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Image restoration using proximal-splitting methods

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Abstract In this paper, we focus on giving two fixed-point-like methods, using proximal operators, called forward-backward and Douglas-Rachford, for solving the restoration problem for grayscale images corrupted with Gaussian noise model. We discuss how to evaluate proximal operators and provide an example in reconstructed image. The main idea is to choose the classic variational model $TVL1$ for recovering a true image u from an observed image f contaminated with Gaussian noise. The objective function is a sum of two convex terms: the ℓ_1-norm data fidelity and the total variational regularization. The first term forces the final image to be not too far away from the initial image and the second term performs actually the noise reduction. Experimental results prove the efficiency of the proposed work by performing some test by changing the noise levels applied to different images. We notice that the Peak Signal-to-Noise Ratio ($PSNR$) is used to evaluate the quality of the restored images.

Keywords Proximal operator · Fixed point · Splitting · Forward–backward algorithm · Douglas–Rachford algorithm · Image restoration · Total variation · ℓ_1-norm · PSNR

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1 Introduction

Total variation [1] represents a powerful regularity measure in image restoration for recovering piecewise homogeneous areas with sharp edges [2], [3]. Image restoration techniques are used to make the corrupted image as similar as that of the original image. Various methods available for image restoration such as inverse filter, Weiner filter, constrained least square filter, blind deconvolution method etc, and Many optimization approaches for regularization-based image inverse problems have been developed [4], [5], [6], our framework is to solve the restoration problem in the form of a convex optimization, we use in particular a proximal algorithm [7], [8], [9] that introduces the proximal operators of the objective terms. There are various restoration techniques for noise removal based on proximal filtering , forward-backward [10], [11], [12], and Douglas-Rachford algorithms [13], [14] fall into the class of proximal splitting algorithms.

The remainder of the paper is organized as follows: first, in Section 2 definition and some properties of proximal operator, fixed points, Forward- backward splitting and Douglas Rachford splitting are provided. Then, in section 3 we present the considered restoration problem using proximal splitting algorithms. Numerical results are reported in Section 4 where beneficial analysis are done. Some conclusions and future works are drawn in Section 5.

2 Proximal Splitting Methods

2.1 Proximal operator

The proximal algorithm or proximal point method can be understood in many different ways, but we will start by thinking about as a regularizer that naturally damps its influence as the iterations proceed, and as a twist on gradient descent.

In proximal algorithms, the base operation is evaluating the proximal operator of a function, which involves solving a small convex optimization problem. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function. The proximal operator $prox_{\gamma f}x : \mathbb{R}^n \to \mathbb{R}^n$ of f with parameter $\gamma > 0$ is defined by:

$$prox_{\gamma f}x = arg\min_{x_0} \left\{ f\left(x_0\right) + \frac{1}{2\gamma} \left\|x_0 - x\right\|_2^2 \right\} \qquad (1)$$

where $\|.\|_2$ is the usual Euclidean norm.

The parameter γ controls the extent to which the proximal operator maps points towards the minimum of f, with larger values of γ associated with mapped points near the minimum, and smaller values giving a smaller movement towards the minimum.

The function minimized on the righthand side is strongly convex and not everywhere infinite, so it has a unique minimizer for every $x \in \mathbb{R}^n$.

$prox_f\left(x\right)$ is a point that compromises between minimizing f and being near

to x. For this reason, $prox_f\,(x)$ is sometimes called a proximal point of x with respect to f. In $prox_{\gamma f}\,(x)$, the parameter γ can be interpreted as a relative weight or trade-off parameter between these terms.
To keep notation light, we write $\|.\|_2$ rather than $\|.\|$.
Throughout this paper, when we refer to the proximal operator of a function, the function will be assumed to be closed proper convex, and it may take on the extended value $+\infty$.

2.2 Fixed points

The fixed points of the proximal operator of f are precisely the minimizers of f. In other words,

$$prox_{\gamma f}\,(x^*) = x^* \tag{2}$$

if and only if x^* minimizes f.
This implies a close connection between proximal operators and fixed point theory, and suggests that proximal algorithms can be interpreted as solving optimization problems by finding fixed points of appropriate operators.

