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Mathematical analysis of contact problem involving fractional derivatives

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Dedications

To those who sowed the seeds of hope in me, and watered them with prayers, support, and love… .

To my dear father ***Noui Boubekeur*** *and my loving mother* ***Gacemi Saida*** *, thank you for your unconditional love, for your reassuring presence in every moment of fatigue and doubt. You are the light that has illuminated my path.*

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And finally… To myself, the one who fought, was patient, and persevered… Thank you, Rima, for not giving up, for moving forward despite everything. Today, you are reaping the rewards of your efforts.

Noui Rima

Dedications

Praise be to Allah for His grace and assistance in completing this research.

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To the one whom Allah has blessed with dignity and reverence....

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To the one whose name I carry with pride my dear ***father****, may Allah protect him.*

To the one whose prayers were the secret to my success... Whose tenderness is the balm for my wounds my dear mother, may Allah preserve you..

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Developed by: ***Sara***

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Table of contents

General introduction

Contact between deformable bodies is a very common phenomenon in industrial and everyday life, the contact between brake pads with wheels, wheel and road tires, pistons with skirts or complex metal forming processes are just a few examples. Significant progress has been made recently in the mathematical modeling and analysis of the various processes involved in contact between deformed bodies. Due to its inherent complexity, communication phenomena lead to new interesting mathematical models, i.e., a system of partial differential equations, associated with boundary conditions and initial conditions, which ultimately describe a specific contact process. Currently, a general mathematical theory of contact mechanics (MTCM) is emerging. It is concerned with the mathematical structures that underlie communication problems with different constitutive laws, i.e., different materials, different techniques and different communication conditions.

The emerging mathematical theory of communication mechanics is concerned with the mathematical structures underlying general contact problems with different constitutive laws, namely various materials, engineering, and different contact conditions. Where the contact problem is essentially how forces are applied to a structure and how these structures react when they experience these forces. Besides the problems mentioned above, there are other real and very important phenomena such as material damage and adhesion of bodies.

One phenomenon will be considered in this thesis: the phenomenon of contact with thermal effects. Contact and friction processes are invariably accompanied by heat production, which can be considerable. For example, sudden braking of a car can result in the dissipation of significant power in the form of heat. The thermal effect in contact processes affects the composition and rigidity of surfaces and causes thermal stresses in the contacting bodies. The way in which heat affects the mechanical properties of a surface can be partially taken into account (assuming that the coefficient of friction depends on temperature).

Mathematical models in thermodynamics require four elements: the heat production condition, the condition describing the heat exchange between the body and the foundation, the constitutive relationship, and the energy equation. These models have recently been developed in (4; 7; 9; 15; 25).

Thermomechanics focuses on the effects of heat on the stresses and deformations of mechanical bodies and vice versa. It is an extension of isothermal mechanics. This extension is due to the fact that stresses and deformations arise not only from mechanical forces but also from temperature variations.

There are general models for contact issues including friction in a viscoelastic body in (2; 6; 21). The quasistatic frictional contact between an electro-viscoelastic body and a deformable conductive base was examined using a mathematical model in (14). In (1), A. Amassad et al. examined the regularized version of the problem and demonstrated the existence and uniqueness of the weak solution while modeling a quasistatic thermoviscoelastic problem with bilateral contact and a slip rate dependent condition.

At the end of the 17th century, Gottfried Leibniz, Guillaume de l'Hôpital, and Johann Bernoulli laid the groundwork for the theory of fractional calculus (16). In the mid-19th century, following the works of Joseph Liouville and Bernhard Riemann, a number of results on this theory were presented, see (19).

The mechanical modeling of materials that resemble rubber is one use for fractional computation. In this regard, the fractional model of Maxwell and the fractional constitutive laws of Kelvin-Finger are used as references for the models that incorporate certain materials with viscoelastic features (10; 18).

In (11), the authors examined a broad quasistatic frictionless contact problem for a viscoelastic body, modeled by the fractional Kelvin-Voigt law, with the contact condition articulated by the Clarke subdifferential of a nonconvex and nonsmooth functional.

Z. Zeng et al. (30; 31) presented a category of generalized differential hemivariational inequalities that incorporate the time fractional order derivative operator in the context of a frictional contact problem.

The memory comprises three chapters and is organized as follows:

In the first chapter, we begin by defining the physical framework, the behavior laws of different materials, the boundary conditions as well as the mechanical formulation of the problem to be studied.

In the second chapter we will provide some introductions and definitions about the fractional Caputo derivative. Then, we review some results concerning functional spaces, variational equations and inequalities, Gronwall's lemma and some theorems that will be of great use for the demonstrations.

In the last chapter, we study a contact problem with friction and normal compliance in thermo-visco-elasticity with time. We present a variational formulation of the problem and prove the existence and uniqueness of a solution using fixed-point techniques and Gronwall's lemma and Caputo derivative.

General Notations

$\mathbb{N}$	Set of natural numbers,
$\mathbb{R}$	The set of real numbers,
c	Strictly positive real constant,
i.e.	That is to say,
$\partial_i\psi$	The partial derivative of ψ with respect to the i^{th} component $x : \partial_i\psi = \frac{\partial\psi}{\partial x_i}$,
$\nabla\psi$	Gradient of the map $\psi : \nabla\psi = (\partial_1\psi, \ldots, \partial_d\psi)$,
$\mathbb{S}^d$	The space of second-order symmetric tensors on $\mathbb{R}^d (d = 2, 3)$,
$\mathrm{Div}\psi$	Divergence of the map,$\psi : \mathrm{Div}\psi = (\partial_1\psi + \ldots + \partial_d\psi)$,
$(.,.)_{\mathrm{X}}$	The scalar product of X,
$\|\cdot\|_{\mathrm{X}}$	The norm of X,
a.e.	Almost everywhere,
Ω^ℓ	Open of $\mathbb{R}^d$, sometimes domain The Hertzian,
$\bar{\Omega}^\ell$	The adherence of Ω^ℓ,
Γ^ℓ	The boundary of $\Omega^\ell : \Gamma^\ell = \partial\Omega^\ell$,
Γ_i^ℓ	The boundary parts $\Gamma^\ell, (i = 1, 2, 3)$,
$mes\Gamma_i^\ell$	Dimensional $(d-1)$ Lebesgue measure of Γ_i^ℓ,
$d\Gamma_i^\ell$	Surface measure on Γ_i^ℓ,
$\boldsymbol{\nu}^\ell$	The outgoing unit normal to Γ^ℓ,
$v_\nu^\ell, \boldsymbol{v}_\tau^\ell$	The normal and tangential components of the vector field v^ℓ defined on $\bar{\Omega}^\ell$,
$\boldsymbol{\sigma}_\nu^\ell, \boldsymbol{\sigma}_\tau^\ell$	The normal and tangential components of the vector field $\boldsymbol{\sigma}$ defined on $\bar{\Omega}^\ell$,
${}_0^C D_t^\alpha f(t)$	the Caputo fractional derivative of f
${}_0 I_t^\alpha f(t)$	The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function f
$L^2(\Omega^\ell)$	Space of functions u^ℓ measurable on Ω^ℓ such that $\int_{\Omega^\ell} \|u^\ell\|^2 dx < +\infty$,
$\|\cdot\|_{L^2(\Omega^\ell)}$	The norm of $L^2(\Omega^\ell)$ defined by $\| u^\ell \|_{L^2(\Omega^\ell)} = (\int_{\Omega^\ell} \|u^\ell\|^2 dx)^{\frac{1}{2}}$,
$L^\infty(\Omega^\ell)$	Space of functions u^ℓ measurable on Ω^ℓ such that,
	$\exists c > 0 : \| u^\ell \| < c$, a.e , on Ω^ℓ,
H^ℓ	The space $L^2(\Omega^\ell)^d$,

H_1^ℓ	The space $\mathrm{H}^1(\Omega^\ell)^d$,
$\mathcal{H}^\ell$	The space $L^2(\Omega^\ell)_s^{d\times d}$,
$\mathcal{H}_1^\ell$	The space $\{\tau^\ell = (\tau_{ij}^\ell) \in \mathcal{H};\ \operatorname{div}\tau^\ell \in \mathrm{H}^\ell\}$,
$\mathrm{H}^{\frac{1}{2}}(\Gamma^\ell)$	The Sobolev space of order $\frac{1}{2}$ on Γ^ℓ,
$\mathrm{H}^{-\frac{1}{2}}(\Gamma^\ell)$	The dual space of $\mathrm{H}^{\frac{1}{2}}(\Gamma^\ell)$,
$\mathrm{H}^1(\Omega^\ell)$	The Sobolev space of order 1 on Ω^ℓ,
H_{Γ^ℓ}	Space$(\mathrm{H}^{\frac{1}{2}}(\Gamma^\ell))^d$,
$\mathrm{H}'_{\Gamma^\ell}$	The dual space of H_{Γ^ℓ}.
$\gamma : \mathrm{H}_1 \to \mathrm{H}_\Gamma$	The trace application for vector functions,

If in addition $[0,T]$ a time interval, $k \in \mathbb{N}$ and $1 \le p \le +\infty$, we note by

$C(0,T;\mathrm{H})$	The space of continuous functions from $[0,T]$ to H,
$C^1(0,T;\mathrm{H})$	The space of continuously differentiable functions on $[0,T]$ in H,
$L^p(0,T;\mathrm{H})$	The space of measurable functions on $[0,T]$ in H,
$\|\cdot\|_{L^p(0,T;\mathrm{H})}$	The norm of $L^p(0,T;\mathrm{H})$,
$\mathrm{W}^{k,p}(0,T;\mathrm{H})$	The Sobolev space of parameters k and p,
$\|\cdot\|_{\mathrm{W}^{k,p}(0,T;\mathrm{H})}$	The norm of $\mathrm{W}^{k,p}(0,T;\mathrm{H})$,
$\Gamma_1^\ell, \Gamma_2^\ell, \Gamma_3$	The parts of $\Gamma^l = \overline{\Gamma_1^\ell \cup \Gamma_2^\ell \cup \Gamma_3}$,
$\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$	The contact interface between bodies Ω^1, Ω^2,
u^ℓ	Vectors of displacements in the domain Ω^ℓ, we write u_i^ℓ the components of the vector in the canonical basis,
σ^ℓ	Stress tensor corresponding to the displacement u^ℓ, we write σ_i^ℓ The components of the tensor in the canonical basis,
σ_ν^ℓ	Normal of the stresses at the boundary of the domain:$\sigma_\nu^\ell = (\sigma^\ell\nu^\ell).\nu^\ell$,
$\boldsymbol{\sigma}_\tau^\ell$	The tangential component of the tensor field $\boldsymbol{\sigma}^\ell$,
$\varepsilon(u^\ell)$	Linearized strain tensor:$\varepsilon(u^\ell)_{ij} = \frac{1}{2}(\partial_i u_j^\ell + \partial_j u_i^\ell)$.

Chapter 1

Modeling and Preliminaries

In this chapter, we first introduce the physical framework of contact problems and briefly review the mechanics of continuum media and the behavior of deformable solids (theromo-viscoelastic laws, thermo-viscoelastic with fractional derivative). Then, we present the different contact boundary conditions with friction which will be used in the following chapters.

1.1 Physical frameworks - Mathematical models

In this section we will introduce a physical framework and their mathematical models of the mechanical problems occurring in this memory.

The **physical framework** is as follows. We consider two thermo-viscoelastic bodies which occupy the bounded domains Ω^1 and Ω^2 of $\mathbb{R}^d$ ($d = 1, 2, 3$ in applications). We put a superscript ℓ to indicate that a quantity or subset is related to the domain $\Omega^\ell, \ell = 1, 2$. For each domain Ω^ℓ, the boundary $\partial\Omega^\ell = \Gamma^\ell$ is assumed to be Lipschitz continuous and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ such that $meas(\Gamma_3^\ell) > 0$. Let $[0; T]$ time interval of interest, where $T > 0$. The bodies are submitted to the action of bodies forces of density f_0^ℓ, and a volume heat flux of density q_0^ℓ. It also submitted to mechanical and thermal constants on the boundary. Indeed, the bodies are assumed to be clamped in Γ_1^ℓ and therefore the displacement filed vanishes there. Moreover, we assume that a density of traction forces, denote by f_1^ℓ, acts on the boundary part Γ_2^ℓ. The contact is modelled by the normal compliance condition and the friction is described by Coulomb's law in a form with a zero-gap function. The behavior of the material is described using the Kelvin-Voigt constitutive law under the assumption that the process is quasistatic. In light of these presumptions, the mechanical problem we're looking at here could be described as follows.

