'(1) = 0. \end{cases}$$

such that $\int_0^1 f(x)dx = 0$.

1) Discrete this problem using the centered finite difference method.

2) Write the discrete problem in matrix form $A_h U_h = b_h$, then deduce that this problem is ill-posed.

Exercise 77. *We consider the following problem:*

$$\begin{cases} -u''(x) + cu(x) = f(x), \; x \in \left]0,1\right[, \\ u'(0) = u(1) = 0. \end{cases}$$

where $f \in \mathcal{C}([0,1])$ and c is a strictly positive real number $(c > 0)$.

1) Discrete this problem using the centered finite difference method.

2) Write the discrete problem in matrix form $A_h U_h = b_h$, then show that it admits a unique solution.

Exercise 78. *We are interested here in the following one-dimensional elliptic problem:*

$$\begin{cases} -u''(x) + 2u(x) = x, \; 0 < x < 1, \\ u(0) = 1, \\ u'(1) + u(1) = 0. \end{cases}$$

1) Write a discretization for this problem using the finite difference method for a uniform mesh.

2) Write the linear system obtained, then study the existence of its solution.

Exercise 79. *Consider the following problem:*

$$\begin{cases} -u''(x) = u(x) + f(x), \ 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where f is a continuous function on $[0,1]$.

Discretize this problem using the centered finite difference method, then study the possibility that it admits a unique solution.

Exercise 80. *We are interested here in the numerical resolution of the following problem:*

$$\begin{cases} \frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0, \ 0 < x < 1, \ 0 < t < T, \\ u(x,0) = u_0(x) \ \textit{given}, \\ u(0,t) = u(1,t) = 0, \ \forall t \in [0,T], \end{cases}$$

and consider the following discrete problem:

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + \lambda \frac{u_i^n - u_{i-1}^n}{\triangle x} = 0.$$

1) Show that this scheme is consistent.

2) Study its stability.

Exercise 81. *Consider the following discrete problem:*

$$\frac{u_i^{n+1}-u_i^n}{\triangle t}+\lambda\frac{u_{i+1}^n-u_{i-1}^n}{2\triangle x}=0,$$

This scheme is called the Lax-Wendroff scheme.

Study the stability of this scheme.

Exercise 82. *Consider the following problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}=0,\ \ 0<x<1,\ \ 0<t<1, \\ u(x,0)=u_0(x)\ \ \textit{given}, \\ u(0,t)=u(1,t)=0,\ \ \forall t\in[0,T]\,. \end{cases}$$

To discretize this problem, we use the following finite difference scheme:

$$\frac{u_j^{n+1}-u_j^n}{\triangle t}-\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\triangle x)^2}=0,$$

where $x_j=j\triangle x,\ j=0,...,M+1,\ t^n=n\triangle t\ ,\ n=0,...,N+1,\ \triangle x=\dfrac{1}{M+1}$, $\triangle t=\dfrac{1}{N+1}$et $u_j^n:=u\left(x_j,t^n\right).$

Study the stability of this scheme in the sense of the norm $\|.\|_2\,.$

Exercise 83. *Consider the following problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}=0,\ \ 0<x<1,\ \ 0<t<T, \\ u(x,0)=u_0(x)\ \ \textit{given}, \\ u(0,t)=u(1,t)=0,\ \ \forall t\in[0,T]\,. \end{cases}$$

To discretize this problem, we use the following finite difference scheme:

$$\frac{u_j^{n+1}-u_j^n}{\triangle t}-\left[\frac{3}{4}\frac{\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)}{\left(\triangle x\right)^2}+\frac{1}{4}\frac{\left(u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}\right)}{\left(\triangle x\right)^2}\right]=0,$$

where $x_j = j\triangle x$, $t^n = n\triangle t$, $\triangle x = \dfrac{1}{M+1}$, $\triangle t = \dfrac{T}{N+1}$ and $u_j^n := u\left(x_j, t^n\right)$, $j = 0, ..., M+1$, $n = 0, ..., N+1$.

Study the stability of this scheme in the sense of the norm $\|.\|_2$.

Exercise 84. *We are looking for a numerical solution of the following heat equation:*

$$\begin{cases} \frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0,\ 0<x<1,\ 0<t<T, \\ u(x,0)=u_0(x)\ \textit{given}, \\ u(0,t)=u(1,t)=0,\ \forall t\in\left[0,T\right], \end{cases}$$

For this, we define the $\theta-$scheme method as follows:

$$\frac{u_i^{n+1}-u_i^n}{\triangle t}-\left[\left(1-\theta\right)\frac{u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}}{\left(\triangle x\right)^2}+\theta\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\left(\triangle x\right)^2}\right]=0,\ \theta\in\left[0,1\right].$$

1) What is the scheme type for $\theta = 0$, $\theta = 1$, $\theta = \frac{1}{2}$?

2) Study the stability of this scheme for $\theta = \dfrac{1}{2}$ (use the Von-Neumann method).

Exercise 85. *Consider the second order wave equation in time and space defined by:*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \ 0 < x < 1, \ 0 < t < T, \ c > 0, \\ u(x,0) = u_0(x) \ \textit{given}, \\ u(0,t) = u(1,t) = 0, \ \forall t \in [0,T]. \end{cases}$$

1) Discretize this problem using the finite difference method centered in time and space.

2) Study the stability of the discrete scheme.

Exercise 86. *Consider the following problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + u = 0, \ 0 < x < 1, \ 0 < t < T, \\ u(x,0) = u_0(x) \ \textit{given}, \\ u(0,t) = u(1,t) = 0, \ \forall t \in [0,T]. \end{cases}$$

To discretize this problem, we use the following finite difference scheme:

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\triangle x)^2} + u_i^n = 0,$$

where $x_i = i\triangle x$, $t^n = n\triangle t$, $\triangle x = \dfrac{1}{M+1}$, $\triangle t = \dfrac{T}{N+1}$ and $u_i^n := u(x_i, t^n)$.
Study the stability of this scheme in the sense of the norm $\|.\|_\infty$.

Conclusion

In this work, we have introduced courses and solved exercises about numerical analysis for master 1 of mathematics. These courses included one part of matrix analysis and another part of the numerical study of models of partial differential equations using the finite difference method. We relied on details and many diverse examples. In addition, we supported all chapters with exercises, some of which were solved and others were unsolved, which could enable students to understand all the courses.

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