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Applications of Orthogonal Polynomials for the Numerical Resolution of Integro-Differential Equations

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شكر وعرفان

الحمد لله رب العالمين الذي وفقنا وأعاننا على إنجاز هذا العمل، والصلاة والسلام على أشرف الأنبياء والمرسلين سيدنا محمد وعلي آله وصحبه أجمعين.

نتقدم بخالص الشكر والتقدير إلى الدكتورة **"دودي نجاة "** على ما قدمته من دعم وتوجيهات قيمة خلال مسار إعداد هذه المذكرة.لقد كان لتوجيهاتها و دعمها الدائم الأثر الكبير في تجاوز التحديات و إتمام هذا العمل.

كما لا يفوتنا أن نعبر عن شكرنا وامتناننا لأعضاء لجنة المناقشة على قبول مناقشتنا هذا العمل وتقديم ملاحظاتهم التي لا تقدر بثمن.

ولا يسعنا إلا أن نوجه بالشكر إلى كافة أساتذتنا في كلية العلوم الدقيقة على ما قدموه لنا من علم ومعرفة، لقد كانت بصائمهم أثر واضح في طيلة مسيرتنا الجامعية، وإلي جميع زملائنا الأعزاء الذين كانوا خير رفيق.

وفي الختام نوجه بأصدق مشاعر الامتنان لعائلتنا العزيزة التي كانت دائما الدعم الأساسي لنا.

الإهداء

إلى من غرس في قلبي حب العلم، وكان لي سندا في كل خطوة،

إلى من تعب وكافح ليضيء دربي بنور المعرفة

"إلي والدي الحبيب"،

الذي علمني أن الصبر والإرادة هما مفتاح النجاح

"إلي والدتي قرة عيني"،

التي لم تدخر جهدا في دعمي بصلاتها ودعواتها التي كانت لي نورا في كل عتمة ،

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الذرة أسماء بودرهم

الإهداء

"وقل إعملو فسيرى الله عملكم ورسوله والمؤمنون وستردون إلى عالم الغيب والشهادة فينبئكم بما كنتم تعملون"

إلى نفسي الطموحة أولا ابتدأت بطموح وانتهت بنجاح

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وللإنسانة العظيمة أمي التي سهلت لي الشدائد بدعائها

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وأخر دعواهم الحمد لله رب العالمين

نور الهدى حرزولي

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Introduction

Integral and integro-differential equations play a fundamental role in various scientific and engineering applications, including fluid mechanics, biological modeling, and signal processing. Due to the complexity of finding analytical solutions for these equations, especially in practical situations, numerical methods have emerged as an efficient approach to obtain accurate approximate solutions.

Among the most effective and robust numerical techniques are orthogonal polynomials, which serve as powerful mathematical tools. This approach leverages the unique properties of orthogonal polynomials, such as Chebyshev, Laguerre Legendre, and Jacobi polynomials, enabling flexible and precise functional representations of solutions. By doing so, complex integro-differential problems are transformed into algebraic systems that can be efficiently solved using numerical computation.

This study focuses on employing collocation methods (Méthodes de Collocation) to numerically solve integro-differential equations. These methods approximate the solution using orthogonal polynomials, taking advantage of their properties to convert the original problem into algebraic equations. This approach is characterized by its accuracy and rapid convergence, making it well-suited for handling complex mathematical problems.

In this work, we begin by providing a theoretical overview of integral and integro-differential equations, followed by a presentation of the most commonly used orthogonal polynomials and their mathematical properties. Subsequently, we will apply the collocation method to some practical examples, analyze the results, and discuss the method's effectiveness and accuracy. The objective of this research is to highlight the importance of using orthogonal polynomials in numerical solutions of integro-differential equations and to demonstrate their efficiency in addressing complex mathematical challenges.

Chapter 1

Review of integral and integro-differential equations

This chapter provides definitions of integral equations and integro-differential equations, along with their classifications. It also explores the connection between these two types of equations and explains how to reduce the order of an integro-differential equation.

1.1 Integral equations

Definition 1.1. *An integral equation (I.E) is an equation where the unknown function, typically dependent on one or more variables, appears inside an integral. The general representation of a linear integral equation is*

$$h(x)u(x) = f(x) + \lambda \int_{\Omega} K(x,t)u(t)dt.$$

In this equation $h(x)$, $f(x)$, and $K(x,t)$ are known functions, while the function $u(x)$, which appears both inside and outside the integral sign, is the unknown to be determined. λ represents a real or complex parameter, different from zero, and Ω is a closed, bounded, and measurable set with in an n-dimensional Euclidean space. The function $K(x,t)$ is referred to as the kernel of the integral equation.

1.2 Classification of integral equations

1.2.1 Volterra integral equations

Linear Volterra integral equations of the first, second, and third kinds are defined similarly to the previously mentioned equations, except that the upper integration limit is variable, i.e., $b = x$. The general form of this equation is

$$h(x)u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt.$$

i. If $h(x) = 0$, the equation is written as

$$f(x) + \lambda \int_a^x k(x,t)u(t)dt = 0,$$

and it is called a first-kind Volterra integral equation.

ii. If $h(x) = 1$, the equation is written as

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt,$$

and it is called a second-kind Volterra integral equation.

iii. If $h(x)$ is continuous and vanishes at certain points but not everywhere in $[a, b]$, the integral equation is called a third-kind Volterra.

Remark 1.1. *The Volterra integral equation is a special case of the Fredholm equation. It is sufficient to take a kernel $k(x,t)$ that satisfies the condition $k(x,t) = 0$ for all $x < t$.*

Exercice 1.1.

1. *First-kind Volterra integral equation*

$$x \exp(2x - 1) = \int_0^x \exp(xt)u(t)dt.$$

2. *Second-kind Volterra integral equation*

$$x^2 - u(x) = \int_0^x x\cos(t)u(t)dt.$$

3. *Third-kind Volterra integral equation*

$$(x^2 - x)u(x) = \int_0^x x\cos(t)u(t)dt.$$

1.2.2 Fredholm integral equations

An integral equation, where the integration limits are fixed, is referred to as a Fredholm linear integral equation. Its general form is

$$h(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt.$$

i. **First kind:** When $h(x) = 0$, the equation simplifies to

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0,$$

and is termed a Fredholm integral equation of the first kind.

ii. **Second kind:** When $h(x) = 1$, the equation becomes

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt,$$

and is referred to as a Fredholm integral equation of the second kind.

iii. **Third Kind:** If $h(x)$ is continuous and vanishes at certain points within the interval $[a,b]$, but not uniformly, the equation is classified as a Fredholm integral equation of the third kind.

Exercice 1.2.

1. *Example of a linear Fredholm integral equation of the first kind*

$$cos(x) = \int_0^1 (x-t)u(t)dt.$$

2. *Example of a linear Fredholm integral equation of the second kind*

$$u(x) = x^2 + x + \int_2^3 xtu(t)dt.$$

3. *Example of a linear Fredholm integral equation of the third kind*

$$xu(x) = \int_0^1 x^2tu(t)dt.$$

1.2.3 Volterra-Fredholm integral equations

A linear Volterra-Fredholm equation is an equation of the form

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt.$$

Exercice 1.3. *The equation*

$$u(x) = 2x^2 + 5 + \int_0^x xu(t)dt - \int_0^1 tu(t)dt,$$

is a linear Volterra-Fredholm integral equation.

1.2.4 Singular integral equations

An integral equation is considered singular if one or both integration limits are infinite. For example

$$u(x) = f(x) + \lambda \int_0^{+\infty} \sin(xt)u(t)dt.$$

Alternatively, the equation can be singular if the kernel becomes infinite near one or more points within the integration interval. For example,
if the kernel $k(x,t)$ of the Fredholm integral equation is given by

$$k(x,t) = \frac{H(x,t)}{|x-t|^{\alpha}}, \quad 0 < \alpha < 1,$$

it is classified as a singular integral equation.

1.3 Integro-differential equations

Definition 1.2. *A linear integro-differential equation (I.D.E) is an equation that combines both integral and differential operations, where the unknown function is u. The linear form of an integro-differential equation (I.D.E) of order n is given by*

$$u^{(n)}(x) + a_1 u^{(n-1)}(x) + \cdots + a_n u(x) + \sum_{m=0}^{s} \int_E k_m(x,t)u^{(m)}(t)dt = f(x),$$

where $a_1, a_2, \ldots, a_n$ are constants, $f(x)$ and $k_m(x)$, (for $m = 0, 1, \ldots, s$) are given functions, and $u(x)$ is the unknown function.
The function $u(x)$ is subject to initial conditions of the form

$$u(0) = u_0, \quad u'(0) = u_0', \quad \ldots, \quad u^{(n-1)}(0) = u_0^{(n-1)}.+$$

1.4 Classification of integro-differential equations

1.4.1 Volterra integro-differential equations

The linear Volterra integro-differential equation appears in the form

$$u^{(n)}(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt.$$

For example

$$u''(x) + u'(x) = 2 - 2x - 2(\sin(x) + \cos(x)) - 2\int_0^x tu(t)dt, \quad u(0) = -1, \quad u'(0) = 1,$$

$$u'(x) = 1 - \frac{1}{2}x^2 - x\exp(x) + \int_0^x tu(t)dt, \quad u(0) = 0.$$

These are linear Volterra integro-differential equations.

1.4.2 Fredholm integro-differential equations

The linear Fredholm integro-differential equation appears in the form

$$u^{(n)}(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt,$$

where $u^{(n)}$ represents the n^{th} derivative of $u(x)$ Lower-order derivatives may also appear alongside $u^{(n)}$ on the left-hand side. For example

$$u'(x) = 2 - \frac{1}{6}x + 2\int_0^1 xu(t)dt, \quad u(0) = 0,$$

$$u''(x) + u'(x) = x - \sin x - \int_0^{\frac{\pi}{2}} xtu(t)dt, \quad u(0) = 0, \quad u'(0) = 1.$$

These are linear Fredholm integro-differential equations.

1.4.3 Volterra-Fredholm integro-differential equations

The linear Volterra-Fredholm integro-differential equation appears in the form

$$u^{(n)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt,$$

where λ_1 and λ_2 are numerical parameters, k_1 and k_2 are the integral kernels, f is the given function, and u is the unknown function. For example

$$u'(x) = 2 - x - \int_0^x tu(t)dt + \int_0^1 (x-2)u(t)dt, \quad u(0) = -1.$$

This is a linear Volterra-Fredholm integro-differential equation.

1.4.4 Singular integro-differential equations

An integro-differential equation is considered singular if one or both of the following conditions hold:

1. One or both limits of integration are infinite.

2. The kernel becomes infinite near one or more points in the integration interval.

Remark 1.2.

1. *The order of an integro-differential equation (I.D.E) is determined by the highest derivative in the differential operator.*

2. *An (I.D.E) is said to be ordinary if the unknown function depends on only one variable. If it depends on two or more variables, the (I.D.E) is called partial.*

1.5 Reduction of a high-order linear differential equation to a lower-order equation

In the next section, we will discuss first order equations, because they are the simplest form that equations can take.

1.5.1 First-order equations

We consider the differential equation

$$\begin{cases} u'(x) = f(x, u), \\ u(x_0) = u_0. \end{cases} \tag{1.1}$$

By integrating both sides from x_0 to x, we obtain

$$\int_{x_0}^{x} u'(t)dt = \int_{x_0}^{x} f(x, u(t))dt,$$

which leads to

$$u(x) = u_0 + \int_{x_0}^{x} f(t, u(t))dt. \tag{1.2}$$

On the other hand, if we assume (1.2), we obtain

$$u'(x) = f(x, u(x)),$$

with the initial condition

$$u(x_0) = u_0,$$

which implies (1.1). Therefore, (1.1) and (1.2) are equivalent.
Sometimes, it is useful to transform the resolution of a differential equation into the resolution of an integral equation, and vice versa.

At follows, we will explain Émile Picard's method, which is one of the most important iterative methods for solving the integral form of the equation (1.2).

Émile Picard's method:

Consider solving the initial value problem

$$\begin{cases} u'(x) = f(x, u), \\ u(x_0) = u_0, \end{cases}$$

or equivalently, solving the integral equation

$$u(x) = A + \int_{x_0}^{x} f(x, u(t))dt.$$

We will solve this integral equation by constructing a sequence of successive approximations to $u(.)$. We begin by choosing an initial approximation, $u_0(.)$ where it is common to take $u_0(x) = u(x_0)$. Then, we define the sequence by $u_1(x), u_2(x), \cdots, u_n(x)$ as follows

$$u_1(x) = A + \int_{x_0}^{x} f(x, u_0(t))dt,$$

$$u_2(x) = A + \int_{x_0}^{x} f(x, u_1(t))dt,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$u_n(x) = A + \int_{x_0}^{x} f(x, u_{n-1}(t))dt.$$

where $A = u_0(x)$.

1.5.2 Second-order equations

The following is an important lemma for transforming second-order to first-order equation.