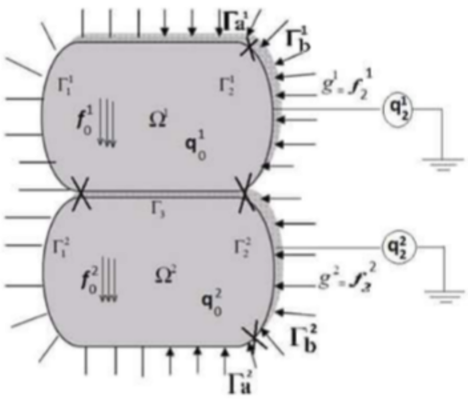

Figure 1.1: Physical framework

We denote by $\mathbb{S}^d$, $\quad (d = 2, 3)$ the space of symmetric tensors of order two on $\mathbb{R}.(,)$ and $\|.\|$ represent respectively the scalar product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$. Thus, we have

$$u.v = u_i.v_i; \qquad \|v\| = (v, v)^{\frac{1}{2}}, \quad \forall u, v \in \mathbb{R}^d$$
$$\sigma.\tau = \sigma_{ij}.\tau_{ij}; \qquad \|\tau\| = (\tau, \tau)^{\frac{1}{2}}, \quad \forall \sigma, \tau \in \mathbb{S}^d$$

with the silent index convention. For a vector V, we use the notation v to denote the trace $\gamma\nu$ of V, on γ. We denote by v_ν and v_τ the normal and tangential components of v on the boundary given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu \tag{1.1}$$

We denote by $\sigma = \sigma(x, t)$ the stress field, by $u = u(x, t)$ the displacement field and by $\varepsilon(u)$ the infinitesimal strain field. To simplify the notations, we do not explicitly indicate the dependence of the functions on $x \in \bar{\Omega}$ and $t \in [0, T]$.
For a stress field Σ we denote by σ_ν and σ_τ the normal and tangential components at the boundary given by

$$\sigma_\nu = \sigma\nu.\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu.\nu \tag{1.2}$$

Using (1.1) and (1.2), we obtain the relation

$$(\sigma\nu).\nu = \sigma_\nu v_\nu + \sigma_\tau v_\tau \tag{1.3}$$

which will be used throughout this thesis in establishing variational formulations of mechanical contact problems.
Before we get to the mathematical models that fit the existing physical framework, there are some notes and conventions that we use throughout this period.
Note that the dot above a function represents the derivation with respect to time, for example

$$\dot{u}^\ell = \frac{du^\ell}{dt} \qquad \ddot{u}^\ell = \frac{d^2u^\ell}{dt^2}$$

where $\dot{u}^\ell$ denotes the velocity field and $\ddot{u}^\ell$ denotes the acceleration field.

The unknown functions of the problem are the displacement fields $\boldsymbol{u}^\ell : \Omega^\ell \times [0, T] \to \mathbb{R}^d$ and the stress fields $\boldsymbol{\sigma}^\ell : \Omega^\ell \times [0, T] \to \mathbb{S}^d$, $\ell = 1.2$. Let us denote the mass density by $\rho^\ell : \Omega^\ell \to \mathbb{R}_+$ and the volume force density by $f_0^\ell : \Omega^\ell \times [0,T] \to \mathbb{R}^d$. The body's evolution is described by the Cauchy equation of motion:

$$\mathrm{Div}\boldsymbol{\sigma}^\ell + f_0^\ell = \rho^\ell \ddot{\boldsymbol{u}}^\ell \qquad \text{in} \quad \Omega^\ell \times [0,T]. \tag{1.4}$$

The evolution processes modeled by the previous equation are called dynamic processes. In some situations, this equation can be further simplified: for example, in the case where $\dot{\boldsymbol{u}}^\ell = 0$, it is an equilibrium problem (static processes), or in the case where the velocity field $\dot{\boldsymbol{u}}^\ell$ varies very slowly with respect to time, i.e. the term $\rho^\ell \ddot{\boldsymbol{u}}^\ell$ can be neglected (quasi-static processes). In these two cases, the equation of motion becomes:

$$\mathrm{Div}\boldsymbol{\sigma}^\ell + f_0^\ell = 0 \qquad \text{in} \quad \Omega^\ell \times [0,T]. \tag{1.5}$$

The equation is equivalent to scalar relations, and mathematically this equation is not sufficient to model the body equilibrium problem because, for example, the $\boldsymbol{u}_i^\ell$ components of the displacement field do not appear in this equation.

We now pass the boundary conditions used in chapter 3.

1.2 The thermal Phenomena

1.2.1 Thermal Conduction

Thermal conduction is a mode of temperature or heat transfer between two media in contact.

The temperature θ^ℓ is defined by a parabolic equation, which represents the conservation of energy as follows

$${}_0^C D_t^\alpha \theta^\ell - \mathcal{K}^\ell \Delta \theta^\ell = q_0^\ell, \tag{1.6}$$

where q_0^ℓ is a datum, which represents the heat source of the volume. Here and below κ_0^ℓ is a strictly positive constant. for more details, see (3)

1.3 Constitutive Laws

Constitutive laws are relationships between the stress tensor and the strain tensor and their derivatives. A whole series of tests must be devised and performed to establish a constitutive law. Physical experiments for one-dimensional materials constitute the starting point in establishing constitutive laws. Here are four classic examples of tests on solids: monotonic loading tests, charge-unloading tests, creep tests, and relaxation tests.

In the description of purely electromechanical phenomena, constitutive laws (or constitutive laws) are understood to mean a relationship between the stress tensor σ^{ℓ}, the infinitesimal strain tensor ε^{ℓ}, and their time derivatives $\dot{\sigma}^{\ell}$ and $\dot{\varepsilon}^{\ell}$.

1.3.1 Viscoelastic constitutive laws with long memory

Here we consider a category of materials where the stress tensor $\boldsymbol{\sigma}^{\ell}$ is related by the constitutive law:

$$\boldsymbol{\sigma}^{\ell} = \mathcal{A}^{\ell}\varepsilon(\boldsymbol{u}^{\ell}) + \int_0^t \mathcal{Q}^{\ell}(t-s, \varepsilon(\boldsymbol{u}^{\ell}(s)))ds \tag{1.7}$$

where $\mathcal{A}^{\ell} : \Omega^{\ell} \times \mathbb{S}^d \to \mathbb{S}^d$ is the nonlinear elasticity operator, , $\mathcal{Q}^{\ell} = \left(\mathcal{Q}^{\ell}_{ij}\right)$ is a relaxation tensor.

1.3.2 Viscoelastic behavior laws with short memory

The behavior law of an electroviscoelastic material with short memory is given by:

$$\boldsymbol{\sigma}^{\ell}(t) = \mathcal{A}^{\ell}(\varepsilon(\dot{\boldsymbol{u}}^{\ell}(t))) + \mathcal{B}^{\ell}(\varepsilon(\boldsymbol{u}^{\ell}(t))) \tag{1.8}$$

where $\boldsymbol{u}^{\ell}$, $\boldsymbol{\sigma}^{\ell}$ represent the displacement field and the stress field respectively, where the dot above denotes the derivative with respect to the time variable. Furthermore, $\boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell})$ denotes the linearized strain tensor, $\mathcal{A}^{\ell}$ and $\mathcal{B}^{\ell}$ are nonlinear operators describing the purely viscous and elastic properties of the material, respectively.

1.3.3 Viscoelastic behavior laws with time-fractional

The behavior law of an viscoelastic material with with time-fractional Kelvin-Voigt is given by:

$$\boldsymbol{\sigma}^{\ell}(t) = \mathcal{A}^{\ell}(\varepsilon({}_0^cD_t^{\alpha}u^{\ell}(t))) + \mathcal{B}^{\ell}(\varepsilon(\boldsymbol{u}^{\ell}(t))) \tag{1.9}$$

1.3.4 Thermo-viscoelastic behavior laws with time-fractional

The behavior law of an thermo-viscoelastic material with with time-fractional Kelvin-Voigt is given by:

$$\sigma^\ell = \mathcal{A}^\ell \varepsilon^\ell(u^\ell) + \mathcal{C}^\ell\Big(\varepsilon^\ell({}_0^cD_t^\alpha u^\ell)\Big) - \mathcal{M}^\ell\Big(\theta^\ell\Big) \tag{1.10}$$

for more details, see (3)

1.3.5 Boundary conditions

Let us now define the boundary conditions on each of the three parts of Γ^ℓ.

The Displacement Boundary Condition

The body is fixed in a position on the part $\Gamma_1^\ell \times [0,T]$, the displacement field $\boldsymbol{u}^\ell$ is therefore zero:

$$\boldsymbol{u}^\ell = 0 \qquad \text{on } \Gamma_1^\ell \times [0,T]. \tag{1.11}$$

Traction boundary condition.

A surface traction of density f_2^ℓ acts on $\Gamma_2^\ell \times [0,T]$ and consequently the Cauchy stress vector $\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell$ satisfies:

$$\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell = f_2^\ell \qquad \text{on } \Gamma_2^\ell \times [0,T]. \tag{1.12}$$

Continuous contact boundary conditions.

The normal displacement is defined by

$$[u_\nu] = u_\nu^1 + u_\nu^2,$$

and the tangent displacement by

$$[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2.$$

The continuity of the constraints on the interface Γ_3 is expressed by:

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu,\ \ \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau \qquad \text{on } \Gamma_3. \tag{1.13}$$

Bilateral Contact Condition

The contact is bilateral, meaning that contact is maintained during the movement and there is no separation between the two bodies.

The normal component of the displacement field is zero on the contact surface and therefore

$$[u_\nu] = 0$$

1.4 Contact Laws

1.4.1 Frictionless Contact

In perfect contact, where there is no friction, the mechanical action transmitted by an obstacle between two solids can only be normal to the contact (perpendicular to the common tangent plane of the contact) at any point. This translates into the relationship

$$\sigma_\tau = 0,$$

which means that the tangential stress is zero. If this is not the case, we say that the tangential motion occurs with friction, which requires us to introduce a friction law that takes into account the tangential component along with the other variables of the system.

1.4.2 Contact with normal compliance

In this case, the bodies are assumed to be deformable and the contact area is not known a priori. The normal stress σ_ν^ℓ satisfies the so-called normal compliance condition.

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ -\sigma_\nu = p_\nu([u_\nu] - g), \end{cases} \tag{1.14}$$

where $[u_\nu]$ is the normal displacement, g represents the gap between the bodies, and p_ν is a given positive function, called the normal compliance function. This condition indicates that the foundation exerts an action on the body as a function of its penetration $[u_\nu] - g$. Let us specify that in chapter 3 of the thesis we consider the case between two bodies, that is to say, the gap is zero, $g = 0$. As an example of the function p_ν we can consider

$$p_\nu(r) = c_\nu r_+, \tag{1.15}$$

where c_ν is a positive constant and $r_+ = \max\{0, r\}$. A second example is given by

$$p_\nu(r) = \begin{cases} c_\nu r_+ & \text{if} \quad r \leq \alpha, \\ c_\nu \alpha & \text{if} \quad r > \alpha, \end{cases} \tag{1.16}$$

where α is a positive coefficient relating to the surface hardness. In this case, the contact condition (1.14) means that when the penetration is too deep, i.e. when it exceeds α, the foundation disintegrates and no longer offers resistance to penetration.