2.3 Forward- backward splitting

Let $f_1 \in \mathbb{R}^n$, let $f_2 : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable with a β Lipschitz continuous gradient ∇f_2 , i.e., $\forall\,(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\|\nabla f_2\,(x) - \nabla f_2\,(y)\| \le \beta\,\|x - y\| \tag{3}$$

Where $\beta \in\,]0, +\infty[$. Suppose that $f_1\,(x) + f_2\,(x)$ as $\|x\| \to +\infty$. The problem is to

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\;\; f_1\,(x) + f_2\,(x) \tag{4}$$

Solution of Eq. (4) is characterized by the fixed point equation:

$$x = prox_{\gamma} f_1\,(x - \gamma\nabla f_2\,(x)) \tag{5}$$

where $\gamma \in\,]0, +\infty[$. Equation 5 can be written as:

$$\begin{cases} x = prox_{\gamma f_1}\,(y) \\ y = x - \gamma\nabla f_2\,(x) \end{cases} \tag{6}$$

which motivates the following scheme (Algorithm. 1)

Algorithm 1 Forward-Backward (Explicit-Implicit)

1: **Fix** $\varepsilon \in \left]0, \min\left\{1, \frac{1}{\beta}\right\}\right[$ **and** $x_0 \in H$
2: **For** $n = 0, 1, \ldots$ **:**

- $\gamma_n \in \left[\varepsilon, \frac{2}{\beta} - \varepsilon\right]$
- **Forward (implict) step :** $y_n = x_n - \gamma_n \nabla f_2(x_n)$
- $\tau_n \in [\varepsilon, 1]$
- **Backward (explict) step :** $x_{n+1} = x_n + \tau_n \left(prox_{\gamma_n f_1} y_n - x_n\right)$

2.4 Douglas Rachford Splitting

Let $f_1 \in \mathbb{R}^n$, $f_2 \in \mathbb{R}^n$ such that $\arg\min\ f_1 + f_2 \neq \varnothing$ and $f_1(x) + f_2(x)$ as $\|x\| \to +\infty$. The problem is to

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ f_1(x) + f_2(x) \tag{7}$$

Solution of Eq. (7) is characterized by the two-level condition:

$$\begin{cases} x = prox_{\gamma f_2} y \\ prox_{\gamma f_2} y = prox_{\gamma f_1}(2prox_{\gamma f2} y - y) \end{cases} \tag{8}$$

which motivates the following scheme (Algorithm. 2)

Algorithm 2 Douglas Rachford

1: **Fix** $\varepsilon \in]0, 1[$**,** $\gamma > 0$ **et** $x_0 \in H$
2: **For** $n = 0, 1, \ldots$ **:**

- $y_n = prox_{\gamma f_2} x_n$
- $\tau_n \in [\varepsilon, 2 - \varepsilon]$
- $x_{n+1} = x_n + \tau_n \left(proy_{\gamma f_1}(2y_n - x_n) - y_n\right)$

3 Reconstructed image using proximal algorithm

3.1 Reconstructed image using forward-backward splitting algorithm

It is unconstrained problem with energy function split in two components:

$$\text{minimise } E = E_{reg}(u) + E_{data}(u) \tag{9}$$

The application of forward- backward algorithm to the problem (9) is a 4 steps process:

– Step 1: Create the energy that describe the quality image u

$$\min_u E(u) = \min_u \left\{ \frac{\lambda}{2} \|Ku - f\|^2 + \|\nabla u\| \right\} \tag{10}$$

with

$$E_{data}(u) = \frac{1}{2} \|K.u - f\|^2 \tag{11}$$

$$E_{reg}(u) = \|\nabla u\| \tag{12}$$

– Step 2: Define the gradient of $E_{data}(u)$:

$$\nabla E_{data}(u) = \lambda K^T (Ku - f) \tag{13}$$

– Step 3: Define the proximal operator of $E_{reg}(u)$

$$prox_{\gamma E_{reg}}(\nabla u) = \max\left(0, 1 - \frac{\gamma}{|\nabla u|}\right) \nabla u \tag{14}$$