Lemma 1.1. *([4]) If f is continuous function, we have*

$$\int_a^x \int_a^s f(y)\, dy\, ds = \int_a^x f(y)(x-y)\, dy.$$

Proof. Let

$$F(s) := \int_a^s f(y).$$

So, we have

$$\int_a^x \left(\int_a^s f(y)dy \right) ds = \int_a^x F(s)ds = \int_a^x 1 \cdot F(s)ds.$$

$$\int_a^x \left(\int_a^s f(y)\, dy \right) ds = [sF(s)]_a^x - \int_a^x sF'(s)ds = xF(x) - aF(a) - \int_a^x sf(s)ds.$$

$$x \int_a^x f(y)dy - \int_a^x yf(y)dy = \int_a^x f(y)(x-y)dy.$$

□

Exercice 1.4. *Consider solving the initial value problem*

$$u''(x) + a(x)u'(x) + b(x)u(x) = g(x);\ u(0) = \alpha;\ u'(0) = \beta.$$

Let's modify the variable to y, then integrate from 0 to z

$$\int_0^z u''(y)dy + \int_0^z a(y)u'(y)dy + \int_0^z b(y)u(y)dy = \int_0^z g(y)dy. \tag{1.3}$$

Using integration by parts on the second term on the left-hand side, we obtain

$$[u'(y)]_0^z + [a(y)u(y)]_0^z - \int_0^z a'(y)u(y)dy + \int_0^z b(y)u(y)dy = \int_0^z g(y)dy.$$

Considering the initial conditions, we get

$$u'(z) - \beta + a(z)u(z) - a(0)\alpha - \int_0^z [a'(y) - b(y)]u(y)dy = \int_0^z g(y)dy.$$

Integrating once more with respect to from 0 to

$$[u(z)]_0^x - \beta x + \int_0^x a(z)u(z)dz - a(0)\alpha x - \int_0^x \int_0^z [a'(y) - b(y)]u(y)dydz = \int_0^x \int_0^z g(y)dydz.$$

This can be simplified using Lemma 1.1 as

$$u(x) - \alpha - \beta x + \int_0^x a(y)u(y)dy - a(0)\alpha x - \int_0^x (x-y)[a'(y) - b(y)]u(y)dy = \int_0^x (x-y)g(y)dy.$$

Rewriting in a more compact form

$$u(x) + \int_0^x [a(y) - (x-y)[a'(y) - b(y)]]u(y)dy = \int_0^x (x-y)g(y)dy + [\beta + a(0)\alpha]x + \alpha.$$

Defining

$$k(x,y) = a(y) - (x-y)[a'(y) - b(y)], f(x) = \int_0^x (x-y)g(y)dy + [\beta + a(0)\alpha]x + \alpha,$$

we reduce the equation to the following form

$$u(x) + \int_0^x k(x,y)u(y)dy = f(x),$$

which is a Volterra integral equation of the second kind.

Exercice 1.5. *Consider the differential equation*

$$u'' + xu' + u = 0,$$

with initial conditions

$$u(0) = 1, \quad u'(0) = 0.$$

Let us define

$$u'' = \phi(x),$$

then

$$u' = \int_0^x \phi(t)dt + u'(0); \ \ \textit{Thus } u = \int_0^x (x-t)\phi(t)dt + 1.$$

Substituting into the given differential equation, we obtain

$$\phi(x) + \int_0^x x\phi(t)dt + \int_0^x (x-t)\phi(t)dt + 1 = 0.$$

Therefore,

$$\phi(x) = -1 - \int_0^x (2x-t)\phi(t)dt.$$

1.5.3 Equations of higher order

We consider the linear integro-differential equation of order

$$u^{(n)}(x) + \sum_{i=1}^{n} a_i(x)u^{(n-i)}(x) + \int_a^b K(x,t)u(t)dt = g(x),$$

with the initial conditions

$$u(a) = \alpha_1, \quad u'(a) = \alpha_2, \quad u''(a) = \alpha_3, \cdots, u^{(n-1)}(a) = \alpha_n.$$

To convert an -th order integro-differential equation (IDE) into a first-order system of IDEs, we define

$$z_1(x) = u(x),\ z_2(x) = u'(x), \cdots, z_n(x) = u^{(n-1)}(x).$$

This transformation leads to the system

$$\begin{cases} z_1'(x) & = z_2(x), \\ z_2'(x) & = z_3(x), \\ z_3'(x) & = z_4(x), \\ & \vdots \\ z_{n-1}'(x) & = z_n(x), \\ z_n'(x) & = g(x) - \sum_{i=1}^{n} a_i(x)z_{n+1-i}(x) - \int_a^b K(x,t)z_1(t)dt, \end{cases}$$

with the initial conditions

$$z_1(a) = \alpha_1, z_2(a) = \alpha_2, \cdots, z_n(a) = \alpha_n.$$

1.6 Existence and uniqueness of solutions for nonlinear integro-differential equations

In this section, we present the fixed-point theorems of Banach, Brouwer, and Schauder. These theorems are essential for proving the existence of numerical solutions to nonlinear integro-differential equations.

The Banach fixed-point theorem is the most well-known and straightforward. It states that a contraction mapping in a complete metric space has a unique fixed point.

The fixed-point theorems of Brouwer and Schauder are more powerful and topological in nature. They assert that a continuous mapping on a compact convex set has at least one fixed point, which is not necessarily unique.

In this section, we apply Banach's fixed-point theorem to integro-differential equations, specifically first-order Volterra equation and Volterra-Fredholm equation of order n.

1.6.1 Some fixed-point theorems

Banach's fixed-point theorem:

Banach's fixed-point theorem guarantees the existence and uniqueness of a fixed point for a contraction mapping in a complete metric space.

Theorem 1.1. *(Banach's theorem) Let (X, d) be a complete metric space, and let $M \subseteq X$ be a non-empty closed subset. If the mapping $T : M \to M$ is a contraction, then T has a unique fixed point in M.*

Brouwer's fixed-point theorem:

Brouwer's fixed-point theorem guarantees the existence of at least one fixed point (though not necessarily uniqueness) for a continuous function on a closed ball in a finite-dimensional space.

Theorem 1.2. *(Brouwer's theorem) Any continuous mapping T of the closed unit ball $B^n \subset \mathbb{R}^n$ into itself has at least one fixed point.*

Schauder's fixed point theorem:

Schauder's Fixed Point Theorem extends Brouwer's theorem by proving the existence of a fixed point for a continuous function on a compact convex set in a Banach space.

Theorem 1.3. *(Schauder's theorem) Let X be a Banach space, and let $M \subseteq X$ be a non-empty, convex, and compact set. Then, any continuous mapping $T : M \to M$ has at least one fixed point.*

To view the demonstration of three previous theorem, see [6] and [12].

1.6.2 Applications of Banach's fixed point theorem

First-order Volterra integro-differential equations:

In this section, we provide some conditions ensuring the existence and uniqueness of the solution to the Volterra integral-differential equation of the form

$$\begin{cases} u'(x) = f(x) + \int_a^x K(x,t,u(t))dt, \; x \in I = [a,b], \\ u(a) = u_0, \end{cases} \tag{1.4}$$

where $f : [a,b] \to \mathbb{R}^n, K : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $u' : [a,b] \to \mathbb{R}^n$ are continuous functions. To establish existence and uniqueness, equation (1.4) must be reformulated as a fixed-point problem.

We consider the Banach space $C_\beta(I, \mathbb{R}^n)$, which consists of continuous functions from I to $\mathbb{R}^n$, endowed with the norm

$$\|u\|_{\beta,\infty} = \sup_{x \in I} \frac{|u(x)|}{e^{\beta(x-a)}}, \text{ with } \beta > 0.$$

Theorem 1.4. *([12]) Suppose that K is L-lipschitz with respect to the third variable, i.e., there exists $L > 0$ such that, for all $x, t \in I$ and $u, v \in \mathbb{R}^n$, we have*

$$|K(x,t,u(t)) - K(x,t,v(t))| < L|u(t) - v(t)|. \tag{1.5}$$

If $\frac{L}{\beta^2} < 1$, then problem (1.4) admits a unique solution.

Proof. Let $u \in C_\beta(I, \mathbb{R}^n)$. By integrating equation (1.4) from a to x, we obtain

$$u(x) = u_0 + \int_a^x f(s)ds + \int_a^x \int_a^s K(s,t,u(t))dtds, \quad x \in I. \tag{1.6}$$

Conversely, if $u \in C^1(I)$, differentiating equation (1.4) recovers equation (1.6), and setting $x = a$ retrieves the initial condition $u(a) = u_0$. We now define the operator T on $C_\beta(I, \mathbb{R}^n)$ by

$$(Tu)(x) = u_0 + \int_a^x f(s)\,ds + \int_a^x \int_a^s K(s,t,u(t))\,dt\,ds, \quad x \in I. \tag{1.7}$$

It is clear that T maps $C_\beta(I, \mathbb{R}^n)$ into itself, since f and K are continuous, ensuring that

$$T(C_\beta(I, \mathbb{R}^n)) \subseteq (C_\beta(I, \mathbb{R}^n))$$

Next, we verify T that is a contraction. For $u, v \in C_\beta(I, \mathbb{R}^n)$, we have

$$|(Tu)(x) - (Tv)(x)| \leq t \int_a^x \int_a^s |K(s,t,u(t)) - K(s,t,v(t))|dtds.$$

Using the Lipschitz condition on K,

$$\int_a^x \int_a^s |K(s,t,u(t)) - K(s,t,v(t))|dtds \leq L \int_a^x \int_a^s |u(t) - v(t)|dtds,$$

it follows that

$$L \int_a^x \int_a^s |u(t) - v(t)|dtds \leq L\|u - v\|_{\beta,\infty} \int_a^x \int_a^s e^{-\beta(t-a)} e^{\beta(t-a)}dtds.$$

Evaluating the integral

$$\|u - v\|_{\beta,\infty} \int_a^x \int_a^s e^{-\beta(t-a)} e^{\beta(t-a)}dtds \leq L\|u - v\|_{\beta,\infty} \left(\frac{e^{\beta(x-a)} - 1}{\beta^2} - \frac{x-a}{\beta} \right),$$

thus

$$\|(Tu)(x) - (Tv)(x)\|_{\beta,\infty} \leqslant \frac{L}{\beta^2}\|u - v\|_{\beta,\infty}, \quad \forall u, v \in C_\beta(I, \mathbb{R}^n), \quad \beta > 0. \tag{1.8}$$

Since $\frac{L}{\beta^2} < 1$, the operator T is a contraction. By Banach's fixed-point theorem, T has a unique fixed point in $C_\beta(I, \mathbb{R}^n)$, which is the solution of equation (1.4). □

E.I.D of Volterra-Fredholm of order n:

We consider the initial value problem for the integro-differential equation of Volterra–Fredholm of order n, given by

$$\begin{cases} u^{(n)}(x) = F\left(x, u(x), ..., u^{(n-1)}(x), (Au)(x), (Bu)(x)\right), \quad x \in I = [a, b], \quad 0 \leq a < b, \\ u^{(k)}(a) = c_k, \quad k = 0, 1, ..., n-1, \end{cases} \tag{1.9}$$

where

$$(Au)(x) = \int_a^x k_1(x,t) h_1(t, u(t), u'(t), \cdots, u^{(n-1)}(t)) dt, \tag{1.10}$$

$$(Bu)(x) = \int_a^b k_2(x,t) h_2(t, u(t), u'(t), ..., u^{(n-1)}(t)) dt. \tag{1.11}$$

In equations (1.9)-(1.11) $F \in C(I \times \mathbb{R}^{n+2}, \mathbb{R})$, $k_i \in C(I^2, \mathbb{R})$ and $h_i \in C(I \times \mathbb{R}^n, \mathbb{R})$ for $i = 1, 2$ are continuous functions, while c_k are given real constants.

Let $E = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) be the product space, and let $y^{(j)}(x) : I \to \mathbb{R}$ for $j = 0, 1, \cdots, n-1$ be continuous functions. We define the norm

$$|y(x)|_E = \sum_{j=0}^{n-1} |y^{(j)}(x)|,$$

for $(y(x), y'(x), \cdots, y^{(n-1)}(x)) \in E, x \in I$.

Let G be the space of functions $(y(x), y'(x), \cdots, y^{(n-1)}(x)) \in E$ that are continuous for $x \in I$ and satisfy the condition

$$|y(x)|_E = \circ(\exp(\lambda x)), \quad x \in I. \tag{1.12}$$

where λ is a positive constant. In the space G, we define the norm

$$|y|_G = \sup_{x \in I} \left\{ ||y(x)|_E \exp(-\lambda x) \right\}. \tag{1.13}$$

It is easy to see that G, equipped with the norm defined by (1.13), forms a Banach space. Moreover, condition (1.12) implies the existence of a constant N_0 such that

$$|y(x)|_E \leqslant N_0 \exp(\lambda x).$$

By substituting this into (1.13), we obtain

$$|y|_G \leqslant N_0. \tag{1.14}$$

It is also easy to observe that the solution $u(x)$ of problem (1.9), along with its derivatives, satisfies the equations

$$u^{(j)}(x) = \sum_{i=j}^{n-1} c_i \frac{(x-a)^{i-j}}{(i-j)!} + \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} F\left(s, u(s), u'(s), ..., u^{(n-1)}(s), (Au)(s), (Bu)(s)\right) ds,$$

for $0 \leqslant j \leqslant n-1$.

The result concerning the existence of a unique solution to the initial value problem (IVP) (1.9) is given in the following theorem. For more details see [8].

Theorem 1.5. *([8]) Suppose that*

1. *The functions F, h_i (for $i = 1, 2$) satisfy the conditions*

$$\begin{aligned} &|F(x, y_0, y_1, \cdots, y_{n-1}, v_1, v_2) - F(x, z_0, z_1, \cdots, z_{n-1}, w_1, w_2)| \\ &\leq p(x) \sum_{j=0}^{n-1} |y_j - z_j| + |v_1 - w_1| + |v_2 - w_2|, \end{aligned} \tag{1.15}$$

$$|h_i(x, y_0, y_1, ..., y_{n-1}) - h_i(x, z_0, z_1, ..., z_{n-1})| \leqslant q_i(x) \sum_{j=0}^{n-1} |y_j - z_j|, \tag{1.16}$$

   where $p, q_i \in C(I, \mathbb{R})$.

2. *There exists a constant α such that $0 \leq \alpha \leq 1$ and*

$$\sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} p(s)[\exp(\lambda s) + h_1^*(s) + h_2^*(s)] ds \leq \alpha \exp(\lambda x), \tag{1.17}$$

   for $x \in I$, where

$$h_1^*(x) = \int_a^x |k_1(x,t)| q_1(t) \exp(\lambda t) dt, \tag{1.18}$$

$$h_2^*(x) = \int_a^b |k_2(x,t)| q_2(t) \exp(\lambda t) dt, \tag{1.19}$$

   and λ is given in (1.12).

3. *There exists a positive constant P such that*

$$h(x) + \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} |F(s, 0, 0, \cdots, 0, (A_0)(s), (B_0)(s))| ds \leq P \exp(\lambda x), \tag{1.20}$$

where

$$h(x) = \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} |c_i| \frac{(x-a)^{i-j}}{i \vec{-} j}. \tag{1.21}$$

Then, the problem (1.9) admits a unique solution $u(x)$ in G over I.