1.4.3 Contact conditions with normal compliance and adhesion.

We will describe the contact condition with normal compliance and adhesion on $\Gamma_3 \times [0, T]$, we introduce an internal state variable defined on $\Gamma_3 \times [0, T]$, which represents the adhesion intensity on the contact surface, such that $0 \leq \beta \leq 1$. When $\beta = 1$ at a point $x \in \Gamma_3$, adhesion is complete and all bonds are active, when $\beta = 0$ all bonds are deactivated and there is no adhesion, and when $0 < \beta < 1$ it is the case of partial adhesion and measures the fraction of bonds. For more details on this section, we refer for example to (**?**).

We assume that the normal stress satisfies the normal compliance condition with adhesion:

$$\sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \beta^2 R_\nu([u_\nu]) \qquad \text{on} \quad \Gamma_3 \times [0, T], \tag{1.17}$$

where σ_ν is the normal displacement, γ_ν is a positive coefficient, $p_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ is a given function called the normal compliance function, and the function $R_\nu : \mathbb{R} \longrightarrow \mathbb{R}_+$ is the truncation operator given by:

$$R_\nu(s) = \begin{cases} \mathrm{L} & \text{if} \quad s < -\mathrm{L}, \\ -s & \text{if} \quad -\mathrm{L} \leq s \leq 0, \\ 0 & \text{if} \quad s > 0. \end{cases} \tag{1.18}$$

Here $\mathrm{L} > 0$ is the characteristic length of the bonds. Condition (1.17) indicates that each body exerts an action on the other body according to its penetration $[u]$, where the second term of the equality is the contribution of adhesion to the surface tension.

Note that the normal compliance condition with adhesion (1.17) has already been used in (8; 24).

When the adhesion field β is zero, (1.17) becomes:

$$\sigma_\nu = -p_\nu([u_\nu]) \qquad \text{on} \quad \Gamma_3 \times (0, T), \tag{1.19}$$

which represents the normal compliance condition.

1.4.4 Condition in the tangent plane

At points of Γ_3, the tangential stiffness generated by the glue is assumed to depend on adhesion and tangential displacement,

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta)\mathbf{R}_\tau([\boldsymbol{u}_\tau])) \qquad \text{on} \quad \Gamma_3 \times (0, T), \tag{1.20}$$

where $\mathbf{R}_\tau$ is a truncation operator defined by the relation

$$\mathbf{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if} \quad \|\boldsymbol{v}\| \leq \mathrm{L}, \\ \mathrm{L}\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} & \text{if} \quad \|\boldsymbol{v}\| > \mathrm{L}. \end{cases} \tag{1.21}$$

Then, $p_\tau(\beta)$ acts as a spring constant, which increases with β; the traction is in the opposite direction to the displacement. The maximum modulus of the tangential traction in (1.20) is $p_\tau(1)\mathrm{L} > 0$. The tangential frictional traction is assumed to be much smaller than the adhesion and is therefore omitted.

1.4.5 Conditions of bilateral contact with friction and wear.

Now let p_i, for $i = \nu, \tau$ be the normal compliance functions satisfying $p_i(r) = 0$ if $r \leq 0$ and other assumptions given in the following and note that wear appears under normal compliance conditions as follows:

$$\begin{cases} \sigma_\nu = -p_\nu([u_\nu] - \omega - g) \\ \boldsymbol{\sigma}_\tau = -p_\tau([u_\nu] - \omega - g)\dfrac{\boldsymbol{v}^*}{\| \boldsymbol{v}^* \|} \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \tag{1.22}$$

where g represents the initial gap between the two bodies.

Friction follows the laws given by:

$$\begin{cases} |\boldsymbol{\sigma}_\tau| = p_\tau([u_\nu] - \omega - g) \\ \boldsymbol{\sigma}_\tau = -\lambda \boldsymbol{v}^*, \quad \lambda \geq 0 \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{1.23}$$

These relationships impose constraints on the evolution of the shear stress; in particular, the shear stress is in the opposite direction to the relative sliding velocity $\boldsymbol{v}^*$. An example of the normal compliance function p_ν is

$$p_\nu(r) = c_\nu r_+, \tag{1.24}$$

or, more generally,

$$p_\nu(r) = c_\nu (r_+)^m, \tag{1.25}$$

Here c_ν is a positive constant, $r_+ = \max\{0, r\}$ and $m > 0$. The normal compliance contact condition was proposed in (27), in the special form of (1.25). Subsequently, it was used in a large number of articles, see e.g. (20) and the references therein. A related form of normal compliance functions is given by

$$p_\tau = \mu p_\nu \tag{1.26}$$

where $\mu \geq$ is the coefficient of friction, and

$$p_\tau = \mu p_\nu (1 - \delta p_\nu)_+ \tag{1.27}$$

where δ is a small positive material constant related to surface wear and hardness and $\mu \geq 0$ is the coefficient of friction. This related form of compliance contact condition has been used in (25; 26; 22).

Chapter 2

Mathematical Tools

In this chapter, we provide some analytical background, such as an overview of the fundamental properties of Banach and Hilbert spaces, as well as Sobolev-type spaces. We also introduce some functional spaces for mechanics that are needed in the study of the contact problems we address in the rest of the manuscript. We also provide a brief overview of strongly monotone operators. We provide some examples and recall an existence and uniqueness results for variational equalities and inequalities with operator type.

2.1 Preliminaries

In this section, we recall some known definitions and properties on nonlinear analysis and fractional calculus, which can be found in (29; 30; 31).

Definition 2.1.1 *(The Riemann–Liouville fractional integral). Let X be a Banach space and $(0,T)$ be a finite time interval. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a given function $f \in L^1(0,T;X)$ is defined by*

$$
{}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds \quad \text{for all } t \in (0,T),
$$

where $\Gamma(\cdot)$ stands for the Gamma function defined by

$$
\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt.
$$

To complement the definition, we set ${}_0I_t^0 = I$, where I is the identity operator, which means that ${}_0I_t^0 f(t) = f(t)$ for a.e $t \in (0,T)$.

Definition 2.1.2 *(The Caputo derivative of order, $0 < \alpha \leq 1$). Let X be a Banach space, $0 < \alpha \leq 1$, and $(0,T)$ be a finite time interval. For a given function $f \in AC(0,T;W)$, the Caputo fractional derivative of f is defined by:*

$$ {}_0^C D_t^\alpha f(t) = 0I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds \quad \text{for all } t \in (0,T). $$

The notation $AC(0,T;X)$ refers to the space of all absolutely continuous functions from $(0,T)$ into X.

It is obvious that if $\alpha = 1$, then the Caputo derivative reduces to the classical first-order derivative, that is, we have:

$$ {}_0^C D_t^1 f(t) = I f'(t) = f'(t) \quad \text{for a.e } t \in (0,T). $$

Proposition 2.1.1 *Let X be a Banach space and $\alpha, \beta > 0$. Then the following statements hold:*

(i) For $y \in L^1(0,T;X)$, we have

$$ 0I_t^\alpha 0I_t^\beta y(t) = 0I_t^{\alpha+\beta} y(t) \quad \text{for } t \in (0,T); $$

(ii) For $y \in AC(0,T;X)$ and $\alpha \in (0,\alpha]$, we have

$$ 0I_t^\alpha {}_0^C D_t^\alpha y(t) = y(t) - y(0) \quad \text{for a.e } t \in (0,T); $$

(iii) For $y \in L^1(0,T;X)$, we have

$$ {}_0^C D_t^\alpha 0I_t^\alpha y(t) = y(t) \quad \text{for a.e } t \in (0,T). $$

Definition 2.1.3 *(The Clarke generalized directional derivative and the generalized gradient). Let $J : X \to \mathbb{R}$ be a locally Lipschitz function. We denote by $J^0(u,v)$ the Clarke generalized directional derivative of J at the point $x \in X$ in the direction $y \in X$ defined by*

$$ J^0(x,y) = \limsup_{\substack{\lambda \to 0^+ \\ z \to x}} \frac{J(z+\lambda y) - J(z)}{\lambda}. $$

The generalized gradient of $J : X \to \mathbb{R}$ at $x \in X$ is defined by

$$ \partial J(x) = \left\{ \xi \in X^* \mid \forall y \in X, J^0(x,y) \geq \langle \xi, y \rangle_{X^*,X} \right\}. $$

2.2 Function Spaces

Let us introduce the following Hilbert spaces, associated with the mechanical unknowns $\boldsymbol{u}^\ell$ and $\boldsymbol{\sigma}^\ell$:

$$\begin{cases} H^\ell = \Big\{\boldsymbol{u}^\ell = (u_i^\ell) \quad | \quad u_i^\ell \in \mathrm{L}^2(\Omega^\ell)\Big\} = (\mathrm{L}^2(\Omega^\ell))^d, \\ \mathcal{H}^\ell = \Big\{\boldsymbol{\sigma}^\ell = (\sigma_{ij}^\ell) \quad | \quad \sigma_{ij}^\ell = \sigma_{ji}^\ell \in \mathrm{L}^2(\Omega^\ell)\Big\} = (\mathrm{L}_s^2(\Omega^\ell))^{d\times d}, \\ H_1^\ell = \Big\{\boldsymbol{u}^\ell = (u_i^\ell) \quad | \quad u_i^\ell \in H^1(\Omega^\ell)\Big\} = (H^1(\Omega^\ell))^d, \\ \mathcal{H}_1^\ell = \Big\{\boldsymbol{\sigma}^\ell \in \mathcal{H}^\ell \quad | \quad \operatorname{div} \boldsymbol{\sigma}^\ell \in H^\ell\Big\} \\ Q^\ell = L^2(\Omega^\ell). \end{cases} \tag{2.1}$$

The spaces $H^\ell, \mathcal{H}^\ell, H_1^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces equipped with the following scalar products:

$$\begin{cases} (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \displaystyle\int_{\Omega^\ell} u_i^\ell v_i^\ell dx, \\ (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \displaystyle\int_{\Omega^\ell} \sigma_{ij}^\ell \tau_{ij}^\ell dx, \\ (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} + (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \\ (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} + (\mathrm{Div}\boldsymbol{\sigma}^\ell, \mathrm{Div}\boldsymbol{\tau}^\ell)_{H^\ell}, \end{cases} \tag{2.2}$$

respectively, where $\varepsilon : H_1^\ell \to \mathcal{H}^\ell$ and $\mathrm{Div} : \mathcal{H}_1^\ell \to H^\ell$ are respectively the deformation and divergence operators, defined by

$$\varepsilon(\boldsymbol{u}^\ell) = (\varepsilon_{ij}(\boldsymbol{u}^\ell)),\ \varepsilon_{ij}(\boldsymbol{u}^\ell) = \frac{1}{2}(u_{i,j}^\ell + u_{j,i}^\ell),\ \mathrm{Div}\boldsymbol{\sigma}^\ell = (\sigma_{ij,j}^\ell).$$

The norms on the spaces $H^\ell, \mathcal{H}^\ell, H_1^\ell$ and $\mathcal{H}_1^\ell$ are denoted by $\|\cdot\|_{H^\ell}, \|\cdot\|_{\mathcal{H}^\ell}, \|\cdot\|_{H_1^\ell}$ and $\|\cdot\|_{\mathcal{H}_1^\ell}$, respectively.

Since the boundary Γ^ℓ is Lipschitzian, the exterior normal vector $\boldsymbol{\nu}^\ell$ to the boundary is defined p.p. For any vector field $\boldsymbol{v}^\ell \in H_1^\ell$ we use the notation $\boldsymbol{v}^\ell$ to denote the trace $\gamma\boldsymbol{v}^\ell$ of $\boldsymbol{v}^\ell$ on Γ^ℓ. Recall that the trace map $\gamma : H_1^\ell \to \mathrm{L}^2(\Gamma^\ell)^d$ is linear and continuous, but is not surjective.