– Step 4: Apply forward backward algorithm. Computes the following sequence of points:

$$u_{i,j}^{n+1} = u_{i,j}^n - \tau_n . prox_{\gamma E_{reg}} \left(u_{i,j}^n - \gamma \left(\nabla E_{data}(u) \right)_{i,j}^{n+1} \right) \tag{15}$$

$\tau_n > 0$ is step size.

3.2 Reconstructed image using Douglas-Rachford splitting algorithm

3.2.1 Problem with constraints DR1

In this section, the Douglas–Rachford splitting method is applied to solve the problem (16). We follow the steps below:

– Step 1: Create an image reconstruction problem with constraints that describe the quality image u

$$\begin{aligned} &\arg\min_x |\nabla u| \\ &such\ that \\ &\|Ku - f\| \leq \varepsilon \end{aligned} \tag{16}$$

with

$$E_{data}(u) = \|K.u - f\|^2 \tag{17}$$

$$E_{reg}(u) = \|\nabla u\| \tag{18}$$

where ϵ is an estimated upper bound on the noise level.

– Step 2: Define the proximal operator of $E_{reg}(u)$

$$prox_{\gamma E_{reg}}(\nabla u) = \max\left(0, 1 - \frac{\gamma}{|\nabla u|}\right) \nabla u \tag{19}$$

- Step 3: Define the proximal operator of $E_{data}(u)$
  $E_{data}(u)$ is the indicator function of the set C define by $\|Ku - f\| \leq \varepsilon$. We define the *prox* of $E_{data}(u)$ as:

$$prox_{E_{data}}(z) = \arg\min_{u} \frac{1}{2}\|u - z\|^2 + i_C(u) \tag{20}$$

  with $i_C(u)$ the indicator function of a set C gicen by the Eq (21):

$$i_C(u) = \begin{cases} 0 & u \in C \\ +\infty & otherwise \end{cases} \tag{21}$$

  This previous problem has an identical solution as:

$$\begin{aligned} &\arg\min_{x} \|u - z\|^2 \\ &such\ that \\ &\|Kz - f\| \leq \varepsilon \end{aligned} \tag{22}$$

  It is simply a projection on the $B2 - ball$.

3.2.2 Problem without constraints DR2

The application of Douglas- Rachford algorithm to the problem without constraints is a 4 steps process:

- Step 1: Create the energy that describe the quality image u

$$\min_{u} E(u) = \min_{u} \left\{ \frac{\lambda}{2}\|Ku - f\|^2 + \|\nabla u\| \right\} \tag{23}$$

  with

$$E_{data}(u) = \frac{1}{2}\|K.u - f\|^2 \tag{24}$$

$$E_{reg}(u) = \|\nabla u\| \tag{25}$$

- Step 2: Define the proximal operator of $E_{data}(u)$

$$\Pr ox_{\gamma E_{data}} u = \frac{\lambda}{2} K^*(I + \gamma K^* K)^{-1} \tag{26}$$

- Step 3: Define the proximal operator of $E_{reg}(u)$

$$prox_{\gamma E_{reg}}(\nabla u) = \max\left(0, 1 - \frac{\gamma}{|\nabla u|}\right)\nabla u \tag{27}$$