Proof. Let $u \in G$, and define the operator T on G by

$$(Tu)(x) = \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} + \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} F\big(s, u(s), u'(s), \ldots, u^{(n-1)}(s), (Au)(s), (Bu)(s)\big)\, ds. \tag{1.22}$$

By differentiating both sides of equation (1.22) with respect to x, we obtain

$$(Tu)^{(j)}(x) = \sum_{i=j}^{n-1} c_i \frac{(x-a)^{i-j}}{(i-j)!} + \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} F\left(s, u(s), u'(s), \ldots, u^{(n-1)}(s), (Au)(s), (Bu)(s)\right) ds. \tag{1.23}$$

For $0 \leq j \leq n+1$, it is evident that $Tu(j)(x)$ is continuous on I.

Now, we need to show that the operator T maps G onto itself. We verify that condition (1.9) is satisfied. From (1.15), (1.20), and (1.23), we have

$$\begin{aligned}
\|(Tu)(x)\|_E &= \sum_{j=0}^{n-1} \|(Tu)^{(j)}(x)\| + h(x) + \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} [F(s, 0, 0, \subseteq, 0, (A0)(s), (B0)(s))ds \\
&+ \sum_{j=0}^{n-1} \|(Tu)^{(j)}(x)\| + h(x) + \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} F(s, u(s), u'(s), \subseteq, 0, (Au)(s), (Bu)(s)) \\
&- F(s, 0, 0, \subseteq, 0, (A0)(s), (B0)(s))]\, ds \\
&\leq P \exp(\lambda x) + \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} p(x)\, [|u(s)|_E + \|(Au)(s) - (A_0)(s)\| \\
&+ \|(Bu)(s) - (B_0)(s)\|]\, ds \\
&\leq P \exp(\lambda x) + \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} p(s) \exp(\lambda s) |u|_G.
\end{aligned} \tag{1.24}$$

From (1.15), (1.20) and (1.23), we obtain

$$\begin{aligned}(Au)(s) - (A_0)(s) &\leq \int_a^s k_1(s,t)\left(h_1\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\right) - h_1(t, 0, 0, \ldots, 0)\right) dt \\ &\leq \int_a^s k_1(s,t) q_1(t) \exp(\lambda t) u(t) E \exp(-\lambda t) dt \\ &\leq \int_a^s k_1(s,t) q_1(t) \exp(\lambda t) u^G dt \\ &= u^G h_1^*(s). \end{aligned} \tag{1.25}$$

Similarly, from (1.11), (1.16), and (1.19), we obtain

$$|(Bu)(s) - (B0)(s)| \leq |u|_G h_2^*(s). \tag{1.26}$$

Using (1.24), (1.25), (1.26), (1.14) and (1.17), we get

$$\begin{aligned}|u(T,x)| &\leq P \exp(\lambda x) + \sum_{j=0}^{n-1} \frac{a}{(n-j-1)!}(x-s)^{n-j-1} \\ &\quad \times \int p(s)\exp(\lambda s) + h_1^*(s) + h_2^*(s)\, ds \cdot |u|_G \\ &\leq P\exp(\lambda x) + N_0 \alpha \exp(\lambda x) \\ &= (P + N_0\alpha)\exp(\lambda x). \end{aligned} \tag{1.27}$$

From (1.27), it follows that $Tx \in G$, proving that the operator T maps G onto itself.

Now, we must verify that the operator T is a contraction. Let $u, v \in G$. From (1.25) and (1.23), we have

$$\begin{aligned}|(Tu)(x) - (Tv)(x)| &\leq \sum_{j=0}^{n-1} \left|(Tu)^{(j)}(x) - (Tv)^{(j)}(x)\right| \\ &\leq \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} \Big| F(s, u(s), u'(s), \ldots, u^{(n-1)}(x), (Au)(s), (Bu)(x)) \\ &\quad - F(s, v(s), v'(s), \ldots, v^{(n-1)}(x), (Av)(s), (Bv)(x)) \Big| ds \\ &\leq \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} p(s) \Big(|u(x) - v(x)|_E + |(Au)(x) - (Av)(x)| \\ &\quad + |(Bu)(x) - (Bv)(x)| \Big) ds \end{aligned}$$

$$\leq \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} p(s) \exp(\lambda s) |u-v|_G$$
$$+ |(Au)(x) - (Av)(x)| + |(Bu)(x) - (Bv)(x)| ds. \tag{1.28}$$

From (1.25) and (1.26), we obtain

$$\|(Au)(x) - (Av)(x)\| \leq |u-v|_G h_1^*(s), \tag{1.29}$$

and

$$\|(Bu)(x) - (Bv)(x)\| \leq |u-v|_G h_2^*(s). \tag{1.30}$$

Substituting (1.29) and (1.30) into (1.28) and using condition (1.27), we get

$$\begin{aligned}
|(Tu)(x) - (Tv)(x)| &\leq \sum_{j=0}^{n-1} \left|(Tu)^{(j)}(x) - (Tv)^{(j)}(x)\right| \\
&\leq \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} \Big| F(s, u(s), u'(s), \ldots, u^{(n-1)}(x), (Au)(s), (Bu)(x)) \\
&\quad - F(s, v(s), v'(s), \ldots, v^{(n-1)}(x), (Av)(s), (Bv)(x)) \Big| ds \\
&\leq \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} p(s) \Big(|u(x) - v(x)|_E + |(Au)(x) - (Av)(x)| \\
&\quad + |(Bu)(x) - (Bv)(x)| \Big) ds \\
&\leq \sum_{j=0}^{n-1} \int_a^x \frac{(x-s)^{n-j-1}}{(n-j-1)!} p(s) \exp(\lambda s) |u-v|_G \\
&\quad + \int_a^x |(Au)(x) - (Av)(x)| ds + \int_a^x |(Bu)(x) - (Bv)(x)| ds.
\end{aligned} \tag{1.31}$$

Consequently, from (1.31), we have

$$\|Tu - Tv\|_G \leq \alpha |u-v|_G. \tag{1.32}$$

Since $\alpha < 1$ is a contraction. Therefore, by Banach's fixed-point theorem, has a unique fixed point in , which is the solution to problem (1.9). □

Chapter 2

Orthogonal polynomials and their properties

Orthogonal polynomials form a fundamental cornerstone in mathematical analysis and approximation theory. They emerge from orthogonality conditions defined by an inner product involving a weight function over a specified interval. Their significance lies in the fact that they constitute complete functional systems within Hilbert spaces, making them powerful tools in solving differential equations, developing spectral expansions, and addressing problems in numerical analysis. Notable examples include Tchebyshev, Legendre, Laguerre, and Jacobi polynomials, all of which are closely related to special solutions of second-order differential equations.

2.1 Tchebyshev polynomials

2.1.1 Definition of Tchebyshev polynomials

Definition 2.1. *The Tchebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation*

$$T_n(x) = \cos(n\theta) \quad \textit{when} \quad x = \cos(\theta). \tag{2.1}$$

If the variable is within the interval $[-1, 1]$, then the corresponding variable can be taken within the range $[0, \pi[$. These ranges are traversed in opposite directions since $x = 1$ corresponds to

$\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$. It is well known, as a consequence of de Moivre's theorem, that $\cos(n\theta)$ is a polynomial of degree n in $\cos\theta$. In fact, we are familiar with the fundamental formulas

$$\cos(0\theta) = 1, \quad \cos(1\theta) = \cos(\theta), \quad \cos(2\theta) = 2\cos^2(\theta) - 1$$

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta), \quad \cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1, \ldots$$

from equation (2.1), we can directly derive the first few Tchebyshev polynomial

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1. \end{aligned}$$

Calculation of T_n for some values of x:

Let $n \in \mathbb{N}$, for $x = -1, 0, 1$, we have

- $T_n(1) = T_n(cos(0)) = cos(n * 0) = 1$.
- $T_n(-1) = (-1)^n T_n(1) = (-1)^n$.
- T_{2n+1} is an odd function, so $T_{2n+1}(0) = 0$ additioonally, $T_{2n}(0) = T_{2n}(cos(\frac{\pi}{2}) = cos(n\pi) = (-1)^n$.

In summary:

- For all $n \in \mathbb{N}, T_n(1) = 1$.
- For all $n \in \mathbb{N}, T_n(-1) = (-1)^n$.
- For all $n \in \mathbb{N}, T_{2n+1}(0) = 0$ and $T_{2n}(0) = (-1)^n$, or equivalently, $T_n(0) = cos(n\pi)$.

2.1.2 Recurrence relations for $T_n(x)$

Theorem 2.1. *Given the first two Tchebyshev polynomials, and all subsequent polynomials for can be generated using the recurrence relation*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Additionally, the derivative of $T_n(x)$ with respect to x is given by

$$(1 - x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x).$$

Proof.

- Let's show that

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$$

We set $x = cos(\theta)$, then

$$T_{n+1}(\cos(\theta)) = \cos((n+1)\theta).$$

Using the angle addition formula

$$\begin{aligned} \cos((n+1)\theta) =& \cos(n\theta + \theta) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \\ =& [2\cos(n\theta)\cos(\theta) - \cos(n\theta)\cos(\theta)] - \sin(n\theta)\sin(\theta) \\ =& 2\cos(\theta)\cos(n\theta) - (\cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta)) \\ =& 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta). \end{aligned}$$

Since $x = cos(\theta)$, we get

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Thus,

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$$

- Let us show that

$$(1 - x^2)T_n'(x) = -nxT_n(x) - nT_{n-1}(x).$$

Let $x = \cos$. Then, we know

$$T_n'(x) = \frac{n\sin(n\cos^{-1}(x))}{\sqrt{1-x^2}} \implies (1-x^2)T_n'(x) = (1-x^2)\cdot\frac{n\sin(n\cos^{-1}(x))}{\sqrt{1-x^2}}.$$

Now, substituting $x = cos(\theta)$, we get

$$(1-\cos^2(\theta))T_n(x) = (1-\cos^2(\theta))\cdot\frac{n\sin(n\theta)}{\sin(\theta)} = n\sin(\theta)\cdot\sin(n\theta).$$

Using the identity for the product of sines

$$n\sin\theta\sin(n\theta) = -\frac{n}{2}[\cos((n+1)\theta) - \cos((n-1)\theta)].$$

This becomes

$$-\frac{n}{2}[T_{n+1}(\cos\theta) - T_{n-1}(\cos\theta)].$$

Then using the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

we simplify to get

$$-\frac{n}{2}[2xT_n(x) - 2T_{n-1}(x)] = -nxT_n(x) - nT_{n-1}(x).$$

Therefore,

$$(1-x^2)T_n'(x) = -nxT_n(x) - nT_{n-1}(x).$$

□

Calculating of some polynomials: Let's use the recurrence relation to compute T_1, T_2, T_3, T_4 and T_5 step by step.

First, we recall the recurrence relation

$$T_{n+1}(x) = 2x\cdot T_n(x) - T_{n-1}(x).$$

Given

$$T_0(x) = 1, \qquad T_1(x) = x,$$

Using the recurrence, we get

$$\begin{aligned}
T_2(x) =& 2x \cdot T_1(x) - T_0(x) = 2x^2 - 1, \\
T_3(x) =& 2x \cdot T_2(x) - T_1(x) = 4x^3 - 3x, \\
T_4(x) =& 2x \cdot T_3(x) - T_2(x) = 8x^4 - 8x^2 + 1, \\
T_5(x) =& 2x \cdot T_4(x) - T_3(x) = 16x^5 - 20x^3 + 5x.
\end{aligned}$$

The following figure represents the first five terms of Tchebyshev's polynomials.

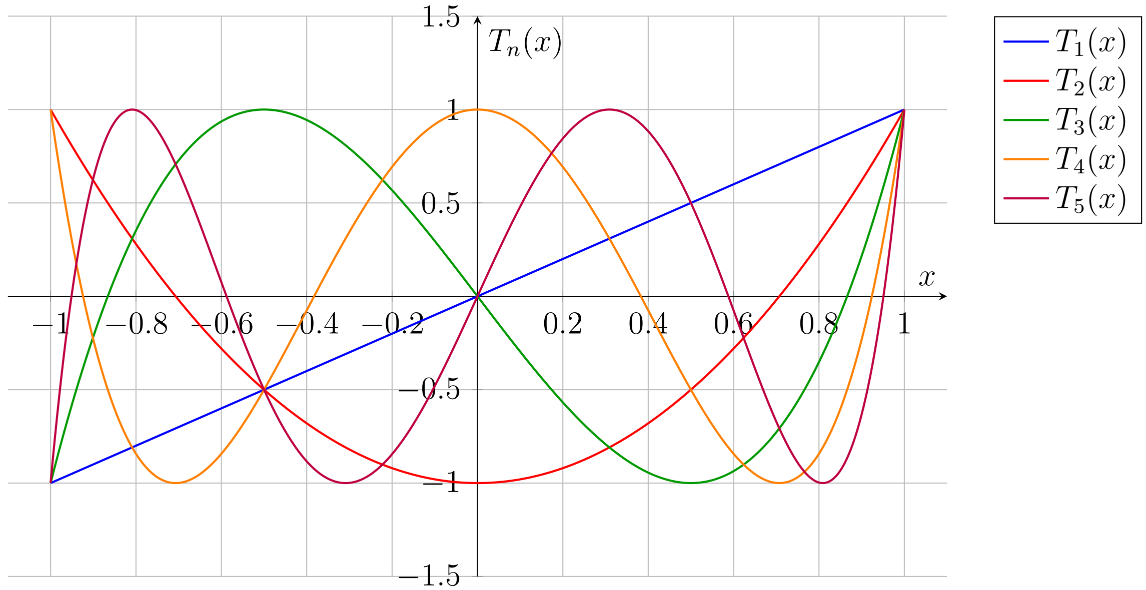

Figure 2.1: Plot of the Tchebychev polynomials T_1 to T_5.