Let H'_{Γ^ℓ} be the dual of H_{Γ^ℓ}, and $(\cdot,\cdot)$ be the duality product between H'_{Γ^ℓ} and H_{Γ^ℓ}. For any $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$, there exists an element $\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell \in H'_{\Gamma^\ell}$ such that:

$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell, \gamma\boldsymbol{v}^\ell) = (\boldsymbol{\sigma}^\ell, \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\mathrm{Div}\boldsymbol{\sigma}^\ell, \boldsymbol{v}^\ell)_{H^\ell} \ \ \forall \boldsymbol{v}^\ell \in H_1^\ell. \tag{2.3}$$

Furthermore, if $\boldsymbol{\sigma}^\ell$ is regular enough (e.x. C^1), we have the formula

$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell, \gamma\boldsymbol{v}^\ell) = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell \cdot \boldsymbol{v}^\ell da \ \ \forall \boldsymbol{v}^\ell \in H_1^\ell. \tag{2.4}$$

So, for $\boldsymbol{\sigma}^\ell$ fairly regular we have the following Green's formula:

$$(\boldsymbol{\sigma}^\ell, \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\mathrm{Div}\boldsymbol{\sigma}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \cdot \boldsymbol{v}^\ell da \ \ \forall \boldsymbol{v}^\ell \in H_1^\ell, \tag{2.5}$$

where da is an element of surface measurement.

$$(\Delta\theta^\ell(t), \delta^\ell)_{L^2(\Omega^\ell)} + (\nabla\theta^\ell(t) \cdot \nabla\delta^\ell)_{L^2(\Omega^\ell)} = \kappa_0^\ell \int_{\Gamma^\ell} \frac{\partial^\ell\theta^\ell(t)}{\partial\nu^\ell} \ \ \delta^\ell dx, \quad \forall \delta^\ell \in \mathbb{L}_1 \tag{2.6}$$

Let K_{ad} the set of admissible displacement

$$K_{ad} = \big\{ v = (v^1, v^2) \in V : [v_\nu] \leq 0 \text{ on } \Gamma_3 \big\}.$$

Since $meas\Gamma_1^\ell > 0$, the following Korn's inequality

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell} \geq c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell, \tag{2.7}$$

holds, where the constant c_K denotes a positive constant which may depends only on Ω^ℓ, Γ_1^ℓ .

Moreover, by the Sobolev trace theorem, there exists a constant c_d, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3, such that

$$\|v^\ell\|_{L^2(\Gamma_3)^d} \leq c_d \|v^\ell\|_{V^\ell} \quad \forall v^\ell \in V^\ell. \tag{2.8}$$

In order to simplify the notations, we define the product spaces

$$\begin{gathered} V = V^1 \times V^2, \quad H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \quad Q = Q^1 \times Q^2 \end{gathered} \tag{2.9}$$

Let $0 < T < \infty$ and let $(X, \|\cdot\|_X)$ be a real Banach space, we use the classical notation for spaces $L^p(0,T;X), \mathrm{W}^{k,p}(0,T;X)$, where ù $1 \leq p \leq \infty$, $k \geq 1$. We denote by $C(0,T;\mathrm{X})$ and $C^1(0,T;\mathrm{X})$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with value on X, respectively, with the norms:

$$\| f \|_{C(0,T;\mathrm{X})} = \max_{t\in[0,T]} \| f(t) \|_{\mathrm{X}},$$

$$\| f \|_{C^1(0,T;\mathrm{X})} = \max_{t\in[0,T]} \| f(t) \|_{\mathrm{X}} + \max_{t\in[0,T]} \| \dot{f}(t) \|_{\mathrm{X}} .$$

We denote by $C_c(0,T;\mathrm{X})$ the set of continuous functions with compact support in $[0,T]$ with values in X .

Definition 2.2.1 *A function $f : [0,T] \to \mathrm{X}$ is said to be measurable if there exists a subset $\mathrm{E} \subset [0,T]$ of measure zero and a sequence $(f_n)_{n\in\mathbb{N}}$ of functions belonging to $C_c(0,T;\mathrm{X})$ such that $\| f_n(t) - f(t) \|_{\mathrm{X}} \longrightarrow 0$ when $n \longrightarrow \infty$, for all $t \in [0,T] \setminus \mathrm{E}$.*

2.3 Reminders on Hilbert spaces

Let H be a real vector space and $(.,.)_H$ be a scalar product on H, that is, $(.,.)_H : H \times H \to \mathbb{R}$ is a symmetric and positive definite bilinear map.

We denote by $\| \cdot \|_H$ the application of $H \to \mathbb{R}_+$ defined by:

$$\| u \|_H = (u,u)_H^{\frac{1}{2}}, \tag{2.10}$$

and we recall that $\| \cdot \|_H$ is a norm on H that satisfies the Cauchy-Schwartz inequality:

$$|(u,v)_H| \leq \| u \|_H \| v \|_H, \ \ \forall u, v \in H. \tag{2.11}$$

We say that H is a Hilbert space if H is complete for the norm defined by (2.10). Let $H^{'}$ be the dual space of H, that is, the space of linear and continuous functionals on H equipped with the norm:

$$\| \eta \|_{H'} = \sup_{v \in H - \{0\}} \frac{< \eta, v >_{H' \times H}}{\| v \|_H},$$

where $\langle .,. \rangle_{H' \times H}$ represents the duality between $H^{'}$ and H.

Theorem 2.3.1 ***(Riesz-Fréchet representation theorem)****: Let H be a Hilbert space and let $H^{'}$ be its dual space. Then, for all $\phi \in H^{'}$ there exists unique $f \in H$ such that*

$$\langle \phi, v \rangle_{H' \times H} = (f, v)_H \quad \forall v \in H.$$

Additionally

$$\| \phi \|_{H'} = \| f \|_H .$$

The importance of this theorem is that any continuous linear form on H can be represented using the scalar product. The map $\phi \mapsto f$ is an isometric isomorphism that allows us to identify H and $H^{'}$.

2.4 Sobolev Spaces

Sobolev spaces were introduced at the beginning of the century and made it possible to solve many problems concerning partial differential equations that had previously been unanswered.

We begin with a brief review of some results on the Sobolev space $H^1(\Omega)$ defined by:

$$H^1(\Omega) = \Big\{ u \in \mathrm{L}^2(\Omega) \quad | \quad \partial_i u \in \mathrm{L}^2(\Omega) \, i = 1, \cdots, d \Big\}.$$

We denote by ∇u the component vector $\partial_i u$. We have $\nabla u \in \mathrm{L}^2(\Omega)^d$ for all $u \in H^1(\Omega)$.

We know that $H^1(\Omega)$ is a Hilbert space for the scalar product:

$$(u,v)_{H^1(\Omega)} = (u,v)_{\mathrm{L}^2(\Omega)} + (\partial_i u, \partial_i v)_{\mathrm{L}^2(\Omega)},$$

and the associated standard:

$$\| u \|_{H^1(\Omega)} = (u,u)^{\frac{1}{2}}_{H^1(\Omega)}, \text{and we write } \| u \|^2_{H^1(\Omega)} = \| u \|^2_{\mathrm{L}^2(\Omega)} + \| \nabla u \|^2_{\mathrm{L}^2(\Omega)^d} .$$

We have the following results:

$$C^1(\bar{\Omega}) \text{ is dense in } H^1(\Omega).$$

Theorem 2.4.1 ***(Rellich)***

$$H^1(\Omega) \subset \mathrm{L}^2(\Omega) \textit{ with compact injection.}$$

Theorem 2.4.2 ***(Sobolev trace)***

There exists a linear and continuous map $\delta : H^1(\Omega) \to \mathrm{L}^2(\Gamma)$ such that $\delta u = u|_\Gamma$ for all $u \in C^1(\bar{\Omega})$.

The space $\mathrm{L}^2(\Gamma)$ above represents the space of real functions on Γ which are L^2 for the superficial measure $\mathrm{d}\Gamma$. The map δ is called a trace map, it is defined as the density extension of the map $u \to u|_\Gamma$ defined for $u \in C^1(\bar{\Omega})$. Note that the trace map $\delta : H^1(\Omega) \to \mathrm{L}^2(\Gamma)$ is a compact operator.

Definition 2.4.1 *For all $k \in \mathbb{N}$ and for all $p \in [1, +\infty]$, we define the Sobolev space $\mathrm{W}^{k,p}(\Omega)$ by*

$$\mathrm{W}^{k,p}(\Omega) = \Big\{ u \in \mathrm{L}^p(\Omega) \;\; \forall \alpha,\, |\alpha| \le k;\; \exists v_\alpha \in \mathrm{L}^p(\Omega),\; \textit{such that}\, v_\alpha = D^\alpha u \Big\}.$$

We will very often misuse the spelling which consists of identifying $D^\alpha u$ and v_α.

The norm on the space $\mathrm{W}^{k,p}(\Omega)$ is given by

$$\| u \|_{\mathrm{W}^{k,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\alpha| \le k} \| D^\alpha u \|_{\mathrm{L}^p(\Omega)} \right)^{\frac{1}{p}} & \text{si} \quad 1 \le p < \infty, \\ \displaystyle\max_{|\alpha| \le k} \| D^\alpha u \|_{\mathrm{L}^\infty(\Omega)} & \text{si} \quad p = \infty. \end{cases}$$

For $p = 2$, we denote by $H^k(\Omega)$ the space $\mathrm{W}^{k,2}(\Omega)$ and the previous norm is derived from a scalar product.

Theorem 2.4.3 *The Sobolev spaces* $\mathrm{W}^{k,p}(\Omega)$*, for* $k \in \mathbb{N}$ *and* $p \in [1, +\infty]$*, equipped with the norm* $\| \cdot \|$*, are Banach spaces. Moreover, the spaces* $H^k(\Omega)$*, for any integer* k*, are Hilbert spaces.*

For further details on Sobolev spaces we refer to (5).

2.5 Reminders of nonlinear analysis in Hilbert spaces

In this section, we review some elements of nonlinear analysis in Hilbert spaces and some results concerning variational evolution equations and inequalities that are used in the study of mechanical problems.

2.5.1 I-Strongly monotone operator

We begin here with a brief review of monotone and Lipschitz friction operators. To do this, we consider a Hilbert space

Definition 2.5.1 *Let* $\mathrm{A} : \mathrm{X} \to \mathrm{X}$ *be a nonlinear operator, the operator* A *is said:*

1. *monotone if*
$$(Au - Av, u - v)_{\mathrm{X}} \geq 0, \forall u, v \in \mathrm{X},$$
2. *strongly monotonic if there exists* $m > 0$ *such that*
$$(Au - Av, u - v)_{\mathrm{X}} \geq m||u - v||^2_{\mathrm{X}}, \forall u, v \in \mathrm{X},$$
3. *Lipschitz if there exists* $L > 0$ *such that*
$$||Au - Av||_{\mathrm{X}} \leq L||u - v||_{\mathrm{X}}, \forall u, v \in \mathrm{X},$$
4. *hemicontinuous if*
$$\forall u, v \in \mathrm{X},\ \textit{theapplication}\ t \to A(u + tv) : \mathbb{R} \to \mathrm{X}'\ \textit{is continuous.}$$

Theorem 2.5.1 ***(Banach Fixed Point Theorem)***

Let K *be a closed and nonempty subset of the Banach space* $(\mathrm{X}, \| \cdot \|_{\mathrm{X}})$*. Suppose that* $\mathbf{\Pi} : \mathrm{K} \to \mathrm{K}$ *is a contraction, i.e. there exists* $c \in]0, 1[$ *such that*

$$\| \mathbf{\Pi}(u) - \mathbf{\Pi}(v) \|_{\mathrm{X}} \leq c \| u - v \|_{\mathrm{X}} \quad \forall u, v \in \mathrm{K}.$$

Then, there exists a unique element $u \in \mathrm{K}$ *such that* $\mathbf{\Pi}(u) = u$*, i.e., has a unique fixed point in* K*.*

For the operator $\mathbf{\Pi}^m : \mathrm{K} \to \mathrm{K}$ defined by the relation

$$\mathbf{\Pi}^m = \mathbf{\Pi}(\mathbf{\Pi}^{m-1}) \qquad m \geq 2,$$

we have the following version of the fixed point theorem.

Theorem 2.5.2 *Let* K *be a closed and nonempty subset of the Banach space* $(\mathrm{X}, \| \cdot \|_{\mathrm{X}})$. *Suppose that* $\mathbf{\Pi}^m : \mathrm{K} \to \mathrm{K}$ *is a contraction for* m *a positive integer. Then* $\mathbf{\Pi}$ *has a unique fixed point in* K.