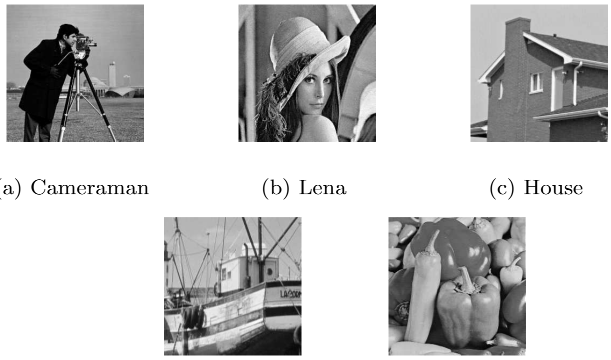
- Step 4: Apply Douglas-Rachford algorithm: computes the following sequence of points, for any fixed $\gamma > 0$ and stepsize $\tau_n \in [\varepsilon, 2 - \varepsilon]$

$$y_n = prox_{\gamma E_{data}} u_n \tag{28}$$

$$u_{n+1} = u_n + \tau_n \left(proy_{\gamma E_{reg}}(2y_n - u_n) - y_n\right) \tag{29}$$

4 Numerical results

We provide numerical results for five test images such as Cameraman, Lena, House, Boat, and Pepper (see Figure 1), the pixel size of five test images is 256×256. (`https://github.com/jianzhangcs/ISTA-Net/tree/master/Test_Image`). A range of noise levels 10% , 20% and 30% are tested.



(a) Cameraman (b) Lena (c) House

(d) Boat (e) Pepper

Fig. 1: True images for Cameraman, Lena, House, Boat, and Pepper

The performance of all the filtering techniques are compared on the basis of the statistical parameter such as $PSNR$. The term Peak Signal-to-Noise Ratio ($PSNR$) is an expression for the ratio between the maximum possible value (power) of a signal and the power of distorting noise that affects the quality of its representation. The mathematical representation of the $PSNR$ is as follows:

$$PSNR = 20\log_{10}\left(\frac{256}{\sqrt{MSE}}\right) \tag{30}$$

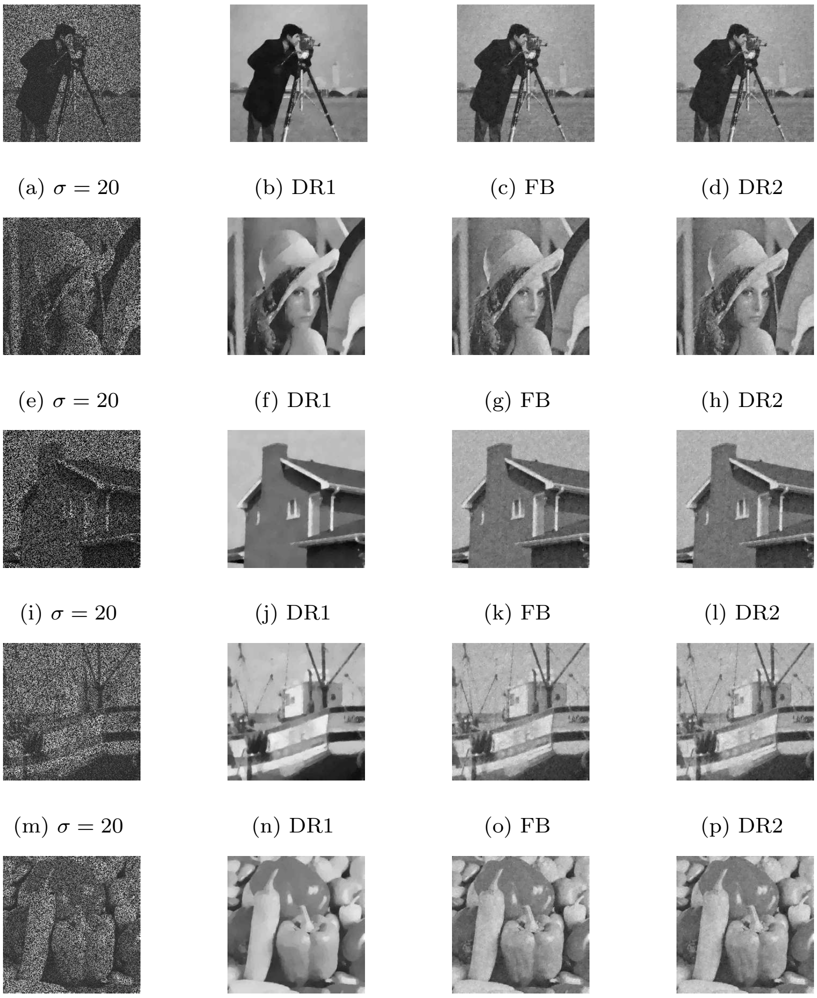
where the MSE (Mean Squared Error) is:

$$MSE = \frac{1}{MN}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\left|u\left(i,j\right) - \hat{u}\left(i,j\right)\right|^2 \tag{31}$$

u represents the matrix data of our original image, $\hat{u}$ represents the matrix data of our degraded image in question, M represents the numbers of rows of pixels of the images and i represents the index of that row, N represents the number of columns of pixels of the image and j represents the index of that column.

The peak signal-to-noise ratio (PSNR) between the restored image and the original image is selected as the performance index (Table 1).

Some visual results of the recovered images for the three algorithms are presented in Fig. 2. $DR2$ and FB not only removes lots of noise, but also preserves more details. We can see that $DR2$ and FB obtain a best compromise between noise-removal and detail-preservation compared with the $DR1$.



(a) $\sigma = 20$ (b) DR1 (c) FB (d) DR2

(e) $\sigma = 20$ (f) DR1 (g) FB (h) DR2

(i) $\sigma = 20$ (j) DR1 (k) FB (l) DR2

(m) $\sigma = 20$ (n) DR1 (o) FB (p) DR2

(q) $\sigma = 20$ (r) DR1 (s) FB (t) DR2

Fig. 2: Restored images with different proximal algorithms ($\sigma = 20$) for 'cameraman', 'lena', 'house', 'boat' and 'peppers'.

Table 1 lists the $PSNR$ values of restored images with random-valued additive gaussian noise level 10%, 20% and 30% for 'cameraman', 'lena', 'house', 'boat' and 'peppers' respectively. It presents the results of the three comparative denoising proximal algorithms FB, $DR1$ and $DR2$ for all test images.

Table 1: PSNR values of restored image for different percentage of additive noise

Image	Method	PSNR		
		10% Noise	20% Noise	30% Noise
cameraman	$DR1$	26.4734	24.9357	23.9520
	FB	26.4920	25.7134	24.2275
	$DR2$	26.5372	25.7134	24.2275
lena	$DR1$	26.8591	25.4343	23.5476
	FB	26.9945	26.1990	23.8520
	$DR2$	26.9116	26.1991	23.8520
house	$DR1$	30.5397	28.8260	25.4867
	FB	31.0293	29.0470	26.6858
	$DR2$	31.0293	29.0470	26.6858
boat	$DR1$	27.3255	24.7160	23.4413
	FB	27.7091	25.8337	23.4856
	$DR2$	27.7285	25.8337	23.4856
pepper	$DR1$	26.5146	25.2657	23.4097
	FB	26.8207	25.6298	23.8551
	$DR2$	26.8208	25.6298	23.8551

Its well-known that a higher $PSNR$ value provides a higher image quality. By comparing the $PSNR$ in table 1, we can infer that $DR2$ and FB gives a better value of $PSNR$ than $DR1$.

5 Conclusion

Restoration techniques are used to improve the quality of corrupted image to ward its original form. In this paper proximal algorithms are addressed to image restoration. Three convex regularization approaches based on proximal algorithms: FB, $DR1$ and $DR2$ has been studied, developed, implemented then compared in order to restore data images degraded by additive noise. To improve the image quality, the performance of all filtering techniques are compared on the basis of the statistical parameters such as Peak Signal to Noise Ratio ($PSNR$) based on Mean Square Error (MSE).
Futures work include the investigation of this study to solve other problems using the proximal splitting algorithm like image denoising if K is identity, image deblurring if K is a blur, image inpainting if K is a diagonal matrix, where whose diagonal entries are either 0 or 1, keeping or killing the corresponding pixels or resolve compressive sensing problem when K is a set of random projections.

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