2.1.3 The generating function for Tchebyshev polynomials of the first kind

Theorem 2.2. *The Tchebyshev polynomials of the first kind can be expressed using the generating function*

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n.$$

Proof. The Tchebyshev polynomials of the first kind are defined by the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

with initial conditions $T_0(x) = 1, \quad T_1(x) = x$. Assume that

$$G(x,t) = \sum_{n=0}^{\infty} T_n(x)t^n.$$

We aim to show that

$$G(x,t) = \frac{1-tx}{1-2tx+t^2}.$$

Multiply both sides of the recurrence relation by t^n and sum over all

$$\sum_{n=1}^{\infty} T_{n+1}(x)t^n = 2x\sum_{n=1}^{\infty} T_n(x)t^n - \sum_{n=1}^{\infty} T_{n-1}(x)t^n.$$

Rewriting the summations by shifting indices

$$\sum_{n=0}^{\infty} T_{n+1}(x)t^n = 2x\sum_{n=0}^{\infty} T_n(x)t^n - t\sum_{n=0}^{\infty} T_n(x)t^n.$$

Using the definition of $G(x,t)$, this simplifies to

$$\frac{G(x,t)-1}{t} = 2xG(x,t) - tG(x,t).$$

$$\Longrightarrow G(x,t) - 1 = t(2xG(x,t) - tG(x,t)).$$

Factor out

$$G(x,t) - t(2xG(x,t) - tG(x,t)) = 1.G(x,t)(1-2tx+t^2) = 1 - tx.$$

Solving for

$$G(x,t) = \frac{1-tx}{1-2tx+t^2}.$$

This confirms that the generating function for Tchebyshev polynomials of the first kind is

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n.$$

□

2.1.4 Products, integrals and derivatives

The Tchebyshev polynomials are essentially disguised forms of the trigonometric functions $cos(nx)$, and they also belong to the class of orthogonal polynomials; as a result, they satisfy many valuable and significant relationships.

Product: Various formulas can be easily derived using the substitution $x = \cos(\theta)$ and trigonometric identities, as follows

$$T_m(x)T_n(x) = \cos(m\theta)\cos(n\theta) = \frac{1}{2}(\cos((m+n)\theta) + \cos(|m-n|\theta))$$

which leads to

$$T_m(x)T_n(x) = \frac{1}{2}(T_{m+n}(x) + T_{|m-n|}(x)). \tag{2.2}$$

Similarly,

$$xT_n(x) = \cos(\theta)\cos(n\theta) = \frac{1}{2}(\cos((n+1)\theta) + \cos(|n-1|\theta))$$

which gives

$$xT_n(x) = \frac{1}{2}(T_{n+1}(x) + T_{|n-1|}(x)).$$

More generally, expressions for $x^m T_n(x)$ for any can be obtained by expressing in terms of Tchebyshev polynomials and then applying equation (2.2).

Along the same lines, we have

$$\begin{aligned}(1-x^2)T_n(x) =& \sin^2(\theta)\cos(n\theta)\\ =& \frac{1}{2}\left(1-\cos(2\theta)\right)\cos(n\theta)\\ =& \frac{1}{2}\cos(n\theta) - \frac{1}{4}\left(\cos((n+2)\theta) + \cos(|n-2|\theta)\right)\end{aligned}$$

which gives

$$(1-x^2)T_n(x) = -\frac{1}{4}T_{n+2}(x) + \frac{1}{2}T_n(x) - \frac{1}{4}T_{|n-2|}(x).$$

It is important to note that the specific cases for $n = 0$ and $n = 1$ are already included in the above formulas. More explicitly

$$\begin{aligned} xT_0(x) =& T_1(x), \\ (1-x^2)T_0(x) =& \frac{1}{2}T_0(x) - \frac{1}{2}T_2(x), \\ (1-x^2)T_1(x) =& \frac{1}{4}T_1(x) - \frac{1}{4}T_3(x). \end{aligned}$$

Derivatives: The properties of differentiation play a crucial role in both analytical and numerical analysis. We start with the following expressions

$$T_{n+1}(x) = \cos[(n+1)\cos^{-1}(x)] \quad \text{and} \quad T_{n-1}(x) = \cos[(n-1)\cos^{-1}(x)].$$

Differentiating both expressions with respect to x, we obtain

$$\frac{1}{n+1}\frac{d[T_{n+1}(x)]}{dx} = \frac{-\sin[(n+1)\cos^{-1}x]}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{1}{n-1}\frac{d[T_{n-1}(x)]}{dx} = \frac{-\sin[(n-1)\cos^{-1}x]}{\sqrt{1-x^2}}.$$

Subtracting these two results yields

$$\frac{1}{n+1}\frac{d[T_{n+1}(x)]}{dx} - \frac{1}{n-1}\frac{d[T_{n-1}(x)]}{dx} = \frac{\sin[(n+1)\theta] - \sin[(n-1)\theta]}{\sin\theta}.$$

This simplifies to

$$\frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} = \frac{2\cos(n\theta)\sin\theta}{\sin\theta} = 2T_n(x), \quad \text{for } n \geq 2,$$

hence

$$T'_0(x) = 0,\ T'_1(x) = T_0,\ T'_2(x) = 4T_1.$$

Integrals: Since we have the differentiation formulas for Tchebyshev polynomials, we can use them to derive the corresponding integration formulas

$$\int T_n(x)\,dx = \frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right) + C, \quad n \geq 2,$$

$$\int T_1(x), dx = \frac{1}{4}T_2(x) + C,$$

$$\int T_0(x)\,dx = T_1(x) + C.$$

2.1.5 Orthogonality of Tchebyshev polynomials

We can determine the orthogonality properties of the Tchebyshev polynomials of the first kind based on the orthogonality of cosine functions. Specifically,

$$\int_0^{\pi} \cos(m\theta)\cos(n\theta)\,d\theta = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\pi}{2}, & \text{if } m = n \neq 0, \\ \pi, & \text{if } m = n = 0. \end{cases}$$

By substituting

$$T_n(x) = \cos(n\theta), \quad \cos(\theta) = x,$$

we obtain the orthogonality properties of the Tchebyshev polynomials

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\pi}{2}, & \text{if } m = n \neq 0, \\ \pi, & \text{if } m = n = 0. \end{cases}$$

This confirms that the Tchebyshev polynomials form an orthogonal set on the interval with the weighting function.

2.2 Laguerre polynomials

In mathematics, Laguerre polynomials are sequences of orthogonal polynomials that are solutions to a certain differential equation, known as the Laguerre equation, named after the French mathematician Edmond Laguerre.

2.2.1 Laguerre's differential equation and its solutions

The Laguerre differential equation is written as

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0. \tag{2.3}$$

In applications, we seek a solution to this equation that remains finite for all finite values of x and such that this solution grows slower than $\exp(\frac{x}{2})$ as x approaches infinity. Thus, if we multiply both sides of equation (2.3) by $x \neq 0$, it then becomes

$$x^2 \frac{d^2y}{dx^2} + x(1-x)\frac{dy}{dx} + nxy = 0, \tag{2.4}$$

that is to say, of the type

$$x^2 \frac{d^2y}{dx^2} + xq(x)\frac{dy}{dx} + R(x)y = 0. \tag{2.5}$$

Given $q(x) = 1 - x$ and $R(x) = nx$, we can apply the Frobenius method [9] to find a solution to equation (2.4), or equivalently equation (2.5). This solution takes the following form

$$z(x,k) = \sum_{r=0}^{r+k} a_r x^{r+k},$$

where k is determined as a root of the characteristic equation for equation (2.5)

$$k^2 + (q_0 - 1)(k + r_0) = 0.$$

The recurrence relation for the coefficients a_r is given by

$$a_{r+1} = a_r \frac{k + r - n}{(k + r + 1)^2}. \tag{2.6}$$

Using this recurrence relation, the solution $z(x,k)$ can be expressed as

$$z(x,k) = a_0 x^k \sum_{r=0}^{\infty} \frac{(-1)^r (n-k)!}{((k+r)!)^2 (n-k-r)!} x^r. \tag{2.7}$$

In our specific case, $q_0 = 1$ and $r_0 = 0$, which implies that the characteristic equation has a double root $k = 0$. Consequently, equation (2.3) admits two linearly independent solutions, expressed as

$$z(x,0) = a_0 \sum_{r=0}^{\infty} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r.$$

Now, to compute $\left[\frac{dz(x,k)}{dk}\right]_{k=0}$, based on relation (2.7), we obtain

$$\frac{\partial z(x,k)}{\partial k} = a_0 (\ln x) x^k \sum_{r=0}^{\infty} \frac{(-1)^r (n-k)!}{((k+r)!)^2 (n-k-r)!} x^r + a_0 x^k \sum_{r=0}^{\infty} \frac{d}{dk}\left[\frac{(-1)^r (n-k)!}{((k+r)!)^2 (n-k-r)!}\right] x^r.$$

Define

$$f_r(k) = \frac{(n-k)!}{((k+r)!)^2(n-k-r)!},$$

then

$$\frac{d}{dk}f_r(k) = f_r(k)\frac{d\ln(f_r(k))}{dk}.$$

Let

$$\begin{aligned}\ln(f_r(k)) &= \ln\frac{(n-k)!}{((k+r)!)^2(n-k-r)!}\\ &= \ln(n-k)! - \ln((k+r)!)^2 - \ln(n-k-r)!\\ &= \ln(n-k)(n-k-1)\cdots\times 1 - 2\ln(k+r)(k+r-1)\cdots\\ &\quad\times 1 - \ln(n-k-r)(n-k-r-1)\cdots\times 1.\end{aligned}$$

Simplify

$$\begin{aligned}\ln(f_r(k)) &= \sum_{i=0}^{r}\ln(n-k-i) - 2\sum_{i=1}^{r}\ln(k+r-i) - \sum_{i=1}^{r}\ln(n-k-r-i)\\ &= \sum_{i=0}^{r}\ln(n-k-i) - 2\ln(k+r-i) - \ln(n-k-r-i).\end{aligned}$$

Thus

$$\begin{aligned}\frac{d\ln f_r(k)}{dk} &= \frac{d}{dk}\left[\sum_{i=0}^{r}\ln(n-k-i) - 2(k+r-i) - \ln(n-k-r-i)\right]\\ &= \sum_{i=0}^{r}\left(-\frac{1}{n-k-i} - 2\frac{1}{k+r-i} + \frac{1}{n-k-r-i}\right).\end{aligned}$$

Hence

$$\frac{d}{dk}f_r(k) = \frac{(n-k)!}{((k+r)!)^2(n-k-r)!}\sum_{i=0}^{r}\left(-\frac{1}{n-k-i} - 2\frac{1}{k+r-i} + \frac{1}{n-k-r-i}\right),$$

and therefore

$$\left[\frac{d}{dk}f_r(k)\right]_{k=0} = \frac{n!}{(r!)^2(n-r)!}\sum_{i=0}^{r}\left(-\frac{1}{(n-i)} - 2\frac{1}{(r-i)} + \frac{1}{(n-r-i)}\right). \tag{2.8}$$

Then

$$\left[\frac{\partial}{\partial k}z(x,k)\right]_{k=0} = a_0(\ln x)\sum_{r=0}^{\infty}\frac{n!(-1)^r}{(r!)^2(n-r)!} + a_0\sum_{i=0}^{r}\left(-\frac{1}{(n-i)} - 2\frac{1}{(r-i)} + \frac{1}{(n-r-i)}\right)(-1)^r x^r,$$

that is

$$\left[\frac{\partial}{\partial k} z(x,k)\right]_{k=0} = (\ln x) z(x,0) + \sum_{r=0}^{\infty} C_r x^r,$$

where

$$C_r = (-1)^r \left(-\frac{1}{(n-i)} - 2\frac{1}{(r-i)} + \frac{1}{(n-r-i)}\right),$$

this particular solution is not acceptable because it diverges at $x = 0$, due to the presence of a term involving $\ln(x)$. Therefore, the solution to equation (2.3) can be expressed as

$$z(x,0) = \sum_{r=0}^{\infty} a_r x^r = a_0 \sum_{r=0}^{\infty} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r.$$

It is important to point out that when n is a natural number, this series becomes a finite sum, this happens because for all $r \geq n+1$, the coefficients a_r become zero. Assuming $a_0 = 1$, we arrive at the standard solution, which is known as the Laguerre polynomial of degree n, denoted by $L_n(x)$, and defined as:

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x_r. \tag{2.9}$$

from equation (2.9), we can directly derive the farst few Laguerre polynomials,

$$\begin{aligned}
L_0 &= 1, \\
L_1 &= 1 - x, \\
L_2 &= \frac{1}{2}(2 - 4x + x^2), \\
L_3 &= \frac{1}{6}(6 - 18x + 9x^2 + x^3), \\
L_4 &= \frac{1}{24}(24 - 96x + 72x^2 - 16x^3 + x^4), \\
L_5 &= \frac{1}{120}(120 - 600x + 6000x^2 - 200x^3 + 25x^4 - x^5).
\end{aligned}$$

The following figure represents the first five terms of Laguerre's polynomials.

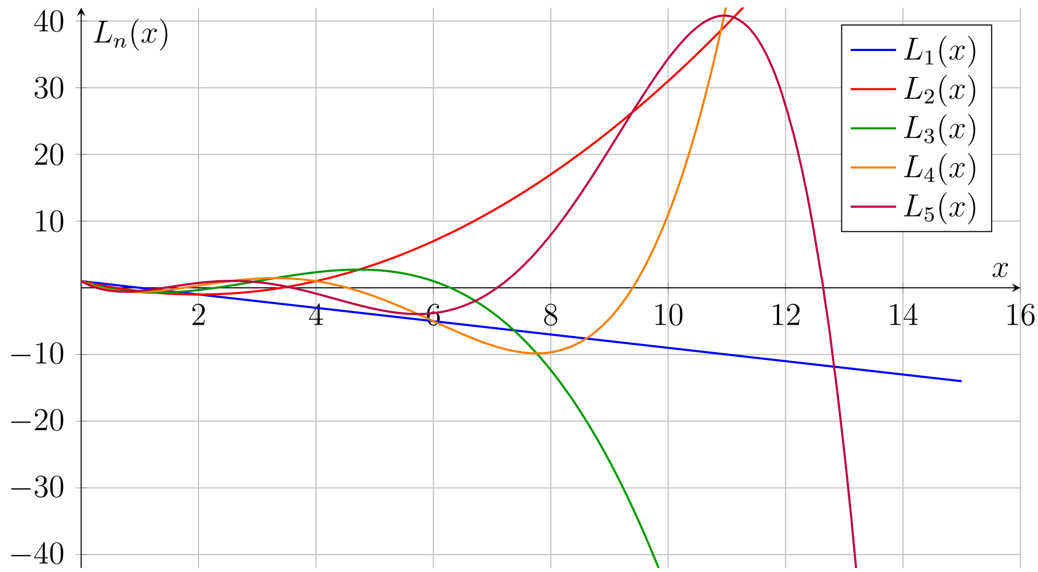

Figure 2.2: Laguerre polynomials of degree 1 to 5.