Definition 2.5.2 *A bilinear form* $a : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbb{R}$ *is continuous if there exists a real* $M > 0$ *such that:*

$$|a(u, v)| \leq M \parallel u \parallel_{\mathrm{X}} \parallel v \parallel_{\mathrm{X}}, \quad \forall u, v \in \mathrm{X}.$$

Definition 2.5.3 *A bilinear form* $a : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbb{R}$ *is said to be coercive if there exists a constant* $m > 0$ *such that:*

$$a(u, u) \geq m \parallel u \parallel_{\mathrm{X}}^2, \quad \forall u \in \mathrm{X}.$$

Theorem 2.5.3 ***(Lax-Milgram Theorem)***

Let H *be a Hilbert space,* $a : \mathrm{H} \times \mathrm{H} \longrightarrow \mathbb{R}$ *a continuous and coercive bilinear form.*

Let $l : \mathrm{H} \longrightarrow \mathbb{R}$ *be a continuous linear form. Then, there exists a unique solution* $u \in \mathrm{H}$ *that satisfies:*

$$a(u, v) = l(v), \quad \forall v \in \mathrm{H}. \tag{2.12}$$

Moreover, if $a(.,.)$ *is symmetric, then* u *is characterized by the property:*

$$\frac{1}{2} a(u, u) - \langle u, u \rangle_{\mathrm{X}} \leq \frac{1}{2} a(v, v) - \langle v, v \rangle_{\mathrm{X}}, \quad \forall v \in \mathrm{X}. \tag{2.13}$$

II- Under differentiability

We consider throughout this paragraph that X is a Hilbert space and K a subset of the space X.

Definition 2.5.4 *The indicator function of* K *is the function* Ψ_{K} *defined by*

$$\Psi_{\mathrm{K}} = \begin{cases} 0 & if \quad u \in \mathrm{K}, \\ +\infty & if \quad u \notin \mathrm{K}. \end{cases}$$

Definition 2.5.5 *Let j be a function: $X \longrightarrow \mathbb{R}$ and u be an element of the space* X *such that $j(u) \neq \pm\infty$. The subdifferential of the function j at u, denoted $\partial j(u)$, is the set defined by*

$$\partial j(u) = \{u' \in \mathrm{X}' \mid \ j(v) \geq j(u) + (u', v - u), \ \forall v \in \mathrm{K}\}. \tag{2.14}$$

The bracket $(\cdot, \cdot)$ denotes the duality between X′ *and* X.

Any element u' of the set $\partial j(u)$ is called a subgradient of the function j at u. The function j is said to be subdifferentiable at u if $\partial j(u) \neq \emptyset$. It is said to be subdifferentiable if it is at every point u of the space X.

We can characterize the subdifferential $\partial\Psi_K$ of an indicator function Ψ_K of a non-empty convex set.

$$\partial\Psi_K = \{u' \in \mathrm{X}' \mid \ (u', v - u) \leq 0, \ \forall v \in \mathrm{K}\}. \tag{2.15}$$

2.6 Gronwall's Lemma

Here we recall the classic Gronwall-type lemmas that appear in many contact problems, in particular to establish the uniqueness of the solution.

Lemma 2.6.1 *Let $m, n \in C([0,T];\mathbb{R})$ be such that $m(t) \geq 0$ and $n(t) \geq 0$ for all $t \in [0,T]$, $a \geq 0$ a constant and $\psi \in C([0,T];\mathbb{R})$.*

(1) If

$$\psi(t) \leq a + \int_0^t m(s)ds + \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T],$$

Then

$$\psi(t) \leq \left(a + \int_0^t m(s)ds\right) exp\left(\int_0^t n(s)ds\right) \quad \forall t \in [0,T].$$

(2) If

$$\psi(t) \leq m(t) + a\int_0^t \psi(s)ds \quad \forall t \in [0,T],$$

Then

$$\int_0^t \psi(s)ds \leq e^{aT}\int_0^t m(s)ds \quad \forall t \in [0,T].$$

for the special case $m = 0$, part (1) of this lemma becomes.

Corollary 2.6.1 *Let $n \in C([0,T];\mathbb{R})$ be such that $n(t) \geq 0$ for all $t \in [0,T]$ and let $a \geq 0$ be. If $\psi \in C([0,T];\mathbb{R})$ is a function such that*

$$\psi(t) \leq a + \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T],$$

Then

$$\psi(t) \leq a\,exp\left(\int_0^t n(s)ds\right), \quad \forall t \in [0,T].$$

Corollary 2.6.1 is often used to show the uniqueness of the solution, as follows. Assuming that there are two solutions, denoting by ψ the norm of the difference between these solutions, we then try to upper bound ψ in the form

$$\psi(t) \leq \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T],$$

with some function $n \geq 0$. Applying the corollary immediately gives the nullity of ψ.

Lemma 2.6.2 *Let $m, n \in C([0,T];\mathbb{R})$ be such that $m(t) \geq 0$ and $n(t) \geq 0$ $\forall t \in [0,T]$, $a \geq 0$. Let also $\phi : [0,T] \to \mathbb{R}$ be a function such that:*

$$\frac{1}{2}\phi^2(s) \leq \frac{1}{2}a^2 + \int_0^s m(t)\phi(t)dt + \int_0^s n(t)\phi^2(t)dt, \quad \forall s \in [0,T].$$

Then

$$|\phi(s)| \leq \left(a + \int_0^s m(t)ds\right) e^{\int_0^s n(t)dt}, \quad \forall s \in [0,T].$$

In the special case $n = 0$, Lemma 2.6.2 becomes:

Corollary 2.6.2 *Let $m \in C([0,T];\mathbb{R})$ such that $m(t) \geq 0$ $\forall t \in [0,T]$ and let $a \geq 0$. Also let $\phi : [0,T] \to \mathbb{R}$ be a function such that:*

$$\frac{1}{2}\phi^2(s) \leq \frac{1}{2}a^2 + \int_0^s m(t)\phi(t)dt, \quad \forall s \in [0,T].$$

Then

$$|\phi(s)| \leq a + \int_0^s m(t)dt, \quad \forall s \in [0,T].$$

Chapter 3

Quasistatic contact problem with time fractional

In this chapter, we consider a mathematical model in a quasistatic process of a contact problem with normal compliance and friction between two thermo-viscoelastic bodies, For this problem, the contact is modeled by friction. Using Green's formula, we propose a variational formulation of the thermo-mechanical problem and then present an existence and uniqueness result of the solution. The proofs are based on the monotone operator argument, the Caputo derivative, the Clarke subdifferential, the Galerkin technique, and the Banach fixed point.

3.1 Position of the thermo-mechanical problem

Considering the previous physical framework, the material behavior is described using the Kelvin-Voigt constitutive law, assuming that the process is quasi-static. Given these assumptions, the mechanical problem studied here could be described as follows.

Problem $\mathcal{P}_1$

For $\ell = 1, 2$, Find a displacement field $u^\ell = (u_i^\ell) : \Omega^\ell \times]0, T[\longrightarrow \mathbb{R}^d$, a stress field $\sigma^\ell = (\sigma_{ij}^\ell) : \Omega \times]0, T[\longrightarrow \mathbb{S}^d$ and a temperature field $\theta^\ell = (\theta_i^\ell) : \Omega^\ell \times]0, T[\longrightarrow \mathbb{R}$ such that

$$\sigma^\ell = \mathcal{A}^\ell \varepsilon^\ell(u^\ell) + \mathcal{C}^\ell\Big(\varepsilon^\ell({}_0^cD_t^\alpha u^\ell)\Big) - \mathcal{M}^\ell\Big(\theta^\ell\Big) \quad \text{in } \Omega^\ell \times (0, T), \tag{3.1}$$

$$\operatorname{Div} \sigma^\ell + f_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0, T), \tag{3.2}$$

$${}_0^CD_t^\alpha \theta^\ell - \mathcal{K}^\ell \Delta \theta^\ell = q_0^\ell \quad \text{in } \Omega^\ell \times (0, T). \tag{3.3}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0, T), \tag{3.4}$$

$$\sigma^\ell \nu^\ell = f_1^\ell \quad \text{on } \Gamma_2^\ell \times (0, T), \tag{3.5}$$

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu \quad \text{where} - \sigma_\nu(u) = p_\nu([u_\nu]) \quad \text{on } \Gamma_3 \times (0, T), \tag{3.6}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = \boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau \\ \|\boldsymbol{\sigma}_\tau\| \le \boldsymbol{p_\tau}([u_\nu]), \\ \|\boldsymbol{\sigma}_\tau\| < \boldsymbol{p}_\tau([u_\nu]) \Rightarrow [\boldsymbol{u}_\tau] = 0 \qquad \text{on } \Gamma_3 \times (0, T), \\ \|\boldsymbol{\sigma}_\tau\| = \boldsymbol{p}_\tau([u_\nu]) \Rightarrow \exists \lambda \ne 0 \\ \text{such that } \boldsymbol{\sigma}_\tau = -\lambda[\boldsymbol{u}_\tau] \end{cases} \tag{3.7}$$

$$\mathcal{K}^\ell \frac{\partial \theta^\ell}{\partial \nu^\ell} + \lambda_0^\ell \theta^\ell = 0 \quad \text{on } \Gamma^\ell \times (0, T), \tag{3.8}$$

$$u^\ell(0, x) = u_0^\ell, \ \theta^\ell(0, x) = \theta_0^\ell \quad \text{in } \Omega^\ell. \tag{3.9}$$

Here (3.1) represents the time-fractional Kelvin-Voigt thermo-viscoelastic constitutive law of Caputo type, see(28). The equilibrium stress and the Fourier law of heat conduction with fractional time are represented by the equations (3.2) and (3.3), respectively. where the displacement and traction boundary conditions are (3.4) and (3.5), respectively. Equation (3.6) represents the normal compliance contact condition,p_ν is a given positive function which will be described below and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction, in this condition the interpenetrability between two bodies , that is $[u_\nu]$ can be positive on Γ_3. The Coulomb law of friction is considered in (3.7), where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. The relation (3.8) represent a Fourier boundary condition for the temperature on Γ^ℓ. Finally, (3.9) represents the initial condition.

To obtain a variational formulation of the mechanical problem $\mathcal{P}_1$, we need to introduce some assumptions about the data

- We define the following operators:

$$a : V \times V \to \mathbb{R}, \qquad a(u,v) := \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(u^\ell), \varepsilon(v^\ell))_{\mathcal{H}^\ell},$$

$$c : V \times V \to \mathbb{R}, \qquad c(u,v) := \sum_{\ell=1}^{2} (\mathcal{C}^\ell \varepsilon(u^\ell), \varepsilon(v^\ell))_{\mathcal{H}^\ell},$$

$$m : Q \times V \to \mathbb{R} \qquad m(\theta,v) := \sum_{\ell=1}^{2} (\mathcal{M}^\ell \theta^\ell, \varepsilon(v^\ell))_{Q^\ell},$$

- The mappings $j_{nc} : V \times V \to \mathbb{R}$, $j_{fr} : V \times V \to \mathbb{R}$, $d : Q \times Q \to \mathbb{R}$ and $\chi : Q \times Q \to \mathbb{R}$ are defined by

$$j_{nc}(u(t),v) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] da, \tag{3.10}$$

$$j_{fr}(u(t),v) = \int_{\Gamma_3} p_\tau([u_\nu]) \|[\boldsymbol{v}_\tau]\| da, \tag{3.11}$$

$$d(\theta(t),\eta) = \sum_{\ell=1}^{2} \mathcal{K}^\ell \int_{\Omega^\ell} \nabla\theta^\ell \nabla\eta^\ell dx, \tag{3.12}$$

$$\chi(\theta(t),\eta) = \sum_{\ell=1}^{2} \lambda_0^\ell \int_{\Gamma^\ell} \theta^\ell \eta^\ell da. \tag{3.13}$$

for all $v \in V$ and η in Q.