Theorem 2.3. *For any natural number n, the Laguerre polynomials can be expressed using the Rodrigues formula as follows*

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right).$$

Proof. To prove this, we apply Leibniz's rule for the n^{th} derivative of a product

$$\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) = \frac{e^x}{n!}\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\frac{d^{n-r}}{dx^{n-r}}x^n \cdot \frac{d^r}{dx^r}e^{-x}.$$

We recall that

$$\frac{d^p}{dx^p}(x^q) = \frac{q!}{(q-p)!}x^{q-p}.$$

Hence, we have

$$\frac{d^{n-r}}{dx^{n-r}}x^n = \frac{n!}{r!}x^r,$$

and

$$\frac{d^2}{dx^2}e^{-x} = (-1)^r e^{-x}.$$

Substituting these into the earlier expression

$$\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) = \frac{e^x}{n!}\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\frac{n!}{r!}x^r \cdot (-1)^r e^{-x}.$$

Simplifying

$$= \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r = L_n(x).$$

□

2.2.2 The generalized Laguerre polynomials

Definition 2.2. *The generalized Laguerre polynomials (GLPs), represented by $L_n^{(\alpha)}(x)$ (with $\alpha > -1$) are orthogonal with respect to the weight function $w_\alpha(x) = x^\alpha e^{-x}$ on the domain $\mathbb{R}_+ = (0, +\infty)$. That is,*

$$\int_0^{+\infty} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) w_\alpha(x)\, dx = \gamma_n^{(\alpha)} \delta_{nm},$$

where the normalization constant is given by

$$\gamma_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{n!}.$$

Specifically, when $\alpha = 0$, the polynomials $L_n^{(0)}(x)$ reduce to the standard Laguerre polynomials, denoted simply as $L_n(x)$. These polynomials are orthogonal with respect to the weight function $w(x) = e^{-x}$, satisfying

$$\int_0^{+\infty} L_n(x) L_m(x) w(x)\, dx = \delta_{nm}.$$

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ satisfy the recurrence relation

$$(n+1) L_{n+1}^{(\alpha)}(x) = (2n + \alpha + 1 - x) L_n^{(\alpha)}(x) - (n+\alpha) L_{n-1}^{(\alpha)}(x),$$

with initial values

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = -x + \alpha + 1.$$

The first few polynomials in this family are

$$\begin{aligned}
L_0^{(\alpha)}(x) &= 1, \\
L_1^{(\alpha)}(x) &= -x + \alpha + 1, \\
L_2^{(\alpha)}(x) &= \frac{1}{2}\left(x^2 - 2(\alpha+2)x + (\alpha+1)(\alpha+2)\right), \\
L_3^{(\alpha)}(x) &= \frac{1}{6}\left(-x^3 + 3(\alpha+3)x^2 - 3(\alpha+2)(\alpha+3)x + (\alpha+1)(\alpha+2)(\alpha+3)\right).
\end{aligned}$$

2.2.3 Some important properties include

1. The GLPs satisfy the second-order linear differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0.$$

2. Rodrigues formula for GLPs is expressed as

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(x^{n+\alpha}e^{-x}\right).$$

3. An explicit formula for GLPs is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n+\alpha}{n-k}x^k.$$

4. The Generalized Laguerre Polynomials (GLPs) satisfy recurrence relations for integer α

$$\partial_x L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x) = -\sum_{k=0}^{n-1} L_k^{(\alpha)}(x),$$

$$L_n^{(\alpha)}(x) = \partial_x L_n^{(\alpha-1)}(x) - \partial_x L_{n+1}^{(\alpha)}(x),$$

$$x\partial_x L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x).$$

Theorem 2.4.

1. *The value of the Laguerre polynomial at zero is*

$$L_n(0) = 1.$$

2. *The first derivative of the Laguerre polynomial at zero is*

$$L_n'(0) = -n.$$

3. *The Laplace transform of the Laguerre polynomial is given by*

$$\int_0^\infty e^{-tx}L_n(x)dx = \frac{1}{t}(1-\frac{1}{t})^n.$$

4. *The second derivative at zero satisfies*

$$L_n''(0) = \frac{1}{2}n(n-1).$$

Proof.

1. We evaluate the generating function for $x = 0$. From equation (2.8), we have

$$\frac{e^0}{1-t} = \frac{1}{1-t},$$

this can be expanded into a power series as

$$\sum_{n=0}^{\infty} L_n(0)t^n,$$

but this is also equal to the geometric series

$$\sum_{n=0}^{\infty} t^n.$$

Comparing both series term by term, it follows that

$$L_n(0) = 1. \tag{2.10}$$

2. Proof that $L_n'(0) = -n$: To show this, we consider the fact that $L_n(x)$ satisfies Laguerre's differential previous equation, and by applying differentiation, the result follows accordingly

$$x\frac{d^2}{dx^2}L_n(x) + (1-x)\frac{d}{dx}L_n(x) + nL_n(x) = 0$$

We substitute $x = 0$ into the given equation, which gives

$$L_n'(0) + nL_n(0) = 0.$$

From equation (2.10), we know

$$L_n(0) = 1.$$

Substituting this into the previous result

$$L_n'(0) + n = 0.$$

Therefore

$$L_n'(0) = -n.$$

3. Let's prove that
$$\int_0^\infty e^{-tx}L_n(x)dx = \frac{1}{t}\left(1-\frac{1}{t}\right)^n.$$
We now replace the expression for $L_n(x)$, defined in equation (2.9), into the integral
$$\int_0^\infty e^{-tx}L_n(x)dx = \int_0^\infty e^{-tx}\sum_{r=0}^{n}\frac{(-1)^r n!}{(r!)^2(n-r)!}x^r dx.$$
Switching the sum and the integral
$$\int_0^\infty e^{-tx}L_n(x)dx = \sum_{r=0}^{n}\frac{(-1)^r n!}{(r!)^2(n-r)!}\int_0^\infty x^r e^{-tx}dx.$$
Now, let's make the substitution $tx = z \Rightarrow dz = tdx$, so the integral becomes
$$\int_0^\infty e^{-tx}L_n(x)dx = \sum_{r=0}^{n}\frac{(-1)^r n!}{(r!)^2(n-r)!}\int_0^\infty (\frac{z}{t})^r e^{-z}\frac{dz}{t}.$$
This simplifies to
$$\int_0^\infty e^{-tx}L_n(x)dx = \sum_{r=0}^{n}\frac{(-1)^r n!}{(r!)^2(n-r)!}\frac{1}{t^{r+1}}\int_0^\infty z^r e^{-z}dz.$$
Recognizing the integral as a Gamma function
$$\begin{aligned}\int_0^\infty e^{-tx}L_n(x)dx &= \sum_{r=0}^{n}\frac{(-1)^r n!}{(r!)^2(n-r)!}\frac{1}{t^{r+1}}\Gamma(r+1)\\ &= \frac{1}{t}\sum_{r=0}^{n}\frac{(-1)^r n!}{r!(n-r)!}\left(\frac{1}{t}\right)^r, \quad (\Gamma(r+1)=r!).\end{aligned}$$
This is simply
$$\int_0^\infty e^{-tx}L_n(x)dx = \frac{1}{t}\sum_{r=0}^{n}\binom{n}{r}\left(-\frac{1}{t}\right)^r.$$
Expanding the binomial sum
$$\begin{aligned}\int_0^\infty e^{-tx}L_n(x)dx &= \frac{1}{t}\left[\binom{n}{0}+\binom{n}{1}\left(-\frac{1}{t}\right)+\binom{n}{2}\left(-\frac{1}{t}\right)^2+\cdots+\binom{n}{n}\left(-\frac{1}{t}\right)^n\right]\\ &= \frac{1}{t}\left[1-\frac{1}{t}\right]^n.\end{aligned}$$

4. We aim to demonstrate the following identity
$$L_n''(0) = \frac{1}{2}n(n-1).$$

In the formula for $L_n(x)$, as given by equation , the only term that contributes to the second derivative at $x = 0$ is the one containing x^2. Thus, we obtain

$$L_n''(0) = \frac{n!}{4(n-2)!}2 = \frac{n(n-1)}{2}.$$

□

Although we will not use Jacobi and Legendre polynomials, we will give a brief overview of their forms and properties.

2.3 Legendre polynomials

Definition 2.3. *A polynomial of degree n, denoted by $P_n(x)$, and defined as*

$$P_n(x) = \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r (2n-2r)!}{r!(n-r)!(n-2r)!} x^{n-2r}.$$

is called Legendre polynomial.

2.3.1 Generating function

Proposition 2.1. *The generating function for the Legendre polynomials is expressed as*

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \quad \text{for } |x| \leq 1 \text{ and } |t| < 1.$$

2.3.2 Recurrence relation

Proposition 2.2. *The Legendre polynomials satisfy the following recurrence relation*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

Proof. We start with the generating function

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{2.11}$$

Differentiating both sides of equation (2.11) with respect to t, we obtain

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t)=\sum_{n=0}^{\infty} nP_n(x)t^{n-1}. \tag{2.12}$$

Multiplying both sides of equation (2.12) by $(1-2xt+t^2)$, we get

$$-\frac{1}{2}(-2x+2t)(1-2xt+t^2)^{-1/2}=(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}. \tag{2.13}$$

Substituting equation (2.11) into the left-hand side of equation (2.13), we have

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n=(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Now, we expand and rearrange the terms

$$x\sum_{n=0}^{\infty} P_n(x)t^n-\sum_{n=0}^{\infty} P_n(x)t^{n+1}=\sum_{n=0}^{\infty} nP_n(x)t^{n-1}-2x\sum_{n=0}^{\infty} nP_n(x)t^n+\sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

By shifting indices

- $\sum_{n=0}^{\infty} P_n(x)t^{n+1}=\sum_{n=1}^{\infty} P_{n-1}(x)t^n$,
- $\sum_{n=0}^{\infty} nP_n(x)t^{n-1}=\sum_{n=1}^{\infty} nP_n(x)t^{n-1}=\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n$,
- $\sum_{n=0}^{\infty} nP_n(x)t^{n+1}=\sum_{n=1}^{\infty}(n-1)P_{n-1}(x)t^n$.

Substituting these into the equation , we get

$$\sum_{n=0}^{\infty}\left(xp_n(x)-P_{n-1}(x)\right)t^n=\sum_{n=0}^{\infty}\left((n+1)P_{n+1}(x)-2xnP_n(x)+(n-1)P_{n-1}(x)\right)t^n.$$

Equating the coefficients of t^n, we obtain

$$xP_n(x)-P_{n-1}(x)=(n+1)P_{n+1}(x)-2xnP_n(x)+(n-1)P_{n-1}(x).$$

Rearranging terms

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0.$$

Therefore, we conclude

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x).$$

□

2.3.3 Rodrigue's Formula

Proposition 2.3. *Rodrigue's formula is given by*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

The first Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}(3x^2-1), \\
P_3(x) &= \frac{1}{2}(5x^3-3x), \\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3), \\
P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x).
\end{aligned}$$

To clarify further, we have drawn the first five terms of the Legendre.

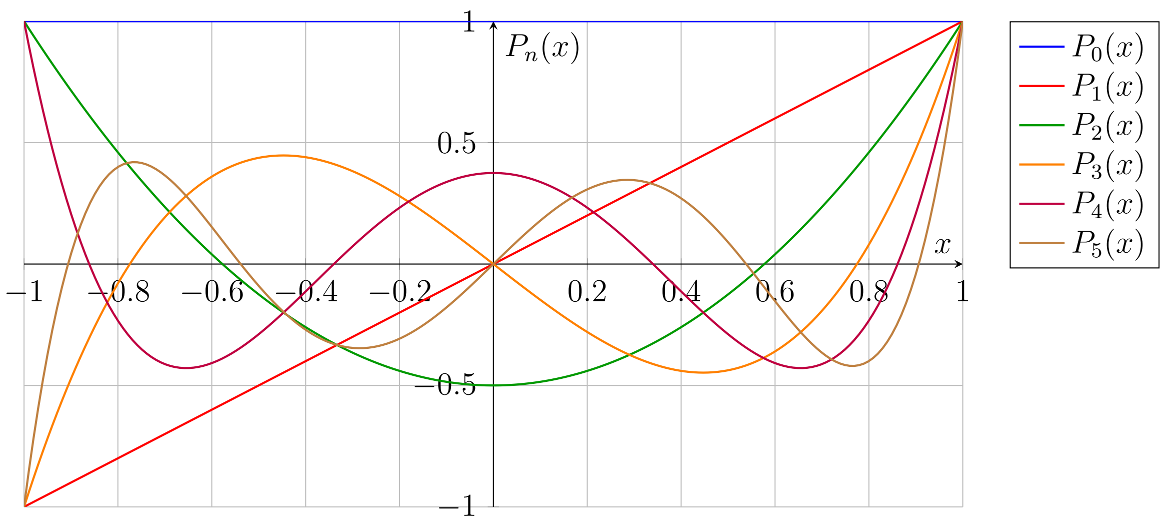

Figure 2.3: Plot of the Legendre polynomials $P_0(x)$ to $P_5(x)$.

2.3.4 Orthogonality of Legendre polynomials

Theorem 2.5. *Legendre polynomials are orthogonal over the interval* $[-1, 1[$ *with respect to the uniform weight. If* $m, n \in \mathbb{N}^*$ *then*

$$\langle P_m(x), P_n(x)\rangle = \int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases}$$

2.4 Jacobi polynomials

Definition 2.4. *The Jacobi polynomial of degree* n*, denoted by* $J_n^{\alpha,\beta}(x)$*, is a polynomial that is orthogonal with respect to the Jacobi weight function*

$$w^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

on the interval $I = (-1, 1)$*. This orthogonality condition is expressed as*

$$\int_{-1}^{+1} J_n^{\alpha,\beta}(x)J_m^{\alpha,\beta}(x)w^{\alpha,\beta}(x)dx = \gamma_n^{\alpha,\beta}\delta_{n,m},$$

where $\gamma_n^{\alpha,\beta} = \| J_n^{\alpha,\beta} \|_{w^{\alpha,\beta}}^2$.
The weight function $w^{\alpha,\beta}$ *is integrable in* $L^1(I)$ *if and only if* $\alpha, \beta > -1$*, (which is assumed throughout this section).*

2.4.1 Fundamental properties

The Jacobi polynomials of degree n satisfy a recurrence relation for parameters $\alpha > -1$ and $\beta > -1$,

$$a_n J_{n+1}^{\alpha,\beta}(x) = (b_n + xc_n)J_n^{\alpha,\beta}(x) - d_n J_{n-1}^{\alpha,\beta}(x).$$

This sequence is initialized with

$$J_0^{\alpha,\beta}(x) = 1, \quad J_1^{\alpha,\beta}(x) = \frac{(\alpha-\beta)}{2} + \left(1 + \frac{(\alpha+\beta)}{2}\right)(x),$$

With the following coefficients

$$\begin{aligned} a_n &= 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta), \\ b_n &= (2n+\alpha+\beta+1)(\alpha^2-\beta^2), \\ c_n &= (2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2), \\ d_n &= 2(n+\alpha)(n+\alpha+\beta)(2n+\alpha+\beta+2), \end{aligned}$$

The first five polynomials obtained for $\alpha = 1$ and $\beta = 0$, are shown in the following figure.

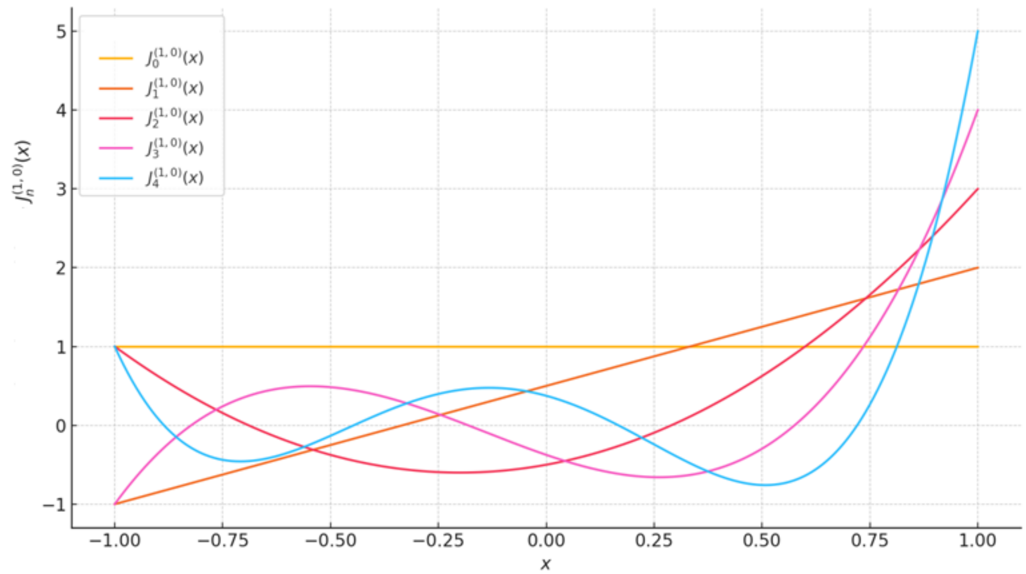

Figure 2.4: First five Jacobi polynomials for $\alpha = 1$ and $\beta = 0$.

1. The Jacobi polynomials satisfy the differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0.$$

2. By setting $\lambda = 2n + \alpha + \beta$ the Jacobi polynomials satisfy the relations

$$\lambda(1-x^2)\partial_x J_n^{\alpha,\beta}(x) = n(\alpha - \beta - \lambda x)J_n^{\alpha,\beta}(x) + 2(n+\alpha)(n+\beta)J_{n-1}^{\alpha,\beta}(x).$$

$$J_n^{\alpha,\beta-1}(x) - J_n^{\alpha-1,\beta}(x) = J_{n-1}^{\alpha,\beta}(x).$$

$$\lambda J_n^{\alpha,\beta-1}(x) = (n+\alpha+\beta)J_n^{\alpha,\beta}(x) + (n+\alpha)J_{n-1}^{\alpha,\beta}(x).$$

$$(1-x)J_n^{\alpha+1,\beta}(x) + (1+x)J_n^{\alpha,\beta+1}(x) = 2J_n^{\alpha,\beta}(x).$$

3. The norm of the Jacobi polynomial is

$$\| J_n^{\alpha,\beta} \|_{w^{\alpha,\beta}}^2 = \gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}.$$

4. Rodrigues formula

$$J_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}\frac{1}{(1-x)^\alpha(1+x)^\beta}\frac{d^n}{dx^n}\left((1-x^2)^n(1-x)^\alpha(1+x)^\beta\right).$$

5. The derivative of the Jacobi polynomial satisfies the relation

$$\partial_x J_n^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1)J_{n-1}^{\alpha+1,\beta+1}(x).$$

$$\partial_x^k J_n^{\alpha,\beta}(x) = \frac{\Gamma(n+k+\alpha+\beta+1)}{2^k\Gamma(n+\alpha+\beta+1)}J_{n-k}^{\alpha+k,\beta+k}(x), \quad n \geq k.$$

6. Integration formula

$$2n\int_0^1 J_n^{\alpha,\beta}(t)(1-t)^\alpha(1+t)^\beta dt = J_{n-1}^{\alpha+1,\beta+1}(0) - h^{\alpha,\beta}(x)J_{n-1}^{\alpha+1,\beta+1}(x),$$

where

$$h^{\alpha,\beta}(x) = (1-x)^{1+\alpha}(1+x)^{1+\beta}.$$

2.4.2 Special cases of Legendre polynomial

1. For $\alpha = \beta = 0$, the resulting polynomial is the Legendre polynomial

$$L_n(x) = J_n^{0,0}(x).$$

2. For $\alpha = \beta = -\frac{1}{2}$, the polynomial obtained is the Tchebyshev polynomial of the first kind

$$T_n(x) = \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}J_n^{-\frac{1}{2},-\frac{1}{2}}(x).$$

3. For $\alpha = \beta = \frac{1}{2}$, we get the Tchebyshev polynomial of the second kind

$$U_n(x) = \frac{(n+1)!\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}J_n^{\frac{-1}{2},\frac{-1}{2}}(x).$$

4. For $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$, we obtain the Tchebyshev polynomial of the third kind

$$V_n(x) = \frac{n!\,\sqrt{\pi}}{\Gamma\big(n+\frac{1}{2}\big)} J_n^{-\frac{1}{2},\frac{-1}{2}}(x).$$

5. For $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, the result is the Tchebyshev polynomial of the fourth kind

$$W_n(x) = \frac{n!\sqrt{\pi}}{\Gamma\big(n+\frac{1}{2}\big)} J_n^{\frac{1}{2},-\frac{1}{2}}(x).$$

6. When $\alpha = \beta$, the Gegenbauer polynomial emerges

$$G_n^{\alpha}(x) = \frac{\Gamma(2\alpha+n)\,\Gamma\big(\alpha+\frac{1}{2}\big)}{\Gamma(2\alpha)\,\Gamma\big(\alpha+n+\frac{1}{2}\big)}\, J_n^{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}(x).$$

For more information on orthogonal polynomials, their properties and applications, see [5], [12], [3], [1] and [11].

Chapter 3

Numerical solution for some integro-differential equations

3.1 Tchebyshev-Galerkin method

3.1.1 Discription of method

Let us consider the following Volterra integro-differential equation

$$u'(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt, \qquad a \le x \le b, \tag{3.1}$$

$$u(a) = \alpha. \tag{3.2}$$

In this equation, $u(x)$ is the unknown function, $k(x,t)$ is a given continuous and square-integrable kernel, $f(x)$ is a known function, and λ is a given real parameter.

The approach employed here is based on Tchebyshev polynomials, as thoroughly discussed in [7]. To approximate the solution on a closed finite interval, we use a basis polynomial approach. Assume that

$$u(x) \approx u_n(x) = \sum_{i=0}^{n} a_i T_i \left(\frac{2x - (b+a)}{b-a} \right), \tag{3.3}$$

where $T_i\left(\frac{2x-(b+a)}{b-a}\right)$ represents the shifted Tchebyshev polynomial over the interval $[a,b]$. Consequently, the derivative is approximated by

$$u'(x) \approx u_n'(x) = \sum_{i=0}^{n} \frac{2}{b-a} a_i T_i'\left(\frac{2x-(b+a)}{b-a}\right).$$

Substituting the expressions for $u_n(x)$ and $u_n'(x)$ into equation (3.1), we get

$$\sum_{i=0}^{n} \frac{2}{b-a} a_i T_i'\left(\frac{2x-(b+a)}{b-a}\right) = f(x) + \lambda \sum_{i=0}^{n} a_i \int_a^x K(x,t) T_i\left(\frac{2t-(b+a)}{b-a}\right) dt, \quad a \le x \le b.$$

To determine the unknown coefficients α_i , we apply the Galerkin method by multiplying both sides of the equation by $T_j\left(\frac{2x-(b+a)}{b-a}\right)$ and integrating with respect to x over the interval $[-1,1]$

$$\sum_{i=0}^{n} \frac{2}{b-a} a_i \int_{-1}^{1} T_i'\left(\frac{2x-(b+a)}{b-a}\right) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx = \int_{-1}^{1} f(x) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx +$$
$$\int_{-1}^{1}\left(\lambda \sum_{i=0}^{n} a_i \int_a^x K(x,t) T_i\left(\frac{2t-(b+a)}{b-a}\right) dt\right) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx, \quad j = 0, 1, \ldots, n. \quad (3.4)$$

Or equivalently, if needed, the integrals can be computed using numerical methods. This leads to a system of linear equations in terms of the unknown coefficients $\{a_i\}_{i=0}^n$. Many researchers incorporate the initial condition

$$u(a) = \alpha \Rightarrow \sum_{i=0}^{n} a_i T_i\left(\frac{2a-(b-a)}{b-a}\right) = \sum_{i=0}^{n} a_i T_i(-1) = \alpha. \quad (3.5)$$

To provide an additional equation, ensuring the number of equations matches the number of unknowns in the system. The unknown parameters are then obtained by solving the system of equations (3.4) and (3.5). Substituting these values into equation (3.3) yields an approximate solution to the integro-differential equation (3.1) This approach can similarly be applied to a Fredholm integro-differential equation of the form

$$\begin{aligned} u'(x) &= f(x) + \lambda \int_a^b K(x,t) u(t)\, dt, \quad a \le x \le b, \\ u(a) &= \alpha. \end{aligned}$$

The equation (3.4) is equivalent to

$$\sum_{i=0}^{n} \frac{2}{b-a} a_i \int_{-1}^{1} T_i'\left(\frac{2x-(b+a)}{b-a}\right) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx = \int_{-1}^{1} f(x) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx +$$
$$\lambda \sum_{i=0}^{n} a_i \int_{-1}^{1}\left(\int_a^x K(x,t) T_i\left(\frac{2t-(b+a)}{b-a}\right) dt\right) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx, \; j = 0, ..., n.$$

The last equations present a linear system of $n+1$ equations and $n+1$ unknown, the matrix of the system is given by

$$\begin{aligned} A_{i,j} =& \frac{2}{b-a}\int_{-1}^{1} T_i'\left(\frac{2x-(b+a)}{b-a}\right) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx \\ &-\lambda \int_{-1}^{1}\left(\int_a^x K(x,t)T_i\left(\frac{2t-(b+a)}{b-a}\right) dt\right) T_j\left(\frac{2x-(b+a)}{b-a}\right) dx, \end{aligned}$$

and the vector B, given by

$$B_j = \int_{-1}^{1} f(x)T_j\left(\frac{2x-(b+a)}{b-a}\right) dx,\ j = 0, ..., n.$$

3.1.2 Detailed problems

Example 3.1.

Let us consider the following fredholem **intergal equation**

$$u(x) + 5\int_0^1 xt^2u(t)dt = x^2 + x. \tag{3.6}$$

Where, the exact solution is given by $u(x) = x^2$.

Using Tchebyshev bases to solve linear fredholem equation

A linear Fredholem equation of the second kind generally has the form

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt. \tag{3.7}$$

Tchebyshev polynomials of the first kind $T_i(x)$ are defined over the interval $[-1, 1]$. The first few are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,\ \ldots.$$

We approximate $u(x)$ as

$$u(x) \approx \sum_{i=0}^{n} a_iT_i(x). \tag{3.8}$$

Substitute thes approximation into the original equation.

Interchange the sum and the integral

$$\sum_{i=0}^{n} a_iT_i(x) = f(x) + \lambda\sum_{n=0}^{n}\int_a^b K(x,t)a_iT_i(x)dt. \tag{3.9}$$

Finding a_i passing through the next steps:

- Product both sides into Tchebyshev polynomials (using orthogonal projection), to obtain a linear system for the coefficients a_i.
- Solve the linear system to find a_i.
- Construct the approximate solution $u(x)$.

To simplify the description of the method, we will do the calculations for $n = 2$.