(H1) a) The elasticity operator $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$, the viscosity tensor $\mathcal{C}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ and the thermal conductivity tensor $\mathcal{K}^\ell = (k_{ij}^\ell) : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the usual properties of symmetry, boundedness, and ellipticity,

$$e_{ijkl}^\ell = e_{jikl}^\ell = e_{lkij}^\ell \in L^\infty(\Omega^\ell), \quad c_{ijkl}^\ell = c_{jikl}^\ell = c_{lkij}^\ell \in L^\infty(\Omega^\ell), \quad k_{ij}^\ell = k_{ji}^\ell \in L^\infty(\Omega^\ell),$$

b) The ellipticity property is satisfied by the operators $c, a,$ and d; that is, there exist positive constants $m_c, m_a,$ and m_d such that

$$c(v,v) \geq m_c \|v\|_V^2, \quad a(v,v) \geq m_c \|v\|_V^2, \quad \text{and} \qquad d(\eta,\eta) \geq m_d \|\eta\|_Q^2$$

(H2) From (H1) we have

$$|a(u,v)| \leq M_a \|u\|_V \|v\|_V, \quad |c(u,v)| \leq M_c \|u\|_V \|v\|_V,$$
$$|d(\theta,\eta)| \leq M_d \|\theta\|_Q \|\eta\|_Q, \quad |m(\theta,v)| \leq M_m \|\theta\|_Q \|v\|_V.$$
$$\text{where} \quad M_a, M_c, M_d, M_m > 0$$

(H3) The forces, the traction and the heat flux satisfy

$$f_0^\ell \in C(0,T;L^2(\Omega^\ell), \qquad f_1^\ell \in C(0,T;L^2(\Gamma_2^\ell)), \qquad \text{and} \quad q_0^\ell \in C(0,T;L^2(\Omega^\ell)$$

(H4) The coefficient of heat exchange $\lambda_0^\ell : \Gamma^\ell \times \mathbb{R} \to \mathbb{R}^+$ satisfies the conditions:
a) there exists $M_{\lambda_0} > 0$ such that $|\lambda_0(x,u)| < M_{\lambda_0}$ for all $u \in \mathbb{R}$ and $x \in \Gamma_3$, such that $x \to \lambda_0(x,u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$ and vanishes for all $u \leq 0$ anda.e $x \in \Gamma^\ell$;
b) there exists $L_{\lambda_0} > 0$ such that

$$|\lambda_0(x,u_1) - \lambda_0(x,u_2)| \leq L_{\lambda_0} |u_1 - u_2| \quad \text{for all} \qquad u_1, u_2 \in \mathbb{R}$$

(H5) The normal compliance function p_ν and the friction bound p_τ satisfy the following hypotheses for $= \nu, \tau$:
a) $p_\delta : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$;
b) $x \in p_\delta(x,u)$ is mesurable on Γ_3 for all $u \in \mathbb{R}$;
c) $x \in p_\delta(x,u) = 0$ for $u \leq 0$ and a.e $x \in \Gamma_3$;
d)there exists $L_{\delta_0} > 0$ such that

$$|p_\delta(.,u) - p_\delta(.,v)| \leq L_\delta |u - v| \quad \text{for all} \qquad u, v \in \mathbb{R}_+$$

.

(H6) The functional j satisfies
$\|\partial j(u(t),v)\|_{V^*} \leq m_j(1 + \|u\|_V + \|v\|_V)$ for all $u \in V, v \in V$ anda.e $t \in (0,T)$ with $m_j \geq 0$.

Using Riesz's representation theorem, we conclude that there exist the elements $f \in V$ and $q_{th} \in Q$ given by

$$(f(t),v)_V = \sum_{\ell=1}^{2} \int_{\Omega^\ell} f_0^\ell(t) \cdot v \; dx + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} f_1^\ell(t) \cdot v \, da \quad \text{for all } v^\ell \in V \tag{3.14}$$

$$(q_{th}(t),\eta)_Q = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t) \cdot \eta^\ell \; dx \quad \text{for all } \eta^\ell \in Q^\ell \tag{3.15}$$

3.2 Formulation variationnelle

Let $(\boldsymbol{u}, \boldsymbol{\sigma}, \theta)$ be a triple of sufficiently regular functions that satisfy (3.1)-(3.9). We use Green's formula (2.5) to obtain:

$$\int_{\Omega^\ell} \boldsymbol{\sigma}^\ell \varepsilon(\boldsymbol{v}^\ell) dx + \int_{\Omega^\ell} Div \boldsymbol{\sigma}^\ell \boldsymbol{v}^\ell dx = \int_{\Gamma_1^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da$$

$$+ \int_{\Gamma_2^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da + \int_{\Gamma_3^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da, \quad \forall \boldsymbol{v}^\ell \in \mathrm{V}^\ell.$$

Green's formula for $\ell = 1$:

$$\int_{\Omega^1} \boldsymbol{\sigma}^1 \varepsilon(\boldsymbol{v}^1) dx + \int_{\Omega^1} Div \boldsymbol{\sigma}^1 \boldsymbol{v}^1 dx = \int_{\Gamma_1^1} \boldsymbol{\sigma}^1 \boldsymbol{\nu}^1 \boldsymbol{v}^1 da$$

$$+ \int_{\Gamma_2^1} \boldsymbol{\sigma}^1 \boldsymbol{\nu}^1 \boldsymbol{v}^1 da + \int_{\Gamma_3^1} \boldsymbol{\sigma}^1 \boldsymbol{\nu}^1 \boldsymbol{v}^1 da, \quad \forall \boldsymbol{v}^1 \in \mathrm{V}^1. \tag{3.16}$$

Green's formula for $\ell = 2$:

$$\int_{\Omega^2} \boldsymbol{\sigma}^2 \varepsilon(\boldsymbol{v}^2) dx + \int_{\Omega^2} Div \boldsymbol{\sigma}^2 \boldsymbol{v}^2 dx = \int_{\Gamma_1^2} \boldsymbol{\sigma}^2 \boldsymbol{\nu}^2 \boldsymbol{v}^2 da$$

$$\int_{\Gamma_2^2} \boldsymbol{\sigma}^2 \boldsymbol{\nu}^2 \boldsymbol{v}^2 da + \int_{\Gamma_3^2} \boldsymbol{\sigma}^2 \boldsymbol{\nu}^2 \boldsymbol{v}^2 da, \quad \forall \boldsymbol{v}^2 \in \mathrm{V}^2. \tag{3.17}$$

to addition (3.16) and (3.17)

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell \varepsilon(\boldsymbol{v}^\ell) dx + \sum_{\ell=1}^{2} \int_{\Omega^\ell} Div \boldsymbol{\sigma}^\ell \boldsymbol{v}^\ell dx = \sum_{\ell=1}^{2} \int_{\Gamma_1^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da$$

$$+ \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da + \sum_{\ell=1}^{2} \int_{\Gamma_3^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da, \quad \forall \boldsymbol{v}^\ell \in \mathrm{V}^\ell.$$

according to (3.2), (3.4) and (3.5) we have:

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell \varepsilon(\boldsymbol{v}^\ell) dx - \sum_{\ell=1}^{2} \int_{\Omega^\ell} f_0^\ell . \boldsymbol{v}^\ell dx = \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} f_2^\ell \boldsymbol{v}^\ell da + \sum_{\ell=1}^{2} \int_{\Gamma_3^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da, \quad \forall \boldsymbol{v}^\ell \in \mathrm{V}^\ell$$

We use (3.14) we find:

$$\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell, \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} = (f(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\ell=1}^{2} \int_{\Gamma_3} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \boldsymbol{v}^\ell da, \quad \forall \boldsymbol{v}^\ell \in \mathrm{V}^\ell. \tag{3.18}$$

We calculate $\sum_{\ell=1}^{2} \int_{\Gamma_3^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell \cdot (\boldsymbol{v}^\ell) da = ?$:

Using relations (1.2) and (1.3) we find

$$\begin{aligned}\sum_{\ell=1}^{2}\int_{\Gamma_3}\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell\boldsymbol{v}^\ell da &= \int_{\Gamma_3}\boldsymbol{\sigma}^1\boldsymbol{\nu}^1\boldsymbol{v}^1 da + \int_{\Gamma_3}\boldsymbol{\sigma}^2\boldsymbol{\nu}^2\boldsymbol{v}^2 da\\ &= \int_{\Gamma_3}\boldsymbol{\sigma}_\tau(\boldsymbol{v}_\tau^1-\boldsymbol{v}_\tau^2)da + \int_{\Gamma_3}\boldsymbol{\sigma}_\nu(\boldsymbol{v}_\nu^1-\boldsymbol{v}_\nu^2)da\\ &= \int_{\Gamma_3}\boldsymbol{\sigma}_\tau([\boldsymbol{v}_\tau])da + \int_{\Gamma_3}\boldsymbol{\sigma}_\nu([\boldsymbol{v}_\nu])da. \qquad (3.19)\end{aligned}$$

We substitute (3.6) into (3.19) to find that

$$\sum_{\ell=1}^{2}\int_{\Gamma_3}\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell\boldsymbol{v}^\ell da = \int_{\Gamma_3}\boldsymbol{\sigma}_\tau([\boldsymbol{v}_\tau])da + \int_{\Gamma_3}(-p_\nu([u_\nu])[v_\nu]da \qquad (3.20)$$

From relation (3.7) we conclude

$$-\boldsymbol{p_\tau}([u_\nu]) \leq \boldsymbol{\sigma}_\tau \leq \boldsymbol{p_\tau}([u_\nu]) \qquad (3.21)$$

We substitute (3.21) into (3.20) to find that

$$\sum_{\ell=1}^{2}\int_{\Gamma_3}\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell\boldsymbol{v}^\ell da \geq \int_{\Gamma_3}(-p_\tau([u_\nu])[\boldsymbol{v}_\tau]da + \int_{\Gamma_3}(-p_\nu([u_\nu])[v_\nu]da \qquad (3.22)$$

according to (3.18) and (3.22) we find:

$$\sum_{\ell=1}^{2}(\boldsymbol{\sigma}^\ell,\varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} \geq (f(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} + \int_{\Gamma_3}(-p_\tau([u_\nu])[\boldsymbol{v}_\tau]da + \int_{\Gamma_3}(-p_\nu([u_\nu])[v_\nu]da \qquad (3.23)$$

according to (3.10) and (3.11) we find

$$\sum_{\ell=1}^{2}(\boldsymbol{\sigma}^\ell,\varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_{nc}(u,v) + j_{fr}(u,v) \geq (f(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \qquad (3.24)$$

Now, for all $t\in[0,T]$ and (3.3), we obtain:

$$({}_0^C D_t^\alpha\theta^\ell(t),\eta^\ell)_{L^2(\Omega^\ell)} - (\kappa_0^\ell\Delta\theta^\ell(t),\eta^\ell)_{L^2(\Omega^\ell)} = \left(q_0^\ell,\eta^\ell\right)_{L^2(\Omega^\ell)}, \qquad \forall\eta^\ell\in\mathbb{L}_1. \quad (3.25)$$

Using Green's formula (2.6) to obtain:

$$-\kappa_0^\ell(\Delta\theta^\ell(t),\eta^\ell)_{L^2(\Omega^\ell)} = \kappa_0^\ell\int_{\Omega^\ell}\nabla\theta^\ell(t)\cdot\nabla\eta^\ell dx - \kappa_0^\ell\int_{\Gamma^\ell}\frac{\partial^\ell\theta^\ell(t)}{\partial\nu^\ell}\ \eta^\ell dx, \quad \forall\eta^\ell\in Q^\ell$$

we have according to (3.8) :

$$-\kappa_0^\ell(\Delta\theta^\ell(t),\eta^\ell)_{L^2(\Omega^\ell)} = \kappa_0^\ell\int_{\Omega^\ell}\nabla\theta^\ell(t)\cdot\nabla\eta^\ell dx + \lambda_0^\ell\int_{\Gamma^\ell}\theta^\ell(t)\ \eta^\ell dx, \quad \forall\eta^\ell\in Q^\ell. \qquad (3.26)$$

We use (3.26) and the equality (3.25), we find:

$$({}_0^C D_t^\alpha \theta^\ell, \eta^\ell)_{L^2(\Omega^\ell)} + \kappa_0^\ell \int_{\Omega^\ell} \nabla\theta^\ell(t) \cdot \nabla\eta^\ell dx + \lambda_0^\ell \int_{\Gamma^\ell} \theta^\ell(t) \cdot \eta^\ell dx$$
$$= \left(q_0^\ell(t), \eta^\ell\right)_{L^2(\Omega^\ell)}, \quad \forall \eta^\ell \in Q^\ell,$$

So :

$$\sum_{\ell=1}^{2} \left({}_0^C D_t^\alpha \theta^\ell(t), \eta^\ell\right)_{L^2(\Omega^\ell)} + \sum_{\ell=1}^{2} \kappa_0^\ell \int_{\Omega^\ell} \nabla\theta^\ell(t) \cdot \nabla\eta^\ell dx + \sum_{\ell=1}^{2} \lambda_0^\ell \int_{\Gamma^\ell} \theta^\ell(t) \cdot \eta^\ell dx$$
$$= \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t) \cdot \eta \, dx, \quad \forall \eta^\ell \in Q^\ell.$$

From (3.12), (3.13) and (3.15), we obtain:

$$\sum_{\ell=1}^{2} \left({}_0^C D_t^\alpha \theta^\ell(t), \eta^\ell\right)_{L^2(\Omega^\ell)} + d(\theta, \eta) + \chi(\theta, \eta) = (q_{th}(t), \eta)_Q \qquad \forall \eta^\ell \in Q^\ell. \quad (3.27)$$

From (3.24) and (3.27) we obtain the variational formulation of Problem $\mathcal{P}_1$.