Approximate $u(x)$

$$\begin{aligned} u(x) &\approx a_0T_0(x) + a_1T_1(x) + a_2T_2(x) \\ &= a_0 + a_1x + a_2(2x^2 - 1) \\ &= (a_0 - a_2) + a_1x + a_2x^2, \end{aligned} \tag{3.10}$$

substitute the Tchebyshev expansion of $u(t)$ in (3.6), we get

$$u(x) = x^2 + x - 5\int_{-1}^{1} xt^2(a_0 + a_1t + a_2(2t^2 - 1))dt.$$

Evaluate the integral

$$\begin{aligned} u(x) =&x^2 + x - 5x\int_{-1}^{1} t^2(a_0 + a_1t + a_2(2t^2 - 1))dt \\ =&x^2 + x - 5x\left[\int_{-1}^{1}(a_0t^2)dt + \int_{-1}^{1}(a_1t^3)dt + \int_{-1}^{1}(a_2(2t^4 - t^2)\right] dt. \end{aligned}$$

After calculating the integration, we will get

$$u(x) = x^2 + x - 5x(\frac{2}{3}a_0 + \frac{2}{5}a_2),$$

$$u(x) = x^2 + (-\frac{10}{15}a_0 - \frac{10}{15}a_2 + 1)x. \tag{3.11}$$

By comparinng equations (3.10) and (3.11) it can be deduced that

$$\begin{cases} a_0 - a_2 = 0, \\ \dfrac{-10}{15}a_0 - \dfrac{-10}{15}a_2 + 1 = a_1, \\ a_2 = 1, \end{cases} \tag{3.12}$$

so $a_0 = a_2 = 1$ and $a_1 = \frac{35}{15}$, then

$$u(x) = \frac{35}{15}x + x^2,$$

this is the approximate solution using Tchebyshev bases up to degree 2.

We created the numerical code for this problem using MATLAB, a program that simulates collocation method algorithme wich based on the Tchebyshev polynomial.

Code of calcul

```
clear all; clc
n=4;
a=-1; b=1;
lamda=5;
k=@(x,t)x.*t.^2;
f=@(x)x.^2+x;
w=@(x)1./(sqrt(1-x.^2));
fun=@(x)w(x).*f(x);
rhs=@(n,x)fun(x).*chebyshevT(n,x);
tfun=@(x,t,i)k(x,t).*chebyshevT(i,t);
for i=0:n
q(i+1)=integral(@(x)rhs(i,x),-1,1);
end
rhs=eye(n+1)*pi/2;
rhs(1,1)=pi;
for i=0:n
for j=0:n
B=@(x,t)tfun(x,t,i).*chebyshevT(j,x).*w(x);
C(i+1,j+1)=integral2(B,-1,1,0,1);
end
end
A=rhs+lamda*C';
```

```
aa=inv(A)*q';
xx=a:0.1:b;
phi=@(x)0;
for j=0:n
phi=@(x)aa(j+1)*chebyshevT(j,x)+phi(x);
end
for i=1:length(xx)
yy(i)=phi(xx(i));
end
plot(xx,xx.^2,xx,yy,'ro')
```

The following figure represents the exact and approximate solutions for $n = 4$, where do we remark that they are identical.

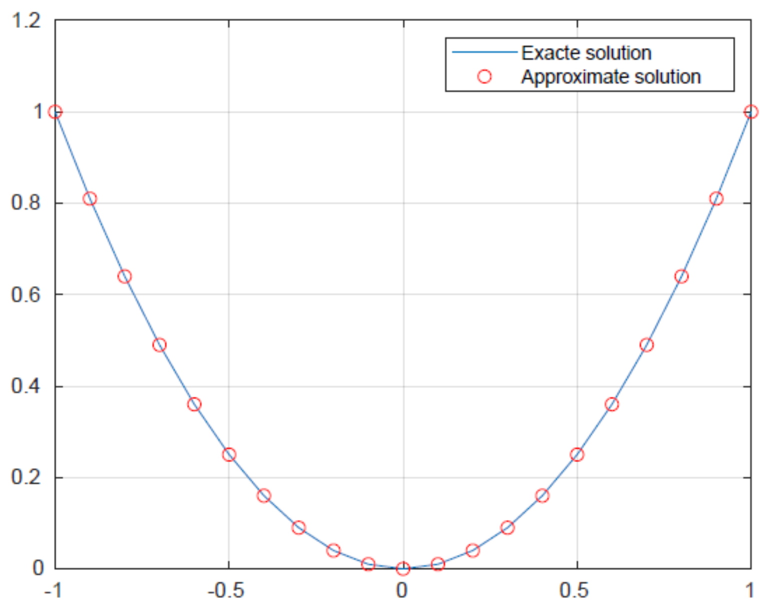

Figure 3.1: Exact and approximate solution Tchebyshev polynomial for $n = 4$.

Value of x	**Exact Solution**	**Approximate Solution**	**Error**
-1.0	1.000000000000000	1.000000000003581	$0.358113538823090\times10^{-11}$
-0.8	0.640000000000000	0.640000000002392	$0.239164243964751\times10^{-11}$
-0.6	0.360000000000000	0.360000000001705	$0.170530256582424\times10^{-11}$
-0.4	0.160000000000000	0.160000000001237	$0.123731580536912\times10^{-11}$
-0.2	0.040000000000000	0.040000000000783	$0.078295009364737\times10^{-11}$
0.0	0.000000000000000	0.000000000000217	$0.021713972140689\times10^{-11}$
0.2	0.040000000000000	0.039999999999495	$0.050525555961300\times10^{-11}$
0.4	0.160000000000000	0.159999999998650	$0.134944833085626\times10^{-11}$
0.6	0.360000000000000	0.359999999997799	$0.220073959056322\times10^{-11}$
0.8	0.640000000000000	0.639999999997136	$0.286459744813783\times10^{-11}$
1.0	1.000000000000000	0.999999999996934	$0.306621394940976\times10^{-11}$

Table 3.1: Approximate solution compared to exact solution, using the Tchebyshev polynomial-based collocation method. The error is calculated for $n = 4$.

We notice from the table that in this example there is high accuracy even for small n (10^{-11}).

Example 3.2.

Consider the following Volterra integro-differential equation

$$u'(x) = 2 - \frac{x^2}{4} + \frac{1}{4}\int_0^x u(t)\,dt, \quad x \in [0,1];\ u(0) = 0. \tag{3.13}$$

The exact solution is given by $u(x) = x\cos(x)$.

In the following, we will explain the steps of the algorithm without performing the numerical computations. To solve equation (3.13), we use a fourth-order approximation for $u(x)$ and $u'(x)$ as follows

$$\begin{aligned} u_4(x) = \sum_{i=0}^{4} a_i T_i(2x-1) =& a_0 + a_1(2x-1) + a_2(8x^2-8x+1) + a_3(32x^3-48x^2+18x-1) \\ &+ a_4(128x^4-256x^3+160x^2-32x+1). \end{aligned} \tag{3.14}$$

The derivative is given by

$$u_4'(x) = 2\sum_{i=0}^{4} a_i T_i'(2x-1) = 2a_1 + a_2(16x-8) + a_3(96x^2-96x+18) + a_4(512x^3-768x^2+320x-32). \tag{3.15}$$

Substituting (3.14) and (3.15) into equation (3.13) gives

$$2a_1 + a_2(16x-8) + a_3(96x^2-96x+18) + a_4(512x^3-768x^2+320x-32) = 2 - \frac{x^2}{4} + \int_0^x [a_0 + a_1(2t-1) + a_2(8t^2-8t+1) + a_3(32t^3-48t^2+18t-1) + a_4(128t^4-256t^3+160t^2-32t+1)],$$

By simple calculations, we find

$$\begin{aligned} 8a_1 - 32a_2 + 72a_3 - 128a_4 =& x(a_0 - a_1 - 63a_2 + 383a_3 - 1279a_4) \\ &+ x^2(a_1 - 4a_2 - 375a_3 + 3056a_4 + 1) \\ &+ x^3(\frac{8}{3}a_2 - 16a_3 - \frac{5984}{3}a_4) \\ &+ x^4(8a_3 - 64a_4) + x^5(\frac{128}{5}a_4). \end{aligned} \tag{3.16}$$

By multiplying both sides of equation by $T_0(2x-1) = 1$ and integrating with respect to x from -1 to 1, we obtain

$$\begin{aligned} \int_{-1}^{1}(8a_1 - 32a_2 + 72a_3 - 128a_4)dx =& \int_{-1}^{1}(x(a_0 - a_1 - 63a_2 + 383a_3 - 1279a_4)dx \\ &+ \int_{-1}^{1}(x^2(a_1 - 4a_2 - 375a_3 + 3056a_4 + 1))dx \\ &+ \int_{-1}^{1}(x^3(\frac{8}{3}a_2 - 16a_3 - \frac{5984}{3}a_4))dx \\ &+ \int_{-1}^{1}(x^4(8a_3 - 64a_4))dx + \int_{-1}^{1}(x^5(\frac{128}{5}a_4))dx. \end{aligned}$$

After computing the integrals, we obtain the following linear equation

$$230a_1 - 920a_2 + 4134a_3 - 34016a_4 = 250. \tag{3.17}$$

In the second step, we multiply both sides of equation (3.16) by $T_1(2x-1) = 2x-1$, and integrate with respect to from -1 to 1, obtaining

$$\begin{aligned}\int_{-1}^{1}(8a_1 - 32a_2 + 72a_3 - 128a_4)(2x-1)dx = &\int_{-1}^{1}(x(a_0 - a_1 - 63a_2 + 383a_3 - 1279a_4)(2x-1)dx \\ &+ \int_{-1}^{1}(x^2 a_1 - 4a_2 - 375a_3 + 3056a_4 + 1)(2x-1)dx \\ &+ \int_{-1}^{1} x^3(\frac{8}{3}a_2 - 16a_3 - \frac{5984}{3}a_4)(2x-1)dx \\ &+ \int_{-1}^{1} x^4(8a_3 - 64a_4))dx + \int_{-1}^{1}(x^5(\frac{128}{5}a_4)(2x-1)dx.\end{aligned} \tag{3.18}$$

In the third step, we multiply equation (3.16) by $T_2(2x-1) = 8x^2 - 8x + 1$, then integrate with respect to x from -1 to 1, yielding

$$\begin{aligned}&\int_{-1}^{1}(8a_1 - 32a_2 + 72a_3 - 128a_4)(8x^2 - 8x + 1)dx \\ =&\int_{-1}^{1}((a_0 - a_1 - 63a_2 + 383a_3 - 1279a_4)(8x^3 - 8x^2 + x)dx \\ &+ \int_{-1}^{1}(a_1 - 4a_2 - 375a_3 + 3056a_4 + 1)(8x^4 - 8x^3 + x^2)dx \\ &+ \int_{-1}^{1}(\frac{8}{3}a_2 - 16a_3 - \frac{5984}{3}a_4)(8x^5 - 8x^4 + x^3)dx \\ &+ \int_{-1}^{1}(8a_3 - 64a_4)(8x^6 - 8x^5 + x^4)dx \\ &+ \int_{-1}^{1}(\frac{128}{5}a_4)(8x^7 - 8x^6 + x^5)dx.\end{aligned} \tag{3.19}$$

By multiplying both sides of equation (3.16) by $T_3(2x-1) = 32x^3 - 48x^2 + 18x - 1$, and integrating with respect to x from -1 to 1, we obtain

$$\int_{-1}^{1}(8a_1-32a_2+72a_3-128a_4)(32x^3-48x^2+18x-1)dx$$
$$=\int_{-1}^{1}(x(a_0-a_1-63a_2+383a_3-1279a_4)(32x^4-48x^3+18x^2-x)dx$$
$$+\int_{-1}^{1}(a_1-4a_2-375a_3+3056a_4+1)(32x^5-48x^4+18x^3-x^2)dx$$
$$+\int_{-1}^{1}(\frac{8}{3}a_2-16a_3-\frac{5984}{3}a_4)(32x^6-48x^5+18x^4-x^3)dx$$
$$+\int_{-1}^{1}(8a_3-64a_4)(32x^7-48x^6+18x^5-x^4)dx$$
$$+\int_{-1}^{1}(\frac{128}{5}a_4)(32x^8-48x^7+18x^6-x^5)dx. \tag{3.20}$$

Next, we multiply both sides of equation (3.16) by $T_4(2x-1)=128x^4-256x^3+160x^2-32x+1$ and integrate with respect to x from -1 to 1, yielding

$$\int_{-1}^{1}(8a_1-32a_2+72a_3-128a_4)(128x^4-256x^3+160x^2-32x+1)dx$$
$$=\int_{-1}^{1}((a_0-a_1-63a_2+383a_3-1279a_4)(128x^5-256x^4+160x^3-32x^2+x)dx$$
$$+\int_{-1}^{1}(a_1-4a_2-375a_3+3056a_4+1)(128x^6-256x^5+160x^4-32x^3+x^2)dx$$
$$+\int_{-1}^{1}(\frac{8}{3}a_2-16a_3-\frac{5984}{3}a_4)(128x^7-256x^6+160x^5-32x^4+x^3)dx$$
$$+\int_{-1}^{1}(8a_3-64a_4)(128x^8-256x^7+160x^6-32x^5+x^4)dx$$
$$+\int_{-1}^{1}(\frac{128}{5}a_4)(128x^9-256x^8+160x^7-32x^6+x^5)dx. \tag{3.21}$$

By straightforward computation of the integrals, equations (3.18), (3.19), (3.20), and (3.21), along with equation (3.17), together form a linear system of five equations with five unknowns.

Using a method for solving linear systems (such as the Gaussian elimination method), the values of a_0, a_1,a_2,a_3 and a_4 are obtained. Therefore, the approximate solution of equation (3.13).

Value of x	**Exact Solution** u_{ex}	**Approximate Solution** u_T	**Error**
0.0000	0.0000	-5.1369e-04	5.1369e-04
0.1250	0.2500	0.2495	4.9144e-04
0.2500	0.5000	0.4995	4.6944e-04
0.3750	0.7500	0.7495	4.4771e-04
0.5000	1.0000	0.9996	4.2625e-04
0.6250	1.2500	1.2496	4.0507e-04
0.7500	1.5000	1.4996	3.8420e-04
0.8750	1.7500	1.7496	3.6366e-04
1.0000	2.0000	1.9997	3.4348e-04

Table 3.2: Exact and approximate solution, using the Chebyshev polynomial-based collocation method. The error is calculated for $n = 8$.