Problem $\mathcal{PV}$. Find the displacement fields $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2) : [0, T] \to V$, the stress fields $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0, T] \to \mathcal{H}$, the temperature fields $\theta = (\theta^1, \theta^2) : [0, T] \to Q$ $t \in]0; T[$, $v \in V$ and $\alpha \in]0, 1[$, we have

$$\begin{aligned} &c({}_0^C D_t^\alpha u(t), v - u(t)) + a(u(t), v - u(t)), -m(\theta(t), v - u(t)) \\ &+ j_{nc}(u(t), v - u(t)) + j_{fr}(u(t), v - u(t)) \geq (f(t), v - u(t)))_V, \end{aligned} \quad (3.28)$$

$$d(\theta(t), \eta) + ({}_0^C D_t^\alpha \theta(t), \eta)_Q + \chi(\theta(t), \eta) = (q_{th}(t), \eta)_Q, \quad \forall \eta \in Q, \quad (3.29)$$

$$u(0, x) = u_0(x), \quad \theta(0, x) = \theta_0(x). \quad (3.30)$$

The solvability of the problem $\mathcal{PV}$ is stated in the following theorem.

Theorem 3.2.1 *Assume that* H(1)–H(6), (3.10)-(3.13) *and*

$$m_a > C_d^2(L_\nu + L_\tau) \qquad \text{and} \qquad m_d > M_{\lambda_0} c_t^2 \quad (3.31)$$

hold. Then problem $\mathcal{PV}$ has a unique solution

$$u \in W^{1,2}(0, T; V), \quad \theta \in W^{1,2}(0, T; Q). \quad (3.32)$$

3.3 Proof of main results

3.3.1 Proof of Theorem 3.2.1

The argument of the monotone operator, the Caputo derivative, the Clarke subdifferential, the Galerkin technique, and the Banach fixed point theorem form the basis of the multi-step proof.

First, Let $\beta \in L^2(0,T;V)$ given by

$$\big(\beta(t), v - u_\beta(t)\big)_V = m\big(\theta_\beta(t), v - u_\beta(t)\big). \tag{3.33}$$

and we consider the following problem.

Problem (PV^{dp})

Find $u_\beta : \Omega\times]0,T[\to \mathbb{R}^d$ such that for a.e. $t \in]0,T[$, $v \in V$, $v \in V$, and $\alpha \in]0,1[$, we have

$$c({}_0^C D_t^\alpha u(t), v - u_\beta(t)) + a(u_\beta(t), v - u_\beta(t)) + (\beta(t), v - u_\beta(t))_V$$
$$+j(u_\beta(t), v) - j(u_\beta(t), u_\beta(t) \geq (f(t), v - u_\beta(t)))_V, \tag{3.34}$$
$$u_\beta(0) = u_0. \tag{3.35}$$

we have the following result.

Lemma 3.3.1 *For all $v \in V$ and for a.e. $t \in]0,T[$, the Problem*(PV^{dp}) *has a unique solution $u_\beta \in W^{1,2}(0,T;V)$.*

Proof. using Riesz's representation theorem, we define the operator representation theorem.

$$(f_\beta(t), v)_V = (f(t), v)_V - (\beta(t), v)_V. \tag{3.36}$$

The Problem (PV^{dp}) can be written

$$c({}_0^C D_t^\alpha u_\beta(t), v - u_\beta(t)) + a(u_\beta(t), v - u_\beta(t))$$
$$+j(u_\beta(t), v - u_\beta(t)) - j(u_\beta(t), u_\beta(t)) \geq (f_\beta(t), v - u_\beta(t)))_V, \tag{3.37}$$
$$u_\beta(0) = u_0.$$

It is clear that the operator c is bilinear continuous and coercive under assumption H(1).

Operator a is continuous and bilinear according to assumptions H(1)a) and H(2).

From assumption H(3), (3.14), (3.36), and the regularity of β, we have that $f_\beta \in L^2(0,T;V)$.

It is evident that j is a locally Lipschitz function from assumption H(5)d).

Using the conclusion from Theorem 19 in (31) and the results of the operators c, a, j, and the function f_β with assumption H(6), we determine that problem (PV^{dp}) has at least one solution, $u_\beta \in W^{1,2}(0,T;V)$.

Here and below c_1, c_2 and c_s denote positive generic constants whose values may change from line to line. ■

The second step is to obtain the following existence and uniqueness conclusion for the temperature field θ_β of the following problem using the displacement field u_β found in Lemma 3.3.1.

Problem (PVth)

Find a temperature field $\theta_\beta : \Omega \times [0,T] \to \mathbb{R}$ such that for a.e $t \in]0,T[, \eta \in Q$, and $\alpha \in]0,1[$, we have

$$d(\theta_\beta(t),\eta) + ({}_0^C D_t^\alpha \theta_\beta(t),\eta)_Q + \chi(\theta_\beta(t),\eta) = (q_{th}(t),\eta)_Q, \quad \forall \eta \in Q, \tag{3.38}$$

$$\theta_\beta(0) = \theta_0. \tag{3.39}$$

Lemma 3.3.2 *For all $\eta \in Q$ and a.e. $t \in]0,T[$, the Problem* (PVth) *has a unique solution $\theta_\beta \in W^{1,2}(0,T;Q)$.*

Proof. We apply the Faedo-Galerkin techniques to demonstrate the above lemma.

For this, we assume the functions $w_k = w_k(t)$, $k = 1,\ldots,m$ consisting of eigenfunctions of $-\Delta$ are smooth

$$[w_k]_{k=1}^{\infty} \text{ is a orthonormal basis of } H^1(\Omega). \tag{3.40}$$

Fix now a positive integer m, we will look for a function $\theta_{z_m} :]0,T[\to H^1(\Omega)$ of the form

$$\theta_{\beta_n} := \sum_{i=1}^{n} x_n^i(t) w_i, \tag{3.41}$$

We denote by F_m the vector space generated by $w_1, w_2, ..., w_m$.

Whence $\theta_{\beta_n} \in F_m$ and $\theta_{\beta_n} \to \theta_n$ in Q.

For each integer

$$n \geq 1$$

consider the following approximate problem: Find $\theta_{\beta_n} \in L^2(0,T;F_n)$ such that ${}_0^C D_t^\alpha \theta_{\beta_n} \in L^2(0,T;F_n)$ and

$$d\big(\theta_{\beta_n}(t), w_k\big) + \big({}_0^C D_t^\alpha \theta_{\beta_n}(t), w_k\big)_Q + \chi\big(u_\beta(t), \theta_{\beta_n}(t), w_k\big) = (q_{th}(t), w_k)_Q, \quad (3.42)$$

$$\theta_{\beta_n}(0) = \big(\theta_0, w_k\big), \quad (k = 1, \ldots, n). \quad (3.43)$$

Using(3.41), we have

$$({}_0^C D_t^\alpha \theta_{\beta_n}(t), w_k)_Q = {}_0^C D_t^\alpha x_n^i(t), \quad (3.44)$$

$$d(\theta_{\beta_n}(t), w_k)_Q = \mathcal{K} x_n^i(t), \quad (3.45)$$

$$\chi(\theta_{\beta_n}(t), w_k) = \chi\big(\sum_{k=i}^{n} x_n^i(t) w_i, w_k\big), \quad (3.46)$$

$$\big(q_{th}(t), w_k\big)_Q = q_{th}^k(t). \quad (3.47)$$

Then (3.42)-(3.43) can be written as follows

$${}_0^C D_t^\alpha x_n^i(t) = h\big(t, x_n^i(t)\big). \quad (3.48)$$

$$x_n^i(0) = (\theta_0, w_i). \quad (3.49)$$

We pose

$$h\big(t, x_n^i(t)\big) = q_{th}^k(t) - \mathcal{K} x_n^i(t) - \chi\Big(\sum_{i=1}^{n} x_n^i(t) w_i, w_k\Big). \quad (3.50)$$

Considering Hypothesis (H2), we find

$$|\mathcal{K} x_{n1}^i(t) - \mathcal{K} x_{n2}^i(t)| \leq M_d \left| x_{n1}^i(t) - x_{n2}^i(t) \right|, \quad (3.51)$$

and by (3.13) , Hypothesis (H4), (H5), we have that

$$\begin{aligned} &\Big|\chi\Big(\sum_{i=1}^{n} x_{n2}^i(t) w_i, w_k\Big) - \chi\Big(\sum_{i=1}^{n} x_{n1}^i(t) w_i, w_k\Big)\Big| \\ &\leq M_{\lambda_0} L_{\lambda_0} \operatorname{meas}(\Gamma_3) |x_{n_1}^i(t) - x_{n_2}^i(t)|. \end{aligned} \quad (3.52)$$

This inequality, when combined with (3.50)–(3.51), reveals the existence of a positive constant c_s such that

$$\big|h\big(t, x_{n2}^i(t)\big) - h\big(t, x_{n1}^i(t)\big)\big| \leq c_s |x_{n_2}^i - x_{n_1}^i|. \quad (3.53)$$

Then, according to a typical approach for fractional calculus (see Proposition 4.6 in (29)), the system of fractional ordinary differential equation (3.48)–(3.49) is satisfied by a unique absolutely continuous function $x_n(t) = (x_n^1(t), \dots, x_n^n(t))$ on $[0, T_*)$. Estimates: Multiply (3.42) by $x_n^i(t)$, sum for $i = 1, \dots, n$ and the fact that $\theta_{\beta_n} \mapsto \frac{1}{2}\|\theta_{\beta_n}\|_Q^2$ is a convex functional to obtain

$$ {}_0^C D_t^\alpha\Big(\frac{1}{2}(\|\theta_{\beta_n}\|_Q^2\Big) \le ({}_0^C D_t^\alpha \theta_{\beta_n}, \theta_{\beta_n}) + d(\theta_{\beta_n}, \theta_{\beta_n}) + \chi(\theta_{\beta_n}, \theta_{\beta_n}) = (q_{th}, \theta_{\beta_n}). \quad (3.54)$$

After some calculus, for all $\epsilon > 0$, we have

$$n_d\|\theta_{\beta_n}(t)\|_Q^2 \le |d(\theta_{\beta_n}, \theta_{\beta_n})|, \quad (3.55)$$

$$|\chi(u_\beta, \theta_{\beta_n}, \theta_{\beta_n})| \le \frac{M_{\lambda_0}^2 M_L^2}{2\epsilon} + \frac{\epsilon c_t^2}{2}\|\theta_{\beta_n}\|_Q^2, \quad (3.56)$$

$$\big|(q_{th}, \theta_{\beta_n}\big)\|_Q \le \frac{1}{2\epsilon}\|q_{th}\|_Q^2 + \frac{\epsilon}{2}\|\theta_{\beta_n}\|_Q^2, \quad (3.57)$$