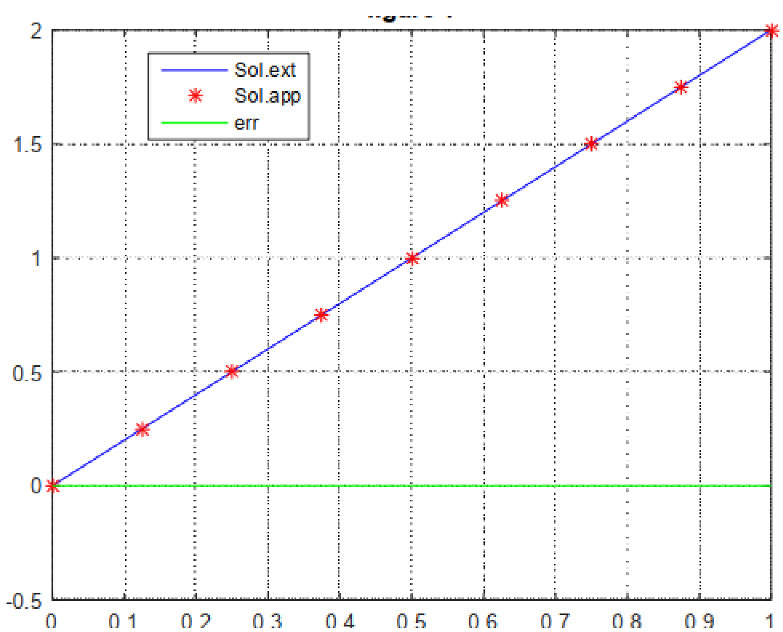

Figure 3.2: Exact and approximate solution, using Tchebyshev polynomial-based collocation method for $n = 8$.

3.2 Numerical solution using collocation method with Laguerre polynomials

3.2.1 Discription of method

Suppose that the function $f(x)$ is approximated using the Laguerre polynomials (LPs) as follows

$$f(x) = a_1 L_1(x) + a_2 L_2(x) + \cdots + a_n L_n(x) = \sum_{i=1}^{n} a_i L_i(x). \tag{3.22}$$

For $r \geq 0$, the function $L_n(x)$ represents the Laguerre basis polynomials of degree n, as defined by

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^n. \tag{3.23}$$

a_r (where $r = 0, 1, \cdots, n$) are the unknown Laguerre coefficients to be determined later. Rewriting equation (3.22) as a dot product

$$f(x) = [L_0(x),\ L_1(x)\,, ..., L_n(x)] \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \tag{3.24}$$

equation (3.24) can be expressed as

$$f(x) = \begin{bmatrix} 1, & x, & x^2, & \ldots, & x^n \end{bmatrix} \begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \ldots & \theta_{0n} \\ 0 & \theta_{11} & \theta_{12} & \ldots & \theta_{1n} \\ 0 & 0 & \theta_{22} & \ldots & \theta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \theta_{nn} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \tag{3.25}$$

where θ_{sr} (with $s, r = 0, 1, 2, \cdots, n$) are the known values of the power basis used to find the Laguerre polynomials (LPs). Additionally, the square matrix is upper triangular and non-singular. For example, when $n = 1$ and $n = 2$, the operational matrices are given in equation

(3.26) and (3.27), respectively

For $n = 1$

$$f(x) = [1, x]\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}a_0\\ a_1\end{bmatrix}. \tag{3.26}$$

For $n = 2$

$$f(x) = [1, x, x^2]\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}. \tag{3.27}$$

Since the derivative of equation (3.23) is

$$L_n'(x) = \frac{d}{dx}\sum_{r=0}^{n}(-1)^r\frac{1}{r!} = \sum_{r=0}^{n}\frac{(-1)^r}{s!(n-r)!}x^{r-1}, \quad n = 1, 2, \ldots, n, \quad x \in [0, \infty[,$$

the derivative of equations (3.24), (3.26) and (3.27) is as follows

$$f(x)' = [0, 1, 2x, \ldots, nx^{(n-1)}]\cdot\begin{bmatrix}\theta_{00} & \theta_{01} & \theta_{02} & \ldots & \theta_{0n}\\ 0 & \theta_{11} & \theta_{12} & \ldots & \theta_{1n}\\ 0 & 0 & \theta_{22} & \ldots & \theta_{2n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & \theta_{nn}\end{bmatrix}\cdot\begin{bmatrix}a_0\\ a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix}, \tag{3.28}$$

for $n = 1$

$$f'(x) = [0, 1]\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}a_0\\ a_1\end{bmatrix}, \tag{3.29}$$

for $n = 2$

$$f'(x) = [0, 1, 2x]\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}. \tag{3.30}$$

3.3 Solution of the second kind Volterra integral (VI) equation using the (LPs)

In this section, the (LPs) method is applied to find the solutions for the (VI) equation

$$f(x) = g(x) + \lambda\int_{b_1}^{b_2} k(x,t)f(t)\,dt, \quad b_1 \le x \le b_2. \tag{3.31}$$

By applying equation (3.22), we assume that

$$f(x) \approx f_n(x) = \sum_{i=0}^{n} a_i L_i(x). \tag{3.32}$$

Substituting equation (3.32) into equation (3.31), it results in

$$\sum_{i=0}^{n} a_i L_i(x) = g(x) + \lambda \int_{b_1}^{b_2} k(x,t) \sum_{i=0}^{n} a_i L_i(t)\, dt. \tag{3.33}$$

By employing equation (3.24), equation (3.33) can be rewritten as

$$\begin{bmatrix} L_0(x) & L_1(x) & \dots & L_n(x) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = g(x) + \lambda \int_{b_1}^{b_2} k(x,t) \begin{bmatrix} L_0(t) & L_1(\tau) & \dots & L_n(t) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} dt. \tag{3.34}$$

Or

$$\begin{aligned} &\begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix} \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \dots & \theta_{1n} \\ 0 & \theta_{22} & \theta_{23} & \dots & \theta_{2n} \\ 0 & 0 & \theta_{33} & \dots & \theta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_{nn} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\ &= g(x) \\ &+ \lambda \int_{x_1}^{x} k(x,t) \begin{bmatrix} 1 & t & t^2 \cdots t^n \end{bmatrix} \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \dots & \theta_{1n} \\ 0 & \theta_{22} & \theta_{23} & \dots & \theta_{2n} \\ 0 & 0 & \theta_{33} & \dots & \theta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_{nn} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} dt. \end{aligned} \tag{3.35}$$

After simplifying equation (3.35), the unknown Laguerre coefficients $a_0, a_1, \cdots, a_n$ are determined by choosing specific points x_β, $(\beta = 0, 1, \cdots, n)$ wich in the interval $[b_1, b_2]$. As a result, equation (3.35) transforms into a system of $(n+1)$ linear algebraic equations with $(n+1)$ unknown coefficients. This system can be solved using the Gaussian elimination method or other numerical method, ensuring unique solutions for the coefficients. Once determined, these coefficients are substituted into equation (3.22) to obtain the approximate numerical solution.

Example 3.3. ([11]) Solve the Volterra integral-differential (VID) equation of the second kind with constant kernel

$$f'(x) = 6 - 3x^2 + \int_0^x f(t)dt, \quad f(0) = 0, \tag{3.36}$$

where the exact solution is $f(x) = 6x$.

Using the Laguerre polynomials, the approximate solution for $n = 2$ can written as

$$f(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \tag{3.37}$$

its derivatives are

$$f'(x) = [0, 1, 2x] \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}. \tag{3.38}$$

By substituting equations (3.37) and (3.38) into (3.36), we get

$$-a_1 - 2a_2 + a_2 x = 6 - 3x^2 + \int_0^x \left[(a_0 + a_1 + a_2) - t(a_1 + 2a_2) + t^2(\frac{1}{2}a_2) \right] dt,$$

after computing the integral, we obtain

$$(-a_1 - 2a_2 - 6) - (a_0 + a_1)x + (\frac{1}{2}a_1 + a_2)x^2 - \frac{1}{6}a_2 x^3.$$

By comparison, we find

$$-a_1 - 2a_2 - 6 = 0 \qquad a_0 + a_1 = 0 \qquad \frac{1}{2}a_1 + a_2 = 0 \qquad \frac{1}{6}a_2, = 0,$$

so

$$a_0 = 6 \qquad a_1 = -6 \qquad a_2 = 0.$$

Finaly, we get

$$f_2(x) = 6L_0(x) - 6L_1(x) = 6x.$$

In the same manner, the exact solution can be obtained for $n = 3$ and $n = 4$.

$$f_3(x) = 6L_0(x) - 6L_1(x) = f(x) = 6x,$$

$$f_4(x) = 6L_0(x - 6L_1(x) = f(x) = 6x.$$

The solutions were approximated in three different degrees and the exact solution was obtained the same and this shows that the error function is zero in this case.

Figure 3.3 displays the comparison of results for $n = 2, 3$ and 4 with exact solution. They seem to be identical.

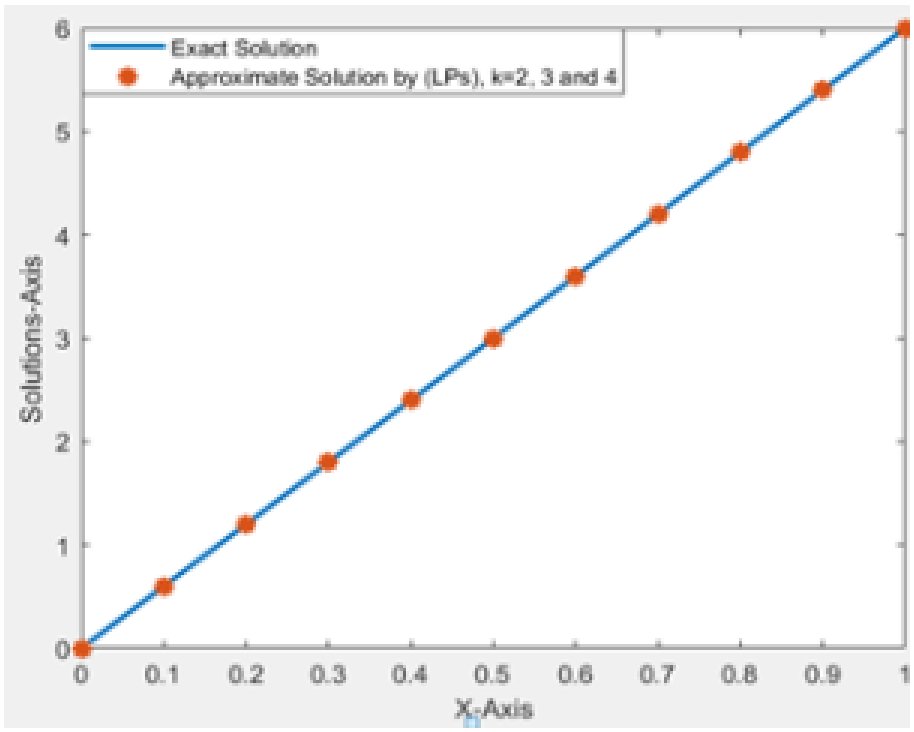

Figure 3.3: The Laguerre polynomials for $n = 2, 3$ and 4.

Conclusion

This study presented an efficient numerical approach to solving integral and integro-differential equations using orthogonal polynomials and the collocation method. By leveraging polynomials like Tchebyshev and Laguerre, complex problems were transformed into manageable algebraic systems.

The proposed method proved effective, demonstrating high accuracy and fast convergence, making it suitable for various applications. Results indicated that orthogonal polynomials enhance solution quality and reduce approximation errors.

In summary, collocation methods with orthogonal polynomials offer a robust framework for solving complex equations, with potential for further applications and algorithmic improvements.

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الملخص

تركز هذه المذكرة على الحل العددي للمعادلات التكاملية والتفاضلية التكاملية باستخدام طرق التجميع المعتمدة على كثيرات الحدود المتعامدة مثل تشيبيشاف, ليجوندر ولاغر. تعتمد الطريقة على تحويل المسائل التفاضلية المعقدة إلى أنظمة جبرية يسهل حلها، مما يحقق دقة عالية وسرعة تقارب. أظهرت النتائج فعالية الأسلوب المقترح، مما يجعله مناسبًا للتطبيقات العلمية و الهندسية، مع إمكانية توسيع استخدامه وتطويره مستقبلاً.

الكلمات المفتاحية: المعادلات التكاملية، المعادلات التكاملية التفاضلية، الحل العددي, معادلة فولتيرا، كثير حدود لاغر، كثير حدود تشيبيشاف.

Abstract

The aim of this work is to focus on the numerical solution of integral and integro-differential equations using collocation methods based on orthogonal polynomials such as Chebyshev, Legendre, and Laguerre. The approach involves transforming complex differential problems into algebraic systems that are easy to solve, ensuring high accuracy and fast convergence. The results demonstrate the efficiency of the proposed method, making it suitable for scientific and engineering applications, with potential for future expansion and development.

Keywords: Integral equations, Integro-differential equations, Numerical solution, Volterra equation, Laguerre polynomials, Chebyshev polynomials.

Résumé

L'objectif de ce travail est de se concentrer sur la résolution numérique d'équations intégrales et intégro-différentielles à l'aide de méthodes de collocation basées sur des polynômes orthogonaux tels que Tchebychev, Legendre et Laguerre. Cette approche consiste à transformer des problèmes différentiels complexes en systèmes algébriques faciles à résoudre, garantissant une grande précision et une convergence rapide. Les résultats démontrent l'efficacité de la méthode proposée, la rendant adaptée aux applications scientifiques et techniques, avec un potentiel d'expansion et de développement futur.

Mots-clés : Équations intégrales, Équations intégro-différentielles, Résolution numérique, Équation de Volterra, Polynômes de Laguerre, polynômes de Tchebychev.