So,

$$ {}_0^C D_t^\alpha\Big(\frac{1}{2}\|\theta_{\beta_n}\|_Q^2\Big) + c_1\|\theta_{\beta_n}\|_Q^2 \le c_2\|q_{th}\|_Q^2 \quad (3.58)$$

Applying proposition2.1.1(ii) to inequality(3.58), we obtain

$$\|\theta_{\beta_n}\|_Q^2 + \frac{2c_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\theta_{\beta_n}\|_Q^2 ds \le c_s(\|q_t\|_Q^2 + \|\theta_0\|_Q^2). \quad (3.59)$$

Consequently, we find that $T_* = +\infty$.
Let $\eta \in Q$, with $\|\eta\|_Q \le 1$, and write $\eta = \eta_1 + \eta_2$, where $\eta_1 \in spam\{w_k\}_{k=1}^m$ are orthogonal in Q,

$$\|\eta_1\|_Q \le \|\eta\|_Q \le 1. \quad (3.60)$$

Using (3.42), we conclude that

$$({}_0^D C_t^\alpha \theta_{\beta_n}, \eta_1) + d(\theta_{\beta_n}, \eta_1) + \chi(\theta_{\beta_n}, \eta_1) = (q_t, \eta_1). \quad (3.61)$$

Similarly to (3.55)–(3.57), we have

$$|d(\theta_{\beta_n}, \eta_1)| \le M_d\|\theta_{\beta_n}(t)\|_Q \quad (3.62)$$

$$|\chi(\theta_{\beta_n}, \eta_1)| \le M_{\lambda_0} c_t, \quad (3.63)$$

$$|(q_t, \eta_1)| \le \|q_{th}\|_Q. \quad (3.64)$$

Thus,

$$\|{}_0^C D_t^\alpha \theta_{\beta_n}\|_{Q^*} \leq c_1 + c_2\|\theta_{\beta_n}\|_Q + \|q_{th}\|_Q. \tag{3.65}$$

A positive constant c_s exists according to inequality (3.59), such that

$$\|{}_0^c D_t^\alpha \theta_{\beta_n}\|_{L^2(0,T,Q^*)} \leq c_s \tag{3.66}$$

Passage to the limite: Let $\{\tau_n\}$ be a sequence such that $\tau_n \to 0$, as $n \to \infty$. Using the previous estimates and applying the compactness result (see Theorem 4.2 in (12)) for the Caputo derivative, there exists a subsequence $\theta_{\beta_{\tau n}}$ and $\theta_\beta \in L^2(0,T;Q)$ such that

$$\theta_{\beta_{\tau n}} \to \theta_\beta \quad \text{strongly in } L^2(0,T;Q) \tag{3.67}$$

and

$${}_0^C D_t^\alpha \theta_{\beta_{\tau n}} \rightharpoonup {}_0^C D_t^\alpha \theta_\beta \text{ weakly in } L^2(0,T;Q^*) \tag{3.68}$$

Then

$$d(\theta_{\beta_{\tau n}}, \eta) \to d(\theta_\beta, \eta) \quad \text{in} \quad \mathbb{R} \tag{3.69}$$

$$({}_0^C D_t^\alpha \theta_{\beta_{\tau n}}, \eta) \to d({}_0^C D_t^\alpha \theta_\beta, \eta) \quad \text{in} \quad \mathbb{R} \tag{3.70}$$

assumption H(4) and (3.13) yield

$$|\theta_{\beta_n}, \eta)| \leq M_{\lambda_0} L_{\lambda_0} \|\eta\|_{L^2(\Gamma_3)}. \tag{3.71}$$

Since $\{\chi(\theta_{\beta_n}, \eta)\}_{n=1}^\infty$ is bounded in $\mathbb{R}$, we may pass to a subsequence if it is necessary. For $\eta = \theta_\beta - \theta_{\beta_{\tau n}}$, by using (3.13) we have

$$|\chi(\theta_\beta, \theta_\beta - \theta_{\beta_{\tau n}}) - \chi(\theta_{\beta_n}, \theta_\beta - \theta_{\beta_{\tau n}})| \leq c_t^2 M_{\lambda_0} L_{\lambda_0} \|\theta_\beta - \theta_{\beta_{\tau n}}\|_Q^2. \tag{3.72}$$

Since trace $\gamma : Q \to L^2(\Gamma_3)$ is compact, the weak convergence of $\theta_{\beta_{\tau n}}$ indicates that

$$\theta_{\beta_{\tau n}} \to \theta_\beta \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.73}$$

Then

$$\chi(\theta_{\beta_{\tau n}}, \eta) \to \chi(\theta_\beta, \eta) \quad \text{in} \quad \mathbb{R}. \tag{3.74}$$

The lemma is proved. ■

The operator $\Pi : L^2(0,T;V) \to L^2(0,T;V)$ is considered in the last step for the functions $\beta \in L^2(0,T;V)$ and θ_β obtained in Lemma 3.3.2, defined by

$$(\Pi\beta(t), v)_V = m(\theta_\beta(t), v), \tag{3.75}$$

for all $v \in V$ and for a.e. $t \in]0,T[$.

We have the following lemma.

Lemma 3.3.3 *There exists a unique $\tilde{\beta} \in L^2(0,T;V)$ such that $\Pi\tilde{\beta} = \tilde{\beta}$.*

Proof. Let $\beta \in \ell^2(0,T;V)$ and $t_1, t_2 \in]0,T[$. By using (3.33) and (H2), we find that

$$\|\Pi\beta(t_1) - \Pi\beta(t_2)\|_V \leq c_s\big(\|\theta_\beta(t_1) - \theta_\beta(t_2)\|_Q\big). \tag{3.76}$$

Since $\theta_\beta \in L^2(0,T;Q)$, we deduce that $\Pi\beta \in L^2(0,T;V)$.

Now let $\beta_1, \beta_2 \in L^2(0,T;V)$. Similarly to (3.76), we get

$$\|\Pi\beta_1(t) - \Pi\beta_2(t)\|_V \leq c_s\big(\|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_Q\big). \tag{3.77}$$

therefore , from (3.34), we obtain

$$\begin{aligned} c({}_0^C D_t^\alpha u_{\beta_1}(t) - {}_0^C D_t^\alpha u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\ + a(u_{\beta_1}(t) - u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\ + j(u_{\beta_1}(t), u_{\beta_1}(t)) - j(u_{\beta_1}(t), u_{\beta_2}(t)) \\ + j(u_{\beta_2}(t), u_{\beta_2}(t)) - j(u_{\beta_2}(t), u_{\beta_1}(t)) \leq 0 \end{aligned} \tag{3.78}$$

From (3.13) and H(5), we have

$$\begin{aligned} |j(u_{\beta_1}(t), u_{\beta_1}(t)) - j(u_{\beta_1}(t), u_{\beta_2}(t)) + j(u_{\beta_2}(t), u_{\beta_2}(t)) \\ - j(u_{\beta_2}(t), u_{\beta_1}(t))| \leq c_d^2(L_\tau + L_\nu)\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \end{aligned} \tag{3.79}$$

By Definition2.1.2, we deduce

$$\|{}_0^C D_t^\alpha u_{\beta_1}(t) - {}_0^C D_t^\alpha u_{\beta_2}(t)\|_V \leq \frac{T^{1-\alpha}}{\Gamma(\alpha)}\|\dot{u}_{\beta_1}(t) - \dot{u}_{\beta_2}(t)\|_V \tag{3.80}$$

Combining (3.78)–(3.80), integrating from 0 to t and using the Gronwall inequality, we conclude that there exists $c_s > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \leq c_s \|\beta_1(t) - \beta_2(t)\|_V \tag{3.81}$$

with the condition$m_a > c_d^2(L_\nu + L_\tau)$. Using (3.38), we have

$$\begin{aligned}({}_0^C D_t^\alpha \theta_{\beta_1}(t) -{}_0^C D_t^\alpha \theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t))_Q + d(\theta_{\beta_1}(t) - \theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) \\ +\chi(\theta_{\beta_1}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) - \chi(\theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) = 0\end{aligned}$$

By (3.13) and (H5), we conclude

In the same way as above,after some calculations we get that there exists $c_s > 0$ such that

$$\|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_{L^2(0,T;Q)} \leq c_s \|\beta_1(t) - \beta_2(t)\|_{L^2(0,T;V)} \tag{3.82}$$

with the condition $m_d > M_{\lambda_0} c_t^2$. Combining (3.77), (3.81)) and (3.82), we obtain

$$\|\Pi\beta_1(t) - \Pi\beta_2(t)\|_{L^2(0,T;V)} \leq c_s \|\beta_1(t) - \beta_2(t)\|_{L^2(0,T;Q)} \tag{3.83}$$

Reiterating this inequality n times leads to

$$\|\Pi^n\beta_1(t) - \Pi^n\beta_2(t)\|_{L^2(0,T;V)} \leq \frac{(c_s)^n}{n!} \|\beta_1(t) - \beta_2(t)\|_{L^2(0,T;V)} \tag{3.84}$$

which implies that for$n$ sufficiently large the power Π^n of Π is a contraction in $L2(0T; V)$. Therefore, there exists a unique element $L^2(0, T; V)$ such that $\Pi\tilde{\beta} = \tilde{\beta}$ ■

We are now ready to prove Theorem3.2.1.

Proof. Proof of theorem3.2.1. let $\tilde{\beta} \in L^2(0, T; V)$ be a fixed point of the operator Π. Denote by $u_{\tilde{\beta}}$ a solution of Problem(PV^{dp}) and let $\theta_{\tilde{\beta}}$ be a solution of Problem(PVth) for $\beta = \tilde{\beta}$. Using the definition of Π, (3.33)–(3.34) and (3.38)–(3.39), we find that $(u_{\tilde{\beta}}, \theta_{\tilde{\beta}})$ is a solution of Problem $\mathcal{PV}$. ■

Conclusion

In this memorandum, we have attempted to present a novel work, so to speak, consisting of a theoretical study of the problem of contact between two mechanical bodies, where the behavior of the two bodies involves a fractional derivative related to time.

We have attempted to study existence and unicity of weak solution. Work in this area is still open to future researchers, who may present or propose new models. Similarly, the same problems and models can be studied by approximating numerical solutions or numerical methods.

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الملخص

في هذه المذكرة، درسنا نموذجًا رياضيًا في عملية شبه ساكنة لمسألة تماس ذات احتكاك بين جسمين حراريين لزجين مرنين. في هذه المسألة، يُنمذج التلامس بالاحتكاك.
باستخدام صيغة غرين، نقترح صياغة متغيرة للمسألة الحرارية الميكانيكية، ثم نعرض نتيجة وجود ووحدة الحل الضعيف.
تستند البراهين إلى ومشتقة كابوتو، وتقنية غاليركين، و نظرية النقطة الثابتة لبناخ.

الكلمات المفتاحية: الاحتكاك, الحل الضعيف, النقطة الثابتة مشتق كابوتو.

Summary

In this memory, we studied a mathematical model in a quasistatic process of a contact problem with normal compliance and friction between two thermo-viscoelastic bodies, For this problem, the contact is modeled by friction. Using Green's formula, we

propose a variational formulation of the thermo-mechanical problem and then present an existence and uniqueness result of the solution. The proofs are based on the monotone operator argument, the Caputo derivative, the Clarke subdifferential, the Galerkin technique, and the Banach fixed point.

Keywords: friction, weak solution, fixed point, Caputo derivative.

Résumé

Dans ce mémoire, nous avons étudié un modèle mathématique, dans un processus quasi-statique, d'un problème de contact avec compliance normale et frottement entre deux corps thermo-viscoélastiques. Pour ce problème, le contact est modélisé par frottement. En utilisant la formule de Green, nous proposons une formulation variationnelle du problème thermo-mécanique, puis présentons un résultat d'existence et d'unicité de la solution. Les preuves reposent sur l'argument de l'opérateur monotone, la dérivée de Caputo, la sous-différentielle de Clarke, la technique de Galerkin et le point fixe de Banach.

Mots-clés : frottement, solution faible, point fixe, dérivée de Caputo.